Problems and Solutions Section 6.6 (6.48 through 6.52)

6.48 Calculate the natural frequencies of the membrane of Example 6.6.1 for the case that one edge $x = 1$ is free.

Solution:

The equation for a square membrane is

$$w_{tt} + w_{yy} = \left(\frac{\rho}{\tau} w_{tt}\right)$$

with boundary condition given by $w(0,y) = 0$, $w_x(l,y) = 0$, $w(x,0) = 0$, $w(x,l) = 0$. Assume separation of variables $w = X(x)Y(y)q(t)$ which yields

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2}\frac{\ddot{q}}{q} = -\omega^2 \quad \text{where } \; c = \sqrt{\rho / \tau}$$

Then

$$\ddot{q} + c^2\omega^2 q = 0$$

is the temporal equation and

$$\frac{X''}{X} = -\omega^2 - \frac{Y''}{Y} = -\alpha^2$$

yields

$$X'' + \alpha^2 X = 0$$
$$Y'' + \gamma^2 Y = 0$$

as the spatial equation where $\gamma^2 = \omega^2 - \alpha^2$ and $\omega^2 = \alpha^2 + \gamma^2$. The separated boundary conditions are $X(0) = 0$, $X'(l) = 0$ and $Y(0) = Y(l) = 0$. These yield

$$X = A\sin\alpha x + B\cos\alpha x$$
$$B = 0$$
$$A\cos\alpha l = 0$$
$$\alpha_n l = \frac{(2n-1)\pi}{2}$$
$$\alpha_n = \frac{(2n-1)\pi}{2l}$$

Next $Y = C \sin \gamma y + D \cos \gamma y$ with boundary conditions which yield $D = 0$ and $C \sin \gamma l = 0$. Thus

$$\gamma_m = m\pi l$$

and for $l = 1$ we get $a_n = \dfrac{(2n-1)\pi}{2}$, for $\gamma_m = m\pi$ $n,m = 1, 2, 3, \ldots$

$$\omega_{nm}^2 = \alpha_n^2 + \gamma_m^2 = \frac{(2n-1)^2\pi^2}{4} + m^2\pi^2 = \left[\frac{(2n-1)^2 + 4m^2}{4}\right]\pi^2$$

$$c^2\omega_{nm}^2 = c^2\left[\frac{(2n-1)^2 + 4m^2}{4}\right]\pi^2$$

So that

$$\omega_{nm} = \sqrt{(2n-1)^2 + 4m^2}\,\frac{c\pi}{2}$$

are the natural frequencies.

6.49 Repeat Example 6.6.1 for a rectangular membrane of size *a* by *b*. What is the effect of *a* and *b* on the natural frequencies?

Solution:

The solution of the rectangular membrane of size $a \times b$ is the same as given in example 6.6.1 for a unit membrane until equation 6.13.1. The boundary condition along $x = a$ becomes

$$A_1 \sin \alpha a \sin \gamma y + A_2 \sin \alpha a \cos \gamma y = 0$$

or

$$\sin \alpha a (A_1 - \sin \gamma y + A_2 \cos \gamma y) = 0$$

Thus $\sin \alpha a = 0$ and $\alpha a = n\pi$ or $\alpha = n\pi/a$, $n = 1, 2, \ldots$ Similarly, the boundary conditions along $y = b$ yields that

$$\gamma = \frac{n\pi}{b} \quad n{=}1,2,3,\ldots$$

Thus the natural frequency becomes

$$\omega_{nm} = \pi\sqrt{a^2 n^2 + b^2 m^2} \quad n, m = 1, 2, 3, \ldots$$

Note that ω_{nm} are no longer repeated, i.e., $\omega_{12} \neq \omega_{21}$, etc.

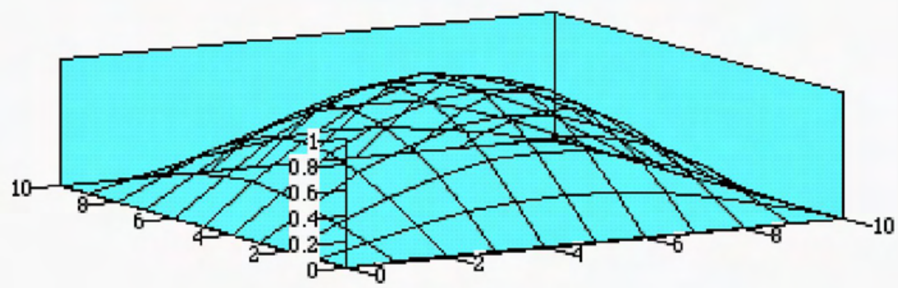
6.50 Plot the first three mode shapes of Example 6.6.1.

