Solutions to the Waves and Electromagnetism exam of August 2016

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Question 1

If we denote the compression of the two individual springs as x_1 and x_2 then then the following must hold:

$$x_1 + x_2 = \frac{F}{k_1} + \frac{F}{k_2} \tag{1}$$

And because $x_1 + x_2 = x_{tot}$ we have that:

$$\frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2} \tag{2}$$

Then we have that:

$$k_{eq} = [\frac{1}{k} + \frac{1}{k}]^{-1} = \frac{k}{2} \tag{3}$$

Thus the radial frequency is:

$$\omega = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2m}} \tag{4}$$

Question 2

During impact, only the conservation of linear momentum holds:

$$m_{bullet}v_{bullet} = (m_{bullet} + m_{block})v_{block} \tag{5}$$

The speed of the block can be derived from the conservation of energy and the known amplitude and spring constant:

$$\frac{1}{2}(m_{bullet} + m_{block})v_{block}^2 = \frac{1}{2}kA^2 \tag{6}$$

Combining above two equations gives that:

$$v_{bullet} = \frac{m_{block} + m_{bullet}}{m_{bullet}}\sqrt{\frac{k}{m_{block} + m_{bullet}}A^2} = \frac{A}{m_{bullet}}\sqrt{k(m_{block} + m_{bullet})} \tag{7}$$

Question 3

Displacement as a function of time for a damped oscillator is given by:

$$x(t) = A_0 e^{-\frac{b}{2m}t} cos(\omega' t) \tag{8}$$

Hence the amplitude as a function of time is:

$$A(t) = A_0 e^{-\frac{b}{2m}t} \tag{9}$$

Solving the above equation for the linear damping coefficient b gives:

$$b = -\frac{2m}{t}\ln(\frac{A(t)}{A_0}) \tag{10}$$

Question 3

Note that any transverse wave equation is of the form:

$$D(x,t) = Acos(kx - \omega t) \tag{11}$$

Where $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi f$. Hence:

$$v = \frac{\omega}{k} \tag{12}$$

Comparison of the wave velocity for all 4 given equations can be done by above equation.

Question 5

For reasons of continuity, the connecting point between the two section*s of the wire oscillates at one frequency, hence the equality:

$$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2} \tag{13}$$

holds. The speed of a wave travelling along a cord is given by:

$$v = \sqrt{\frac{F_T}{\mu}} \tag{14}$$

Hence we can combine above equations to find an expression for λ_2:

$$\lambda_2 = \lambda_1 \sqrt{\frac{\mu_1}{\mu_2}} \tag{15}$$

And λ_1 can be found by use of the given wave equation as:

$$\lambda_1 = \frac{2\pi}{k} \tag{16}$$

Where the wave number k was already discussed in the solution of question 3.

Question 6

This question is simply checking whether:

$$\frac{\partial^2 z}{\partial t^2} = C\frac{\partial^2 z}{\partial t^2} \tag{17}$$

Question 7

This question relies on double application of the Doppler effect. Denote the frequency emitted by the bat as f_0, the frequency reflected by the wall as f_1 and that received by the bat as f_2. Then we have that due to the fact that the bat is a moving source:

$$f_1 = \frac{f_0}{1 - \frac{v_{bat}}{v_{sound}}} \tag{18}$$

And due to the fact that the bat is moving in on the wall as source:

$$f_2 = f_1(1 + \frac{v_{bat}}{v_{sound}}) \tag{19}$$

Combining above two equations gives:

$$f_2 = f_0 \frac{1 + \frac{v_{bat}}{v_{sound}}}{1 - \frac{v_{bat}}{v_{sound}}} = f_0 \frac{v_{sound} + v_{bat}}{v_{sound} - v_{bat}} \tag{20}$$

Question 8

If half of the engines are working, the intensity at a given distance is always half of that than before. Hence the relative sound level becomes:

$$\beta_{rel} = 10 \log(\frac{1}{2}) \tag{21}$$

Which uses $4I_{engine}$ as reference level. Hence the sound level becomes:

$$\beta_2 = \beta_1 + \beta_{rel} = 125 + 10 \log(\frac{1}{2}) \tag{22}$$

Question 9

Important to note here is that charge is conserved and that the voltage over both capacitors must be equal as the final situation is electrostatic. The total charge in the circuit is:

$$Q = C_{eq}V = (C_1 + C_2)V = Q_1 + Q_2 \tag{23}$$

Where Q_1 and Q_2 are the charge levels after the dielectric has been inserted. The voltage over both capacitors is the same and knowing that for a capacitor $V = \frac{Q}{C}$:

$$\frac{Q_1}{KC_1} = \frac{Q_2}{C_2} \tag{24}$$

Thus by use of these equations we can find that for the charge over C_1:

$$Q_1 = \frac{C_1 + C_2}{1 + \frac{C_2}{KC_1}} V \tag{25}$$

Now let us divide this equation on both sides by KC_1 to find the voltage over this capacitor:

$$V_1 = \frac{1}{KC_1} \frac{C_1 + C_2}{1 + \frac{C_2}{KC_1}} V = \frac{1}{3} \frac{5+1}{1+\frac{5}{3}} V = \frac{3}{4} V \tag{26}$$

