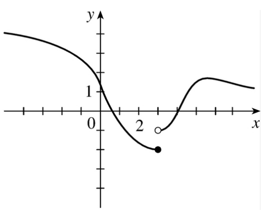
1. From Definition 1, $\lim_{x \to 4} f(x) = f(4)$.

2. The graph of f has no hole, jump, or vertical asymptote.

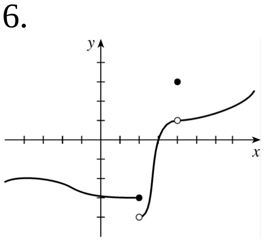
3. **(a)** The following are the numbers at which f is discontinuous and the type of discontinuity at that number: –4 (removable), –2 (jump), 2 (jump), 4 (infinite).
(b) f is continuous from the left at –2 since $\lim_{x \to -2^-} f(x) = f(-2)$. f is continuous from the right at 2 and 4 since $\lim_{x \to 2^+} f(x) = f(2)$ and $\lim_{x \to 4^+} f(x) = f(4)$. It is continuous from neither side at –4 since $f(-4)$ is undefined.

4. g is continuous on $[-4,-2)$, $(-2,2)$, $[2,4)$, $(4,6)$, and $(6,8)$.

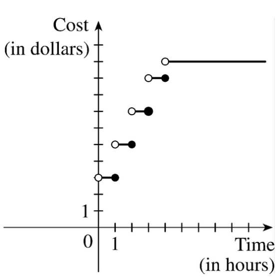
5. The graph of $y=f(x)$ must have a discontinuity at $x=3$ and must show that $\lim_{x \to 3^-} f(x) = f(3)$.



6.



7. **(a)**



(b) There are discontinuities at times $t=1$, 2 , 3 , and 4 . A person parking in the lot would want to keep in mind that the charge will jump at the beginning of each hour.

8. **(a)** Continuous; at the location in question, the temperature changes smoothly as time passes, without any instantaneous jumps from one temperature to another.

(b) Continuous; the temperature at a specific time changes smoothly as the distance due west from New York City increases, without any instantaneous jumps.
(c) Discontinuous; as the distance due west from New York City increases, the altitude above sea level may jump from one height to another without going through all of the intermediate values ––– at a cliff, for example.
(d) Discontinuous; as the distance traveled increases, the cost of the ride jumps in small increments.
(e) Discontinuous; when the lights are switched on (or off), the current suddenly changes between 0 and some nonzero value, without passing through all of the intermediate values. This is debatable, though, depending on your definition of current.

9. Since f and g are continuous functions,

$$\begin{aligned}\lim_{x\to 3}[2f(x)-g(x)] &= 2\lim_{x\to 3} f(x)-\lim_{x\to 3} g(x) && \text{[by Limit Laws 2 and 3]}\\ &= 2f(3)-g(3) && \text{[by continuity of } f \text{ and } g \text{ at } x=3\text{]}\\ &= 2\cdot 5-g(3)=10-g(3)\end{aligned}$$

Since it is given that $\lim_{x\to 3}[2f(x)-g(x)]=4$, we have $10-g(3)=4$, so $g(3)=6$.

10. $\lim_{x\to 4} f(x)=\lim_{x\to 4}\left(x^2+\sqrt{7-x}\right)=\lim_{x\to 4} x^2+\sqrt{\lim_{x\to 4} 7-\lim_{x\to 4} x}=4^2+\sqrt{7-4}=16+\sqrt{3}=f(4)$.
By the definition of continuity, f is continuous at $a=4$.

11. $\lim_{x\to -1} f(x)=\lim_{x\to -1}\left(x+2x^3\right)^4=\left(\lim_{x\to -1} x+2\lim_{x\to -1} x^3\right)^4=\left[-1+2(-1)^3\right]^4=(-3)^4=81=f(-1)$.
By the definition of continuity, f is continuous at $a=-1$.

