1. **(a)** A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.
(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.
(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. **(a)** From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\left\{1/2^n\right\}$
(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n=1-(0.2)^n$, so the sequence is $\{0.8,0.96,0.992,0.9984,0.99968,\dots\}$.

4. $a_n=\dfrac{n+1}{3n-1}$, so the sequence is $\left\{\dfrac{2}{2},\dfrac{3}{5},\dfrac{4}{8},\dfrac{5}{11},\dfrac{6}{14},\dots\right\}=\left\{1,\dfrac{3}{5},\dfrac{1}{2},\dfrac{5}{11},\dfrac{3}{7},\dots\right\}$.

5. $a_n=\dfrac{3(-1)^n}{\text{n!}}$, so the sequence is $\left\{\dfrac{-3}{1},\dfrac{3}{2},\dfrac{-3}{6},\dfrac{3}{24},\dfrac{-3}{120},\dots\right\}=\left\{-3,\dfrac{3}{2},-\dfrac{1}{2},\dfrac{1}{8},-\dfrac{1}{40},\dots\right\}$.

6. $a_n=2\cdot 4\cdot 6\cdot \cdot\cdot\cdot\cdot (2n)$, so the sequence is
$\{2,2\cdot 4,2\cdot 4\cdot 6,2\cdot 4\cdot 6\cdot 8,2\cdot 4\cdot 6\cdot 8\cdot 10,\dots\}=\{2,8,48,384,3840,\dots\}$.

7. $a_1=3$, $a_{n+1}=2a_n-1$. Each term is defined in terms of the preceding term.
$a_2=2a_1-1=2(3)-1=5$. $a_3=2a_2-1=2(5)-1=9$. $a_4=2a_3-1=2(9)-1=17$. $a_5=2a_4-1=2(17)-1=33$. The sequence is $\{3,5,9,17,33,\dots\}$.

8. $a_1=4$, $a_{n+1}=\dfrac{a_n}{a_n-1}$. Each term is defined in terms of the preceding term.

$a_2=\dfrac{a_1}{a_1-1}=\dfrac{4}{4-1}=\dfrac{4}{3}$. $a_3=\dfrac{a_2}{a_2-1}=\dfrac{4/3}{\frac{4}{3}-1}=\dfrac{4/3}{1/3}=4$. Since $a_3=a_1$, we can see that the terms of the sequence will alternately equal 4 and 4/3 , so the sequence is $\left\{4,\dfrac{4}{3},4,\dfrac{4}{3},4,\dots\right\}$.

9. The numerators are all 1 and the denominators are powers of 2 , so $a_n = \dfrac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2 , so $a_n = \dfrac{1}{2n}$.

11. $\{2,7,12,17,\dots\}$. Each term is larger than the preceding one by 5 , so $a_n = a_1 + d(n-1) = 2+5(n-1) = 5n-3$.

12. $\left\{-\dfrac{1}{4}, \dfrac{2}{9}, -\dfrac{3}{16}, \dfrac{4}{25}, \dots\right\}$. The numerator of the n th term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

13. $\left\{1, -\dfrac{2}{3}, \dfrac{4}{9}, -\dfrac{8}{27}, \dots\right\}$. Each term is $-\dfrac{2}{3}$ times the preceding one, so $a_n = \left(-\dfrac{2}{3}\right)^{n-1}$.

14. $\{5,1,5,1,5,1,\dots\}$. The average of 5 and 1 is 3 , so we can think of the sequence as alternately adding 2 and -2 to 3 . Thus, $a_n = 3+(-1)^{n+1} \cdot 2$.

15. $a_n = n(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

16. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{\left(3+5n^2\right)/n^2}{\left(n+n^2\right)/n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}} = \dfrac{1}{1/\sqrt{n}+1}$, so $a_n \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$, so $\lim_{n\to\infty} a_n = \dfrac{1}{3} \lim_{n\to\infty} \left(\dfrac{2}{3}\right)^n = \dfrac{1}{3} \cdot 0 = 0$ by (8) with $r = \dfrac{2}{3}$. Converges

20. $a_n = \dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as

$n\to\infty$, so $a_n\to\infty$ as $n\to\infty$ and the sequence diverges.

21. $a_n=\dfrac{(-1)^{n-1}n}{n^2+1}=\dfrac{(-1)^{n-1}}{n+1/n}$, so $0\le\left|a_n\right|=\dfrac{1}{n+1/n}\le\dfrac{1}{n}\to 0$ as $n\to\infty$, so $a_n\to 0$ by the Squeeze Theorem and Theorem 6. Converges

22. $a_n=\dfrac{(-1)^n n^3}{n^3+2n^2+1}$. Now $\left|a_n\right|=\dfrac{n^3}{n^3+2n^2+1}=\dfrac{1}{1+\dfrac{2}{n}+\dfrac{1}{n^3}}\to 1$ as $n\to\infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence $a_1,a_3,a_{5,}\dots$ converges to -1 and the sequence $a_2,a_4,a_6,\dots$ converges to $+1$. This shows that the given sequence diverges since its terms don't approach a single real number.

