Aircraft Structural Analysis Summary

1. Basic Equations

1.1 Definitions, conventions and basic relations

Before we can start throwing around equations, we have to define some variables and state some conventions. That is what we will be doing in this part.

1.1.1 Definitions

Let's suppose we have an object, that's subject to forces. There can be two kinds of forces it is subject to. First there are **surface forces** acting on the outside of the object. An example of a surface force is pressure. We can resolve surface forces into three components, along the axes. These components are denoted as $\overline{X}$, $\overline{Y}$ and $\overline{Z}$. There are also **body forces**, acting on every particle of the object. An example is the gravitational force. When we resolve body forces into three components, we write it as X, Y and Z.

The forces acting on the object cause internal stresses. (Stress is force per unit area.) Let's suppose we make a cut through the object and examine a point O on the cut. There is a component of the stress normal to the cut, called the **direct (tensile) stress**. This is denoted by the sign σ. There are also two components of the stress parallel to the cut. These components are called **shear stresses** and are denoted by τ.

1.1.2 Notation and sign conventions

Now let's discuss some notation and sign conventions. Often direct stresses are examined along the three basic axes. (The x, y and z-axes.) We then say that σ_x is the direct stress along the x-axis, σ_y is the stress along the y-axis and σ_z is the stress along the z axis. If a certain stress is directed away from its related surface, then we define it as a positive stress. Otherwise it is negative.

We can have a similar notation for shear stresses. However, shear stresses don't only have a direction. They also have a plane in which they act. They therefore have two subscript, like τ_{xy}. The x (the first part of the subscript) denotes the plane in which the shear stress acts. In this case it is the plane orthogonal to the x-axis. The y (the second part of the subscript) then denotes the direction of the shear stress.

The sign convention of shear stress is also a bit difficult. We have to examine two arrows for that. First there is the direction of the shear stress itself. Then there is also the normal vector to the plane on which the shear stress is acting. (This normal vector always points outward.) If they both point in a positive direction, or both in a negative direction, then we say that the shear stress is positive. If one points in a positive direction, and the other in a negative direction, then it is negative.

1.1.3 Basic equations

We can examining the stresses acting on a small part inside an object. By doing so, we can derive a few relations. First, by taking moments, we can derive relations for the shear stresses. These relations are

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx} \qquad \text{and} \qquad \tau_{yz} = \tau_{zy}. \tag{1.1.1}$$

By examining forces in certain directions, we can derive three equilibrium equations, being

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0, \tag{1.1.2}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{yx}}{\partial x} + Y = 0, \tag{1.1.3}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0. \tag{1.1.4}$$

Instead of examining a particle on the inside of an object, we can also examine a particle on the edge. Now surface forces come into play. We can once more derive three equilibrium equations, being

$$\overline{X} = \sigma_x l + \tau_{yx} m + \tau_{zx} n, \tag{1.1.5}$$

$$\overline{Y} = \sigma_y m + \tau_{zy} n + \tau_{xy} l, \tag{1.1.6}$$

$$\overline{Z} = \sigma_z n + \tau_{xz} l + \tau_{yz} m. \tag{1.1.7}$$

The three parameters l, m and n are **direction cosines**. They are added to the equation to compensate for the direction of the surface. To find their values, examine the normal vector of the surface (still pointing outward). l, m and n are the cosines of the angles which this normal vector makes with respect to the x, y and z axis, respectively.

1.2 Stresses in different coordinate systems

We don't always evaluate stresses along the x, y and z-axes. We can also examine them in different coordinate systems. What happens when we start shifting coordinate systems?

1.2.1 Mohr's circle

Let's suppose we know all the stresses in the normal (x, y, z)-coordinate system. When we shift the coordinate system, the normal stresses and the shear stresses change. The way in which this occurs is described by **Mohr's circle**. Mohr stated that if you plot the direct stresses and the shear stresses, you would get a circle. Such a circle is shown in figure 1.1.

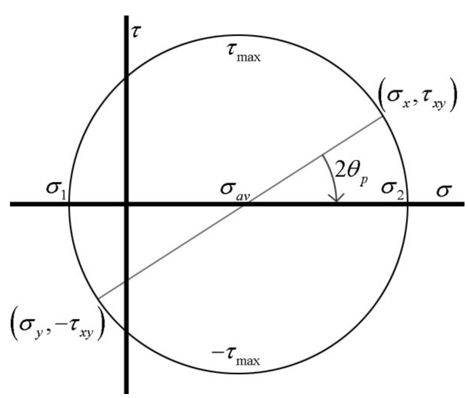

Figure 1.1: Mohr's Circle

How does this work? Suppose we know the stress in x-direction σ_x, the stress in y-direction σ_y and the shear stress τ_{xy}. Let's draw the points (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$ in a coordinate system. We then draw a line between them. The point where this line crosses the x-axis denotes the **average stress** σ_{av}. It can be found using

$$\sigma_{av} = \frac{\sigma_x + \sigma_y}{2}. \tag{1.2.1}$$

The radius of the circle is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \tag{1.2.2}$$

Now if we rotate our coordinate system by an angle θ, then the line in our circle rotates by an angle 2θ. From this, the new stresses can be found.

1.2.2 Directions of maximum stress

It would be nice to know when maximum stress occurs. Maximum normal (direct) stress occurs when we rotate our coordinate system over an angle $\theta_{m\sigma}$. $\theta_{m\sigma}$ can be found using

$$\tan 2\theta_{m\sigma} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \tag{1.2.3}$$

The corresponding stresses are called **principal stresses**. The planes on which they act are the **principal planes**. The **maximum stress** (also called the **major principal stress**) σ_I and the **minimum stress** (also called the **minor principal stress**) σ_{II} can now be found using

$$\sigma_I = \sigma_{av} + R \qquad \text{and} \qquad \sigma_{II} = \sigma_{av} - R. \tag{1.2.4}$$

Similarly, maximum shear stress occurs when we rotate our coordinate system by an angle $\theta_{m\tau}$, where $\theta_{m\tau}$ now satisfies

$$\tan 2\theta_{m\tau} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}. \tag{1.2.5}$$

This angle will always be 45° bigger or smaller than the angle at which maximum direct stresses occur. (This can also be seen from Mohr's circle.) The corresponding maximum shear stress now is

$$\tau_{max} = R = \frac{\sigma_I - \sigma_{II}}{2}. \tag{1.2.6}$$

1.3 Strains

When an object is subject to forces, there will be displacements. These displacements relate to strains. Let's take a look at what kind of strains there are, and how we can find them.

