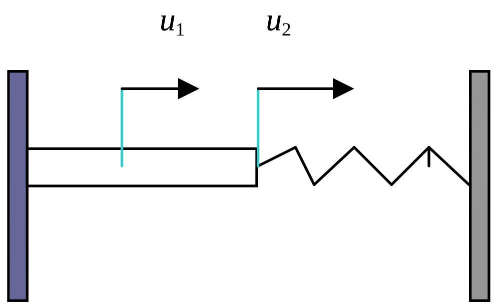
Problems and Solutions Section 8.2 (8.8 through 8.20)

8.8 Consider the bar of Figure P8.3 and model the bar with two elements. Calculate the frequencies and compare them with the solution obtained in Problem 8.3. Assume material properties of aluminum, a cross-sectional area of 1 m, and a spring stiffness of 1×10^6 N/m.

Solution: The finite element model for the two-element bar is

$$M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{0}$$

where $\mathbf{u}(t) = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$



$$M = \frac{\rho A l}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \quad K = \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

As in problem 8.3, the spring adds a stiffness K to degree of freedom 2. The equation of motion is then

$$\frac{\rho A l}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\ddot{\mathbf{u}}(t) + \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 + \dfrac{Kl}{2EA} \end{bmatrix}\mathbf{u}(t) = \mathbf{0}$$

The natural frequencies can be found by eigenanalysis. Using the material properties of aluminum

$\rho = 2700 \text{kg/m}^3$, $E = 7 \times 10^{10}$Pa

$\omega_1 = 129.0$ rad/s

$\omega_2 = 368.4$ rad/s

The solution obtained in problem 8.4 is $\omega_1 = 149.1$ rad/s.

8.9 Repeat Problem 8.8 with a three-element model. Calculate the frequencies and compare them with those of Problem 8.8.

Solution:

The finite element model of the 3 element rod for equal length elements is (from equation (8.25))

$$\frac{\rho A l}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \frac{3EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\mathbf{u} = \mathbf{0}$$

With the spring stiffness included, the global stiffness becomes

$$K = \frac{3EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1+\dfrac{Kl}{3EA} \end{bmatrix}$$

Solving for the natural frequencies gives ω_1 = 125.85 rad/s, ω_2=333.1 rad/s, and ω_3 = 591.7 rad/s

The natural frequencies predicted in 8.9 should be better than those predicted in 8.8. You can compare them to the results of 2 element model by using VTB8_2 and loading the file p8_3_10.con.

8.10 Consider Example 8.2.2. Repeat this example with node 2 moved to $\ell/2$ so that the mesh is uniform. Calculate the natural frequencies and compare them to those obtained in the example. What happens to the mass matrix?

Solution: (8.10, 8.11)

The equation of motion can be shown to be

$$\frac{\rho Al}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{u} = \mathbf{0}$$

$$\omega_1 = \frac{1.16114}{l}\sqrt{\frac{E}{\rho}} = 8204.8 \text{ rad/s}$$

$$\omega_2 = \frac{5.6293}{l}\sqrt{\frac{E}{\rho}} = 28663 \text{ rad/s}$$

The first natural frequency is slightly improved (closer to the distributed parameter 'true' value) while the second natural frequency has become worse.

	Truth	Example 8.22	Problem 8.10	Example 8.2.1
ω_1	$1.571\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.643\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.611\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.579\frac{1}{l}\sqrt{\frac{E}{\rho}}$
ω_2	$4.712\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.196\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.629\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.167\frac{1}{l}\sqrt{\frac{E}{\rho}}$

The natural frequencies found using the 3 element model are much better than the 2 element model.

8.11 Compare the frequencies obtained in Problem 8.10 with those obtained in Section 8.2 using three elements.

Solution:

See the solution for problem 8.10.

8.12 As mentioned in the text, the usefulness of the finite element method rests in problems that cannot readily be solved in closed form. To this end, consider a section of an air frame sketched in Figure P8.13 and calculate a two-element finite model of this structure (i.e., find *M* and *K*) for a bar with

Solution:

$$A(x) = \frac{\pi}{4}\left[h_1^2 + \left(\frac{h_2 - h_1}{l}\right)^2 x^2 + 2h_1\left(\frac{h_2 - h_1}{l}\right)x\right]$$

Two methods exist for creating a finite element model for this wing. The first is to assume each element has a constant cross section. The second is to derive elements based on the variable cross section. If enough elements are used, constant cross section elements can yield acceptable results. However, since in this example only two elements are used, it is better to use a variable cross section element. Both solutions are given.

