Chapter 8

Problems and Solutions Section 8.1 (8.1 through 8.7)

8.1 Consider the one-element model of a bar discussed in Section 8.1. Calculate the finite element of the bar for the case that it is free at both ends rather than clamped.

Solution: The finite element for a rod is derived in section 8.1. Since u_1 is not restrained equations (8.7) and (8.11) are the finite element matrices.

8.2 Calculate the natural frequencies of the free-free bar of Problem 8.1. To what does the first natural frequency correspond? How do these values compare with the exact values obtained from methods of Chapter 6?

Solution:

$$K = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad M = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M^{-1}K = \frac{6E}{\rho l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\lambda_{1,2} = 0, \dfrac{12E}{\rho l^2}$ and the corresponding eigenvectors are

$\mathbf{x}_1 = [1 \quad 1]^T/\sqrt{2}$ and $\mathbf{x}_2 = [1 \quad -1]^T/\sqrt{2}$

Therefore, $\omega_1 = 0,\ \omega_2 = \sqrt{\dfrac{12E}{\rho l^2}}$

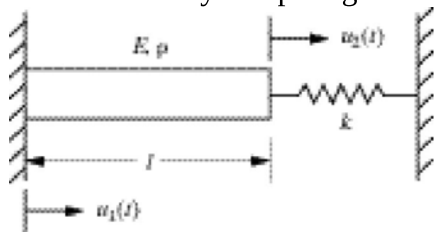
The first natural frequency corresponds to the rigid body mode, or pure translation.

From the solution to problem 6.8,

$$\omega_1 = 0,\ \omega_2 = \sqrt{\frac{\pi^2 E}{\rho l^2}}$$

The first natural frequency is predicted exactly while the second is 10.2% high. A point of interest is that, due to symmetry, the first mode of a clamped-free rod of length $l/2$ has the same natural frequency as the second mode of a free-free rod of length l.

8.3 Consider the system of Figure P8.3, consisting of a spring connected to a clamped-free bar. Calculate the finite element model and discuss the accuracy of the frequency prediction of this model by comparing it with the method of Chapter 6.



Solution:

The finite element for the clamped-free rod is given by (8.14) as

$$\frac{\rho A l}{3}\ddot{u}_2(t) + \frac{EA}{l}u_2(t) = 0$$

The spring has the effect of adding stiffness K at u_2. Thus,

$$\frac{\rho A l}{3}\ddot{u}_2(t) + \left(\frac{EA}{l} + K\right)u_2(t) = 0$$

From (1.16)

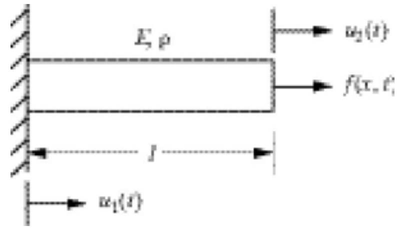
$$\omega = \sqrt{\frac{3(Kl + EA)}{\rho A l}}$$

Next consider the first natural frequency as predicted from the distributed parameter approach of chapter 6. In particular Table 6.1 gives the frequency equation for this system as $\lambda_n \cot\lambda_n = -(Kl/EA)$ where $\lambda_n = \omega_n l/c$, $c^2 = E/\rho$. Approximating $\cot x = 1/x - x/3$ the frequency equation of Table 6.1 becomes

$\lambda_n\,(1/\lambda_n - \lambda_n/3) = -(kl/EA)$ or for n=1 $\omega^2 l^2/c^2 = 3(1 + kl/EA)$

which upon solving for ω is identical to the one element FEM frequency derived above.

8.4 Consider a clamped-free bar with a force *f(t)* applied in the axial direction at the free end as illustrated in Figure P8.4. Calculate the equations of motion using a single-element finite element model.



