Problems and Solutions Section 6.5 (6.40 through 6.47)

6.40 Calculate the natural frequencies and mode shapes of a clamped-free beam. Express your solution in terms of *E*, *I*, ρ, and *l*. This is called the cantilevered beam problem.

Solution:

Clamped-free boundary conditions are

$$w(0,t) = w_x(0,t) = 0 \text{ and } w_{xx}(l,t) = w_{xxx}(l,t) = 0$$

assume *E, I, ρ, l* constant. The equation of motion is

$$\frac{\partial^2 w}{\partial t^2} + \left(\frac{EI}{\rho A}\right)\frac{\partial^4 w}{\partial x^4} = 0$$

assume separation of variables $w(x,t) = \phi(x)q(t)$ to get

$$\left(\frac{EI}{\rho A}\right)\frac{\phi''''}{\phi} = -\frac{\ddot{q}}{q} = \omega^2$$

The spatial equation becomes

$$\phi'''' - \left(\frac{\rho A}{EI}\right)\omega^2\phi = 0$$

define $\beta^4 = \dfrac{\rho A\omega^2}{EI}$ so that $\phi'''' - \beta^4\phi = 0$ which has the solution:

$$\phi = C_1 \sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x$$

Applying the boundary conditions
$w(0,t) = w_x(0,t) = 0$ and $w_{xx}(l,t) = w_{xxx}(l,t) = 0 \Rightarrow$
$\phi(0) = \phi'(0) = 0$ and $\phi''(l) = \phi'''(l) = 0$
Substitution of the expression for ϕ into these yields:
$C_2 + C_4 = 0$
$C_1 + C_3 = 0$
$-C_1\sin\beta l - C_2\cos\beta l + C_3\sinh\beta l + C_4\cosh\beta l = 0$
$-C_1\cos\beta l + C_2\sin\beta l + C_3\cosh\beta l + C_4\sinh\beta l = 0$
Writing these four equations in four unknowns in matrix form yields:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\beta l & -\cos\beta l & \sinh\beta l & \cosh\beta l \\ -\cos\beta l & \sin\beta l & \cosh\beta l & \sinh\beta l \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$$

For a nonzero solution, the determinant must be zero to that (after expansion)

$$\begin{vmatrix} -\sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta \\ -\cos\beta l - \cosh\beta & \sin\beta l - \sinh\beta \end{vmatrix} =$$

$$(-\sin\beta l - \sinh\beta l)(\sin\beta l - \sinh\beta l) -$$

$$(-\cos\beta l - \cosh\beta l)(-\cos\beta l - \cosh\beta l) = 0$$

Thus the frequency equation is cos βl cosh βl = -1 or $\cos\beta_n l = -\dfrac{1}{\cosh\beta_n l}$ and frequencies are $\omega_n = \sqrt{\dfrac{\beta_n^4 EI}{\rho A}}$. The mode shapes are

$$\phi_n = C_{1n}\sin\beta_n x + C_{2n}\cos\beta_n x + C_{3n}\sinh\beta_n x + C_{4n}\cosh\beta_n x$$

Using the boundary condition information that $C_4 = -C_2$ and $C_3 = -C_1$ yields

$$-C_1\sin\beta l - C_2\cos\beta l - C_1\sinh\beta l - C_2\cosh\beta l$$
$$-C_1(\sin\beta l + \sinh\beta l) = C_2(\cos\beta l + \cosh\beta l)$$

so that

$$C_1 = -C_2\left(\frac{\cos\beta l + \cosh\beta l}{\sin\beta l + \sinh\beta l}\right)$$

and the mode shapes can be expressed as:

$$\phi_n = -C_{2n}\left[-\left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sin\beta_n x + \cos\beta_n x\right.$$

$$\left. + \left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sinh\beta_n x - \cosh\beta_n x\right]$$

6.41 Plot the first three mode shapes calculated in Problem 6.40. Next calculate the strain mode shape [i.e., $X'(x)$], and plot these next to the displacement mode shapes $X(x)$. Where is the strain the largest?

Solution: The following Mathcad session yields the plots using the values of β taken from Table 6.4.

