Problems and Solutions Section 8.3 (8.21 through 8.33)

8.21 Use equations (8.47) and (8.46) to derive equation (8.48) and hence make sure that the author and reviewer have not cheated you.

Solution:

$$u(x,t) = C_1(t)x^3 + C_2(t)x^2 + C_3(t)x + C_4(t) \qquad (8.46)$$

$$\begin{aligned} u(0,t) &= u_1(t) \qquad & u_x(0,t) &= u_2(t) \\ u(l,t) &= u_3(t) \qquad & u_x(l,t) &= u_4(t) \end{aligned} \qquad (8.47)$$

Substituting(8.46) into (8.47)

$$u(0,t) = C_4(t) = u_1(t)$$
$$u_x(0,t) = C_3(t) = u_2(t)$$
$$u(l,t) = C_1(t)l^3 + C_2(t)l^2 + C_3(t)l + C_4(t) = u_3(t)$$
$$u_x(l,t) = 3C_1(t)l + 2C_2(t)l + C_3(t) = u_4(t)$$

This gives

$$C_1 = \frac{1}{l^3}\left(2(u_1 - u_3) + l(u_2 + u_4)\right)$$
$$C_2 = \frac{1}{l^2}\left(3(u_3 - u_1) - l(u_4 + 2u_2)\right)$$
$$C_3 = u_2$$
$$C_4 = u_1$$

8.22 It is instructive, though tedious, to derive the beam element deflection given by equation (8.49). Hence derive the beam shape functions.

Solution:

Substituting (8.48) into (8.46) gives

$$u(x,t) = \left[1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right]u_1(t) + l\left[\frac{x}{l} - 2\frac{x^2}{l^2} + \frac{x^3}{l^3}\right]u_2(t)$$
$$+\left[3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right]u_3(t) + l\left[\frac{-x^2}{l^2} + \frac{x^3}{l^3}\right]u_4(t)$$

8.23 Using the shape functions of Problem 8.22, calculate the mass and stiffness matrices given by equations (8.53) and (8.56). Although tedious, this involves only simple integration of polynomials in *x*.

Solution:

$$T(t) = \frac{1}{2}\int_0^l \rho A\big(u_t(x,t)\big)^2 dx$$

$$= \frac{1}{2}\dot{\mathbf{u}}^T M \dot{\mathbf{u}}$$

where

$$\mathbf{u} = \big[u_1(t) \quad u_2(t) \quad u_3(t) \quad u_4(t)\big]^T$$

And M is given by equation (8.35).

Similarly

$$V(t) = \frac{1}{2}\int_0^l EI\big[u_{xx}(x,t)\big]^2 dx$$

$$= \frac{1}{2}\mathbf{u}^T K \mathbf{u}$$

where K is given by (8.56)

8.24 Calculate the natural frequencies of the cantilevered beam given in equation (8.69) using l = 1 m and compare your results with those listed in Table 6.1.

Solution:

$$M = \frac{\rho A}{840}\begin{bmatrix} 312 & 0 & 54 & -6.5 \\ 0 & 2 & 6.5 & -.75 \\ 54 & 6.5 & 156 & -11 \\ -6.5 & -.75 & -11 & 1 \end{bmatrix}$$

$$K = 8EI\begin{bmatrix} 24 & 0 & -12 & 3 \\ 0 & 2 & -3 & \frac{1}{2} \\ -12 & -3 & 12 & -3 \\ 3 & \frac{1}{2} & -3 & 1 \end{bmatrix}$$

Following the procedures of section 4.2

$$\omega_1 = 3.5177\sqrt{\frac{EI}{\rho A}},\ \omega_2 = 22.2215\sqrt{\frac{EI}{\rho A}}$$

