Chapter 6

Problems and Solutions Section 6.2 (6.1 through 6.7)

6.1 Prove the orthogonality condition of equation (6.28).

Solution:

Calculate the integrals directly. For *n* = *n*, let $u = n\pi x/l$ so that $du = (n\pi/l)dx$ and the integral becomes

$$\frac{l}{n\pi}\int_0^{n\pi}\sin^2 u du = \frac{l}{n\pi}\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)\Bigg|_0^{n\pi}$$

$$= \frac{l}{n\pi}\left(\frac{1}{2}n\pi - \frac{1}{4}\sin 4n\pi\right) - 0 = \frac{l}{2}$$

where the first step used a table of integrals. For $n \neq m$ let $u = \pi x/l$ so that $d\text{u} = (\pi/l)dx$ and

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx = \frac{l}{\pi}\int_0^l \sin mu \sin nu du$$

which upon consulting a table of integrals is

$$\frac{l}{\pi}\left\{\frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(n+m)\pi}{2(n+m)}\right\} = 0\,.$$

6.2 Calculate the orthogonality of the modes in Example 6.2.3.

Solution:

One needs to show that $\int_0^1 X_n(x)X_m(x)dx = 0$ for $m \neq n,$ where $X_m(t) = a_n \sin \sigma_n x.$ But each mode $X_n(x)$ must satisfy equation (6.14), i.e.

$$X_n'' = -\sigma_n^2 X_n \qquad (1)$$

Likewise

$$X_m'' = \sigma_m^2 X_m \qquad (2)$$

Multiply (1) by X_m and integrate from 0 to l. Then multiply (2) by $X_n(x)$ and integrate from 0 to l. This yields

$$\int_0^l X_n'' X_m dx = -\sigma_n^2 \int_0^l X_n X_m dx$$
$$\int_0^l X_m'' X_n dx = -\sigma_m^2 \int_0^l X_m X_n dx$$

Subtracting these two equations yields

$$\int_0^l \left(X_n'' X_m - X_m'' X_n \right) dx = \left(\sigma_n^2 - \sigma_m^2 \right) \int_0^l X_n(x) X_m(x) dx$$

Integrate by parts on the left side to get

$$\int_0^l X_n' X_m' dx - \int_0^l X_m' X_n' dx + X_n' X_m \Big|_0^l - X_m X_n' \Big|_0^l$$
$$= X_m(l)kX_n(l) - X_n(l)kX_m(l) = 0$$

from the boundary condition given by eq. (6.50). Thus

$$\left(\sigma_n^2 - \sigma_m^2 \right) \int_0^l X_n X_m dx = 0.$$

But from fig. 6.4, $\sigma_n \neq \sigma_m$ for $m \neq n$ so that

$$\int_0^l X_n X_m dx = a_n^2 \int_0^l \sin \sigma_n x \sin \sigma_m x dx = 0$$

and the modes are orthogonal.

6.3. Plot the first four modes of Example 6.2.3, for the case l = 1 m, k = 800 N/m and τ = 800 N/m.

Solution:

The mode shapes are given as $\sin\sigma_n x$ where σ_n satisfies eq. (6.51). To solve this numerically values of l, k and τ must be given. For example chose l = 1 m, k = 800 N/m, and τ = 800 N/m the equation (6.51) becomes

$$\tan \sigma = -\sigma$$

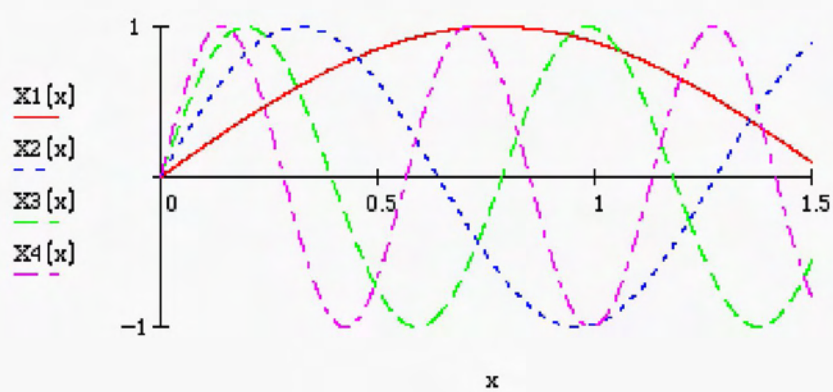
Solving using MATLAB for the first 4 values yields