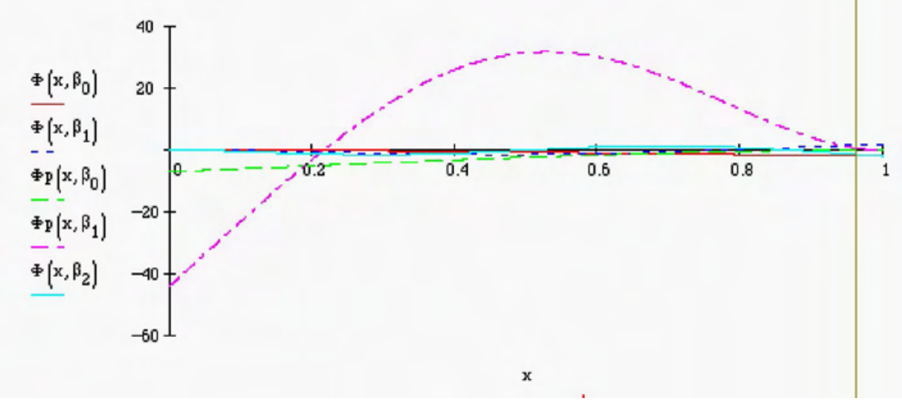
$$\beta := \begin{bmatrix} 1.87510407 \\ 4.69409113 \\ 7.85475744 \end{bmatrix}$$

$$\Phi(x,b) := \left[\left(-\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\cdot\sin(b\cdot x)\right)+\cos(b\cdot x)+\left(\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\right)\cdot\sinh(b\cdot x)-\cosh(b\cdot x)\right]$$

$$\left[\left(-\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\cdot\sin(b\cdot x)\right)+\cos(b\cdot x)+\left(\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\right)\cdot\sinh(b\cdot x)-\cosh(b\cdot x)\right]$$

$$\left[\left[\frac{-(\cos(b)+\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\cos(b\cdot x)\cdot b-\sin(b\cdot x)\cdot b+\frac{(\cos(b)+\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\cosh(b\cdot x)\cdot b\right]-\sinh(b\cdot x)\cdot b\right]$$

$$\Phi p(x,b) := \frac{-(-\cos(b)-\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\sin(b\cdot x)\cdot b^2-\cos(b\cdot x)\cdot b^2+\frac{(\cos(b)+\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\sinh(b\cdot x)\cdot b^2-\cosh(b\cdot x)\cdot b^2$$



The strain is largest at the free end.

6.42 Derive the general solution to a fourth-order ordinary differential equation with constant coefficients of equation (6.100) given by equation (6.102).

Solution:

From equation (6.100) with $\beta^4 = \rho A\omega^2 / EI$, the problem is to solve $X'''' - \beta^4 X = 0$. Following the procedure for the second order equations suggested in example 6.2.1 let $X(x) = Ae^{\lambda t}$ which yields

$$\left(\lambda^4 - \beta^4\right)Ae^{\lambda x} = 0 \text{ or } \lambda^4 = \beta^4$$

This characteristic equation in λ has 4 roots

$$\lambda = -\beta, \beta, -\beta j, \text{ and } \beta j$$

each of which corresponds to a solution, namely $A_1e^{-\beta x}$, $A_2e^{\beta x}$, $A_3e^{-\beta jx}$ and $A_4e^{\beta jx}$. The most general solution is the sum of each of these or

$$X(x) = A_1e^{-\beta x} + A_2e^{\beta x} + A_3e^{-\beta jx} + A_4e^{\beta jx} \qquad \text{(a)}$$

Now recall equation (A.19), i.e., $e^{\beta x} = \cos\beta x + j\sin\beta x$, and add equations (A.21) to yield $e^{\beta jx} = \sinh\beta x + \cosh\beta x$. Substitution of these two expressions into (a) yields

$$X(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

where *A*, *B*, *C*, and *D* are combinations of the constants A_1, A_2, A_3 and A_4 and may be complex valued.

6.43 Derive the natural frequencies and mode shapes of a pinned-pinned beam in transverse vibration. Calculate the solution for $w_0(x) = \sin 2\pi x/l$ min and $\dot{w}_0(x) = 0$.

Solution: Use $w(x,t) = \phi(x)q(t)$ in equation (6.29) with $\dot{w}(x,0) = 0$ or $\dot{q}(0) = 0$. Then the temporal solution $q = A \sin \omega t + B \cos \omega t$ with $\dot{q}(0) = 0$ yields $A = 0$. The spatial solution is $\phi = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x$ where $\beta^4 = \dfrac{\rho A \omega^2}{EI}$. The boundary conditions become

$$\phi(0) = \phi''(0) = \phi(l) = \phi''(l) = 0$$

Applied to $\phi(x)$ these yield the matrix equation

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \sin \beta l & \cos \beta l & \sinh \beta l & \cosh \beta l \\ -\sin \beta l & -\cos \beta l & \sinh \beta l & \cosh \beta l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0$$

