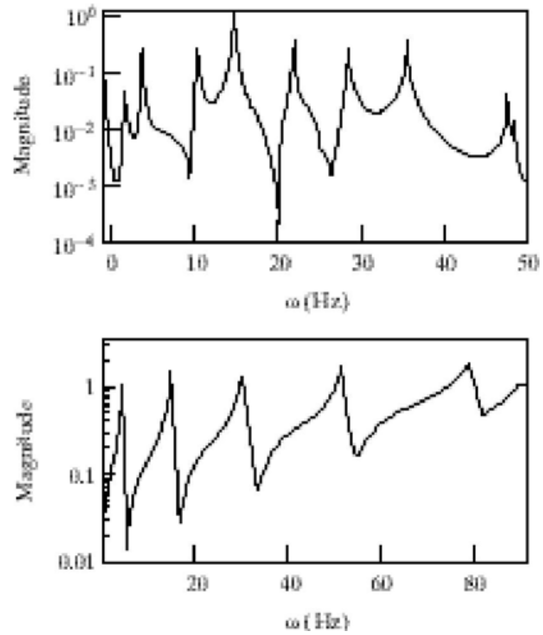
Problems and Solutions for Section 7.4 (7.10-7.19)

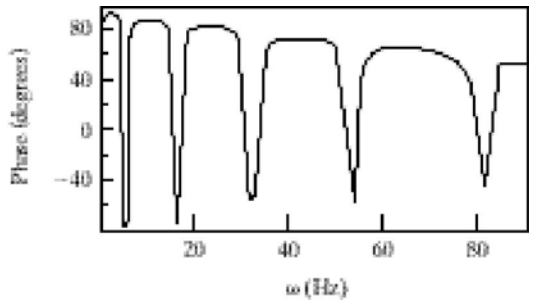
7.10 Consider the magnitude plot of Figure P7.10. How many natural frequencies does this system have, and what are their approximate values?



Solution:

The system looks to have 8 modes with approximate natural frequencies of 2, 4, 10, 15, 22, 29, 36, and 47 Hz.

7.11 Consider the experimental transfer function plot of Figure P7.11. Use the methods of Example 7.4.1 to determine ζ_i and ω_i.



Solution:

For each mode:

$$\zeta_i = \frac{\omega_{bi} - \omega_{ai}}{2\omega_i}$$

where ω_{bi} and ω_{ai} are the frequencies where the magnitude is $1/\sqrt{2}$ of the resonant magnitude. All values given in the following table are approximate.

Mode	ω_i (Hz)	$\lvert H(\omega_i)\rvert$	$\frac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (Hz)	ω_{bi} (Hz)	ζ_i
1	4.80	0.089	0.063	4.56	5.04	0.049
2	15.20	1.050	0.742	14.76	15.48	0.024
3	30.95	1.800	1.270	30.47	31.19	0.012
4	52.62	2.000	1.414	52.14	52.85	0.007
5	80.00	2.100	1.480	79.05	80.48	0.009

7.12 Consider a two-degree-of-freedom system with frequencies ω_1 = 10 rad/s, ω_2 = 15 rad/s, and damping ratios $\zeta_1 = \zeta_2$ = 0.01. With modal $s = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, calculate the transfer function of this system for an input at x_1 and a response measurement at x_2.

Solution:

Since the natural frequencies, damping ratios and mode shapes are given, the system can be expressed in modal coordinates as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{r}} + \begin{bmatrix} 2(.01)10 & 0 \\ 0 & 2(.01)15 \end{bmatrix}\dot{\mathbf{r}} + \begin{bmatrix} 10^2 & 0 \\ 0 & 15^2 \end{bmatrix}\mathbf{r} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} f(t) = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} f(t)$$

$$y = \frac{1}{\sqrt{2}}\{0 \quad 1\}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{r} = \frac{1}{\sqrt{2}}\{1 \quad 1\}\mathbf{r}$$

