

1. (a)

$$\begin{aligned}\lim_{x \rightarrow a} [f(x)+h(x)] &= \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} h(x) \\ &= -3 + 8 = 5\end{aligned}$$

$$(b) \lim_{x \rightarrow a} [f(x)]^2 = \left[\lim_{x \rightarrow a} f(x) \right]^2 = (-3)^2 = 9$$

$$(c) \lim_{x \rightarrow a} \sqrt[3]{h(x)} = \sqrt[3]{\lim_{x \rightarrow a} h(x)} = \sqrt[3]{8} = 2$$

$$(d) \lim_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{\lim_{x \rightarrow a} f(x)} = \frac{1}{-3} = -\frac{1}{3}$$

$$(e) \lim_{x \rightarrow a} \frac{f(x)}{h(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} h(x)} = \frac{-3}{8} = -\frac{3}{8}$$

$$(f) \lim_{x \rightarrow a} \frac{g(x)}{f(x)} = \frac{\lim_{x \rightarrow a} g(x)}{\lim_{x \rightarrow a} f(x)} = \frac{0}{-3} = 0$$

(g) The limit does not exist, since $\lim_{x \rightarrow a} g(x) = 0$ but $\lim_{x \rightarrow a} f(x) \neq 0$.

$$(h) \lim_{x \rightarrow a} \frac{2f(x)}{h(x)-f(x)} = \frac{2\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} h(x) - \lim_{x \rightarrow a} f(x)} = \frac{2(-3)}{8 - (-3)} = -\frac{6}{11}$$

$$2. (a) \lim_{x \rightarrow 2} [f(x)+g(x)] = \lim_{x \rightarrow 2} f(x) + \lim_{x \rightarrow 2} g(x) = 2 + 0 = 2$$

(b) $\lim_{x \rightarrow 1} g(x)$ does not exist since its left- and right-hand limits are not equal, so the given limit does not exist.

$$(c) \lim_{x \rightarrow 0} [f(x)g(x)] = \lim_{x \rightarrow 0} f(x) \cdot \lim_{x \rightarrow 0} g(x) = 0 \cdot 1.3 = 0$$

(d) Since $\lim_{x \rightarrow -1} g(x) = 0$ and g is in the denominator, but $\lim_{x \rightarrow -1} f(x) = -1 \neq 0$, the given limit does not exist.

$$(e) \lim_{x \rightarrow 2} x^3 f(x) = \left[\lim_{x \rightarrow 2} x^3 \right] \left[\lim_{x \rightarrow 2} f(x) \right] = 2^3 \cdot 2 = 16$$

$$(f) \lim_{x \rightarrow 1} \sqrt{3+f(x)} = \sqrt{3 + \lim_{x \rightarrow 1} f(x)} = \sqrt{3+1} = 2$$

3.

$$\begin{aligned}
\lim_{x \rightarrow -2} (3x^4 + 2x^2 - x + 1) &= \lim_{x \rightarrow -2} 3x^4 + \lim_{x \rightarrow -2} 2x^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \quad [\text{Limit Laws 1 and 2}] \\
&= 3 \lim_{x \rightarrow -2} x^4 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} x + \lim_{x \rightarrow -2} 1 \quad [3] \\
&= 3(-2)^4 + 2(-2)^2 - (-2) + (1) \quad [9, 8, \text{ and } 7] \\
&= 48 + 8 + 2 + 1 = 59
\end{aligned}$$

4.

$$\begin{aligned}
\lim_{x \rightarrow 2} \frac{2x^2 + 1}{x^2 + 6x - 4} &= \frac{\lim_{x \rightarrow 2} (2x^2 + 1)}{\lim_{x \rightarrow 2} (x^2 + 6x - 4)} \quad [\text{Limit Law 5}] \\
&= \frac{2 \lim_{x \rightarrow 2} x^2 + \lim_{x \rightarrow 2} 1}{\lim_{x \rightarrow 2} x^2 + 6 \lim_{x \rightarrow 2} x - \lim_{x \rightarrow 2} 4} \quad [2, 1, \text{ and } 3] \\
&= \frac{2(2)^2 + 1}{(2)^2 + 6(2) - 4} = \frac{9}{12} = \frac{3}{4} \quad [9, 7, \text{ and } 8]
\end{aligned}$$

5.

$$\begin{aligned}
\lim_{x \rightarrow 3} (x^2 - 4)(x^3 + 5x - 1) &= \lim_{x \rightarrow 3} (x^2 - 4) \cdot \lim_{x \rightarrow 3} (x^3 + 5x - 1) \quad [\text{Limit Law 4}] \\
&= \left(\lim_{x \rightarrow 3} x^2 - \lim_{x \rightarrow 3} 4 \right) \cdot \left(\lim_{x \rightarrow 3} x^3 + 5 \lim_{x \rightarrow 3} x - \lim_{x \rightarrow 3} 1 \right) \quad [2, 1, \text{ and } 3] \\
&= (3^2 - 4) \cdot (3^3 + 5 \cdot 3 - 1) \quad [7, 8, \text{ and } 9] \\
&= 5 \cdot 41 = 205
\end{aligned}$$