A: <u>Variable cross section elements</u>

Following the procedure of section 8.1, the shape function of the first element is given by

$$u(x,t) = \left(1 - \frac{2x}{l}\right)u_1(t) + \frac{2x}{l}u_2(t)$$

The strain energy for element 1 is given by

$$\begin{aligned} V_1(t) &= \int_0^{l/2} EA(x)\left[\frac{\partial u_1(x,t)}{\partial x}\right]^2 dx \\ &= \frac{E\pi}{48l}[(7h_1^2 + 4h_1h_2 + h_2^2)u_1^2(t) - (14h_1^2 + 8h_1h_2 + 2h_2^2)u_1(t)u_2(t) \\ &\qquad + (7h_1^2 + 4h_1h_2 + h_2^2)u_2^2(t)] \end{aligned}$$

However, since $u_1(t) = 0$,

$$V_1(t) = \frac{E\pi}{48l}(7h_1^2 + 4h_1h_2 + h_2^2)u_2^2(t)$$

For element 2, the shape function is

$$u_2(x,t) = 2\left(1 - \frac{x}{l}\right)u_2(t) + \left(\frac{2x}{l} - 1\right)u_3(t)$$

The strain energy for element 2 is then given by

$$V_2(t) = \frac{1}{2}\int_{l/2}^{l} EA(x)\left[\frac{\partial u_2(x,t)}{\partial x}\right]^2 dx$$
$$= \frac{E\pi}{48l}\left(h_1^2 + 4h_1h_2 + 7h_2^2\right)\left(u_2^2(t) + 2u_2(t)u_3(t) + u_3^2(t)\right)$$

The total strain energy is then

$$V(t) = \frac{E\pi}{48l}\left((f_1 + f_2)u_2^2(t) - 2f_2u_2(t)u_3(t) + f_2u_3^2(t)\right)$$

where $f_1 = 7h_1^2 + 4h_1h_2 + h_2^2$ and $f_2 = h_1^2 + 4h_1h_2 + 7h_2^2$

In matrix form this is

$$V(t) = \frac{1}{2}\begin{bmatrix} u_2(t) & u_3(t)\end{bmatrix} K \begin{bmatrix} u_2(t) & u_3(t)\end{bmatrix}^T$$

where

$$K = \frac{E\pi}{24l}\begin{bmatrix} f_1 \times f_2 & -f_2 \\ -f_2 & f_2 \end{bmatrix}$$

The kinetic energy of element 1 is given by

$$T_1(t) = \int_0^{l/2} A(x)\rho\left[\frac{\partial u_1(x,t)}{\partial x}\right]^2 dx$$
$$= \frac{l\pi\rho}{1920}(16h_1^2 + 18h_1h_2 + 6h_2^2)\dot{u}_2^2(t)$$

(since $\dot{u}_1(t) = 0$, terms including $\dot{u}_1(t)$ have been dropped)

Similarly, the kinetic energy of element 2 is

$$T_2(t) = \int_{l/2}^{l} A(x)\rho\left[\frac{\partial u_2(x,t)}{\partial x}\right]^2 dx = \frac{l\pi\rho}{1920}[(6h_1^2 + 18h_1h_2 + 16h_2^2)\dot{u}_2^2$$
$$+ (3h_1^2 + 8h_1h_2 + 31h_2^2)\dot{u}_2\dot{u}_3 + (h_1^2 + 8h_1h_2 + 31h_2^2)\dot{u}_3^2]$$