1.3.1 Strain relations

We generally distinguish two types of strains. The **longitudinal** or **direct strains** (denoted by ε) relate to changes in length. **Shear strains** (denoted by γ) relate to changes in angles.

Let's examine a point O of an object. Due to the deformation of this object, this point O moves. It moves a distance u along the x-axis, a distance v along the y-axis and a distance w along the z-axis. It can now be shown that the direct strains in x, y and z-direction satisfy

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \varepsilon_z = \frac{\partial w}{\partial z}. \tag{1.3.1}$$

The orientations of lines passing through point O have also changed. For example, let's consider two lines in the xy-plane that were perpendicular. (There was an angle of $\pi/2$ between them.) Now they aren't perpendicular anymore. Their relative angle now is $\pi/2 - \gamma_{xy}$. This works the same for the xz-plane and the yz-plane. So we have three shear strains γ_{xy}, γ_{xz} and γ_{yz}. If the displacements are small, then it can be shown that

$$\gamma_{xy} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \qquad \gamma_{xz} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad \text{and} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}. \tag{1.3.2}$$

Now we have six kinds of displacements. It seems like a lot of unknowns. Luckily there are relations between them. There are 6 compatibility equations. These equations are

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}, \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \tag{1.3.3}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}, \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yx}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right), \tag{1.3.4}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}, \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zy}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right). \tag{1.3.5}$$

1.3.2 Relations between stress and strain

Currently, we've got quite a couple of equations. But we got even more unknowns. So we need more equations. Where do we get those equations from?

We can try to describe the relationship between stress and strain. For that, we first have to make a few assumptions. First, we assume that the object we're looking at is **homogeneous**. This means that the material properties are the same at every point in the object. We also assume that the object is **isotropic**, meaning that the properties are the same in every direction. It also means that the stress and the strain are proportional.

From these assumptions we can derive that

$$\varepsilon_x = \frac{\sigma_x - \nu\left(\sigma_y + \sigma_z\right)}{E}, \qquad \varepsilon_y = \frac{\sigma_y - \nu\left(\sigma_z + \sigma_x\right)}{E} \qquad \text{and} \qquad \varepsilon_z = \frac{\sigma_z - \nu\left(\sigma_x + \sigma_y\right)}{E}. \tag{1.3.6}$$

Here ν is the **Poisson ratio**. There are also a relations between the shear stresses and shear strains. These relations are

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \text{and} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}. \tag{1.3.7}$$

The variable G is called the **modulus of rigidity**. It is related to E and ν according to

$$G = \frac{E}{2\left(1+\nu\right)}. \tag{1.3.8}$$

1.3.3 Changes of volume

When an object deforms, its volume changes. It would be interesting to know at what rate this happens. If V is the volume of a particle, then the **volumetric strain** e of that particle is

$$e = \frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right). \tag{1.3.9}$$

If an object is compressed at a constant pressure p, then $\sigma_x = \sigma_y = \sigma_z = -p$. We then have

$$e = -\frac{3\left(1-2\nu\right)}{E}p = -\frac{p}{K}, \qquad \text{with } K = \frac{E}{3\left(1-2\nu\right)}. \tag{1.3.10}$$

The constant K is known as the **bulk modulus** or the **modulus of volume expansion**.

1.3.4 Thermal effects

When an object is heated, it expands. It does this according to

$$\varepsilon = \alpha \, \Delta T, \tag{1.3.11}$$

where α is the **coefficient of thermal expansion**. If also stresses are involved, then we get new equations for the strains. We simply add $\alpha \, \Delta T$ up to the old equations. We then get

$$\varepsilon_x = \frac{\sigma_x - \nu \, (\sigma_y + \sigma_z)}{E} + \alpha \, \Delta T, \tag{1.3.12}$$

$$\varepsilon_y = \frac{\sigma_y - \nu \, (\sigma_z + \sigma_x)}{E} + \alpha \, \Delta T, \tag{1.3.13}$$

$$\varepsilon_z = \frac{\sigma_z - \nu \, (\sigma_x + \sigma_y)}{E} + \alpha \, \Delta T. \tag{1.3.14}$$

2. Stress Functions

2.1 The Airy Stress Function

Previously we have examined general equations. However, solving them can be very hard. So let's look for tools with which we can apply them. In this chapter, we'll be looking at stress functions. The first one to introduce is the Airy stress function.

2.1.1 Stress state conditions

Before we start defining things, we will make some simplifications. First of all, we assume there are no body forces, so $X = Y = Z = 0$. Second, we will only deal with two-dimensional problems. For that, we have to assume that $\sigma_z = 0$. If this is the case, we have **plane stress** (the stress only occurs in a plane). Together, these two assumptions turn the equilibrium conditions of the previous chapter into

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{2.1.1}$$

Next to equilibrium conditions, we also had compatibility conditions. Based on our assumptions, we can simplify those as well. We then get only one equation, being

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \tag{2.1.2}$$

And finally there were the boundary conditions. Adjusting those will give

$$\overline{X} = \sigma_x l + \tau_{xy} m \qquad \text{and} \qquad \overline{Y} = \sigma_y m + \tau_{xy} l. \tag{2.1.3}$$

Now we have derived the new conditions for the stress state. Let's see how we can apply them.

2.1.2 The Airy stress function

It is time to talk about stress functions. A **stress function** is a function from which the stress can be derived at any given point x, y. These stresses then automatically satisfy the equilibrium conditions.

Now let's examine such a stress function. The **Airy stress function** ϕ is defined by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad \text{and} \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{2.1.4}$$

We can insert these stresses in the equilibrium conditions (2.1.1). We then directly see that they are satisfied for every ϕ! How convenient... However, if we insert the above definitions into the compatibility condition (2.1.2), we get

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0. \tag{2.1.5}$$

This equation is called the **biharmonic equation**. It needs to be satisfied by every valid Airy stress function as well.