But $C_2 + C_4 = 0$ and -$C_2 + C_4 = 0$ so $C_2 = C_4 = 0$ and this reduces to

$$\begin{bmatrix} \sin \beta l & \sinh \beta \\ -\sin \beta & \sinh \beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = 0$$

or $\sin \beta l \sinh \beta l + \sin \beta l \sinh \beta l = 0$,

$C_3 = -\dfrac{C_1 \sin \beta l}{\sinh \beta l}$, and $-C_1 \sin \beta l - C_1 \sin \beta l = 0$ so that the frequency equation

becomes $\sin \beta l = 0$ and thus $\beta_n l = n\pi$, $n = 1,2,3,\ldots$ and $\beta_n = \dfrac{n\pi}{l}$, $n = 1,2,3,\ldots$so

that $C_3 = 0$ and the frequencies are $\omega_n = \left(\dfrac{n\pi}{l}\right)^2 \sqrt{\dfrac{EI}{\rho A}}$ with mode shapes $\phi_n(x) =$

$C_{1n} \sin \beta_n x$. The total solution is the series $w(x,t) = \sum_{n=1}^{\infty} \{\beta_n \cos \omega_n t \sin \beta_n x\}$.

Applying the second initial condition yields $w(x,0) = \sin \dfrac{2\pi x}{l} = \sum_{n=1}^{\infty} \beta_n \sin \dfrac{n\pi x}{l}$

and therefore

$$B_n = \begin{cases} 0 & n = 1 \\ & n = 3,4,\ldots \\ 1 & n = 2 \end{cases}$$

so that

$$w(x,t) = \cos\omega_2 t \sin\frac{2\pi x}{l}$$

6.44 Derive the natural frequencies and mode shapes of a fixed-fixed beam in transverse vibration.

Solution: Follow example 6.5.1 to get the solution in the 5th entry of table 6.4. The spatial equation for the transverse vibration of a beam has solution of the form (6.102)

$$X(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x$$

where $\beta^4 = \rho A \omega^2 / EI$. The clamped boundary conditions are given by equation (6.94) as $X(0) = X'(0) = X(l) = X'(l) = 0$. Applying these boundary conditions to the solution yields

$$X(0) = 0 = a_1(0) + a_2(1) + a_3(0) + a_4(1) \qquad (1)$$

$$X'(0) = 0 = \beta a_1(1) - \beta a_2(0) + \beta a_3(1) + \beta a_4(0) \qquad (2)$$

$$X(l) = 0 = a_1 \sin \beta l + a_2 \cos \beta l + a_3 \sinh \beta l + a_4 \cosh \beta l \qquad (3)$$

$$X'(l) = 0 = \beta a_1 \cos \beta l - \beta a_2 \sin \beta l + \beta a_3 \cosh \beta l + \beta a_4 \sinh \beta l$$
(4)

dividing (2) and (3) by $\beta \neq 0$ and writing in matrix form yields

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sin \beta l & \cos \beta l & \sinh \beta l & \cosh \beta l \\ \cos \beta l & -\sin \beta l & \cosh \beta l & \sinh \beta l \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix must have zero determinant for a nonzero solution for the a_n. Taking the determinant yields (expanding by minors across the top row).

$$\sinh^2 \beta l - \cosh^2 \beta l - \sin \beta l \sinh \beta l + \cos \beta l \cosh \beta l +$$
$$\cos \beta l \cosh \beta l \sin \beta l \sinh \beta l - \sin^2 \beta l - \cos^2 \beta l = 0$$

which reduces to

$$-1 + 2 \cos \beta l \cosh \beta l - 1 = 0 \quad \text{or} \quad \cos \beta l \cosh \beta l = 1$$

since $\sinh^2 \beta l - \cosh^2 \beta l = -1$ and $\sin^2 x + \cos^2 x = 1$. The solutions of this characteristic equation are given in table 6.4. Next from equation (1) $a_2 = -a_4$ and from equation (2) $a_1 = -a_3$ so equation (3) can be written as

$$-a_3 \sin \beta l - a_4 \cos \beta l + a_3 \sinh \beta l + a_4 \cosh \beta l = 4$$

Solving this for a_3 yields

$$a_3 = a_4 \left(\frac{\cos \beta l - \cosh \beta l}{\sinh \beta l - \sin \beta l} \right)$$

Recall also that $a_1 = -a_3$. Substitution into the solution $X(x)$ and factoring out a_4 yields

$$X(x) = a_4 \cosh \beta x - \cosh \beta x - \left(\frac{\cos \beta l - \cosh \beta l}{\sin \beta l - \sinh \beta l} \right) \left(\sinh \beta x - \sin \beta x \right)$$

in agreement with table 6.4. Note that a_4 is arbitrary as it should be.

