Performance Summary

1 Thermodynamics

1.1 An Ideal Gas

For a perfect gas, the **gas law** holds:

$$p = \rho R T \qquad \Leftrightarrow \qquad pv = RT \tag{1.1}$$

where p $[Pa]$ is the **pressure** of the air, ρ $[kg/m^3]$ is the **density**, R $[J/(kg\,K)]$ is the **specific gas constant**, T $[K]$ is the **temperature** and v $[m^3/kg]$ is the **specific volume**. Per definition $\rho v = 1$. For an ideal gas, the **internal energy** only depends on its temperature: $u = c_v\,T$, where u $[J/kg]$ is the **specific internal energy** and c_v is a constant (depending on the type of gas) about which more will be explained in paragraph 1.3.

1.2 First Law of Thermodynamics

Let us consider 1 kg of gas. The change in energy, represented by the **heat** δq $[J/kg]$ added, is equal to the increase in internal energy du $[J/kg]$ plus the **external work** done δw $[J/kg]$:

$$\delta q = du + \delta w \tag{1.2}$$

This is the **first law of thermodynamics**. The work can be expressed as $\delta w = p\,dv$ such that the first law of thermodynamics becomes $\delta q = du + p\,dv$.

1.3 Specific Heat

If the 1 kg of gas is heated, and if the volume is kept constant (such that the pressure increases), then the **specific heat at constant volume**, c_v $[J/(kg\,K)]$, is defined such that:

$$\delta q = c_v\,dT \tag{1.3}$$

where dT $[K]$ is the rise in temperature. In the case of constant volume, no work can be done, so $\delta q = du$ and thus also $du = c_v\,dT$. Integrating gives the equation for the internal energy:

$$u = c_v\,T \tag{1.4}$$

However, if the 1 kg is heated such that the pressure stays constant (such that the volume increases), then the **specific heat at constant pressure**, c_p $[J/(kg\,K)]$, is defined such that:

$$\delta q = c_p\,dT \tag{1.5}$$

Using the gas law, it can be found that $\delta q = c_v\,dT + R\,dT$. So we get the important relation:

$$R = c_p - c_v \tag{1.6}$$

1.4 Enthalpy

Let's not consider a system at which gas is locked an at rest, but let's consider a system at which gas enters with a velocity V_1 $[m/s]$ and exits (with different pressure, temperature, etcetera) with a velocity

V_2 $[m/s]$. The kinetic energy of the gas (when it enters) is $\frac{1}{2}V_1^2$ (since we consider only 1 kg of the gas). The work done can be shown to be $w_1 = p_1 v_1$ (note the difference between the specific volume v and the velocity V). If we also keep the internal energy in mind, we can show that the total energy in state 1 is:

$$e_1 = u_1 + p_1 v_1 + \frac{1}{2}V_1^2 \tag{1.7}$$

Since the combination $u + pv$ often occurs, it is given a name. The **enthalpy** h $[J/kg]$ is defined as:

$$h = u + pv \tag{1.8}$$

In case of a perfect gas, the enthalpy can also be written as $h = c_p T$ (since $pv = RT$ and $c_v + R = c_p$). So for a perfect gas, the enthalpy is only a function of the temperature. So the energy entering the system can be found by measuring the temperature and the velocity. To simplify this, the **total enthalpy** is introduced and defined as:

$$h_t = h + \frac{1}{2}V^2 = u + pv + \frac{1}{2}V^2 \tag{1.9}$$

In the same way the **total temperature** is defined:

$$T_t = T + \frac{V^2}{2c_p} \qquad \Rightarrow \qquad h_t = c_p T_t \tag{1.10}$$

This makes the **balance of energy** the following:

$$q_{in} + w_{in} = h_{t_2} - h_{t_1} = c_p(T_{t_2} - T_{t_1}) = c_p(T_2 - T_1) + \frac{1}{2}(V_2^2 - V_1^2) \tag{1.11}$$

where q_{in} is the added heat and w_{in} is the added work. The balance of energy is always valid, whether there is friction or not.

1.5 Entropy

The **entropy** s $[J/(kg\,K)]$ can be calculated using the following equations:

$$s = c_v \ln\left(\frac{T}{T_{ref}}\right) + R\ln\left(\frac{v}{v_{ref}}\right) = c_p \ln\left(\frac{T}{T_{ref}}\right) - R\ln\left(\frac{p}{p_{ref}}\right) = c_v \ln\left(\frac{p}{p_{ref}}\right) - c_p \ln\left(\frac{\rho}{\rho_{ref}}\right) \tag{1.12}$$

where the first part is the definition of the entropy. All parts in equation 1.12 are equivalent due to the gas law. Using the entropy, the first law of thermodynamics can be written as:

$$\delta q = T\,ds \tag{1.13}$$

For diabatic processes (heat is supplied from the environment) or for processes with friction (heat is supplied by friction) $\delta q > 0$ and so $ds > 0$. Only for an **adiabatic** and **frictionless** process (no heat is supplied to the gas) $ds = 0$. Such a process is called an **isentropic** process (or sometimes an adiabatic reversible process). It means entropy is reserved, which gets expressed in the following important equation:

$$s_2 - s_1 = 0 \tag{1.14}$$

1.6 Mollier Diagram

The enthalpy and the entropy are state variables. If (h, s) is known, the entire state of the gas is known. Using the definitions of h and s all variables concerning the gas can be determined. It is also possible to make a diagram with on the horizontal axis the entropy s and on the vertical axis the enthalpy h. Such a diagram is called a **Mollier diagram**. The state of a gas can be represented by a point in a Mollier diagram.

