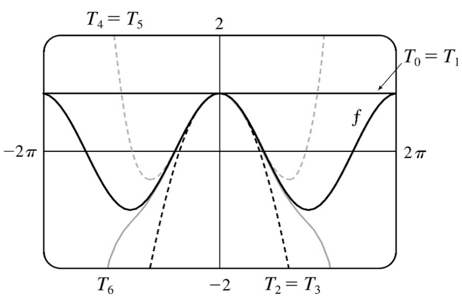
1. **(a)**

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



(b)

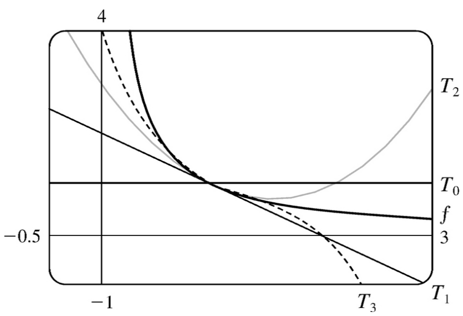
x	f	$T_0=T_1$	$T_2=T_3$	$T_4=T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. **(a)**

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$

0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



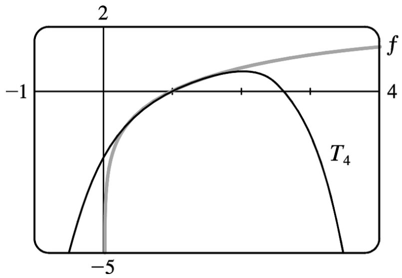
(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

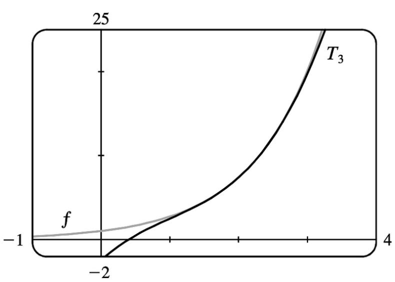
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}(1)}{n!}(x-1)^n=0+(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4$$

4.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2

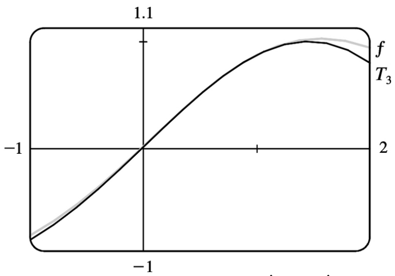


$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(2)}{n!}(x-2)^n=e^2+e^2(x-2)+\frac{e^2}{2}(x-2)^2+\frac{e^2}{6}(x-2)^3$$

5.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$

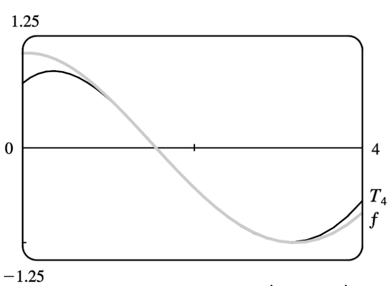
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x-\frac{\pi}{6}\right)^n=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3$$

6.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{2\pi}{3}\right)$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$

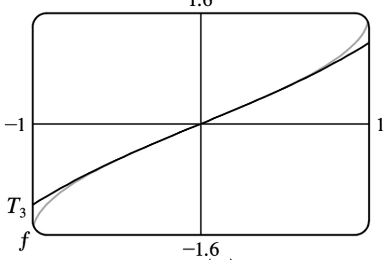


$$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}\left(\frac{2\pi}{3}\right)}{n!}\left(x-\frac{2\pi}{3}\right)^n$$

$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{2\pi}{3}\right)+\frac{1}{4}\left(x-\frac{2\pi}{3}\right)^{2}+\frac{\sqrt{3}}{12}\left(x-\frac{2\pi}{3}\right)^{3}-\frac{1}{48}\left(x-\frac{2\pi}{3}\right)^{4}$$

7.

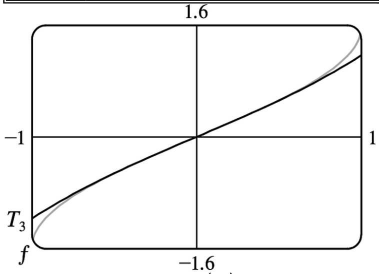
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arcsin x$	0
1	$1/\sqrt{1-x^2}$	1
2	$x/(1-x^2)^{3/2}$	0
3	$(2x^2+1)/(1-x^2)^{5/2}$	1



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=x+\frac{x^3}{6}$$

8.

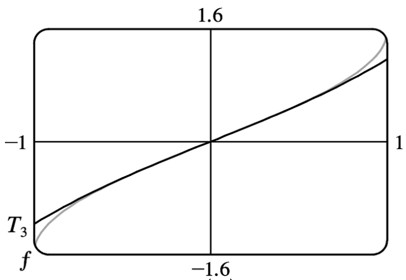
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(\ln x)/x$	0
1	$(1-\ln x)/x^2$	1
2	$(-3+2\ln x)/x^3$	-3
3	$(11-6\ln x)/x^4$	11



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(1)}{n!}(x-1)^n=(x-1)-\frac{3}{2}(x-1)^2+\frac{11}{6}(x-1)^3$$

9.