This is the representation of the system in modal coordinates, if proportional damping is assumed. The transfer function is:

$$Y(s) = \frac{1}{\sqrt{2}}\{1 \quad 1\}R(s)$$

where

$$R(s) = \frac{1}{\sqrt{2}}\begin{Bmatrix} \dfrac{1}{s^2 + 0.2s + 100} \\ \dfrac{-1}{s^2 + 0.3s + 225} \end{Bmatrix} F(s)$$

Combining the previous two expressions yields

$$\frac{Y(s)}{F(s)} = \frac{(0.1)(s + 1250)}{(s^2 + 0.2s + 100)(s^2 + 0.3s + 225)}$$

7.13 Plot the magnitude and phase of the transfer function of Problem 7.12 and see if you can reconstruct the modal data (ω_1, ω_2, ζ_1, and ζ_2) from your plot.

Solution:

For each mode:

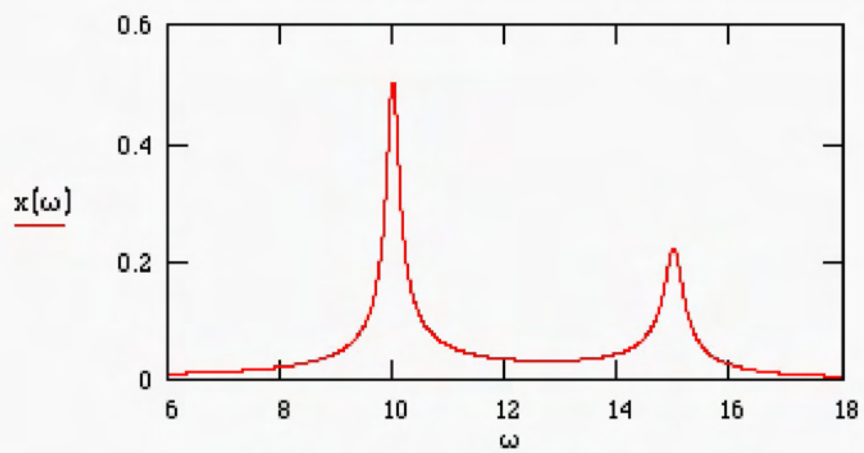
$$\zeta_i = \frac{\omega_{bi} - \omega_{ai}}{2\omega_i}$$

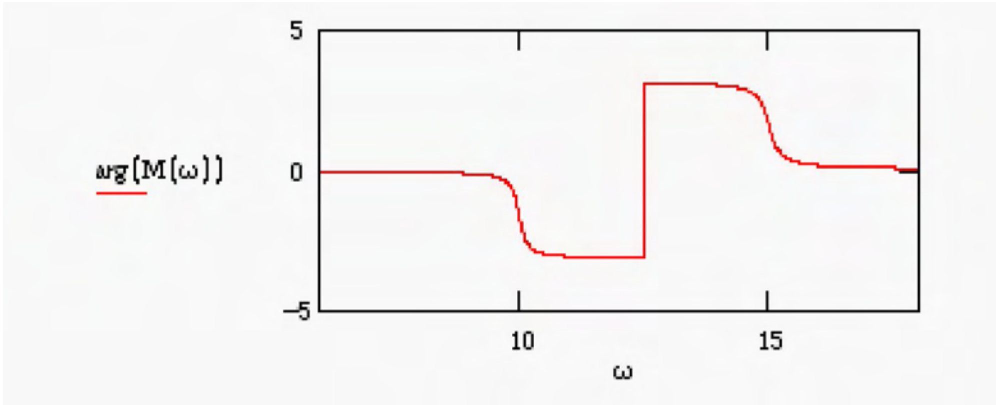
where ω_{bi} and ω_{ai} are the frequencies where the magnitude is $1/\sqrt{2}$ of the resonant magnitude. All values in the following table are approximate.