6.

$$\begin{aligned}
\lim_{t \rightarrow -1} (t^2 + 1)^3 (t + 3)^5 &= \lim_{t \rightarrow -1} (t^2 + 1)^3 \cdot \lim_{t \rightarrow -1} (t + 3)^5 \quad [\text{Limit Law 4}] \\
&= \left[\lim_{t \rightarrow -1} (t^2 + 1) \right]^3 \cdot \left[\lim_{t \rightarrow -1} (t + 3) \right]^5 \quad [6]
\end{aligned}$$

$$\begin{aligned}
 &= \left[\lim_{t \rightarrow -1} t^2 + \lim_{t \rightarrow -1} 1 \right]^3 \cdot \left[\lim_{t \rightarrow -1} t + \lim_{t \rightarrow -1} 3 \right]^5 \quad [1] \\
 &= [(-1)^2 + 1]^3 \cdot [-1 + 3]^5 = 8 \cdot 32 = 256 \quad [9, 7, \text{ and } 8]
 \end{aligned}$$

7.

$$\lim_{x \rightarrow 1} \left(\frac{1+3x}{1+4x^2+3x^4} \right)^3 = \left(\lim_{x \rightarrow 1} \frac{1+3x}{1+4x^2+3x^4} \right)^3 \quad [6]$$

$$= \left[\frac{\lim_{x \rightarrow 1} (1+3x)}{\lim_{x \rightarrow 1} (1+4x^2+3x^4)} \right]^3 \quad [5]$$

$$= \left[\frac{\lim_{x \rightarrow 1} 1 + 3 \lim_{x \rightarrow 1} x}{\lim_{x \rightarrow 1} 1 + 4 \lim_{x \rightarrow 1} x^2 + 3 \lim_{x \rightarrow 1} x^4} \right]^3 \quad [2, 1, \text{ and } 3]$$

$$= \left[\frac{1+3(1)}{1+4(1)^2+3(1)^4} \right]^3 = \left[\frac{4}{8} \right]^3 = \left(\frac{1}{2} \right)^3 = \frac{1}{8} \quad [7, 8, \text{ and } 9]$$

8.

$$\lim_{u \rightarrow -2} \sqrt{u^4 + 3u + 6} = \sqrt{\lim_{u \rightarrow -2} (u^4 + 3u + 6)} \quad [11]$$

$$= \sqrt{\lim_{u \rightarrow -2} u^4 + 3 \lim_{u \rightarrow -2} u + \lim_{u \rightarrow -2} 6} \quad [1, 2, \text{ and } 3]$$

$$= \sqrt{(-2)^4 + 3(-2) + 6} \quad [9, 8, \text{ and } 7]$$

$$= \sqrt{16 - 6 + 6} = \sqrt{16} = 4$$

9.

$$\lim_{x \rightarrow 4^-} \sqrt{16-x^2} = \sqrt{\lim_{x \rightarrow 4^-} (16-x^2)} \quad [11]$$

$$= \sqrt{\lim_{x \rightarrow 4^-} 16 - \lim_{x \rightarrow 4^-} x^2} \quad [2]$$

$$=\sqrt{16-(4)^2}=0 \quad [7 \text{ and } 9]$$

10. (a) The left-hand side of the equation is not defined for $x=2$, but the right-hand side is.

(b) Since the equation holds for all $x \neq 2$, it follows that both sides of the equation approach the same limit as $x \rightarrow 2$, just as in Example 3. Remember that in finding $\lim_{x \rightarrow a} f(x)$, we never consider $x=a$.

$$11. \lim_{x \rightarrow 2} \frac{x^2+x-6}{x-2} = \lim_{x \rightarrow 2} \frac{(x+3)(x-2)}{x-2} = \lim_{x \rightarrow 2} (x+3) = 2+3=5$$

$$12. \lim_{x \rightarrow -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{x \rightarrow -4} \frac{(x+4)(x+1)}{(x+4)(x-1)} = \lim_{x \rightarrow -4} \frac{x+1}{x-1} = \frac{-4+1}{-4-1} = \frac{-3}{-5} = \frac{3}{5}$$

$$13. \lim_{x \rightarrow 2} \frac{x^2-x+6}{x-2} \text{ does not exist since } x-2 \rightarrow 0 \text{ but } x^2-x+6 \rightarrow 8 \text{ as } x \rightarrow 2.$$

$$14. \lim_{x \rightarrow 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x \rightarrow 4} \frac{x(x-4)}{(x-4)(x+1)} = \lim_{x \rightarrow 4} \frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}$$

$$15. \lim_{t \rightarrow -3} \frac{t^2-9}{2t^2+7t+3} = \lim_{t \rightarrow -3} \frac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim_{t \rightarrow -3} \frac{t-3}{2t+1} = \frac{-3-3}{2(-3)+1} = \frac{-6}{-5} = \frac{6}{5}$$

