Problems and Solutions Section 5.4 (5.37 through 5.52)

5.37 A machine, largely made of aluminum, is modeled as a simple mass (of 100 kg) attached to ground through a spring of 2000 N/m. The machine is subjected to a 100-N harmonic force at 20 rad/s. Design an undamped tuned absorber system (i.e., calculate m_a and k_a) so that the machine is stationary at steady state. Aluminum, of course, is not completely undamped and has internal damping that gives rise to a damping ratio of about $\zeta = 0.001$. Similarly, the steel spring for the absorber gives rise to internal damping of about $\zeta_a = 0.0015$. Calculate how much this spoils the absorber design by determining the magnitude X using equation (5.32).

Solution:

From equation (5.21), the steady-state vibration will be zero when

$$\omega^2 = \frac{k_a}{m_a}$$

Choosing $\mu = 0.2$ yields

$$m_a = \mu m = (0.2)(100) = 20 \text{ kg}$$
$$\mathrm{k_a} = m_a \omega_a^2 = (20)(20)^2 = 8000 \text{ N/m}$$

With damping of $\zeta = 0.001$ and $\zeta_a = 0.0015$, the values of c and c_a are

$$c = 2\zeta\sqrt{km} = 2(0.001)\sqrt{(2000)(100)} = 0.894 \text{ kg/s}$$
$$c_a = 2\zeta_a\sqrt{k_a m_a} = 2(0.0015)\sqrt{(8000)(20)} = 1.2 \text{ kg/s}$$

From equation (5.32),

$$X = \frac{(k_a - m_a\omega^2)F_0 + c_a\omega F_0 j}{\det(K - \omega^2 M + \omega jC)}$$

Since

$$M = \begin{bmatrix} 100 & 0 \\ 0 & 20 \end{bmatrix} \quad C = \begin{bmatrix} 2.0944 & -1.2 \\ -1.2 & 1.2 \end{bmatrix} \quad K = \begin{bmatrix} 10,000 & -8000 \\ -8000 & 8000 \end{bmatrix}$$

the denominator is -6.4×10^7-$1.104\times10^6 j$, so the value of X is

$$X = \frac{\left(k_a m_a \omega^2\right)\left(F_0 + c_a \omega F_0 j\right)}{\det\left(K - \omega^2 M + \omega jC\right)}$$

Using Window 5.4, the magnitude is

$$\left|X\right| = 3.75 \times 10^{-5} \text{ m}$$

This is a very small displacement, so the addition of internal damping will not affect the design very much.

5.38 Plot the magnitude of the primary system calculated in Problem 5.37 with and without the internal damping. Discuss how the damping affects the bandwidth and performance of the absorber designed without knowledge of internal damping.

Solution: From Problem 5.37, the values are

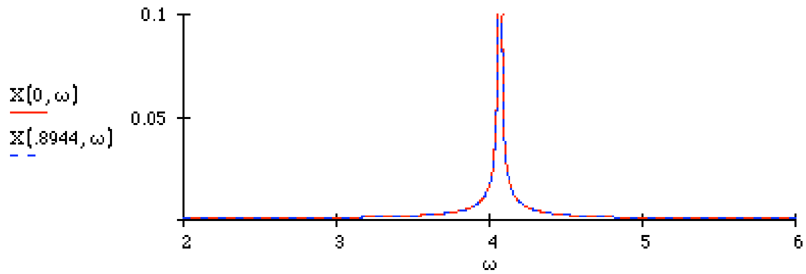
$$\begin{aligned} &m = 100 \text{ kg} && m_a = 20 \text{ kg} \\ &c = 0.8944 \text{ kg/s} && c_a = 1.2 \text{ kg/s} \\ &k = 2000 \text{ N/m} && k_a = 8000 \text{ N/m} \\ &F_0 = 100 \text{ N} && \omega = 20 \text{ rad/s} \end{aligned}$$

Using Equation (5.32), the magnitude of X is plotted versus ω with and without the internal damping (c). Note that X is reduced when $X < F_0/k = 0.05$ m and magnified when $X > 0.05$ m. The plots of the two values of X show that there is no observable difference when internal damping is added. In this case, knowledge of internal damping is not necessary.