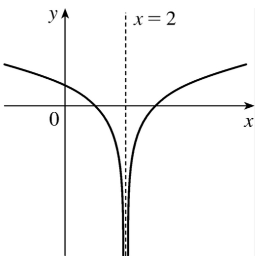
12. $\lim_{x\to 4} g(x)=\lim_{x\to 4}\dfrac{x+1}{2x^2-1}=\dfrac{\lim_{x\to 4} x+\lim_{x\to 4} 1}{2\lim_{x\to 4} x^2-\lim_{x\to 4} 1}=\dfrac{4+1}{2(4)^2-1}=\dfrac{5}{31}=g(4)$. So g is continuous at 4 .

13. For $a>2$, we have $\lim_{x\to a} f(x)=\lim_{x\to a}\dfrac{2x+3}{x-2}=\dfrac{\lim_{x\to a}(2x+3)}{\lim_{x\to a}(x-2)}$ [Limit Law 5] $=\dfrac{2\lim_{x\to a} x+\lim_{x\to a} 3}{\lim_{x\to a} x-\lim_{x\to a} 2}$ [1, 2, and 3] $=\dfrac{2a+3}{a-2}$ [7 and 8] $=f(a)$. Thus, f is continuous at $x=a$ for every a in $(2,\infty)$; that is, f is continuous on $(2,\infty)$.

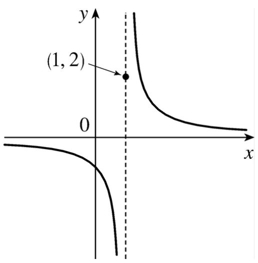
14. For $a<3$, we have $\lim_{x\to a} g(x)=\lim_{x\to a} 2\sqrt{3-x}=2\lim_{x\to a}\sqrt{3-x}$ [Limit Law 3] $=2\sqrt{\lim_{x\to a}(3-x)}$ [11]

$=2\sqrt{\lim_{x\to a} 3-\lim_{x\to a} x}$ [2] $=2\sqrt{3-a}$ [7 and 8] $=g(a)$, so g is continuous at $x=a$ for every a in $(-\infty,3)$. Also, $\lim_{x\to 3^-} g(x)=0=g(3)$, so g is continuous from the left at 3 . Thus, g is continuous on $(-\infty,3]$.

15. $f(x)=\ln|x-2|$ is discontinuous at 2 since $f(2)=\ln 0$ is not defined.

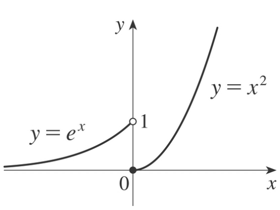


16. $f(x)=\begin{cases} 1/(x-1) & \text{if } x\neq 1 \\ 2 & \text{if } x=1 \end{cases}$ is discontinuous at 1 because $\lim_{x\to 1} f(x)$ does not exist.



17. $f(x)=\begin{cases} e^x & \text{if} x<0 \\ x^2 & \text{if} x\geq 0 \end{cases}$

The left–hand limit of f at $a=0$ is $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} e^x=1$. The right–hand limit of f at $a=0$ is $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x^2=0$. Since these limits are not equal, $\lim_{x\to 0} f(x)$ does not exist and f is discontinuous at 0.



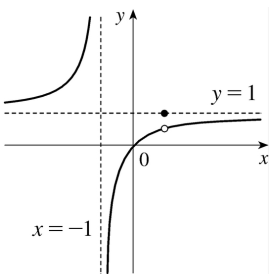
18.

$$f(x)=\begin{cases} \dfrac{x^2-x}{x^2-1} & \text{if } x\neq 1 \\ 1 & \text{if } x=1 \end{cases}$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2} \ ,$$

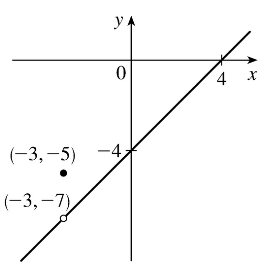
but $f(1)=1$, so f is discontinous at 1.



