Problems and Solutions Section 6.8 (6.64 through 6.68)

6.64 Calculate the response of the damped string of Example 6.8.1 to a disturbance force of $f(x,t) = (\sin \pi x/l) \sin 10t$.

Solution:

$f(x,t) = \sin\left(\frac{\pi x}{l}\right)\sin 10t$. Assume a solution of the form:

$$w_n(x,t) = T_n(t)X_n(x)$$

where

$$X_n(x) = \sin\frac{n\pi x}{l}$$

Substitute into (6.158)

$$\left\{\rho\ddot{T}_n + \gamma\dot{T}_n - \tau\left[-\left(\frac{n\pi}{l}\right)^2\right]T_n\right\}\sin\frac{n\pi x}{l} = \sin\left(\frac{\pi x}{l}\right)\sin 10t$$

Multiply by $\sin\frac{n\pi x}{l}$ and integrate over the length of the string:

$$\left\{\rho\ddot{T}_n + \gamma\dot{T}_n + \tau\left(\frac{n\pi}{l}\right)^2 T_n\right\}\frac{l}{2} = \begin{cases} 0 & \text{for } n = 1 \\ \sin 10t & \text{for } n > 1 \end{cases}$$

Only the particular solution is of interest since we are looking for the response to the disturbance force. Therefore, dropping the subscripts:

$$\rho\ddot{T} + \gamma\dot{T} + \tau\left(\frac{\pi}{l}\right)^2 T = \sin 10t$$

$$\ddot{T} + \left(\frac{\gamma}{\rho}\right)\dot{T} + \left(\frac{c\pi}{l}\right)^2 T = \frac{\sin 10t}{\rho} \quad \text{where } c = \sqrt{\frac{\tau}{\rho}}$$

Solution is

$$T = A\sin(20t - \phi)$$

where

$$A=\frac{1}{\rho\sqrt{\left(\frac{c^2\pi^2}{l^2}-100\right)^2+100\frac{\gamma^2}{\rho^2}}}=\frac{l^2}{\sqrt{\rho^2\left(c^2\pi^2-100l^2\right)^2+100\gamma^2l^4}}$$

$$\phi=\tan^{-1}\left[\frac{10\frac{\gamma}{\rho}}{\frac{c^2\pi^2}{l^2}-100}\right]=\tan^{-1}\left[\frac{10\gamma l^2}{\rho c^2\pi^2-100\rho l^2}\right]$$

$$w(x,t)=A\sin(10t-\phi)\sin\frac{\pi x}{l}$$

where A and ϕ are given above.

6.65 Consider the clamped-free bar of Example 6.3.2. The bar can be used to model a truck bed frame. If the truck hits an object (at the free end) causing an impulsive force of 100 N, calculate the resulting vibration of the frame. Note here that the truck cab is so massive compared to the bed frame that the end with the cab is modeled as clamped. This is illustrated in Figure P6.65.

Solution: Assume constant area and constant material properties. Equation of motion:

$$\rho A w_{tt} - EA w_{xx} = f(x,t) = -100\delta(x-l)\delta(t)$$

Mode shapes (eigenvalues) of a fixed-free bar are (Table 6.1)

$$X_n(x) = \sin\frac{(2n-1)\pi x}{2l}$$

Assume a solution of the form: $w_n(x,t) = X_n(x)T_n(t)$. Substitute into the equation of motion:

$$\left\{\ddot{T}_n - \left[-\left(\frac{(2n-1)\pi}{2l}\right)^2\right]c^2 T_n\right\}\sin\frac{(2n-1)\pi x}{2l} = -\frac{100}{\rho A}\delta(x-l)\delta(t)dx$$

$$\left\{\ddot{T}_n + \omega_n^2 T_n\right\}\sin\frac{(2n-1)\pi x}{2l} = -\frac{100}{\rho A}\delta(x-l)\delta(t)$$

