Problems and Solutions Section 6.7 (6.53 through 6.63)

6.53 Calculate the response of Example 6.7.1 for l = 1 m, $E = 2.6 \times 10^{10}$ N/m^2 and $\rho = 8.5 \times 10^3$ kg/m^3. Plot the response using the first three modes at $x = l/2$, $l/4$, and $3l/4$. How many modes are needed to represent accurately the response at the point $x = l/2$?

Solution:

$$w(x,t) = \sum_{n=1}^{\infty}\left(\frac{0.02}{l^2\sigma_n^2}(-1)^{n+1}\right)e^{-0.01\omega_n t}\cos\omega_{2n}t\sin\sigma_n x$$

Where

$$\sigma_n = \frac{(2n-1)\pi}{2l}$$
$$\omega_n = \sigma_n\sqrt{\frac{E}{\rho}}$$
$$\omega_{dn} = 0.9999\omega_n$$

For l = 1 m

$$E = 2.6\times10^{10}\ \text{N/m}^2$$
$$\rho = 8.5\times10^3\ \text{kg/m}^3$$

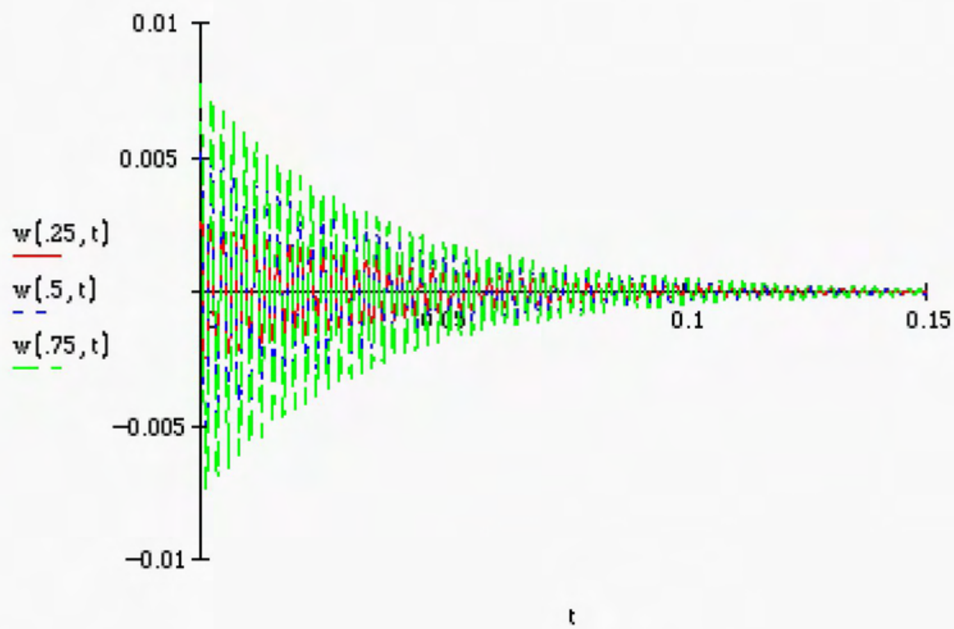
Response using first three modes at $x = \frac{l}{2}, \frac{l}{4}, \frac{3l}{4}$ plotted below.

Three modes accurately represents the response at $x = \frac{l}{2}$. The error between a three and higher mode approximation is less than 0.2%.

$$\sigma(n) := \frac{2\cdot n - 1}{2}\cdot\pi \qquad E := 2.6\cdot 10^{10} \qquad \rho := 8.5\cdot 10^{3}$$

$$\omega(n) := \sigma(n)\cdot\sqrt{\frac{E}{\rho}}$$

$$w(x,t) := \sum_{n=1}^{3} \frac{.02}{\sigma(n)^2}(-1)^{n+1}\cdot e^{-0.01\cdot\omega(n)\cdot t}\cdot\cos(\omega(n)\cdot t)\cdot\sin(\sigma(n)\cdot x)$$



6.54 Repeat Example 6.7.1 for a modal damping ratio of $\zeta_n = 0.01$.

Solution: Using $\zeta_n = 0.01$ and the frequency given in the example

$$\omega_{dn} = \omega_n\sqrt{1-\zeta_n^2} = 0.995\omega_n, \quad \omega_n = \frac{2n-1}{2l}\sqrt{\frac{E}{\rho}} = \sigma_n\sqrt{\frac{E}{\rho}}$$

