Problems and Solutions Section 8.4 (8.34 through 8.43)

8.34 Refer to the tapered bar of Figure P8.13. Calculate a lumped-mass matrix for this system and compare it to the solution of Problem 8.13. Since the beam is tapered, be careful how you divide up the mass.

Solution: The lumped mass at node 2 should be the total mass between $x = .25$ and $x = .75$. Therefore

$$M_2 = 2700 \int_{.25}^{.75} \frac{\pi}{4}\left[h_1^2 + \left(\frac{h_2 - h_1}{l}\right)^2 x^2 + 2h_1\left(\frac{h_2 - h_1}{l}\right)x\right]dx$$
$$= 26.5$$

likewise for node 3

$$M_3 = 2700 \int_{.75}^{1} \frac{\pi}{4}\left[h_1^2 + \left(\frac{h_2 - h_1}{l}\right)^2 x^2 + 2h_1\left(\frac{h_2 - h_1}{l}\right)x\right]dx$$
$$= 7.289$$

The mass matrix is then

$$M = \begin{bmatrix} 26.5 & 0 \\ 0 & 7.289 \end{bmatrix}$$

and the natural frequencies are

$\omega_1 = 6670$ rad/s and $\omega_2 = 13106$ rad/s.

For the distributed mass system

$\omega_1 = 7414$ rad/s and $\omega_2 = 20368$ rad/s.

The first natural frequency found by the distributed mass model is slightly better than the lumped mass model when compared to the three element distributed mass model derived in problem 13.

8.35 Calculate and compare the natural frequencies obtained for a tapered bar by using first, the consistent-mass matrix (Problem 8.12), and second, the lumped-mass matrix (Problem 8.34).

Solution:

See solution for Problem 8.34.

8.36 Consider again the machine punch of Problem 8.16 and Figure P8.15. Calculate the natural frequencies of this system using a lumped-mass matrix and compare the results to those obtained with the consistent-mass matrix.

Solution:

The lumped mass matrix is

$$M = \begin{bmatrix} \frac{\rho_1 A_1 l_1}{2} + \frac{\rho_2 A_2 l_2}{2} & 0 \\ 0 & \frac{\rho_2 A_2 l_2}{2} \end{bmatrix}$$
$$= rl \begin{bmatrix} A_1 + A_2 & 0 \\ 0 & A_2 \end{bmatrix}$$
$$= \begin{bmatrix} .078 & 0 \\ 0 & .039 \end{bmatrix}$$

The natural frequencies are

ω_1 = 38756 rad/s and ω_2 = 93565 rad/s.

The results for the consistent mass matrix were

ω_1 = 40798.6 rad/s and ω_2 = 142525 rad/s.

The first natural frequency is within 5% for both predictions. For this case, the inconsistent mass matrix is adequate for the 1st mode.

8.37 Consider again the bridge support of Figure P8.17 discussed in connection with Problem 8.17. Develop a four-element finite element model of this structure using a lumped-mass approximation and calculate the natural frequencies. Use constant area elements.

Solution:

We will use elements which each have constant cross section by finding the average area for each element. Elements are numbered from one to four from bottom to top.

$$A_1 = \frac{1}{.25l}\int_0^{.25l} A(x)dx = \frac{A_0}{.25l}\left(-le^{-\frac{x}{l}}\right)\Bigg|_0^{.25l}$$

$$= -4A_0\left(e^{-.25} - 1\right) = .8848A_0$$

likewise

$$A_2 = .6891A_0,\ A_3 = .5367A_0,\ A_4 = .4179A_0$$

Assembling the stiffness matrix yields

$$K = \frac{EA_0}{.25l}\begin{bmatrix} 1.5739 & -.6891 & 0 & 0 \\ -.6891 & 1.2258 & -.5367 & 0 \\ 0 & -.5367 & .9546 & -.4179 \\ 0 & 0 & -.4179 & .4179 \end{bmatrix}$$

To find the mass matrix, we will assume again that the elements have constant cross section. This yields

$$M = \frac{\rho A_0 l}{8}\begin{bmatrix} 1.5739 & 0 & 0 & 0 \\ 0 & 1.2258 & 0 & 0 \\ 0 & 0 & .9546 & 0 \\ 0 & 0 & 0 & .4179 \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = 1.86\frac{1}{l}\sqrt{\frac{E}{\rho}},\ \omega_2 = 4.50\frac{1}{l}\sqrt{\frac{E}{\rho}},\ \omega_3 = 6.62\frac{1}{l}\sqrt{\frac{E}{\rho}},\ \omega_4 = 7.78\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

8.38 Consider the torsional vibration problem illustrated in Figure P8.20 and discussed in Problem 8.20. Calculate a lumped-mass matrix for the single element.

Solution:

The total mass moment of inertia would be divided between the two degrees of freedom.

Therefore

$$M = \frac{1}{2}\begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}$$

8.39 Estimate the first three natural frequencies of a clamped-free bar of length l in torsional vibration by using a lumped-mass model and four elements.

Solution:

The stiffness matrix is

$$K = \frac{4G\gamma}{l}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The mass matrix is

$$M = \frac{\rho J l}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = 1.56\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}},\ \omega_2 = 4.445\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}},\ \omega_3 = 6.65\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}},\ \omega_4 = 7.8463\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}}$$

From table 6.3, it can be seen that the first two natural frequencies predicted by the finite element model are good approximations.

8.40 Calculate the natural frequencies of a pinned-pinned beam of length l using one element and the consistent-mass matrix of equation (8.73).

Solution:

The mass matrix is

$$M = \frac{\rho A l^3}{48}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the stiffness matrix is

$$K = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

Finding the natural frequencies gives

$$\omega_1 = 9.798\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = 16.971\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

The first natural frequency from distributed theory is

$$\omega_n = 9.869\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.41 Calculate the natural frequencies of a pinned-pinned beam of length l using one element and the lumped-mass matrix of equation (8.73). Compare your results to those obtained with at consistent-mass matrix of Problem 8.40.

Solution:

The consistent mass matrix is

$$M = \frac{\rho A l^3}{420}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$

which gives

$$\omega_1 = 10.96\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}, \quad \omega_2 = 50.20\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

which is worse than the inconsistent mass matrix results. (See solution 8.40)

8.42 Calculate a three-element finite element model of a cantilevered beam (see Problem 8.25) using a lumped mass that includes rotational inertia. Also calculate the system's natural frequencies and compare them with those obtained with a consistent-mass matrix of Problem 8.25 and with the values obtained by the methods of Chapter 6.

Solution:

The mass matrix is $M = \rho Al \,\text{diag}\left(1, \frac{1}{24}, 1, \frac{1}{24}, \frac{1}{2}, \frac{1}{48}\right)$ using the $[u_1 \; lu_2]$ convention for the displacement vector.

The natural frequencies are then

$$\omega_i = a_i \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$a_i = .368,\; 2.00,\; 4.98,\; 10.7,\; 14.5,\; 17.1$$

This is not as good as the consistent mass matrix results. From distributed parameter theory $a_1 = .3911$.

8.43 Repeat Problem 8.42 using a lumped-mass matrix that neglects the rotational degree of freedom. Discuss any problems you encounter when trying to solve the related eigenvalue problem.

Solution:

$$M = \rho Al \,\text{diag}\left(1, 0, 1, 0, \frac{1}{2}, 0\right)$$

The singularity of the mass matrix does not allow a solution to be found.