$$16. \lim_{x \rightarrow -1} \frac{x^2-4x}{x^2-3x-4} \text{ does not exist since } x^2-3x-4 \rightarrow 0 \text{ but } x^2-4x \rightarrow 5 \text{ as } x \rightarrow -1.$$

$$17. \lim_{h \rightarrow 0} \frac{(4+h)^2-16}{h} = \lim_{h \rightarrow 0} \frac{(16+8h+h^2)-16}{h} = \lim_{h \rightarrow 0} \frac{8h+h^2}{h} = \lim_{h \rightarrow 0} \frac{h(8+h)}{h} = \lim_{h \rightarrow 0} (8+h) = 8+0=8$$

$$18. \lim_{x \rightarrow 1} \frac{x^3-1}{x^2-1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x \rightarrow 1} \frac{x^2+x+1}{x+1} = \frac{1^2+1+1}{1+1} = \frac{3}{2}$$

19.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(1+h)^4 - 1}{h} &= \lim_{h \rightarrow 0} \frac{(1+4h+6h^2+4h^3+h^4) - 1}{h} = \lim_{h \rightarrow 0} \frac{4h+6h^2+4h^3+h^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(4+6h+4h^2+h^3)}{h} = \lim_{h \rightarrow 0} (4+6h+4h^2+h^3) = 4+0+0+0=4\end{aligned}$$

20.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} &= \lim_{h \rightarrow 0} \frac{(8+12h+6h^2+h^3) - 8}{h} = \lim_{h \rightarrow 0} \frac{12h+6h^2+h^3}{h} \\ &= \lim_{h \rightarrow 0} (12+6h+h^2) = 12+0+0=12\end{aligned}$$

$$21. \lim_{t \rightarrow 9} \frac{9-t}{3-\sqrt{t}} = \lim_{t \rightarrow 9} \frac{(3+\sqrt{t})(3-\sqrt{t})}{3-\sqrt{t}} = \lim_{t \rightarrow 9} (3+\sqrt{t}) = 3+\sqrt{9}=6$$

22.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h} &= \lim_{h \rightarrow 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \lim_{h \rightarrow 0} \frac{(1+h)-1}{h(\sqrt{1+h}+1)} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{1+h}+1)} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}\end{aligned}$$

23.

$$\begin{aligned}\lim_{x \rightarrow 7} \frac{\sqrt{x+2}-3}{x-7} &= \lim_{x \rightarrow 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x \rightarrow 7} \frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} \\ &= \lim_{x \rightarrow 7} \frac{x-7}{(x-7)(\sqrt{x+2}+3)} = \lim_{x \rightarrow 7} \frac{1}{\sqrt{x+2}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}\end{aligned}$$

24.

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^4 - 16}{x-2} &= \lim_{x \rightarrow 2} \frac{(x+2)(x-2)(x^2+4)}{x-2} = \lim_{x \rightarrow 2} (x+2)(x^2+4) = \lim_{x \rightarrow 2} (x+2) \lim_{x \rightarrow 2} (x^2+4) \\ &= (2+2)(2^2+4) = 32\end{aligned}$$

25.

$$\lim_{x \rightarrow -4} \frac{\frac{1}{4} + \frac{1}{x}}{4+x} = \lim_{x \rightarrow -4} \frac{\frac{x+4}{4x}}{4+x} = \lim_{x \rightarrow -4} \frac{x+4}{4x(4+x)} = \lim_{x \rightarrow -4} \frac{1}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}$$

$$26. \lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2+t} \right) = \lim_{t \rightarrow 0} \frac{(t^2+t)-t}{t(t^2+t)} = \lim_{t \rightarrow 0} \frac{t^2}{t \cdot t(t+1)} = \lim_{t \rightarrow 0} \frac{1}{t+1} = \frac{1}{0+1} = 1$$

27.

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{x^2-81}{\sqrt{x}-3} &= \lim_{x \rightarrow 9} \frac{(x-9)(x+9)}{\sqrt{x}-3} = \lim_{x \rightarrow 9} \frac{(\sqrt{x}-3)(\sqrt{x}+3)(x+9)}{\sqrt{x}-3} \quad \left[\begin{array}{l} \text{factor } x-9 \text{ as a} \\ \text{difference of squares} \end{array} \right] \\ &= \lim_{x \rightarrow 9} [(\sqrt{x}+3)(x+9)] = (\sqrt{9}+3)(9+9) = 6 \cdot 18 = 108 \end{aligned}$$

28.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(3+h)^{-1} - 3^{-1}}{h} &= \lim_{h \rightarrow 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h} = \lim_{h \rightarrow 0} \frac{3-(3+h)}{h(3+h)3} = \lim_{h \rightarrow 0} \frac{-h}{h(3+h)3} \\ &= \lim_{h \rightarrow 0} \left[-\frac{1}{3(3+h)} \right] = -\frac{1}{\lim_{h \rightarrow 0} [3(3+h)]} = -\frac{1}{3(3+0)} = -\frac{1}{9} \end{aligned}$$

29.