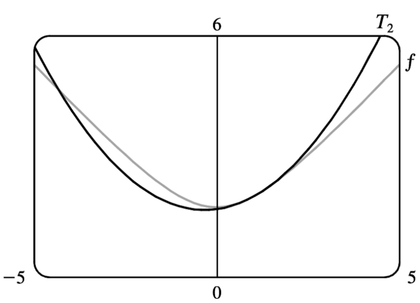
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=\frac{0}{1}\cdot 1+\frac{1}{1}x^1+\frac{-4}{2}x^2+\frac{12}{6}x^3=x-2x^2+2x^3$$

10.

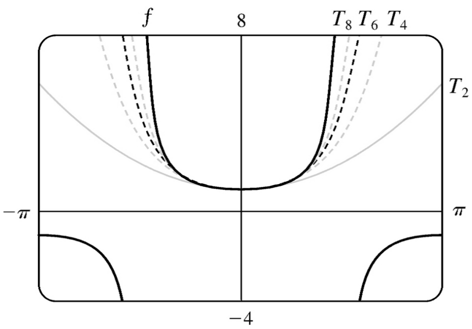
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\left(3+x^2\right)^{1/2}$	2
1	$x\left(3+x^2\right)^{-1/2}$	$\frac{1}{2}$
2	$3\left(3+x^2\right)^{-3/2}$	$\frac{3}{8}$



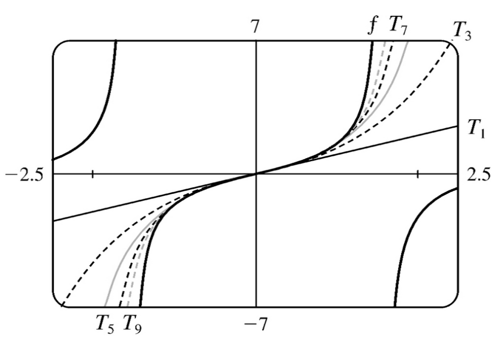
$$T_2(x)=\sum_{n=0}^{2}\frac{f^{(n)}(1)}{n!}(x-1)^n=2+\frac{1}{2}(x-1)+\frac{3/8}{2}(x-1)^2=2+\frac{1}{2}(x-1)+\frac{3}{16}(x-1)^2$$

11. In Maple, we can find the Taylor polynomials by the following method: first define `f:=sec(x);` and then set

T2:=convert(taylor(f,x=0,3),polynom);, T4:=convert(taylor(f,x=0,5),polynom);, etc. (The third argument in the taylor function is one more than the degree of the desired polynomial). We must convert to the type polynom because the output of the taylor function contains an error term which we do not want. In Mathematica, we use Tn:=Normal[Series[f,{x,0,n}]] , with n=2,4, etc. Note that in Mathematica, the "degree" argument is the same as the degree of the desired polynomial. In Derive, author sec x , then enter Calculus ,Taylor,8,0; and then simplify the expression. The eighth Taylor polynomial is $T_8(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+\frac{277}{8064}x^8$.



12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for tan x is $T_9(x)=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9$.



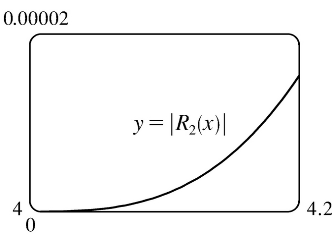
13.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	

(a) $f(x)=\sqrt{x}\approx T_2(x)=2+\frac{1}{4}(x-4)-\frac{1/32}{2!}(x-4)^2=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2$

(b) $\left|R_2(x)\right|\le\frac{M}{3!}|x-4|^3$, where $\left|f'''(x)\right|\le M$. Now $4\le x\le 4.2\Rightarrow|x-4|\le 0.2\Rightarrow|x-4|^3\le 0.008$.
Since $f'''(x)$ is decreasing on $[4,4.2]$, we can take $M=\left|f'''(4)\right|=\frac{3}{8}4^{-5/2}=\frac{3}{256}$, so
$\left|R_2(x)\right|\le\frac{3/256}{6}(0.008)=\frac{0.008}{512}=0.000015625$.

(c) From the graph of $\left|R_2(x)\right|=\left|\sqrt{x}-T_2(x)\right|$, it seems that the error is less than 1.52×10^{-5} on $[4,4.2]$.