Solution: A three mesh routine from any of the programs can be used. Mathcad results follow for the 11, 12, 21 and 31 modes:

$N := 10 \quad i := 0..N \quad j := 0..N \quad x_i := i \cdot 0.1 \quad y_j := j \cdot 0.1$

$w(x, y) := (\sin(\pi \cdot x)) \cdot \sin(\pi \cdot y)$

$M_{(i,j)} := w(x_i, y_j)$

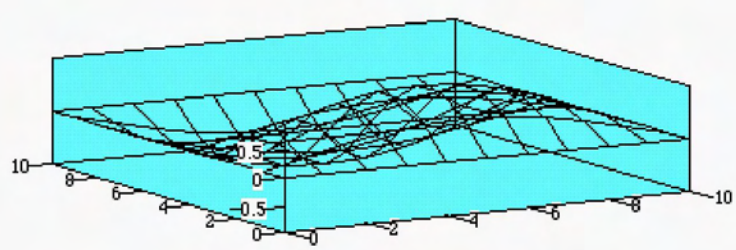


M

$N := 10 \quad i := 0..N \quad j := 0..N \quad x_i := i \cdot 0.1 \quad y_j := j \cdot 0.1$

$w(x, y) := (\sin(\pi \cdot x)) \cdot \sin(2\ \pi \cdot y)$

$M_{(i,j)} := w(x_i, y_j)$

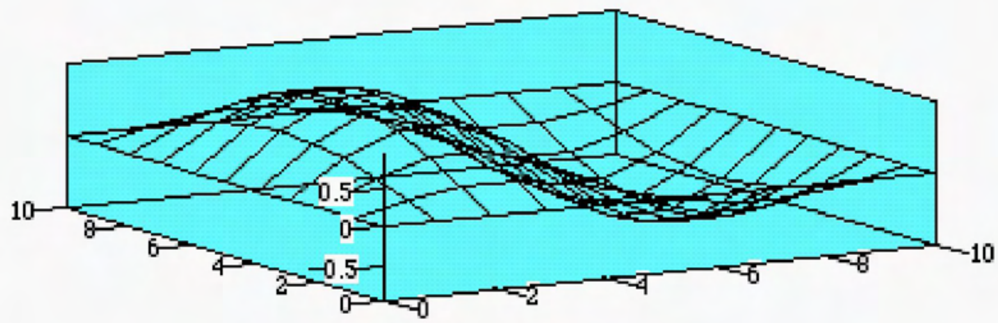


M

$N := 10 \quad i := 0..N \quad j := 0..N \quad x_i := i \cdot 0.1 \quad y_j := j \cdot 0.1$

$w(x, y) := (\sin(2\ \pi \cdot x)) \cdot \sin(\pi \cdot y)$

$M_{(i, j)} := w(x_i, y_j)$



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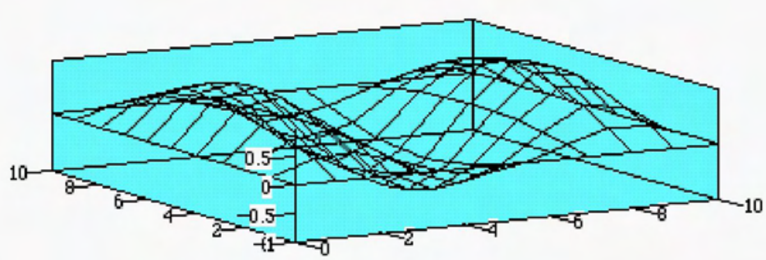
$N := 10 \quad i := 0..N \quad j := 0..N \quad x_i := i \cdot 0.1 \quad y_j := j \cdot 0.1$

$w(x, y) := (\sin(3\ \pi \cdot x)) \cdot \sin(\pi \cdot y)$

$M_{(i, j)} := w(x_i, y_j)$



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6.51 The lateral vibrations of a circular membrane are given by

$$\frac{\partial^2 \omega(r,\phi,t)}{\partial r^2} + \frac{1}{r}\frac{\partial \omega(r,\phi,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \omega(r,\phi,t)}{\partial \phi \partial r} = \frac{\rho}{\tau}\frac{\partial^2 \omega(r,\phi,t)}{\partial t^2}$$

where r is the distance form the center point of the membrane along a radius and ϕ is the angle around the center. Calculate the natural frequencies if the membrane is clamped around its boundary at $r = R$.