The state of a gas usually changes along certain lines in a Mollier diagram. For an isentropic process, the line in a Mollier Diagram is vertical (as the entropy - the horizontal axis - stays constant). Sometimes lines are drawn in Mollier diagrams for a gas at a constant pressure. Such a line is called a **isobar**. In a jet engine certain processes occur at constant pressure. The state of a gas then changes along one of the isobars of the Mollier diagram. When calculating gas states for piston engines, it is often handy to use **isochors** - lines at which the gas has constant density.

1.7 Equation of Poisson

Let's define γ as $\gamma = c_p/c_v$. This variable can come in handy in many equations. For isentropic processes ($s_2 - s_1 = 0$), the equations of Poisson can be derived. These equations state that the following quantities remain constant during isentropic processes:

$$\frac{p}{\rho^\gamma} = \text{constant} \qquad \frac{T}{\rho^{\gamma-1}} = \text{constant} \qquad \frac{T}{p^{\left(\frac{\gamma}{\gamma-1}\right)}} = \text{constant} \tag{1.15}$$

1.8 Speed of Sound

The crossing of a sound wave may assumed to be isentropic. From that assumption can be derived that the **speed of sound** a $[m/s]$ is:

$$a = \sqrt{\gamma R T} = \sqrt{\gamma p v} = \sqrt{\gamma \frac{p}{\rho}} \tag{1.16}$$

The **Mach number** M is now defined as:

$$M = \frac{V}{a} \tag{1.17}$$

Using this Mach number, it can be derived that for an isentropic process the total pressure can be calculated using:

$$p_t = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{1.18}$$

2 Propulsion

2.1 Engine Thrust and Heat

Let's look at an engine flying at a velocity V_0 $[m/s]$ (with respect to the air). The air enters the engine with a velocity V_0. Suppose it exits the engine with a velocity V_j $[m/s]$. Also suppose that the mass of the air that goes through the engine every second (the **mass flow**) is m $[kg/s]$. The **thrust** can now be found using the change in momentum:

$$T = I_{out} - I_{in} = m(V_j - V_0) \tag{2.1}$$

Now suppose the engine uses m_f $[kg/s]$ kilograms of fuel per second. The **fuel flow** F $[N/s]$ is now defined as:

$$F = m_f \cdot g \tag{2.2}$$

If we know the **heating value** H $[J/kg]$ (the amount of joules in one kilogram of fuel), we can also find the added energy Q $[J/s]$ every second:

$$Q = m_f \cdot H = \frac{F \cdot H}{g} \tag{2.3}$$

And if we want to find the heat q_{in} $[J/kg]$ added to one kilogram of air, we can calculate it using:

$$q_{in} = \frac{Q}{m} = H\frac{m_f}{m} \tag{2.4}$$

2.2 Engine Power and Efficiency

When we know the velocity of the air prior to and after the engine, and if we now the mass flow of the air, we can calculate the energy put into the air every second - the **jet power** P_j $[J/s]$:

$$P_j = \frac{1}{2}mV_j^2 - \frac{1}{2}mV_0^2 = \frac{1}{2}m(V_j^2 - V_0^2) = \frac{1}{2}m(V_j - V_0)(V_j + V_0) = \frac{1}{2}T(V_j + V_0) \tag{2.5}$$

The **power available** P_a $[J/s]$, which is the work added every second, can be calculated using:

$$P_a = T \cdot V_0 = m_f \cdot V_0(V_j - V_0) \tag{2.6}$$

Using these equations for the power, we can calculate various efficiencies. The **propulsive efficiency** η_j is defined as:

$$\eta_j = \frac{P_a}{P_j} = \frac{T \cdot V_0}{\frac{1}{2}T(V_j + V_0)} = \frac{2V_0}{V_j + V_0} = \frac{2}{1 + \frac{V_j}{V_0}} \tag{2.7}$$

The **thermal efficiency** η_{th} is defined as:

$$\eta_{th} = \frac{P_j}{Q} = \frac{\frac{1}{2}m(V_j^2 - V_0^2)}{m_f \cdot H} = \frac{V_j^2 - V_0^2}{2q_{in}} \tag{2.8}$$

The **total efficiency** η_{tot} is defined as:

$$\eta_{tot} = \frac{P_a}{Q} = \frac{T \cdot V_0}{Q} = \eta_j \cdot \eta_{th} \tag{2.9}$$

There are multiple ways to increase efficiency. To increase propulsive efficiency, it is necessary to keep V_j close to V_0 (instead of giving a little bit of air a lot of acceleration, give a lot of air a little bit of acceleration). To increase thermal efficiency, the **pressure ratio** ε (the ratio of the pressure before and after the compressor) should be high. So the air should be compressed as much as possible.

Using the equation $Q = m\,q_{in}$, the thrust can also be rewritten as:

$$T = Q\frac{\eta_{tot}}{V_0} = m\,q_{in}\frac{2}{V_0 + V_j}\eta_{th} \tag{2.10}$$

This equation also shows that if the thrust increases (thus the engine gets a higher RPM and thus a higher compression ratio) also the thermal efficiency increases.

2.3 The Perfect Jet Engine

The **perfect jet engine** is a jet engine at which most steps are isentropic. Only when the air goes through the combustion chamber, the entropy rises. At that step, the temperature rises isobaric. From this fact the entropy after the combustion chamber can be found (by following the isobar lines in a Mollier diagram, for example). It is also assumed that all the energy taken out by the turbine is put back in the system by the compressor.