14.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	

(a)

$f(x)=x^{-2}\approx T_2(x)$

$=1-2(x-1)+\frac{6}{2!}(x-1)^2$

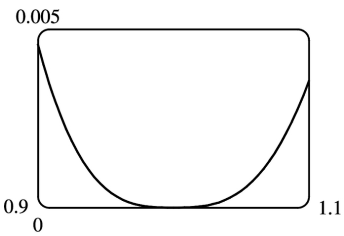
$=1-2(x-1)+3(x-1)^2$

(b) $\left|R_2(x)\right|\le\frac{M}{3!}|x-1|^3$, where $\left|f'''(x)\right|\le M$. Now $0.9\le x\le 1.1\Rightarrow|x-1|\le 0.1\Rightarrow|x-1|^3\le 0.001$.
Since $f'''(x)$ is decreasing on $[0.9,1.1]$, we can take

$M = \left| f^{\prime\prime\prime}(0.9) \right| = \dfrac{24}{(0.9)^5}$, so

$\left| R_2(x) \right| \leq \dfrac{24/(0.9)^5}{6}(0.001) = \dfrac{0.004}{0.59049}$

≈ 0.00677404

(c) 

From the graph of $\left| R_2(x) \right| = \left| x^{-2} - T_2(x) \right|$, it seems that the error is less than 0.0046 on $[0.9,1.1]$.

15.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

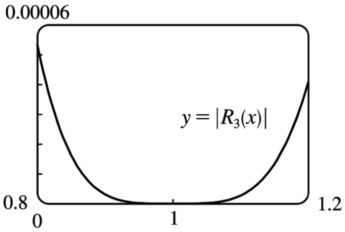
(a) $f(x) = x^{2/3} \approx T_3(x) = 1 + \frac{2}{3}(x-1) - \frac{2/9}{2!}(x-1)^2 + \frac{8/27}{3!}(x-1)^3 = 1 + \frac{2}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{4}{81}(x-1)^3$

(b) $\left| R_3(x) \right| \leq \frac{M}{4!}|x-1|^4$, where $\left| f^{(4)}(x) \right| \leq M$. Now $0.8 \leq x \leq 1.2 \Rightarrow |x-1| \leq 0.2 \Rightarrow |x-1|^4 \leq 0.0016$.

Since $\left| f^{(4)}(x) \right|$ is decreasing on $[0.8,1.2]$, we can take $M = \left| f^{(4)}(0.8) \right| = \frac{56}{81}(0.8)^{-10/3}$, so

$\left| R_3(x) \right| \leq \dfrac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016) \approx 0.00009697$.

(c) From the graph of $\left|R_3(x)\right|=\left|x^{2/3}-T_3(x)\right|$, it seems that the error is less than 0.0000533 on $[0.8,1.2]$.



16.

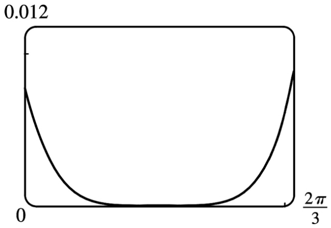
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	cos x	$\frac{1}{2}$
1	–sin x	$-\frac{\sqrt{3}}{2}$
2	–cos x	$-\frac{1}{2}$
3	sin x	$\frac{\sqrt{3}}{2}$
4	cos x	$\frac{1}{2}$
5	–sin x	

(a)

$$f(x)=\cos x\approx T_4(x)$$
$$=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4$$

(b) $\left|R_4(x)\right|\le\frac{M}{5!}\left|x-\frac{\pi}{3}\right|^5$, where $\left|f^{(5)}(x)\right|\le M$. Now $0\le x\le\frac{2\pi}{3}\Rightarrow\left(x-\frac{\pi}{3}\right)^5\le\left(\frac{\pi}{3}\right)^5$, and letting $x=\frac{\pi}{2}$ gives $M=1$, so $\left|R_4(x)\right|\le\frac{1}{5!}\left(\frac{\pi}{3}\right)^5\approx0.0105$.

(c)



From the graph of $|R_4(x)| = |\cos x - T_4(x)|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	

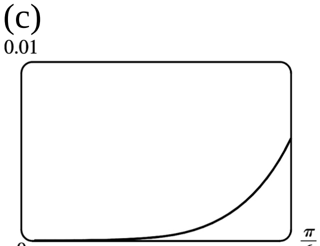
(a) $f(x) = \tan x \approx T_3(x) = x + \frac{1}{3}x^3$

(b) $|R_3(x)| \le \frac{M}{4!}|x|^4$, where $|f^{(4)}(x)| \le M$. Now $0 \le x \le \frac{\pi}{6} \Rightarrow x^4 \le \left(\frac{\pi}{6}\right)^4$, and letting $x = \frac{\pi}{6}$ $\left[\text{since } f^{(4)} \text{ is increasing on } \left(0, \frac{\pi}{6}\right)\right]$ gives

$$|R_3(x)| \le \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \frac{4\sqrt{3}}{9}\left(\frac{\pi}{6}\right)^4 \approx 0.057859$$

(c)