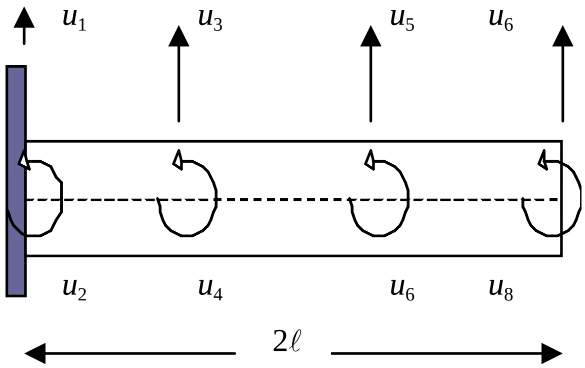
$$\omega_3 = 75.1571\sqrt{\frac{EI}{\rho A}},\ \omega_4 = 218.138\sqrt{\frac{EI}{\rho A}}$$

From continuous theory, the natural frequencies of a cantilevered beam are $\omega_i = \beta_i\sqrt{\frac{EI}{\rho A}}$ where $\beta_1 = 3.51601$, $\beta_2 = 22.0345$, $\beta_3 = 61.6972$, $\beta_4 = 120.9019$.

The predictions of the first two natural frequencies are quite accurate while the predictions of the third and fourth natural frequencies are terrible.

8.25 Calculate the finite element model of a cantilevered beam one meter in length using three elements. Calculate the natural frequencies and compare them to those obtained in Problem 8.23 and with the exact values listed in Table 6.4.

Solution: Define u_i using the following figure;



The equation for element one is

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \mathbf{0}$$

The equation for element two is

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \mathbf{0}$$

The equation for element 3 is the same as for element 2 but with the vector

$[u_3\ u_4\ u_5\ u_6]^T$ replaced with $[u_5\ u_6\ u_7\ u_8]^T$.

Combining the elemental equation using the superposition of the like coordinates yields

$$
\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13l & 0 & 0 \\ 0 & 8l^2 & 13l & -3l^2 & 0 & 0 \\ 54 & 13l & 312 & 0 & 54 & -13l \\ -13l & -3l^2 & 0 & 8l^2 & 13l & -3l^2 \\ 0 & 0 & 54 & 13l & 156 & -22l \\ 0 & 0 & -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \\ \ddot{u}_7 \\ \ddot{u}_8 \end{bmatrix}
+\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l & 0 & 0 \\ 0 & 8l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \mathbf{0}
$$

which can also be written in the form

$$
\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \\ l\ddot{u}_6 \\ \ddot{u}_7 \\ l\ddot{u}_8 \end{bmatrix}
+\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} u_3 \\ lu_4 \\ u_5 \\ lu_6 \\ u_7 \\ lu_8 \end{bmatrix} = \mathbf{0}
$$

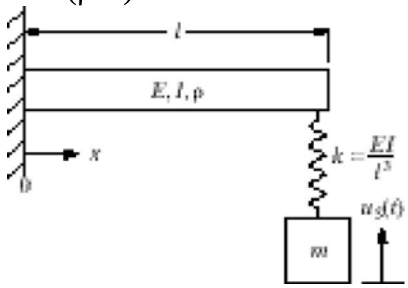
Following the procedure of example 8.3.3

$$\omega_1 = .3907\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\ ,\ \omega_2 = 2.456\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

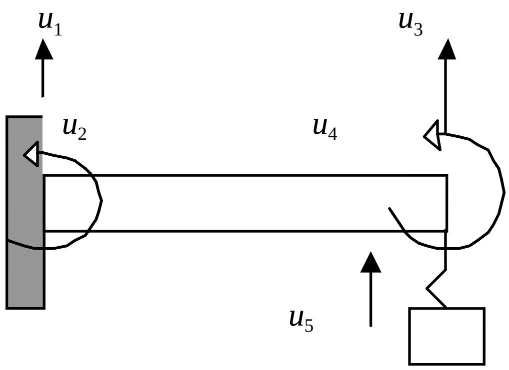
$$\omega_3 = 6.941\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\ ,\ \omega_4 = 15.63\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_5 = 29.42\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\ ,\ \omega_6 = 58.64\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.26 Consider the cantilevered beam of Figure P8.26 attached to a lumped spring-mass system. Model this system using a single finite element and calculate the natural frequencies. Assume $m = (\rho A l)/420$.