23. $a_n=\cos(n/2)$. This sequence diverges since the terms don't approach any particular real number as $n\to\infty$. The terms take on values between -1 and 1 .

24. $a_n=\cos(2/n)$. As $n\to\infty$, $2/n\to 0$, so $\cos(2/n)\to\cos 0=1$. Converges

25. $a_n=\dfrac{(2n-1)!}{(2n+1)!}=\dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!}=\dfrac{1}{(2n+1)(2n)}\to 0$ as $n\to\infty$. Converges

26. $2n\to\infty$ as $n\to\infty$, so since $\lim\limits_{x\to\infty}\arctan x=\dfrac{\pi}{2}$, we have $\lim\limits_{n\to\infty}\arctan 2n=\dfrac{\pi}{2}$. Converges

27. $a_n=\dfrac{e^n+e^{-n}}{e^{2n}-1}\cdot\dfrac{e^{-n}}{e^{-n}}=\dfrac{1+e^{-2n}}{e^n-e^{-n}}\to\dfrac{1+0}{e^n-0}\to 0$ as $n\to\infty$. Converges

28. $a_n=\dfrac{\ln n}{\ln 2n}=\dfrac{\ln n}{\ln 2+\ln n}=\dfrac{1}{\dfrac{\ln 2}{\ln n}+1}\to\dfrac{1}{0+1}\to 1$ as $n\to\infty$. Converges

29. $a_n=n^2e^{-n}=\dfrac{n^2}{e^n}$. Since $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x}=\lim\limits_{x\to\infty}\dfrac{2x}{e^x}=\lim\limits_{x\to\infty}\dfrac{2}{e^x}=0$, it follows from Theorem 3 that $\lim\limits_{n\to\infty}a_n=0$. Converges

30. $a_n = n\cos n\pi = n(-1)^n$. Since $\left|a_n\right| = n \to \infty$ as $n \to \infty$, the given sequence diverges.

31. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$, so since $\lim\limits_{n\to\infty} \dfrac{1}{2^n} = 0$, $\left\{ \dfrac{\cos^2 n}{2^n} \right\}$ converges to 0 by the Squeeze Theorem.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1 + \dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Converges

33. $a_n = n\sin(1/n) = \dfrac{\sin(1/n)}{1/n}$. Since $\lim\limits_{x\to\infty} \dfrac{\sin(1/x)}{1/x} = \lim\limits_{t\to 0^+} \dfrac{\sin t}{t}$ [] where $t = 1/x$]$=1$, it follows from Theorem 3 that $\left\{a_n\right\}$ converges to 1.

34. $a_n = \sqrt{n} - \sqrt{n^2 - 1} = \sqrt{n^2 \cdot \dfrac{1}{n}} - \sqrt{n^2\left(1 - \dfrac{1}{n^2}\right)} = n\left(\dfrac{1}{\sqrt{n}} - \sqrt{1 - \dfrac{1}{n^2}}\right) \to n(0-1) \to -n$ as $n \to \infty$, so $a_n \to -\infty$ as $n \to \infty$. Diverges

35. $a_n = \left(1 + \dfrac{2}{n}\right)^{1/n} \Rightarrow \ln a_n = \dfrac{1}{n}\ln\left(1 + \dfrac{2}{n}\right)$. As $n \to \infty$, $\dfrac{1}{n} \to 0$ and $\ln\left(1 + \dfrac{2}{n}\right) \to 0$, so $\ln a_n \to 0$. Thus, $a_n \to e^0 = 1$ as $n \to \infty$. Converges

36. $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$. $\left|a_n\right| \le \dfrac{1}{1+\sqrt{n}}$ and $\lim\limits_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$, so $\dfrac{-1}{1+\sqrt{n}} \le a_n \le \dfrac{1}{1+\sqrt{n}} \Rightarrow \lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem. Converges

37. $\{0,1,0,0,1,0,0,0,1,\ldots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.

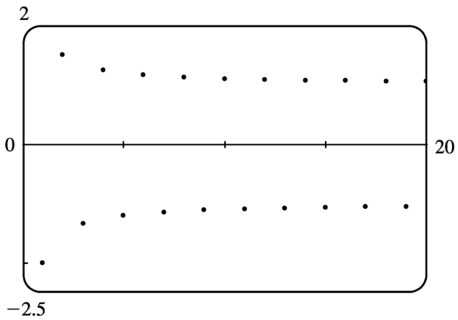
38. $\left\{ \dfrac{1}{1}, \dfrac{1}{3}, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{4}, \dfrac{1}{6}, \ldots \right\}$. $a_{2n-1} = \dfrac{1}{n}$ and $a_{2n} = \dfrac{1}{n+2}$ for all positive integers n . $\lim\limits_{n\to\infty} a_n = 0$ since $\lim\limits_{n\to\infty} a_{2n-1} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ and $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} \dfrac{1}{n+2} = 0$. For n sufficiently large, a_n can be made as close to 0 as we like. Converges

39. $a_n = \dfrac{n!}{2^n} = \dfrac{1}{2} \cdot \dfrac{2}{2} \cdot \dfrac{3}{2} \cdot \cdot \cdot \cdot \cdot \dfrac{(n-1)}{2} \cdot \dfrac{n}{2} \ge \dfrac{1}{2} \cdot \dfrac{n}{2} = \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\left\{a_n\right\}$ diverges.

