Problems and Solutions Section 7.5 (7.20-7.24)

7.20 Using the definition of the mobility transfer function of Window 7.4, calculate the Re and Im parts of the frequency response function and hence verify equations (7.15) and (7.16).

Solution:

From Window 7.4:

$$\frac{sX(s)}{F(s)} = \frac{s}{ms^2 + cs + k}$$

$$\alpha(\omega) = \frac{j\omega}{(k - m\omega^2) + j\omega c}$$

$$\alpha(\omega) = \frac{j\omega\left[(k - m\omega^2) - j\omega c\right]}{(k - m\omega^2)^2 + (\omega c)^2}$$

$$\alpha(\omega) = \frac{\omega c^2 + j\omega(k - m\omega^2)}{(k - m\omega^2)^2 + (\omega c)^2}$$

The previous expression can be separated into real and imaginary parts:

$$Re\left[\alpha(\omega)\right] = \frac{\omega^2 c}{(k - \omega^2 m)^2 + (\omega c)^2} \qquad Im\left[\alpha(\omega)\right] = \frac{\omega(k - \omega^2 m)}{(k - \omega^2 m)^2 + (\omega c)^2}$$

7.21 Using equations (7.15) and (7.16), verify that the Nyquist plot of the mobility frequency response function does in fact form a circle.

Solution:

Define
$$A = \frac{\omega^2 c}{(k-\omega^2 m)^2 + (\omega c)^2} - \frac{1}{2c} = \mathrm{Re}(\alpha) - \frac{1}{2c}$$

$$B = \frac{\omega(k - \omega^2 m)}{(k-\omega^2 m)^2 + (\omega c)^2} = \mathrm{Im}(\alpha)$$

Show that

$$A^2 + B^2 = \left(\frac{1}{2c}\right)^2$$

which is a circle of radius $\frac{1}{2c}$ with center at $Re(\alpha) = \frac{1}{2c}$, $Im(\alpha) = 0$.

$$A^2 + B^2 = \left[\frac{\omega^2 c}{(k-\omega^2 m)^2 (\omega c)^2} - \frac{1}{2c}\right]^2 + \left[\frac{\omega(k-\omega^2 m)}{(k-\omega^2 m)^2 + (\omega c)^2}\right]^2$$

$$A^2 + B^2 = \frac{(\omega^2 c)^2}{\left[(k-\omega^2 m)^2 + (\omega c)^2\right]^2} + \frac{\omega^2\,(k-\omega^2 m)^2}{\left[(k-\omega^2 m)^2 + (\omega c)^2\right]^2} - \frac{\omega^2}{(k-\omega^2 m)^2 + (\omega c)^2} + \left(\frac{1}{2c}\right)$$

$$A^2 + B^2 = \frac{\omega^2}{(k-\omega^2 m)^2 + (\omega c)^2}\left[\frac{(k-\omega^2 m)^2 + (\omega c)^2}{(k-\omega^2 m)^2 + (\omega c)^2}\right] - \frac{\omega^2}{(k-\omega^2 m)^2 + (\omega c)^2} + \left(\frac{1}{2c}\right)^2$$

$$A^2 + B^2 = \left(\frac{1}{2c}\right)^2$$

Which is the equation of a circle.

7.22 Consider a single-degree-of-freedom system of mass 10 kg, stiffness 1000 N/m, and damping ratio of 0.01. Pick five values of ω between 0 and 20 rad/s and plot five points of the Nyquist circle using equations (7.15) and (7.16). Do these form a circle?

Solution:

SDOF oscillator:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m = 10 \text{ kg} \qquad k = 1000 \text{ N/m} \quad \zeta = 0.01$$

First, calculate the damping constant c.

$$\omega_n^2 = \frac{k}{m} = 100$$

$$c = 2\zeta\omega_n m = 2(0.01)(10)(10) = 2 \text{ Ns/m}$$

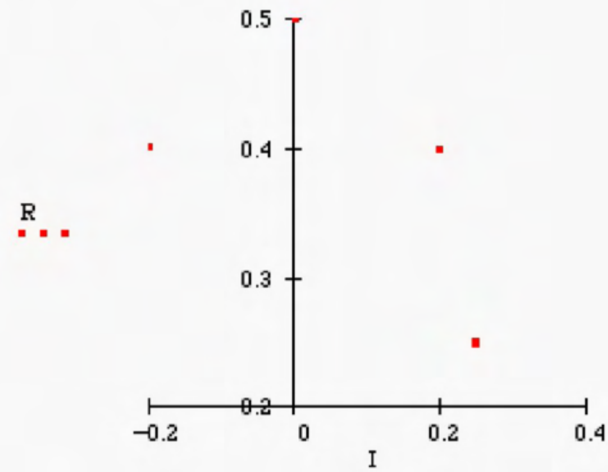
$$Re[\alpha] = \frac{2\omega^2}{(1000 - 10\omega^2)^2 + (2\omega)^2}$$

$$Im[\alpha] = \frac{\omega(1000 - 10\omega^2)}{(1000 - 10\omega^2)^2 + (2\omega)^2}$$

ω	$Re(\alpha)$	$Im(\alpha)$
9.90	0.2487	0.2500
9.95	0.3996	0.2003
10.00	0.5000	0.0000
10.05	0.4004	-0.1997
10.10	0.2512	0.2500

