**Problems and Solutions Section 6.3 (6.8 through 6.29)**

**6.8** Calculate the natural frequencies and mode shapes for a free-free bar. Calculate the temporal solution of the first mode.

**Solution:**

Following example 6.31 (with different B.C.'s), the spatial response of the bar will be

$$X(x) = a\sin\sigma x + b\cos\sigma x$$

The boundary conditions are $X'(0) = X'(l) = 0$. The expression for $X'$ is $X'(x) = \sigma a\cos\sigma x - \sigma b\sin\sigma x$ so at 0:

$$0 = \sigma a \Rightarrow a = 0$$

at $l$

$$0 = -\sigma b\sin\sigma l, \quad b \neq 0$$

so that $\sigma l = n\pi$ or $\sigma = n\pi/l$ where $n$ starts a *zero*. Hence the mode shapes are of the form

$$X_n(x) = b_n\cos\frac{n\pi x}{l} \text{ for } n = 1, 2, 3, \ldots \text{ and for } n = 0,$$

$$X_0(x) = b_0\cos\left(\frac{0\pi}{l}x\right) = b_0 \text{ a constant.}$$

The temporal solution is given by eq. (6.15) to be

$$\frac{\ddot{T}_n(t)}{c^2T_n(t)} = -\sigma^2$$

so that the temporal solution of the first mode:

$$\ddot{T}_0(t) + 0c^2T_0(t) = 0 = \ddot{T}_0(t) \quad \underline{T_0(t) = b + ct}$$

**6.9** Calculate the natural frequencies and mode shapes of a clamped-clamped bar.

**Solution:** The calculation of the natural frequencies and mode shapes of a clamped-clamped bar is identical to that of the fixed-fixed string since the equations of motion are mathematically the same. The solution of this problem is thus given at the beginning of section 6.2, but is repeated here: Applying separation of variable to eq. (6.56) yields that the spatial variable must satisfy eq. (6.59) of example 6.3.1, i.e., $X(x) = a\sin\sigma x + b\cos\sigma x$ where $a$ and $b$ are constants to be determined. The clamped boundary conditions require that $X(0) = X(l) =$ or

$0 = b$ or $X = a\sin\sigma x$

$0 = a\sin\sigma l$ or $\sigma = n\pi/l$

Hence the mode shapes will be of the form

$X_n = a_n \sin\sigma_n x$

Where $\sigma_\nu = n\pi/l$. The frequencies are determined from the temporal solution and become

$$\omega_n = \sigma_n c = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}, \quad n = 1, 2, 3, \ldots$$

**6.10** It is desired to design a 4.5 m, clamped-free bar such that the first natural frequency is 1878 Hz. Of what material should it be made?

**Solution:** First change the frequency into radians:

1878 Hz =1878x2π rad/s=11800 rad/s

The first natural frequency is given computed in Example 6.3.1, Equation (6.63) as

$$\omega_1 = \frac{2\pi}{l}\sqrt{\frac{E}{\rho}} \Rightarrow \frac{E}{\rho} = \omega_1^2 \frac{4l^2}{\pi^2} = (11800)^2 \frac{4l^2}{\pi^2}$$

$$\Rightarrow \frac{E}{\rho} = 7.143 \times 10^7$$

in Nm/kg. Examining the ratios from Table 2.1 for the values given yields that for Steel:

$$\frac{E}{\rho} = \frac{2 \times 10^{11}}{2.8 \times 10^3} = 7.143 \times 10^7 \text{ Nm/kg}$$

Thus a **steel** bar with a length 4.5 meters will have a first natural frequency of 1878 Hz. This is something like a truck chassis.

**6.11** Compare the natural frequencies of a clamped-free 1-m aluminum bar to that of a 1-m bar made of steel, a carbon composite, and a piece of wood.

**Solution:**

For a clamped-free bar the natural frequencies are given by eq. (6.6.3) as

$$\omega_n = \frac{(2n-1)\pi}{2l}\sqrt{\frac{E}{\rho}}$$

Referring to values of r and E from table 1.2 yields (for $\omega_1$):
Steel

$$\frac{\pi}{(2)(1)}\sqrt{\frac{2.0\times10^{11}}{7.8\times10^{3}}} = 7{,}954 \text{ rad/s (1266Hz)}$$

Aluminum

$$\frac{\pi}{(2)(1)}\sqrt{\frac{7.1\times10^{10}}{2.7\times10^{3}}} = 8{,}055 \text{ rad/s (1282 Hz)}$$

Wood

$$\frac{\pi}{(2)(1)}\sqrt{\frac{5.4\times10^{9}}{6.0\times10^{2}}} = 4{,}712 \text{ rad/s (750 Hz)}$$

Carbon composite (student must hunt for $E/\rho$ and guess a little) from Vinson and Sierakowski's book on composites $\sqrt{E/\rho}$ = 3118 and

$$\frac{\pi}{2}(3118) = 4897 \text{ rad/s (780 Hz)}$$

**6.12** Derive the boundary conditions for a clamped-free bar with a solid lumped mass, of mass M attached to free end.

**Solution:** At the clamped end, $x = 0$, the boundary condition is $w(0,t) = 0$ or $X(x) = 0$. At the end $x = l$ the tensile force in the bar must be equal to the inertia force of the attached mass. For an attached mass of value $M$, this becomes

$$EA\left.\frac{\partial w(x,t)}{\partial x}\right|_{x=l} = -M\left.\frac{\partial^2 w(x,t)}{\partial t^2}\right|_{x=l}$$

**6.13** Calculate the mode shapes and natural frequencies of the bar of Problem 6.12. State how the lumped mass affects the natural frequencies and the mode shapes.

