

# Calculus Formula and Data Overview

## Calculus - Period 1

### Differentiation and Integration

#### Chain Rule:

$$(f(g(x)))' = f'(g(x))g'(x) \quad (1)$$

#### Implicit Differentiation:

When applying implicit differentiation for a function  $y$  of  $x$ , every term with a  $y$  should, after normal differentiation (often involving the product rule), be multiplied by  $y'$  because of the chain rule. After that, the equation should be solved for  $y'$ .

#### Linear Approximations:

$$f(x) - f(a) \approx f'(a)(x - a) \quad (2)$$

#### Mean Value Theorem:

If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there is a  $c$  in  $(a, b)$  such that:

$$f(b) - f(a) = f'(c)(b - a) \quad (3)$$

#### Integration:

$$\int_a^b f(x)dx = F(b) - F(a) \quad (4)$$

Where  $F$  is any antiderivative/primitive function of  $f$ , that is,  $F' = f$ .

#### Substitution Rule:

If  $u = g(x)$  then:

$$\int f(g(x))g'(x)dx = \int f(u)du \quad (5)$$

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du \quad (6)$$

#### Integration By Parts:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \quad (7)$$

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx \quad (8)$$

#### Improper Integrals:

$$\int_1^{\infty} \frac{1}{x^p}dx \quad (9)$$

This function is convergent for  $p > 1$  and divergent for  $p \leq 1$ .

#### Comparison Theorem:

If  $f(x) \geq g(x) \geq 0$  for  $x \geq a$  then:

- If  $\int_a^{\infty} f(x)dx$  is convergent, then  $\int_a^{\infty} g(x)dx$  is convergent.
- If  $\int_a^{\infty} g(x)dx$  is divergent, then  $\int_a^{\infty} f(x)dx$  is divergent.

### Complex Numbers

#### Complex Number Notations:

$$i^2 = -1 \quad (10)$$

$$z = a + bi = r(\cos \theta + i \sin \theta) = re^{i\theta} \quad (11)$$

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta \quad (12)$$

$$|z| = r = \sqrt{a^2 + b^2} \quad (13)$$

$$\theta = \arctan \frac{b}{a} \quad \text{or} \quad \theta = \arctan \frac{b}{a} + \pi$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (14)$$

#### Complex Number Calculation:

$$(a + bi) + (c + di) = (a + c) + (b + d)i \quad (15)$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \quad (16)$$

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

#### Complex Conjugates:

$$z = a + bi \Rightarrow \bar{z} = a - bi \quad (17)$$

$$\overline{z + w} = \bar{z} + \bar{w}$$

$$\overline{z\bar{w}} = \bar{z}w$$

$$\overline{z^n} = \bar{z}^n$$

$$z\bar{z} = |z|^2$$

### Differential Equations

#### Separable Differential Equations:

$$\text{Form : } \frac{dy}{dx} = y' = P(x)Q(y)$$

$$\text{Solution : } \int \frac{1}{Q(y)} dy = \int P(x) dx \quad (19)$$

### First-Order Differential Equations

$$\text{Form : } y' + P(x)y = Q(x)$$

Let  $\Upsilon(x)$  be any integral of  $P(x)$ . Solution is:

$$y = e^{-\Upsilon(x)} \left( \int e^{\Upsilon(x)} Q(x) dx + C \right) \quad (20)$$

### Homogeneous second-order linear differential equations:

$$\text{Form : } ay'' + by' + cy = 0$$

- $b^2 - 4ac > 0$ :

Define  $r$  such that  $ar^2 + br + c = 0$ .

$$\text{Solution : } y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \quad (21)$$

- $b^2 - 4ac = 0$ :

Define  $r$  such that  $ar^2 + br + c = 0$ .

$$\text{Solution : } y = c_1 e^{rx} + c_2 x e^{rx} \quad (22)$$

- $b^2 - 4ac < 0$ :

