

1. (a) To have $5x+3$ within a distance of 0.1 of 13, we must have $12.9 \leq 5x+3 \leq 13.1 \Rightarrow 9.9 \leq 5x \leq 10.1 \Rightarrow 1.98 \leq x \leq 2.02$. Thus, x must be within 0.02 units of 2 so that $5x+3$ is within 0.1 of 13.

(b) Use 0.01 in place of 0.1 in part (a) to obtain 0.002.

2. (a) To have $6x-1$ within a distance of 0.01 of 29, we must have $28.99 \leq 6x-1 \leq 29.01 \Rightarrow 29.99 \leq 6x \leq 30.01 \Rightarrow 4.998\bar{3} \leq x \leq 5.001\bar{6}$. Thus, x must be within 0.0016 units of 5 so that $6x-1$ is within 0.01 of 29.

(b) As in part (a) with 0.001 in place of 0.01, we obtain 0.00016.

(c) As in part (a) with 0.0001 in place of 0.01, we obtain 0.000016.

3. On the left side of $x=2$, we need $|x-2| < \left| \frac{10}{7} - 2 \right| = \frac{4}{7}$. On the right side, we need

$|x-2| < \left| \frac{10}{3} - 2 \right| = \frac{4}{3}$. For both of these conditions to be satisfied at once, we need the more

restrictive of the two to hold, that is, $|x-2| < \frac{4}{7}$. So we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

4. On the left side, we need $|x-5| < |4-5| = 1$. On the right side, we need $|x-5| < |5.7-5| = 0.7$. For both conditions to be satisfied at once, we need the more restrictive condition to hold; that is, $|x-5| < 0.7$. So we can choose $\delta = 0.7$, or any smaller positive number.

5. The leftmost question mark is the solution of $\sqrt{x} = 1.6$ and the rightmost, $\sqrt{x} = 2.4$. So the values are $1.6^2 = 2.56$ and $2.4^2 = 5.76$. On the left side, we need $|x-4| < |2.56-4| = 1.44$. On the right side, we need $|x-4| < |5.76-4| = 1.76$. To satisfy both conditions, we need the more restrictive condition to hold — namely, $|x-4| < 1.44$. Thus, we can choose $\delta = 1.44$, or any smaller positive number.

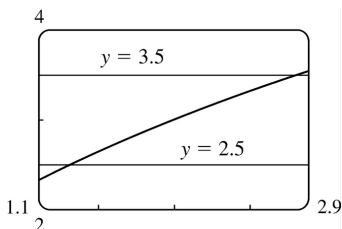
6. The left-hand question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the right-hand question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$. On the left side, we need

$|x-1| < \left| \frac{1}{\sqrt{2}} - 1 \right| \approx 0.292$ (rounding down to be safe). On the right side, we need

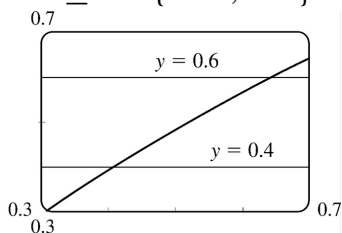
$|x-1| < \left| \sqrt{\frac{3}{2}} - 1 \right| \approx 0.224$. The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

7. $|\sqrt{4x+1} - 3| < 0.5 \Leftrightarrow 2.5 < \sqrt{4x+1} < 3.5$. We plot the three parts of this inequality on the same screen

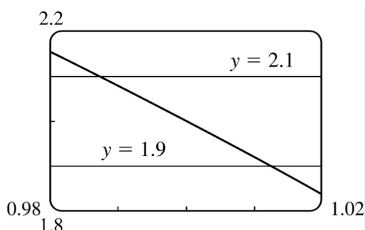
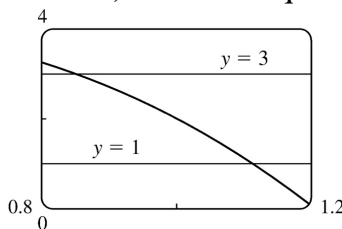
and identify the x -coordinates of the points of intersection using the cursor. It appears that the inequality holds for $1.3125 \leq x \leq 2.8125$. Since $|2 - 1.3125| = 0.6875$ and $|2 - 2.8125| = 0.8125$, we choose $0 < \delta < \min\{0.6875, 0.8125\} = 0.6875$.