$m := 100$ $\quad ma := 20$ $\quad ca := 1.2$ $\quad k := 2000$

$ka := 8000$ $\quad F0 := 100$

$$X(c,\omega) := \sqrt{\frac{\left(ka - ma\cdot\omega^2\right)\cdot F0^2 + \left[ca\cdot\left(\omega\cdot F0\right)\right]^2}{\left[\left(k - m\cdot\omega^2\right)\cdot\left(ka - ma\cdot\omega^2\right) - (ma\cdot ka + ca\cdot c)\cdot\omega^2\right]^2 + \left[ka\cdot c + k\,ca - \left[ca\cdot(m + ma) + c\cdot ma\right]\cdot\omega^2\right]^2\cdot\omega^2}}$$



5.39 Derive Equation (5.35) for the damped absorber from Eqs. (5.34) and (5.32) along with Window 5.4. Also derive the nondimensional form of Equation (5.37) from Equation (5.35). Note the definition of ζ given in Equation (5.36) is not the same as the ζ values used in Problems 5.37 and 5.38.

Solution:

Substituting Equation (5.34) into the denominator of Equation (5.32) yields

$$\frac{X}{F_0} = \frac{\left(k_a - m_a\omega^2\right) + c_a\omega j}{\left[\left(-m\omega^2 + k\right)\left(-m_a\omega^2 + k_a\right)\right] + \left[\left(k - \left(m + m_a\right)\omega^2\right)c_a\omega\right]j}$$

Referring to Window 5.4, the value of $\left|\frac{X}{F_0}\right|$ can be found by noting that

$$\begin{aligned} A_1 &= k_a - m_a\omega \\ B_1 &= c_a\omega \\ A_2 &= \left(-m\omega^2 + k\right)\left(-m_a\omega^2 + k_a\right) - m_a k_a\omega^2 \\ B_2 &= \left(k - \left(m + m_a\right)\omega^2\right)c_a\omega \end{aligned}$$

Since

$$\left|\frac{X}{F_0}\right| = \sqrt{\frac{A_1^2 + B_1^2}{A_2^2 + B_2^2}}$$

then

$$\frac{X^2}{F_0^2} = \frac{\left(k_a - m_a\omega^2\right)^2 + c_a^2\omega^2}{\left[\left(-m\omega^2 + k\right)\left(-m_a\omega^2 + k_a\right) - m_a k_a\omega^2\right]^2 + \left[k - \left(m + m_a\right)\omega^2\right]^2 c_a^2\omega 2}$$

which is Equation (5.35)

To derive Equation (5.37), substitute $c_a = 2\zeta m_a\omega_p, k_a = m_a\omega_a^2$, and $m_a = \mu m$, then multiply by k^2 to get

$$\frac{X^2k^2}{F_0^2} = \frac{k^2\left(\omega_a^2 - \omega^2\right)^2 + 4\zeta^2\omega_p^2\omega_{dr}^2k^2}{\left[\left(k - m\omega^2\right)\left(\omega_a^2 - \omega^2\right) - \mu m_a^2\omega^2\right]^2 + \left[k - \left(1 - \mu\right)m\omega^2\right]^2\left(4\right)\zeta^2\omega_p^2\omega^2}$$

Substituting $k = m\omega_p^2, \omega = r\omega_p$, and $\omega_a = \beta\omega_p$ yields

$$\frac{X^2k^2}{F_0^2} = \frac{m^2\omega_p^4\left(\beta^2\omega_p^2 - r^2\omega_p^2\right) + 4\zeta\ \omega_p^2\omega_{dr}^2k^2}{\left[\left(\omega_p^2 - r^2\omega_p^2\right)\left(\beta^2\omega_p^2 - r^2\omega_p^2\right)m - \mu m\beta^2r^2\omega_p^4\right]^2 + \left[m\omega_p^2 - \left(1 - \mu\right)mr^2\omega_p^2\right]^2\left(4\right)\zeta^2r^2\omega_p^2}$$

Canceling m^2 and ω_p^8 yields

$$\frac{X^2k^2}{F_0^2} = \frac{\left(\beta^2 - r^2\right)^2 + \left(2\zeta r\right)^2}{\left[\left(1 - r^2\right)\left(\beta^2 - r^2\right) - \mu r^2\beta^2\right]^2 + \left(2\zeta r\right)^2\left(1 - r^2 - \mu r^2\right)^2}$$

