Problems and Solutions Section 6.4 (6.30 through 6.39)

6.30 Calculate the first three natural frequencies of torsional vibration of a shaft of Figure 6.7 clamped at $x = 0$, if a disk of inertia $J_0 = 10$ kg m^2/rad is attached to the end of the shaft at $x = l$. Assume that $l = 0.5$ m, $J = 5$ m^4, $G = 2.5 \times 10^9$ Pa, $\rho =$ 2700 kg/m^3.

Solution: The equation of motion is $\ddot{\theta} = \dfrac{G}{\rho}\theta''$. Assume separation of variables:

$\theta = \phi(X)q(t)$ to get $\phi\ddot{q} = \dfrac{G}{\rho}\phi'' q$ or $\dfrac{\rho}{G}\dfrac{\ddot{q}}{q} = \dfrac{\phi''}{\phi} = -\sigma^2$ so that

$$\ddot{q} + \frac{G}{\rho}\sigma^2 q = 0 \text{ and } \phi'' + \sigma^2\phi = 0$$

where $\omega^2 = \dfrac{G}{\rho}\sigma^2$. The clamped-inertia boundary condition is $\theta(0,t) = 0$, and $-GJ\theta'(l,t) = J_0\ddot{\theta}(l,t)$. This yields that $\phi(0) = 0$ and

$$GJ\phi'(l)q(t) = J_0\phi(l)\ddot{q}(t) = J_0\phi(l)\frac{G}{\rho}\sigma^2 q(t)$$

or $$J\phi'(l) = J_0\frac{\sigma^2}{\rho}\phi(l)$$

The solution of the spatial equation is of the form

$$\phi(x) = A\sin\sigma x + B\cos\sigma x$$

but the clamped boundary condition yields $B = 0$. The inertia boundary condition

$$JA\sigma\cos\sigma l = J_0\frac{\sigma^2}{\rho}A\sin\sigma l$$

yields

$$\tan\sigma l = \frac{J}{J_0}\frac{\rho l}{\sigma l} = \frac{1}{\sigma l}\left(\frac{5\text{ m}^4}{10\text{kg m}^2}\right)\quad(2700\text{kg/m}^3)(0.5\text{m})$$

So the frequency equation is

$$\tan\sigma l = \frac{675}{\sigma l}$$

Using the MATLAB function **fsolve**; this has the solutions

$$\left.\begin{matrix}\sigma_1 l = 1.5685\\ \sigma_2 l = 4.7054\\ \sigma_3 l = 7.8424\end{matrix}\right\} \text{ or } \left\{\begin{matrix}\sigma_1 = 3.1369\\ \sigma_2 = 9.4108\\ \sigma_3 = 15.6847\end{matrix}\right.$$

Thus $\omega_1 = 3018.5$ rad/s $\Rightarrow$ $f_1 = 480.4$ Hz
$\omega_2 = 9055.6$ rad/s $\Rightarrow$ $f_2 = 1441.2$ Hz
$\omega_3 = 15092.6$ rad/s $\Rightarrow$ $f_3 = 2402.1$ Hz

6.31 Compare the frequencies calculated in the previous problem to the frequencies of the lumped-mass single-degree-of-freedom approximation of the same system.

Solution:

First calculate the equivalent torsional stiffness of the rod.

$$k = \frac{GJ}{l} = \frac{(2.5\times 10^9)(5)}{0.5} = 2.5\times 10^{10}$$

$$J_0\ddot{\theta} = -k\theta$$

$$J_0\ddot{\theta} + k\theta = 0$$

$$10\ddot{\theta} + 2.5\times 10^{10}\theta = 0 \quad \text{or} \quad \ddot{\theta} + 2.5\times 10^{9}\theta = 0$$

so that $\omega^2 = 2.5 \times 10^9$, $\omega = 5 \times 10^5$ rad/s or about 80,000 Hz, far from the 482 Hz of problem 6.30.

6.32 Calculate the natural frequencies and mode shapes of a shaft in torsion of shear modulus G, length l, polar inertia J, and density ρ that is free at $x = 0$ and connected to a disk of inertia J_0 at $x = l$.