where $c^2 = \dfrac{E}{\rho}$ and $\omega_n = \dfrac{(2n-1)\pi c}{2l}$. Multiply by $\sin\dfrac{(2n-1)\pi x}{2l}$ and integrate over the length of the rod:

$$\ddot{T}_n + \omega_n^2 T_n = -\frac{2}{l}\int_0^l \frac{100}{\rho A}\sin\left(\frac{(2n-1)\pi x}{2l}\right)\delta(x-l)\delta(t)$$

$$= -\frac{200}{\rho A l}\sin\left(\frac{(2n-1)\pi}{2}\right)\delta(t)$$

which has the solution:

$$T_n(t) = -\frac{200}{\rho A l \omega_n}\sin\left(\frac{(2n-1)\pi}{2}\right)\sin\omega_n t$$

The total solution is:

$$w_n(x,t) = -\sum_{n=1}^{\infty}\left\{\left[\frac{400}{\rho A(2n-1)\pi c}\right]\sin\left(\frac{(2n-1)\pi}{2}\right)\sin\left(\frac{(2n-1)\pi ct}{2l}\right)\sin\left(\frac{(2n-1)\pi x}{2l}\right)\right\}$$

6.66 A rotating machine sits on the second floor of a building just above a support column as indicated in Figure P6.66. Calculate the response of the column in terms of *E*, *A*, and ρ of the column modeled as a bar.

Solution: Referring to equation (6.55) for the equation of a bar and summing forces to get the effect of the applied force yields

$$\rho A w_{tt} - EAw_{xx} = \delta(x-l)F_0 \sin\omega t$$

subject to the boundary conditions $w(0,t) = w_x(0,t) = 0$. Following the method of example 6.8.1, use separation of variables where the spatial function is the clamped-free mode shapes used in example 6.3.1:

$$w(x,t) = X_n(x)T_n(t) = (a_n \sin\sigma_n x)T_n(t), \quad \sigma_n = \frac{2n-1}{2l}\pi$$

Substitution into the equation of motion yields

$$\left(\rho A\ddot{T}_n(t) + EA\sigma_n^2 T_n(t)\right)a_n \sin\sigma_n x = \delta(x-l)F_0 \sin\omega t$$

(the minus sign in front of *EA* goes away because of the second derivative of sine being negative). Next, let $a_n = 1$ (recalling that eigenvectors have arbitrary magnitude) and multiply by sin $\sigma_n x$ and integrate over the length of the beam to get:

$$\left(\rho A\ddot{T}_n(t) + EA\sigma_n^2 T_n(t)\right)\frac{l}{2} = F_0 \sin\omega t \int_0^l \delta(x-l)\sin\sigma_n x dx$$

The integral on the right is a bit tricky as the delta function acts at the end of the interval. The details are below, however integrating yields

$$\left(\rho A\ddot{T}_n(t) + EA\sigma_n^2 T_n(t)\right)\frac{l}{2} = \frac{1}{2}F_0 \sin\omega t \frac{\sin\sigma_n l}{2} = (-1)^{n-1}\frac{F_0}{2}\sin\omega t$$

Dividing by the appropriate constants this simplifies to

$$\ddot{T}_n(t) + \frac{E}{\rho}\sigma_n^2 T_n(t) = \frac{(-1)^{n-1}F_0}{\rho A}\sin\omega t$$

This has particular solution

$$T_{np}(t) = \frac{(-1)^{n-1}}{\rho A}\left(\frac{F_0}{\omega_n^2 - \omega^2}\right)\sin\omega t \quad \text{where } \omega_n = \sqrt{\frac{E}{\rho}}\frac{(2n-1)\pi}{2l}$$

Combined with the homogenous solution, the total temporal solution is

$$T_n(t) = C_{1n}\sin\omega_n t + C_{2n}\cos\omega_n t + \left(\frac{(-1)^{n-1}}{\rho A}\frac{F_0}{\omega_n^2 - \omega^2}\right)\sin\omega t$$