The total kinetic energy can be written

$$T(t) = \frac{l\pi\rho}{1920}[(22h_1^2 + 36h_1h_2 + 22h_2^2)\dot{u}_2^2 + (3h_1^2 + 14h_1h_2 + 23h_2^2)\dot{u}_2\dot{u}_3$$
$$+ (h_1^2 + 8h_1h_2 + 31h_2^2)\dot{u}_3^2] = \frac{1}{2}\left[\dot{u}_2\dot{u}_3\right]M\left[\dot{u}_2\dot{u}_3\right]^T$$

where

$$M = \frac{l\pi\rho}{1920}\begin{bmatrix} 44h_1^2 + 72h_1h_2 + 44h_2^2 & 3h_1^2 + 14h_1h_2 + 23h_2^2 \\ 3h_1^2 + 14h_1h_2 + 23h_2^2 & 2h_1^2 + 16h_1h_2 + 62h_2^2 \end{bmatrix}$$

B: <u>Constant cross section elements</u>

The average cross section area of element 1 is

$$A_1 = \frac{\pi}{48}(7h_1^2 + 4h_1h_2 + h_2^2)$$

and the average cross section area of element 2 is

$$A_2 = \frac{\pi}{48}(h_1^2 + 4h_1h_2 + 7h_2^2)$$

Finding the potential energy again yields the same global stiffness matrix as for the variable cross section model.

The kinetic energy can then be found by

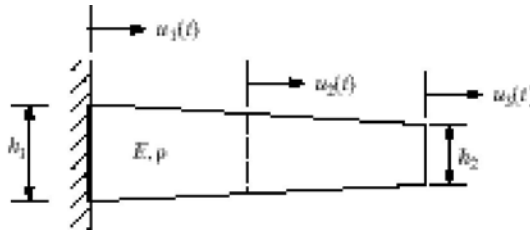
$$T(t) = \frac{1}{2}\int_0^{l/2} A_1\rho\left[\frac{\partial u_1(x,t)}{\partial x}\right]^2 dx + \frac{1}{2}\int_{l/2}^{l} A_2\rho\left[\frac{\partial u_2(x,t)}{\partial x}\right]^2 dx$$
$$= \frac{1}{2}\left[\dot{u}_2\dot{u}_3\right]M\left[\dot{u}_2\dot{u}_3\right]^T$$

where

$$M = \frac{\rho l}{12}\begin{bmatrix} 2(A_1 + A_2) & A_2 \\ A_2 & 2A_2 \end{bmatrix}$$

which is not identical to the mass matrix derived using variable cross section elements.

8.13 Let the bar in Figure P8.13 be made of aluminum 1 m in length with h_1 = 20 cm and h_2 = 10 cm. Calculate the natural frequencies using the finite element model of Problem 8.12.



Solution:

$E = 7 \times 10^{10}$Pa, $\rho = 2700$ kg/m^3

$h_1 = .2$m, $h_2 = .1$m, $l = 1$m

Using the variable cross section elements

$$K = \begin{bmatrix} 2.566 \times 10^9 & -8.705 \times 10^8 \\ -8.705 \times 10^8 & 8.705 \times 10^8 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 16.081 & 2.783 \\ 2.783 & 4.506 \end{bmatrix}$$

The natural frequencies are then $\omega_1 = 7414$ rad/s and $\omega_2 = 20368$ rad/s

The constant cross sectional area mass matrix is

$$M = \begin{bmatrix} 16.493 & 2.798 \\ 2.798 & 5.596 \end{bmatrix}$$

which give $\omega_1 = 7092$ rad/s, $\omega_2 = 18636$ rad/s

8.14 Repeat Problems 8.12 and 8.13 using a three-element four-node finite element model.

Solution:

The shape functions for 3 evenly spaced elements are

$$u_1(x,t) = \left(1 - \frac{3x}{2l}\right)u_1(t) + \frac{3x}{l}u_2(t)$$

$$u_2(x,t) = 2\left(1 - \frac{3x}{2l}\right)u_2(t) + \left(\frac{3x}{l} - 1\right)u_3(t)$$

$$u_3(x,t) = 3\left(1 - \frac{x}{l}\right)u_3(t) + 2\left(\frac{3x}{2l} - 1\right)u_4(t)$$