From the graph of

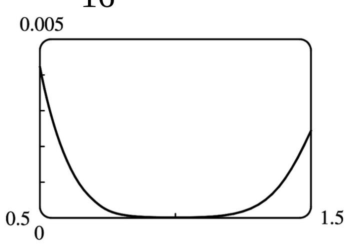
$\left|R_3(x)\right| = \left|\tan x - T_3(3)\right|$, it seems that the error is less than 0.006 on $[0, \pi/6]$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x) = \ln(1+2x) \approx T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{4/9}{2!}(x-1)^2 + \frac{16/27}{3!}(x-1)^3$

(b) $\left|R_3(x)\right| \le \frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.5 \le x \le 1.5 \Rightarrow -0.5 \le x-1 \le 0.5 \Rightarrow |x-1| \le 0.5 \Rightarrow$ $|x-1|^4 \le \frac{1}{16}$, and letting $x = 0.5$ gives $M = 6$, so $\left|R_3(x)\right| \le \frac{6}{4!} \cdot \frac{1}{16} = \frac{1}{64} = 0.015625$.



(c)
From the graph of $\left|R_3(x)\right| = \left|\ln(1+2x) - T_3(x)\right|$, it seems that the error is less than 0.005 on $[0.5, 1.5]$.

19.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

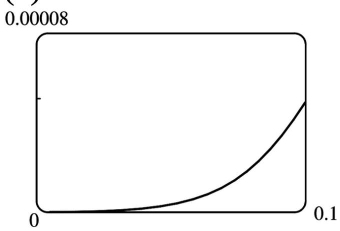
(a) $f(x)=e^{x^2}\approx T_3(x)=1+\frac{2}{2!}x^2=1+x^2$

(b) $\left|R_3(x)\right|\le\frac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right|\le M$.

Now $0\le x\le 0.1\Rightarrow x^4\le(0.1)^4$, and
letting $x=0.1$ gives

$$\left|R_3(x)\right|\le\frac{e^{0.01}(12+0.48+0.0016)}{24}(0.1)^4\approx 0.00006 .$$

(c)



From the graph of $\left|R_3(x)\right|=\left|e^{x^2}-\left(1+x^2\right)\right|$, it appears that the error is less than 0.000051 on $[0,0.1]$.

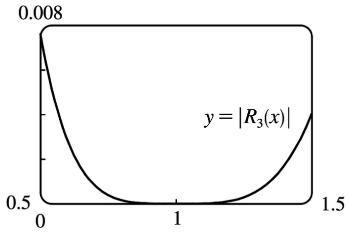
20.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x\ln x$	0
1	$\ln x+1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	

(a) $f(x)=x\ln x\approx T_3(x)=(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3$

(b) $\left|R_3(x)\right|\le\frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right|\le M$. Now $0.5\le x\le 1.5\Rightarrow|x-1|\le\frac{1}{2}\Rightarrow|x-1|^4\le\frac{1}{16}$.
Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.5,1.5]$, we can take $M=\left|f^{(4)}(0.5)\right|=2/(0.5)^3=16$, so
$\left|R_3(x)\right|\le\frac{16}{24}(1/16)=\frac{1}{24}=0.041\overline{6}$.

(c) From the graph of $\left|R_3(x)\right|=\left|x\ln x-T_3(x)\right|$, it seems that the error is less than 0.0076 on $[0.5,1.5]$.



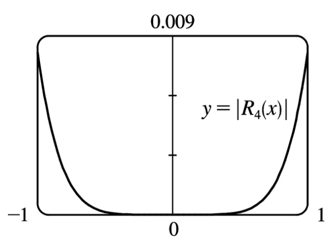
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x\sin x$	0
1	$\sin x+x\cos x$	0
2	$2\cos x-x\sin x$	2
3	$-3\sin x-x\cos x$	0
4	$-4\cos x+x\sin x$	-4
5	$5\sin x+x\cos x$	

(a) $f(x)=x\sin x\approx T_4(x)=\frac{2}{2!}(x-0)^2+\frac{-4}{4!}(x-0)^4=x^2-\frac{1}{6}x^4$

(b) $\left|R_4(x)\right|\le\frac{M}{5!}|x|^5$, where $\left|f^{(5)}(x)\right|\le M$. Now $-1\le x\le 1\Rightarrow|x|\le 1$, and a graph of $f^{(5)}(x)$ shows that $\left|f^{(5)}(x)\right|\le 5$ for $-1\le x\le 1$. Thus, we can take $M=5$ and get $\left|R_4(x)\right|\le\frac{5}{5!}\cdot 1^5=\frac{1}{24}=0.041\overline{6}$.

(c) From the graph of $\left|R_4(x)\right|=\left|x\sin x-T_4(x)\right|$, it seems that the error is less than 0.0082 on $[-1,1]$.



22.