Define  $\alpha = -\frac{b}{2a}$  and  $\beta = \frac{\sqrt{4ac - b^2}}{2a}$ . Solution:

$$y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x) \quad (23)$$

### Nonhomogeneous second-order linear differential equations:

$$\text{Form : } ay'' + by' + cy = P(x)$$

First solve  $ay_c'' + by_c' + cy_c = 0$ . Then use an auxiliary equation to find one solution  $y_p$  for the given differential equation. The solution is:

$$y = y_c + y_p \quad (24)$$

## Calculus - Period 2

### Testing Series

#### Convergence/Divergence:

Suppose  $a$  is a series of numbers  $a_1, a_2, \dots$ , and  $s_n = \sum_{k=1}^n a_k$ . A series  $s_n$  converges if  $\lim_{n \rightarrow \infty} s_n = s$  exists as a real number. The limit  $s$  is then called the sum of series  $a$ . If  $s$  doesn't exist as a finite

number, the series is divergent. Be careful not to confuse the series  $a_n$  with the series  $\sum a_n = s$ .

#### Monotonic Sequence Theorem

If a sequence is either increasing ( $a_{n+1} > a_n$  for all  $n \geq 1$ ) or decreasing ( $a_{n+1} < a_n$  for all  $n \geq 1$ ), it is called a monotonic sequence. If there are  $c_1$  and  $c_2$  such that  $c_1 < a_n < c_2$  for all  $n \geq 1$ , it is called bounded. Every bounded monotonic sequence is convergent.

#### Test for divergence:

If  $\lim_{n \rightarrow \infty} a_n$  does not exist, or if  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then the series  $s_n$  is divergent.

#### Integral test:

If  $f$  is a continuous positive decreasing function on  $[1, \infty)$  and  $a_n = f(n)$  for integer  $n$ , then the series  $s_n$  is convergent if, and only if, the integral  $\int_1^{\infty} f(x) dx$  is convergent.

#### Comparison test:

Suppose  $a_n$  and  $b_n$  are series with positive terms and  $a_n \leq b_n$  for all  $n$ , then:

- If  $\sum b_n$  is convergent, then  $\sum a_n$  is convergent.
- If  $\sum a_n$  is divergent, then  $\sum b_n$  is divergent.

#### Limit comparison test:

Suppose  $a_n$  and  $b_n$  are series with positive terms. If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$  and  $0 < c < \infty$ , then either both series are convergent or divergent.

#### Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \dots$$

satisfies  $a_{n+1} \leq a_n$  for all  $n$  and  $\lim_{n \rightarrow \infty} a_n = 0$ , then the series is convergent.

#### Absolute convergence:

A series  $\sum a_n$  is called absolutely convergent if the series  $\sum |a_n|$  is convergent. A series  $\sum a_n$  is called conditionally convergent if it is convergent but not absolutely convergent. If a series  $\sum a_n$  is absolutely convergent, then it is convergent.

#### Ratio test:

- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then the series  $\sum a_n$  is absolutely convergent.

- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ , then the series  $\sum a_n$  is divergent.

**Root test:**

- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ , then the series  $\sum a_n$  is absolutely convergent.
- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$ , then the series  $\sum a_n$  is divergent.

## Power Series

**Radius of convergence:**

Power series are written as

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad (25)$$

where  $x$  is a variable and the  $c_n$ 's are constant coefficients of the series. When tested for convergence, there are only three possibilities:

- The series converges only if  $x = a$ . ( $R = 0$ )
- The series converges for all  $x$ . ( $R = \infty$ )
- The series converges for  $|x - a| < R$  and diverges for  $|x - a| > R$ . For  $|x - a| = R$  other means must point out whether convergence or divergence occurs.

The number  $R$  is called the radius of convergence, and can often be found using the ratio test.