So the total solution is

$$w(x,t) = \sum_{n=1}^{\infty}\left\{C_{1n}\sin\omega_n t + C_{2n}\cos\omega_n t + \frac{(-1)^{n-1}}{\rho A}\left(\frac{F_0}{\omega_n^2 - \omega^2}\right)\sin\omega t\right\}\sin\left(\frac{(2n-1)\pi x}{2l}\right)$$

The following it the evaluation of the Dirac integral used about (courtesy of Jamil Renno)
Start with the integral at hand

$$\int_0^l \delta\left(x-l\right)\sin\left(\sigma_m x\right)dx = \lim_{\tau\to 0}\left[\int_0^l d_\tau\left(x-l\right)\sin\left(\sigma_m x\right)dx\right]$$

where $d_\tau\left(x-l\right)=\begin{cases}\dfrac{1}{2\tau} & l-\tau<x<l+\tau \\ 0 & x\le l-\tau \text{ or } x\ge l+\tau\end{cases}$ is the pulse over the interval $\left[l-\tau, l+\tau\right]$.

Hence, the integral can be subdivided over two intervals

$$\int_0^l \delta\left(x-l\right)\sin\left(\sigma_m x\right)dx = \lim_{\tau\to 0}\left[\int_0^{l-\tau} d_\tau\left(x-l\right)\sin\left(\sigma_m x\right)dx + \int_{l-\tau}^{l} d_\tau\left(x-l\right)\sin\left(\sigma_m x\right)dx\right]$$

$$= \lim_{\tau\to 0}\left[\int_0^{l-\tau} 0\sin\left(\sigma_m x\right)dx + \int_{l-\tau}^{l}\frac{1}{2\tau}\sin\left(\sigma_m x\right)dx\right] = \lim_{\tau\to 0}\left[\int_{l-\tau}^{l}\frac{1}{2\tau}\sin\left(\sigma_m x\right)dx\right]$$

$$= \lim_{\tau\to 0}\left[\frac{1}{2\tau}\frac{1}{\sigma_m}\left[-\cos\left(\sigma_m x\right)\right]_{l-\tau}^{l}\right] = \lim_{\tau\to 0}\frac{\cos\left[\sigma_m\left(l-\tau\right)\right]-\cos\left[\sigma_m l\right]}{2\tau\sigma_m}$$

$$\overset{\text{L'Hopital's Rule}}{=} \lim_{\tau\to 0}\frac{\dfrac{d}{d\tau}\left\{\cos\left[\sigma_m\left(l-\tau\right)\right]-\cos\left[\sigma_m l\right]\right\}}{\dfrac{d}{d\tau}\left\{2\tau\sigma_m\right\}}$$

$$= \lim_{\tau\to 0}\frac{\left(\dfrac{-1}{\sigma_m}\right)\left(-\sin\left[\sigma_m\left(l-\tau\right)\right]\right)}{2\sigma_m} = \frac{\sin\left(\sigma_m l\right)}{2}$$

6.67 Recall Example 6.8.2, which models the vibration of a building due to a rotating machine imbalance on the second floor. Suppose that the floor is constructed so that the beam is clamped at one end and pinned at the other, and recalculate the response (recall Example 6.5.1). Compare your solution and that of Example 6.8.2, and discuss the difference.

Solution:

Clamped-pinned beam conditions yield mode shapes (eigenfunctions) of the form:

$$X_n(x) = a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n(\sinh\beta_n x - \sin\beta_n x)\right]$$

where $\tan\beta_n l = \tanh\beta_n l$ and

$$\sigma_n = \begin{cases} 1.0008 & \text{for } n = 1 \\ 1 & \text{for } n > 1 \end{cases}$$

Normalize the mode shape as follows:

$$\int_0^l X_n^2\,dx = 1 \Rightarrow$$

$$a_n^2 \int_0^l [\cosh\beta_n x - \cos\beta_n x - \sigma_n(\sinh\beta_n x - \sin\beta_n x)]^2\,dx = 1$$