2.1.3 Applying the Airy stress function

Now you may be wondering, how can we apply the Airy stress function? To be honest, that is kind of a problem. Given the loading condition of an object, it's rather difficult to determine a corresponding stress function. On the other hand, if we have a stress function ϕ, it is often possible to find a corresponding loading condition. This idea is called the **inverse method**.

So how do we apply this inverse method? We first have to assume a certain form of ϕ with a number of unknown coefficients $A, B, C, \ldots$. We know ϕ has to satisfy the biharmonic equation (2.1.5) and the boundary conditions (2.1.3). From these conditions, the unknown coefficients can (hopefully) be solved. The most difficult step in this process is to choose a form for ϕ. Sadly, that part is beyond the scope of this summary.

2.1.4 St. Venant's principle

Sometimes a problem occurs when applying the boundary conditions. For example, if the object we are considering is subject to a concentrated (local) force, there will be huge local variations in the stress. It is hard to adjust the boundary conditions to these **local effects**.

In this case, use can be made of **St. Venant's principle**. It states that local variations eventually average out. You just 'cut' the part with local effects out of your object. For the rest of the object, you can then assume loading conditions with which you are able to make calculations.

2.1.5 Displacements

Let's suppose we have found the stress function ϕ for an object. We can now find the stresses σ_x, σ_y and τ_{xy} at every position in the object. These stresses will thus be functions of x and y.

Using these stresses, we can find the displacements u, v and γ_{xy}. To do this, we first need to adjust the stress-strain relations from the previous chapter to the two-dimensional world. For the direct strain we find

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x - \nu\sigma_y}{E} \qquad \text{and} \qquad \varepsilon_y = \frac{\partial v}{\partial y} = \frac{\sigma_y - \nu\sigma_x}{E}. \tag{2.1.6}$$

So first we can find ε_x and ε_y, as functions of the position x, y. We then integrate those strains to find the displacements u and v. Don't underestimate these integrals. They are often quite difficult, since ε_x and ε_y are functions of both x and y.

After we have found u and v, we can use them to find γ_{xy}. This goes according to

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G}. \tag{2.1.7}$$

And now everything is known about the object!

2.2 The Prandtl stress function

The Airy stress function is quite suitable when a force is applied to a two-dimensional object. Similarly, the Prandtl function is useful when torsion is present. Let's take a look at it.

2.2.1 Conditions

Let's examine a rod with a constant cross-section. Its axis lies on the z-axis. We can apply a torsion T to both its sides. This torsion T is said to be positive when it is directed counterclockwise about the z-axis (according to the right-hand rule). Since we only apply torsion, we can assume there are no normal (direct) stresses, so $\sigma_x = \sigma_y = \sigma_z = 0$. The same goes for the shear stress τ_{xy}, so $\tau_{xy} = 0$. From this follows that also $\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$. We also assume no body forces are present.

So most of the stresses are zero. We only have two non-zero stresses left, being τ_{zy} and τ_{zx}. The **Prandtl stress function** ϕ is now defined by

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} \qquad \text{and} \qquad \tau_{zx} = \frac{\partial \phi}{\partial y}. \tag{2.2.1}$$

It can be shown that τ_{zy} and τ_{zx} only depend on the x and y-coordinates. They don't vary along the z-axis.

We know that ϕ should satisfy the conditions from the first chapter. We can find that ϕ automatically satisfies the equilibrium equations. We can reduce all compatibility equations to one equation, being

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \text{constant}, \tag{2.2.2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the **two-dimensional Laplace operator**.

Finally there are the boundary conditions. We can derive two things from that. First, we can derive that, along the outer surface of the rod, we have $\partial \phi/ds = 0$. So ϕ is constant along the rod surface. Since this constant doesn't really matter, we usually assume that $\phi = 0$ along the outer surface of the rod.

Second, we can also look at the two rod ends, where the torsion T is being applied. If we sum up the shear stresses in this region, we can find the relation between the torsion T and the function ϕ. This relation states that

$$T = 2 \iint \phi \, dx \, dy. \tag{2.2.3}$$

2.2.2 Displacements

With all the conditions we just derived, we often can't find ϕ just yet. We also need to look at the displacements. Let's call θ the **angle of twist** and $d\theta/dz$ the **rate of twist**. It follows that, for the displacements u and v, we have

$$u = -\theta y \qquad \text{and} \qquad v = \theta x. \tag{2.2.4}$$

Previously we have also seen that $\nabla^2 \phi$ is constant. However, we didn't know what constant it was equal to. Now we do. It can be shown that

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\frac{d\theta}{dz}. \tag{2.2.5}$$

And finally we have all the equations that ϕ must satisfy. That's great! However, we can simplify matters slightly. Let's introduce the **torsion constant** J. It is defined by

$$T = GJ\frac{d\theta}{dz}. \tag{2.2.6}$$

By the way, the product GJ is called the **torsional rigidity**. From the above two equations, and the relation between T and ϕ, we can find that

$$GJ = -\frac{4G}{\nabla^2 \phi} \int \int \phi \, dx \, dy. \tag{2.2.7}$$

2.2.3 Finding the Prandtl stress function

We now know all the conditions which ϕ must satisfy. However, finding ϕ is still a bit difficult. Just like for the Airy stress function, we first have to assume a form for ϕ. This form should be such that it satisfies all the above conditions.

The first condition you should pay attention to, is the condition that $\phi = 0$ around the edge. Then we multiply this relation by a constant, to find our stress function. Using the other conditions, we can then find the value of our constant. For example, if our cross-section is a circle, we would have $x^2 + y^2 = R^2$ around the edge. A suitable function for ϕ would then be $\phi = C\left(x^2 + y^2 - R^2\right)$. Find C using the remaining conditions, and you've found ϕ.

2.2.4 Warping

We know that the rod will twist. But that's not the only way in which it will deform. There is also **warping**, being the displacement of points in the z-direction. To know how an object warps, we have to find an expression for w. For that, we have to use the relations

$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} + \frac{d\theta}{dz} y \qquad \text{and} \qquad \frac{\partial w}{\partial y} = \frac{\tau_{zy}}{G} - \frac{d\theta}{tz} x. \tag{2.2.8}$$

Integrating the above expressions should give you w: the displacement in z-direction.