**Differentiation and integration:**

Differentiation and integration of power functions is possible in the interval  $(a - R, a + R)$ , where the function does not diverge. It goes as follows:

$$\left( \sum_{n=0}^{\infty} c_n(x-a)^n \right)' = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1} \quad (26)$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \quad (27)$$

**Representation of functions as power series:**

The first way to represent functions as power series is simple, but doesn't always work. To find the representation of  $f(x)$ , first find a function  $g(x)$  such that  $f(x) = ax^b \frac{1}{1-g(x)}$ , where  $a$  and  $b$  are constants. The power series is then equal to:

$$f(x) = \sum_{n=0}^{\infty} a \cdot g(x)^{n+b} \quad (28)$$

The second way to represent functions as power series goes as follows. Let  $f^{(n)}(x)$  be the  $n$ 'th derivative of  $f(x)$ . Supposing the function  $f(x)$  has a power series (this sometimes still has to be proven), the following function must be true:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad (29)$$

This representation is called the Taylor series of  $f(x)$  at  $a$ . For the special case that  $a = 0$ , it is called the Maclaurin series.

**Binomial series:**

If  $k$  is any real number and  $|x| < 1$ , the power function representation of  $(1+x)^k$  is:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \quad (30)$$

$$\text{where } \binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} \quad (31)$$

for  $n \geq 1$ , and  $\binom{k}{0} = 1$ .

## Vectors

**Notation:**

A vector  $\mathbf{a}$  is often written as:

$$\mathbf{a} = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \quad (32)$$

Where  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are unit vectors.

**Vector length:**

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \quad (33)$$

**Vector addition and subtraction:**

$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\mathbf{i} + (a_y + b_y)\mathbf{j} + (a_z + b_z)\mathbf{k} \quad (34)$$

$$\mathbf{a} - \mathbf{b} = (a_x - b_x)\mathbf{i} + (a_y - b_y)\mathbf{j} + (a_z - b_z)\mathbf{k} \quad (35)$$

**Dot product:**

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \quad (36)$$

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \quad (37)$$

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \quad (38)$$

**Cross product:**

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad (39)$$

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \quad (40)$$

## Vector Functions:

**Notation:**

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad (41)$$

**Differentiation and integration:**

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k} \quad (42)$$

$$\mathbf{R}(t) = F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k} + \mathbf{D} \quad (43)$$

**Function dependant unit vectors:**

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad (44)$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \quad (45)$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \quad (46)$$

**Trajectory length:**

$$ds(t) = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = |\mathbf{r}'(t)|dt \quad (47)$$

$$s(t) = \int_a^t |\mathbf{r}'(t)|dt \quad (48)$$

**Trajectory velocity and acceleration:**

$$|\mathbf{v}(t)| = \frac{ds(t)}{dt} = |\mathbf{r}'(t)| \quad (49)$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) \quad (50)$$

**Trajectory curvature:**

$$\kappa(t) = \left| \frac{d\mathbf{T}(t)}{ds(t)} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \quad (51)$$

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \quad (52)$$

**Expressing acceleration in unit vectors:**

$$\mathbf{a}(t) = |\mathbf{v}(t)|'\mathbf{T}(t) + \kappa|\mathbf{v}(t)|^2\mathbf{N}(t) \quad (53)$$

$$\mathbf{a}(t) = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}\mathbf{T}(t) + \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\mathbf{N}(t) \quad (54)$$

## Calculus - Period 3

### Functions of Multiple Variables

**Definitions:**

The domain  $D$  is the set  $(x, y)$  for which  $f(x, y)$  exists. The range is the set of values  $z$  for which there are  $x, y$  such that  $z = f(x, y)$ . The level curves are the curves with equations  $f(x, y) = k$  where  $k$  is a constant.

**Checking for Limits:**

If  $f(x, y) \rightarrow L_1$  as  $(x, y) \rightarrow (a, b)$  along a path  $C_1$  and  $f(x, y) \rightarrow L_2$  as  $(x, y) \rightarrow (a, b)$  along a path  $C_2$ , where  $L_1 \neq L_2$  then  $\lim_{(x,y) \rightarrow (a,b)} f(x, y)$  does not exist. Also  $f$  is continuous at  $(a, b)$  if  $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

**Partial Derivatives:**

The partial derivative of  $f$  with respect to  $x$  at  $(a, b)$  is:

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b) \quad (55)$$

In words, to find  $f_x$ , regard  $y$  as constant and differentiate  $f(x, y)$  with respect to  $x$ .  $f_y$  is defined similarly. If  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , then  $f_{xy} = f_{yx}$ .