Solution:

This is a tough problem. Assign it only if you want to introduce Bessel functions. The differential equation of a circular membrane is:

$$\frac{\partial^2 W(r,\phi)}{\partial r^2} + \frac{1}{r}\frac{\partial W(r,\phi)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W(r,\phi)}{\partial \phi^2} + \beta^2 W(r,\phi) = 0$$

$$\beta^2 = \left(\frac{\omega}{c}\right)^2 \qquad c = \frac{T}{\rho}$$

Assume:

$$W(r,\phi) = F(r)G(\phi)$$

The differential equation separates into:

$$\frac{d^2 G}{d\phi^2} + m^2 G = 0$$

$$\frac{d^2 F}{dr^2} + \frac{1}{r}\frac{dF_1}{dr} + \left(\beta^2 - \frac{m^2}{r^2}\right)F = 0$$

Since the solution in ϕ must be continuous, m must be an integer. Therefore

$$G_m(\phi) = B_{1m}\sin m\phi + B_{2m}\cos m\phi$$

The equation in r is a Bessel equation and has the solution

$$F_m(r) = B_{3m}J_m(\beta r) + B_{4m}Y_m(\beta r)$$

Where $J_m(\beta r) + Y_m(\beta r)$ are the m^{th} order Bessel functions of the first and second kind, respectively. Writing the general solution $F(r)G(\phi)$ as

$$W_m(r,\phi) = A_{1m}J_m(\beta r)\sin m\phi + A_{2m}J_m(\beta r)\cos m\phi$$
$$+ A_{3m}Y_m(\beta r)\sin m\phi + A_{4m}Y_m(\beta r)\cos m\phi$$

Enforcing the boundary condition

$$W_m(R,\phi) = 0 \quad m = 0,1,2,...$$

Since every interior point must be finite and $Y_m(\beta r)$ tends to infinity as r $\rightarrow$ 0, A_{3m} = A_{4m} = 0. At $r = R$

$$W_m(R,\phi) = A_{1m} J_m(\beta R)\sin m\phi + A_{2m} J_m(\beta R)\cos m\phi = 0$$

This can only be satisfied if

$$J_m(\beta R) = 0 \quad m = 1,2,...$$

For each m, $J_m(\beta R) = 0$ has an infinite number of solutions. Denote β_{mn} as the nth root of the mth order Bessel function of the first kind, normalized by R. Then the natural frequencies are:

$$\omega_{mn} = c\beta_{mn}$$

6.52 Discuss the orthogonality condition for Example 6.6.1.

Solution:

The eigenfuncitons of example 6.6.1 are given as

$$X_n(x)Y_n(y) = A_{nm} \sin m\pi x \sin n\pi y$$

Orthogonality in this case is generalized to two dimensions and becomes

$$\int_0^1 \int_0^1 A_{nm} A_{pq} \sin m\pi x \sin n\pi y \sin p\pi y \sin q\pi y dx dy = 0 \quad mn \neq pq$$

Integrating yields

$$A_{nm} A_{pq} \int_0^1 \sin n\pi x \sin p\pi x dx \int_0^1 \sin m\pi g \sin g y dy$$
$$= A_{nm} A_{pq} \left[\frac{\sin(n-p)\pi x}{2(n-p)} - \frac{\sin(n+p)\pi x}{2(m+p)} \right] \left[\frac{\sin(m-1)\pi x}{2(m-q)} - \frac{\sin(m+p)\pi x}{2(m+p)} \right]$$

Evaluating at $x = 0$ and $x = 1$ this expression is zero. The expression is also zero provided $n = p$ and $n \neq q$ illustrating that the modes are in fact orthogonal.