Question 10

By definition:

$$E = -\nabla V = -2xy^3\mathbf{i} - 3x^2y^2\mathbf{j} + \mathbf{k} \tag{27}$$

Hence:

$$E(1,1,1) = -2i - 3\mathbf{j} + \mathbf{k} \tag{28}$$

Question 11

Note how the charge on each capacitor is the same due to the isolation of the interconnecting wires. Hence the voltage on the second capacitor becomes:

$$V = \frac{Q}{2C} = \frac{C_{eq}}{2C} V = \frac{[\frac{1}{C} + \frac{1}{2C} + \frac{1}{3C}]^{-1}}{2C} V = \frac{\frac{6}{11} C}{2C} V = \frac{3}{11} V \tag{29}$$

Question 12

Let us find the resistance of each light bulb as a function of its power rating:

$$R = \frac{V^2}{P} \tag{30}$$

As $P = \frac{V^2}{R}$. Hence, if they are connected in series, the current through each is the same and the power dissipated is given by:

$$P = I^2 R = \frac{V^2 I^2}{P} \tag{31}$$

Thus the light bulb with the lowest power rating burns the brightest in this set up. Now if we consider the parallel set up, we know that the voltage drop over each light bulb must be the same. Hence the power drop over a light bulb is given by:

$$P = \frac{V^2}{R} \tag{32}$$

Which is simply its initial power rating. Hence the bulb with the highest power rating burns the brightest in the parallel set up.

Question 13

Let us use Kirchoff's loop rules:

$$\mathcal{E}_1 = I_1 r + IR \tag{33}$$
$$\mathcal{E}_2 = I_2 r + IR \tag{34}$$

By the conservation of charge:

$$I = I_1 + I_2 \tag{35}$$

Hence we can combine and find that:

$$I = \frac{\mathcal{E}_1 + \mathcal{E}_2}{2R + r} \tag{36}$$

The known current I can be used to find either I_1 or I_2. Then the voltage delivered is:

$$\Delta V = \mathcal{E}_2 - I_2 r \tag{37}$$

Question 14

By use of the relation between current density j and electric field E ($j = \frac{1}{\rho}E$) we find that:

$$E = \rho j \tag{38}$$

Question 15

Apply Gauss' Law:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \tag{39}$$

Such that:

$$Q = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{A} \tag{40}$$

Only the left-most side of the cube in the y-z plane and the top side of the cube are exposed to an electric field of magnitude L. Both sides are of area L^2. Hence, the total flux and thus charge becomes:

$$Q = 2\epsilon_0 L^3 \tag{41}$$

Question 16

The assumption that both the volt- and ammeter do not influence the circuit is very important here. The internal resistance can be found by considering the difference in emf (when only voltmeter is connected) and terminal voltage (when ammeter is connected) of the battery:

$$r = \frac{\Delta V}{I} = \frac{\mathcal{E} - V}{I} \tag{42}$$

Question 17

The force on a current-carrying wire in a magnetic field is:

$$F = BIl \tag{43}$$

By use of the kinematic equation $v^2 = v_0^2 + 2as$ we find that the desired acceleration is:

$$a = \frac{V^2}{2s} \tag{44}$$

By application of Newton's second law $F = ma$ to the magnetic force we can find that the following equality must hold:

$$\frac{BIl}{m} = \frac{v^2}{2s} \tag{45}$$

If we solve for the current I:

$$I = \frac{mv^2}{2Blms} \tag{46}$$

Question 18

The two sides of the loop perpendicular to the wire experience opposite but equal forces. We thus only consider the loop section*s parallel to the wire. The section* closer to the wire will experience a greater force as the magnitude of the magnetic field caused by a wire is given by:

$$B = \frac{\mu_0 I}{2\pi r} \tag{47}$$

The force due to a magnetic field is:

$$F = BIl \tag{48}$$

We are asked for the magnitude and thus have no sense of direction to account for in our final answer. However, the force on the section* closer to the wire is in opposite direction of that on the section* further away and hence we need to consider the difference between the two forces. For the magnitude of the magnetic fields we have that:

$$B_1 = \frac{\mu_0 I_2}{2\pi(d - \frac{a}{2})} \tag{49}$$

$$B_2 = \frac{\mu_0 I_2}{2\pi(d + \frac{a}{2})} \tag{50}$$

And then the forces become:

$$F_1 = \frac{\mu_0 I_1 I_2 a}{2\pi(d - \frac{a}{2})} F_1 = \frac{\mu_0 I_1 I_2 a}{2\pi(d + \frac{a}{2})} \tag{51}$$