8. $\left| \sin x - \frac{1}{2} \right| < 0.1 \Leftrightarrow 0.4 < \sin x < 0.6$. From the graph, we see that for this inequality to hold, we need $0.42 \leq x \leq 0.64$. So since $|0.5 - 0.42| = 0.08$ and $|0.5 - 0.64| = 0.14$, we choose $0 < \delta \leq \min\{0.08, 0.14\} = 0.08$.



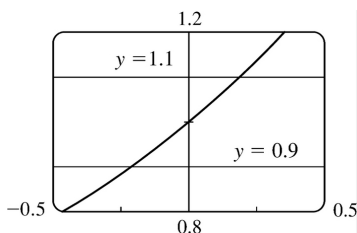
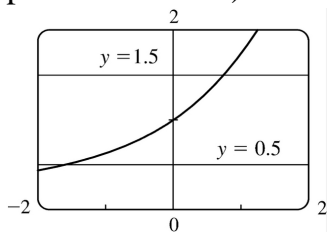
9. For $\varepsilon = 1$, the definition of a limit requires that we find δ such that $\left| (4+x-3x^3) - 2 \right| < 1 \Leftrightarrow 1 < 4+x-3x^3 < 3$ whenever $0 < |x-1| < \delta$. If we plot the graphs of $y=1$, $y=4+x-3x^3$ and $y=3$ on the same screen, we see that we need $0.86 \leq x \leq 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\varepsilon = 0.1$, we must find δ such that $\left| (4+x-3x^3) - 2 \right| < 0.1 \Leftrightarrow 1.9 < 4+x-3x^3 < 2.1$ whenever $0 < |x-1| < \delta$. From the graph, we see that we need $0.988 \leq x \leq 1.012$. So since $|1 - 0.988| = 0.012$ and $|1 - 1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



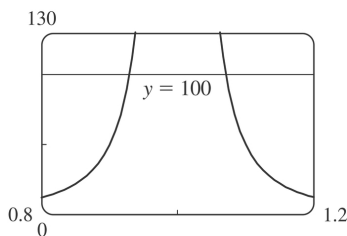
10. For $\varepsilon=0.5$, the definition of a limit requires that we find δ such that $\left| \frac{e^x-1}{x} - 1 \right| < 0.5 \Leftrightarrow$

$0.5 < \frac{e^x-1}{x} < 1.5$ whenever $0 < |x-0| < \delta$. If we plot the graphs of $y=0.5$, $y=\frac{e^x-1}{x}$, and $y=1.5$ on the same screen, we see that we need $-1.59 \leq x \leq 0.76$. So since $|0 - (-1.59)| = 1.59$ and $|0 - 0.76| = 0.76$, we choose $\delta=0.76$ (or any smaller positive number). For $\varepsilon=0.1$, we must find δ such that

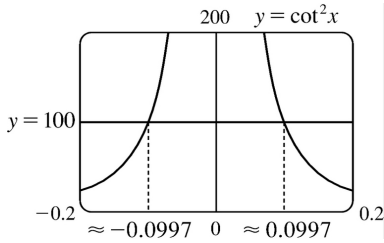
$\left| \frac{e^x-1}{x} - 1 \right| < 0.1 \Leftrightarrow 0.9 < \frac{e^x-1}{x} < 1.1$ whenever $0 < |x-0| < \delta$. From the graph, we see that we need $-0.21 \leq x \leq 0.18$. So since $|0 - (-0.21)| = 0.21$ and $|0 - 0.18| = 0.18$, we choose $\delta=0.18$ (or any smaller positive number) for the inequality to hold.



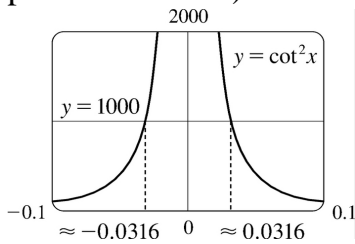
11. From the graph, we see that $\frac{x}{(x^2+1)(x-1)^2} > 100$ whenever $0.93 \leq x \leq 1.07$. So since $|1 - 0.93| = 0.07$ and $|1 - 1.07| = 0.07$, we can take $\delta = 0.07$ (or any smaller positive number).



12. For $M=100$, we need $-0.0997 < x < 0$ or $0 < x < 0.0997$. Thus, we choose $\delta = 0.0997$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 100$.



For $M=1000$, we need $-0.0316 < x < 0$ or $0 < x < 0.0316$. Thus, we choose $\delta=0.0316$ (or any smaller positive number) so that if $0 < |x| < \delta$, then $\cot^2 x > 1000$.