**Tangent Planes:**

For points close to  $z_0 = f(x_0, y_0)$  the curve of  $f(x, y)$  can be approximated by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad (56)$$

The plane described by this equation is the plane tangent to the curve of  $f(x, y)$  at  $(x_0, y_0)$ .

**Differentials:**

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad (57)$$

If  $z = f(x, y)$ ,  $x = g(s, t)$  and  $y = h(s, t)$  then:

$$\frac{dz}{ds} = \frac{\partial z}{\partial x} \frac{dx}{ds} + \frac{\partial z}{\partial y} \frac{dy}{ds} \quad (58)$$

**Directional Derivatives:**

The directional derivative of  $f$  at  $(x_0, y_0)$  in the direction of a unit vector (meaning,  $|\mathbf{u}| = 1$ )  $\mathbf{u} = \langle a, b \rangle$  is:

$$D_{\mathbf{u}}f(x_0, y_0) = f_x(x, y)a + f_y(x, y)b = \nabla f \cdot \mathbf{u} \quad (59)$$

$$\mathbf{grad} f = \nabla f = \langle f_x(x, y), f_y(x, y) \rangle \quad (60)$$

The maximum value of  $D_{\mathbf{u}}f(x, y)$  is  $|\nabla f(x, y)|$  and occurs when the vector  $\mathbf{u} = \langle a, b \rangle$  has the same direction as  $\nabla f(x, y)$ .

**Local Maxima and Minima:**

If  $f$  has a local maximum or minimum at  $(a, b)$ , then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ . If  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$  then  $(a, b)$  is a critical point. If  $(a, b)$  is a critical point, then let  $D$  be defined as:

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2 \quad (61)$$

- If  $D > 0$  then:
  - If  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a minimum.
  - If  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a maximum.
- If  $D < 0$ , then  $f(a, b)$  is a saddle point.

**Absolute Maxima and Minima:**

To find the absolute maximum and minimum values of a continuous function  $f$  on a closed bounded set  $D$ , first find the values of  $f$  at the critical points of  $f$  in  $D$ . Then find the extreme values of  $f$  on the boundary of  $D$ . The largest of these values is the absolute maximum. The lowest is the minimum.

**Multiple Integrals****Integrals over Rectangles:**

If  $R$  is the rectangle such that  $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ , then:

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx \quad (62)$$

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy \quad (63)$$

**Integrals over Regions:**

If  $D_1$  is the region such that  $D_1 = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ , then:

$$\iint_{D_1} f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (64)$$

If  $D_2$  is the region such that  $D_2 = \{(x, y) | a \leq y \leq b, h_1(y) \leq x \leq h_2(y)\}$ , then:

$$\iint_{D_2} f(x, y) dA = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy \quad (65)$$

**Integrating over Polar Coordinates**

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad (66)$$

$$x = r \cos \theta \quad y = r \sin \theta \quad (67)$$

If  $R$  is the polar rectangle such that  $R = \{(r, \theta) | 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\}$  where  $0 \leq \beta - \alpha \leq 2\pi$ , then:

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta \quad (68)$$

If  $D$  is the polar rectangle such that  $D = \{(r, \theta) | 0 \leq h_1(\theta) \leq r \leq h_2(\theta), \alpha \leq \theta \leq \beta\}$  where  $0 \leq \beta - \alpha \leq 2\pi$ , then:

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta \quad (69)$$

**Applications:**

If  $m$  is the mass, and  $\rho(x, y)$  the density, then:

$$m = \iint_D \rho(x, y) dA \quad (70)$$

The  $x$ -coordinate of the center of mass is:

$$\bar{x} = \frac{\iint_D x \rho(x, y) dA}{\iint_D \rho(x, y) dA} \quad (71)$$

The moment of inertia about the  $x$ -axis is:

$$I_x = \iint_D y^2 \rho(x, y) dA \quad (72)$$

The moment of inertia about the origin is:

$$I_0 = \iint_D (x^2 + y^2) \rho(x, y) dA = I_x + I_y \quad (73)$$

**Triple Integrals**

If  $E$  is the volume such that  $E = \{(x, y, z) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y)\}$ , then:

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x, y)}^{h_2(x, y)} f(x, y, z) dz dy dx \quad (74)$$

# Calculus - Period 4

## Three-Dimensional Integrals

### Cylindrical Coordinates:

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad (75)$$

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z \quad (76)$$

### Integrating Over Cylindrical Coordinates:

$$\int \int \int_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \dots \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} r f(r \cos \theta, r \sin \theta, z) dz dr d\theta \quad (77)$$

### Spherical Coordinates:

$$x = \rho \cos \theta \sin \phi \quad y = \rho \sin \theta \sin \phi \quad z = \rho \cos \phi \quad (78)$$

$$\rho^2 = x^2 + y^2 + z^2 \quad (79)$$

### Integrating Over Spherical Coordinates:

If  $E$  is the spherical wedge given by  $E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$ , then:

$$\int \int \int_E f(x, y, z) dV = \int_a^b \int_{\alpha}^{\beta} \int_c^d \rho^2 \sin \phi \dots \dots f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) d\rho d\theta d\phi \quad (80)$$

### Change of Variables:

The Jacobian of the transformation  $T$  given by  $x = g(u, v)$  and  $y = h(u, v)$  is:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \quad (81)$$

If the Jacobian is nonzero and the transformation is one-to-one, then:

$$\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \quad (82)$$

This method is similar to the one for triple integrals, for which the Jacobian has a bigger matrix and the change-of-variable equation has some more terms.

## Basic Vector Field Theorems

### Definitions

- A piecewise-smooth curve - A union of a finite number of smooth curves.
- A closed curve - A curve of which its terminal point coincides with its initial point.
- A simple curve - A curve that doesn't intersect itself anywhere between its endpoints.
- An open region - A region which doesn't contain any of its boundary points.
- A connected region - A region  $D$  for which any two points in  $D$  can be connected by a path that lies in  $D$ .
- A simply-connected region - A region  $D$  such that every simple closed curve in  $D$  encloses only points that are in  $D$ . It contains no holes and consists of only one piece.
- Positive orientation - The positive orientation of a simple closed curve  $C$  refers to a single counterclockwise traversal of  $C$ .

### Vector Field:

A vector field on  $\mathbb{R}^n$  is a function  $\mathbf{F}$  that assigns to each point  $(x, y)$  in an  $n$ -dimensional set an  $n$ -dimensional vector  $\mathbf{F}(x, y)$ . The gradient  $\nabla f$  is defined by:

$$\nabla f(x, y, \dots) = f_x \mathbf{i} + f_y \mathbf{j} + \dots \quad (83)$$

and is called the gradient vector field. A vector field  $\mathbf{F}$  is called a conservative vector field if it is the gradient of some scalar function.

### Line Integrals:

The line integral of  $f$  along  $C$  is:

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (84)$$

The line integral of  $f$  along  $C$  with respect to  $x$  is:

$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) \frac{dx}{dt} dt \quad (85)$$

The line integral of a vector field  $\mathbf{F}$  along  $C$  is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds \quad (86)$$

Where  $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|}$  is the unit tangent vector.

### Conservative Vector Fields:

If  $C$  is the curve given by  $\mathbf{r}(t)$  ( $a \leq t \leq b$ ), then:

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \quad (87)$$

The integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is independent of path in  $D$  if and only if  $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$  for every closed path  $C$  in  $D$ .

If  $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$  is a conservative vector field, then:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad (88)$$

Also, if  $D$  is an open simply-connected region, and if  $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ , then  $\mathbf{F}$  is conservative in  $D$ .