Rearranging and taking the square root gives the form of Equation (5.37):

$$\frac{Xk}{F_0} = \sqrt{\frac{\left(2\zeta r\right)^2 + \left(r^2 - \beta^2\right)^2}{\left(2\zeta r\right)^2\left(r^2 - 1 + \mu r^2\right)^2 + \left[\mu r^2\beta^2 - \left(r^2 - 1\right)\left(r^2 - \beta^2\right)\right]^2}}$$

5.40 (Project) If you have a three-dimensional graphics routine available, plot Equation (5.37) [i.e., plot (X/Δ) versus both r and ζ for $0 < \zeta < 1$ and $0 < r < 3$, and a fixed μ and β.] Discuss the nature of your results. Does this plot indicate any obvious design choices? How does it compare to the information obtained by the series of plots given in Figures 5.19 to 5.21? (Three-dimensional plots such as these are becoming commonplace and have not yet been taken advantage of fully in vibration absorber design.)

Solution: To compare to Figure 5.18, the values $\mu = 0.25$ and $\beta = 0.8$ in Equation (5.37) yield

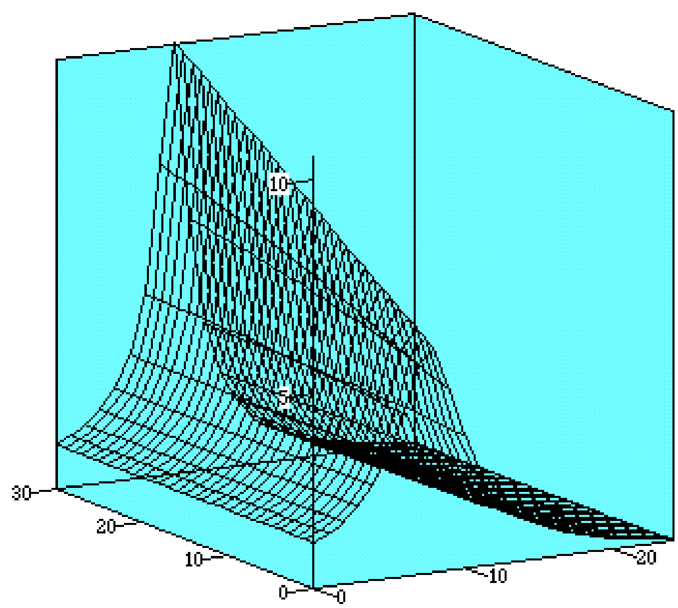
$$\frac{X}{\Delta} = \sqrt{\frac{(2\zeta r)^2 + (r^2 - 0.64)^2}{(2\zeta r)^2 (1.25r^2 - 1)^2 + \left[0.16r^2 - (r^2 - 1)(r^2 - 0.64)\right]^2}}$$

This is plotted for $0.5 < r < 2$ and $0.5 < \zeta < 1$. A Mathcad plot is given.

$N := 30 \quad i := 0 .. N\cdot .8 \quad j := 0 .. N\cdot 1 \quad r_i := 0.5 + i\cdot 0.05 \quad \zeta_j := 0.5 + j\cdot 0.015$

$$X(r,\zeta) := \sqrt{\frac{(2\cdot\zeta\cdot r)^2 + (r^2 - 0.64)^2}{(2\cdot r\cdot\zeta)^2\cdot(1.25\cdot r^2 - 1^2)^2 + \left[0.16\cdot r^2 - (r^2 - 1)\cdot(r^2 - 0.64)\right]^2}}$$

$M_{[i,j]} := X(r_i, \zeta_j)$



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This supplies much more information than two-dimensional plots.