Solution: Define u_i using the following figure:



The model for the spring mass system is

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_5 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The single element model for the beam is

$$\frac{\rho A l}{420} \begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \mathbf{0}$$

Superimposing like coordinates yields

$$\frac{\rho A l}{420} \begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The equation of motion may also be written

$$\frac{\rho A l^4}{420 EI} \begin{bmatrix} 156 & -22 & 0 \\ -22 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \end{bmatrix} + \begin{bmatrix} 13 & -6 & -1 \\ -6 & 4 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_3 \\ lu_4 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The eigenvalue/eigenvector problem is then

$(A-\lambda I)\mathbf{v}=0$

where

$$A = M^{-1}K\frac{\rho A l^4}{420EI},\ \lambda = \frac{\rho A l^4}{420EI}\omega^2$$

$$A = \begin{bmatrix} -.5714 & .4571 & -.0286 \\ -4.6429 & 3.5143 & -.1571 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\lambda_1 = .0294,\ \lambda_2 = 1,\ \lambda_3 = 2.9134$$

$$\omega_1 = 3.52\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = 20.49\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_3 = 34.98\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.27 Repeat Problem 8.26 using two finite elements for the beam and compare the frequencies.

Solution:

A two element model of a cantilevered beam has been created in example 8.3.3.

Superimposing like coordinates for this example with the spring mass model yields

$$\frac{\rho A l}{840}\begin{bmatrix} 312 & 0 & 54 & -6.5l & 0 \\ 0 & 2l^2 & 6.5l & -.75l^2 & 0 \\ 54 & 6.5l & 156 & -11l & 0 \\ -6.5l & -.75l^2 & -11l & l^2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \\ \ddot{u}_7 \end{bmatrix} + \frac{8EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 3l & 0 \\ 0 & 2l^2 & -3l & \frac{1}{2}l^2 & 0 \\ -12 & -3l & 12+\frac{1}{8} & -3l & -\frac{1}{8} \\ 3l & \frac{1}{2}l^2 & -3l & l^2 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \mathbf{0}$$

Note that the coordinate vector for the spring mass system has changed from $[u_3\ u_5]^T$ to $[u_5\ u_6]^T$.

As in (8.26), the equations may be written in the form

$$
\frac{\rho A l^4}{6720 EI}\begin{bmatrix} 312 & 0 & 54 & -6.5 & 0 \\ 0 & 2 & 6.5 & -.75 & 0 \\ 54 & 6.5 & 156 & -11 & 0 \\ -6.5 & -.75 & -11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \\ l\ddot{u}_6 \\ \ddot{u}_7 \end{bmatrix}
$$

$$
+\frac{8EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 3 & 0 \\ 0 & 2 & -3 & \frac{1}{2} & 0 \\ -12 & -3 & 12+\frac{1}{8} & -3 & -\frac{1}{8} \\ 3 & \frac{1}{2} & -3 & 1 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}\begin{bmatrix} u_3 \\ lu_4 \\ u_5 \\ lu_6 \\ u_7 \end{bmatrix} = \mathbf{0}
$$

The eigenvalue/eigenvector problem is then

$$(A-\lambda I)\mathbf{v} = 0$$

where

$$A = M^{-1}K\frac{\rho A l^4}{6720 EI}, \ \lambda = \frac{\rho A l^4}{6720 EI}\omega^2$$

$$
A = \begin{bmatrix} .2878 & .0640 & -.2907 & .0868 & -.0004 \\ 3.6700 & 2.1247 & -5.9000 & 1.5919 & -.0062 \\ .9274 & .2094 & -1.0368 & .3516 & -.0041 \\ 17.8241 & 4.8163 & -20.7187 & 6.6253 & -.0519 \\ 0 & 0 & -.0625 & 0 & .0625 \end{bmatrix}
$$

$$\lambda_1 = .0427, \ \lambda_2 = .2455, \ \lambda_3 = .2772, \ \lambda_4 = .9173, \ \lambda_5 = 2.6614$$

$$\omega_1 = \frac{3.50}{l^2}\sqrt{\frac{EI}{\rho A}}, \ \omega_2 = \frac{20.12}{l^2}\sqrt{\frac{EI}{\rho A}}, \ \omega_3 = \frac{22.73}{l^2}\sqrt{\frac{EI}{\rho A}}$$

The one element (3 DOF) model predicted the first 2 natural frequencies well. The prediction of the third natural frequency was extremely poor using only one element.