The perfect jet engine consists of 5 parts. So when air goes through the jet engine, it has 6 consecutive states. These states change as follows:

- The inlet (1-2): $w_{in} = q_{in} = 0$ $\quad V_2 \approx 0 \quad \Rightarrow \quad \Delta h = \frac{1}{2}V_1^2$
- The compressor (2-3): $q_{in} = 0$ $\quad V_3 \approx V_2 \approx 0 \quad \Rightarrow \quad \Delta h = w_{in}$
- The combustion chamber (3-4): $w_{in} = 0$ $\quad V_4 \approx V_3 \approx 0 \quad \Rightarrow \quad \Delta h = q_{in}$
- The turbine (4-5): $w_{out} = -w_{in} \quad q_{in} = 0$ $\quad V_5 \approx V_4 \approx 0 \quad \Rightarrow \quad \Delta h = -w_{in}$
- The exhaust (5-6): $w_{in} = q_{in} = 0$ $\quad V_5 \approx 0 \quad \Rightarrow \quad \Delta h = -\frac{1}{2}V_6^2$

Note that the free stream velocity is $V_0 = V_1$ and the exhaust velocity is $V_j = V_6$.

2.4 The Turbo-Prop Engine

The **turbo-prop engine** is very similar to the jet engine. The turbo-prop is in fact a jet engine with a propeller in front of it. This propeller accelerates air just slightly, giving the engine a higher propulsive efficiency. An indication for this is the by-pass ratio, defined as:

$$\lambda = \frac{M_{cold}}{M_{hot}} \tag{2.11}$$

where M_{cold} is the cold airflow (passing through the propeller) and M_{hot} is the hot airflow (passing through the compressor). In a turbo-prop engine, there is one additional step. Between the (first) turbine and the exhaust, there is a second turbine. This second turbine is connected to the propeller. However, since the turbo-prop engine isn't a jet engine, there also isn't a jet power. Instead, there is the shaft power (break power) P_{br} $[J/s]$. The propulsive efficiency η_p and the thermal efficiency η_{th} are now defined as:

$$\eta_p = \frac{P_a}{P_{br}} \qquad \eta_{th} = \frac{P_{br}}{Q} \tag{2.12}$$

The propeller in a turbo-prop engine doesn't give the air a lot of acceleration. So let's assume that $V = V_j$. The thermal efficiency now is:

$$\eta_{th} = \frac{P_{br}}{Q} = \frac{m(h_5 - h_6)}{Q} = \frac{m(h_5 - h_{exhaust}) - \frac{1}{2}mV^2}{Q} \tag{2.13}$$

For a jet engine, the thermal efficiency can be expressed as:

$$\eta_{th} = \frac{P_j}{Q} = \frac{\frac{1}{2}mV_j^2 - \frac{1}{2}mV_0^2}{Q} = \frac{m(h_5 - h_{exhaust}) - \frac{1}{2}mV^2}{Q} \tag{2.14}$$

These equations are equal! (Note that for a jet engine $h_{exhaust} = h_6$ but for a turbo-prop engine $h_{exhaust} = h_7$, due to the extra step in a turbo-prop engine.)

There is another similarity between the turbo-prop engine and the jet engine. For both engines the efficiency decreases if the temperature increases, or if the aircraft flies higher (lower density).

2.5 Nozzles

When engines fly at (nearly) supersonic speeds, their engines often have a nozzle with a throat. For the sonic regions $V \uparrow$ as $A \downarrow$, where A is the cross-section of the engine. However, for supersonic speed $V \uparrow$ as $A \uparrow$. The throat (where A is smallest) is the point at which the transition from sonic to supersonic velocity occurs. Thus $M_t = 1 \Leftrightarrow V_t = a$ at the throat.

The nozzle should be made in such a way that at the exhaust the pressure in the engine p_e is equal to the outside pressure p_0. If $p_e > p_0$ (the airplane is flying at a height higher than it was designed for) the exhausted air continues to expend and accelerate after it has exited the nozzle, but this acceleration does not contribute to the aircraft thrust. So there is a waste of energy. If $p_e < p_0$ (the airplane is flying at a height lower than it was designed for) the exhausted air causes shock waves, which also mean a waste of energy.

3 Flight Mechanics

3.1 Two-Dimensional Airfoils

The **Reynolds number** is defined as:

$$Re = \frac{\rho V c}{\mu} \tag{3.1}$$

where c $[m]$ is the chord length and μ $[Pa\,s]$ is the viscosity of the air. The **lift** L $[N]$ and **drag** D $[N]$ can be calculated using:

$$L = c_l \frac{1}{2}\rho V^2 S \qquad\qquad D = c_d \frac{1}{2}\rho V^2 S \tag{3.2}$$

where the **lift coefficient** c_l and the **drag coefficient** c_d depend on the **angle of attack** α [rad] (being the angle between the longitudinal axis of the aircraft and the direction of flight), the shape of the wing, the Mach number and the Reynolds number. Also S $[m^2]$ is the wing surface. It is possible to plot the lift and drag coefficients with respect to the angle of attack. However, it is also possible to plot the lift coefficient with respect to the drag coefficient. The diagram that results is called a **lift-drag polar**.