**Solution:** Via separation of variables [i.e., $w(x,t) = X(x)T(t)$], the spatial equation becomes (following example 6.3.1 for instance)

$$X(x) = a\sin\sigma x + b\cos\sigma x$$

Applying the boundary condition at $x = 0$ yields

$$X(0) = 0 = a\sin(0) + b\cos(0) \Rightarrow b = 0 \quad 0 = b$$

so the spatial solution reduces to $X(x) = a\sin\sigma x$. Now the second boundary condition (see 6.12) involves time deviates so that $w(x,t) = X(x)T(t)$ substituted into the boundary condition $EAW_x = -Mw_{tt}(l,t)$ becomes:

$$EAX'(l)T(t) = -MX(l)\ddot{T}(t) \Rightarrow \frac{EAX'(l)}{MX(l)} = -\frac{\ddot{T}(t)}{T(t)}$$

From equation (6.15) $\ddot{T}/T = -\sigma^2 c^2$, so this boundary condition becomes

$$\frac{EA}{M}\cdot\frac{X'(l)}{X(l)} = \sigma^2 c^2 \qquad (1)$$

Substitution of $X(x) = a\sin\sigma x$ and $X'(x) = a\sigma\cos\sigma x$ into (1) yields

$$\frac{EA}{M}\frac{a\sigma\cos\sigma l}{a\sin\sigma l} = \sigma^2 c^2$$

or

$$\cot\sigma l = \frac{\sigma c^2 M}{EA}$$

which describes multiple values of $\sigma = \sigma_n$, $n = 1, 2, 3,\ldots$ The frequency of oscillation is related to $\sigma_n$ by $\omega_n = \sigma_n c$, where $c = \sqrt{E/\rho}$. Let $\rho Al = m$ be the mass of the beam and rewrite $\cot(\sigma l)$ as

$$\cot\sigma l = \cot\left(\frac{\omega_n l}{c}\right) = \frac{\sigma(E/\rho)M}{EA} = \frac{(\omega_n l/c)}{A\rho l}\cdot M = \frac{\omega_n l}{c}\frac{M}{m}.$$

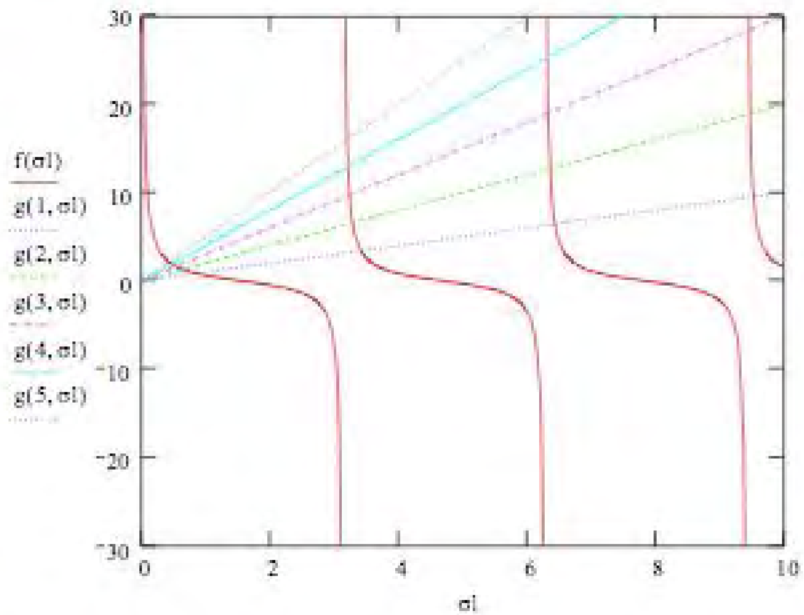
This can be rewritten as

$$\alpha\cot\alpha = \beta$$

where $\beta = m/M$ and $\alpha = \omega_n l/c$. As the mass ratio $\beta$ increases (tip mass increases) the frequency increases. The mode shapes are proportional to $\sin\sigma_n x$, where $\sigma_n$ is calculated numerically from $\cot(\sigma l) = (M/m)\sigma l$, similar to the calculation showing in Figure 6.4. This is illustrated in the following Mathcad session.

$f(\sigma l) := \cot(\sigma l)$ $\qquad$ $g(\beta, \sigma l) := \beta \cdot \sigma l$

Here, $\beta$ is the end mass - to - bar mass ratio ($\beta = M/\rho AL$). Hence the transcendental equation $f(\sigma l) = g(\sigma l)$ is totally nondimensional which makes it possible to study the effect of end mass - to - bar mass ratio in a nondimesional basis. That is what you see in the following figure where 5 different end mass - to - bar mass ratios are investigated.




For each value of $\beta$ (from 1 to 5), the intersection of f and g gives the frequency parameter $\sigma L$ which is directly proportional to the natural frequency of the mode of interest. Therefore, it is obvious from the figure that **increasing the end mass reduces the natural frequencies of the bar** (and looks like the first natural frequency is the most sensitive one). This makes perfect physical sense: if you add a large end mass, the structure becomes much more flexible.

The limiting behavoirs are $\beta = 0$ and $\beta = \text{inf.}$. The former case ($\beta = 0$) gives the natural frequencies of a clamped-free bar without tip mass (from the roots of $\cot(\sigma l) = 0$ or just $\cos(\sigma l) = 0$ if you like) whereas the latter case ($\beta = \text{inf.}$) gives the natural frequencies of a clamped-clamped bar (from the roots of $\sin(\sigma l) = 0$) which is also evident from the above graph.