Solution:

The finite element equation of motion for an unforced clamped-free bar is given by equation (8.14). Using (8.13) it can be seen that the forced equation is

$$\frac{\rho A l}{3}\ddot{u}_2(t)+\frac{EA}{l}u_2(t)=f(t)$$

8.5 Compare the solution of a cantilevered bar modeled as a single finite element with that of the distributed-parameter method summarized in Figure 8.1 truncated at three modes by calculating (a) $u(x,t)$ and (b) $u(l/2,t)$ for a 1-m aluminum beam at t = 0.1, 1, and 10s using both methods. Use the initial condition $u(x,0) = 0.1x$ m and $u_t(x,0) = 0$.

Solution: (8.5, 8.6)

For the finite element of the bar

$\rho = 2700$ kg/m^3, $E = 7 \times 10^{10}$ N/m^2

The unforced equation of motion is then

$$\ddot{u}_2(t) + 7.78 \times 10^7 u_2(t) = 0$$

From window 8.2

$u_2(t) = .1\cos(8.819\times10^3 t)$

Using the shape functions for the bar

$u(\mathrm{x,t}) = u_2(t)x = .1x\cos(8.819\times10^3 t)$

For the continuous model truncated at 3 modes, (see example6.3.1)

$\omega_{1,2,3}$ = 8000 rad/s, 24000 rad/s, 40000 rad/s and the mode shapes are

$$X_1(x) = \sin\left(\frac{\pi}{2}\frac{x}{l}\right) \quad , \; X_4(x) = \sin\left(\frac{7\pi}{2}\frac{x}{l}\right)$$
$$X_2(x) = \sin\left(\frac{3\pi}{2}\frac{x}{l}\right) \quad , \; X_5(x) = \sin\left(\frac{9\pi}{2}\frac{x}{l}\right)$$
$$X_3(x) = \sin\left(\frac{5\pi}{2}\frac{x}{l}\right)$$

The solution is given by (6.27) as

$$u(x,t) = \sum_{n=1}^{\infty}(c_n \sin\omega_n t + d_n \cos\omega_n t)X_n(x)$$

Since we are given $\dot{u}(x,t) = 0, \; c_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\omega_n t)X_n(x)$$

Considering the initial condition $u(x,0) = .1x$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)\pi}{2}\frac{x}{l} = .1x$$

Multiplying by $\sin\dfrac{(2m-1)\pi}{2}\dfrac{x}{l}$ and integrating from $x = 0$ to $x = l$,

$$\int_0^l .1x\sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx = a_n\int_0^l \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)\sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx$$

$$\int_0^l .1x\sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx = a_n\left(\frac{l^2}{2}\right)$$

$$a_n = \frac{2}{l^2}\int_0^l .1x\sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)dx$$

a_1=.08106, a_2=-.009006, a_3=.003242, a_4=0.001654, a_5=.001001

from (6.63)

$$\omega_n = \frac{(2n-1)\pi}{2}\sqrt{\frac{E}{\rho}},\ \omega_1 = \frac{\pi}{2}\sqrt{\frac{E}{\rho}} = 7998\text{rad/s},\ \omega_2 = \frac{3\pi}{2}\sqrt{\frac{E}{\rho}} = 23994\text{rad/s},$$

$$\omega_3 = \frac{5\pi}{2}\sqrt{\frac{E}{\rho}} = 39990\,\text{rad/s},\ \omega_4 = \frac{7\pi}{2}\sqrt{\frac{E}{\rho}} = 55987\text{rad/s}$$

$$\omega_5 = \frac{9\pi}{2}\sqrt{\frac{E}{\rho}} = 71982\,\text{rad/s}$$

Substitution into 6.27 yields

$$\begin{aligned}\omega(x,t) = {} & .08106\cos(7998t)\sin\left(\frac{\pi}{2}\frac{x}{l}\right) - .00901\cos(23994t)\sin\left(\frac{3\pi}{2}\frac{x}{l}\right) \\ & + .00324\cos(39990t)\sin\left(\frac{5\pi}{2}\frac{x}{l}\right) - .00165\cos(55987t)\sin\left(\frac{7\pi}{2}\frac{x}{l}\right) \\ & + .001001\cos(71982t)\sin\left(\frac{9\pi}{2}\frac{x}{l}\right)\end{aligned}$$

Note that for problem 8.5 the last two terms are neglected.