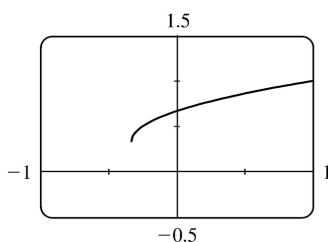
$$\begin{aligned} \lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) &= \lim_{t \rightarrow 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t \rightarrow 0} \frac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{1+t}(1+\sqrt{1+t})} = \lim_{t \rightarrow 0} \frac{-t}{t\sqrt{1+t}(1+\sqrt{1+t})} \\ &= \lim_{t \rightarrow 0} \frac{-1}{\sqrt{1+t}(1+\sqrt{1+t})} = \frac{-1}{\sqrt{1+0}(1+\sqrt{1+0})} = -\frac{1}{2} \end{aligned}$$

30.

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x-x^2}}{1-\sqrt{x}} &= \lim_{x \rightarrow 1} \frac{\sqrt{x}(1-x^{3/2})}{1-\sqrt{x}} = \lim_{x \rightarrow 1} \frac{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x+x})}{1-\sqrt{x}} \quad [\text{difference of cubes}] \\ &= \lim_{x \rightarrow 1} [\sqrt{x}(1+\sqrt{x+x})] = \lim_{x \rightarrow 1} [1(1+1+1)] = 3 \end{aligned}$$

Another method: We "add and subtract" 1 in the numerator, and then split up the fraction:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x-x^2}}{1-\sqrt{x}} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x}-1)+(1-x^2)}{1-\sqrt{x}} = \lim_{x \rightarrow 1} \left[-1 + \frac{(1-x)(1+x)}{1-\sqrt{x}} \right] \\ &= \lim_{x \rightarrow 1} \left[-1 + \frac{(1-\sqrt{x})(1+\sqrt{x})(1+x)}{1-\sqrt{x}} \right] = -1 + (1+\sqrt{1})(1+1) = 3 \end{aligned}$$



31. (a)

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+3x}-1} \approx \frac{2}{3}$$

(b)

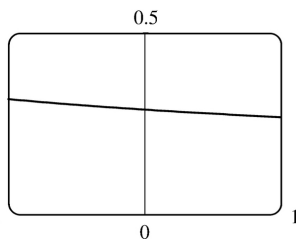
x	$f(x)$
-0.001	0.6661663
-0.0001	0.6666167
-0.00001	0.6666617
-0.000001	0.6666662
0.000001	0.6666672
0.00001	0.6666717
0.0001	0.6667167
0.001	0.6671663

 The limit appears to be $\frac{2}{3}$.

(c)

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{x}{\sqrt{1+3x}-1} \cdot \frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1} \right) &= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{(1+3x)-1} = \lim_{x \rightarrow 0} \frac{x(\sqrt{1+3x}+1)}{3x} \\ &= \frac{1}{3} \lim_{x \rightarrow 0} (\sqrt{1+3x}+1) && \text{[Limit Law 3]} \\ &= \frac{1}{3} \left[\sqrt{\lim_{x \rightarrow 0} (1+3x)} + \lim_{x \rightarrow 0} 1 \right] && \text{[1 and 11]} \\ &= \frac{1}{3} \left(\sqrt{\lim_{x \rightarrow 0} 1 + 3 \lim_{x \rightarrow 0} x} + 1 \right) && \text{[1, 3, and 7]} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{3} (\sqrt{1+3 \cdot 0} + 1) && [7 \text{ and } 8] \\
 &= \frac{1}{3} (1+1) = \frac{2}{3}
 \end{aligned}$$



32. (a) $\lim_{x \rightarrow 0} \frac{\sqrt{3+x} - \sqrt{3}}{x} \approx 0.29$

(b)

x	$f(x)$
-0.001	0.2886992
-0.0001	0.2886775
-0.00001	0.2886754
-0.000001	0.2886752
0.000001	0.2886751
0.00001	0.2886749
0.0001	0.2886727
0.001	0.2886511

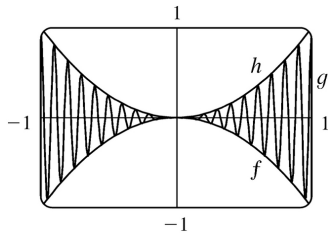
The limit appears to be approximately 0.2887 .