Solution:

Assume zero initial conditions, i.e. $\theta(x,0) = \dot{\theta}(x,0) = 0$. From equation 6.66

$$\frac{\partial^2\theta(x,t)}{\partial t^2} = \left(\frac{G}{\rho}\right)\frac{\partial^2\theta(x,t)}{\partial x^2} \tag{1}$$

The boundary condition at $x = l$ and at $x = 0$ is

$$GJ\frac{\partial\theta(l,t)}{\partial x} = -J_0\frac{\partial^2\theta(l,t)}{\partial t^2} \qquad \frac{\partial\theta(0,t)}{\partial x} = 0$$

Using separation of variable in (1) of form $\theta(x,t) = \Theta(x)T(t)$ yields:

$$\frac{\Theta''(x)}{\Theta(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\sigma^2 \tag{2}$$

where $c^2 = \dfrac{G}{\rho}$ and σ^2 is a separation constant. (2) can now be rewritten as 2 equations

$$\Theta''(x) + \sigma^2\Theta(x) = 0$$

$$\ddot{T}(t) + c^2\sigma^2 T(t) = 0 \rightarrow \omega = \sigma = \sigma\sqrt{\frac{G}{\rho}}$$

from the boundary condition at $x = l$

$$GJ\Theta'(l)T(t) = -J_0\Theta(l)\ddot{T}(t)$$

$$-\frac{GJ}{J_0}\frac{\Theta'(l)}{\Theta(l)} = \frac{\ddot{T}(t)}{T(t)} = -c^2\sigma^2$$

$$\Theta'(l) = \frac{J_0}{GJ}\frac{G}{\rho}\sigma^2\Theta = \frac{J_0\sigma^2}{J\rho}\Theta(l)$$

The boundary condition at $x = 0$ yields simply $\Theta'(0) = 0$. The general solution is of the form

$$\Theta(x) = a_1 \sin \sigma x + a_2 \cos \sigma x \text{ so that } \Theta'(x) = a_1 \sigma \cos \sigma x - a_2 \sigma \sin \sigma x$$

The boundary conditions applied to these solutions yield:

$$\Theta'(l) = a_1 \sigma \cos \sigma l - a_2 \sigma \sin \sigma l = \frac{J_0 \sigma^2}{J\rho}[a_1 \sin \sigma l + a_2 \cos \sigma l]$$
$$a_1 \left[\cos \sigma l - \frac{J_0 \sigma^2}{J\rho} \sin \sigma l \right] = a_2 \left[\sin \sigma l + \frac{J_0 \sigma}{J\rho} \cos \sigma l \right]$$
$$\Theta'(0) = a_1 \sigma = 0 \rightarrow a_1 = 0$$
$$a_2 \left[\sin \sigma l + \frac{J_0 \sigma}{J\rho} \cos \sigma l \right] = 0$$

For the non-trivial solution of this last expression, the coefficients of a_2 must vanish, which yields

$$\tan \sigma l = -\frac{J_0}{J\rho}\sigma$$

This must be solved numerically for σ (except for the rigid body case of σ = 0) and the frequency is calculated from $\omega = \sigma\sqrt{\frac{G}{\rho}}$. The mode shapes are $\Theta(x) = a_2 \cos \sigma x$. Note the solution for σ is illustrated in figure 6.4 page 479 of the text.

6.33 Consider the lumped-mass model of Figure 4.21 and the corresponding three-degree-of-freedom model of Example 4.8.1. Let $J_1 = k_1 = 0$ in this model and collapse it to a two-degree-of-freedom model. Comparing this to Example 6.4.1, it is seen that they are a lumped-mass model and a distributed mass model of the same physical device. Referring to Chapter 1 for the effects of lumped stiffness on a rod in torsion (k_2), compare the frequencies of the lumped-mass two-degree-of-freedom model with those of Example 6.4.1.

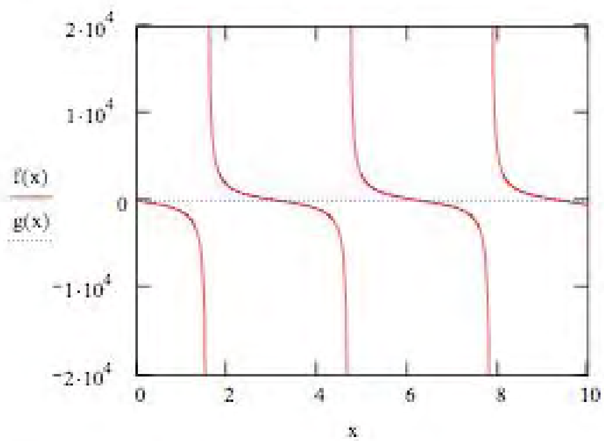
Solution: From Mathcad:

$ro := 7800 \qquad J := 5 \qquad J_1 := 10 \qquad J_2 := 10 \qquad L := 0.425 \qquad G := 80 \cdot 10^9$

$$a := \frac{ro \cdot J \cdot L}{J_1 + J_2} \qquad b := \frac{J_1 \cdot J_2}{(J_1 + J_2) \cdot ro \cdot J \cdot L}$$ from p.492