5.41 Repeat Problem 5.40 by plotting $\left|X/\Delta\right|$ versus r and β for a fixed ζ and μ.

Solution: Using Equation (5.37) with μ = 0.25 and ζ 0.1 yields

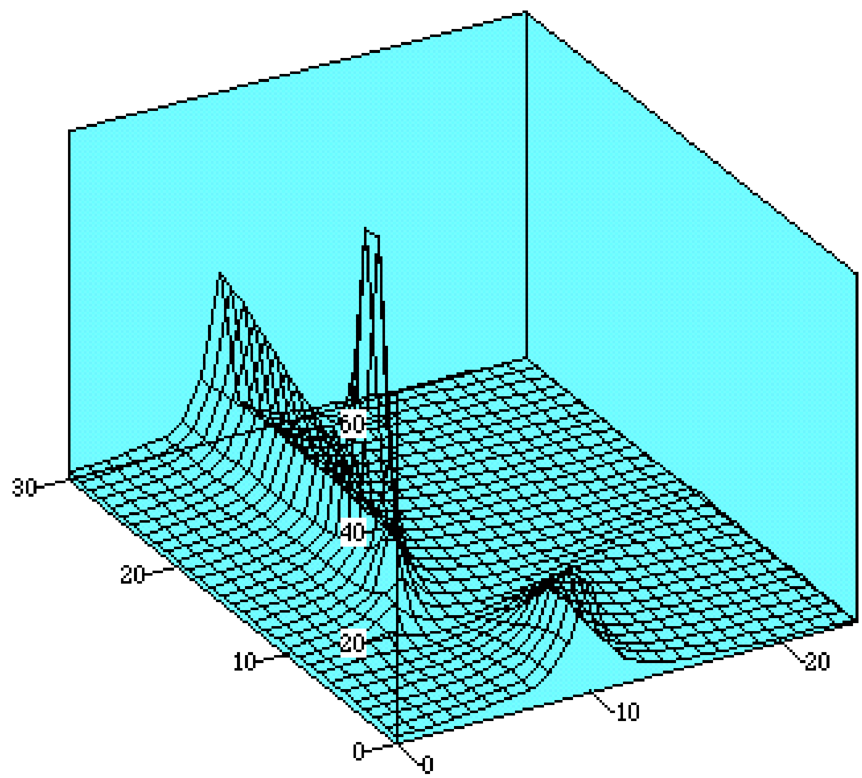
$$\frac{X}{\Delta}=\sqrt{\frac{0.04r^2+\left(r^2-\beta^2\right)^2}{0.04r^2\left(1.25r^2-1\right)^2+\left[0.25r^2\beta^2-\left(r^2-1\right)\left(r^2-\beta^2\right)\right]^2}}$$

This is plotted for 0.5 < r < 1.25 and 0 < β < 3.

$N := 30 \quad i := 0..N\cdot .8 \quad j := 0..N\cdot 1 \quad r_i := 0.5 + i\cdot 0.05 \quad \beta_j := 0 + j\cdot 0.1$

$$X(r,\beta) := \sqrt{\frac{0.04\cdot r^2+\left(r^2-\beta^2\right)^2}{0.04\cdot r^2\cdot\left(1.25\cdot r^2-1^2\right)^2+\left[0.25\cdot r^2\cdot\beta^2-\left(r^2-1\right)\cdot\left(r^2-\beta^2\right)\right]^2}}$$

$M_{[i,j]} := X\left(r_i,\beta_j\right)$



5.42 (Project) The full damped vibration absorber equations (5.32) and (5.33) have not historically been used in absorber design because of the complicated nature of the complex arithmetic involved. However, if you have a symbolic manipulation code available to you, calculate an expression for the magnitude X by using the code to calculate the magnitude and phase of Equation (5.32). Apply your results to the absorber design indicated in Problem 5.37 by using m_a, k_a and ζ_a as design variables (i.e., design the absorber).

Solution:

Equation (5.32):

$$X = \frac{\left(k_a - m_a\omega^2\right)F_0 + c_a\omega F_0}{\det\left(K - \omega^2 M + \omega jC\right)}$$

where M, C and K are defined above Equation (5.32).

Using Equation (5.34) for the denominator, then calculating the magnitude yields

$$\left|X\right| = \sqrt{\frac{\left(k_a - m_a\omega^2\right)F_0^2 + c_a^2\omega^2 F_0^2}{\left[\left(k - m\omega^2\right)\left(k_a - m_a\omega^2\right) - \left(m_a k_a + c_a c\right)\omega^2\right]^2 + \left[k_a c + kc_a - \left(c_a\left(m + m_a\right) + cm_a\right)\omega^2\right]^2\omega^2}}$$

The phase is

$$\phi = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

where the imaginary part, denoted Im, is

$$\mathrm{Im} = -ck_a^2 l + \left(2k_a m_a - 2k_a km_a - k_a^2 m_a\right)\omega^2$$

and the real part, denoted Re, is

$$\mathrm{Re} = k_a^2 k + \left(c_a^2 - k_a^2 m - 2k_a km_a - k_a^2 m_a\right)\omega^2$$
$$+\left(\left(k + k_a\right)m_a^2 + 2k_a mm_a - c_a^2\left(m + m_a\right)\right)\omega^4 - mm_a^2\omega^6$$

From Problem 5.37 and its solution, the values are

m = 100 kg	m_a = 20 kg
c = 0.8944 kg/s	c_a = 1.2 kg/s
k = 2000 N/m	k_a = 8000 N/m
F_0 = 100 N	ω = 20 rad/s

Substituting these values into the magnitude equation yields

$$\left|X\right| = 3.75 \times 10^{-5}\ \text{m}$$

This is the same result as given in Problem 5.37.