40.

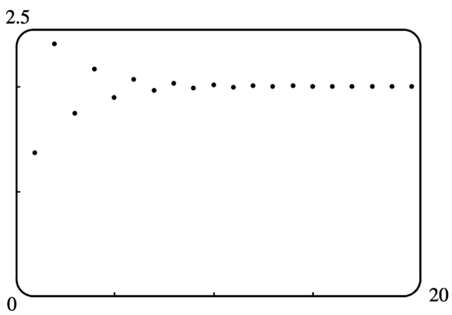
$0 < \left| a_n \right| = \dfrac{3^n}{n!} = \dfrac{3}{1} \cdot \dfrac{3}{2} \cdot \dfrac{3}{3} \cdot \cdot \cdot \cdot \cdot \dfrac{3}{(n-1)} \cdot \dfrac{3}{n} \le \dfrac{3}{1} \cdot \dfrac{3}{2} \cdot \dfrac{3}{n} = \dfrac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\left\{ (-3)^n / n \right\}$ converges to 0 .

41.



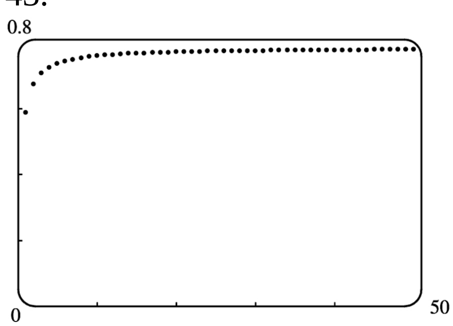
From the graph, we see that the sequence $\left\{ (-1)^n \dfrac{n+1}{n} \right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

42.



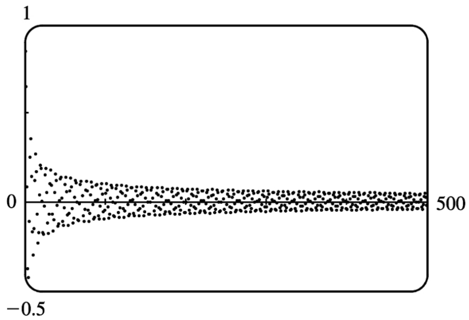
From the graph, it appears that the sequence converges to 2 .
$\left\{ \left(-\dfrac{2}{\pi} \right)^n \right\}$ converges to 0 by (6), and hence $\left\{ 2 + \left(-\dfrac{2}{\pi} \right)^n \right\}$ converges to $2+0=2$.

43.



From the graph, it appears that the sequence converges to about 0.78 .
$\lim_{n\to\infty} \dfrac{2n}{2n+1} = \lim_{n\to\infty} \dfrac{2}{2+1/n} = 1$, so $\lim_{n\to\infty} \arctan\left(\dfrac{2n}{2n+1} \right) = \arctan 1 = \dfrac{\pi}{4}$.

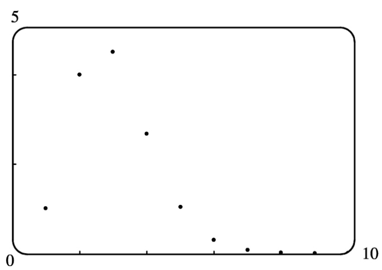
44.



From the graph, it appears that the sequence converges (slowly) to 0 .

$0 \le \frac{|\sin n|}{\sqrt{n}} \le \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\left\{ \frac{\sin n}{\sqrt{n}} \right\}$ converges to 0 .

45.



From the graph, it appears that the sequence converges to 0 .

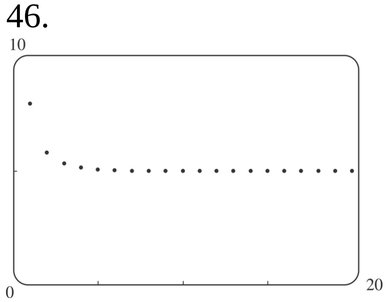
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdot \;\cdot\;\cdot\;\cdot\;\cdot\; \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\le \frac{n^2}{(n-1)(n-2)(n-3)} \quad [\text{for } n \ge 4]$$

$$= \frac{1/n}{(1-1/n)\,(1-2/n)\,(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{ n^3/\text{n!} \right\}$ converges to 0 .

46.



From the graph, it appears that the sequence converges to 5.

$$5=\sqrt[n]{5^n} \le \sqrt[n]{3^n+5^n} \le \sqrt[n]{5^n+5^n} = \sqrt[n]{2}\sqrt[n]{5^n}$$
$$= \sqrt[n]{2}\cdot 5 \to 5 \text{ as } n\to\infty \ \lim_{n\to\infty} 2^{1/n}=2^0=1]$$

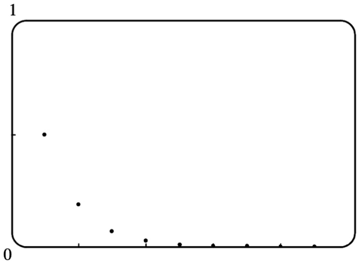
Hence, $a_n \to 5$ by the Squeeze Theorem.