3.2 Three-Dimensional Airfoils

In reality wings aren't two-dimensional but three-dimensional. And in three dimensions also wing vortices occur, causing induced drag. The drag coefficient now consists of two parts. The part being present when there is zero lift C_{D_0}, which is thus called the **zero lift drag coefficient**, and the part belonging to the induced drag C_{D_i}. (Note that since we're talking about three-dimensional airfoils, we use capital letters to denote the coefficients.) The induced drag coefficient is:

$$C_{D_i} = \frac{C_L^2}{\pi A e} \tag{3.3}$$

where A is the aspect ratio of the wing, defined as b^2/S (with $b\,[m]$ the wing span), and e is Oswald's factor, depending on the lift distribution of the wing. The drag coefficient now is:

$$C_D = C_{D_0} + C_{D_i} = C_{D_0} + \frac{C_L^2}{\pi A e} \tag{3.4}$$

3.3 Flight Types

There are multiple ways of flying. Some of them have gotten a specific name. These are their definitions:

- **Gliding Flight** - Flight in which the thrust is 0: $T = 0$.
- **Steady Flight** - Flight in which the forces and moments do not vary in time, neither in magnitude nor direction.
- **Straight Flight** - Flight in which the center of gravity of the aircraft travels along a straight line.
- **Symmetric Flight** - Flight in which both the angle of sideslip (angle between the direction of motion and the longitudinal axis of the airplane) is zero, and the plane of symmetry of the airplane is perpendicular to the normal plane of the earth.

Let's look at a **steady horizontal flight**. The lift is equal to the weight: $L = W$. It follows that:

$$V = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}} \tag{3.5}$$

with V_{min} as $C_{L_{max}}$. The factor W/S is called the **wing loading**. So the minimum velocity is low when there is a low wing loading, or when the lift coefficient is high. The latter is often achieved by using slats.

3.4 Straight Symmetric Flight

Now let's look at a **straight symmetric flight** in which the aircraft is climbing. The angle between the direction of flight and the ground plane is the **flight path angle** γ [rad]. The angle between the longitudinal axis of the aircraft and the direction of flight is the angle of attack. The angle between the longitudinal axis of the aircraft and the ground plane is the **pitch angle** θ [rad]. Note that $\theta = \gamma + \alpha$. Figure 1 visualizes the definitions of these angles.

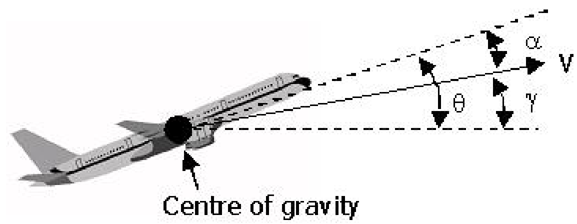

Figure 1: Visualization of angles in symmetric flight.

By drawing a free body diagram of the aircraft, the sum of the forces in multiple directions can be calculated. If we assume that the thrust vector is in the direction of flight, we get:

$$T - D - W \sin\gamma = \frac{W}{g}\frac{dV}{dt} \qquad\qquad L - W\cos\gamma = 0 \tag{3.6}$$

Now let's define the power required P_r $[J/s]$ as follows:

$$P_r = DV = C_D \frac{1}{2}\rho V^3 S = \frac{C_D}{C_L} LV = \frac{C_D}{C_L} WV \tag{3.7}$$

In the last part of this equation the assumption $\cos\gamma = 1$ was used, which is accurate for normal climb angles. Multiplying the first part of equation 3.6 by V, and by using the relation $V(dV/dt) = (1/2)\,(d(V^2)/dt)$, it can be rewritten as:

$$\frac{1}{2}\frac{W}{g}\frac{dV^2}{dt} = TV - DV - WV\sin\gamma = P_a - P_r - P_c \tag{3.8}$$

where P_c $[J/s]$ is the **climb power**. (Note that $P_c = WV\sin\gamma = W\,dh/dt$, with h $[m]$ as the height.) So the quantity $P_a - P_r$ can be seen as the power left to climb and accelerate - to increase the potential/kinetic energy of the aircraft.

3.5 Steady Gliding

When the engines of an airplane aren't active (or if the airplane doesn't have any engines), the airplane is gliding. So $T = 0$ and $P_a = 0$ and thus:

$$-P_r = W\frac{dh}{dt} + \frac{W}{g}\frac{dV^2}{dt} \tag{3.9}$$

If the airplane still follows a horizontal path ($dh/dt = 0$), the velocity decreases. In most situations this isn't favorable, so pilots usually try to keep a constant velocity, at the cost of height. Thus the aircraft descends. The **descend angle** $\bar{\gamma}$ [rad] is defined as $\bar{\gamma} = -\gamma$. Now the following applies:

$$C_D\frac{1}{2}\rho V^2 S = W\sin\bar{\gamma} \quad \text{and} \quad C_L\frac{1}{2}\rho V^2 S = W\cos\bar{\gamma} \quad \Rightarrow \quad \tan\bar{\gamma} = \frac{C_D}{C_L} \tag{3.10}$$

Suppose we want to travel as much distance as possible. To accomplish this, we should minimize $\sin\bar{\gamma}$. It can be shown that $\sin\bar{\gamma} = D/W$, so $\sin\bar{\gamma} \downarrow$ as $D \downarrow$. So we want to chose our V and C_L such that the drag D is minimal. The drag can be calculated with equation 3.2. However, this equation still has the velocity V in it, which is a function of the lift coefficient C_L . To solve this problem, we use equation 3.5. From this follows that the drag is $D = \frac{C_D}{C_L}W$. Since the weight W is constant, the drag is minimal if C_D/C_L is minimal. This is the case if:

$$\frac{d\left(\frac{C_D}{C_L}\right)}{dC_L} = 0 \quad \Rightarrow \quad \frac{C_L\left(\frac{dC_D}{dC_L}\right) - C_D}{C_L^2} = 0 \quad \Rightarrow \quad \frac{dC_D}{dC_L} = \frac{C_D}{C_L} \tag{3.11}$$

This can be solved using equation 3.4:

$$\frac{C_D}{C_L} = \frac{dC_D}{dC_L} = \frac{d\left(C_{D_0} + \frac{C_L^2}{\pi Ae}\right)}{dC_L} = 0 + \frac{2C_L}{\pi Ae} \quad \Rightarrow \quad \frac{C_{D_0}}{C_L} + \frac{C_L}{\pi Ae} = 2\frac{C_L}{\pi Ae} \quad \Rightarrow \quad C_L = \sqrt{C_{D_0}\pi Ae} \tag{3.12}$$

Using equation 3.5 the corresponding velocity can be found. So if we know our zero lift drag coefficient, we know how to fly to get as far as possible.