## Surfaces

### Parametric Surfaces:

A surface described by  $\mathbf{r}(u, v)$  is called a parametric surface.  $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$  and  $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$ . For smooth surfaces ( $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$  for every  $u$  and  $v$ ) the tangent plane is the plane that contains the tangent vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$ , and the vector  $\mathbf{r}_u \times \mathbf{r}_v$  is the normal vector to the tangent plane.

### Surface Areas:

For a parametric surface, the surface area is given by:

$$A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA \quad (89)$$

For a surface graph of  $g(x, y)$ , the surface area is given by:

$$A = \iint_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dA \quad (90)$$

### Surface Integrals:

For a parametric surface, the surface integral is given by:

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA \quad (91)$$

For a surface graph of  $g(x, y)$ , the surface integral is given by:

$$\begin{aligned} \iint_S f(x, y, z) dS &= \\ \iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} dA & \end{aligned} \quad (92)$$

### Normal Vectors:

For a parametric surface, the normal vector is given by:

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \quad (93)$$

For a surface graph of  $g(x, y)$ , the normal vector is given by:

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} \quad (94)$$

### Flux:

If  $\mathbf{F}$  is a vector field on a surface  $S$  with unit normal vector  $\mathbf{n}$ , then the surface integral of  $\mathbf{F}$  over  $S$  is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS \quad (95)$$

This integral is also called the flux of  $\mathbf{F}$  across  $S$ . For a parametric surface, the flux is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA \quad (96)$$

For a surface graph of  $g(x, y)$ , the flux is given by:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( -P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA \quad (97)$$

## Advanced Vector Field Theorems

### Curl:

If  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ , then the curl of  $\mathbf{F}$ , denoted by  $\text{curl } \mathbf{F}$  or also  $\nabla \times \mathbf{F}$ , is defined by:

$$\left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \quad (98)$$

If  $f$  is a function of three variables, then:

$$\text{curl}(\nabla f) = \mathbf{0} \quad (99)$$

This implies that if  $\mathbf{F}$  is conservative, then  $\text{curl } \mathbf{F} = \mathbf{0}$ . The converse is only true if  $\mathbf{F}$  is defined on all of  $\mathbb{R}^n$ . So if  $\mathbf{F}$  is defined on all of  $\mathbb{R}^n$  and if  $\text{curl } \mathbf{F} = \mathbf{0}$ , then  $\mathbf{F}$  is a conservative vector field.

### Divergence:

If  $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ , then the divergence of  $\mathbf{F}$ , denoted by  $\text{div } \mathbf{F}$  or also  $\nabla \cdot \mathbf{F}$ , is defined by:

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad (100)$$

If  $\mathbf{F}$  is a vector field on  $\mathbb{R}^n$ , then  $\text{div } \text{curl } \mathbf{F} = 0$ . If  $\text{div } \mathbf{F} = 0$ , then  $\mathbf{F}$  is said to be incompressible. Note that  $\text{curl } \mathbf{F}$  returns a vector field and  $\text{div } \mathbf{F}$  returns a scalar field.

**Green's Theorem:**

Let  $C$  be a positively oriented piecewise-smooth simple closed curve in the plane and  $D$  be the region bounded by  $C$ . Now:

$$\int_C P dx + Q dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad (101)$$

This can also be useful for calculating areas. To calculate an area, take functions  $P$  and  $Q$  such that  $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$  and then apply Green's theorem.

In vector form, Green's theorem can also be written as:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA \quad (102)$$

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \iint_D \text{div } \mathbf{F}(x, y) dA \quad (103)$$

**Stoke's Theorem:**

Let  $S$  be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve  $C$  with positive orientation. Let  $\mathbf{F}$  be a vector field that contains  $S$ . Then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \quad (104)$$

**The Divergence Theorem:**

Let  $E$  be a simple solid region and let  $S$  be the boundary surface of  $E$ , given with positive (outward) orientation. Let  $\mathbf{F}$  be a vector field on an open region that contains  $E$ . Then:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV \quad (105)$$