Alternate Solution: Let $y=\left(3^x+5^x\right)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(3^x+5^x\right)}{x} = \lim_{x\to\infty} \frac{3^x \ln 3+5^x \ln 5}{3^x+5^x}$$
$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3+\ln 5}{\left(\frac{3}{5}\right)^x+1} = \ln 5$$

so $\lim_{x\to\infty} y=e^{\ln 5}=5$, and so $\left\{\sqrt[n]{3^n+5^n}\right\}$ converges to 5 .

47.

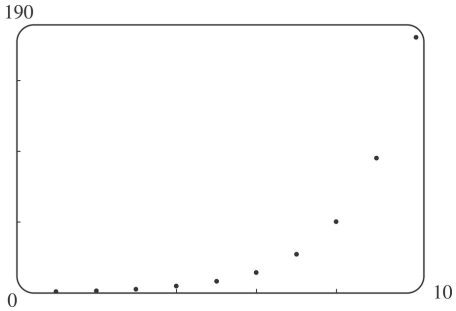


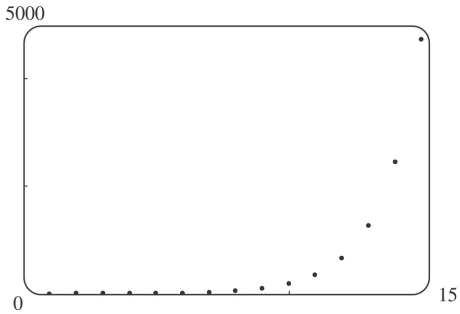
From the graph, it appears that the sequence approaches 0 .

$$0<a_n = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n}\cdot\frac{3}{2n}\cdot\frac{5}{2n}\cdot \cdots \cdot \frac{2n-1}{2n}$$
$$\le \frac{1}{2n}\cdot(1)\cdot(1)\cdot \cdots \cdot(1)=\frac{1}{2n}\to 0 \quad \text{as} \quad n\to\infty$$

So by the Squeeze Theorem, $\left\{\frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2n)^n}\right\}$ converges to 0 .

48.





From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdot \cdot \cdot \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\dfrac{3}{2}\right)^{n-1}$ for all n . This is clearly true for $n=1$, so let $P(n)$ be the statement that the above is true for n . We must show it is then true for $n+1$.

$a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis). But $\dfrac{2n+1}{n+1} \geq \dfrac{3}{2}$, and so we get that $a_{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{3}{2} = \left(\dfrac{3}{2}\right)^{n}$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{\left(\dfrac{3}{2}\right)^{n-1}\right\}$ diverges (by (8)), so does the given sequence $\{a_n\}$.

49. **(a)** $a_n = 1000(1.06)^n \Rightarrow a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim_{n\to\infty} a_n = 1000 \lim_{n\to\infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

50. $a_{n+1} = \begin{cases} \frac{1}{2} a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11 , 34 , 17 , 52 , 26 , 13 , 40 , 20 , 10 , 5 , 16 , 8 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 . When $a_1 = 25$, the first 40 terms are 25 , 76 , 38 , 19 , 58 , 29 , 88 , 44 , 22 , 11 , 34 , 17 , 52 , 26 , 13 , 40 , 20 , 10 , 5 , 16 , 8 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 , 2 , 1 , 4 . The famous Collatz conjecture is that this sequence always reaches 1 , regardless of the

starting point a_1 .

51. If $|r|\geq 1$, then $\{r^n\}$ diverges by (8), so $\{nr^n\}$ diverges also, since $|nr^n|=n|r^n|\geq|r^n|$. If $|r|<1$ then $\lim_{x\to\infty} xr^x=\lim_{x\to\infty}\frac{x}{r^{-x}}=\lim_{x\to\infty}\frac{1}{(-\ln r)r^{-x}}=\lim_{x\to\infty}\frac{r^x}{-\ln r}=0$, so $\lim_{n\to\infty} nr^n=0$, and hence $\{nr^n\}$ converges whenever $|r|<1$.

52. **(a)** Let $\lim_{n\to\infty} a_n=L$. By Definition 1, this means that for every $\varepsilon>0$ there is an integer N such that $|a_n-L|<\varepsilon$ whenever $n>N$. Thus, $|a_{n+1}-L|<\varepsilon$ whenever $n+1>N\Leftrightarrow n>N-1$. It follows that $\lim_{n\to\infty} a_{n+1}=L$ and so $\lim_{n\to\infty} a_n=\lim_{n\to\infty} a_{n+1}$.

(b) If $L=\lim_{n\to\infty} a_n$ then $\lim_{n\to\infty} a_{n+1}=L$ also, so L must satisfy $L=1/(1+L)\Rightarrow L^2+L-1=0\Rightarrow L=\frac{-1+\sqrt{5}}{2}$ (since L has to be non–negative if it exists).