But now suppose we do not want to go as far as possible, but just want to stay in the air as long as possible. For that, we first introduce the **rate of descent** RD $[m/s]$, which is defined as:

$$RD = -\frac{dh}{dt} = -V\sin\gamma = V\sin\bar{\gamma} \tag{3.13}$$

The aircraft stays as long as possible in the air if RD is minimal. We know that:

$$RD = -\frac{dh}{dt} = V\sin\bar{\gamma} = V\frac{C_D}{C_L}\cos\bar{\gamma} = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{C_D^2}{C_L^3}\cos^3\bar{\gamma}} \tag{3.14}$$

Since W, S and ρ are constants, the rate of descend is minimal if C_D^2/C_L^3 is minimal. Using a method analogue to what we just did, we find that:

$$0 = \frac{d\left(\frac{C_D^2}{C_L^3}\right)}{dC_L} = \frac{C_L^3 \cdot 2C_D \cdot 2\frac{C_L}{\pi Ae} - 3C_L^2 \cdot C_D^2}{C_L^6} \quad \Rightarrow \quad 4\frac{C_L^2}{\pi Ae} = 3C_D \quad \Rightarrow \quad C_L = \sqrt{3C_{D_0}\pi Ae} \tag{3.15}$$

The time until we reach the ground (the **endurance**) t $[s]$ and the traveled distance (the **range**) s $[m]$ can then be calculated using:

$$t = \frac{h}{RD} \qquad\qquad s = \frac{h}{\tan\bar{\gamma}} \tag{3.16}$$

3.6 Propeller Aircraft Range and Endurance

The jet engine and the propeller are very different. So their thrust develops differently. In a propeller aircraft the thrust T decreases as the velocity increases. This happens in such a way that the power available P_a is (approximately) constant. In a jet aircraft the thrust T simply stays constant for any velocity.

Suppose we have a propeller aircraft that wants to fly as far as possible with the fuel it has. Let's assume that the velocity and height stay constant, and thus $P_a = P_r$. The distance per amount of fuel should be maximized. If W_f is the weight of the fuel that is left, the following quantity should be maximized:

$$\frac{s}{W_f} = \frac{ds}{dW_f} = \frac{ds/dt}{dW_f/dt} = \frac{V}{F} \tag{3.17}$$

So the quantity V/F should be maximized. If we define the **power coefficient** C_P $[N/J]$ (which can assumed to be constant for propeller aircrafts) such that $F = C_P P_{br}$, we can find that:

$$\frac{V}{F} = \frac{V}{C_P P_{br}} = \frac{V\eta_j}{C_P P_a} = \frac{V\eta_j}{C_P P_r} = \frac{V\eta_j}{C_P DV} = \frac{\eta_j}{C_P}\frac{1}{D} \tag{3.18}$$

Since η_j and C_P are constants for the aircraft, the quantity V/F is at a maximum if the drag is at a minimum. In the last paragraph we already found out when this was the case. So the aircraft has a maximum range if $C_L = \sqrt{C_{D_0}\pi Ae}$.

But what if we want to stay in the air as long as possible with the amount of fuel we have? Then we ought to minimize the fuel flow F. We can find that:

$$F = VD\frac{C_P}{\eta_j} = \frac{C_P}{\eta_j}W\sqrt{\frac{W}{S}\frac{2}{\rho}\frac{C_D^2}{C_L^3}} \tag{3.19}$$

Since C_P, η_j, W, S and ρ are all constants, this is minimal if C_D^2/C_L^3 is minimal. So the aircraft has maximum endurance if $C_L = \sqrt{3C_{D_0}\pi Ae}$.

3.7 Jet Aircraft Range and Endurance

Suppose we have a jet aircraft that wants to fly as far as possible with the fuel it has. Once more we assume $P_a = P_r$ and thus $T = D$. We should once more maximize V/F. If we define the **thrust coefficient** C_T (which can assumed to be constant for jet aircrafts) such that $F = C_T T$, we can find that:

$$\frac{V}{F} = \frac{V}{C_T T} = \frac{V}{C_T D} = \frac{1}{C_T W}\sqrt{\frac{W}{S}\frac{2}{\rho}\frac{C_L}{C_D^2}} \tag{3.20}$$

So the aircraft has maximum range if C_L/C_D^2 is minimal. It can be derived that this is the case if $C_D = 4C_{D_i}$ and thus $C_L = \sqrt{\frac{1}{3}C_{D_0}\pi Ae}$.

If the jet aircraft wants to stay in the air as long as possible, the quantity F should be minimized. This is the case if C_D/C_L is minimal. Now it's easy to see that the aircraft has maximum endurance if $C_L = \sqrt{C_{D_0}\pi Ae}$.