From Mathematica

$$\begin{aligned} a_n^2 = 4\beta_n \,/\, \Big\{ & 4\beta_n l + 2\sigma_n\cos(2\beta_n l) - 2\sigma_n\cosh(2\beta_n l) - 4\cosh(\beta_n l)\sin(\beta_n l) \\ & - 4\sigma_n^2\cosh(\beta_n l)\sin(\beta_n l) + \sin(2\beta_n l) - \sigma_n^2\sin(2\beta_n l) \\ & - 4\cos(\beta_n l)\sinh(\beta_n l) + 4\sigma_n^2\cos(\beta_n l)\sinh(\beta_n l) \\ & + 8\sigma_n\sin(\beta_n l)\sinh(\beta_n l) + \sinh(2\beta_n l) + \sigma_n^2\sinh(2\beta_n l) \Big\} \end{aligned}$$

The equation of motion for the system is: (constant properties)

$$\rho A w_{tt} + EI w_{xxxx} = f(x,t) = 100\sin 3t\,\delta\left(x - \frac{l}{2}\right)$$

Assume a solution of the form: $w_n(x,t) = X_n(x)T_n(t)$

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n'''' = \frac{100}{\rho A}\sin 3t\,\delta\left(x - \frac{l}{2}\right)$$

Using the mode shapes given above:

$$X_n'''' = \beta_n^4 X_n = \frac{\omega_n^2}{c^2} X_n$$

where

$$\beta_n^4 = \frac{\rho A}{EI}\omega_n^2, \quad c^2 = \frac{EI}{\rho A}$$

The equation of motion reduces to:

$$\left\{\ddot{T}_n + \omega_n^2\right\} X_n = \frac{100}{\rho A}\sin 3t\,\delta\left(x - \frac{l}{2}\right)$$

Multiply by X_n and integrate over the length of the beam:

$$\ddot{T}_n + \omega_n^2 T_n = \frac{100}{\rho A}\sin 3t \int_0^l X_n(x)\,\delta\left(x - \frac{l}{2}\right) dx$$

$$= \frac{100}{\rho A}\sin 3t X_n\left(\frac{l}{2}\right)$$

$$= \frac{100 a_n}{\rho A}\sin 3t\left[\cosh\frac{\beta_n l}{2} - \cos\frac{\beta_n l}{2} - \sigma_n\left(\sinh\frac{\beta_n l}{2} - \sin\frac{\beta_n l}{2}\right)\right]$$

or:

$$T_n(t) = \left[\frac{100 X_n\left(\frac{l}{2}\right)}{\rho A\left(\omega_n^2 - 9\right)}\right]\sin 3t$$

The solution is then:

$$w(x,t) = \sum_{n=1}^{\infty}\left\{a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]\right.$$
$$\left.\left[\frac{100}{\rho A\left(\omega_n^2 - 9\right)}\right] X_n\left(\frac{l}{2}\right)\right\}\sin 3t$$

where a_n, ω_n, and β_n are given above. The free time response is stiffer for the clamped case as the frequencies are higher (See Table 6.4).

The comparison of the solution between the two models (one with a pinned end and one with a fixed or clamped end) had two purposes: design and modeling. From the design point of view it is important to know how to construct the floor for a minimum value of response. From the modeling point of view it is important to know how much the solution is effected by the choice of boundary conditions as part of the modeling.

Here the comparison can be made by calculating the response and then evaluating it and plotting it using a truncated solution (say 3 modes, as given in Equation 6.181) at a given point of interest (i.e. for a particular value of x). This gives an accurate comparison.

Next you can compare the differences in the details. For instance the clamped-pinned natural frequencies are lower then the clamped-clamped frequencies (just look at Table 6.4) because the clamped-clamped system is stiffer. Next, one of these sets of frequencies is going to have a natural frequency that is closer to the driving frequency, and hence produce a larger response. To make such comparisons, pick a value for the physical parameters (let omega = beta squared for instance) and check. In this case the clamped-pinned frequency is about 3.9 rad/s, which is much closer to the driving frequency of 3 rad/s then the clamped-clamped first natural frequency of 4.7 rad/s. Thus the first term in the series solution for the example will be larger then the corresponding term in the series solution for the clamped-clamped case.