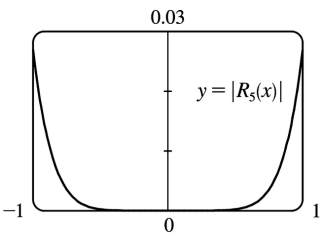
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh 2x$	0
1	$2\cosh 2x$	2
2	$4\sinh 2x$	0
3	$8\cosh 2x$	8
4	$16\sinh 2x$	0
5	$32\cosh 2x$	32

6	64sinh $2x$	

(a) $f(x)=\sinh 2x\approx T_5(x)=2x+\frac{8}{3!}x^3+\frac{32}{5!}x^5=2x+\frac{4}{3}x^3+\frac{4}{15}x^5$

(b) $\left|R_5(x)\right|\le\frac{M}{6!}|x|^6$, where $\left|f^{(6)}(x)\right|\le M$. For x in $[-1,1]$, we have $|x|\le 1$. Since $f^{(6)}(x)$ is an increasing odd function on $[-1,1]$, we see that $\left|f^{(6)}(x)\right|\le f^{(6)}(1)=64\sinh 2=32(e^2-e^{-2})\approx 232.119$, so we can take $M=232.12$ and get $\left|R_5(x)\right|\le\frac{232.12}{720}\cdot 1^6\approx 0.3224$.

(c) From the graph of $\left|R_5(x)\right|=\left|\sinh 2x-T_5(x)\right|$, it seems that the error is less than 0.027 on $[-1,1]$.



23. From Exercise 5, $\sin x=\frac{1}{2}+\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right)-\frac{1}{4}\left(x-\frac{\pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{6}\right)^3+R_3(x)$, where $\left|R_3(x)\right|\le\frac{M}{4!}\left|x-\frac{\pi}{6}\right|^4$ with $\left|f^{(4)}(x)\right|=|\sin x|\le M=1$. Now $x=35^\circ=(30^\circ+5^\circ)=\left(\frac{\pi}{6}+\frac{\pi}{36}\right)$ radians, so the error is

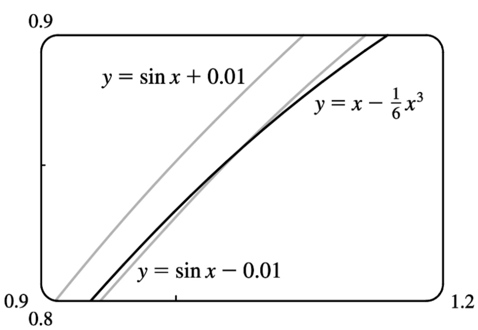
$\left|R_3\left(\frac{\pi}{36}\right)\right|\le\frac{\left(\frac{\pi}{36}\right)^4}{4!}<0.000003$. Therefore, to five decimal places,
$\sin 35^\circ\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right)-\frac{1}{4}\left(\frac{\pi}{36}\right)^2-\frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3\approx 0.57358$.

24. From Exercise 16,
$\cos x=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{4}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{12}\left(x-\frac{\pi}{3}\right)^3+\frac{1}{48}\left(x-\frac{\pi}{3}\right)^4+R_4(x)$. Now since $x=69^\circ=(60^\circ+9^\circ)=\left(\frac{\pi}{3}+\frac{\pi}{20}\right)$ radians, the error is $\left|R_4(x)\right|\le\frac{\left(\frac{\pi}{20}\right)^5}{5!}<8\times 10^{-7}$. Therefore, to five decimal places, $\cos 69^\circ\approx\frac{1}{2}-\frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right)-\frac{1}{4}\left(\frac{\pi}{20}\right)^2+\frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3+\frac{1}{48}\left(\frac{\pi}{20}\right)^4\approx 0.35837$.

25. All derivatives of e^x are e^x , so $\left|R_n(x)\right| \leq \frac{e^x}{(n+1)!}\ |x|^{n+1}$, where $0<x<0.1$. Letting $x=0.1$, $R_n(0.1) \leq \frac{e^{0.1}}{(n+1)!}\ (0.1)^{n+1} < 0.00001$, and by trial and error we find that $n=3$ satisfies this inequality since $R_3(0.1)<0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n=0$, 1 , 2 , and 3 , we can estimate $e^{0.1}$ to within 0.00001 . (In fact, this sum is $1.1051\overline{6}$ and $e^{0.1} \approx 1.10517$.)

26. Example 6 in Section .9 gives the Maclaurin series for $\ln(1-x)$ as $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ for $|x|<1$. Thus, $\ln 1.4 = \ln[1-(-0.4)] = -\sum_{n=1}^{\infty} \frac{(-0.4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.4)^n}{n}$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $\left|a_6\right| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (non–zero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4 \approx 0.33647$.)