2.2.5 The membrane analogy

Let's consider the lines along the cross-section for which ϕ is constant. These special lines are called **lines of shear stress** or **shear lines**. You may wonder, why are they special? Well, to see that, let's look at the shear stresses τ_{zx} and τ_{zy} at some point. We find that the resultant shear stress (the sum of τ_{zx} and τ_{zy}) is tangential to the shear line. Furthermore, the magnitude of this stress is equal to $-\partial\phi/\partial n$, where the vector $\mathbf{n}$ is the normal vector of the shear line (pointing outward).

This may be a bit hard to visualize. Luckily, there is a tool that can help you. It's called the **membrane analogy** (also called the **soap film analogy**). Let's suppose we have a membrane (or a soap film) with as shape the cross-section of our rod. We can apply a pressure p to this membrane from below. It then deflects upwards by a distance w. This deflection w now corresponds to our stress function ϕ, so $w(x, y) = \phi(x, y)$. Note that we have $w = 0$ at the edges of our membrane, just like we had $\phi = 0$ at the edges of our rod.

We can also look at the volume beneath our soap bubble. We then find that

$$\text{Volume} = \iint w \, dx \, dy, \qquad \text{which implies that} \qquad T = 2 \times \text{Volume}. \tag{2.2.9}$$

2.2.6 Torsion of narrow rectangular strips

Let's examine a narrow rectangular strip. Its height (in y-direction) is s, while its thickness (in x-direction) is t. Normally it is very hard to find the Prandtl stress function ϕ for this rod. However, if t is much smaller than s, we can simplify things. In this case, we can assume that ϕ doesn't vary with y. So we find that

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = -2G\frac{d\theta}{dz}. \tag{2.2.10}$$

By integrating this twice, the stress function ϕ can be obtained relatively easily. (Okay, you still have to find the two constants that show up in the integration, but that isn't very hard.) And once the stress function is known, all the other data will follow.

3. Bending, Shear and Torsion

It is time to examine some basic loads that beams can be subject to. We especially look at thin-walled beams, as they frequently occur in Aerospace Engineering. We can then derive general methods and equations. With those, we can find the stresses that are present in the beam.

3.1 Bending of Beams

We start by examining bending. This is because we need the bending equations when we examine shear.

3.1.1 Definitions and conventions for bending

Let's examine a beam of any shape. Just like in the previous chapter, its longitudinal axis lies on the z-axis. Now the beam is subject to a bending moment M. We can dissolve this bending moment M into a component M_x about the x-axis and a component M_y about the y-axis.

Let's discuss the sign convention of these moments. We say a moment M_x is positive, if it causes (positive) tensile stresses in the region $y > 0$. Similarly, M_y is positive, if it causes tensile stresses in the region $x > 0$. We can see that M_x satisfies the right hand rule (it is directed counterclockwise if you look at it from the positive x-direction). However, the moment M_y does not satisfy this rule. If you look at it from the positive y-axis, it is directed clockwise.

When evaluating bending, we will have to use moments of inertia. There are the **moment of inertia about the x-axis** I_{xx}, the **moment of inertia about the y-axis** I_{yy} and the **product of inertia** I_{xy}. They are defined as

$$I_{xx} = \int_A y^2\, dA, \qquad I_{yy} = \int_A x^2\, dA \qquad \text{and} \qquad I_{xy} = \int_A xy\, dA. \tag{3.1.1}$$

3.1.2 The general bending equation

The bending moments M_x and M_y cause the beam to bend. Now let's look at the cross-section of the beam. Part of the beam is subject to tensile stresses, while the other part is in compression. The line separating these two regions is called the **neutral axis**. It can be shown that this is a straight line. It always goes through the center of gravity of the cross-section.

It would be great to know what stresses are present in the beam. And the good part is, a general equation can be derived for that. What we wind up with is

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y = \left(\frac{I_{yy} y - I_{xy} x}{I_{xx} I_{yy} - I_{xy}^2}\right) M_x + \left(\frac{I_{xx} x - I_{xy} y}{I_{xx} I_{yy} - I_{xy}^2}\right) M_y. \tag{3.1.2}$$

In the above equation, you find two relations for σ_z. As you can see, they are equivalent. You can use either one of them. Which one is the most convenient depends on the circumstances.

If the cross-section of the beam is symmetric about the x-axis or about the y-axis (or both), then we have $I_{xy} = 0$. This simplifies the above equation drastically. We then remain with

$$\sigma_z = \frac{M_x}{I_{xx}} y + \frac{M_y}{I_{yy}} x. \tag{3.1.3}$$

3.1.3 Beams of multiple materials

Sometimes beams are made of multiple materials. Different materials generally have different stiffnesses, so also different values of E. How do we take this into account? Well, to do that, we define the **weighted cross-sectional area** A^* as

$$dA^* = \frac{E}{E_{ref}} dA, \qquad \text{which implies that} \qquad A^* = \int_A \frac{E}{E_{ref}} dA. \tag{3.1.4}$$

Here E_{ref} is just some reference E-modulus. Although it can be anything, it's usually taken to be the E-modulus of one of the present materials. We can now also define the weighted moment of inertias as

$$I_{xx}^* = \int_A y^2 \, dA^*, \qquad I_{yy}^* = \int_A x^2 \, dA^* \quad \text{and} \quad I_{xy}^* = \int_A xy \, dA^*. \tag{3.1.5}$$

Based on these definitions, we can derive a new expression for σ_z. We find that

$$\sigma_z = \frac{E}{E_{ref}} \left(\left(\frac{M_y I_{xx}^* - M_x I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}} \right) x + \left(\frac{M_x I_{yy}^* - M_y I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}} \right) y \right). \tag{3.1.6}$$

Note that E in the above equation is the E-modulus at the position where you want to know σ_z.

3.1.4 The neutral axis

We already know that the neutral axis is a straight line that passes through the COG of the cross-section. What we don't know, is its orientation. We define α as the clockwise angle between the x-axis and the neutral axis. (So if the neutral axis is pointing 30° upwards, then $\alpha = -30°$.)