53. Since $\{a_n\}$ is a decreasing sequence, $a_n>a_{n+1}$ for all $n\geq 1$. Because all of its terms lie between 5 and 8 , $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L . L must be less than 8 since $\{a_n\}$ is decreasing, so $5\leq L<8$.

54. $a_n=1/5^n$ defines a decreasing geometric sequence since $a_{n+1}=\frac{1}{5}a_n<a_n$ for each $n\geq 1$. The sequence is bounded since $0<a_n\leq\frac{1}{5}$ for all $n\geq 1$.

55. $a_n=\frac{1}{2n+3}$ is decreasing since $a_{n+1}=\frac{1}{2(n+1)+3}=\frac{1}{2n+5}<\frac{1}{2n+3}=a_n$ for each $n\geq 1$. The sequence is bounded since $0<a_n\leq\frac{1}{5}$ for all $n\geq 1$. Note that $a_1=\frac{1}{5}$.

56. $a_n=\frac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x)=\frac{2x-3}{3x+4}$, $f'(x)=\frac{(3x+4)(2)-(2x-3)(3)}{(3x+4)^2}=\frac{17}{(3x+4)^2}>0$. The sequence is bounded since $a_n\geq a_1=-\frac{1}{7}$ for $n\geq 1$, and $a_n<\frac{2n-3}{3n}<\frac{2n}{3n}=\frac{2}{3}$ for $n\geq 1$.

57. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $\left| a_n \right| \leq 1$ for all $n \geq 1$.

58. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$. [$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0$] for $x > 1$.] The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

59. $a_n = \frac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \frac{x}{x^2+1}$,
$f'(x) = \frac{\left(x^2+1\right)(1) - x(2x)}{\left(x^2+1\right)^2} = \frac{1-x^2}{\left(x^2+1\right)^2} \leq 0$ for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

60. $a_n = n + \frac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \frac{1}{x}$ is increasing for $x > 1$. [$g'(x) = 1 - 1/x^2 > 0$] for $x > 1$.] The sequence is unbounded since $a_n \to \infty$ as $n \to \infty$. (It is, however, bounded below by $a_1 = 2$.)

61. $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, $\ldots$, so $a_n = 2^{\left(2^n - 1\right)/2^n} = 2^{1 - \left(1/2^n\right)}$. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1-\left(1/2^n\right)} = 2^1 = 2$.
Alternate solution : Let $L = \lim_{n\to\infty} a_n$. (We could show the limit exists by showing that $\left\{ a_n \right\}$ is bounded and increasing.) Then L must satisfy $L = \sqrt{2 \cdot L} \Rightarrow L^2 = 2L \Rightarrow L(L-2) = 0$. $L \neq 0$ since the sequence increases, so $L = 2$.

62. **(a)** Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \Leftrightarrow \sqrt{2+a_{n+1}} \geq \sqrt{2+a_n} \Leftrightarrow 2 + a_{n+1} \geq 2 + a_n \Leftrightarrow a_{n+1} \geq a_n$, which is the induction hypothesis. $a_{n+1} \leq 3 \Leftrightarrow \sqrt{2+a_n} \leq 3 \Leftrightarrow 2 + a_n \leq 9 \Leftrightarrow a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n , and so $a_1 \leq a_n \leq 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim_{n\to\infty} a_n$ exists.
(b) If

$L=\lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1}=L$ also, so $L=\sqrt{2+L} \Rightarrow L^2=2+L \Leftrightarrow L^2-L-2=0 \Leftrightarrow (L+1)(L-2)=0 \Leftrightarrow L=2$ (since L can't be negative).

63. We show by induction that $\{a_n\}$ is increasing and bounded above by 3 .
Let P_n be the proposition that $a_{n+1}>a_n$ and $0<a_n<3$. Clearly P_1 is true. Assume that P_n is true. Then $a_{n+1}>a_n \Rightarrow \frac{1}{a_{n+1}}<\frac{1}{a_n} \Rightarrow -\frac{1}{a_{n+1}}>-\frac{1}{a_n}$.
Now $a_{n+2}=3-\frac{1}{a_{n+1}}>3-\frac{1}{a_n}=a_{n+1} \Leftrightarrow P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3 , so $1=a_1<a_n<3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem.
If $L=\lim_{n\to\infty} a_n$, then $\lim_{n\to\infty} a_{n+1}=L$ also, so L must satisfy $L=3-1/L \Rightarrow L^2-3L+1=0 \Rightarrow L=\frac{3\pm\sqrt{5}}{2}$. But $L>1$, so $L=\frac{3+\sqrt{5}}{2}$.