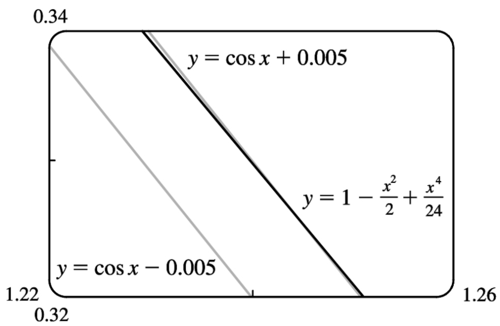
27. $\sin x = x - \frac{1}{3!}\ x^3 + \frac{1}{5!}\ x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}\ x^3$ is less than $\left|\frac{1}{5!}\ x^5\right| < 0.01 \Leftrightarrow \left|x^5\right| < 120(0.01) \Leftrightarrow |x| < (1.2)^{1/5} \approx 1.037$. The curves $y=x-\frac{1}{6}\ x^3$ and $y=\sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and



the given approximation are odd functions, we need to check the estimate only for $x>0$. Thus, the desired range of values for x is $-1.037<x<1.037$.

28.

$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left| -\frac{1}{6!}x^6 \right| < 0.005 \Leftrightarrow x^6 < 720(0.005) \Leftrightarrow |x| < (3.6)^{1/6} \approx 1.238$. The curves $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $y = \cos x + 0.005$ intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation



are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.

29. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is

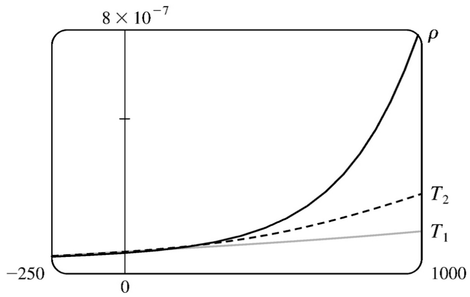
$v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance travelled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of $2\ \mathrm{m/s}^2$ for that long (if it did, its final speed would be $140\ \mathrm{m/s} \approx 313\ \mathrm{mi/h}$!)

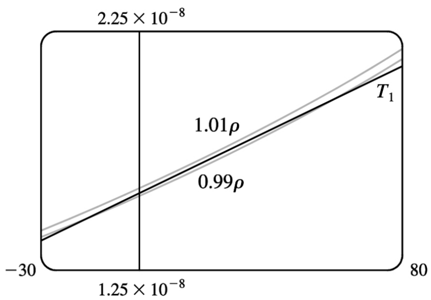
30. **(a)**

n	$\rho^{(n)}(t)$	$\rho^{(n)}(20)$
0	$\rho_{20} e^{\alpha(t-20)}$	ρ_{20}
1	$\alpha \rho_{20} e^{\alpha(t-20)}$	$\alpha \rho_{20}$
2	$\alpha^2 \rho_{20} e^{\alpha(t-20)}$	$\alpha^2 \rho_{20}$

The linear approximation is $T_1(t) = \rho(20) + \rho'(20)(t-20) = \rho_{20}[1 + \alpha(t-20)]$. The quadratic approximation is

$$T_2(t)=\rho(20)+\rho'(20)(t-20)+\frac{\rho''(20)}{2}(t-20)^2=\rho_{20}\left[1+\alpha(t-20)+\frac{1}{2}\alpha^2(t-20)^2\right]$$

(b) 

(c) 

From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t)\le T_1(t)\le 1.01\rho(t)$, for -14° C $\le t\le 58^\circ$ C.

31. $E=\frac{q}{D^2}-\frac{q}{(D+d)^2}=\frac{q}{D^2}-\frac{q}{D^2(1+d/D)^2}=\frac{q}{D^2}\left[1-\left(1+\frac{d}{D}\right)^{-2}\right]$.

We use the Binomial Series to expand $(1+d/D)^{-2}$:

$$E=\frac{q}{D^2}\left[1-\left(1-2\left(\frac{d}{D}\right)+\frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2-\frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3+\cdots\right)\right]$$

$$=\frac{q}{D^2}\left[2\left(\frac{d}{D}\right)-3\left(\frac{d}{D}\right)^2+4\left(\frac{d}{D}\right)^3-\cdots\right]\approx\frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right)=2qd\cdot\frac{1}{D^3}$$

when D is much larger than d ; that is, when P is far away from the dipole.

32. **(a)** $\frac{n_1}{\ell_o}+\frac{n_2}{\ell_i}=\frac{1}{R}\left(\frac{n_2 s_i}{\ell_i}-\frac{n_1 s_o}{\ell_o}\right)$ (Equation 1) where

$$\ell_o=\sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)\cos\phi}\ \text{ and } \ell_i=\sqrt{R^2+\left(s_i-R\right)^2+2R\left(s_i-R\right)\cos\phi}\quad (2)$$

Using $\cos\phi\approx 1$ gives

$$\ell_o=\sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)}=\sqrt{R^2+s_o^2+2Rs_o+R^2-2Rs_o-2R^2}=\sqrt{s_o^2}=s_o$$

and similarly, $\ell_i=s_i$. Thus, Equation 1 becomes

$$\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{1}{R}\left(\frac{n_2s_i}{s_i}-\frac{n_1s_o}{s_o}\right)\Rightarrow\frac{n_1}{s_o}+\frac{n_2}{s_i}=\frac{n_2-n_1}{R}$$