(c)

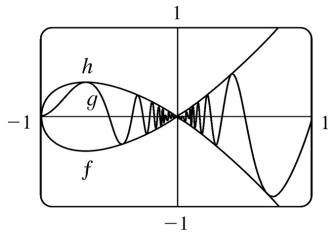
$$\begin{aligned}
 \lim_{x \rightarrow 0} \left(\frac{\sqrt{3+x} - \sqrt{3}}{x} \cdot \frac{\sqrt{3+x} + \sqrt{3}}{\sqrt{3+x} + \sqrt{3}} \right) &= \lim_{x \rightarrow 0} \frac{(3+x) - 3}{x(\sqrt{3+x} + \sqrt{3})} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{3+x} + \sqrt{3}} \\
 &= \frac{\lim_{x \rightarrow 0} 1}{\lim_{x \rightarrow 0} \sqrt{3+x} + \lim_{x \rightarrow 0} \sqrt{3}} && [\text{Limit Laws 5 and 1}] \\
 &= \frac{1}{\sqrt{\lim_{x \rightarrow 0} (3+x)} + \sqrt{3}} && [7 \text{ and } 11] \\
 &= \frac{1}{\sqrt{3+0} + \sqrt{3}} && [1, 7, \text{ and } 8]
 \end{aligned}$$

$$= \frac{1}{2\sqrt{3}}$$

33. Let $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$ and $h(x) = x^2$. Then $-1 \leq \cos 20\pi x \leq 1 \Rightarrow -x^2 \leq x^2 \cos 20\pi x \leq x^2 \Rightarrow f(x) \leq g(x) \leq h(x)$. So since $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$, by the Squeeze Theorem we have $\lim_{x \rightarrow 0} g(x) = 0$.



34. Let $f(x) = -\sqrt{x^3 + x^2}$, $g(x) = \sqrt{x^3 + x^2} \sin(\pi/x)$, and $h(x) = \sqrt{x^3 + x^2}$. Then $-1 \leq \sin(\pi/x) \leq 1 \Rightarrow -\sqrt{x^3 + x^2} \leq \sqrt{x^3 + x^2} \sin(\pi/x) \leq \sqrt{x^3 + x^2} \Rightarrow f(x) \leq g(x) \leq h(x)$. So since $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$, by the Squeeze Theorem we have $\lim_{x \rightarrow 0} g(x) = 0$.



35. $1 \leq f(x) \leq x^2 + 2x + 2$ for all x . Now $\lim_{x \rightarrow -1} 1 = 1$ and

$\lim_{x \rightarrow -1} (x^2 + 2x + 2) = \lim_{x \rightarrow -1} x^2 + 2 \lim_{x \rightarrow -1} x + \lim_{x \rightarrow -1} 2 = (-1)^2 + 2(-1) + 2 = 1$. Therefore, by the Squeeze Theorem,

$\lim_{x \rightarrow -1} f(x) = 1$.

36. $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$. Now $\lim_{x \rightarrow 1} 3x = 3$ and $\lim_{x \rightarrow 1} (x^3 + 2) = \lim_{x \rightarrow 1} x^3 + \lim_{x \rightarrow 1} 2 = 1^3 + 2 = 3$. Therefore,

by the Squeeze Theorem, $\lim_{x \rightarrow 1} f(x) = 3$.

37. $-1 \leq \cos(2/x) \leq 1 \Rightarrow -x^4 \leq x^4 \cos(2/x) \leq x^4$. Since $\lim_{x \rightarrow 0} (-x^4) = 0$ and $\lim_{x \rightarrow 0} x^4 = 0$, we have

$\lim_{x \rightarrow 0} [x^4 \cos(2/x)] = 0$ by the Squeeze Theorem.

38. $-1 \leq \sin(\pi/x) \leq 1 \Rightarrow e^{-1} \leq e^{\sin(\pi/x)} \leq e^1 \Rightarrow \sqrt{x}/e \leq \sqrt{x} e^{\sin(\pi/x)} \leq \sqrt{x} e$. Since $\lim_{x \rightarrow 0^+} (\sqrt{x}/e) = 0$ and

$\lim_{x \rightarrow 0^+} (\sqrt{x} e) = 0$, we have $\lim_{x \rightarrow 0^+} [\sqrt{x} e^{\sin(\pi/x)}] = 0$ by the Squeeze Theorem.

39. If $x > -4$, then $|x+4| = x+4$, so $\lim_{x \rightarrow -4^+} |x+4| = \lim_{x \rightarrow -4^+} (x+4) = -4+4 = 0$.

If $x < -4$, then $|x+4| = -(x+4)$, so $\lim_{x \rightarrow -4^-} |x+4| = \lim_{x \rightarrow -4^-} -(x+4) = -(-4+4) = 0$.

Since the right and left limits are equal, $\lim_{x \rightarrow -4} |x+4| = 0$.

40. If $x < -4$, then $|x+4| = -(x+4)$, so $\lim_{x \rightarrow -4^-} \frac{|x+4|}{x+4} = \lim_{x \rightarrow -4^-} \frac{-(x+4)}{x+4} = \lim_{x \rightarrow -4^-} (-1) = -1$.

41. If $x > 2$, then $|x-2| = x-2$, so $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^+} \frac{x-2}{x-2} = \lim_{x \rightarrow 2^+} 1 = 1$. If $x < 2$, then $|x-2| = -(x-2)$, so

$\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} = \lim_{x \rightarrow 2^-} \frac{-(x-2)}{x-2} = \lim_{x \rightarrow 2^-} -1 = -1$. The right and left limits are different, so $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ does not exist.