Integrating to find the strain energy, the strain energies in matrix notation are

$$V_1(t) = \frac{1}{2}\begin{bmatrix} u_1 & u_2 \end{bmatrix} K_1 \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$$

$$V_2(t) = \frac{1}{2}\begin{bmatrix} u_2 & u_3 \end{bmatrix} K_2 \begin{bmatrix} u_2 & u_3 \end{bmatrix}^T$$

$$V_3(t) = \frac{1}{2}\begin{bmatrix} u_3 & u_4 \end{bmatrix} K_3 \begin{bmatrix} u_3 & u_4 \end{bmatrix}^T$$

where

$$K_1 = \frac{E\pi}{36l}(19h_1^2 + 7h_1h_2 + h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_2 = \frac{E\pi}{36l}(7h_1^2 + 13h_1h_2 + 7h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_3 = \frac{E\pi}{36l}(h_1^2 + 7h_1h_2 + 19h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Writing the total strain energy in matrix form, the global stiffness matrix is

$$K = \frac{E\pi}{36l}\begin{bmatrix} f_1 + f_2 & -f_2 & 0 \\ -f_2 & f_2 + f_3 & -f_3 \\ 0 & -f_3 & f_3 \end{bmatrix}$$

where

$f_1 = 19h_1^2 + 7h_1h_2 + h_2^2$, $f_2 = 7h_1^2 + 13h_1h_2 + 7h_2^2$ and $f_3 = h_1^2 + 7h_1h_2 + 19h_2^2$

The kinetic energy of each element in matrix form is

$$T_1(t) = \frac{1}{2}\begin{bmatrix}\dot{u}_1 & \dot{u}_2\end{bmatrix} M_1 \begin{bmatrix}\dot{u}_1 & \dot{u}_2\end{bmatrix}^T, \quad T_2(t) = \frac{1}{2}\begin{bmatrix}\dot{u}_2 & \dot{u}_3\end{bmatrix} M_2 \begin{bmatrix}\dot{u}_2 & \dot{u}_3\end{bmatrix}^T,$$

$$T_3(t) = \frac{1}{2}\begin{bmatrix}\dot{u}_3 & \dot{u}_4\end{bmatrix} M_3 \begin{bmatrix}\dot{u}_3 & \dot{u}_4\end{bmatrix}^T$$

where

$$M_1 = \frac{l\pi\rho}{3240}\begin{bmatrix} 76h_1^2 + 13h_1h_2 + h_2^2 & \frac{1}{2}\left(63h_1^2 + 24h_1h_2 + 3h_2^2\right) \\ \frac{1}{2}\left(63h_1^2 + 24h_1h_2 + 3h_2^2\right) & 51h_1^2 + 33h_1h_2 + 6h_2^2 \end{bmatrix}$$

$$M_2 = \frac{l\pi\rho}{3240}\begin{bmatrix} 31h_1^2 + 43h_1h_2 + 16h_2^2 & \frac{1}{2}\left(23h_1^2 + 44h_1h_2 + 23h_2^2\right) \\ \frac{1}{2}\left(23h_1^2 + 44h_1h_2 + 23h_2^2\right) & 16h_1^2 + 43h_1h_2 + 31h_2^2 \end{bmatrix}$$

$$M_3 = \frac{l\pi\rho}{3240}\begin{bmatrix} 6h_1^2 + 33h_1h_2 + 51h_2^2 & \frac{1}{2}\left(3h_1^2 + 24h_1h_2 + 63h_2^2\right) \\ \frac{1}{2}\left(3h_1^2 + 24h_1h_2 + 63h_2^2\right) & h_1^2 + 13h_1h_2 + 76h_2^2 \end{bmatrix}$$

Evaluating and assembling the mass and stiffness matrices gives:

$$K = \begin{bmatrix} 9.285 & -3.726 & 0 \\ -3.729 & 5.987 & -2.2602 \\ 0 & -2.2602 & 2.2602 \end{bmatrix} \times 10^9$$