(b) Using $\cos\phi\approx 1-\frac{1}{2}\phi^2$ in (2) gives us

$$\begin{aligned}\ell_o&=\sqrt{R^2+\left(s_o+R\right)^2-2R\left(s_o+R\right)\left(1-\frac{1}{2}\phi^2\right)}\\&=\sqrt{R^2+s_o^2+2Rs_o+R^2-2Rs_o+Rs_o\phi^2-2R^2+R^2\phi^2}=\sqrt{s_o^2+Rs_o\phi^2+R^2\phi^2}\end{aligned}$$

Anticipating that we will use the binomial series expansion $(1+x)^k\approx 1+kx$, we can write the last expression for ℓ_o as $s_o\sqrt{1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i=s_i\sqrt{1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)}$. Thus, from Equation 1, $\frac{n_1}{\ell_o}+\frac{n_2}{\ell_i}=\frac{1}{R}\left(\frac{n_2s_i}{\ell_i}-\frac{n_1s_o}{\ell_o}\right)\Leftrightarrow n_1\ell_o^{-1}+n_2\ell_i^{-1}=\frac{n_2}{R}\cdot\frac{s_i}{\ell_i}-\frac{n_1}{R}\cdot\frac{s_o}{\ell_o}\Leftrightarrow$

$$\frac{n_1}{s_o}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}+\frac{n_2}{s_i}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}$$
$$=\frac{n_2}{R}\left[1-\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]^{-1/2}-\frac{n_1}{R}\left[1+\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]^{-1/2}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_i^{-1} by the first two terms in their binomial series, we get

$$\frac{n_1}{s_o}\left[1-\frac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]+\frac{n_2}{s_i}\left[1+\frac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]$$
$$=\frac{n_2}{R}\left[1+\frac{1}{2}\phi^2\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\right]-\frac{n_1}{R}\left[1-\frac{1}{2}\phi^2\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\right]\Leftrightarrow$$
$$\frac{n_1}{s_o}-\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2}{s_i}+\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)$$
$$=\frac{n_2}{R}+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_1}{R}+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\Leftrightarrow$$
$$\frac{n_1}{s_o}+\frac{n_2}{s_i}$$
$$=\frac{n_2}{R}-\frac{n_1}{R}+\frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_1\phi^2}{2R}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)+\frac{n_2\phi^2}{2R}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)-\frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)$$
$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2}{2}\left(\frac{R}{s_o}+\frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o}+\frac{1}{R}\right)+\frac{n_2\phi^2}{2}\left(\frac{R}{s_i}-\frac{R^2}{s_i^2}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$
$$=\frac{n_2-n_1}{R}+\frac{n_1\phi^2R^2}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)\left(\frac{1}{R}+\frac{1}{s_o}\right)+\frac{n_2\phi^2R^2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)\left(\frac{1}{R}-\frac{1}{s_i}\right)$$
$$=\frac{n_2-n_1}{R}+\phi^2R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R}+\frac{1}{s_o}\right)^2+\frac{n_2}{2s_i}\left(\frac{1}{R}-\frac{1}{s_i}\right)^2\right]$$

From Figure 8, we see that $\sin\phi=h/R$. So if we approximate $\sin\phi$ with ϕ, we get $h=R\phi$ and $h^2=\phi^2R^2$ and hence, Equation 4, as desired.

33. **(a)** If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x\to 1$ as $x\to\infty$. So we can approximate $\tanh(2\pi d/L)\approx 1$, and so $v^2\approx gL/(2\pi)\Leftrightarrow v\approx\sqrt{gL/(2\pi)}$.
(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate

$\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$, and so $v^2 \approx \frac{gL}{2\pi} \cdot \frac{2\pi d}{L} \Leftrightarrow v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x \tanh x$	0
3	$2\operatorname{sech}^2 x(3\tanh^2 x-1)$	-2

(c) Since tanh x is an odd function, its Maclaurin series is alternating, so the error in the approximation $\tanh \frac{2\pi d}{L} \approx \frac{2\pi d}{L}$ is less than the first neglected term, which is

$$\frac{|f'''(0)|}{3!}\left(\frac{2\pi d}{L}\right)^3 = \frac{1}{3}\left(\frac{2\pi d}{L}\right)^3 .$$

If $L>10d$, then $\frac{1}{3}\left(\frac{2\pi d}{L}\right)^3 < \frac{1}{3}\left(2\pi \cdot \frac{1}{10}\right)^3 = \frac{\pi^3}{375}$, so the error in the approximation $v^2=gd$ is less than $\frac{gL}{2\pi} \cdot \frac{\pi^3}{375} \approx 0.0132gL$.