4 Aircraft Limits

4.1 Velocities at Different Altitudes

The **flight envelope** is more or less defined as the combinations of velocity and height at which the airplane can fly in a normal way. For a certain height, an aircraft has a minimum and a maximum velocity. However, this minimum and maximum velocity differs for different heights. First let's look at the minimum flight velocity. This minimum velocity is:

$$V_{min} = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}} \tag{4.1}$$

So if $h \uparrow$ then $\rho \downarrow$ and thus $V_{min} \uparrow$. For higher altitudes the minimum velocity increases due to a decrease in air density.

The maximum velocity depends on the power that is available. It is the velocity at which $P_{a_{max}} = P_r$. However, at different altitudes the aircraft usually has a different $P_{a_{max}}$, so it's kind of hard to calculate the maximum velocity. But this maximum velocity can be exceeded in a dive. When doing this, the airplane exits its flight envelope, which is usually considered to be a rather dangerous thing.

4.2 Equivalent Airspeed

The airspeed indicator of an aircraft doesn't indicate the **true airspeed** (the velocity of the aircraft with respect to the surrounding air). Instead, it indicates the **equivalent airspeed**, which is the airspeed that gives the same dynamic pressure $q = \frac{1}{2}\rho V^2$ at sea level, as the true airspeed in the current atmosphere. So at sea level the true airspeed and the equivalent airspeed are equal. But if the altitude increases, and thus the density decreases, a higher velocity is needed to reach the same dynamic pressure. Therefore the equivalent airspeed is generally lower than the true airspeed (and the difference increases with increasing altitudes). The relation between the true airspeed V and the equivalent airspeed V_e can be found as follows:

$$\frac{1}{2}\rho_0 V_e^2 = \frac{1}{2}\rho V^2 \qquad \Rightarrow \qquad V_e = V\sqrt{\frac{\rho}{\rho_0}} \tag{4.2}$$

where $\rho_0 = 1.225\ kg/m^3$ is the air density at sea-level. An interesting to note is that the minimum equivalent airspeed is:

$$V_{e_{min}} = V_{min}\sqrt{\frac{\rho}{\rho_0}} = \sqrt{\frac{W}{S}\frac{2}{\rho_0}\frac{1}{C_L}} \tag{4.3}$$

So the minimum equivalent airspeed is constant at different altitudes. This saves the pilot a lot of calculations, since the airspeed indicator of an airplane also indicates the equivalent airspeed.

4.3 Maximum Height

The aircraft can not fly at infinite heights. The higher you go, the less air you find, and air is something airplanes need for thrust and lift. So there must be a ceiling. This ceiling is the place at which the airplane can not go any higher. Let's define the rate of climb RC as $-RD$. So the maximum rate of climb is 0 at the ceiling. The rate of climb can be calculated using:

$$RC = \frac{P_a - P_r}{W} \tag{4.4}$$

So at the theoretical ceiling, when $RC_{max} = 0$ also $(P_a - P_r)_{max} = 0$. Thus $P_a \leq P_r$ and the airplane can only fly in the ceiling if $P_a = P_r$.

However, this ceiling is only a **theoretical ceiling**. Since if the rate of climb is 0 m/s in the theoretical ceiling, how could you get there? There is also a **service ceiling**, which is in practice about the highest point at which aircrafts can fly. The service ceiling is the height at which the maximum rate of climb of the airplane is 0.5 m/s.

4.4 Supersonic Limits

When an aircraft is flying at supersonic velocities, shock waves occur. The shape of the shock wave can be either oblique or blunt. Oblique shock waves are caused by sharp edges and are relatively weak, while blunt shock waves are caused by rounded edges and are relatively strong. Oblique shock waves have an angle, called the **Mach angle**, which can be calculated using:

$$\mu = \arcsin \frac{a}{V} = \arcsin \frac{1}{M} \tag{4.5}$$

When the air passes through a shock wave, a lot of things happen. To make a list: $V \downarrow$, $p \uparrow$, $T \uparrow$, $M \downarrow$, $s \uparrow$. The entropy s increases due to a loss in energy, which is caused by additional drag called **wave drag**. This wave drag is also caused by shock waves.

When flying at high Mach numbers, buffeting can occur. This can be dangerous, and to prevent this, the airworthiness regulations define a **maximum Mach number** M_D for an airplane after several tests. This results in a maximum velocity of:

$$V_D = M_D \sqrt{\gamma R T} \tag{4.6}$$

Since the temperature deceases as the height increases, also the maximum velocity due to the maximum Mach number decreases as the height increases (until the stratosphere is reached where T is constant). To increase safety even more, an extra margin gets taken into account, which results in the **maximum operating Mach number** M_{M0}. This is the highest Mach number at which the aircraft is allowed to fly.

4.5 Gusts

If an aircraft encounters a sudden upward gust, the angle of attack (with respect to the airflow) will increase. If the air gusts travels upward with a velocity u, the change in angle of attack is:

$$\Delta\alpha = \tan \frac{u}{V} \approx \frac{u}{V} \tag{4.7}$$

The change in lift coefficient now is:

$$\Delta C_L = \frac{dC_L}{d\alpha} \Delta\alpha = \frac{dC_L}{d\alpha} \frac{u}{V} \tag{4.8}$$

This makes the change in lift the following:

$$\Delta L = (\Delta C_L) \frac{1}{2} \rho \left(V^2 + u^2\right) S = \frac{dC_L}{d\alpha} \frac{1}{2} \rho \left(uV + 2u^2 + \frac{u^3}{V}\right) S \approx \frac{dC_L}{d\alpha} \frac{1}{2} \rho u V S \tag{4.9}$$

In the last step the assumption was made that $u << V$. Since a sudden huge increase in lift can be dangerous (high G-forces and breaking wings may occur), the airworthiness regulations have set another limit, being the **maximum equivalent airspeed due to gust loading** V_{e_d} $[m/s]$. This makes the maximum airspeed due to gust loading:

$$V_D = V_{e_d} \sqrt{\rho} \rho_0 \tag{4.10}$$

However, to increase safety, there is an additional margin to this maximum allowed airspeed, being the **maximum operational velocity** V_{M0}. This is the highest velocity at which an aircraft is allowed to fly.