$a = 828.75 \qquad b = 3.017 \times 10^{-4}$

$f(x) := (b \cdot x^2 - a) \cdot \tan(x) \qquad g(x) := x$



$x_1 := 0 \qquad x_2 := 3.138$

$$\omega_1 := x_1 \cdot \sqrt{\frac{G}{ro \cdot L^2}} \qquad \boxed{\omega_1 = 0}$$

$$\omega_2 := x_2 \cdot \sqrt{\frac{G}{ro \cdot L^2}} \qquad \omega_2 = 2.365 \times 10^4 \qquad f_2 := \frac{\omega_2}{2\pi} \qquad \boxed{f_2 = 3.763 \times 10^3}$$

2 dof model

$$k := \frac{G \cdot J}{L} \qquad k = 9.412 \times 10^{11}$$

$$M := \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \qquad K := \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

$$M = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \qquad K = \begin{pmatrix} 9.412 \times 10^{11} & -9.412 \times 10^{11} \\ -9.412 \times 10^{11} & 9.412 \times 10^{11} \end{pmatrix}$$

$$\mathrm{sort}\left(\mathrm{eigenvals}\left(M^{-1} \cdot K\right)\right) = \begin{pmatrix} 1.526 \times 10^{-5} \\ 1.882 \times 10^{11} \end{pmatrix}$$

Natural frequencies: $\mathrm{nat\_freqs} := \mathrm{sort}\left(\sqrt{\mathrm{eigenvals}\left(M^{-1} \cdot K\right)}\right)$

$$\frac{\mathrm{nat\_freqs}}{2\pi} = \begin{pmatrix} 6.217 \times 10^{-4} \\ 6.905 \times 10^{4} \end{pmatrix}$$

6.34 The modulus and density of a 1-m aluminum rod are $E = 7.1 \times 10^{10}$ N/m^2, $G = 2.7 \times 10^{10}$ N/m^2, and $\rho = 2.7 \times 10^3$ kg/m^2. Compare the torsional natural frequencies with the longitudinal natural frequencies for a free-clamped rod.

Solution:

The appropriate boundary conditions are: $\theta'(0,t) = 0$ and $\theta(l,t) = 0$ for the rod and $w'(0,t) = 0 = w(l,t)$ for the bar. The separated equations are

$$\ddot{\theta} = \left(\frac{G}{\rho}\right)\theta'' \text{ and } \phi\ddot{q} = \left(\frac{G}{\rho}\right)\phi'' q$$

$$\ddot{q} + \left(\frac{G}{\rho}\right)\sigma^2 q = 0 \text{ and } \phi'' + \sigma^2\phi = 0$$

Solutions are

$$q_n = A_n \sin\omega_n t + B_n \cos\omega_n t \text{ and } \phi_n = C_n \sin\sigma_n x + D_n \cos\sigma_n x$$

where $\omega_n^2 = \dfrac{G}{\rho}\sigma_n^2$. But $\phi'(0) = 0$ so that $C_n = 0$. The other boundary condition yields $\phi_n(l) = D_n \cos\sigma_n l = 0$ so that

$$\sigma_n l = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, \ldots$$

Thus the torsional frequencies are

$$\omega_n = \sqrt{\frac{G}{\rho}}\sigma_n$$

and the longitudinal frequencies are

$$\omega_n = \sqrt{\frac{E}{\rho}}\sigma_n$$

where

$$\sigma_n = \frac{(2n-1)\pi}{2l}$$

From the values given $\sqrt{\dfrac{G}{\rho}}$ = 3162 m/s and $\sqrt{\dfrac{E}{\rho}}$ = 5128 m/s. Thus the natural frequencies of the longitudinal vibration are *1.6 times larger* than the torsional vibrations.

6.35 Consider the aluminum shaft of Problem 6.32. Add a disk of inertia J_0 to the free end of the shaft. Plot the torsional natural frequencies versus increasing the tip inertia J_0 of a single-degree-of-freedom model and for the first natural frequency of the distributed-parameter model in the same plot. Are there any values of J_0 for which the single-degree–of-freedom model gives the same frequency as the full distributed model?

Solution:

Refer to problem 6.32 of the rod clamped at $x = 0$ with inertia J_0 at $x = l$. The *sdof* model of the frequency is given in example 1.5.1 as

$$\omega = \sqrt{\frac{GJ}{lJ_0}}$$

where G = torsional rigidity, J = polar moment of inertia of the rod of length l and J_0 is the disc inertia. The first natural frequency according to distributed parameter theory is given in problem 6.30 as the solution of

$$\tan\sigma / 2 = -\frac{\rho}{\sigma J_0}, \quad \omega = \sigma\frac{G}{\rho}$$

which will have a solution for a given value of J_0 equivalent to that of the *sdof* system.