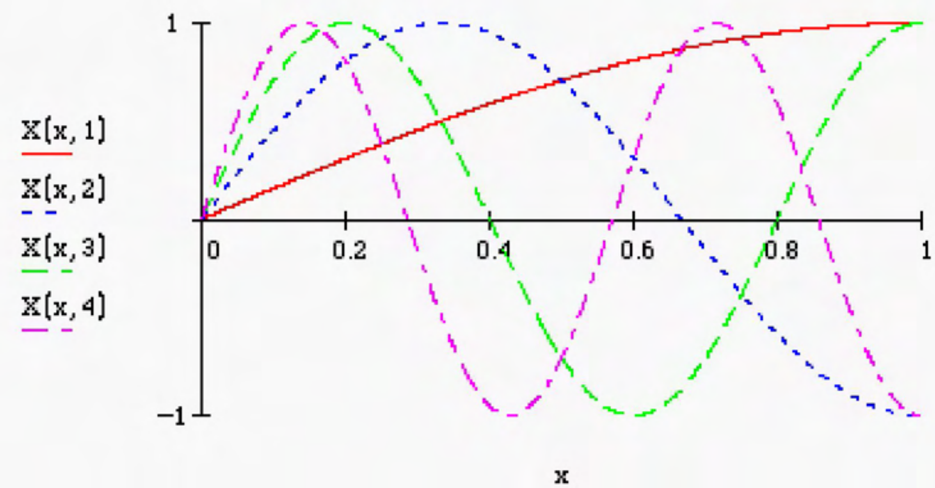
**6.14** Calculate and plot the first three mode shapes of a clamed-free bar.

**Solution:** The second entry of Table 6.1 yields the solution

$$X_n(x) = \sin\frac{(2n-1)}{2\ell}\pi x$$

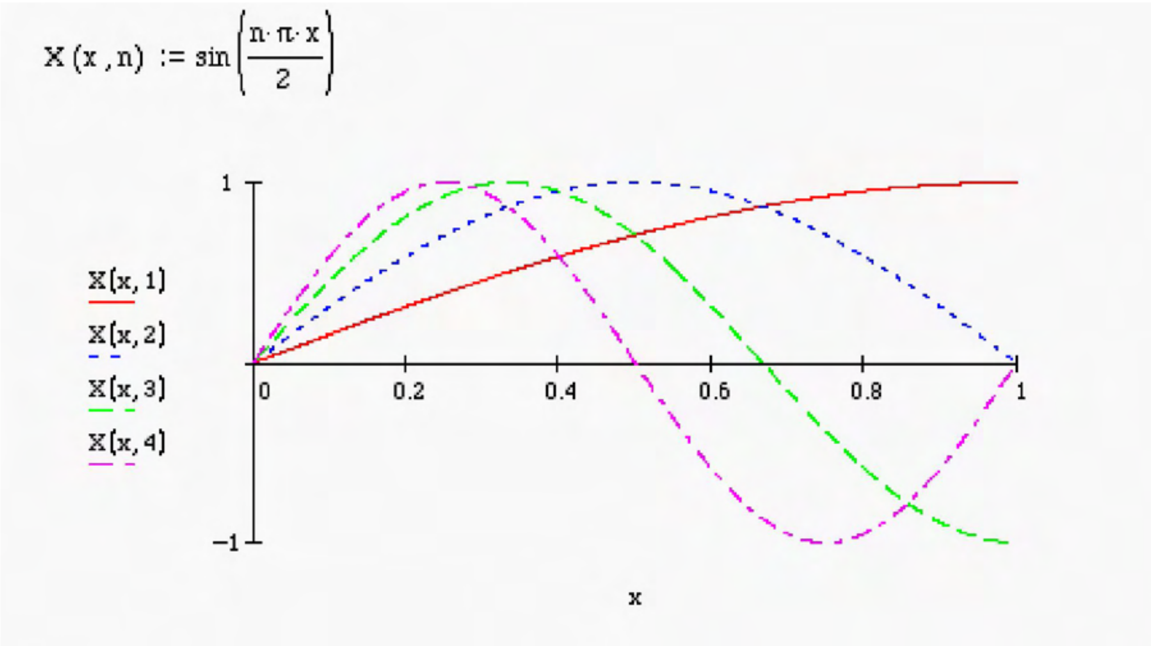
which is calculated following the procedures out lined in Example 6.3.1. The plot is given in Mathcad for the case $\ell$ = 1m.

$$X(x,n) := \sin\left[\frac{\left[(2n-1)\cdot\pi\cdot x\right]}{2}\right]$$



**6.15** Calculate and plot the first three mode shapes of a clamed-clamped bar and compare them to the plots of Problem 6.14.

**Solution:** As in problem 6.14 the solution is given in table 6.1. The important item here is to notice the difference between mode shapes from the plots of $\sin\frac{(2n-1)\pi}{2l}x$ and sin ($n\pi x/l$). In particular notice the difference at the free end.




**6.16** Calculate and compare the eigenvalues of the free-free, clamped-free, and the clamped-clamed bar. Are the related? What does this state about the system's natural frequencies?

**Solution:**

Students can calculate these or just use the results listed in table 6.1. Note for $l$ = 1

free-free $0, \pi c, 2\pi c\ldots$

clamped-free $\frac{\pi c}{2}, \frac{3\pi c}{2}, \frac{5\pi c}{2}\ldots$

clamped-clamped $\pi c, 2\pi c, 3\pi c\ldots$

so that the free-free and clamped-clamped values are a $\pi$ shift from one another with the clamped-free values falling in between: as the number of constraints increases, the frequency increases.

**6.17** Consider the nonuniform bar of Figure P6.17, which changes cross-sectional area as indicated in the figure. In the figure $A_1$, $E_1$, $\rho_1$, and $l_1$ are the cross-sectional area, modulus, density and length of the first segment, respectively, and $A_2$, $E_2$, $\rho_2$, and $l_2$ are the corresponding physical parameters of the second segment. Determine the characteristic equation.