64. We use induction. Let P_n be the statement that $0<a_{n+1}\le a_n\le 2$. Clearly P_1 is true, since $a_2=1/(3-2)=1$. Now assume that P_n is true. Then $a_{n+1}\le a_n \Rightarrow -a_{n+1}\ge -a_n \Rightarrow 3-a_{n+1}\ge 3-a_n \Rightarrow$
$a_{n+2}=\frac{1}{3-a_{n+1}}\le\frac{1}{3-a_n}=a_{n+1}$. Also $a_{n+2}>0$ (since $3-a_{n+1}$ is positive) and $a_{n+1}\le 2$ by the induction hypothesis, so P_{n+1} is true.
To find the limit, we use the fact that $\lim_{n\to\infty} a_n=\lim_{n\to\infty} a_{n+1} \Rightarrow L=\frac{1}{3-L} \Rightarrow L^2-3L+1=0 \Rightarrow L=\frac{3\pm\sqrt{5}}{2}$. But $L\le 2$, so we must have $L=\frac{3-\sqrt{5}}{2}$.

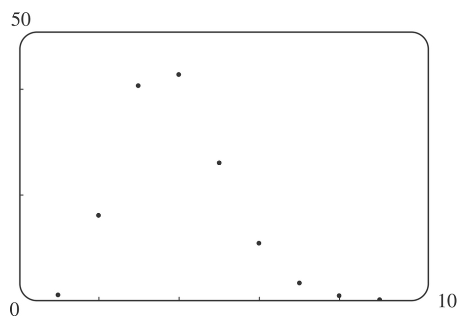
65. **(a)** Let a_n be the number of rabbit pairs in the n th month. Clearly $a_1=1=a_2$. In the n th month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n=a_{n-1}+a_{n-2}$, so that $\{a_n\}=\{f_n\}$, the Fibonacci sequence.

(b) $a_n=\frac{f_{n+1}}{f_n} \Rightarrow a_{n-1}=\frac{f_n}{f_{n-1}}=\frac{f_{n-1}+f_{n-2}}{f_{n-1}}=1+\frac{f_{n-2}}{f_{n-1}}=1+\frac{1}{f_{n-1}/f_{n-2}}=1+\frac{1}{a_{n-2}}$. If $L=\lim_{n\to\infty} a_n$, then

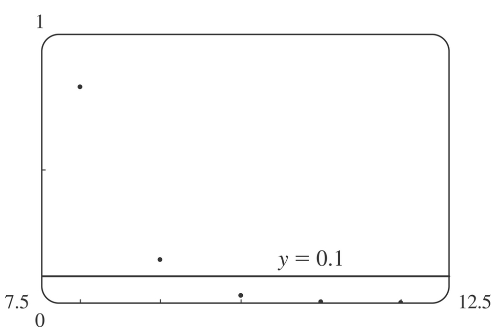
$L=\lim_{n\to\infty} a_{n-1}$ and $L=\lim_{n\to\infty} a_{n-2}$, so L must satisfy $L=1+\frac{1}{L} \Rightarrow L^2-L-1=0 \Rightarrow L=\frac{1+\sqrt{5}}{2}$ (since L must be positive).

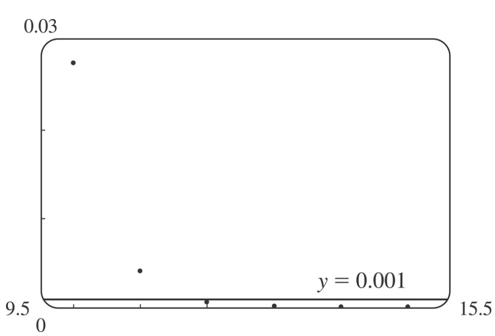
66. **(a)** If f is continuous, then $f(L)=f\left(\lim_{n\to\infty} a_n\right)=\lim_{n\to\infty} f\left(a_n\right)=\lim_{n\to\infty} a_{n+1}=L$ by Exercise 52(a).

(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L\approx 0.73909$.

67. **(a)** 

From the graph, it appears that the sequence $\left\{ \frac{n^5}{n!} \right\}$ converges to 0 , that is, $\lim_{n\to\infty} \frac{n^5}{n!}=0$.

(b) 



From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon=0.1$ is 9 , since $n^5/n!<0.1$ whenever $n\geq 10$, but $9^5/9!>0.1$. From the second graph, it seems that for $\varepsilon=0.001$, the smallest possible value for N is 11 .

68. Let $\varepsilon>0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n>N$ then $n>\ln(\varepsilon)/\ln|r|$ $\Rightarrow n\ln|r|<\ln\varepsilon \Rightarrow \ln\left(|r|^n\right)<\ln\varepsilon \Rightarrow |r|^n<\varepsilon \Rightarrow \left|r^n-0\right|<\varepsilon$, and so by Definition 1, $\lim_{n\to\infty} r^n=0$.

69. If $\lim_{n\to\infty}\left|a_n\right|=0$ then $\lim_{n\to\infty}-\left|a_n\right|=0$, and since $-\left|a_n\right|\le a_n\le\left|a_n\right|$, we have that $\lim_{n\to\infty}a_n=0$ by the Squeeze Theorem.