Let's find α. We know that for every point on the neutral axis x_{na}, y_{na} we have $\sigma_z = 0$. We can insert this into the previously derived equation for σ_z. We then find that

$$\frac{y_{na}}{x_{na}} = -\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}. \tag{3.1.7}$$

We can also see that $\tan \alpha = -y_{na}/x_{na}$. It follows that

$$\alpha = \arctan \left(\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}} \right). \tag{3.1.8}$$

We haven't considered the case where the beam consists of multiple materials. However, that case works exactly the same. Just add stars (*) to the I_{xx}, I_{yy} and I_{xy} in the above equation.

3.2 Shear Forces and Thin-Walled Beams

In aerospace engineering, thin-walled beams often occur. Just think of stringers, stiffeners, or even whole fuselages. How do those beams cope with shear stresses? Let's see if we can find that out.

3.2.1 Conditions for thin-walled beams

Let's examine a thin-walled beam (a beam with very small thickness). Its cross-section is just a curving line with thickness t. It can be either a closed curve (giving a **closed section beam**) or an open curve (resulting in an **open section beam**).

A shear force S is acting on our beam. We can split this force S up in a part S_x (pointing in the x-direction) and a part S_y (pointing in the y-direction). This shear force causes stresses in the beam. First of all, there is the stress in z-direction σ_z. There are also stresses in x and y-direction. However, this time we don't write them as such. Instead, we only consider the so-called **hoop stress** σ_s. This is the stress in circumferential direction (so the stress along the curve). Similarly, we only deal with one shear stress, being $\tau_{zs} = \tau_{sz} = \tau$. So the only stresses we are considering are σ_z, σ_s and τ.

We're almost ready to examine stresses in the beam. But first we need to make another definition. The **shear flow** q is defined as $q = \tau t$. Now it's time to derive the equilibrium equations for our beam. We find that

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial q}{\partial z} + t\frac{\partial \sigma_s}{\partial s} = 0. \tag{3.2.1}$$

We can also examine the displacements. The displacement of a point in z-direction is denoted by w. There are also the displacement in circumferential (tangential) direction v_t and the displacement in normal direction v_n. Corresponding to these displacements are the strains ε_z, ε_s and γ. The strain ε_s isn't important, so we ignore that one. The relations for the remaining strains are

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \text{and} \qquad \gamma = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z}. \tag{3.2.2}$$

3.2.2 Deriving an equation for the shear flow

Let's see if we can find the shear flow q caused by the shear forces S_x and S_y. In equation (3.2.1) we saw q. However, we also saw $\partial\sigma_z/\partial z$. Let's examine this σ_z a bit closer. What causes it?

The shear force S_x causes a bending moment M_y. Similarly, S_y causes M_x. These bending moments then cause the stress σ_z. From basic mechanics we know that $S_x = \partial M_y/\partial z$ and $S_y = \partial M_x/\partial z$. If we apply this to the bending equation (3.1.2), we find that

$$\frac{\partial \sigma_z}{\partial z} = \left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x + \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y. \tag{3.2.3}$$

By inserting this relation into the equilibrium equation (3.2.1), and by integrating, we find that

$$q(s) - q_0 = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx\,ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty\,ds. \tag{3.2.4}$$

Here s is the (counterclockwise) distance along the cross-section, from some point 0 with shear flow q_0. This expression for $q(s)$ is quite important, so keep it in mind.

3.2.3 Finding the shear center

When we apply a shear force S somewhere on the cross-section, then the beam will most likely twist. There is only one point where we can apply S such that the beam does not twist. This point is called the **shear center**, and has coordinates ξ_s, η_s. How do we find the position of this shear center? Let's look at that now. (We will only discuss how to find the x-coordinate ξ_s, since finding the y-coordinate η_s goes similar.)

To find ξ_s, we assume a certain position where the shear force S_y applies. If S_y is actually acting on the shear center, then the rate of twist $d\theta/dz$ must be zero. This is the condition we need to calculate the shear flow $q(s)$.

We can now evaluate moments about any point. The moment caused by the force S_y should then be equal to the moment caused by the shear flow. From this the position of S_y (and thus also ξ_s) can be derived.

You may be wondering, how do we find the moment caused by the shear flow? To do that, we replace the shear flow by forces.

If the cross-section consists of straight lines, we can split it up in parts. For every part i of the cross-section, we can evaluate the integral

$$F_{q_i} = \int_0^{s_i} q(s_i)\, ds_i. \tag{3.2.5}$$

This F_{q_i} is then the resultant force of the shear flow in part i. Once we have replaced the shear flow by the forces F_{q_i}, it isn't hard to calculate the moment anymore.

Things are slightly more difficult if the cross-section is curved. Splitting the cross-section up in parts isn't possible anymore. However, we can also find the moment caused by the shear forces directly. Let's suppose we take moments about some point B. We then have

$$M_q = \int_0^{s} q(s) p\, ds, \tag{3.2.6}$$

where the variable p is the shortest distance between point B and the line tangential to the part ds of the cross-section.

3.2.4 Shear flow

Let's suppose we have an open section beam. We can now find $q(s)$ quite easily. Since the cross-section is not a closed curve, it must have two edges. At those two edges the shear flow q is zero. We can now apply equation (3.2.4). If we take one of the edges as point 0, we have $q_0 = 0$. Since we also know the shape of our cross-section, we can solve for $q(s)$. And from this we can find the shear stress τ. Sounds simple, doesn't it?

There is one small addition we have to make though. When you apply a shear stress S to a beam, it can also twist. (Like it does when it is subject to torsion.) Open section beams can't support twist. So to prevent them from twisting, you must apply the shear force S in the shear center. Then the above method works. And luckily, we already know how find this shear center.