8.28 Calculate the natural frequencies of a clamped-clamped beam for the physical parameters l = 1m, $E = 2 \times 10^{11}$ N/m^2, ρ = 7800 kg/m^3, $I = 10^{-6}$ m^4, and $A = 10^{-2}$ m^2, using the beam theory of Chapter 6 and a four-element finite element model of the beam.

Solution:

Using VTB8_1

$$M = \begin{bmatrix} 14.49 & 0 & 2.5071 & -.151 & 0 & 0 \\ 0 & .0232 & .0151 & -.0087 & 0 & 0 \\ 2.507 & .151 & 14.49 & 0 & 2.507 & -.151 \\ -.151 & -.0087 & 0 & .0232 & .151 & -.0087 \\ 0 & 0 & 2.5071 & .151 & 14.49 & 0 \\ 0 & 0 & -.151 & -.0087 & 0 & .0232 \end{bmatrix}$$

and

$$K = 1 \times 10^5 \begin{bmatrix} 3072 & 0 & -1536 & 192 & 0 & 0 \\ 0 & 64 & -192 & 16 & 0 & 0 \\ -1536 & -192 & 3072 & 0 & -1536 & 192 \\ 192 & 16 & 0 & 64 & -192 & 16 \\ 0 & 0 & -1536 & -192 & 3072 & 0 \\ 0 & 0 & 192 & 16 & 0 & 64 \end{bmatrix}$$

Remember to zero the x translations since we are not interested in the extensional deformations. The natural frequencies are then found to be

ω_1= 1134 rad/s, ω_2 = 3152 rad/s, ω_3 = 6253 rad/s, ω_4 = 11830 rad/s, ω_5 = 19565 rad/s, ω_6 = 31524 rad/s

From distributed theory

ω_1= 1132.9 rad/s, ω_2 = 3122.9 rad/s, ω_3 = 6122.2 rad/s, ω_4 = 10120 rad/s, ω_5 = 15118 rad/s, ω_6 = 21115 rad/s

8.29 Repeat Problem 8.28 with two elements and compare the frequencies with the four-element model. Calculate the frequencies of a clamped-clamped beam using one element. Any comment?

Solution:

Since only two of the six degrees of freedom are free, the mass and stiffness matrices are simply

$$M = \frac{2\rho A \frac{l}{2}}{420}\begin{bmatrix} 156 & 0 \\ 0 & 4\left(\frac{l}{2}\right)^2 \end{bmatrix}$$

and

$$K = \frac{2EI}{\left(\frac{l}{2}\right)^3}\begin{bmatrix} 12 & 0 \\ 0 & 4\left(\frac{l}{2}\right)^2 \end{bmatrix}$$

where $l = 1$ m. The natural frequencies are then

$$\omega_1 = \sqrt{\frac{\frac{192EI}{l^3}}{\frac{156\rho Al}{420}}} = 22.736\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 1151 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{\frac{16EI}{l^3}}{\frac{\rho Al}{420}}} = 81.96\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 4151 \text{ rad/s}$$

If you are only interested in the first natural frequency, a two degree of freedom model is adequate. However, the six degree of freedom model is much more accurate and can better predict the second mode. (In general, a finite element model must have twice as many degrees of freedom as the number of modes you want to predict).