The time response is then $T_n(t) = A_n e^{-0.1\omega_n t}\sin(\omega_{dn}t + \phi_n)$ and the total solution is:

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-0.1\omega_n t}\sin(\omega_{dn}t+\phi_n)\sin\frac{(2n-1)}{2l}\pi x$$

The initial conditions are:

$$w(x,0) = 0.01\frac{x}{l}\text{ m} \text{ and } w_t(x,0) = 0$$

Therefore:

$$0.01\frac{x}{l} = A_n \sin\phi_n \sin\sigma_n x$$

Multiply by $\sin\sigma_m x$ and integrate over the length of the bar to get

$$0.01\frac{(-1)^{m+1}}{l\,\sigma_m^2} = A_m \sin\phi_m \frac{l}{2} \quad m = 1,2,3,\ldots$$

From the velocity initial condition

$$w_t(x,0) = 0 = \sum_{n=1}^{\infty} A_n\left[-0.1\omega_n\sin\phi_n + \omega_{dn}\cos\phi_n\right]\sin\sigma_n x$$

Again, multiply by $\sin\sigma_m x$ and integrate over the length of the bar to get

$$A_m(-0.1\omega_n\sin\phi_n + \omega_{dn}\cos\phi_n)\frac{l}{2} = 0$$

Since A_m is not zero this yields:

$$\tan\phi_n = \frac{\sin\phi_n}{\cos\phi_n} = \frac{\sqrt{1-\zeta_n^3}}{0.1} = 9.9499 \Rightarrow \phi_n = 1.4706 \text{ rad } = 84.3°$$

Substitution into the equation from the displacement initial condition yields:

$$A_m = \frac{0.01}{l^2\sigma_m^2}(-1)^{m+1}\frac{1}{\sin\phi_n} = \frac{0.0201}{l^2\sigma_m^2}(-1)^{m+1}$$

The solution is then

$$w(x,t) = \sum_{n=1}^{\infty}\frac{0.01}{l^2\sigma_m^2}(-1)^{m+1}e^{-0.1\omega_n t}\sin(\omega_{dn}t+\phi_n)\sin\frac{(2n-1)}{2l}\pi x$$

6.55 Repeat Problem 6.53 for the case of Problem 6.54. Does it take more or fewer modes to accurately represent the response at $l/2$?

Solution: Use the result given in 6.54 and

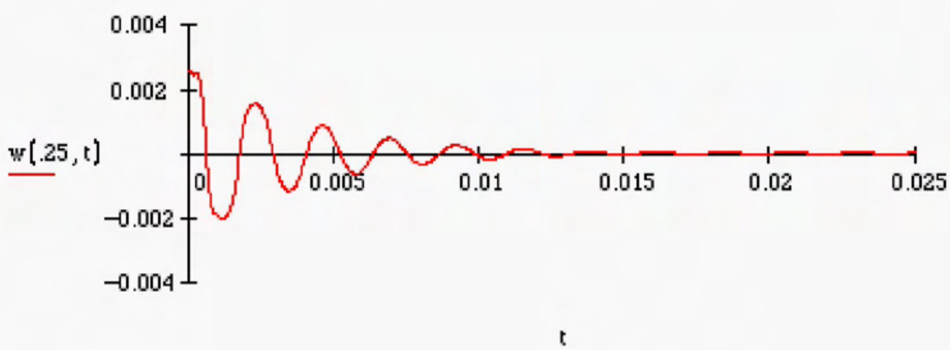
$$l = 1 \text{ m}$$
$$E = 2.6 \times 10^{10} \text{ N/m}^2$$
$$\rho = 8.5 \times 10^3 \text{ kg/m}^3$$

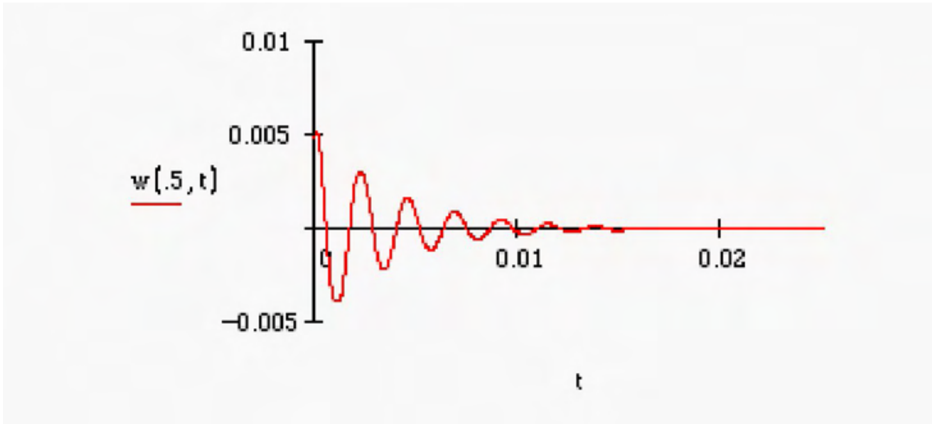
The response is plotted below at $x = \frac{l}{4}, \frac{l}{2}, \frac{3l}{4}$. An accurate representation of the response is obtained with three modes. The error between a three mode and a higher mode representation is always less than 0.2%. The results here are from Mathcad:

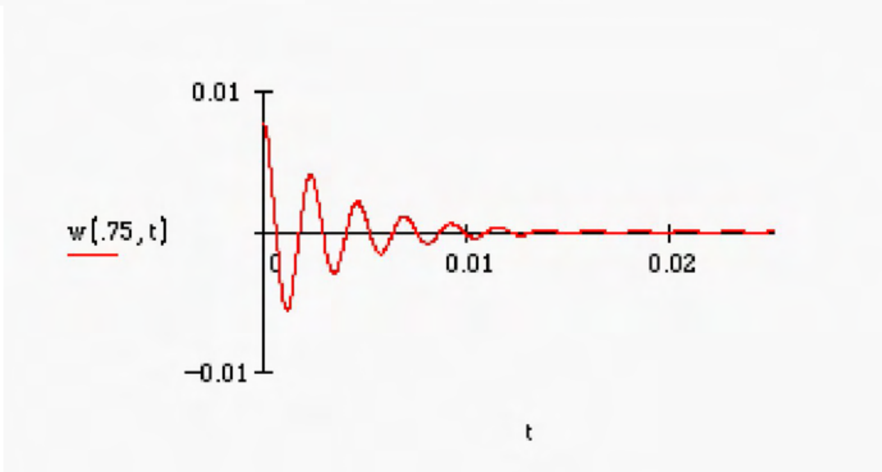
$$\sigma(n) := \frac{2 \cdot n - 1}{2} \cdot \pi \qquad E := 2.6 \cdot 10^{10} \qquad \rho := 8.5 \cdot 10^3$$

$$\omega(n) := \sigma(n) \cdot \sqrt{\frac{E}{\rho}} \qquad \omega d(n) := 0.995 \cdot \omega(n)$$

$$w(x,t) := \sum_{n=1}^{3} \frac{.0201}{\sigma(n)^2} (-1)^{n+1} \cdot e^{-0.1 \cdot \omega(n) \cdot t} \cdot \sin(\omega d(n) \cdot t + 1.4706) \cdot \sin(\sigma(n) \cdot x)$$




0.01
0.005
w(.5,t)
0
0.01
0.02
−0.005
t


0.01
w(.75,t)
0
0.01
0.02
−0.01
t

6.56 Calculate the form of modal damping ratios for the clamped string of equation (6.151) and the clamped membrane of equation (6.152).

Solution:

(a) For the string:

$$\rho w_{tt} + \gamma w_t - \tau w_{xx} = 0$$
$$\rho \phi \ddot{q} + \gamma \phi \dot{q} - \tau \phi'' q = 0$$
$$\frac{\rho}{\tau}\frac{\ddot{q}}{q} + \frac{\gamma}{\tau}\frac{\dot{q}}{q} = \frac{\phi''}{\phi} = -\sigma^2$$
$$\ddot{q} + \left(\frac{\gamma}{\rho}\right)\dot{q} + \left(\frac{\tau}{\rho}\right)\sigma^2 q = 0$$

$\phi'' + \sigma^2 \phi = 0$ which has the solution $\phi = A\sin\sigma x + B\cos\sigma x$. The boundary conditions $\phi(0) = \phi(l) = 0$ yield $\sigma_n = \dfrac{n\pi}{l}, \quad n = 1, 2, 3, \ldots$

$$\omega_n^2 = \left(\frac{\tau}{\rho}\right)\sigma_n^2 = \frac{\tau}{\rho}\left(\frac{n\pi}{l}\right)^2$$
$$2\zeta_n \omega_n = \frac{\gamma}{\rho}$$
$$\zeta_n = \frac{\gamma}{2\rho}\sqrt{\frac{\rho}{\pi}}\left(\frac{n\pi}{l}\right)$$
$$\zeta_n = \frac{\gamma}{2\sqrt{\rho\tau}}\left(\frac{n\pi}{l}\right)$$