13. (a) $A = \pi r^2$ and $A = 1000 \text{ cm}^2 \Rightarrow \pi r^2 = 1000 \Rightarrow r^2 = \frac{1000}{\pi} \Rightarrow$

$r = \sqrt{\frac{1000}{\pi}} \quad [r > 0] \approx 17.8412 \text{ cm.}$

(b) $|A - 1000| \leq 5 \Rightarrow -5 \leq \pi r^2 - 1000 \leq 5 \Rightarrow 1000 - 5 \leq \pi r^2 \leq 1000 + 5 \Rightarrow$

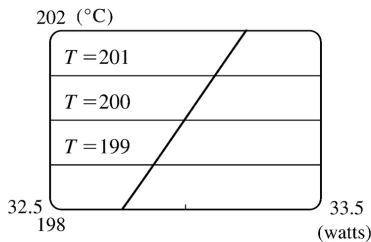
$\sqrt{\frac{995}{\pi}} \leq r \leq \sqrt{\frac{1005}{\pi}} \Rightarrow 17.7966 \leq r \leq 17.8858. \quad \sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}} \approx 0.04466$ and

$\sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}} \approx 0.04455$. So if the machinist gets the radius within 0.0445 cm of 17.8412,

the area will be within 5 cm² of 1000.

(c) x is the radius, $f(x)$ is the area, a is the target radius given in part (a), L is the target area (1000), ϵ is the tolerance in the area (5), and δ is the tolerance in the radius given in part (b).

14. (a) $T = 0.1w^2 + 2.155w + 20$ and $T = 200 \Rightarrow 0.1w^2 + 2.155w + 20 = 200 \Rightarrow$ [by the quadratic formula or from the graph] $w \approx 33.0$ watts ($w > 0$)

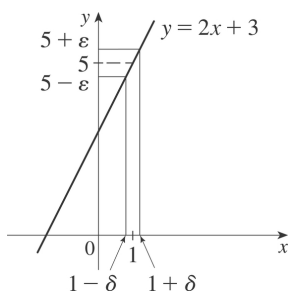


(b) From the graph, $199 \leq T \leq 201 \Rightarrow 32.89 < w < 33.11$.

(c) x is the input power, $f(x)$ is the temperature, a is the target input power given in part (a), L is the target temperature (200), ε is the tolerance in the temperature (1), and δ is the tolerance in the power input in watts indicated in part (b) (0.11 watts).

15. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-1| < \delta$, then $|(2x+3)-5| < \varepsilon$. But $|(2x+3)-5| < \varepsilon \Leftrightarrow |2x-2| < \varepsilon \Leftrightarrow 2|x-1| < \varepsilon \Leftrightarrow |x-1| < \varepsilon/2$. So if we choose $\delta = \varepsilon/2$, then $0 < |x-1| < \delta \Rightarrow |(2x+3)-5| < \varepsilon$. Thus, $\lim_{x \rightarrow 1} (2x+3) = 5$ by the definition of a limit.

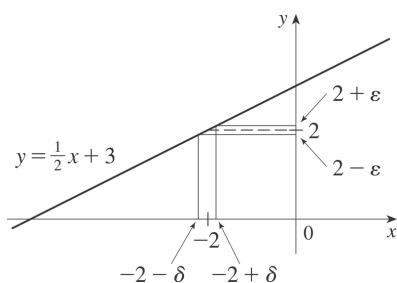
$x \rightarrow 1$



16. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-2)| < \delta$, then $\left| \left(\frac{1}{2}x+3 \right) - 2 \right| < \varepsilon$. But

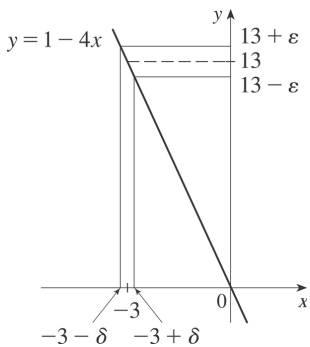
$\left| \left(\frac{1}{2}x+3 \right) - 2 \right| < \varepsilon \Leftrightarrow \left| \frac{1}{2}x+1 \right| < \varepsilon \Leftrightarrow \frac{1}{2}|x+2| < \varepsilon \Leftrightarrow |x-(-2)| < 2\varepsilon$. So if we choose $\delta = 2\varepsilon$, then

$0 < |x-(-2)| < \delta \Rightarrow \left| \left(\frac{1}{2}x+3 \right) - 2 \right| < \varepsilon$. Thus, $\lim_{x \rightarrow -2} \left(\frac{1}{2}x+3 \right) = 2$ by the definition of a limit.