19. $f(x)=\begin{cases} \dfrac{x^2-x-12}{x+3} & \text{if } x\neq -3 \\ -5 & \text{if } x=-3 \end{cases} = \begin{cases} x-4 & \text{if } x\neq -3 \\ -5 & \text{if } x=-3 \end{cases}$

So $\lim_{x\to -3} f(x)=\lim_{x\to -3} (x-4)=-7$ and $f(-3)=-5$.

Since $\lim_{x\to -3} f(x)\neq f(-3)$, f is discontinuous at -3 .

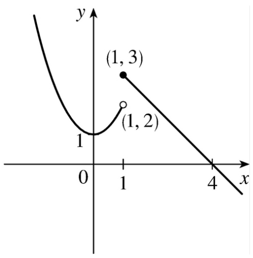


20. $f(x)=\begin{cases} 1+x^2 & \text{if } x<1 \\ 4-x & \text{if } x\geq 1 \end{cases}$

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} (1+x^2)=1+1^2=2$ and

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} (4-x)=4-1=3$.

Thus, f is discontinuous at 1 because $\lim_{x\to 1} f(x)$ does not exist.



21. $F(x)=\dfrac{x}{x^2+5x+6}$ is a rational function. So by Theorem 5 (or Theorem 7), F is continuous at every number in its domain, $\left\{x\,|\,x^2+5x+6\neq 0\right\}=\{x\,|\,(x+3)(x+2)\neq 0\}=\{x\,|\,x\neq -3,-2\}$ or $(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$.

22. By Theorem 7, the root function $\sqrt[3]{x}$ and the polynomial function $1+x^3$ are continuous on R . By part 4 of Theorem 4, the product $G(x)=\sqrt[3]{x}\left(1+x^3\right)$ is continuous on its domain, R .

23. By Theorem 5, the polynomials x^2 and $2x-1$ are continuous on $(-\infty,\infty)$. By Theorem 7, the root function $\sqrt{x}$ is continuous on $[0,\infty)]$. By Theorem 9, the composite function $\sqrt{2x-1}$ is continuous on its domain, $[\frac{1}{2},\infty)]$. By part 1 of Theorem 4, the sum $R(x)=x^2+\sqrt{2x-1}$ is continuous on $[\frac{1}{2},\infty)]$.

24. By Theorem 7, the trigonometric function $\sin x$ and the polynomial function $x+1$ are continuous on R . By part 5 of Theorem 4, $h(x)=\dfrac{\sin x}{x+1}$ is continuous on its domain, $\{x\,|\,x\neq -1\}$.

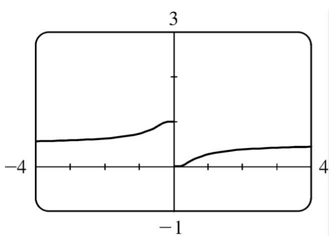
25. By Theorem 5, the polynomial $5x$ is continuous on $(-\infty,\infty)$. By Theorems 9 and 7, $\sin 5x$ is continuous on $(-\infty,\infty)$. By Theorem 7, e^x is continuous on $(-\infty,\infty)$. By part 4 of Theorem 4, the product of e^x and $\sin 5x$ is continuous at all numbers which are in both of their domains, that is, on $(-\infty,\infty)$.

26. By Theorem 5, the polynomial x^2-1 is continuous on $(-\infty,\infty)$. By Theorem 7, $\sin^{-1}$ is continuous on its domain, $[-1,1]$. By Theorem 9, $\sin^{-1}\left(x^2-1\right)$ is continuous on its domain, which is $\left\{x\,|\,-1\leq x^2-1\leq 1\right\}=\left\{x\,|\,0\leq x^2\leq 2\right\}=\{x\,|\,|x|\leq\sqrt{2}\}=[-\sqrt{2},\sqrt{2}]$.