**Solution:** Let the subscript 1 denote the first part of the beam and 2 the second part of the beam. The bar equation must be satisfied in each part so that equation of motion is in two parts:

$$E_1\frac{\partial^2 w_1(x,t)}{\partial x^2}=\rho_1\frac{\partial^2 w_1(x,t)}{\partial t^2}\quad 0<x<\ell_1$$

$$E_2\frac{\partial^2 w_2(x,t)}{\partial x^2}=\rho_2\frac{\partial^2 w_2(x,t)}{\partial t^2}\quad \ell_1<x<\ell_1+\ell_2=\ell$$

The boundary conditions are the two from the clamped-free configuration then there are two more conditions expressing force and displacement continuity at the point where the two beams join ($x = \ell_1$). Follow the procedure of separation of variables but this time keep the constant $c$ in the spatial equation so that we may write: $w_1(x,t) = X_1(x)T(t)$ and $w_2(x,t) = X_2(x)T(t)$ where the function of time is common to both beams. Then denoting $\sigma^2$ as the separation constant and substituting the separated forms into the equation of motion yields:

$$\frac{c_1^2X_1''(x)}{X_1(x)}=\frac{\ddot{T}(t)}{T(t)}=-\sigma^2\quad 0<x<\ell_1\ \text{ and } c_1=\sqrt{\frac{E_1}{\rho_1}}\quad (1)$$

$$\frac{c_2^2X_2''(x)}{X_2(x)}=\frac{\ddot{T}(t)}{T(t)}=-\sigma^2\quad \ell_1<x<\ell\ \text{ and } c_2=\sqrt{\frac{E_2}{\rho_2}}\quad (2)$$

In this way the temporal equation for both parts is the same ($\sigma$ does not depend on which part of the beam and will show up in the characteristic equation). Solving the two spatial equations yields:

$$(1)\Rightarrow X_1=a_1\sin\frac{\sigma}{c_1}x+a_2\cos\frac{\sigma}{c_1}x\quad 0<x<\ell_1$$

$$(2)\Rightarrow X_2=a_3\sin\frac{\sigma}{c_2}x+a_4\cos\frac{\sigma}{c_2}x\quad \ell_1<x<\ell$$

There are now 4 boundary conditions (one at each end and two in the middle) which will yield 4 equations in the 4 coefficients $a_i$. This set of equations must be singular yielding the characteristic equation for $\sigma$.

From the clamped end:

$$X_1(0)=0\Rightarrow a_1\sin(0)+a_2\cos(0)=0 \qquad (3)$$

From the free end:

$$X_2'(\ell)=0\Rightarrow\frac{\sigma}{c_2}a_3\cos\frac{\sigma\ell}{c_2}-\frac{\sigma}{c_2}a_4\sin\frac{\sigma\ell}{c_2}=0 \qquad (4)$$

From the middle and enforcing displacement continuity at $x = \ell_1$:

$$a_1\sin\frac{\sigma}{c_1}\ell_1+a_2\cos\frac{\sigma}{c_1}\ell_1=a_3\sin\frac{\sigma}{c_2}\ell_1+a_4\cos\frac{\sigma}{c_2}\ell_1 \qquad (5)$$

From the middle and enforcing force, equation (6.54) continuity at $x = \ell_1$:

$$E_1 A_1 X_1'(\ell_1) = E_2 A_2 X'(\ell_1)$$

$$\Rightarrow E_1 A_1 \frac{\sigma}{c_1}\left(a_1 \cos\frac{\sigma\ell_1}{c_1} - a_2 \sin\frac{\sigma\ell_1}{c_1}\right) = E_2 A_2 \frac{\sigma}{c_2}\left(a_3 \cos\frac{\sigma\ell_1}{c_2} - \frac{\sigma}{c_2} a_4 \sin\frac{\sigma\ell_1}{c_2}\right) \tag{6}$$

Equations (3) through (6) are 4 equations in the 4 unknowns $a_i$. Writing these in matrix form as a homogeneous algebraic equation yields:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\sigma l}{c_2} & -\sin\frac{\sigma l}{c_2} \\ \sin\frac{\sigma}{c_1} l_1 & \cos\frac{\sigma}{c_1} l_1 & -\sin\frac{\sigma}{c_2} l_1 & -\cos\frac{\sigma}{c_2} l_1 \\ \frac{E_1 A_1}{c_1}\cos\frac{\sigma l_1}{c_1} & -\frac{E_1 A_1}{c_1}\sin\frac{\sigma l_1}{c_1} & -\frac{E_2 A_2}{c_2}\cos\frac{\sigma l_1}{c_2} & \frac{E_2 A_2}{c_2}\sin\frac{\sigma l_1}{c_2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order for the vector **a** to be nonzero, the determinant of the matrix coefficient must be zero (recall chapter 4). This yields the characteristic equation (computed using Mathcad):

$$E_2 A_2 c_1 \sin\frac{\sigma l_1}{c_1}\left[\sin\frac{\sigma l}{c_2}\cos\frac{\sigma l_1}{c_2} - \sin\frac{\sigma l_1}{c_2}\cos\frac{\sigma l}{c_2}\right]$$

$$= E_1 A_1 c_2 \cos\frac{\sigma l_1}{c_1}\left[\sin\frac{\sigma l_1}{c_2}\sin\frac{\sigma l}{c_2} + \cos\frac{\sigma l_1}{c_2}\cos\frac{\sigma l}{c_2}\right] \tag{7}$$