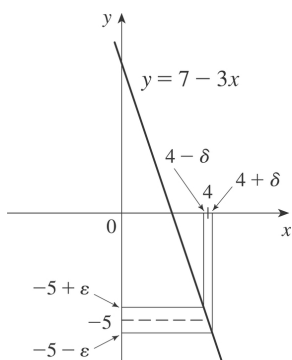


17. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-(-3)| < \delta$, then $|(1-4x)-13| < \varepsilon$. But $|(1-4x)-13| < \varepsilon \Leftrightarrow |-4x-12| < \varepsilon \Leftrightarrow |-4||x+3| < \varepsilon \Leftrightarrow |x-(-3)| < \varepsilon/4$. So if we choose $\delta = \varepsilon/4$, then $0 < |x-(-3)| < \delta \Rightarrow |(1-4x)-13| < \varepsilon$. Thus, $\lim_{x \rightarrow -3} (1-4x) = 13$ by the definition of a limit.

$x \rightarrow -3$



18. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 4| < \delta$, then $|(7 - 3x) - (-5)| < \varepsilon$. But $|(7 - 3x) - (-5)| < \varepsilon \Leftrightarrow |-3x + 12| < \varepsilon \Leftrightarrow |-3| |x - 4| < \varepsilon \Leftrightarrow |x - 4| < \varepsilon / 3$. So if we choose $\delta = \varepsilon / 3$, then $0 < |x - 4| < \delta \Rightarrow |(7 - 3x) - (-5)| < \varepsilon$. Thus, $\lim_{x \rightarrow 4} (7 - 3x) = -5$ by the definition of a limit.



19. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 3| < \delta$, then $\left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon \Leftrightarrow \frac{1}{5} |x - 3| < \varepsilon \Leftrightarrow |x - 3| < 5\varepsilon$. So choose $\delta = 5\varepsilon$. Then $0 < |x - 3| < \delta \Rightarrow |x - 3| < 5\varepsilon \Rightarrow \frac{|x - 3|}{5} < \varepsilon \Rightarrow \left| \frac{x}{5} - \frac{3}{5} \right| < \varepsilon$. By the definition of a limit, $\lim_{x \rightarrow 3} \frac{x}{5} = \frac{3}{5}$.

20. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - 6| < \delta$, then $\left| \left(\frac{x}{4} + 3 \right) - \frac{9}{2} \right| < \varepsilon \Leftrightarrow \left| \frac{x}{4} - \frac{3}{2} \right| < \varepsilon \Leftrightarrow \frac{1}{4} |x - 6| < \varepsilon \Leftrightarrow |x - 6| < 4\varepsilon$. So choose $\delta = 4\varepsilon$. Then $0 < |x - 6| < \delta \Rightarrow |x - 6| < 4\varepsilon \Rightarrow \frac{|x - 6|}{4} < \varepsilon \Rightarrow \left| \frac{x}{4} - \frac{3}{2} \right| < \varepsilon \Rightarrow \left| \left(\frac{x}{4} + 3 \right) - \frac{9}{2} \right| < \varepsilon$. By the definition of a limit, $\lim_{x \rightarrow 6} \left(\frac{x}{4} + 3 \right) = \frac{9}{2}$.

21. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-5)| < \delta$, then $\left| \left(4 - \frac{3}{5}x \right) - 7 \right| < \varepsilon \Leftrightarrow$

$$\left| -\frac{3}{5}x-3 \right| < \varepsilon \Leftrightarrow \frac{3}{5}|x+5| < \varepsilon \Leftrightarrow |x-(-5)| < \frac{5}{3}\varepsilon. \text{ So choose } \delta = \frac{5}{3}\varepsilon. \text{ Then } |x-(-5)| < \delta \Rightarrow$$

$$\left| \left(4 - \frac{3}{5}x\right) - 7 \right| < \varepsilon. \text{ Thus, } \lim_{x \rightarrow -5} \left(4 - \frac{3}{5}x\right) = 7 \text{ by the definition of a limit.}$$

22. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $\left| \frac{x^2+x-12}{x-3} - 7 \right| < \varepsilon$. Notice that if

$0 < |x-3|$, then $x \neq 3$, so $\frac{x^2+x-12}{x-3} = \frac{(x+4)(x-3)}{x-3} = x+4$. Thus, when $0 < |x-3|$, we have

$$\left| \frac{x^2+x-12}{x-3} - 7 \right| < \varepsilon \Leftrightarrow |(x+4)-7| < \varepsilon \Leftrightarrow |x-3| < \varepsilon. \text{ We take } \delta = \varepsilon \text{ and see that } 0 < |x-3| < \delta \Rightarrow$$

$$\left| \frac{x^2+x-12}{x-3} - 7 \right| < \varepsilon. \text{ By the definition of a limit, } \lim_{x \rightarrow 3} \frac{x^2+x-12}{x-3} = 7.$$

23. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|x-a| < \varepsilon$. So $\delta = \varepsilon$ will work.

24. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|c-c| < \varepsilon$. But $|c-c| = 0$, so this will be true no matter what δ we pick.

25. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $|x^2-0| < \varepsilon \Leftrightarrow x^2 < \varepsilon \Leftrightarrow |x| < \sqrt{\varepsilon}$. Take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-0| < \delta \Rightarrow |x^2-0| < \varepsilon$. Thus, $\lim_{x \rightarrow 0} x^2 = 0$ by the definition of a limit.

26. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $|x^3-0| < \varepsilon \Leftrightarrow |x|^3 < \varepsilon \Leftrightarrow |x| < \sqrt[3]{\varepsilon}$. Take $\delta = \sqrt[3]{\varepsilon}$. Then $0 < |x-0| < \delta \Rightarrow |x^3-0| < \delta^3 = \varepsilon$. Thus, $\lim_{x \rightarrow 0} x^3 = 0$ by the definition of a limit.

27. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-0| < \delta$, then $||x|-0| < \varepsilon$. But $||x|| = |x|$. So this is true if we pick $\delta = \varepsilon$. Thus, $\lim_{x \rightarrow 0} |x| = 0$ by the definition of a limit.

28. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $9-\delta < x < 9$, then $\left| \sqrt[4]{9-x}-0 \right| < \varepsilon \Leftrightarrow \sqrt[4]{9-x} < \varepsilon \Leftrightarrow 9-x < \varepsilon^4 \Leftrightarrow 9-\varepsilon^4 < x < 9$. So take $\delta = \varepsilon^4$. Then $9-\delta < x < 9 \Rightarrow \left| \sqrt[4]{9-x}-0 \right| < \varepsilon$. Thus, $\lim_{x \rightarrow 9^-} \sqrt[4]{9-x} = 0$ by the definition of a limit.

29. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $\left| (x^2-4x+5)-1 \right| < \varepsilon \Leftrightarrow |x^2-4x+4| < \varepsilon \Leftrightarrow$

$|(x-2)^2| < \varepsilon$. So take $\delta = \sqrt{\varepsilon}$. Then $0 < |x-2| < \delta \Leftrightarrow |x-2| < \sqrt{\varepsilon} \Leftrightarrow |(x-2)^2| < \varepsilon$. Thus, $\lim_{x \rightarrow 2} (x^2 - 4x + 5) = 1$ by the definition of a limit.

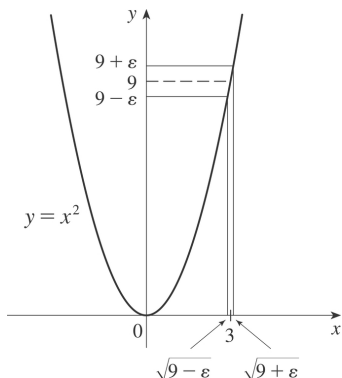
30. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-3| < \delta$, then $|(x^2 + x - 4) - 8| < \varepsilon \Leftrightarrow |x^2 + x - 12| < \varepsilon \Leftrightarrow |(x-3)(x+4)| < \varepsilon$. Notice that if $|x-3| < 1$, then $-1 < x-3 < 1 \Rightarrow 6 < x+4 < 8 \Rightarrow |x+4| < 8$. So take $\delta = \min\{1, \varepsilon/8\}$. Then $0 < |x-3| < \delta \Leftrightarrow |(x-3)(x+4)| \leq |8(x-3)| = 8 \cdot |x-3| < 8\delta \leq \varepsilon$. Thus, $\lim_{x \rightarrow 3} (x^2 + x - 4) = 8$ by the definition of a limit.

31. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x - (-2)| < \delta$, then $|(x^2 - 1) - 3| < \varepsilon$ or upon simplifying we need $|x^2 - 4| < \varepsilon$ whenever $0 < |x+2| < \delta$. Notice that if $|x+2| < 1$, then $-1 < x+2 < 1 \Rightarrow -5 < x-2 < -3 \Rightarrow |x-2| < 5$. So take $\delta = \min\{\varepsilon/5, 1\}$. Then $0 < |x+2| < \delta \Rightarrow |x-2| < 5$ and $|x+2| < \varepsilon/5$, so $|(x^2 - 1) - 3| = |(x+2)(x-2)| = |x+2| |x-2| < (\varepsilon/5)(5) = \varepsilon$. Thus, by the definition of a limit, $\lim_{x \rightarrow -2} (x^2 - 1) = 3$.