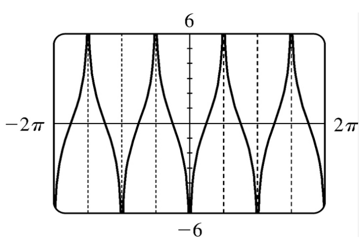
27. By Theorem 5, the polynomial t^4-1 is continuous on $(-\infty,\infty)$. By Theorem 7, ln x is continuous on its domain, $(0,\infty)$. By Theorem 9, $\ln\left(t^4-1\right)$ is continuous on its domain, which is $\left\{t\,|\,t^4-1>0\right\}=\left\{t\,|\,t^4>1\right\}=\{t\,|\,|t|>1\}=(-\infty,-1)\cup(1,\infty)$.

28. By Theorem 7, $\sqrt{x}$ is continuous on $[0,\infty)$. By Theorems 7 and 9, $e^{\sqrt{x}}$ is continuous on $[0,\infty)$. Also by Theorems 7 and 9, $\cos\left(e^{\sqrt{x}}\right)$ is continuous on $[0,\infty)$.

29. The function $y=\dfrac{1}{1+e^{1/x}}$ is discontinuous at $x=0$ because the left– and right–hand limits at $x=0$ are different.



30. The function $y=\tan^2 x$ is discontinuous at $x=\dfrac{\pi}{2}+\pi k$, where k is any integer. The function $y=\ln\left(\tan^2 x\right)$ is also discontinuous where $\tan^2 x$ is 0 , that is, at $x=\pi k$. So $y=\ln\left(\tan^2 x\right)$ is discontinuous at $x=\dfrac{\pi}{2}n$, n any integer.



31. Because we are dealing with root functions, $5+\sqrt{x}$ is continuous on $[0,\infty)$, $\sqrt{x+5}$ is continuous on $[-5,\infty)$, so the quotient $f(x)=\dfrac{5+\sqrt{x}}{\sqrt{5+x}}$ is continuous on $[0,\infty)$. Since f is continuous at $x=4$, $\lim_{x\to 4} f(x)=f(4)=\dfrac{7}{3}$.

32. Because x is continuous on R , sin x is continuous on R , and $x+\sin x$ is continuous on R , the

composite function $f(x)=\sin(x+\sin x)$ is continuous on R, so $\lim_{x\to\pi} f(x)=f(\pi)=\sin(\pi+\sin\pi)=\sin\pi=0$.

33. Because x^2-x is continuous on R, the composite function $f(x)=e^{x^2-x}$ is continuous on R, so $\lim_{x\to 1} f(x)=f(1)=e^{1-1}=e^0=1$.

34. Because arctan is a continuous function, we can apply Theorem 8.

$$\lim_{x\to 2}\arctan\left(\frac{x^2-4}{3x^2-6x}\right)=\arctan\left(\lim_{x\to 2}\frac{(x+2)(x-2)}{3x(x-2)}\right)=\arctan\left(\lim_{x\to 2}\frac{x+2}{3x}\right)=\arctan\frac{2}{3}\approx 0.588$$

35. $f(x)=\begin{cases} x^2 & \text{if } x<1 \\ \sqrt{x} & \text{if } x\geq 1\end{cases}$

By Theorem 5, since $f(x)$ equals the polynomial x^2 on $(-\infty,1)$, f is continuous on $(-\infty,1)$. By Theorem 7, since $f(x)$ equals the root function $\sqrt{x}$ on $(1,\infty)$, f is continuous on $(1,\infty)$. At $x=1$, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x^2=1$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}\sqrt{x}=1$. Thus, $\lim_{x\to 1} f(x)$ exists and equals 1. Also, $f(1)=\sqrt{1}=1$. Thus, f is continuous at $x=1$. We conclude that f is continuous on $(-\infty,\infty)$.