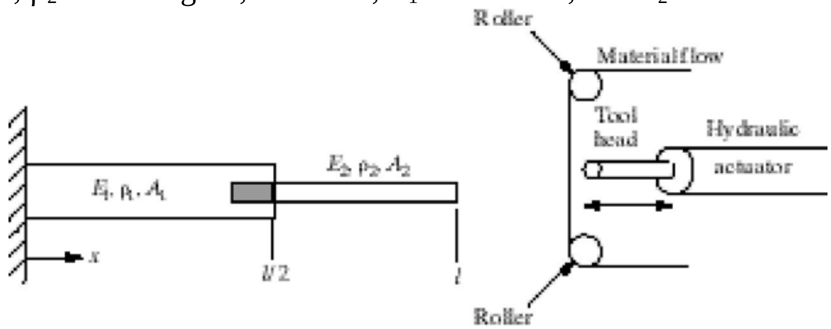
$$M = \begin{bmatrix} 13.1423 & 2.6573 & 0 \\ 2.6573 & 8.4299 & 1.6101 \\ 0 & 1.6101 & 2.7751 \end{bmatrix}$$

ω_1 = 10406 rad/s, ω_2 = 27309 rad/s, ω_3 = 47797 rad/s

Note that a ten element model yields

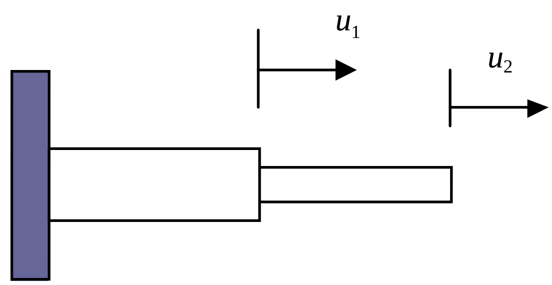
ω_1 = 10316 rad/s, ω_2 = 25183 rad/s

8.15 Consider the machine punch of Figure P8.15. This punch is made of two materials and is subject to an impact in the axial direction. Use the finite element method with two elements to model this system and estimate (calculate) the first two natural frequencies. Assume $E_1 = 8 \times 10^{10}$ Pa, $E_2 = 2.0 \times 10^{11}$ Pa, $\rho_1 = 7200$ kg/m^3, $\rho_2 = 7800$ kg/m^3, $l = 0.2$ m, $A_1 = 0.009$ m^2, and $A_2 = 0.0009$ m^2.



Solution: The total strain energy of the system is

$$V(t) = \frac{1}{2}\left\{u_1^2 \frac{2E_1A_1}{l} + \frac{2E_1A_1}{l}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right\}$$



The vector of derivatives of the potential energy gives

$$\begin{bmatrix} \dfrac{\partial V}{\partial u_1} \\ \dfrac{\partial V}{\partial u_2} \end{bmatrix} = \frac{2}{l}\begin{bmatrix} E_1A_1 + E_2A_2 & -E_2A_2 \\ -E_2A_2 & E_2A_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The stiffness matrix is then

$$K = \frac{2}{l}\begin{bmatrix} E_1A_1 + E_2A_2 & -E_2A_2 \\ -E_2A_2 & E_2A_2 \end{bmatrix}$$

In similar fashion, the total kinetic energy is

$$T(t) = \frac{1}{2}\left\{\dot{u}_1^2 \frac{\rho_1A_1l}{6} + \frac{l}{12}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}^T \begin{bmatrix} 2\rho_2A_2 & \rho_2A_2 \\ \rho_2A_2 & 2\rho_2A_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}\right\}$$

The mass matrix is then

$$M = \frac{l}{12}\begin{bmatrix} 2(\rho_2 A_2 + \rho_1 A_1) & \rho_2 A_2 \\ \rho_2 A_2 & 2\rho_2 A_2 \end{bmatrix}$$

$E_1 = 8\times 10^{10}\,\text{Pa},\ \rho_1 = 7200\text{kg/m}^3,\ E_2 = 2.0\times 10^{11}\,\text{Pa},\ \rho_2 = 7800\text{kg/m}^3,$
$l = .2 A_1 = .0009,\ A_2 = .0001$

$$K = \begin{bmatrix} 9.2 & -2 \\ -2 & 2 \end{bmatrix}\times 10^8 \quad M = \begin{bmatrix} .242 & .013 \\ .013 & .026 \end{bmatrix}$$

ω_1 = 47556.1 rad/s, ω_2 = 101975 rad/s

8.16 Recalculate the frequencies of Problem 8.15 assuming that it is made entirely of one material and size (i.e., $E_1 = E_2$, $\rho_1 = \rho_2$, and $A_1 = A_2$), say steel, and compare your results to those of Problem 8.15.