5.43 A machine of mass 200 kg is driven harmonically by a 100-N force at 10 rad/s. The stiffness of the machine is 20,000 N/m. Design a broadband vibration absorber [i.e., Equation (5.37)] to limit the machine's motion as much as possible over the frequency range 8 to 12 rad/s. Note that other physical constraints limit the added absorber mass to be at most 50 kg.

Solution:

Since $\omega_p = \sqrt{\dfrac{k}{m}}$ = 10 rad/s, then r ranges from

$$\frac{8}{10} \le r \le \frac{12}{10}$$

$$0.8 \le r \le 1.2$$

By observing Figure 5.21, the values of $\mu = 0.25$, $\beta = 0.8$, and $\zeta = 0.27$ yield a reasonable solution for the required range of r. So the values of m_a, c_a, and k_a are

$m_a = \mu m = (0.25)(200) = 50$ kg
$c_a = 2\zeta m_a \omega_a = 2(0.27)(50)(10) = 270$ kg/s
$k_a = m_a \omega_a \beta^2 \omega_p^2 = (50)(10)(0.8)^2(10)^2 = 32000$ N/m

Note that an extensive optimization could have been used to solve for μ, β, and ζ, but this is not covered until section 5.5.

5.44 Often absorber designs are afterthoughts such as indicated in example 5.3.1. Add a damper to the absorber design of Figure 5.17 to increase the useful bandwidth of operation of the absorber system in the event the driving frequency drifts beyond the range indicated in Example 5.3.2.

Solution:

From Examples 5.3.1 and 5.3.2,

$$m = 73.16 \text{ kg} \quad m_a = 18.29 \text{ kg}$$
$$k = 2600 \text{ N/m} \quad k_a = 6500 \text{ N/m}$$

$$7.4059 < \omega < 21.0821 \text{ rad/s}$$

The values μ and β are

$$\mu = \frac{m_a}{m} = 0.25$$

$$\beta = \frac{\omega_a}{\omega_p} = \frac{\sqrt{k_a / m_a}}{\sqrt{k / m}} = 3.1623$$

Choosing ζ = 0.2 (by trial and error) will allow ω to go beyond 21.0821 rad/s without $\frac{X_k}{F_0}$ going above 1. However, it will not prevent $\frac{Xk}{F_0}$ from going above 1 when ω < 7.4089 rad/s. The value of c_a is

$$c_a = 2\zeta m_a \omega_p = 2(0.2)(18.29)\sqrt{\frac{2600}{73.16}} = 43.61 \text{ kg/s}$$

5.45 Again consider the absorber design of Example 5.3.1. If the absorber spring is made of aluminum and introduces a damping ratio of $\zeta = 0.001$, calculate the effect of this on the deflection of the saw (primary system) with the design given in Example 5.3.1.

Solution:

From Examples 5.3.1 and 5.3.2,

$$X = \frac{\left(k_a - m_a\omega^2\right)F_0 + c_a\omega F_0 j}{\det\left(K - \omega^2 M + \omega jC\right)}$$

where $c_a = 2\zeta\sqrt{k_a m_a} = 2\left(0.001\right)\sqrt{\left(6500\right)\left(18.29\right)} = 0.6896$ kg/s

Since

$$M = \begin{bmatrix} 73.16 & 0 \\ 0 & 18.29 \end{bmatrix} \quad C = \begin{bmatrix} 0.6896 & -0.6896 \\ -0.6896 & 0.6896 \end{bmatrix} \quad K = \begin{bmatrix} 9100 & -6500 \\ -6500 & 6500 \end{bmatrix}$$

The denominator is $-1.4131\times10^7-12{,}363j$ when $\omega = 7.4089$ rad/s,

$$\left|X_1\right| = 0.00499 \text{ m}$$

and when $\omega = 21.0821$ rad/s,

$$\left|X_2\right| = 0.00512 \text{ m}$$

The nondimensional values become

$$\left|\frac{X_j k}{F_0}\right| = 0.999$$

$$\left|\frac{X_2 k}{F_0}\right| = 1.023$$

There is very little effect on the saw deflection since the values of $\left|\frac{Xk}{F_0}\right|$ are still approximately 1 at the endpoints of the driving frequency range.