3.2.5 Shear of closed section beams

Now let's look at closed section beams. This time we run into a problem. There isn't any point 0 for which we know the shear flow q_0. To solve this problem, we first rewrite the shear flow $q(s)$ as

$$q(s) = q_b + q_0, \qquad \text{where } q_b = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx\, ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty\, ds. \tag{3.2.7}$$

We now examine the rate of twist $d\theta/dz$. It can be shown that

$$2A\frac{d\theta}{dz} = \oint \frac{q(s)}{Gt} ds = \oint \frac{q_b + q_0}{Gt} ds, \tag{3.2.8}$$

where A is the area enclosed by the cross-section. The integral $\oint$ means we integrate over the entire curved cross-section. Solving the above equation for q_0 gives

$$q_0 = \frac{2A\frac{d\theta}{dz} - \oint \frac{q_b}{Gt} ds}{\oint \frac{1}{Gt} ds}. \tag{3.2.9}$$

Often the above integral can be simplified. If the shear force S acts in the shear center, then the rate of twist $d\theta/dz$ is zero. It also often occurs that G or t (or both) are constant. In both cases the above equation simplifies greatly.

3.3 Torsion and Thin-Walled Beams

Previously we saw that a shear force can cause twist in beams. Torsion causes twist as well. How do thin-walled beams react to pure torsion? Let's find that out.

3.3.1 The center of twist

Let's suppose we have a thin-walled beam (open section or closed section). We can apply a torsion T to both its sides. The beam will then twist by an angle θ. Every point of the beam will have a displacement u in x-direction and v in y-direction.

Now comes the interesting part. The beam will always twist in such a way, that it appears to be rotating about some point R. This point R is called the **center of twist**. If u, v and θ are known for the cross-section, then its position x_R, y_R can be found using

$$x_R = -\frac{dv/dz}{d\theta/dz} \qquad \text{and} \qquad y_R = \frac{du/dz}{d\theta/dz}. \tag{3.3.1}$$

There is one surprising thing though. If the beam is only subject to torsion, then the center of twist is equal to the shear center! So if we know the shear center, we also know the center of twist when the beam is subject to torsion.

3.3.2 Torsion of closed section beams

Now let's look at a closed section beam. Since we only apply torsion, no direct stresses are present. This reduces the equilibrium equations to $\partial q/\partial z = 0$ and $\partial q/\partial s = 0$. This means that q is constant everywhere. It now follows that the torsion T is

$$T = 2Aq, \tag{3.3.2}$$

with A still the area enclosed by the cross-section. The above equation is known as the **Bredt-Bahto formula**.

What about displacements? Well, it can be shown that both θ, u and v vary linearly with z. So the rate of twist $d\theta/dz$ is constant. And the nice part is, we even got an equation for $d\theta/dz$. This equation is

$$\frac{d\theta}{dz} = \frac{q}{2A} \oint \frac{1}{Gt} ds = \frac{T}{4A^2} \oint \frac{1}{Gt} ds. \tag{3.3.3}$$

3.3.3 Warping in closed section beams

When a beam twists, there is usually also warping (meaning $w \neq 0$). It can be shown that the warping w stays constant for different z. However, within a cross-section the warping w is generally not constant. But, calculating it requires a couple of difficult integrations, so we won't elaborate on it further in this summary.

However, there is one important rule you do need to know. Let's define p_R as the shortest distance between the center of twist R, and the line tangential to some part ds of the beam. If we have

$$p_R G t = \text{constant}, \tag{3.3.4}$$

then the beam does not warp under pure torsion. Such kind of beams are known as **Neuber beams**. Examples are circular beams of constant thickness and triangular beams of constant thickness. But there are plenty more Neuber beams.

3.3.4 Torsion of open section beams

We have previously said that open section beams can't take torsion. This wasn't entirely true. They can take a bit of torsion. However, the stresses and deformations are, in this case, usually quite big.

When a closed-section beam is subject to torsion, the shear flow can flow all around the cross-section. We saw that in this case the shear flow q was constant. Along the thickness, also the shear stress τ_{zs} was constant. However, this doesn't work for open section beams. Instead, for open section beams, $q = 0$ everywhere. But now the shear stress τ_{zs} varies (linearly) along the thickness of the cross-section.

Let's examine a small piece ds of the cross-section. Let's look at the line in the middle of this piece. We call n the distance from this line. It can now be shown that the shear stress τ_{zs} varies according to

$$\tau_{zs} = \frac{2n}{J}T = 2Gn\frac{d\theta}{dz}. \tag{3.3.5}$$

The maximum shear stress occurs at maximum n. So this occurs at the edge of the cross-section, where $n = t/2$. By the way, the torsion constant J can be found using

$$J = \frac{1}{3}\int t^3\,ds, \qquad \text{where also} \qquad T = GJ\frac{d\theta}{dz}. \tag{3.3.6}$$

3.4 Combined open and closed section beams

Previously we have only considered beams that were either open or closed. But what do we do if a beam has both an open and a closed part?

3.4.1 Shear

Let's suppose we have a thin-walled beam which is both open and closed. On it is acting a shear force S, acting in the shear center. To find the shear flow q in the beam, we can still use the known equation

$$q(s) - q_0 = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s tx\,ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s ty\,ds. \tag{3.4.1}$$

However, we will often have to evaluate this equation multiple times, for different parts. And every time, a new value q_0 shows up. But, if we are clever, we choose point 0 such that every time $q_0 = 0$. We can then use the above equation to find the shear force $q(s)$ at every point in the beam.

You might be wondering, how do you know where $q = 0$? Well, we always have $q = 0$ at the end of a cross-section. We can often also deduce points of $q = 0$ from symmetry. Let's look at the line L through which the force S is acting. It often occurs that L is an axis of symmetry of the cross-section. In this case every point on L generally also has $q = 0$.

3.4.2 Torsion

Now suppose we have a thin-walled beam that is subject to torsion. In this case it is often wise to first find the total torsional rigidity GJ_{tot}. To find this, we first need to find the torsional rigidity GJ_i of every sub-part i. We can find this using

$$GJ_i = \frac{4A^2G}{\oint \frac{1}{t}ds} \quad \text{for closed sections, and} \quad GJ_i = \frac{G}{3}\int t^3\,ds \quad \text{for open sections.} \tag{3.4.2}$$

To find the total torsional rigidity GJ_{tot}, just add up all the separate torsional rigidities GJ_i. The torsional rigidity of closed sections is generally much bigger than that of open sections. So often the value GJ of open sections can be neglected.