36. $f(x)=\begin{cases} \sin x & \text{if } x<\pi/4 \\ \cos x & \text{if } x\geq \pi/4\end{cases}$

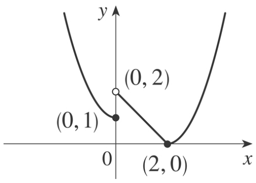
By Theorem 7, the trigonometric functions are continuous. Since $f(x)=\sin x$ on $(-\infty,\pi/4)$ and $f(x)=\cos x$ on $(\pi/4,\infty)$, f is continuous on $(-\infty,\pi/4)\cup(\pi/4,\infty)$.

$\lim_{x\to(\pi/4)^-} f(x)=\lim_{x\to(\pi/4)^-}\sin x=\sin\frac{\pi}{4}=1/\sqrt{2}$ since the sine function is continuous at $\pi/4$. Similarly, $\lim_{x\to(\pi/4)^+} f(x)=\lim_{x\to(\pi/4)^+}\cos x=1/\sqrt{2}$ by continuity of the cosine function at $\pi/4$. Thus, $\lim_{x\to(\pi/4)} f(x)$ exists and equals $1/\sqrt{2}$, which agrees with the value $f(\pi/4)$. Therefore, f is continuous at $\pi/4$, so f is continuous on $(-\infty,\infty)$.

37. $f(x)=\begin{cases} 1+x^2 & \text{if } x\leq 0 \\ 2-x & \text{if } 0<x\leq 2 \\ (x-2)^2 & \text{if } x>2\end{cases}$

f is continuous on $(-\infty,0)$, $(0,2)$, and $(2,\infty)$ since it is a polynomial on each of these intervals. Now

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1+x^2) = 1$ and

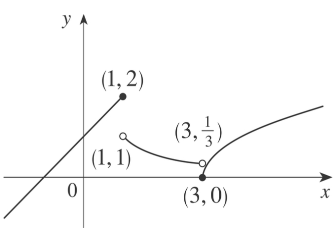


$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2-x) = 2$, so f is discontinuous at 0 . Since $f(0)=1$, f is continuous from the left at 0. Also, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2-x) = 0$, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-2)^2 = 0$, and $f(2)=0$, so f is continuous at 2. The only number at which f is discontinuous is 0 .

38. $f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 1/x & \text{if } 1 < x < 3 \\ \sqrt{x-3} & \text{if } x \ge 3 \end{cases}$

f is continuous on $(-\infty, 1)$, $(1,3)$, and $(3,\infty)$, where it is a polynomial, a rational function, and a composite of a root function with a polynomial, respectively. Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+1) = 2$ and

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (1/x) = 1$, so f is discontinuous at 1 .



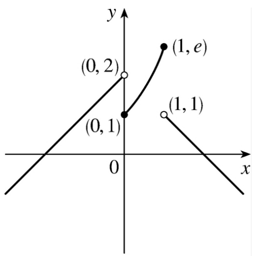
Since $f(1)=2$, f is continuous from the left at 1 . Also, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (1/x) = 1/3$, and

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x-3} = 0 = f(3)$, so f is discontinuous at 3 , but it is continuous from the right at 3.

39. $f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ e^x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } x > 1 \end{cases}$

f is continuous on $(-\infty, 0)$ and $(1,\infty)$ since on each of these intervals it is a polynomial; it is continuous on $(0,1)$ since it is an exponential. Now

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(x+2)=2$ and $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} e^x=1$, so f is discontinuous at 0. Since $f(0)=1$, f is continuous from the right at 0. Also $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} e^x=e$ and $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2-x)=1$, so f is discontinuous at 1. Since $f(1)=e$, f is continuous from the left at 1.



40. By Theorem 5, each piece of F is continuous on its domain. We need to check for continuity at $r=R$.
$\lim_{r\to R^-} F(r)=\lim_{r\to R^-}\frac{GMr}{R^3}=\frac{GM}{R^2}$ and $\lim_{r\to R^+} F(r)=\lim_{r\to R^+}\frac{GM}{r^2}=\frac{GM}{R^2}$, so $\lim_{r\to R} F(r)=\frac{GM}{R^2}$. Since $F(R)=\frac{GM}{R^2}$, F is continuous at R. Therefore, F is a continuous function of r.