5.46 Consider the undamped primary system with a viscous absorber as modeled in Figure 5.22 and the rotational counterpart of Figure 5.23. Calculate the magnification factor $\left|Xk / M_O\right|$ for a 400 kg compressor having a natural frequency of 16.2 Hz if driven at resonance, for an absorber system defined by μ = 0.133 and ζ = 0.025.

Solution:

From Eqs. (5.39), with μ = 0.133, ζ = 0.025, and r = 1:

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2}} = 150.6$$

The design with ζ = 0.1 produces the smallest displacement.

5.47 Recalculate the magnification factor $\left|Xk / M_O\right|$ for the compressor of Problem 5.46 if the damping factor is changed to ζ = 0.1. Which absorber design produces the smallest displacement of the primary system ζ = 0.025 or ζ = 0.1?

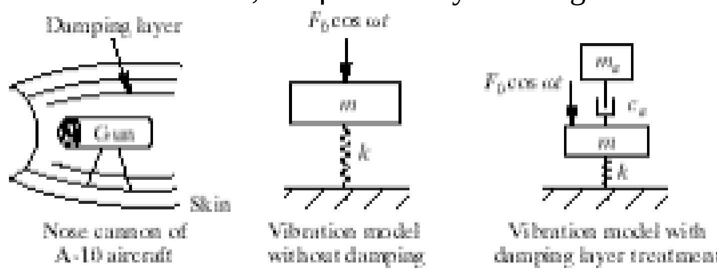
Solution:

From Equation (5.39), with μ = 0.133, ζ = 0.1, and r = 1:

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2}} = 38.34$$

The design with ζ = 0.1 produces the smallest displacement.

5.48 Consider a one-degree-of-freedom model of the nose of an aircraft (A-10) as illustrated in Figure P5.48. The nose cracked under fatigue during battle conditions. This problem has been fixed by adding a viscoelastic material to the inside of the skin to act as a damped vibration absorber as illustrated in Figure P5.48. This fixed the problem and the vibration fatigue cracking disappeared in the A-10's after they were retrofitted with viscoelastic damping treatments. While the actual values remain classified, use the following data to calculate the required damping ratio given M = 100 kg, f_a = 3 Hz, and $k = 3.533 \times 10^6$ N/m, such that the maximum response is less than 0.25 mm. Note that since mass always needs to be limited in an aircraft, use μ = 0.1 in your design.



Solution:

From Equation (5.39), with μ = 0.1, and $r = \dfrac{30(2\pi)}{\sqrt{k/m}}$ = 1.885, and M_0 replaced by F_0,

$$\frac{Xk}{F_0} = \sqrt{\frac{4\zeta^2 + (1.885)^2}{4\zeta^2\left[(1.1)(1.885)^2 - 1\right]^2\left[(1.885)^2 - 1\right]^2(1.885)^2}}$$

$$= \sqrt{\frac{4\zeta^2 + 3.553}{33.834\zeta + 23.159}}$$

With no damping $\dfrac{Xk}{F_0} = 0.392$. This value must be reduced. Choose a "high" damping ratio of ζ = 0.7 so that

$$\frac{Xk}{F_0} = 0.372$$

The value of c_a is

$$c_a = 2\zeta\mu m\sqrt{\frac{k}{m}} = 2(0.7)(0.1)(100)\sqrt{\frac{10^6}{100}} = 1400 \text{ kg/s}$$

5.49 Plot an amplification curve such as Figure 5.24 by using Equation (5.39) for ζ = 0.02 after several values of μ (μ = 0.1, 0.25, 0.5, and 1). Can you form any conclusions about the effect of the mass ratio on the response of the primary system? Note that as μ gets large $\left|Xk/M_o\right|$ gets very small. What is wrong with using very large μ in absorber design?