6.36 Calculate the mode shapes and natural frequencies of a bar with circular cross section in torsional vibration with free-free boundary conditions. Express your answer in terms of G, l, and ρ.

Solution:

The separated equations are $\ddot{q}+\left(\frac{G}{\rho}\sigma^2\right)q=0$ and $\phi''+\sigma^2\phi=0$

where $\omega_n=\sqrt{\frac{G}{\rho}}\sigma_n$. Thus

$$q_n=A_n\sin\omega_n t+B_n\cos\omega_n t \quad \text{and} \quad \phi_n=C_n\sin\sigma_n x+D_n\cos\sigma_n x$$

The boundary conditions are

$$\phi_n'(0)=0$$
$$\phi_n'(l)=0$$

But $\phi_n'=C_n\sigma_n\cos\sigma_n x-D_n\sigma_n\sin\sigma_n x$ so that $\phi_n'(0)=0\Rightarrow C_n=0$ and the frequency equation becomes $\phi_n'(l)=0=-D_n\sigma_n\sin\sigma_n 0$. This has the solution $\sigma_n l=n\pi$ or $\sigma_n=\frac{n\pi}{l}$. Hence

$$\omega_n=\sqrt{\frac{G}{\rho}}\frac{n\pi}{l} \quad \text{and} \quad \phi_n(x)=\cos\frac{n\pi x}{l}.$$

6.37 Calculate the mode shapes and natural frequencies of a bar with circular cross section in torsional vibration with fixed boundary conditions. Express you answer in terms of *G*, *l*, and ρ,

Solution: From equation 6.66

$$\frac{\partial^2\theta(x,t)}{\partial t^2} = \left(\frac{G}{\rho}\right)\frac{\partial^2\theta(x,t)}{\partial x^2}$$

Assume a solution of the form $\theta(x,t) = \Theta(x)T(t)$ so that

$$\Theta(x)\ddot{T}(t) = \frac{G}{\rho}\Theta''(x)T(t)$$

Separate where σ^2 is the separation constant and $c^2 = \dfrac{G}{\rho}$

$$\frac{\Theta''(x)}{\Theta(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\sigma^2$$

or $\Theta''(x) + \sigma^2\Theta(x) = 0$ and $\ddot{T}(t) + \sigma^2 c^2 T(t) = 0$ where $\omega = \sqrt{\dfrac{G}{\rho}}\sigma.$ The boundary conditions for a fixed-fixed rod are $\Theta(0) = 0$ and $\Theta(l) = 0$ from the solution of the spatial equations

$$\Theta(0) = a_2 = 0$$
$$\Theta(l) = a_1 \sin\sigma l = 0.$$

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For the non-trivial solution

$$\sin\sigma \mathrm{l} = 0$$
$$\sigma = \frac{\mathrm{n}\pi}{\mathrm{l}}, \quad n = 0,1,2,..$$

natural frequency

$$\omega = \sqrt{\frac{G}{\rho}}\frac{n\pi}{l}, \quad n = 1,2,...$$

mode shape

$$\Theta(x) = a_1 \sin\frac{n\pi}{l}x, \quad n = 0,1,2,...$$

6.38 Calculate the eigenfunctions of Example 6.4.1.

Solution:

From example 6.4.1 the eigenfunctions are

$$\Theta_n(x) = a_1 \sin\sigma_n x + a_2 \cos\sigma_n x \text{ or } \Theta_n(x) = A_n\left(-\frac{\sigma J_1}{\rho J}\sin\sigma_n x + \cos\sigma_n x\right)$$

where σ_n are determined by equation 6.8.4.

6.39 Show that the eigenfunctions of Problem 6.38 are orthogonal.

Solution:

Orthogonality requires $\int_0^1 \theta_n(x)\theta_m(x)dx = 0, \quad m \neq n.$ From direct calculation

$$\int_0^l\left(-\frac{\sigma J_1}{\rho J}\sin\sigma_n x + \cos\sigma_n x\right)\left(-\frac{\sigma J_1}{\rho J}\sin\sigma_m x + \cos\sigma_m x\right)dx$$

$$= \left(\frac{\sigma J_1}{\rho J}\right)^2 \int_0^l \sin\sigma_m x \sin\sigma_n x dx$$

$$-\frac{\sigma J_1}{\rho J}\int_0^l \sin\sigma_n x \sin\sigma_m x dx - \frac{\sigma J_1}{\rho J}\int_0^l \sin\sigma_m x \sin\sigma_n x dx$$

$$+\int_0^l \cos\sigma_n x \cos\sigma_m x dx$$

where each integral vanishes. Also one can use the same calculation as problem 6.3 since the natural frequencies have distinct values.