70. **(a)**

$$\frac{b^{n+1}-a^{n+1}}{b-a}=b^n+b^{n-1}a+b^{n-2}a^2+b^{n-3}a^3+\cdot\cdot\cdot+ba^{n-1}+a^n$$
$$<b^n+b^{n-1}b+b^{n-2}b^2+b^{n-3}b^3+\cdot\cdot\cdot+bb^{n-1}+b^n=(n+1)b^n$$

(b) Since $b-a>0$, we have $b^{n+1}-a^{n+1}<(n+1)b^n(b-a)\Rightarrow b^{n+1}-(n+1)b^n(b-a)<a^{n+1}\Rightarrow b^n\left[(n+1)a-nb\right]<a^{n+1}$.

(c) With this substitution, $(n+1)a-nb=1$, and so $b^n=\left(1+\frac{1}{n}\right)^n<a^{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1+\frac{1}{2n}\right)^n\left(\frac{1}{2}\right)<1\Rightarrow\left(1+\frac{1}{2n}\right)^n<2\Rightarrow\left(1+\frac{1}{2n}\right)^{2n}<4$.

(e) $a_n<a_{2n}$ since $\left\{a_n\right\}$ is increasing, so $a_n<a_{2n}<4$.

(f) Since $\left\{a_n\right\}$ is increasing and bounded above by 4 , $a_1\le a_n\le 4$, and so $\left\{a_n\right\}$ is bounded and monotonic, and hence has a limit by Theorem 11.

71. **(a)** First we show that $a>a_1>b_1>b$.

$a_1-b_1=\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}\left(a-2\sqrt{ab}+b\right)=\frac{1}{2}\left(\sqrt{a}-\sqrt{b}\right)^2>0$ (since $a>b$) $\Rightarrow a_1>b_1$. Also $a-a_1=a-\frac{1}{2}(a+b)=\frac{1}{2}(a-b)>0$ and $b-b_1=b-\sqrt{ab}=\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)<0$, so $a>a_1>b_1>b$. In the same way we can show that $a_1>a_2>b_2>b_1$ and so the given assertion is true for $n=1$. Suppose it is true for $n=k$, that is, $a_k>a_{k+1}>b_{k+1}>b_k$. Then

$$a_{k+2}-b_{k+2}=\frac{1}{2}\left(a_{k+1}+b_{k+1}\right)-\sqrt{a_{k+1}b_{k+1}}=\frac{1}{2}\left(a_{k+1}-2\sqrt{a_{k+1}b_{k+1}}+b_{k+1}\right)$$
$$=\frac{1}{2}\left(\sqrt{a_{k+1}}-\sqrt{b_{k+1}}\right)^2>0$$
$$a_{k+1}-a_{k+2}=a_{k+1}-\frac{1}{2}(a_{k+1}+b_{k+1})=\frac{1}{2}\left(a_{k+1}-b_{k+1}\right)>0$$

and $b_{k+1}-b_{k+2}=b_{k+1}-\sqrt{a_{k+1}b_{k+1}}=\sqrt{b_{k+1}}\left(\sqrt{b_{k+1}}-\sqrt{a_{k+1}}\right)<0\Rightarrow a_{k+1}>a_{k+2}>b_{k+2}>b_{k+1}$, so the assertion is true for $n=k+1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a>a_n>a_{n+1}>b_{n+1}>b_n>b$, which shows that both sequences, $\left\{a_n\right\}$ and $\left\{b_n\right\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim_{n\to\infty} a_n=\alpha$ and $\lim_{n\to\infty} b_n=\beta$. Then $\lim_{n\to\infty} a_{n+1}=\lim_{n\to\infty} \dfrac{a_n+b_n}{2}\Rightarrow\alpha=\dfrac{\alpha+\beta}{2}\Rightarrow$ $2\alpha=\alpha+\beta\Rightarrow\alpha=\beta$.

72. **(a)** Let $\varepsilon>0$. Since $\lim_{n\to\infty} a_{2n}=L$, there exists N_1 such that $\left|a_{2n}-L\right|<\varepsilon$ for $n>N_1$. Since $\lim_{n\to\infty} a_{2n+1}=L$, there exists N_2 such that $\left|a_{2n+1}-L\right|<\varepsilon$ for $n>N_2$. Let $N=\max\left\{2N_1,2N_2+1\right\}$ and let $n>N$. If n is even, then $n=2m$ where $m>N_1$, so $\left|a_n-L\right|=\left|a_{2m}-L\right|<\varepsilon$. If n is odd, then $n=2m+1$, where $m>N_2$, so $\left|a_n-L\right|=\left|a_{2m+1}-L\right|<\varepsilon$. Therefore $\lim_{n\to\infty} a_n=L$.