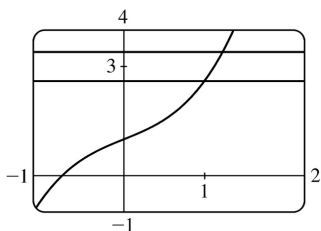
32. Given $\varepsilon > 0$, we need $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|x^3 - 8| < \varepsilon$. Now $|x^3 - 8| = |(x-2)(x^2 + 2x + 4)|$. If $|x-2| < 1$, that is, $1 < x < 3$, then $x^2 + 2x + 4 < 3^2 + 2(3) + 4 = 19$ and so $|x^3 - 8| = |x-2| (x^2 + 2x + 4) < 19|x-2|$. So if we take $\delta = \min\left\{1, \frac{\varepsilon}{19}\right\}$, then $0 < |x-2| < \delta \Rightarrow |x^3 - 8| = |x-2| (x^2 + 2x + 4) < \frac{\varepsilon}{19} \cdot 19 = \varepsilon$. Thus, by the definition of a limit, $\lim_{x \rightarrow 2} x^3 = 8$.

33. Given $\varepsilon > 0$, we let $\delta = \min\left\{2, \frac{\varepsilon}{8}\right\}$. If $0 < |x-3| < \delta$, then $|x-3| < 2 \Rightarrow -2 < x-3 < 2 \Rightarrow 4 < x+3 < 8 \Rightarrow |x+3| < 8$. Also $|x-3| < \frac{\varepsilon}{8}$, so $|x^2 - 9| = |x+3| |x-3| < 8 \cdot \frac{\varepsilon}{8} = \varepsilon$. Thus, $\lim_{x \rightarrow 3} x^2 = 9$.

34. From the figure, our choices for δ are $\delta_1 = 3 - \sqrt{9 - \varepsilon}$ and $\delta_2 = \sqrt{9 + \varepsilon} - 3$. The *largest* possible choice for δ is the minimum value of $\{\delta_1, \delta_2\}$; that is, $\delta = \min\{\delta_1, \delta_2\} = \delta_2 = \sqrt{9 + \varepsilon} - 3$.



35. (a) The points of intersection in the graph are $(x_1, 2.6)$ and $(x_2, 3.4)$ with $x_1 \approx 0.891$ and $x_2 \approx 1.093$. Thus, we can take δ to be the smaller of $1-x_1$ and x_2-1 . So $\delta = x_2 - 1 \approx 0.093$.



(b) Solving $x^3 + x + 1 = 3 + \epsilon$ gives us two nonreal complex roots and one real root, which is

$$x(\epsilon) = \frac{\left(216 + 108\epsilon + 12\sqrt{336 + 324\epsilon + 81\epsilon^2}\right)^{2/3} - 12}{6\left(216 + 108\epsilon + 12\sqrt{336 + 324\epsilon + 81\epsilon^2}\right)^{1/3}}. \text{ Thus, } \delta = x(\epsilon) - 1.$$

(c) If $\epsilon = 0.4$, then $x(\epsilon) \approx 1.093272342$ and $\delta = x(\epsilon) - 1 \approx 0.093$, which agrees with our answer in part (a).

36. 1. *Guessing a value for δ* Let $\epsilon > 0$ be given. We have to find a number $\delta > 0$ such that

$\left| \frac{1}{x} - \frac{1}{2} \right| < \epsilon$ whenever $0 < |x-2| < \delta$. But $\left| \frac{1}{x} - \frac{1}{2} \right| = \left| \frac{2-x}{2x} \right| = \frac{|x-2|}{|2x|} < \epsilon$. We find a positive constant C such that $\frac{1}{|2x|} < C \Rightarrow \frac{|x-2|}{|2x|} < C|x-2|$ and we can make $C|x-2| < \epsilon$ by taking $|x-2| < \frac{\epsilon}{C} = \delta$.

We restrict x to lie in the interval $|x-2| < 1 \Rightarrow 1 < x < 3$ so $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow \frac{1}{6} < \frac{1}{2x} < \frac{1}{2} \Rightarrow \frac{1}{|2x|} < \frac{1}{2}$. So

$C = \frac{1}{2}$ is suitable. Thus, we should choose $\delta = \min\{1, 2\epsilon\}$.