And the difference is:

$$\begin{aligned} F_1 - F_2 =& \frac{\mu_0 I_1 I_2 a}{2\pi}(\frac{1}{d-\frac{a}{2}} - \frac{1}{d+\frac{a}{2}}) \\ =& \frac{\mu_0 I_1 I_2 a}{2\pi}(\frac{d+\frac{a}{2}}{d^2-\frac{a^2}{4}} - \frac{d-\frac{a}{2}}{d^2-\frac{a^2}{4}}) \\ =& \frac{\mu_0 I_1 I_2 a^2}{2\pi(d^2-\frac{a^2}{4})} \end{aligned} \quad (52)$$

Question 19

We can imagine the point to lie on a circular loop of radius x. Hence, when the flux changes through this loop, an emf is induced proportional to the rate of change of the flux. Hence, by Faraday's law we have that:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi}{dt} \quad (53)$$

Thus in our case this becomes:

$$2\pi x E = -\pi x^2 \frac{dB}{dt} \quad (54)$$

Thus, solving for the electric field E and multiplying with the charge q we have that the magnitude of the force equals:

$$F = qE = \frac{qr}{2}\frac{dB}{dt} \quad (55)$$

Question 20

The vertical components of the two magnetic fields cancel each other. We only consider the horizontal component and note that the total magnetic field is twice the horizontal component of one magnetic field as they are both equal in magnitude. The distance from the point L to one of the wires is:

$$r = \sqrt{d^2 + (\sqrt{2}d)^2} = \sqrt{3}d \quad (56)$$

The sine of the angle θ the line connecting L and the wire makes with the horizontal, which we need for computation of the horizontal component, is given by:

$$\sin(\theta) = \frac{d}{\sqrt{3}d} = \frac{1}{\sqrt{3}} \quad (57)$$

Now the total magnetic field becomes:

$$B = 2B_x = 2\frac{\mu_0 I}{2\pi\sqrt{3}d}\frac{1}{\sqrt{3}} = \frac{\mu_0 I}{3\pi d} \quad (58)$$

Question 21

The charge Q on the capacitor as a function of time in a LC circuit is given by:

$$Q = Q_0 \cos(\omega t) \tag{59}$$

The current as a function of time is thus:

$$I = -\frac{dQ}{dt} = \omega Q_0 sin(\omega t) \tag{60}$$

Hence the maximum current $I_0 = \omega Q_0 = \omega C V_0$. Now to find the current in inductor 2 we must consider the relation between the currents in the two inductors. Let us first compute the equivalent inductance. We know that the sum of the currents over the two inductors must equal the current supplied by the capacitor. Hence, denoting X_L as the impedance offered by a resistor:

$$\frac{V}{X_{L_{eq}}} = \frac{V}{X_{L_1}} + \frac{V}{X_{L_2}} \tag{61}$$

However, $X_L = \omega L$. Thus the equivalent inductance is:

$$L_{eq} = [\frac{1}{L_1} + \frac{1}{L_2}]^{-1} = \frac{2}{3}L \tag{62}$$

The ratio of the two currents through the inductors is:

$$I_1 = 2I_2 \tag{63}$$

which can be both mathematically and intuitively reasoned. We now have the following equality:

$$I_0 = I_1 + I_2 = 3I_2 \tag{64}$$

Thus:

$$I_2 = \frac{1}{3}I_0 = \frac{1}{3}\omega CV = \frac{1}{3}\sqrt{\frac{1}{L_{eq}C}} = \frac{1}{3}\sqrt{\frac{3C}{2L}}V = V\sqrt{\frac{C}{6L}} \tag{65}$$

Question 22

Let us first find the the magnetic field at the center of loop due to the current flowing through the loop. We make use of the Biot-Savart law and notice that the infinitesimal line element dl is always perpendicular to r, omitting the cross product:

$$B_{loop} = \int dB = \int \frac{\mu_0 I_2}{4\pi}\frac{dl}{r^2} \tag{66}$$

Moreover, $dl = Rd\theta$ and $r = R$:

$$B_{loop} = \frac{\mu_0 I_2}{4\pi R}\int_0^{2\pi} d\theta = \frac{\mu_0 I_2}{2R} \tag{67}$$

The magnetic field strength due to the wire is:

$$B_{wire} = \frac{\mu_0 I_1}{4\pi R} \tag{68}$$

Equating B_{wire} and B_{loop} gives:

$$I_1 = 2\pi I_2 \tag{69}$$

Question 23

Decrease in upwards flux leads to current that wants to oppose the change (Lenz's law) and thus produces an upward magnetic field: hence the current is counterclockwise. The opposite is true for the second loop, as the decrease in downwards flux leads to a downward magnetic field caused by the current: it is clockwise.

Question 24

If all impedance is caused by R then all voltage drops over R and hence there is no output voltage. This occurs if the circuit is at resonance (i.e. the capacitor and inductor cancel each other's impedance). This occurs if:

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \tag{70}$$

Question 25

The average value of the Poynting vector **S** (which is the intensity) in terms of the magnetic field amplitude is given by:

$$\bar{S} = \frac{1}{2}\frac{c}{\mu_0}B_0^2 \tag{71}$$