Mode	ω_i (rad/s)	$\lvert H(\omega_i)\rvert$	$\frac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	10	0.50	0.354	9.89	10.07	0.009
2	15	0.22	0.156	14.83	15.16	0.011

$$M(\omega) := \frac{(0.1)\cdot(\omega\cdot j + 1250)}{\left(-\omega^2 + 0.2\cdot\omega\cdot j + 100\right)\cdot\left(-\omega^2 + 0.3\cdot\omega\cdot j + 225\right)}$$

$$x(\omega) := \left|M(\omega)\right|$$





arg(M(ω))
5
0
−5
10
15
ω

7.14 Consider equation (7.14) for determining the damping ratio of a single mode. If the measurement in frequency varies by 1%, how much will the value of ζ change?

Solution:

$$\zeta = \frac{\omega_b - \omega_a}{2\omega_d}$$

If $\omega_d = \omega_{do}(1 \pm 0.01)$ where ω_{do} is the measured natural frequency, then the damping ratio is

$$\zeta = \frac{\omega_b - \omega_a}{2\omega_{do}}\left\{\frac{1}{1 \pm 0.01}\right\} = \zeta_o\left\{\frac{1}{1 \pm 0.01}\right\}$$

If ω_d is 0.99 ω_{do}, then ζ = 1.01 ζ_o

If ω_d is 1.01 ω_{do}, then ζ = 0.99 ζ_o

Thus, 1 percent changes in the measured natural frequency produce similar changes in the measured damping ratio.

7.15 Discuss the problems of using equation (7.14) if the natural frequencies of the structure are very close together.

Solution:

Equation (7.14) assumes that the response at resonance is due to a single degree of freedom system. If the natural frequencies are very close together, this assumption is not valid. This will introduce error into the damping ratio calculation since the peak response at each resonant frequency will be due to a combination of responses from each of the closely spaced modes.

7.16 Discuss the limitation of using equation (7.15) if ζ is very small. What happens if ζ is very large?

Solution: When ζ is very small (<0.01), it is difficult to determine where $R(\alpha)$ is the largest since equation (7.15) is changing very rapidly in the vicinity of resonance. When ζ is very large (>0.707), the frequency response near resonance is very flat, again making it difficult to determine the damped natural frequency. In either case, experimentally determined damping ratios will contain error since they depend on an accurate determination of the resonant frequency. Problem 7.18 contains plots that illustrate these ideas.

7.17 Consider the two-degree-of-freedom system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_0 \sin \omega t \\ 0 \end{bmatrix}$$

and calculate the transfer function $|X/F|$ as a function of the damping parameter c.

Solution:

The equations of motion for the system are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{x} = \begin{Bmatrix} f_o \\ 0 \end{Bmatrix} f(t)$$

Taking the Laplace transform yields

$$\begin{bmatrix} s^2+2 & -1 \\ -1 & s^2+cs+2 \end{bmatrix} X(s) = \begin{Bmatrix} f_o \\ 0 \end{Bmatrix} F(s)$$

Inverting the matrix on the left hand side leads to an expression for $X(s)$:

$$X(s) = \frac{1}{(s^2+2)(s^2+cs+2)-1}\begin{bmatrix} s^2+cs+2 & 1 \\ 1 & s^2+2 \end{bmatrix}\begin{Bmatrix} f_o \\ 0 \end{Bmatrix} F(s)$$