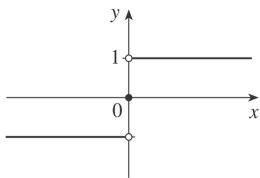
42. If $x > \frac{3}{2}$, then $|2x-3| = 2x-3$, so $\lim_{x \rightarrow 1.5^+} \frac{2x^2-3x}{|2x-3|} = \lim_{x \rightarrow 1.5^+} \frac{2x^2-3x}{2x-3} = \lim_{x \rightarrow 1.5^+} \frac{x(2x-3)}{2x-3} = \lim_{x \rightarrow 1.5^+} x = 1.5$. If

$x < \frac{3}{2}$, then $|2x-3| = 3-2x$, so $\lim_{x \rightarrow 1.5^-} \frac{2x^2-3x}{|2x-3|} = \lim_{x \rightarrow 1.5^-} \frac{2x^2-3x}{-(2x-3)} = \lim_{x \rightarrow 1.5^-} \frac{x(2x-3)}{-(2x-3)} = \lim_{x \rightarrow 1.5^-} -x = -1.5$. The

right and left limits are different, so $\lim_{x \rightarrow 1.5} \frac{2x^2-3x}{|2x-3|}$ does not exist.

43. Since $|x| = -x$ for $x < 0$, we have $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{-x} \right) = \lim_{x \rightarrow 0^-} \frac{2}{x}$, which does not exist since the denominator approaches 0 and the numerator does not.

44. Since $|x|=x$ for $x>0$, we have $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right) = \lim_{x \rightarrow 0^+} \left(\frac{1}{x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0^+} 0 = 0$.



45. (a)

(b)

(i) Since $\operatorname{sgn}x=1$ for $x>0$, $\lim_{x \rightarrow 0^+} \operatorname{sgn}x = \lim_{x \rightarrow 0^+} 1 = 1$.

(ii) Since $\operatorname{sgn}x=-1$ for $x<0$, $\lim_{x \rightarrow 0^-} \operatorname{sgn}x = \lim_{x \rightarrow 0^-} -1 = -1$.

(iii) Since $\lim_{x \rightarrow 0^-} \operatorname{sgn}x \neq \lim_{x \rightarrow 0^+} \operatorname{sgn}x$, $\lim_{x \rightarrow 0} \operatorname{sgn}x$ does not exist.

(iv) Since $|\operatorname{sgn}x|=1$ for $x \neq 0$, $\lim_{x \rightarrow 0} |\operatorname{sgn}x| = \lim_{x \rightarrow 0} 1 = 1$.

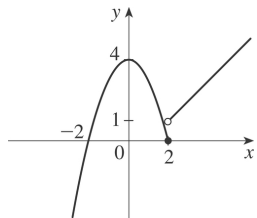
46. (a)

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (4-x^2) = \lim_{x \rightarrow 2^-} 4 - \lim_{x \rightarrow 2^-} x^2 \\ &= 4 - 4 = 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (x-1) = \lim_{x \rightarrow 2^+} x - \lim_{x \rightarrow 2^+} 1 \\ &= 2 - 1 = 1 \end{aligned}$$

(b) No, $\lim_{x \rightarrow 2} f(x)$ does not exist since $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$.

(c)

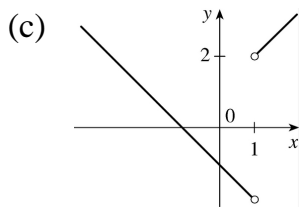


47. (a)

$$(i) \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1^+} (x + 1) = 2$$

$$(ii) \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{|x - 1|} = \lim_{x \rightarrow 1^-} \frac{x^2 - 1}{-(x - 1)} = \lim_{x \rightarrow 1^-} -(x + 1) = -2$$

(b) No, $\lim_{x \rightarrow 1} F(x)$ does not exist since $\lim_{x \rightarrow 1^+} F(x) \neq \lim_{x \rightarrow 1^-} F(x)$.



48. (a)

$$(i) \lim_{x \rightarrow 0^+} h(x) = \lim_{x \rightarrow 0^+} x^2 = 0^2 = 0$$

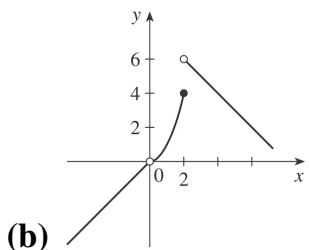
$$(ii) \lim_{x \rightarrow 0^-} h(x) = \lim_{x \rightarrow 0^-} x = 0, \text{ so } \lim_{x \rightarrow 0} h(x) = 0.$$

$$(iii) \lim_{x \rightarrow 1} h(x) = \lim_{x \rightarrow 1} x^2 = 1^2 = 1$$

$$(iv) \lim_{x \rightarrow 2^-} h(x) = \lim_{x \rightarrow 2^-} x^2 = 2^2 = 4$$

$$(v) \lim_{x \rightarrow 2^+} h(x) = \lim_{x \rightarrow 2^+} (8 - x) = 8 - 2 = 6$$

(vi) Since $\lim_{x \rightarrow 2^-} h(x) \neq \lim_{x \rightarrow 2^+} h(x)$, $\lim_{x \rightarrow 2} h(x)$ does not exist.