$\Rightarrow$

$$\frac{E_2 A_2 c_1}{E_1 A_1 c_2}\tan\frac{\sigma l_1}{c_1}\left[\sin\frac{\sigma l}{c_2}\cos\frac{\sigma l_1}{c_2} - \cos\frac{\sigma l}{c_2}\sin\frac{\sigma l_1}{c_2}\right]$$

$$= \sin\frac{\sigma l}{c_2}\sin\frac{\sigma l_1}{c_2} + \cos\frac{\sigma l}{c_2}\cos\frac{\sigma l_1}{c_2} \tag{8}$$

Further simplifying yields

$$\frac{E_2 A_2 c_1}{E_1 A_1 c_2}\tan\frac{\sigma l_1}{c_1}\sin\frac{\sigma(l - l_1)}{c_2} = -\cos\frac{\sigma(l - l_1)}{c_2}$$

$$\Rightarrow \frac{E_2 A_2 c_1}{E_1 A_1 c_2}\tan\frac{\sigma l_1}{c_1}\tan\frac{\sigma(l - l_1)}{c_2} = -1$$

Given the parameter values, equation (9) must be solved numerically for σ, yielding the natural frequencies.

**6.18** Show that the solution obtained to Problem 6.17 is consistent with that of a uniform bar.

**Solution:**

If the bar is the same, then $E_1 = E_2 = E$, $\rho_1 = \rho_2 = \rho$ etc. and the characteristic equation from (1) in the solution to Problem 6.17 becomes ($l = l_1$)

$$\sin\frac{\sigma\ell}{c}\left[\sin\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c} - \sin\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c}\right] = \cos\frac{\sigma\ell}{c}\left[\sin\frac{\sigma\ell}{c}\sin\frac{\sigma\ell}{c} + \cos\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c}\right]$$

$$\Rightarrow \sin\frac{\sigma\ell}{c}(0) = \cos\frac{\sigma\ell}{c}\left[\sin^2\frac{\sigma\ell}{c} + \cos^2\frac{\sigma\ell}{c}\right]$$

$$\Rightarrow 0 = \cos\frac{\sigma\ell}{c}(1) \Rightarrow \frac{\sigma\ell}{c} = \frac{2n-1}{2}\pi$$

so that $\sigma_n = \omega_n = \dfrac{(2n-1)\pi}{2l}\sqrt{\dfrac{E}{\rho}}$ which according to table 6.1 entry 2 is the frequency of a clamped-free bar of length $l$ .

**6.19** Calculate the first three natural frequencies for the cable and spring system of Example 6.2.3 for $l = 1$, $k = 100$, $\tau = 100$ (SI units).

**Solution:**

For $l = 1$, $k = 100$ and $\tau = 100$ the frequency equation (6.51) becomes

$\tan\sigma = -\sigma$

Using MATLAB the first 3 solutions are

$\sigma_1 = 0$, $\sigma_2 = 2.029$, $\sigma_3 = 4.913$. But zero is not allowed because of the boundary conditions.

**6.20** Calculate the first three natural frequencies of a clamped-free cable with a mass of value $m$ attached to the free end. Compare these to the frequencies obtained in Problem 6.17.

**Solution:**

Recall example 6.1.1. The force balance at the boundary $x = l$ yields

$$\tau w_x(x,t)\big|_{x=l} = -m w_{tt}(l,t)$$

The boundary condition at $x = 3$ remains $w(0,t) = 0$. The equation of motion is (6.8) or

$$c^2 w_{xx}(x,t) = w_{tt}(x,t)$$

Again, separation of variable $w(x,t) = X(x)T(t)$ yields eq. (6.12) or

$$\frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = -\sigma^2$$

The spatial equation is

$$X'' + \sigma^2 X(x) = 0$$

which has solution $X(x) = a_1 \sin \sigma x + a_2 \cos \sigma x$. Applying the boundary conditions yields $X(0) = 0$ or $a_2 = 0$. Substitution of $X(x) = a_1 \sin \sigma_2 x$ into the boundary condition at $x = l$ yields

$$[a_1 \tau\sigma \cos \sigma t]T(t) = -m\ddot{T}(t) a_1 \sin \sigma l$$

But $\ddot{T}(t)/T(t) = \sigma^2 c^2$ so this becomes

$$\tau\sigma \cos \sigma l = m\sigma^2 c^2$$

or that

$$\tan \sigma l = \frac{\tau}{m\sigma c^2} \quad (\text{or } \cot \sigma l = \frac{n\sigma}{\rho})$$

is the characteristic equation (see also table 6.1) with mode shape $\sin \sigma_n x$. A plot of their characteristic equation $\cos(\sigma l) = \frac{mc^2}{lr}\sigma l = \frac{m}{lp}(\sigma l)$ yields the value of the frequencies relative to those of problem 6.16.

**6.21** Calculate the boundary conditions of a bar fixed at $x = 0$ and connected to ground through a mass and a spring as illustrated in Figure P6.21.

**Solution:**

A free body diagram of the boundary is shown in Figure 1.

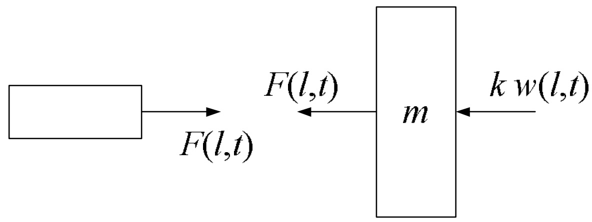



Figure 1

Consider first the end of the rod, the force is related to the axial extension of the rod though

$$F(l,t) = \sigma \mathrm{A}\big|_{x=l} = EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l}$$

On the other hand, applying Newton's second law to the mass yields

$$-F(l,t) - kw(x,t)\big|_{x=l} = m\frac{\partial w(x,t)}{\partial t^2}\bigg|_{x=l}$$