The rate of twist now follows from

$$\frac{d\theta}{dz} = \frac{T}{GJ_{tot}}. \tag{3.4.3}$$

To find the shear flow (and thus also the shear stress) for a closed section i, we can use

$$q = \frac{GJ_i}{2A}\frac{d\theta}{dz}, \tag{3.4.4}$$

where GJ_i is the torsional rigidity of that closed section i. To find the shear stress in an open section, we still have

$$\tau = \frac{2n}{J}T = 2Gn\frac{d\theta}{dz}. \tag{3.4.5}$$

4. Application of Theory to Aircraft

4.1 Structural Idealization

It's time to apply some of our theory into practice. Let's look at airplanes. In an airplane are many parts that have a rather complicated shape. Let's find a way to examine them.

4.1.1 Simplifying a shape

Let's examine an aircraft fuselage. It often consists of a shell with a couple of stringers. Altogether, we have got a complicated shape. We need to make some assumptions and simplifications, such that we can evaluate it.

First we do something about the stringers. We replace them by concentrations of area (so-called **booms**). These booms have the same cross-sectional area as the original stringer.

Now let's look at a piece of fuselage skin with width b and (effective) thickness t_D. The normal stress σ varies along this piece. On the left side is a stress σ_1 and on the right a stress σ_2. We want to replace this piece of skin by two booms at the edges. This should be done, such that the effects are the same. So the two booms should take the same force and the same moment as the piece of skin. From this, we can derive that these two booms have area B_1 (left) and B_2 (right), where

$$B_1 = \frac{t_D b}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right) \qquad \text{and} \qquad B_2 = \frac{t_D b}{6}\left(2 + \frac{\sigma_1}{\sigma_2}\right). \tag{4.1.1}$$

Since we have replaced the skin by two booms, the remaining effective thickness t_D of the skin is 0.

Let's take a closer look at the ratio σ_2/σ_1 in the above equation. This ratio depends on the loads which our fuselage is subject to. And thus so do B_1 and B_2. This means that if we load our fuselage differently, our booms will have different areas.

We now make an important assumption. We assume that the booms take all the normal stresses, while the skin takes all the shear stresses. This makes our analysis a lot simpler. To examine normal stresses, we only have to evaluate a set of points with known areas. Also examining shear stress is a bit easier now.

4.1.2 Normal stress

In our new fuselage, how do we calculate the normal stress? For that, we can still use the general equation we derived for bending. Let's just repeat it. It was

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)y = \left(\frac{I_{yy}y - I_{xy}x}{I_{xx}I_{yy} - I_{xy}^2}\right)M_x + \left(\frac{I_{xx}x - I_{xy}y}{I_{xx}I_{yy} - I_{xy}^2}\right)M_y. \tag{4.1.2}$$

Finding the moments of inertia is now quite easy. For the booms $B_1, \ldots, B_n$, just use

$$I_{xx} = \sum_{i=1}^{n} y_i^2 B_i, \qquad I_{yy} = \sum_{i=1}^{n} x_i^2 B_i \quad \text{and} \quad I_{xy} = \sum_{i=1}^{n} x_i y_i B_i. \tag{4.1.3}$$

4.1.3 Shear flow

Let's examine a beam subject to shear stresses S_x and S_y. We have assumed that the skin takes all the shear stresses. We stick to this assumption. However, it turns out that the booms do effect the shear stress. Let's suppose we have two pieces of skin with shear flow q_1 and q_2. In between these pieces is a boom with area B_r, coordinates x_r, y_r and direct stress σ_z. It can now be shown that

$$q_2 - q_1 = -\frac{\partial \sigma_z}{\partial z} B_r = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) B_r x_r - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) B_r y_r. \tag{4.1.4}$$

From this, we can derive that the shear stress $q(s) = q_b + q_0$ is given by

$$q(s) = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x\, ds + \sum_{i=1}^{n} B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y\, ds + \sum_{i=1}^{n} B_r y_r\right) + q_0. \tag{4.1.5}$$

The two sums in the above equation sum over all the booms between point 0 and s. Also note that if we have replaced our skin by booms as well, then the remaining effective thickness t_D is zero. This would mean that the integrals in the above equation vanish.

For open section beams, we should take point 0 as some point where $q_0 = 0$, just like we're used to. For closed section beams, we have to find the value for q_0. This can still be done using known methods. Just take moments about some point. The moment caused by the shear stresses should then be equal to the moment caused by the shear force S.

4.2 Tapered Sections

We have previously always assumed that the cross-section of a beam stays constant for varying z. What happens if it doesn't? Let's find that out.

4.2.1 Tapered wing spars

Let's consider an I-shaped wing spar, whose height h changes. We assume that the web takes all the shear stress. Similarly, the flanges take all the direct stresses. We thus replace these flanges by two booms with areas B_1 (top) and B_2 (bottom).

When the beam is subject to a shear force S_y (and thus also a bending moment M_x), the flanges will be subject to forces P_1 and P_2. However, only the components in z-direction ($P_{z,1}$ and $P_{z,2}$) counteract the bending moment M_x. This goes according to $P_{z,1} = \sigma_1 B_1$. (σ_1 can be found by using the bending equation.)

But now comes the surprising part, the part of P_1 acting in y-direction (being $P_{y,1}$) effects the shear flow in the web. In fact, the **effective shear force** $S_{y,w}$ acting on the web can be found using

$$S_{y,w} = S_y - P_{y,1} - P_{y,2} = S_y - P_{z,1}\frac{\delta y_1}{\delta z} - P_{z,2}\frac{\delta y_2}{\delta z}. \tag{4.2.1}$$

Here the parameters y_1 and y_2 denote the y-coordinate of the flanges. When using the above equation, special care should be payed to the direction of the forces $P_{y,1}$ and $P_{y,2}$. Using the effective shear force, the shear flow in the web can be calculated, exactly in the way you are normally used to.

It is sometimes sligthly difficult to see whether the effective shear stress increases, or whether it decreases. There is a rule of thumb for that. If the cross-section is widening, then the effective shear stress is usually lower than the actual shear stress. And similarly, if the cross-section is getting smaller, then the effective shear stress is higher than the actual shear stress.