49. (a)

(i) $[x] = -2$ for $-2 \leq x < -1$, so $\lim_{x \rightarrow -2^+} [x] = \lim_{x \rightarrow -2^+} (-2) = -2$

(ii) $[x] = -3$ for $-3 \leq x < -2$, so $\lim_{x \rightarrow -2^-} [x] = \lim_{x \rightarrow -2^-} (-3) = -3$.

The right and left limits are different, so $\lim_{x \rightarrow -2} [x]$ does not exist.

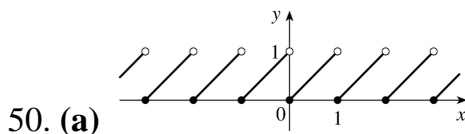
(iii) $[x] = -3$ for $-3 \leq x < -2$, so $\lim_{x \rightarrow -2.4} [x] = \lim_{x \rightarrow -2.4} (-3) = -3$.

(b)

(i) $[x] = n-1$ for $n-1 \leq x < n$, so $\lim_{x \rightarrow n^-} [x] = \lim_{x \rightarrow n^-} (n-1) = n-1$.

(ii) $[x] = n$ for $n \leq x < n+1$, so $\lim_{x \rightarrow n^+} [x] = \lim_{x \rightarrow n^+} n = n$.

(c) $\lim_{x \rightarrow a} [x]$ exists $\Leftrightarrow a$ is not an integer.



(b)

(i) $\lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} (x - [x]) = \lim_{x \rightarrow n^-} [x - (n-1)] = n - (n-1) = 1$

(ii) $\lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} (x - [x]) = \lim_{x \rightarrow n^+} (x - n) = n - n = 0$

(c) $\lim_{x \rightarrow a} f(x)$ exists $\Leftrightarrow a$ is not an integer.

51. The graph of $f(x) = [x] + [-x]$ is the same as the graph of $g(x) = -1$ with holes at each integer, since $f(a) = 0$ for any integer a . Thus, $\lim_{x \rightarrow 2^-} f(x) = -1$ and $\lim_{x \rightarrow 2^+} f(x) = -1$, so $\lim_{x \rightarrow 2} f(x) = -1$. However,

$f(2) = [2] + [-2] = 2 + (-2) = 0$, so $\lim_{x \rightarrow 2} f(x) \neq f(2)$.

52. $\lim_{v \rightarrow c^-} \left(L_0 \sqrt{1 - \frac{v^2}{c^2}} \right) = L_0 \sqrt{1 - 1} = 0$. As the velocity approaches the speed of light, the length

approaches 0 .

A left-hand limit is necessary since L is not defined for $v > c$.

53. Since $p(x)$ is a polynomial, $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Thus, by the Limit Laws,

$$\begin{aligned} \lim_{x \rightarrow a} p(x) &= \lim_{x \rightarrow a} \left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \right) \\ &= a_0 + a_1 \lim_{x \rightarrow a} x + a_2 \lim_{x \rightarrow a} x^2 + \cdots + a_n \lim_{x \rightarrow a} x^n \\ &= a_0 + a_1a + a_2a^2 + \cdots + a_na^n = p(a) \end{aligned}$$

Thus, for any polynomial p , $\lim_{x \rightarrow a} p(x) = p(a)$.

54. Let $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are any polynomials, and suppose that $q(a) \neq 0$. Thus,

$$\lim_{x \rightarrow a} r(x) = \lim_{x \rightarrow a} \frac{p(x)}{q(x)} = \frac{\lim_{x \rightarrow a} p(x)}{\lim_{x \rightarrow a} q(x)} \quad [\text{Limit Law 5}] = \frac{p(a)}{q(a)} \quad [\text{Exercise 53}] = r(a) .$$

55. Observe that $0 \leq f(x) \leq x^2$ for all x , and $\lim_{x \rightarrow 0} 0 = 0 = \lim_{x \rightarrow 0} x^2$. So, by the Squeeze Theorem,

$$\lim_{x \rightarrow 0} f(x) = 0 .$$

56. Let $f(x) = [x]$ and $g(x) = -[x]$. Then $\lim_{x \rightarrow 3} f(x)$ and $\lim_{x \rightarrow 3} g(x)$ do not exist (Example 10) but

$$\lim_{x \rightarrow 3} [f(x) + g(x)] = \lim_{x \rightarrow 3} ([x] - [x]) = \lim_{x \rightarrow 3} 0 = 0 .$$