2. *Showing that δ works* Given $\epsilon > 0$ we let $\delta = \min\{1, 2\epsilon\}$. If $0 < |x-2| < \delta$, then $|x-2| < 1 \Rightarrow 1 < x < 3 \Rightarrow$

$\frac{1}{|2x|} < \frac{1}{2}$ (as in part 1). Also $|x-2| < 2\epsilon$, so $\left| \frac{1}{x} - \frac{1}{2} \right| = \frac{|x-2|}{|2x|} < \frac{1}{2} \cdot 2\epsilon = \epsilon$. This shows that

$$\lim_{x \rightarrow 2} (1/x) = \frac{1}{2} .$$

37. 1. *Guessing a value for δ* Given $\varepsilon > 0$, we must find $\delta > 0$ such that $|\sqrt{x} - \sqrt{a}| < \varepsilon$ whenever $0 < |x-a| < \delta$. But $|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$ (from the hint). Now if we can find a positive constant C such that $\sqrt{x} + \sqrt{a} > C$ then $\frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \frac{|x-a|}{C} < \varepsilon$, and we take $|x-a| < C\varepsilon$. We can find this number by restricting x to lie in some interval centered at a . If $|x-a| < \frac{1}{2}a$, then $-\frac{1}{2}a < x-a < \frac{1}{2}a \Rightarrow \frac{1}{2}a < x < \frac{3}{2}a$

$\Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$, and so $C = \sqrt{\frac{1}{2}a} + \sqrt{a}$ is a suitable choice for the constant. So

$|x-a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a} \right) \varepsilon$. This suggests that we let $\delta = \min \left\{ \frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a} \right) \varepsilon \right\}$.

2. *Showing that δ works* Given $\varepsilon > 0$, we let $\delta = \min \left\{ \frac{1}{2}a, \left(\sqrt{\frac{1}{2}a} + \sqrt{a} \right) \varepsilon \right\}$. If $0 < |x-a| < \delta$,

then $|x-a| < \frac{1}{2}a \Rightarrow \sqrt{x} + \sqrt{a} > \sqrt{\frac{1}{2}a} + \sqrt{a}$ (as in part 1). Also $|x-a| < \left(\sqrt{\frac{1}{2}a} + \sqrt{a} \right) \varepsilon$, so

$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \frac{\left(\sqrt{\frac{1}{2}a} + \sqrt{a} \right) \varepsilon}{\left(\sqrt{\frac{1}{2}a} + \sqrt{a} \right)} = \varepsilon$. Therefore, $\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$ by the definition of a limit.

38. Suppose that $\lim_{t \rightarrow 0} H(t) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |t| < \delta \Rightarrow |H(t) - L| < \frac{1}{2} \Leftrightarrow$

$L - \frac{1}{2} < H(t) < L + \frac{1}{2}$. For $0 < t < \delta$, $H(t) = 1$, so $1 < L + \frac{1}{2} \Rightarrow L > \frac{1}{2}$. For $-\delta < t < 0$, $H(t) = 0$, so $L - \frac{1}{2} < 0 \Rightarrow$

$L < \frac{1}{2}$. This contradicts $L > \frac{1}{2}$. Therefore, $\lim_{t \rightarrow 0} H(t)$ does not exist.

39. Suppose that $\lim_{x \rightarrow 0} f(x) = L$. Given $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that $0 < |x| < \delta \Rightarrow |f(x) - L| < \frac{1}{2}$.

Take any rational number r with $0 < |r| < \delta$. Then $f(r) = 0$, so $|0 - L| < \frac{1}{2}$, so $L \leq |L| < \frac{1}{2}$. Now take

any irrational number s with $0 < |s| < \delta$. Then $f(s) = 1$, so $|1 - L| < \frac{1}{2}$. Hence, $1 - L < \frac{1}{2}$, so $L > \frac{1}{2}$. This

contradicts $L < \frac{1}{2}$, so $\lim_{x \rightarrow 0} f(x)$ does not exist.

40. First suppose that $\lim_{x \rightarrow a} f(x) = L$. Then, given $\varepsilon > 0$ there exists $\delta > 0$ so that $0 < |x-a| < \delta \Rightarrow$

$|f(x) - L| < \varepsilon$. Then $a - \delta < x < a \Rightarrow 0 < |x-a| < \delta$ so $|f(x) - L| < \varepsilon$. Thus,

$\lim_{x \rightarrow a^-} f(x) = L$. Also $a < x < a + \delta \Rightarrow 0 < |x - a| < \delta$ so $|f(x) - L| < \varepsilon$. Hence, $\lim_{x \rightarrow a^+} f(x) = L$.