(b) For the membrane

$$\frac{\rho}{\tau} w_{tt} + \frac{\gamma}{\tau} w_t = w_{xx} + w_{yy}$$
$$\left(\frac{\rho}{\tau}\right) XY\ddot{q} + \left(\frac{\gamma}{\tau}\right) XY\dot{q} = X''Yq + XY''q$$
$$\left(\frac{\rho}{\tau}\right)\frac{\ddot{q}}{q} + \left(\frac{\gamma}{\tau}\right)\frac{\dot{q}}{q} = \frac{X''}{X} + \frac{Y''}{Y} = -\beta^2$$
$$\ddot{q} + \left(\frac{\gamma}{\rho}\right)\dot{q} + \left(\frac{\tau}{\rho}\right)\beta^2 q = 0$$

$\frac{X''}{X} = -\frac{Y''}{Y} - \beta^2 = -\alpha^2$ .The boundary conditions are $X(0) = X(l) = 0$ and $Y(0) = Y(l) = 0$. The two spatial solutions become

$$X'' + \alpha^2 = 0 \qquad Y'' + \gamma^2 Y = 0$$

$$X = A\sin\alpha x + B\cos\alpha x \qquad Y = C\sin\gamma x + D\cos\gamma x$$

$$B = 0 \qquad D = 0$$

$$\alpha_n + \frac{n\pi}{l} \quad n = 1,2,3,\ldots \qquad \gamma_m = \frac{n\pi}{l} \quad m = 1,2,3,\ldots$$

Thus

$$\beta_{mn}^2 = \left(n^2 + m^2\right)\left(\frac{\pi}{l}\right)^2$$

$$\omega_{mn}^2 = \frac{\tau}{\rho}\left(n^2 + m^2\right)\left(\frac{\pi}{l}\right)^2$$

$$2\zeta_m n\omega_m n = \frac{\gamma}{\rho}$$

$$\zeta_{mn} = \frac{\gamma}{2\rho\omega_{mn}} = \frac{\gamma}{2\rho}\frac{1}{\sqrt{\frac{\tau}{\rho}\left(n^2 + m^2\right)}}\frac{l}{\pi}$$

$$\zeta_{mn} = \frac{\gamma l}{2\sqrt{\rho\tau\left(n^2 + m^2\right)}}$$

6.57 Calculate the units on γ and β in equation (6.153).

Solution: The units are found from

$$\frac{\text{mg}}{\text{m}^3}\text{m}^2\frac{\text{m}}{\text{s}^2} = \gamma\frac{\text{m}}{\text{s}}$$

$$\frac{\text{kg}}{\text{s}^2}\frac{\text{s}}{\text{m}} = \gamma$$

$$\gamma = \frac{\text{kg}}{\text{m}\bullet\text{s}}$$

6.58 Assume that E, I, and ρ are constant in equations (6.153) and (6.154) and calculate the form of the modal damping ratio ζ_n.

Solution:

If E, I, and ρ are constant in equation 6.153 and 6.154. Then separation of variables works and the mode shapes become those given in table 6.4, which can be normalized so that $\int_0^l X_n X_m \, dx = \delta_{nm}$. Substitution of $w(x,t) = a_n(t)X_n(x)$ into equation (6.153) multiplying by $X_m(x)$ and integrating over x yield the mth modal equation:

$$\rho A \ddot{a}_n(t) + \gamma \dot{a}_n(t) + \beta I\left(\frac{\omega_n^2}{c^2}\right)\dot{a}_n(t) + EI\frac{\omega_n^2}{c^2}a_n(t) = 0$$

where equation (6.93) has been used to evaluate X'''' and $c^2 = EI/\rho A$. Dividing by ρA yields

$$\ddot{a}_n(t) + \left(\frac{\gamma}{\rho A} + \frac{\beta}{E}\omega_n^2\right)\dot{a}_n(t) + \omega_n^2 a_n(t) = 0$$

which is the *sdof* form of windows 6.4. Thus the coefficients of must be and hence

$$2\zeta_n\omega_n = \frac{\gamma}{\rho A} + \frac{\beta}{E}\omega_n^2$$

and

$$\omega_n = \beta_n^2\sqrt{\frac{EI}{\rho A}}$$

$$\zeta_n = \frac{\gamma}{2\rho A\omega_n} + \frac{\beta}{E}\omega_n$$

where β_n are given in table 6.4.