34. **(a)** $4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(-k^2\sin^2 x\right)\right]^{-1/2}dx$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1-\frac{1}{2}\left(-k^2\sin^2 x\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(-k^2\sin^2 x\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(-k^2\sin^2 x\right)^3+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(\frac{1}{2}\right)k^2\sin^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x+\cdots\right]dx$$

$$=4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4\right.$$
$$\left.+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6+\cdots\right]$$

[split up the integral and use the result from Exercise 8.1.44]

$$=2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

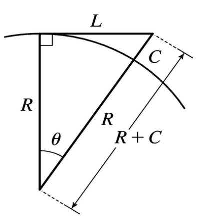
$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8+\cdots\right]$$
$$\le 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{1}{4}k^4+\frac{1}{4}k^6+\frac{1}{4}k^8+\cdots\right]$$

The terms in brackets (after the first) form a geometric series with $a=\frac{1}{4}k^2$ and $r=k^2=\sin^2\left(\frac{1}{2}\theta_0\right)<1$. So $T\le 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{k^2/4}{1-k^2}\right]=2\pi\sqrt{\frac{L}{g}}\,\frac{4-3k^2}{4-4k^2}$.

(c) We substitute $L=1$, $g=9.8$, and $k=\sin\left(10^\circ/2\right)\approx 0.08716$, and the inequality from part (b) becomes $2.01090\le T\le 2.01093$, so $T\approx 2.0109$. The estimate $T\approx 2\pi\sqrt{L/g}\approx 2.0071$ differs by about 0.2%.

If $\theta_0=42^\circ$, then $k\approx 0.35837$ and the inequality becomes $2.07153\le T\le 2.08103$, so $T\approx 2.0763$. The one–term estimate is the same, and the discrepancy between the two estimates increases to about 3.4%.

35. **(a)** L is the length of the arc subtended by the angle θ, so $L=R\theta\Rightarrow\theta=L/R$. Now $\sec\theta=(R+C)/R$ $\Rightarrow R\sec\theta=R+C\Rightarrow C=R\sec\theta-R=R\sec(L/R)-R$.



(b) From Exercise 11, $\sec x\approx T_4(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4$. By part (a),

$$C\approx R\left[1+\frac{1}{2}\left(\frac{L}{R}\right)^2+\frac{5}{24}\left(\frac{L}{R}\right)^4\right]-R=R+\frac{1}{2}R\cdot\frac{L^2}{R^2}+\frac{5}{24}R\cdot\frac{L^4}{R^4}-R=\frac{L^2}{2R}+\frac{5L^4}{24R^3}.$$

(c) Taking $L=100$ km and $R=6370$ km, the formula in part (a) says that $C=R\sec(L/R)-R=6370\sec(100/6370)-6370\approx 0.78500996544$ km

The formula in part (b) says that $C\approx\frac{L^2}{2R}+\frac{5L^4}{24R^3}=\frac{100^2}{2\cdot 6370}+\frac{5\cdot 100^4}{24\cdot 6370^3}\approx 0.78500995736$ km.

The difference between these two results is only $0.000000008 08$ km, or $0.000008 08$ m!

36. $T_n(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0\le m\le n$. Then
$T_n^{(m)}(x)=m!\frac{f^{(m)}(a)}{m!}(x-a)^0+(m+1)(m)\cdots(2)\frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1+\cdots$
$+n(n-1)\cdots(n-m+1)\frac{f^{(n)}(a)}{n!}(x-a)^{n-m}$

For $x=a$, all terms in this sum except the first one are 0 , so $T_n^{(m)}(a)=\frac{m!\,f^{(m)}(a)}{m!}=f^{(m)}(a)$.

37. Using $f(x)=T_n(x)+R_n(x)$ with $n=1$ and $x=r$, we have $f(r)=T_1(r)+R_1(r)$, where T_1 is the first–degree Taylor polynomial of f at a . Because $a=x_n$, $f(r)=f\left(x_n\right)+f'\left(x_n\right)\left(r-x_n\right)+R_1(r)$. But r is a root of f , so $f(r)=0$ and we have $0=f\left(x_n\right)+f'\left(x_n\right)\left(r-x_n\right)+R_1(r)$. Taking the first two terms to the left side gives us $f'\left(x_n\right)\left(x_n-r\right)-f\left(x_n\right)=R_1(r)$. Dividing by $f'\left(x_n\right)$, we get
$x_n-r-\frac{f\left(x_n\right)}{f'\left(x_n\right)}=\frac{R_1(r)}{f'\left(x_n\right)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1}-r$, so $\left|x_{n+1}-r\right|=\left|\frac{R_1(r)}{f'\left(x_n\right)}\right|$. Taylor's Inequality gives us
$\left|R_1(r)\right|\le\frac{\left|f''(r)\right|}{2!}\left|r-x_n\right|^2$. Combining this inequality with the facts $\left|f''(x)\right|\le M$ and $\left|f'(x)\right|\ge K$ gives us $\left|x_{n+1}-r\right|\le\frac{M}{2K}\left|x_n-r\right|^2$.