57. Let $f(x) = H(x)$ and $g(x) = 1 - H(x)$, where H is the Heaviside function defined in Exercise 1.3.59.

Thus, either f or g is 0 for any value of x . Then $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 0} g(x)$ do not exist, but

$$\lim_{x \rightarrow 0} [f(x)g(x)] = \lim_{x \rightarrow 0} 0 = 0 .$$

58.

$$\lim_{x \rightarrow 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \lim_{x \rightarrow 2} \left(\frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} \right)$$

$$\begin{aligned}
&= \lim_{x \rightarrow 2} \left[\frac{(\sqrt{6-x})^2 - 2^2}{(\sqrt{3-x})^2 - 1^2} \cdot \frac{\sqrt{3-x+1}}{\sqrt{6-x+2}} \right] = \lim_{x \rightarrow 2} \left(\frac{6-x-4}{3-x-1} \cdot \frac{\sqrt{3-x+1}}{\sqrt{6-x+2}} \right) \\
&= \lim_{x \rightarrow 2} \frac{(2-x)(\sqrt{3-x+1})}{(2-x)(\sqrt{6-x+2})} = \lim_{x \rightarrow 2} \frac{\sqrt{3-x+1}}{\sqrt{6-x+2}} = \frac{1}{2}
\end{aligned}$$

59. Since the denominator approaches 0 as $x \rightarrow -2$, the limit will exist only if the numerator also approaches 0 as $x \rightarrow -2$. In order for this to happen, we need $\lim_{x \rightarrow -2} (3x^2 + ax + a + 3) = 0 \Leftrightarrow$

$3(-2)^2 + a(-2) + a + 3 = 0 \Leftrightarrow 12 - 2a + a + 3 = 0 \Leftrightarrow a = 15$. With $a = 15$, the limit becomes

$$\lim_{x \rightarrow -2} \frac{3x^2 + 15x + 18}{x^2 + x - 2} = \lim_{x \rightarrow -2} \frac{3(x+2)(x+3)}{(x-1)(x+2)} = \lim_{x \rightarrow -2} \frac{3(x+3)}{x-1} = \frac{3(-2+3)}{-2-1} = \frac{3}{-3} = -1.$$

60. *Solution 1:* First, we find the coordinates of P and Q as functions of r . Then we can find the equation of the line determined by these two points, and thus find the x -intercept (the point R), and take the limit as $r \rightarrow 0$.

The coordinates of P are $(0, r)$. The point Q is the point of intersection of the two circles $x^2 + y^2 = r^2$ and $(x-1)^2 + y^2 = 1$. Eliminating y from these equations, we get $r^2 - x^2 = 1 - (x-1)^2 \Leftrightarrow r^2 = 1 + 2x - 1 \Leftrightarrow x = \frac{1}{2}r^2$.

Substituting back into the equation of the shrinking circle to find the y -coordinate, we get

$$\left(\frac{1}{2}r^2\right)^2 + y^2 = r^2 \Leftrightarrow y^2 = r^2 \left(1 - \frac{1}{4}r^2\right) \Leftrightarrow y = r\sqrt{1 - \frac{1}{4}r^2} \text{ (the positive } y\text{-value)}. \text{ So the coordinates of }$$

Q are $\left(\frac{1}{2}r^2, r\sqrt{1 - \frac{1}{4}r^2}\right)$. The equation of the line joining P and Q is thus

$$y - r = \frac{r\sqrt{1 - \frac{1}{4}r^2} - r}{\frac{1}{2}r^2 - 0} (x - 0). \text{ We set } y = 0 \text{ in order to find the } x\text{-intercept, and get}$$

$$x = -r \frac{\frac{1}{2}r^2}{r\left(\sqrt{1 - \frac{1}{4}r^2} - 1\right)} = \frac{-\frac{1}{2}r^2 \left(\sqrt{1 - \frac{1}{4}r^2} + 1\right)}{1 - \frac{1}{4}r^2 - 1} = 2 \left(\sqrt{1 - \frac{1}{4}r^2} + 1\right).$$

Now we take the limit as $r \rightarrow 0^+$: $\lim_{r \rightarrow 0^+} x = \lim_{r \rightarrow 0^+} 2 \left(\sqrt{1 - \frac{1}{4}r^2} + 1\right) = \lim_{r \rightarrow 0^+} 2(\sqrt{1} + 1) = 4$.

So the limiting position of R is the point $(4, 0)$.

Solution 2: We add a few lines to the diagram, as shown. Note that $\angle PQS=90^\circ$ (subtended by diameter PS).

So $\angle SQR=90^\circ=\angle OQT$ (subtended by diameter OT). It follows that $\angle OQS=\angle TQR$. Also $\angle PSQ=90^\circ-\angle SPQ=\angle ORP$. Since $\triangle QOS$ is isosceles, so is $\triangle QTR$, implying that $QT=TR$. As the circle C_2 shrinks, the point Q plainly approaches the origin, so the point R must approach a point twice as far from the origin as T , that is, the point $(4,0)$, as above.