6.59 Calculate the form of the solution $w(x,t)$ for the system of Problem 6.58.

Solution:

The form of the solution of the m time equation is just

$$A_n e^{-\zeta_n \omega_n t} \sin\left(\omega_{dn} t + \phi_n\right)$$

where ζ_n and ω_n are as given in problem 6.58, $\omega_{dn} = \omega_n \sqrt{1-\zeta_n^2}$, and A_n and ϕ_n are constants determined by initial conditions. The total solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-\zeta_n \omega_n t} \sin\left(\omega_{dn} t + \phi_n\right) X_n(x)$$

where $X_n(t)$ are the eigenfunctions given in table 6.4.

6.60 For a given cantilevered composite beam, the following values have been measured for bending vibration:

$E = 2.71\text{x}10^{10}$ N/m^2 $\quad$ $\rho = 1710$ kg/m^3
$A = 0.597\text{x}10^{-3}$ m^2 $\quad$ $l = 1$ m
$I = 1.64\text{x}10^{-9}$ m^4 $\quad$ $\gamma = 1.75$ N s/m^2
$\beta = 20{,}500$ Ns/m^2

Calculate the solution for the beam to an initial displacement of $w_t(x,0) = 0$ and $w(x,0) = 3\sin \pi x$.

Solution:

Using the values given and the formulas for $a_n(t)$ from problem 6.58 the temporal equation becomes

$$\ddot{a}_n + \left(1.714 \times .00000075 G\omega_n^2\right)\dot{a}_n + \omega_n^2 a_n = 0$$

from problem 6.59,

$$w_t(x,t)\Big|_{t=0} = 0 = \sum A_n\left[\left(-\zeta_n\omega_n\right)\sin\phi_n + \omega_{dn}\cos\phi_n\right]X_n(x)$$

and

$$w(x,0) = 3\sin \pi x = \sum A_n \sin\phi_n X_n(x)$$

Multiplying by $X_n(x)$ and integrating yields that

$$\zeta_n\omega_n \sin\phi_n = \omega_{dn}\cos\phi_n \quad \text{or} \quad \tan\phi_n = \frac{\omega_{dn}}{\zeta_n\omega_n}$$

and $3\int_0^l \sin \pi x X_n(x)dx = A_n \sin\phi_n$ so that

$$A_n = \frac{3\int_0^l \sin\pi x X_n(x)dx}{\sin\phi_n} = \frac{3}{\sqrt{1-\zeta_n^2}}\int_0^l \sin\pi x X_n(x)dx$$

where $X_n(x)$ is given in table 6.4.

6.61 Plot the solution of Example 6.7.2 for the case $w_t(x,0) = 0$, $w(x,0)=\sin(nx/\ell)$, $\gamma=10$ Ns/m^2, $\tau = 10^4$ N, $\ell = 1$ m and $\rho = 0.01$ kg/m^3.

Solution: From equation (6.156) and the values given, $\zeta_1 = 0.159/n$ or $\zeta_n\omega_n = 500$ and $\omega_{dn} = \sqrt{1 = 0.159^2}$, so that:

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-500t} \sin(\omega_{dn} t + \phi_n) \sin n\pi x$$

Applying the initial conditions yields

$$\int_0^l \sin n\pi x \sin m\pi x dx = \sum_{n=1}^{\infty} A_n \sin(\phi_n) \int_0^l \sin m\pi x \sin n\pi x dx$$

So that $A_n \sin\phi_n = 0$ for all n except $n = 1$, and $A_1 \sin\phi_1 = 1$. So either $\phi_n = 0$ or $A_n = 0$ for n not zero. The other initial condition yields that $\phi_n = \tan^{-1}\left(\frac{-\sqrt{1-\zeta_n^2}}{\zeta_n}\right)$ so that $A_n = 0$ for n not zero. Thus the system is only excited in the first mode. Then

$$w(x,t) = A_1 e^{-500t} \sin(\omega_1 \sqrt{1-\zeta_n^2}\, t + \phi_n) \sin \pi x$$
$$= -1.001 e^{-500t} \sin(3137.7t - 1.50) \sin \pi x$$