Now suppose $\lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$. Let $\varepsilon > 0$ be given. Since $\lim_{x \rightarrow a^-} f(x) = L$, there exists $\delta_1 > 0$ so that

$a - \delta_1 < x < a \Rightarrow |f(x) - L| < \varepsilon$. Since $\lim_{x \rightarrow a^+} f(x) = L$, there exists $\delta_2 > 0$ so that $a < x < a + \delta_2 \Rightarrow |f(x) - L| < \varepsilon$. Let

δ be the smaller of δ_1 and δ_2 . Then $0 < |x - a| < \delta \Rightarrow a - \delta_1 < x < a$ or $a < x < a + \delta_2$ so $|f(x) - L| < \varepsilon$. Hence,

$\lim_{x \rightarrow a} f(x) = L$. So we have proved that $\lim_{x \rightarrow a} f(x) = L \Leftrightarrow \lim_{x \rightarrow a^-} f(x) = L = \lim_{x \rightarrow a^+} f(x)$.

$$41. \frac{1}{(x+3)^4} > 10,000 \Leftrightarrow (x+3)^4 < \frac{1}{10,000} \Leftrightarrow |x+3| < \frac{1}{\sqrt[4]{10,000}} \Leftrightarrow |x - (-3)| < \frac{1}{10}$$

42. Given $M > 0$, we need $\delta > 0$ such that $0 < |x+3| < \delta \Rightarrow 1/(x+3)^4 > M$. Now $\frac{1}{(x+3)^4} > M \Leftrightarrow (x+3)^4 < \frac{1}{M} \Leftrightarrow$

$$|x+3| < \frac{1}{\sqrt[4]{M}}. \text{ So take } \delta = \frac{1}{\sqrt[4]{M}}. \text{ Then } 0 < |x+3| < \delta = \frac{1}{\sqrt[4]{M}} \Rightarrow \frac{1}{(x+3)^4} > M, \text{ so } \lim_{x \rightarrow -3} \frac{1}{(x+3)^4} = \infty.$$

43. Given $M < 0$ we need $\delta > 0$ so that $\ln x < M$ whenever $0 < x < \delta$; that is, $x = e^{\ln x} < e^M$ whenever $0 < x < \delta$.

This suggests that we take $\delta = e^M$. If $0 < x < e^M$, then $\ln x < \ln e^M = M$. By the definition of a limit,

$\lim_{x \rightarrow 0^+} \ln x = -\infty$.

44. (a) Let M be given. Since $\lim_{x \rightarrow a} f(x) = \infty$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow f(x) > M + 1 - c$.

Since $\lim_{x \rightarrow a} g(x) = c$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow |g(x) - c| < 1 \Rightarrow g(x) > c - 1$. Let δ be the

smaller of δ_1 and δ_2 . Then $0 < |x - a| < \delta \Rightarrow f(x) + g(x) > (M + 1 - c) + (c - 1) = M$. Thus, $\lim_{x \rightarrow a} [f(x) + g(x)] = \infty$.

(b) Let $M > 0$ be given. Since $\lim_{x \rightarrow a} g(x) = c > 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow |g(x) - c| < c/2$

$\Rightarrow g(x) > c/2$. Since $\lim_{x \rightarrow a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow f(x) > 2M/c$. Let

$\delta = \min \{ \delta_1, \delta_2 \}$. Then $0 < |x - a| < \delta \Rightarrow f(x)g(x) > \frac{2M}{c} \cdot \frac{c}{2} = M$, so $\lim_{x \rightarrow a} f(x)g(x) = \infty$.

(c) Let $N < 0$ be given. Since $\lim_{x \rightarrow a} g(x) = c < 0$, there exists $\delta_1 > 0$ such that $0 < |x - a| < \delta_1 \Rightarrow |g(x) - c| < -c/2$

$\Rightarrow g(x) < c/2$. Since $\lim_{x \rightarrow a} f(x) = \infty$, there exists $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow f(x) > 2N/c$. (Note that

$c < 0$ and $N < 0 \Rightarrow 2N/c > 0$.) Let $\delta = \min \{ \delta_1, \delta_2 \}$. Then $0 < |x-a| < \delta \Rightarrow f(x) > 2N/c \Rightarrow$
 $f(x)g(x) < \frac{2N}{c} \cdot \frac{c}{2} = N$, so $\lim_{x \rightarrow a} f(x)g(x) = -\infty$.