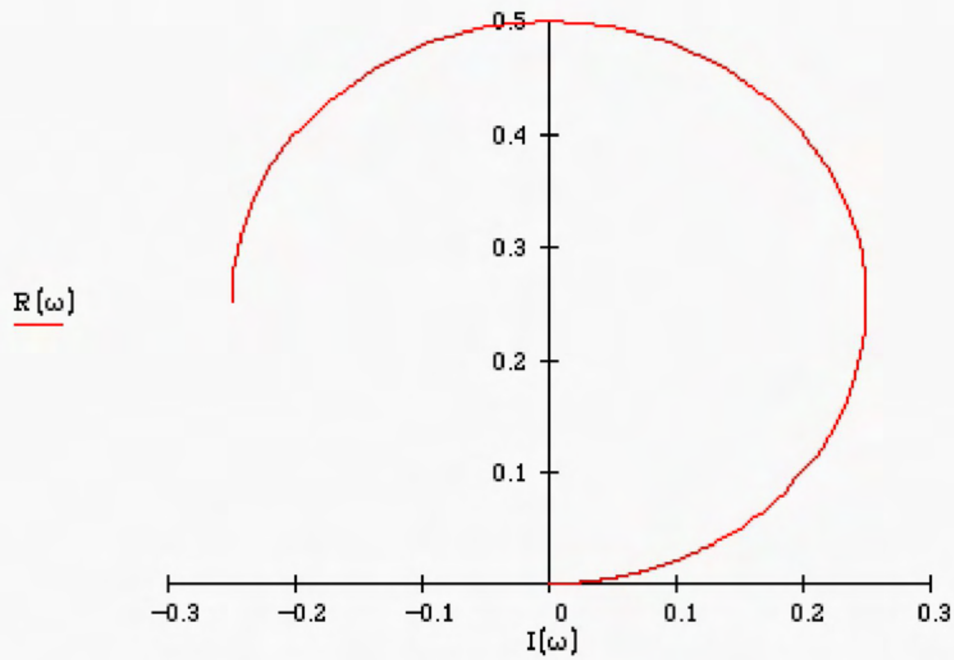
The following plot displays the 5 points listed in the table, as well as the same plot with a fine discretization of the driving frequency ω.

$$R := \begin{bmatrix} 0.2487 \\ 0.3996 \\ 0.5 \\ 0.4004 \\ 0.2512 \end{bmatrix} \qquad I := \begin{bmatrix} 0.25 \\ 0.2003 \\ 0 \\ -0.1997 \\ 0.25 \end{bmatrix}$$



$\omega := 0, 0.01 .. 10.1$

$$R(\omega) := \frac{2\cdot\omega^2}{\left(1000 - 10\cdot\omega^2\right)^2 + (2\ \omega)^2} \qquad I(\omega) := \frac{\omega\cdot\left(1000 - 10\cdot\omega^2\right)}{\left(1000 - 10\cdot\omega^2\right)^2 + (2\ \omega)^2}$$



7.23 Derive equation (7.20) for the damping ratio from equations (7.18) and (7.19). Then verify that equation (7.20) reduces to equation (7.21) at the half-power points.

Solution: Begin with equations (7.18) and (7.19)

$$\tan\left(\alpha/2\right)=\frac{\left(\omega_a/\omega_3\right)^2-1}{2\zeta_3\omega_a/\omega_3}$$

$$\tan\left(\alpha/2\right)=\frac{\left(\omega_b/\omega_3\right)^2-1}{2\zeta_3\omega_b/\omega_3}$$

Multiplying the right hand side of each expression by $\dfrac{\omega_3^2}{\omega_3^2}$ yields

$$\tan\left(\alpha/2\right)=\frac{\omega_a^2-\omega_3^2}{2\zeta_3\omega_a\omega_3}$$

$$\tan\left(\alpha/2\right)=\frac{\omega_3^2-\omega_b^2}{2\zeta_3\omega_b\omega_3}$$

After a suitable multiplication, these expressions are:

$$(2\zeta_3\omega_a\omega_3)\tan\left(\alpha/2\right)=\omega_a^2-\omega_3^2$$

$$(2\zeta_3\omega_b\omega_3)\tan\left(\alpha/2\right)=\omega_3^2-\omega_b^2$$

Adding the previous two equations results in:

$$2\zeta_3(\omega_a+\omega_b)\tan\left(\alpha/2\right)=\omega_a^2-\omega_b^2$$

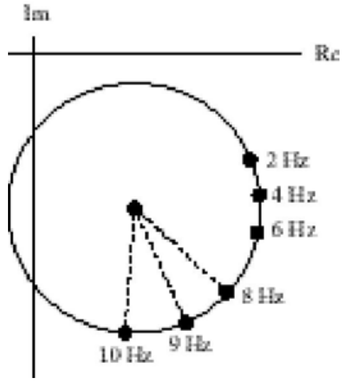
Which can be manipulated to yield equation (7.20)

$$\zeta_3=\frac{\omega_a^2-\omega_b^2}{2\omega_3\left[\omega_a\tan\left(\alpha/2\right)+\omega_b\tan\left(\alpha/2\right)\right]}$$

At the half-power points, α = 90° and $\tan\left(\alpha/2\right)=1$, so (7.20) reduces to:

$$\zeta_3=\frac{\omega_a-\omega_b}{2\omega_3}$$

7.24 Consider the experimental curve fit Nyquist circle of Figure P7.24. Determine the modal damping ratio for this mode



Solution:

From Figure 7.18,

$$\alpha \approx 45^\circ$$
$$\omega_3 = 9\,\text{Hz}$$
$$\omega_b = 10\,\text{Hz}$$
$$\omega_a = 8\,\text{Hz}$$

Using (7.20)

$$\zeta_3 = \frac{10^2 - 8^2}{2(9)\left[8\tan\left(45^\circ/2\right) + 10\tan\left(45^\circ/2\right)\right]}$$

$$\zeta_3 = 0.27$$