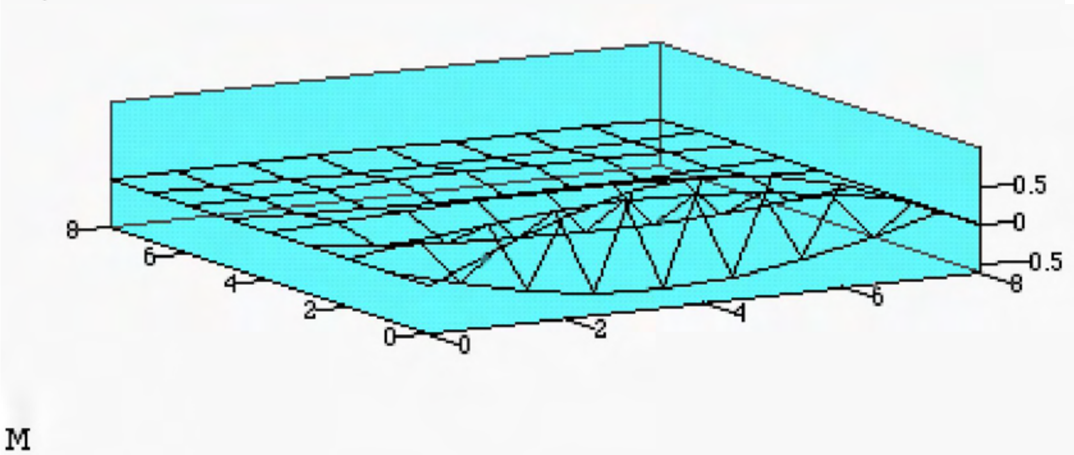
This is plotted in Mathcad below:

$N := 8 \quad i := 0..N \quad j := 0..N \quad x_i := i \cdot \frac{1}{8} \quad t_j := j \cdot 0.001$

$\zeta := 0.159 \quad \omega := \pi \cdot 10^3 \quad \phi := -1.52 \quad \omega d := \omega \cdot \sqrt{1 - \zeta^2}$

$w(x,t) := -1.001\, e^{-\zeta \cdot \omega \cdot t} \cdot (\sin(\pi \cdot x)) \cdot \sin(\omega d \cdot t + \phi)$

$M_{i,j} := w(x_i, t_j)$



6.62 Calculate the orthogonality condition for the system of Example 6.7.2. Then calculate the form of the temporal solution.

Solution: Problem is to fill in the details of example 6.7.2 by checking the coefficients. Equation (6.155) by performing the integration.

6.63 Calculate the form of modal damping for the longitudinal vibration of the beam of Figure 6.14 with boundary conditions specified by equation (6.157).

Solution: This is a discussion problem. The boundary condition given in equation (6.157)

$$AEw_x(0,t) = kw(0,t) + c\frac{\partial w(0,t)}{\partial t}$$

$$AEw_x(l,t) = -kw(l,t) - c\frac{\partial w(l,t)}{\partial t}$$

Do not conform readily to separation of variables and lead to time dependent boundary conditions. However one approach is to treat the damper as applied forces of the bar $cw_t(0,t)$ and $-cw_t(l,t)$. Following this approach the boundary conditions become

$$AEX'(0) = kX(0) \text{ and } AEX'(l) = -kX(l)$$

The general solution of the spatial equation of a bar has the form

$$X(x) = a\sin(\sigma x + b)$$

Where σ is the usual separation constant and a and b are constants. The first boundary condition yields that $\phi = \tan^{-1}(AE/k)$. The second boundary condition yields the characteristic equation

$$-(AE/k)\sigma_n = \tan(\sigma_n l + \phi)$$

Which can be solved for σ_n numerically. Note that σ_n are distinct so that from problem 6.39 the eigenfunctions are orthogonal, i.e. an can be calculated such that

$$X_n(x) = a_n \sin(\sigma_n x + \phi)$$

Are orthonormal. Following the procedure of example 6.8.11, the temporal solution for the forced response is

$$\ddot{T}_n(t) + \omega_n^2 = \int_0^l \left[cw_t(0,t) - cw(l,t)\right] X_r(x)dx$$

$$= \left\{\int_0^l \left[cX_n(0) - cX_n(l)\right] X_n(x)dx\right\}\dot{T}_n(t)$$

Bring the $\dot{T}_n$ term to the left side and comparing its coefficient to $2\zeta_n\omega_n$ yields

$$2\zeta_n\omega_n = \left\{\int_0^l c\left[X_n(l) - X_n(0)\right] X_n(x)\right\}dx$$

The form of the modal damping ratio is thus

$$\zeta_n - \frac{ca_n^2}{2\omega_n\sigma_n}\left[\cos\left(\sigma_n l + \phi\right) - \cos\phi\right]$$

Where a_n^2 is the normalization factor, σ_n are the eigenvalues $\omega_n^2 = c^2\sigma_n^2$ and $\tan^{-1}(AE / k)$.