41. f is continuous on $(-\infty,3)$ and $(3,\infty)$. Now $\lim_{x\to 3^-} f(x)=\lim_{x\to 3^-}(cx+1)=3c+1$ and $\lim_{x\to 3^+} f(x)=\lim_{x\to 3^+}\left(cx^2-1\right)=9c-1$. So f is continuous $\Leftrightarrow 3c+1=9c-1 \Leftrightarrow 6c=2 \Leftrightarrow c=\frac{1}{3}$. Thus, for f to be continuous on $(-\infty,\infty)$, $c=\frac{1}{3}$.

42. The functions x^2-c^2 and $cx+20$, considered on the intervals $(-\infty,4)$ and $[4,\infty)$ respectively, are continuous for any value of c. So the only possible discontinuity is at $x=4$. For the function to be continuous at $x=4$, the left–hand and right–hand limits must be the same. Now $\lim_{x\to 4^-} g(x)=\lim_{x\to 4^-}\left(x^2-c^2\right)=16-c^2$ and $\lim_{x\to 4^+} g(x)=\lim_{x\to 4^+}(cx+20)=4c+20=g(4)$. Thus, $16-c^2=4c+20 \Leftrightarrow c^2+4c+4=0 \Leftrightarrow c=-2$.

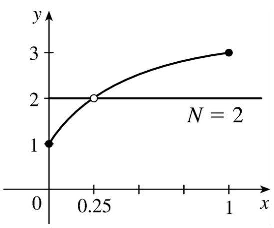
43. **(a)** $f(x)=\frac{x^2-2x-8}{x+2}=\frac{(x-4)(x+2)}{x+2}$ has a removable discontinuity at -2 because $g(x)=x-4$ is continuous on R and $f(x)=g(x)$ for $x\neq -2$. [The discontinuity is removed by defining $f(-2)=-6$.]
(b)

$f(x)=\frac{x-7}{|x-7|} \Rightarrow \lim_{x\to 7^-} f(x)=-1$ and $\lim_{x\to 7^+} f(x)=1$. Thus, $\lim_{x\to 7} f(x)$ does not exist, so the discontinuity is not removable. (It is a jump discontinuity.)

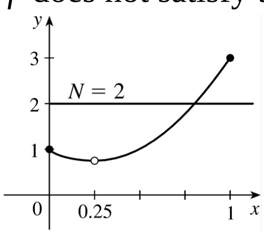
(c) $f(x)=\frac{x^3+64}{x+4}=\frac{(x+4)(x^2-4x+16)}{x+4}$ has a removable discontinuity at -4 because $g(x)=x^2-4x+16$ is continuous on R and $f(x)=g(x)$ for $x\neq -4$.[The discontinuity is removed by defining $f(-4)=48$.]

(d) $f(x)=\frac{3-\sqrt{x}}{9-x}=\frac{3-\sqrt{x}}{\left(3-\sqrt{x}\right)\left(3+\sqrt{x}\right)}$ has a removable discontinuity at 9 because $g(x)=\frac{1}{3+\sqrt{x}}$ is continuous on R and $f(x)=g(x)$ for $x\neq 9$. [The discontinuity is removed by defining $f(9)=\frac{1}{6}$.]

44.



f does not satisfy the conclusion of the Intermediate Value Theorem.



f does satisfy the conclusion of the Intermediate Value Theorem.

45. $f(x)=x^3-x^2+x$ is continuous on the interval $[2,3]$, $f(2)=6$, and $f(3)=21$. Since $6<10<21$, there is a number c in $(2,3)$ such that $f(c)=10$ by the Intermediate Value Theorem.

46. $f(x)=x^2$ is continuous on the interval $[1,2]$, $f(1)=1$, and $f(2)=4$. Since $1<2<4$, there is a number c in $(1,2)$ such that $f(c)=c^2=2$ by the Intermediate Value Theorem.