4.6 Limit Overview

Next to the limits we just saw, there is one additional limit to the flight envelope of an airplane. This is the maximum pressure difference. The pressure cabin can only take a maximum pressure difference, which may not be exceeded. This is the last limit that will be discussed.

It's time to make a graph out of all the limits we have just talked about. This graph can be seen in figure 2. It gives an impression on the flight envelope of a normal aircraft.

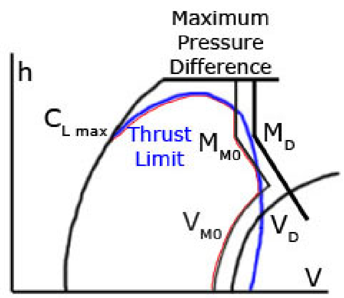

Figure 2: Visualization of the flight envelope.

5 Helicopters

5.1 The Tail Rotor

Most (traditional) helicopters have a termmain rotor and a **tail rotor**. The main rotor provides the lift. To do that, it is rotating with an angular velocity Ω [rad]. (Most helicopters have blades rotating counter-clockwise, when seen from the top, so we will assume the blades are rotating counter-clockwise.) This rotation gives it a velocity, and since the rotor blades have the shape of a wing foil, it causes lift. However, next to lift, it also causes drag. This drag results in a (clockwise) torque. To prevent the rotor from slowing down, the helicopter engine causes an equal but opposite (and thus counter-clockwise) torque. However, Newton's third law implies that every action has an equal and opposite reaction. So if the helicopter causes a counter-clockwise torque on the blades, the blades cause a clockwise torque on the helicopter, which would make it rotate. This is an undesirable effect, and that's why a tail rotor is added.

Suppose a torque Q $[Nm]$ is necessary for the main rotor to have a constant angular velocity. If l_t $[m]$ is the distance between the main rotor and the tail rotor, and if T_t $[N]$ is the thrust from the tail rotor, then to prevent the helicopter from rotating, the following condition should be true:

$$Q = l_t \cdot T_t \tag{5.1}$$

However, tail rotors have a lot of disadvantages. They consume a lot of power, are dangerous, noisy, expensive and under bad wind conditions they give only a marginal control authority. That's why a lot of alternatives for a tail rotor have been introduced over the years.

5.2 Helicopter Performance

A helicopter stays in the air by thrusting air downward. Far above the main rotor the air is still standing still. When the air passes through the main rotor, it has an induced velocity V_i $[m/s]$. We can now express the amount of air passing through the rotor blades every second. If the rotor blades have length R $[m]$, then the **mass flow** m $[kg/s]$ is:

$$m = \rho \pi R^2 V_i \tag{5.2}$$

But when the air has just passed the disk, it is still accelerating. When the air is relatively far below the main rotor, it has a velocity of $2V_i$. Using the momentum equation, we can calculate the thrust of the main rotor:

$$T = I_{out} - I_{in} = 2mV_i - 0 = 2mV_i = 2\rho\pi R^2 V_i^2 \tag{5.3}$$

For a hovering helicopter, the thrust equals the weight ($T = W$). The **induced power** P_i $[J/s]$ is equal to the change in kinetic energy of the flow:

$$P_i = \frac{1}{2} m \left(2V_i\right)^2 = 2mV_i^2 = TV_i = WV_i = W\sqrt{\frac{W}{2\rho\pi R^2}} \tag{5.4}$$

Next to the induced power, there is also the **hover power** P_{hov}, which can be calculated using:

$$P_{hov} = Q\Omega = l_t T_t \Omega \tag{5.5}$$

The real power necessary for hovering, the hover power, is usually not the same as the ideal induced power. The **Figure of Merit** M is defined as:

$$M = \frac{P_i}{P_{hov}} \tag{5.6}$$

For the tail rotor there is also a (different) Figure of Merit M_t.

5.3 Helicopter Control

Helicopters can do more than simple hovering. They can also fly in multiple directions. First of all, helicopters can go up and down. To achieve this either the angular velocity Ω or the rotor blade pitch is increased. The latter can be done by using a **swash plate**. This is a plate below the rotor blades. A small bar is connected to the front of every rotor blade. So if the swash plate goes up, the front of the rotor blades go up as well, increasing the pitch.

The swash plate can not only go up/down. It can also rotate slightly. When this happens, one of the rotor blades gets an increased pitch, while the other (being on the opposite side) gets a decreased pitch. This causes a moment, which causes the helicopter to rotate. By rotating the helicopter, the direction of the thrust changes, which causes the helicopter to go forward/backward or left/right. When using this trick, it is important not to forget the gyroscopic effect. This is present since the rotor blades are spinning rather fast.

The rotor blades are generating lift. If they would be entirely fixed to the rotor axis, there would be large bending moments. This is why they are usually able to rotate in multiple directions. The reason why they still point outward is because of the centrifugal effect. This method offers various advantages, but can sometimes be slightly dangerous. One example is that it can cause ground resonance if the helicopter is hovering close to the ground.