4.2.2 General shapes

In the previous paragraph we considered a vertical web, with two booms at the ends. Now let's consider a general (thin-walled) shape, consisting of a skin with booms. Every boom r with coordinates x_r, y_r has an internal force P_r, with components $P_{x,r}$, $P_{y,r}$ and $P_{z,r}$. These relate to each other according to

$$P_{x,r} = P_{z,r}\frac{\delta x_r}{\delta z}, \qquad P_{y,r} = P_{z,r}\frac{\delta y_r}{\delta z} \qquad \text{and also} \qquad P_r = P_{z,r}\frac{\sqrt{\delta x_r^2 + \delta y_r^2 + \delta z_r^2}}{\delta z}. \tag{4.2.2}$$

The effective shear forces in x and y-direction can now be found using

$$S_{x,w} = S_x - \sum_{r=1}^{n} P_{z,r}\frac{\delta x_r}{\delta z} \qquad \text{and} \qquad S_{y,w} = S_y - \sum_{r=1}^{n} P_{z,r}\frac{\delta y_r}{\delta z}. \tag{4.2.3}$$

Again, the rest of the analysis goes exactly as you're used to. There's one slight exception though. Suppose you have a closed cross-section. Then at some time you need find a function $q(s) = q_b + q_0$. To find q_0, you can take moments about a certain point. These moments should then be equal to the moment caused by S_x and S_y. However, this time the moments caused by $P_{x,r}$ and $P_{y,r}$ should also be taken into account. Do not forget that.

4.3 Aircraft Wings

Aircaft wings often have a rather characteristic shape. Examining wings is therefore an art itself — An art we will delve into now.

4.3.1 The wing shape

Let's consider the cross-section of a wing. It consists of the top and bottom skin of the wing, plus several vertical spars. The wing thus consists of a number N "boxes." Due to the (sometimes large) amount of spars, we have a high amount of redundancy. That is why wings are difficult to analyze.

Soon we will be putting torsion and shear forces on the wing. This causes a certain amount of counter-clockwise (assumed) shear flow q_R in box R. In this case the top of box R has a shear flow q_R, pointed to the left. The bottom has q_R pointed to the right.

But what about the shear stress in the spar to the right of box R? (We call it spar R.) Box R causes a shear flow q_R upward. However, box $R+1$ causes a shear flow q_{R+1} downward. So the shear flow in spar R is $q_R - q_{R+1}$ (upward). In this way the shear flow in every spar can be determined.

4.3.2 Torsion

Let's subject a wing to a torsion T. The torsion T will be divided over the several boxes. Every box R now supports an amount of torsion T_R, where

$$T_R = 2A_R q_R, \qquad \text{with also} \qquad \sum_{R=1}^{N} T_R = T. \tag{4.3.1}$$

The area A_R is the area enclosed by box R. There is just one slight problem. In the above equation, we don't know q_R, nor T_R. So we have $N+1$ equations, but $2N$ unknowns. We need more equations.

We now assume that the rate of twist $d\theta/dz$ of the boxes are all equal. We can find the rate of twist of box R using

$$\frac{d\theta}{dz} = \frac{1}{2A_R G}\oint_R \frac{q}{t}\,ds, \tag{4.3.2}$$

where we integrate around the entire box. (Note that in this case q is not always q_R. Previously we saw that the shear flow in spar R isn't q_R.) Although we have one extra unknown (being $d\theta/dz$), we have N extra equations. So we can solve our system of $2N+1$ equations.

Sometimes the spars have different shear moduli G. In this case we set a reference modulus G_{ref} and define the **modulus-weighted thickness** t^*, such that

$$t^* = \frac{G}{G_{ref}} t, \qquad \text{after which we use} \qquad \frac{d\theta}{dz} = \frac{1}{2A_R G_{ref}} \oint \frac{q}{t^*} ds. \tag{4.3.3}$$

4.3.3 Shear

Now let's apply a shear stress S to our wing. This makes things a bit more complicated. The shear stress $q_R(s)$ in every box is now given by $q_R(s) = q_{b,R} + q_{0,R}$. The value of $q_{b,R}$ around the box can be determined from

$$q_{b,R} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x \, ds + \sum_{i=1}^n B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y \, ds + \sum_{i=1}^n B_r y_r\right). \tag{4.3.4}$$

So $q_{b,R}$ is known. However, $q_{0,R}$ is not. To find it, we once more look at the rate of twist $d\theta/dz$, which is (assumed) equal for all boxes. It is now given by

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} \oint_R \frac{q}{t} ds = \frac{1}{2A_R G} \oint_R \frac{q_{b,R} + q_{0,R}}{t} ds \tag{4.3.5}$$

This gives us N extra equations, but also one extra unknown. We thus need one more equation. We now look at moments. All the shear flows together cause a moment (about a certain point). This moment must be equal to the moment caused by the shear force S (about that same point).

4.3.4 Cut-outs in wings

The last subject in this summary is a rather difficult problem. Let's look at a simple wing box, consisting of two spars with two pieces of skin in between. (If we look at the idealized cross-section, we simply see a rectangle, with booms at the corners.) If we look at the 3D wing box, we can split it up in three identical parts. Now we make a cut-out in the middle part (part 2). We remove the entire bottom skin of this part. This severely weakens the structure. We can now ask ourselves, what will happen if the wing box is subjected to loads?

This is, in fact, quite a difficult problem. Many things happen at the same time. In part 1, the shear forces are gradually being transferred (as normal forces) into the spar flanges (the booms). This causes normal forces P in the flanges.

Let's now look at the cross-section between parts 1 and 2. At this cross-section, the torsion has "translated" itself into two shear forces S. These forces are positioned at (and also supported by) the spars. They result in the same moment as the torsion T. Using this fact, you can find the magnitude of S.

The shear forces S cause certain shear flows q in the spars of part 2. These shear flows change the magnitude of the normal forces P in the flanges. By evaluating moments about certain points, the magnitude of these forces P can be determined at certain positions. Once the normal forces P are known, also the shear flows in part 1 can be found.

The above plan of approach might sound a bit short. However, this is a problem of which the solution can't be explained briefly in a summary. For a clear explanation of the problem, you would have to consult a book on this subject.