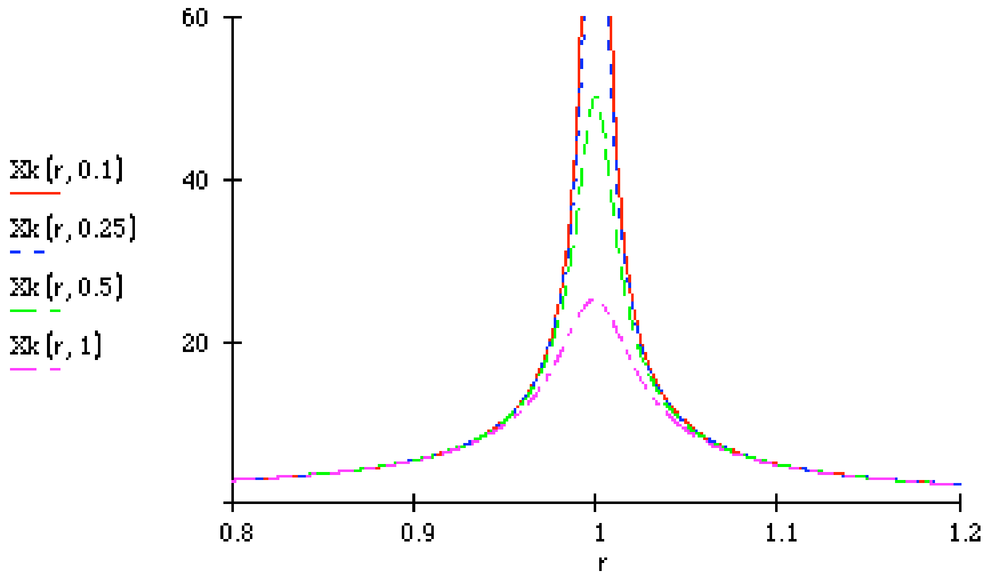
Solution:

From Equation (5.39), with ζ = 0.1:

$$\frac{Xk}{M_0} = \sqrt{\frac{0.0016 + r^2}{0.0016\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 r^2}}$$

The following plot shows amplitude curves for μ = 0.1, 0.25, 0.5, and 1.

$$Xk(r,\mu) := \sqrt{\frac{0.0016 + r^2}{0.0016\cdot\left(r^2 + \mu\cdot r^2 - 1\right)^2 + \left(r^2 - 1\right)^2\cdot r^2}}$$



Note that as the mass ratio, μ, increases, the response of the primary system decreases, particularly in the region near resonance. A higher mass ratio, however, indicates a poor design (and can be quite expensive).

5.50 A Houdaille damper is to be designed for an automobile engine. Choose a value for ζ and μ if the magnification $\left| Xk / M_O \right|$ is to be limited to 4 at resonance. (One solution is μ = 1, ζ = 0.129.)

Solution:

From Equation (5.39), with $r = 1$:

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + 1}{4\zeta^2 \mu^2}}$$

For $\frac{Xk}{M_0} = 4$,

$$64\zeta^2\mu^2 = 4\zeta^2 + 1$$

If μ is limited to 0.3, then the value of ζ is

$$64\zeta^2 (0.3)^2 = 4\zeta^2 + 1$$

$$\zeta = 0.754$$

5.51 Determine the amplitude of vibration for the various dampers of Problem 5.46 if ζ = 0.1, and F_0 = 100 N.

Solution:

From Problem 5.46,

$$k = m\omega_n^2 = (400)\left[(16.2)(2\pi)\right]^2 = 4.144 \times 10^6 \text{ N/m}$$

Also, μ = 0.1, r = 1, and F_0 = 100 N. So, from Equation (5.39), with M_0 replaced by F_0,

$$X = \frac{F_0}{k}\sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2\left(r^2 + \mu r^2 - 1\right)^2 + \left(r^2 - 1\right)^2 r^2}} = 0.00123 \text{ m}$$

5.52 (Project) Use your knowledge of absorbers and isolation to design a device that will protect a mass from both shock inputs and harmonic inputs. It may help to have a particular device in mind such as the module discussed in Figure 5.6.

Solution:

One way to approach this problem would be to design an isolator to protect the mass from shock inputs, and an absorber to protect the mass from harmonic disturbances. An absorber would be particularly useful if the frequency of the harmonic disturbance(s) is well known.

This is a very general approach to such a problem, and solutions will vary greatly depending on the particular parameters involved in an actual system.