$$\sigma_1 = 2.029,\ \sigma_2 = 4.913,\ \sigma_3 = 7.979,\ \sigma_4 = 11.0855$$

So that the mode shapes are sin(2.029)x, sin(4.913)x, sin(7.979)x and sin(11.0855)x. These are plotted below using Mathcad.

$$\sigma := \begin{bmatrix} 2.029 \\ 4.913 \\ 7.979 \\ 11.0855 \end{bmatrix}$$

$$X1(x) := \sin(\sigma_0 \cdot x) \quad X2(x) := \sin(\sigma_1 \cdot x) \quad X3(x) := \sin(\sigma_2 \cdot x) \quad X4(x) := \sin(\sigma_3 \cdot x)$$



6.4 Consider a cable that has one end fixed and one end free. The free end cannot support a transverse force, so that $w_x(l,t) = 0$. Calculate the natural frequencies and mode shapes.

Solution:

The cable equation results in (6.17). The boundary conditions are

$$w(x,t) = X(x)T(t) = 0 \text{ at } x = 0 \text{ (fixed end)}$$

so that $X(0) = 0$ and

$$w_x(x,t) = X'(x)T(t) = 0 \text{ at } x = l \text{ (free end)}$$

so that $X(l) = 0$. Applying these to equation (6.17) yields

$$0 = a_1 \sin(0) + a_2 \cos(0) \text{ so that } a_2 = 0$$
$$0 = a_1 \sigma \cos \sigma(l)$$

so that cos $\sigma l = 0$ or $\sigma l = n$ for odd n and the natural frequency $\sigma_n = \frac{n\pi}{2l}$, n = 1, 3, 5… or $\sigma_n = \frac{2n-1\pi}{2l}$, n = 1, 2, 3…Since $a_2 = 0$, and a_1 is arbitrary the mode shapes are

$$a_n \sin\left(\frac{(2n-1)\pi x}{2l}\right), \quad n = 1,2,3...$$

the natural frequencies are from (6.15) and (6.24):

$$\omega_n = \sqrt{\sigma_n^2 c^2} = c\sigma_n = \frac{(2n-1)\pi c}{2l} = \frac{(2n-1)\pi}{2l}\sqrt{\tau / \rho}$$

6.5 Calculate the coefficients c_n and d_n of equation (6.27) for the system of a clamped-clamped string to the initial displacement given in Figure P6.5 and an initial velocity of $w_t(x,0) = 0$.

Solution:

For the clamped-clamped string the solution is given by eq. (6.27) as

$$w(x,t) = \sum_{i=1}^{\infty} (c_n \sin\sigma_n x \sin\sigma_n ct + d_n \sin\sigma_n x \cos\sigma_n ct)$$

Series $w_t(x,0) = 0$, equation (6.33) yields that $c_n = 0$ for all n. The coefficients d_n are given by eq. (6.31) as

$$d_n = \frac{2}{l}\int_0^l \omega_0(x)\sin\frac{m\pi x}{l}dx \quad m = 1,2,\ldots$$

From fig. 6.16 $\omega_0(x) = \begin{cases} 2x/l & 0 \le x \le l/2 \\ 2(l-x)/l & l/2 \le x \le l \end{cases}$ cm. Calculation yields

$$\begin{aligned} d_n &= \frac{2}{l}\left\{\int_0^{l/2}\frac{2x}{l}\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}\frac{2}{l}(l-x)\sin\frac{n\pi x}{l}dx\right\} \\ &= \frac{8}{\pi^2 n^2}\sin\frac{n\pi}{2} \qquad n = 1,3,5\ldots \end{aligned}$$

and d_n is zero for even values of n.

6.6 Plot the response of the string in Problem 6.5 for the piano string of Example 6.2.2 (l = 1.4 m, m = 110 g, τ =11.1x10^4 N) at $x = l/4$ and $x = l/2$, using 3, 5, and 10 terms in the solution.