6.68 Use the modal analysis procedure suggested at the end of Section 6.8 to calculate the response of a clamped free beam with a sinusoidal loading $F_0 \sin\omega t$ at its free end.

Solution:

The equation of motion is:

$$\rho A w_{tt} + EI w_{xxxx} = f(x,t) = F_0 \delta(x-l)\sin\omega t$$

Assume a solution of the form $w_n(x,t) = X_n(x)T_n(t)$

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n'''' = \frac{F_0}{\rho A}\delta(x,l)\sin\omega t$$

The mode shapes are given in Table 6.4 for a fixed-free beam:

$$X_n(x) = a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]$$

Where

$$\sigma_n = \frac{\sinh\beta_n l - \sin\beta_n l}{\cosh\beta_n l + \cos\beta_n l}$$

$$\beta_n^4 = \frac{\rho A}{EI}\omega_n^2$$

And

$$\cos\beta_n l \cosh\beta_n l = -1$$

From the unforced vibration problem:

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n'''' = 0$$

$$\frac{\ddot{T}_n}{T_n} = -\left(\frac{EI}{\rho A}\right)\frac{X_n''''}{X_n} = -\omega_n^2$$

Therefore

$$X_n'''' = \frac{\rho A}{EI}\omega_n^2 X_n = \beta_n^4 X_n$$

Substitute into the equation of motion and rearrange:

$$\left\{\ddot{T}_n + \omega_n^2 T_n\right\} X_n = \frac{F_0}{\rho A}\delta(x-l)\sin\omega t$$

Normalize the mode shapes as follows:

$$\int_0^l X_n^2 dx = 1$$

$$a_n^2 \int_0^l \left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]^2 dx = 1$$

$$\begin{aligned} a_n^2 = 4\beta_n \,/\, \Big\{ & 4\beta_n l + 2\sigma_n \cos\left(2\beta_n l\right) - 2\sigma_n \cosh\left(2\beta_n l\right) - 4\cosh\left(\beta_n l\right)\sin\left(\beta_n l\right) \\ & - 4\sigma_n^2 \cosh\left(\beta_n l\right) + \sin\left(2\beta_n l\right) - \sigma_n^2 \sin\left(2\beta_n l\right) \\ & - 4\cos\left(\beta_n l\right)\sinh\left(\beta_n l\right) + 4\sigma_n^2 \cos\left(\beta_n l\right)\sinh\left(\beta_n l\right) \\ & + 8\sigma_n \sin\left(\beta_n l\right)\sinh\left(\beta_n l\right) + \sinh\left(2\beta_n l\right) + \sigma_n^2 \sinh\left(2\beta_n l\right) \Big\} \end{aligned}$$

Multiply the equation of motion $X_n(x)$ and integrate over the length of the beam:

$$\begin{aligned} \ddot{T}_n + \omega_n^2 T_n &= \frac{F_0}{\rho A}\int_0^l X_n(x)\delta(x-l)dx\sin\omega t \\ &= \frac{F_0}{\rho A} X_n(l)\sin\omega t \end{aligned}$$

Solving:

$$T_n(t) = \left(\frac{F_0}{\rho A}\right)\left(\frac{X_n(l)}{\omega_n^2 - \omega^2}\right)\sin\omega t$$

The total solution is:

$$w(x,t) = \sum_{n=1}^{\infty}\left\{ a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right] \left(\frac{F_0}{\rho A}\right)\left[\frac{X_n(l)}{\omega_n^2 - \omega^2}\right]\right\}\sin\omega t$$

Where σ_n, w_n are given above and $\cos\beta_n l \cosh\beta_n l = -1$.