6 Measurement Devices

6.1 Airspeed Indication

An airspeed indicator doesn't indicate the **true airspeed** (TAS), but measures the difference between the total and the static pressure. For low velocities this is the dynamic pressure $q = \frac{1}{2}\rho V^2$. So for low velocities this works well. However, for high velocities ($M > 0.3$) compressibility effects need to be taken into account, and this equation can not be applied anymore. Instead, we use thermodynamics to derive an alternative equation for the velocity. We start by noting the following:

$$c_p T + \frac{1}{2}V^2 = c_p T_t \quad \Rightarrow \quad \frac{T_t}{T} = 1 + \frac{V^2}{2c_pT} = 1 + \frac{V^2 \rho R}{2c_p p} = 1 + V^2 \frac{\rho}{p}\frac{c_p - c_v}{2c_p} = 1 + V^2\frac{\rho}{p}\frac{\gamma - 1}{2\gamma} \tag{6.1}$$

Now we can use the formula of Poisson to get:

$$\frac{p_t}{p} = \left(\frac{T_t}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + V^2\frac{\rho}{p}\frac{\gamma-1}{2\gamma}\right)^{\frac{\gamma}{\gamma-1}} \quad \Rightarrow \quad \left(\frac{p_t}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1 = V^2\frac{\rho}{p}\frac{\gamma-1}{2\gamma} \tag{6.2}$$

Solving for V gives:

$$V = \sqrt{\frac{2\gamma}{\gamma-1}\frac{p}{\rho}\left(\left(\frac{p_t}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right)} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{p}{\rho}\left(\left(\frac{q_c}{p} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right)} \tag{6.3}$$

where the pressure difference q_c is defined as $q_c = p_t - p$. So for low speeds $q_c = q$. The last equation indicates that the velocity is a function of q_c, p and T. q_c can be measured, but p and T are very difficult to measure accurately. Therefore the **calibrated airspeed** (CAS) V_c $[m/s]$ is introduced. The indicator assumes that the airplane is flying at 0 m altitude in **International Standard Atmosphere** (ISA), such that $p = p_0$ and $T = T_0$. Thus:

$$V_c = \sqrt{\frac{2\gamma}{\gamma-1}RT_0\left(\left(\frac{q_c}{p_0} + 1\right)^{\frac{\gamma-1}{\gamma}} - 1\right)} \tag{6.4}$$

For $M << 1$ the calibrated airspeed is (approximately) equal to the equivalent airspeed (but not the true airspeed!). And finally there is one more airspeed, the **indicated airspeed** (IAS). This airspeed is equal to the calibrated airspeed, accept for (usually minor) errors caused by the airspeed indicator. However, if, for example, the dial of the airspeed indicator gets stuck, the indicated airspeed is definitely not equal to the calibrated airspeed.

6.2 International Standard Atmosphere (ISA)

The hydrostatic equation is:

$$\frac{dp}{dh} = -\rho g \tag{6.5}$$

Since g isn't constant at high altitudes, we define the geopotential height H $[m]$ such that $g\,dh = g_0 dH$ where g_0 $[m/s^2]$ is the gravitational acceleration at zero altitude. So this gives:

$$\frac{dp}{dH} = -\rho g_0 \tag{6.6}$$

Combining this with the gas law, we have three unknowns and two equations. Therefore we assume that for the troposphere $T = T_0 + \lambda H$ with $T_0 = 288.15$ K and $\lambda = -0.0065$ K/m. If we now combine this assumption with equation 6.6 and with the gas law, we get:

$$\frac{dp}{p} = -\frac{g_0\,dH}{R(T + \lambda H)} \tag{6.7}$$

Integrating and working out the result gives:

$$\frac{p}{p_0} = \left(1 + \frac{\gamma H}{T_0}\right)^{-\frac{g_0}{\lambda R}} = \left(\frac{T}{T_0}\right)^{-\frac{g_0}{\lambda R}} \qquad \Leftrightarrow \qquad \frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{-\frac{g_0}{\lambda R}-1} \tag{6.8}$$

Solving for H gives an equation which is used in altimeters to find the **pressure altitude**:

$$H = \left(\left(\frac{p}{p_0}\right)^{-\frac{\lambda R}{g_0}} - 1\right)\frac{T_0}{\lambda} \tag{6.9}$$

where p can be found by measuring the pressure difference between the static pressure and the pressure of a vacuum. In this way a pilot can find out how high he approximately is.

If every pilot would use the same reference values p_0 and T_0, there is sufficient data to avoid collisions in the air. However, when an airplane has its reference pressure set at ISA-pressure, and if the real pressure is different, he will get wrong readings. Since a pilot would like to know whether his plane is 5 meters or 5 kilometers above the ground, there is a system which gets rid of these disadvantages.

When an airplane lifts off, the pilot asks the control tower for the **QNH** pressure, which is the actual pressure around the airfield. He sets the reference pressure of his altimeter to that value, so he knows his actual altitude quite accurately. When he gets above a certain **transition altitude** (which can differ per region, but is usually a few thousand feet), he sets the reference pressure of his altimeter to the ISA-pressure, such that all airplanes flying in that region have the same reference pressure. When the aircraft gets close to his arrival destination, the pilot asks for the local QNH. When the aircraft passes the local transition altitude, the pilot adjusts the reference pressure of his altimeter to the local QNH. He then has the right altitude reading on his altimeter, so he can land his airplane safely.

Next to an airspeed indicator and an altimeter, a pilot usually also has a thermometer, measuring the temperature outside the airplane.