Solution:

For the piano string of example 6.22, l = 1.4m and c = 11.89. From problem 6.5 the solution has the form

$$w(x,t) = \frac{8}{\pi^2}\left\{\sum_{m,\text{odd}=1}^{\infty}\frac{1}{m^2}\sin\frac{m\pi}{2}\sin\frac{m\pi x}{l}\cos\frac{m\pi c}{l}t\right\}$$

For 3 terms at $x = l/4 = 3.5$, this series becomes

$$w_3(3.5,t) = 0.81\left\{0.24\cos 26.68t + 0.07858\cos 80.04t - 0.02828\cos 133.40t\right\}$$

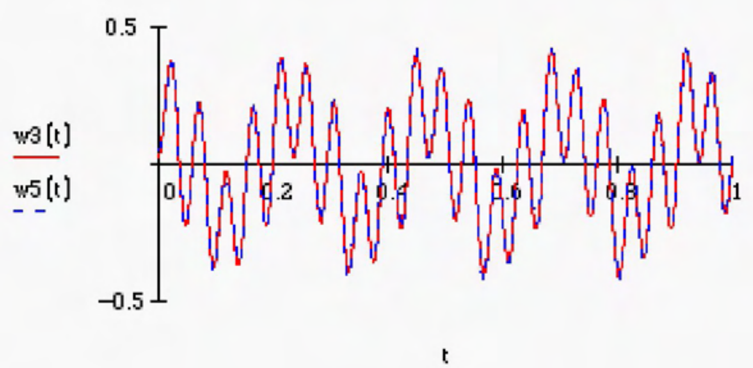
for 5 terms this becomes

$$w_5(3.5,t) = w_3 + 0.01442\cos 182t + 0.00873\cos 240.13t$$

The next terms have coefficients 0.00584, 0.00418, 0.00314, 0.00244 and 0.00195 respectively. Any of the codes can be used to easily plot these. Plot of w_3 and w_5 at $l/4$ are given below in Mathcad:

$$w3(t) := 0.81\cdot(0.24\cos(26.68\cdot t) + 0.07858\cdot\cos(80.84\cdot t) - 0.2828\cdot\cos(133.4\cdot t))$$

$$w5(t) := w3(t) + 0.01441\cdot\cos(182\cdot t) + 0.00873\cdot\cos(240.13\cdot t)$$



6.7 Consider the clamped string of Problem 6.5. Calculate the response of the string to the initial condition

$$w(x,0)=\sin\frac{3\pi x}{l}\quad w_t(x,0)=0$$

Plot the response at $x = l/2$ and $x = l/4$, for the parameters of Example 6.2.2.

Solution:

Since $w_t = 0$ each if the coefficients c_n is zero in equation (6.33). Thus the solution is of the form

$$w(x,t)=\sum_{i=1}^{\infty} d_n \sin\frac{n\pi x}{l}\cos\frac{n\pi c}{l}t$$

as given in problem 6.5. Equation (6.31) for the initial position yields

$$d_n=\frac{2}{l}\int_0^l \sin\frac{3\pi x}{l}\sin\frac{m\pi x}{l}dx\quad m=1,2,...$$

Because of the orthogonality all the $d_n = 0$ except d_3 and from the above integral $d_3 = 1$. Hence the solution collapses to the single term

$$w(x,t)=\sin\frac{3\pi x}{l}\sin\frac{3\pi c}{l}t$$

At $x = l/2$ this becomes

$$w\left(\frac{l}{2},t\right)=\sin\frac{3\pi}{2}\cos\frac{3\pi c}{l}t=-\cos\frac{3\pi c}{l}t$$

At $x = l/4$

$$w\left(\frac{l}{4},t\right)=\sin\frac{3\pi}{4}\cos\frac{3\pi c}{l}t=0.707\cos\frac{3\pi c}{l}t$$

Using the values for the piano string ($l = 1.4$, $c = 1188$ m/s) $w(l/4,t)$ is simply a cosine of frequency 8000 rad/s and amplitude 0.707.