Hence, this yields the following boundary condition

$$m\frac{\partial w(x,t)}{\partial t^2}\bigg|_{x=l} = -EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l} - kw(x,t)\big|_{x=l}$$

**6.22** Calculate the natural frequency equation for the system of Problem 6.21.

**Solution:**

The boundary condition at $x = 0$ is just $w(x,t)|_{x=0} = 0$. Again from separation of variables

$$\ddot{T}(t)/T(t) = -c^2\sigma^2, \quad X(x) = a\sin\sigma x + b\cos\sigma x$$

Applying the boundary condition at 0 yields $X(0) = 0 = b$, so the spatial solution will be of the form $X(x) = a \sin \sigma x$. Substitution of the separated form $w(x,t) = X(x)T(t)$ into the boundary condition at $l$ yields (from problem 6.21)

$$mX(l)\ddot{T}(t) = -kX(l)T(t) - EAX'(l)T(t)$$

Dividing by $T(t)$, and substitution of $\ddot{T}/T = -\sigma^2 c^2$ and $X = a \sin \sigma l$ yields

$$-EA\sigma\cos\sigma l = (-m\sigma^2 c^2 + k)\sin\sigma l \quad \text{or} \quad \tan\sigma l = -\frac{EA\sigma}{k - m\sigma^2 c^2}$$ is the frequency or characteristic equation. Note that this reduces to the values given in Table 6.1 for the special case $m = 0$ and for the case $k = 0$.

**6.23** Estimate the natural frequencies of an automobile frame for vibration in its longitudinal direction (i.e., along the length of the car) by modeling the frame as a (one-dimensional) steel bar.

**Solution:**

Note: The fundamental frequency of an automobile is of primary importance in assuming the quality of an automobile. While an automobile certainly has numerous modes, its fundamental frequency apparently has a large correlation with the occupants perception of quality. The fundamental frequency of a Mercedes 300 series is 25 Hz. Infinity and Lexus have frequencies in the low twenties. This problem has no straightforward answer. Students should think about their own cars or that of their family. For steel $\rho = 7.8 \times 10^3$ kg/m$^2$, $E = 2.0 \times 10^{11}$ N/m. For a Ford Taurus $l = 4.5$ m and assume the width to be 1 meter. The frequency equation in Hertz of a free-free beam is (excluding the rigid body mode)

$$f_n = \frac{n}{2\pi}\frac{\pi}{l}\sqrt{\frac{E}{\rho}} = 562 \text{ Hz, } 1125\text{Hz}\ldots$$

where $n = 1,2,\ldots$ The frequency measured by auto engineers is from a 3 dimensional finite element model and modal test data. The frequency most felt is probably a transverse frequency.

**6.24** Consider the first natural frequency of the bar of Problem 6.21 with $k = 0$ and Table 6.2, which is fixed at one end and has a lumped-mass, $M$, attached at the free end. Compare this to the natural frequency of the same system modeled as a single-degree-of-freedom spring-mass system given in Figure 1.21. What happens to the comparison as $M$ becomes small and goes to zero?

**Solution:**

From figure 1.21, $k = EA/l$ is the stiffness of a cantilevered bar. Hence the frequency is

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{EA}{lm}}$$

for the bar with tip mass $m$ modeled as a single degree of freedom system. Now consider the first natural frequency of the distributed mass model of the same structure given in the last entry of table 6.1.

$$\omega_1 = \frac{\lambda_1 c}{l} = \frac{\lambda_1}{l}\sqrt{\frac{E}{\rho}}$$

where $\lambda_1$ satisfies $\cot\lambda_1 = \left(\frac{m}{\rho A l}\right)\lambda_1$. This last expression can be written as $\lambda_1 \tan\lambda_1\left(\frac{\rho c l}{m}\right)$ since $\lambda_1 = \omega_1 l/c$,

$$\frac{\omega_1 l}{c}\tan\left(\frac{\omega_1 l}{c}\right) = \frac{\rho A l}{m}$$

Now for small, or negligible beam mass, $c$ becomes very large $\left(c = \sqrt{E/\rho}\right)$ and $\omega_1 l/c$ becomes small so that $\tan\theta$ can be approximated as $\theta$. Then this last expression becomes

$$\left(\frac{\omega_1 l}{c}\right)^2 = \frac{\rho A l}{m}, \text{ or } \omega_1 = \sqrt{\frac{EA}{lm}}$$

in agreement with the single degree of freedom values of figure 1.21. As the tip mass goes to zero, the equation for figure 1.21 does not appear to make sense. The equation for $\omega_1$ however reduces to that of a cantilevered beam, i.e., $\omega_1 = \pi c/2l$ since the frequency equation returns to $\omega_1(l/c) = 0$.

**6.25** Following the line of thought suggested in Problem 6.24, model the system of Problem 6.21 as a lumped-mass single-degree-of-freedom system and compare this frequency to the first natural frequency obtained in Problem 6.22.

**Solution:** Note that the system of figure P6.21 is a mass connected to two springs in parallel if the bar is modeled as spring. The stiffness of a bar is given in Chapter 1 to be

$$k_{\text{bar}} = \frac{EA}{\ell}$$

The equivalent stiffness is just the sum, so that the equation of motion is

$$m\ddot{x} + \left(\frac{EA}{\ell} + k\right)x = 0$$

Thus the natural frequency of the bar and spring of figure P6.21 modeled as a single degree of freedom system is just

$$\omega_n = \sqrt{\frac{EA}{m\ell} + \frac{k}{m}}$$

The first natural frequency of the system treated as a distributed mass systems is given by the characteristic equation given in the solution to problem 6.22. To make a comparison, chose some specific values. For a 4 m aluminum beam connected to 1000 kg mass through a 100,000 N/m spring the value is given in the following Mathcad session:

$$k := 100000 \qquad m := 1000 \qquad \rho := 7.8\cdot 10^{3} \qquad E := 2\cdot 10^{11}$$

$$L := 4 \qquad A := 0.2\cdot 0.5 \qquad \rho\cdot A = 780$$

$$c := \sqrt{\frac{E}{\rho}} \qquad \omega n := \sqrt{\frac{E\cdot A}{m\cdot L} + \frac{k}{m}}$$

$$\omega n = 2.236\cdot 10^{3}$$

$$\sigma := \frac{\pi}{6}$$

$$f(\sigma) := \tan(\sigma\cdot L) + \frac{E\cdot A\cdot \sigma}{\left(m\cdot \sigma^{2}\cdot c^{2}\right) + k}$$

$$\text{root}(f(\sigma), \sigma) = 0.545$$

$$\omega 1 := c\cdot \sigma$$

$$\omega 1 = 2.651\cdot 10^{3}$$

> Note for the parameter values chose the frequency of the lumped mass model is a little less then the actual value.

**6.26** Calculate the response of a clamped-free bar to an initial displacement 1 cm at the free end and a zero initial velocity. Assume that $\rho = 7.8 \times 10^3$ kg/m$^3$, $A = 0.001$ m$^2$, $E=10^{10}$ N/m$^2$, and $l = 0.5$ m. Plot the response at $x = l$ and $x = l/2$ using the first three modes.

**Solution:**

The initial conditions are $w(x,t) = 0.01\delta(x\text{-}l)$ and $w_t(x,0) = 0$ and the boundary conditions are $w(0,t) = 0$ and $w_x(l,t) = 0$. From example 6.3.1 the mode shapes are $\sin\left(\frac{2n-1}{2l}\right)\pi x$ and the natural frequencies are

$$\omega_n = \left(\frac{2n-1}{2l}\right)\sqrt{\frac{E}{\rho}} = (2n-1)(1132.38)$$

The solution is given in example 6.3.2 as

$$w(x,t) = \sum (c_n \sin \omega_n t + d_n \cos \omega_n t)\sin\left(\frac{2n-1}{2l}\right)\pi x$$

so that the velocity is

$$w_t(x,t) = \sum_{n=1}^{\infty} (\omega_n c_n \cos \omega_n t - d_n \omega_n \sin \omega_n t)\sin\left(\frac{2n-1}{2l}\right)\pi x$$

Using $w_t(x,0) = 0$ then yields $c_n = 0$ for $n = 1, 2, \ldots$, so that

$$0.01\delta(x-l) = \sum d_n \cos \omega_n t \sin \frac{2n-1}{2l}\pi x$$

Multiplying by $\sin \frac{2m-1}{2l}\pi x$ and integrating from 0 to $l$ yields

$$0.01\int_0^l \delta(x-l)\sin\left(\frac{2m-1}{2l}\right)\pi x dx = c_m \int_0^l \sin^2\left(\frac{2m-1}{2l}\right)\pi x dx$$

using the orthogonality of $\sin \sigma_n x$..

$$0.01 \sin \frac{2m-1}{2}\pi = c_m \frac{l}{2},\ m = 1,2,3\ldots$$

so that $c_m = (.02)(-1)^{m+1}/l = (.004)(-1)^{m+1}$ and the solution is

$$w(x,t) = \sum_{n=1}^{\infty} (.004)(-1)^{n+1} \sin[(2n-1)(1132.28)t]\sin(2n-1)\pi x$$

For $n$ = 3 and $x$ = 0.5,

$$w(0.5,t) = (.004)[\sin 1132.28t - 0 - \sin 33968t]$$

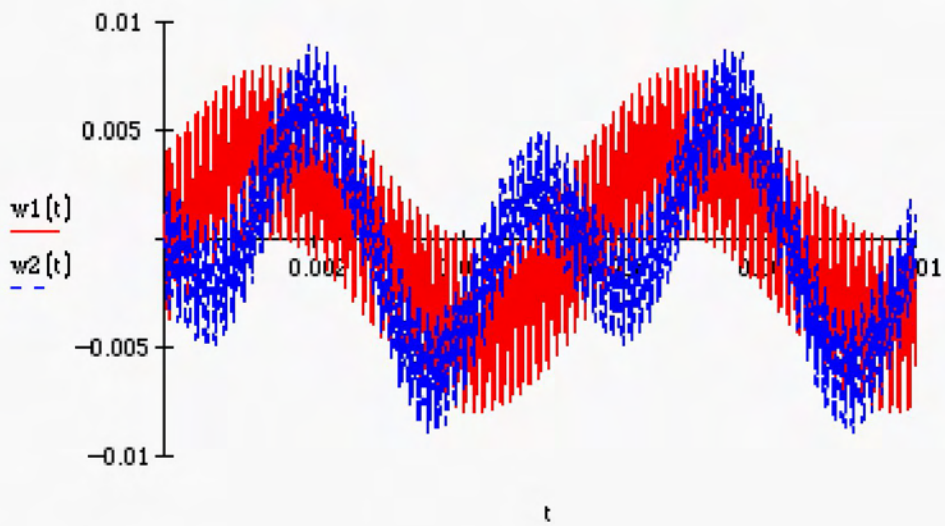
For $n$ = 3 and $x$ = $l$/2 = 0.25

$$w(.25,t) = (.004)[.707\sin 1132.28 - \sin 2264.56t + .707\sin 339684t]$$

These are plotted below using Mathcad:

$$w1(t) := 0.004\cdot(\sin(1132.28\cdot t) - \sin(339684\cdot t))$$

$$w2(t) := 0.004\cdot(0.707\cdot\sin(1132.28\cdot t) - \sin(2264.56\cdot t) + 0.707\cdot\sin(339684\cdot t))$$