8.30 Estimate the first natural frequency of a clamped-simply supported beam. Use a single finite element.

Solution: Since we are using only one element, we need only take the finite element matrix for a single element and strike out the rows and columns corresponding to the fixed degrees of freedom to get the global matrices. This yields

$$M = \frac{4l^3 \rho A}{420}, \quad K = \frac{4l^2 EI}{l^3}$$

Since there is only a single degree of freedom

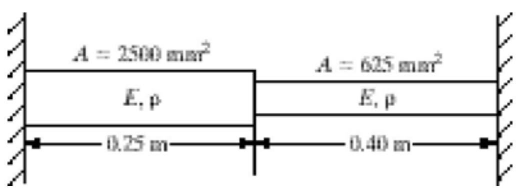
$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{420}\,\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 20.49\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} \text{ rad/s}$$

Distributed theory yields

$$\omega_n = 15.42\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

One degree of freedom is not enough to predict the first natural frequency.

8.31 Consider the stepped beam of Figure P8.31 clamped at each end. Both pieces are made of aluminum. Use two elements, one for each step, and calculate the natural frequencies.



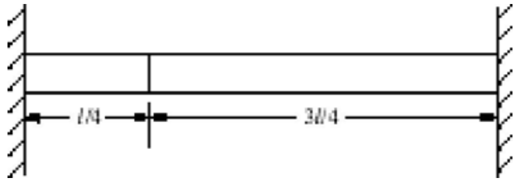
Solution: Only a single degree of freedom is free. The mass and stiffness matrices are therefore scalars.

$$K = \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} = 809375000 \text{ N/m}$$

$$M = \frac{1}{3}\left(\frac{\rho_1 A_1}{l_1} + \frac{\rho_2 A_2}{l_2}\right) = 10.41 \text{ kg}$$

$$\omega = \sqrt{\frac{K}{M}} = 8819.2 \text{ rad/s}$$

8.32 Use a two-element model of nonuniform length to estimate the first few natural frequencies of a clamped-clamped beam. Use the spacing indicated in Figure P8.32. Compare the result to the actual frequencies and to those of Problem 8.28 and 8.29.



Solution: Since it has been shown in example 8.3.3 that the variable l can be factored outside of the mass and stiffness matrices, we can substitute the percentage of total length of each element into the mass and stiffness matrices and get the correct natural frequencies.

$$M = \frac{\rho A(.25l)}{420}\begin{bmatrix} 156 & -22\times.25 \\ -22\times.25 & 4\times.25^2 \end{bmatrix} + \frac{\rho A(.75l)}{420}\begin{bmatrix} 156 & 22\times.75 \\ 22\times.75 & 4\times.75^2 \end{bmatrix}$$

$$= \frac{\rho Al}{420}\begin{bmatrix} 156 & 11 \\ 11 & 1.75 \end{bmatrix}$$

Similarly,

$$K = \frac{EI}{(.25l)^3}\begin{bmatrix} 12 & -6\times.25 \\ -6\times.25 & 4\times.25^2 \end{bmatrix} + \frac{EI}{(.75l)^3}\begin{bmatrix} 12 & 6\times.75 \\ 6\times.75 & 4\times.75^2 \end{bmatrix}$$

$$= \frac{EI}{l^3}\begin{bmatrix} 796.4 & -85.3 \\ -85.3 & 21.33 \end{bmatrix}$$

$$\omega = \sqrt{eig\left(\tilde{M}^{-1}\tilde{K}\right)}\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

where $\tilde{M}$ and $\tilde{K}$ represent the mass and stiffness matrices with the variables E, I, l, ρ and A factored out.

$$\omega_1 = 25.31\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},\ \omega_2 = 132.6\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

This is not nearly as good as the two element model where ω_1 was found to be

$$\omega_1 = 22.74\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

as opposed to the "actual" (from distributed parameter theory) value of

$$\omega_1 = 22.37\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.33 Calculate the first natural frequency of a clamped-pinned beam using first one, then two elements.

Solution:

From problem 8.30, using one element yields

$$\omega_1 = 20.49\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

Using the vibration toolbox and the method described in 8.3.3 (also in the README.8 file) the two element model yields

$$\omega_1 = 15.56\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = 58.41\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_3 = 155.6\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$