8.5

$$u(x,t)\Big|_{t=.1} = -.021205\sin\left(\frac{\pi x}{2}\right) - .00643\sin\left(\frac{3\pi x}{2}\right) - .00314\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\Big|_{t=1} = .07133\sin\left(\frac{\pi x}{2}\right) - .00077\sin\left(\frac{3\pi x}{2}\right) - .00255\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\Big|_{t=10} = .01900\sin\left(\frac{\pi x}{2}\right) - .00587\sin\left(\frac{3\pi x}{2}\right) - .003\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\Big|_{t=.1,\,x=.5} = -.01732$$

$$u(x,t)\Big|_{t=1,\,x=.5} = .05169$$

$$u(x,t)\Big|_{t=10,\,x=.5} = .01546$$

8.6 Repeat Problem 8.5 using a five-mode model. Can you draw any conclusions?

Solution:

$$u(x,t)\big|_{t=.1} = -.0212\sin\left(\frac{\pi x}{2}\right) - .00643\sin\left(\frac{3\pi x}{2}\right) - .00314\left(\frac{5\pi x}{2}\right)$$
$$- .00153\sin\left(\frac{7\pi x}{2}\right) - .00069\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=1} = .07133\sin\left(\frac{\pi x}{2}\right) - .0077\sin\left(\frac{3\pi x}{2}\right) - .00255\left(\frac{5\pi x}{2}\right)$$
$$+ .00129\sin\left(\frac{7\pi x}{2}\right) - .00026\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=10} = .01900\sin\left(\frac{\pi x}{2}\right) + .00587\sin\left(\frac{3\pi x}{2}\right) - .00300\left(\frac{5\pi x}{2}\right)$$
$$- .00146\sin\left(\frac{7\pi x}{2}\right) + .00085\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=.1,\,x=.5} = -.01672$$
$$u(x,t)\big|_{t=1,\,x=.5} = .05060$$
$$u(x,t)\big|_{t=10,\,x=.5} = .01709$$

For the finite element solution from (8.17)

$$u(x,t) = .1x\cos(8819.2t)$$

$$u(x,t)\big|_{t=.1} = -.06445x \qquad u(x,t)\big|_{t=.1,\,x=.5} = -.03222$$
$$u(x,t)\big|_{t=1} = -.07515x \qquad u(x,t)\big|_{t=1,\,x=.5} = -.03758$$
$$u(x,t)\big|_{t=10} = .06047x \qquad u(x,t)\big|_{t=10,\,x=.5} = .03024$$

Conclusion: Not nearly enough elements were used to accurately determine the 1st natural frequency. Since the 1st mode dominates the response (this can be seen by comparing the coefficients, a_n), it must be determined well in order to predict the rod's response.

8.7 Repeat Problem 8.5 using only the first mode in the series solution and the initial condition $u(x,0) = 0.1\sin(\pi x/2l)$, $u_t(x,0) = 0$. For this initial condition, the first mode is exact. Why?

Solution:

Using the same procedure as in problem 8.5, the solution is

$$u(x,t) = .1\sin\left(\frac{\pi x}{2}\right)\cos(7998t)$$

$$u(x,t)\big|_{t=.1} = -.02616\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=.1,x=.5} = -.01850$$

$$u(x,t)\big|_{t=1} = .08800\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=1,x=.5} = .06223$$

$$u(x,t)\big|_{t=10} = .02344\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=10,x=.5} = .01657$$

The finite element solution is unchanged. Again there is horrible agreement between the finite element model and the distributed parameter model.

The fist mode is exact because the initial condition is in the first mode. All coefficients, a_n, for modes other than the first mode are zero.