6.45 Show that the eigenfunctions or mode shapes of Example 6.5.1 are orthogonal. Make them normal.

Solution:

The easiest way to show the orthogonality is to use the fact that the eigenvalues are not repeated and follow the solution to problem 6.2. The eigenfunctions are (table 6.4 or example 6.5).

$$X_n(x) = a_n \left\{\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right\}$$

Note that the constant a_n is arbitrary (a constant times a mode shape is still a mode shape) and normalizing involves choosing the constant a_n so that $\int X_n X_n dx = 1$. Calculating this integral yields:

$$\begin{aligned} a_n^2\int_0^l &\left\{\cosh^2\beta_n x - 2\cos\beta_n x\cosh\beta_n x + \cos^2\beta_n x\right. \\ &- 2\sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\left(\cosh\beta_n x - \cos\beta_n x\right) \\ &\left.+\sigma_n^2\left(\sinh^2\beta_n x - 2\sin\beta_n x\sinh\beta_n x + \sin^2\beta_n x\right)\right\}dx \end{aligned}$$

so

$$\begin{aligned} 1 = a_n^2 &\left[\frac{1}{\beta_n}\left(\frac{\sinh 2\beta_n l + \sin 2\beta_n l}{4}\right) + \beta_n l\right] \\ &- \frac{1}{\beta_n}\left(\sinh\beta_n l\sin\beta_n l + \cos\beta_n l\cosh\beta_n l\right) - \frac{\sigma_n}{\beta_n}\cos^2\beta_n l + \cosh 2\beta_n l \\ &+ \sinh\beta_n l\left(\sin\beta_n l + \cos\beta_n l\right) - \cosh\beta_n l\left(\cos\beta_n - \sin\beta_n l\right) \\ &+ \frac{\sigma_n^2}{\beta_n}\left[\frac{\sinh^2\beta_n l - \sin 2\beta_n l}{4} - 1 - \sin\beta_n l\sinh\beta_n l + \cosh\beta_n l\cos\beta_n l\right] \end{aligned}$$

So denoting the term in [ ] as γ_n and solving for $a_n = 1/\sqrt{\gamma_n}$ yields the normalization constant.

6.46 Derive equation (6.109) from equations (6.107) and (6.108).

Solution:

Using subscript notation for the partial derivatives, equation (6.108) with $f = 0$ yields an expression for φ_x, i.e.

$$\varphi_x = (\kappa AGW_{xx} - \rho Aw_{tt}) / \kappa^2 AG \tag{a}$$

Equation (6.107) can be differentiated once with respect to x to yield a middle term identical to the first term of equation (6.108). Substitution yields

$$EI\varphi_{xx} + \rho Aw_{tt} = \rho I\varphi_{xtt} \tag{b}$$

Equation (a) can be differentiated twice with respect to time to get an expression for $\rho I\varphi_{xx}$ in terms of $w(x,t)$ which when substituted into (b) yields

$$EI\varphi_{xxx} + \rho Aw_{tt} = \rho Iw_{xxtt} - \left(\rho^2 I / \kappa^2 G\right)w_{tttt}$$

The first term $EI\varphi_{xxx}$ can be eliminated by differentiating (a) twice with respect to x to yield

$$EI\left(\kappa^2 AGw_{xxxx} - \rho Aw_{ttxx}\right) + \rho Aw_{tt} = \kappa^2 AGw_{xxtt} - \rho AEIw_{tttt}$$

when substituted into (c). This is an expression in $w(x,t)$ only. Rearranging terms and dividing by $\kappa^2 AG$ yields equation (6.109).

6.47 Show that if shear deformation and rotary inertia are neglected, the Timoshenko equation reduces to the Euler-Bernoulli equation and the boundary conditions for each model become the same.

Solution:

This is a bit of a discussion problem. Since ρI is the inertia of the beam in rotation about φ the term ρIw_{xxxtt} represents rotary inertia. The term $(\rho IE/\kappa^2 G)w_{tttt}$ is the shear distortion and the term $(\rho^2 I/\kappa^2 G)w_{xxtt}$ is a combination of shear distortion and rotary inertia. Removing these terms from equation (6.109) results in equation (6.92).