Performing the multiplication leads to

$$\frac{X_1(s)}{f_o F(s)} = \frac{s^2+cs+2}{s^4+cs^3+4s^2+2cs+3}$$

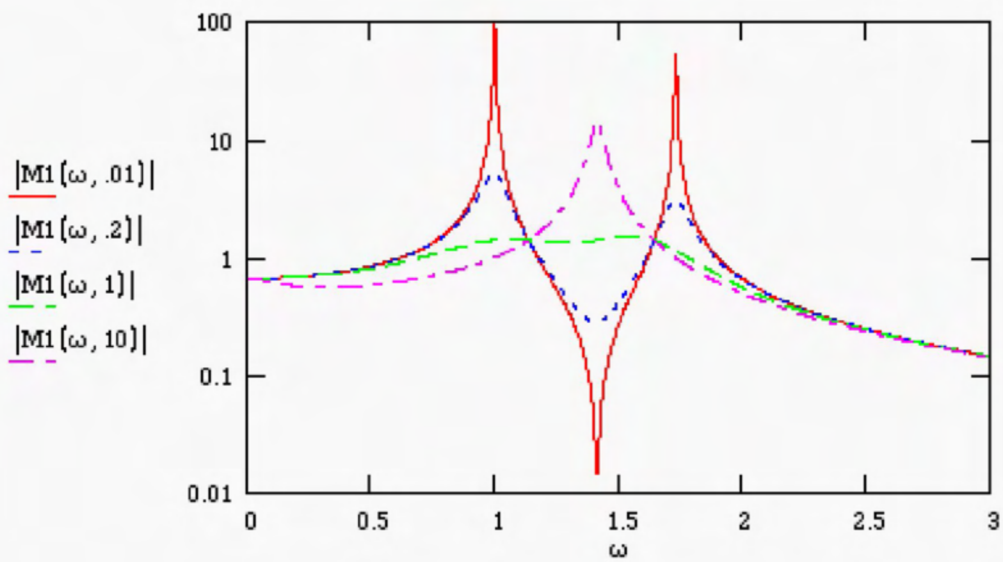
$$\frac{X_2(s)}{f_o F(s)} = \frac{1}{s^4+cs^3+4s^2+2cs+3}$$

7.18 Plot the transfer function of Problem 7.17 for the four cases: $c = 0.01$, $c = 0.2$, $c = 1$, and $c = 10$. Discuss the difficulty in using these plots to measure ζ_i and ω_i for each value of c.

Solution:

For $c = 0.01$, the resonant peaks are very sharp, making an accurate determination of ζ_i difficult. In the case $c = 0.2$, ζ_i and ω_i could be determined fairly easily using the techniques of section 7.4. Increasing c to 1.0 makes the frequency response very flat, which again makes finding ζ_i and ω_i difficult. Finally, when $c = 10$, it almost looks as if there is one resonant peak, which would lead to a completely erroneous result.

$$M1(\omega, c) := \frac{-\omega^2 + c\cdot\omega\cdot j + 2}{\omega^4 - c\cdot\omega^3\cdot j - 4\cdot\omega^2 + 2\cdot c\cdot\omega\cdot j + 3}$$



7.19 Use a numerical procedure to calculate the natural frequencies and damping ratios of the system of Problem 7.18. Label these on your plots from Problem 7.18 and discuss the possibility of measuring these values using the methods of Section 7.4

Solution:

For the case where $c = 0.01$

Mode	ω_i (rad/s)	$\lvert H(\omega_i)\rvert$	$\dfrac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	1.0	59	41.72	0.99	1.02	0.015
2	1.7	48	33.94	1.71	1.69	0.006

Actual values: $\omega_1 = 1.00$ $\zeta_1 = 0.003$

$\omega_2 = 1.73$ $\zeta_2 = 0.001$

The actual values are calculated directly from the equations.

For the case where $c = 0.2$

Mode	ω_i (rad/s)	$\lvert H(\omega_i)\rvert$	$\dfrac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	1.0	5.1	3.61	0.93	1.06	0.064
2	1.7	2.9	2.05	1.69	1.79	0.030

Actual values: $\omega_1 = 1.00$ $\zeta_1 = 0.050$

$\omega_2 = 1.73$ $\zeta_2 = 0.029$

For the case $c = 0.01$, there is more error in the measured parameters than for the case $c = 0.2$ due to the sharpness of the resonant peak.

$$M1(\omega, c) := \frac{-\omega^2 + c\cdot\omega\cdot j + 2}{\omega^4 - c\cdot\omega^3\cdot j - 4\cdot\omega^2 + 2\cdot c\cdot\omega\cdot j + 3}$$