Solution:

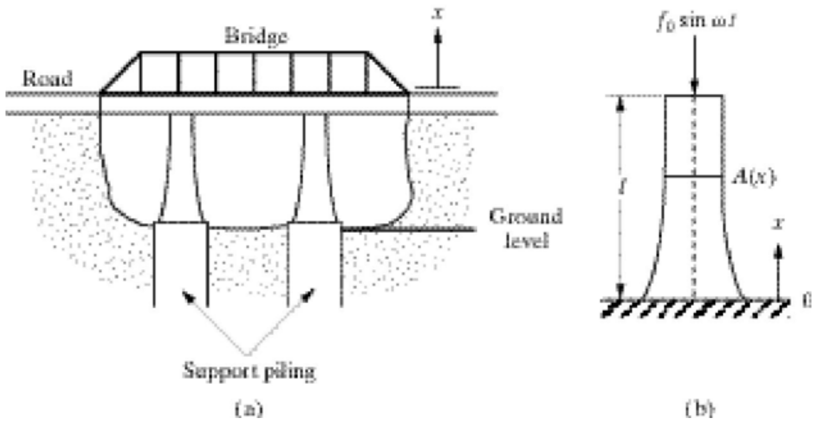
Assume $A_1 = A_2$, $E_1 = E_2$, $\rho_1 = \rho_2$

$$K = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}\times 10^8 \quad M = \begin{bmatrix} .052 & .013 \\ .013 & .026 \end{bmatrix}$$

ω_1 = 40798.6 rad/s, ω_2 = 142525 rad/s

The first natural frequency decreased. This example illustrates how a punch can be modified to raise the first natural frequency by changing the base material.

8.17 A bridge support column is illustrated in Figure P8.17. The column is made of concrete with a cross-sectioned area defined by $A(x) = A_0e^{-x/l}$, where A_0 is the area of the column at ground. Consider this pillar to be cantilevered (i.e., fixed) at ground level and to be excited sinusoidally at its tip in the longitudinal direction due to traffic over the bridge. Calculate a single-element finite element model of this system and compute its approximate natural frequency.



Solution:

$A(x) = A_0e^{-x/l}$

The potential energy is

$$V(t) = \frac{E}{2}\int_0^l A(x)\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx$$

where $u(x,t) = \left(1 - \frac{x}{l}\right)u_1(t) + \frac{x}{l}u_2(t)$

$$V(t) = \frac{EA_0}{2l}\frac{e-1}{e}\left(u_1(t) - u_2(t)\right)^2$$

$$= \frac{EA}{2l}\frac{e-1}{e}u_2^2(t)$$

The stiffness is then

$$K = \frac{EA}{l}\frac{(e-1)}{e}$$

Likewise, the kinetic energy is

$$T(t) = \frac{1}{2}\int_0^l \rho A\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx = \frac{Al\rho}{2e}(2e-5)\dot{u}_2^2(t)$$

The mass is then

$$M = \frac{Al\rho}{e}(2e-5)$$

The first natural frequency is then approximately

$$\omega_1 = \sqrt{\frac{K}{M}} = \sqrt{\frac{E(e-1)}{(2e-5)l^2\rho}} = \frac{1.984}{l}\sqrt{\frac{E}{\rho}}$$

8.18 Redo Problem 8.17 using two elements. What would happen if the "traffic" frequency corresponds with one of the natural frequencies of the support column?

Solution: The shape functions for a 2 element model are

$$u_1(x,t) = \left(1 - \frac{2x}{l}\right)u_1(t) + \frac{2x}{l}u_2(t)$$

$$u_2(x,t) = 2\left(1 - \frac{x}{l}\right)u_2(t) + \left(\frac{2x}{l} - 1\right)u_3(t)$$

The total stain energy in matrix form is

$$V(t) = \frac{1}{2}\begin{bmatrix}u_2 & u_3\end{bmatrix}K\begin{bmatrix}u_2 & u_3\end{bmatrix}^T$$

where

$$K = \frac{4A\left(\sqrt{e} - 1\right)E_0}{el}\begin{bmatrix}1 + \sqrt{e} & -1 \\ -1 & 1\end{bmatrix}$$