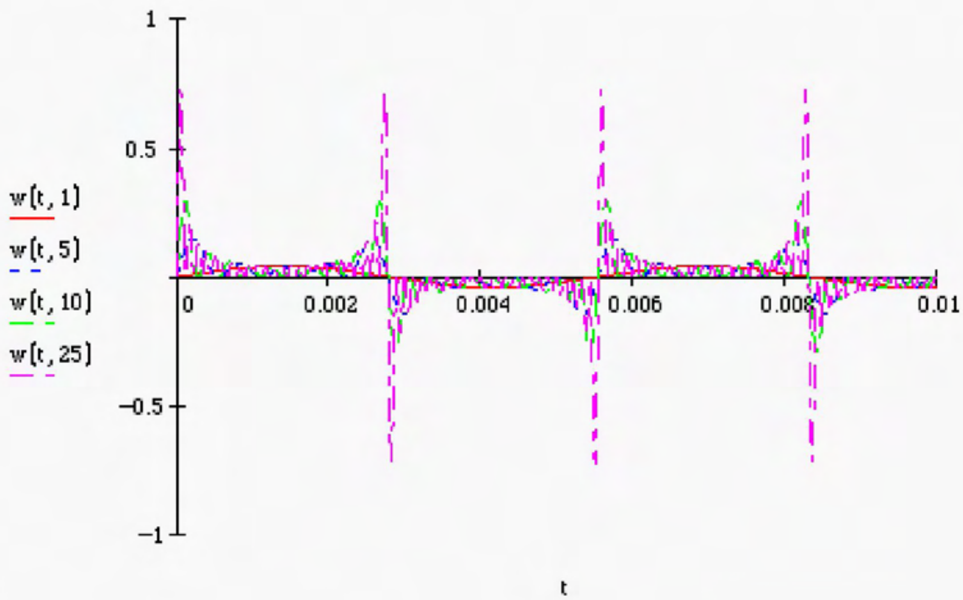


**6.27** Repeat the plots of Problem 6.26 for 5 modes, 10 modes, 15 modes, and so on, to answer the question of how many modes are needed in the summation of equation (6.27) in order to yield an accurate plot of the response for this system.

**Solution:** The following plots in Mathcad illustrate that it takes 10 modes to capture the behavior of this series, by plotting the formula of 6.26.

$x := .5$

$$w(t,N) := \sum_{n=1}^{N} (0.04)\cdot(-1)^{n+1}\cdot\left[\sin\left[(2\cdot n-1)\cdot 1132.28\cdot t\right]\right]\cdot\sin\left[(2\cdot n-1)\cdot\pi\cdot x\right]$$



**6.28** A moving bar is traveling along the *x* axis with constant velocity and is suddenly stopped at the end at $x = 0$, so that the initial conditions are $(x,0) = 0$ and $w(x,0) = v$. Calculate the vibration response.

**Solution:**

Model the bar as a free-free bar. Then from Table 6.2 the natural frequencies are $n\pi c/l$ and the mode shapes are $\cos(n\pi x/l)$. Thus the solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \cos(n\pi x/l)$$

Using the initial condition $w(x,t) = 0$ yields that $B_n = 0$ for $n = 1, 2, 3, \ldots$, i.e.

$$w(x,0) = 0 = \sum B_n \cos(n\pi x/l)$$

which is multiplied by $\cos(n\pi x/l)$ and integrated over (0,l) using orthogonality to get $B_n = 0$. Next differentiate

$$w(x,t) = \sum A_n \sin \omega_n t \cos n\pi x/l$$

to get $w_z(x,t)$, then set $t = 0$ to use the second initial condition.

$$w_t(x,0) = \sum A_n \omega_n \cos(0) \cos(n\pi x/l)$$

Modeling the initial velocity as $v\delta(x)$, multiplying by cos $m\pi x/l$ and integrating yields

$$\int_0^l \delta(x) v \cos(n\pi x/l) dx = \omega_n \left(\frac{l}{2}\right) A_n, \quad \text{or} \quad A_n = \frac{\partial V}{l\omega_n}$$

so that

$$w(x,t) = \frac{2v}{\pi c} \sum_{n=1}^{\infty} \left(\frac{1}{n}\right) \sin\left(\frac{n\pi ct}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

Note that Thomson uses a form of this problem as example 3 of section 5.3, but he models the moving beam as having a clamped free rather than free-free boundary. What do you think?

**6.29** Calculate the response of the clamped-clamped string of Section 6.2 to a zero initial velocity and an initial displacement of $w_0(x) = \sin(2\pi x/l)$. Plot the response at $x = l/2$.

**Solution:**

The clamped-clamped string has eigenfunction sin $n\pi x/l$ and solution given by equation (6.27) where the unknown coefficients $c_n$ and $d_n$ are given by equation (6.31) and (6.33) respectively. Since $\dot{w}_0 = 0$, equation 6.33 yields $c_n = 0$, $n$ = 1,2,3.. with $w_0 = \sin(2\pi x/l)$,

$$d_n = \frac{2}{l}\int_0^l \sin(2\pi x/l)\sin(n\pi x/l)dx$$

which is zero for each $n$ except $n$ =2, in which case $d_n = 1$. Hence

$$w(x,t) = \sin(2\pi ct/l)\sin(2\pi x/l)$$

For $x = l/2$

$$w(l/2,t) = \sin(2\pi ct/l)$$

which has a well known plot given in the following Mathcad session using the values for a piano wire.

$c := 11.89 \quad L := 1.4$

$$w(x,t) := \left[\sin\left[(2)\cdot\pi\cdot c\cdot\frac{t}{L}\right]\right]\cdot\sin\left(2\cdot\pi\cdot\frac{x}{L}\right)$$