(b) $a_1=1$, $a_2=1+\dfrac{1}{1+1}=\dfrac{3}{2}=1.5$, $a_3=1+\dfrac{1}{5/2}=\dfrac{7}{5}=1.4$, $a_4=1+\dfrac{1}{12/5}=\dfrac{17}{12}=1.41\overline{6}$, $a_5=1+\dfrac{1}{29/12}=\dfrac{41}{29}\approx1.413793$, $a_6=1+\dfrac{1}{70/29}=\dfrac{99}{70}\approx1.414286$, $a_7=1+\dfrac{1}{169/70}=\dfrac{239}{169}\approx1.414201$, $a_8=1+\dfrac{1}{408/169}=\dfrac{577}{408}\approx1.414216$. Notice that $a_1<a_3<a_5<a_7$ and $a_2>a_4>a_6>a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2}>a_{2n}$ and $a_{2n-1}<a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2}>a_{2k}$. Then $1+a_{2k-2}>1+a_{2k}\Rightarrow\dfrac{1}{1+a_{2k-2}}<\dfrac{1}{1+a_{2k}}\Rightarrow$ $1+\dfrac{1}{1+a_{2k-2}}<1+\dfrac{1}{1+a_{2k}}\Rightarrow a_{2k-1}<a_{2k+1}\Rightarrow 1+a_{2k-1}<1+a_{2k+1}\Rightarrow\dfrac{1}{1+a_{2k-1}}>\dfrac{1}{1+a_{2k+1}}\Rightarrow$ $1+\dfrac{1}{1+a_{2k-1}}>1+\dfrac{1}{1+a_{2k+1}}\Rightarrow a_{2k}>a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2 , so both $\left\{a_n\right\}$ and $\left\{b_n\right\}$ are bounded monotonic sequences and are therefore convergent by Theorem 11. Let $\lim_{n\to\infty} a_{2n}=L$. Then $\lim_{n\to\infty} a_{2n+2}=L$ also. We have $a_{n+2}=1+\dfrac{1}{1+1+1/(1+a_n)}=1+\dfrac{1}{(3+2a_n)/(1+a_n)}=\dfrac{4+3a_n}{3+2a_n}$, so

$a_{2n+2}=\dfrac{4+3a_{2n}}{3+2a_{2n}}$. Taking limits of both sides, we get $L=\dfrac{4+3L}{3+2L}\Rightarrow 3L+2L^2=4+3L\Rightarrow L^2=2\Rightarrow L=\sqrt{2}$ (since $L>0$). Thus, $\lim\limits_{n\to\infty} a_{2n}=\sqrt{2}$. Similarly we find that $\lim\limits_{n\to\infty} a_{2n+1}=\sqrt{2}$. So, by part (a), $\lim\limits_{n\to\infty} a_n=\sqrt{2}$.

73. **(a)** Suppose $\{p_n\}$ converges to p . Then

$$p_{n+1}=\frac{bp_n}{a+p_n}\Rightarrow \lim_{n\to\infty} p_{n+1}=\frac{b\lim\limits_{n\to\infty} p_n}{a+\lim\limits_{n\to\infty} p_n}\Rightarrow p=\frac{bp}{a+p}\Rightarrow p^2+ap=bp\Rightarrow p(p+a-b)=0\Rightarrow p=0 \text{ or } p=b-a .$$

(b) $p_{n+1}=\dfrac{bp_n}{a+p_n}=\dfrac{\frac{b}{a}p_n}{1+\frac{p_n}{a}}<\dfrac{b}{a}p_n$ since $1+\dfrac{p_n}{a}>1$.

(c) By part (b), $p_1<\left(\dfrac{b}{a}\right)p_0$, $p_2<\left(\dfrac{b}{a}\right)p_1<\left(\dfrac{b}{a}\right)^2 p_0$, $p_3<\left(\dfrac{b}{a}\right)p_2<\left(\dfrac{b}{a}\right)^3 p_0$, etc. In general, $p_n<\left(\dfrac{b}{a}\right)^n p_0$, so $\lim\limits_{n\to\infty} p_n\le\lim\limits_{n\to\infty}\left(\dfrac{b}{a}\right)^n\cdot p_0=0$ since $b<a$.

(d) Let $a<b$. We first show, by induction, that if $p_0<b-a$, then $p_n<b-a$ and $p_{n+1}>p_n$.

For $n=0$, we have $p_1-p_0=\dfrac{bp_0}{a+p_0}-p_0=\dfrac{p_0(b-a-p_0)}{a+p_0}>0$ since $p_0<b-a$. So $p_1>p_0$.

Now we suppose the assertion is true for $n=k$, that is, $p_k<b-a$ and $p_{k+1}>p_k$. Then

$b-a-p_{k+1}=b-a-\dfrac{bp_k}{a+p_k}=\dfrac{a(b-a)+bp_k-ap_k-bp_k}{a+p_k}=\dfrac{a(b-a-p_k)}{a+p_k}>0$ because $p_k<b-a$. So $p_{k+1}<b-a$. And $p_{k+2}-p_{k+1}=\dfrac{bp_{k+1}}{a+p_{k+1}}-p_{k+1}=\dfrac{p_{k+1}(b-a-p_{k+1})}{a+p_{k+1}}>0$ since $p_{k+1}<b-a$. Therefore, $p_{k+2}>p_{k+1}$. Thus, the assertion is true for $n=k+1$. It is therefore true for all n by mathematical induction. A similar proof by induction shows that if $p_0>b-a$, then $p_n>b-a$ and $\{p_n\}$ is decreasing. In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem. It then follows from part (a) that $\lim\limits_{n\to\infty} p_n=b-a$.