Likewise the mass matrix can be found from the total potential energy to be

$$M = \frac{Al\rho}{e}\begin{bmatrix}8\left(e - 1 - \sqrt{e}\right) & 10 - 6\sqrt{e} \\ 10 - 6\sqrt{e} & 8 - 13\sqrt{e}\end{bmatrix}$$

and the natural frequencies are then

$$\omega_1 = \frac{1.939}{l}\sqrt{\frac{E}{\rho}}\text{ rad/s},\quad \omega_2 = \frac{5.605}{l}\sqrt{\frac{E}{\rho}}\text{ rad/s}$$

If the traffic frequency corresponds to a natural frequency of a pillar, the bridge might fail.

8.19 Problems 8.17 and 8.18 represent approximations. As pointed out in Problem 8.18, it is important to know the natural frequencies of this column as precisely as possible. Hence consider modeling this column as a uniform bar of average cross section, calculate the first few natural frequencies, and compare them to the results in Problem 8.17 and 8.18. Which model do you think is closest to reality?

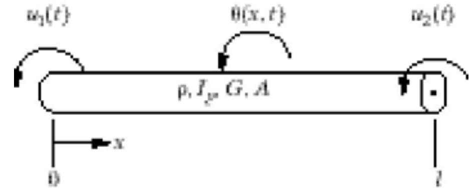
Solution:

The natural frequencies of a rod with constant cross sectional area are independent of the area. Therefore the first 2 natural frequencies are

$$\omega_1 \frac{1.571}{l}\sqrt{\frac{E}{\rho}}\text{ rad/s},\quad \omega_2 = \frac{4.712}{l}\sqrt{\frac{E}{\rho}}\text{ rad/s}$$

It is doubtful that these results are better since we know from the finite element model that the varying cross sectional area does have an effect.

8.20 Torsional vibration can also be modeled by finite elements. Referring to Figure P8.20, calculate a single-element mass and stiffness matrix for the torsional vibration following the steps of Section 8.1. (*Hint*: $\theta(x,t) = c_1(t)\theta + c_2(t)$, $T(t) = \frac{1}{2}\int_0^l \rho I_\rho \left[\theta_t(x,t)\right]^2 dx$ and $V(t) = \frac{1}{2}\int_0^l GI_\rho \left[\theta_t(x,t)\right]^2 dx$.)



Solution:

From equation (6.64), The static (time independent) displacement of the torsional rod element must satisfy

$$\frac{\partial \tau}{\partial x} = 0 = GJ\frac{\partial^2 \theta(x,t)}{\partial x^2}$$

which has the same form as equation (8.1). This can be integrated to yield

$\theta(x) = C_1\theta + C_2$

At $x = 0$

$\theta(0) = \theta_1(t) = C_2$

Likewise, at $x = l$

$\theta(l) = \theta_2(t) = C_1 l + C_2$

$$C_1 = \frac{\theta_2(t) - C_2}{l} = \frac{\theta_2(t) - \theta_1(t)}{l}$$

Substituting the values of C_1 and C_2 into the shape function yields

$$\theta(x,t) = \left(1 - \frac{x}{l}\right)\theta_1(t) + \left(\frac{x}{l}\right)\theta_2(t)$$

Evaluating the strain energy yields

$$\begin{aligned} V(t) &= \frac{GJ}{2l}\left(\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2\right) \\ &= \frac{1}{2}\left[\theta_1(t) \quad \theta_2(t)\right] K \left[\theta_1(t) \quad \theta_2(t)\right]^T \end{aligned}$$

where the stiffness matrix is defined by

$$K = \frac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Likewise, evaluating the kinetic energy yields

$$\begin{aligned} T(t) &= \frac{1}{2}\frac{\rho A l}{3}\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ &= \frac{1}{2}\left[\dot{\theta}_1(t) \quad \dot{\theta}_2(t)\right] M \left[\dot{\theta}_1(t) \quad \dot{\theta}_2(t)\right]^T \end{aligned}$$

where the mass matrix is defined by

$$M = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$