47. $f(x)=x^4+x-3$ is continuous on the interval $[1,2]$, $f(1)=-1$, and $f(2)=15$. Since $-1<0<15$, there is a number c in $(1,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $x^4+x-3=0$ in the interval $(1,2)$.

48. $f(x)=\sqrt[3]{x}+x-1$ is continuous on the interval $[0,1]$, $f(0)=-1$, and $f(1)=1$. Since $-1<0<1$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the

equation $\sqrt[3]{x}+x-1=0$, or $\sqrt[3]{x}=1-x$, in the interval (0,1).

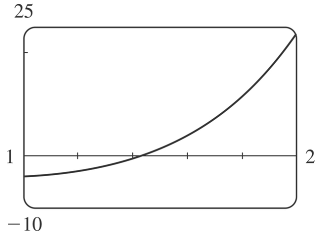
49. $f(x)=\cos x-x$ is continuous on the interval $[0,1]$, $f(0)=1$, and $f(1)=\cos 1-1\approx-0.46$. Since $-0.46<0<1$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\cos x-x=0$, or $\cos x=x$, in the interval $(0,1)$.

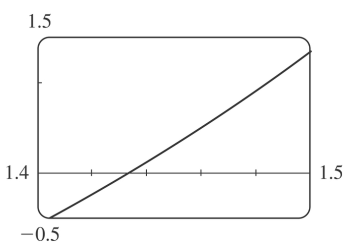
50. $f(x)=\ln x-e^{-x}$ is continuous on the interval $[1,2]$, $f(1)=-e^{-1}\approx-0.37$, and $f(2)=\ln 2-e^{-2}\approx 0.56$. Since $-0.37<0<0.56$, there is a number c in $(1,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\ln x-e^{-x}=0$, or $\ln x=e^{-x}$, in the interval $(1,2)$.

51. **(a)** $f(x)=e^{x}+x-2$ is continuous on the interval $[0,1]$, $f(0)=-1<0$, and $f(1)=e-1\approx 1.72>0$. Since $-1<0<1.72$, there is a number c in $(0,1)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $e^{x}+x-2=0$, or $e^{x}=2-x$, in the interval $(0,1)$.
(b) $f(0.44)\approx-0.007<0$ and $f(0.45)\approx 0.018>0$, so there is a root between 0.44 and 0.45 .

52. **(a)** $f(x)=\sin x-2+x$ is continuous on $[0,2]$, $f(0)=-2$, and $f(2)=\sin 2\approx 0.91$. Since $-2<0<0.91$, there is a number c in $(0,2)$ such that $f(c)=0$ by the Intermediate Value Theorem. Thus, there is a root of the equation $\sin x-2+x=0$, or $\sin x=2-x$, in the interval $(0,2)$.
(b) $f(1.10)\approx-0.009<0$ and $f(1.11)\approx 0.006>0$, so there is a root between 1.10 and 1.11 .

53. **(a)** Let $f(x)=x^{5}-x^{2}-4$. Then $f(1)=1^{5}-1^{2}-4=-4<0$ and $f(2)=2^{5}-2^{2}-4=24>0$. So by the Intermediate Value Theorem, there is a number c in $(1,2)$ such that $f(c)=c^{5}-c^{2}-4=0$.
(b) We can see from the graphs that, correct to three decimal places, the root is $x\approx 1.434$.

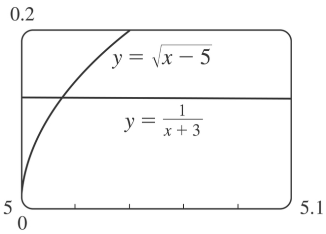




54. **(a)** Let $f(x)=\sqrt{x-5}-\frac{1}{x+3}$. Then $f(5)=-\frac{1}{8}<0$ and $f(6)=\frac{8}{9}>0$, and f is continuous on $[5,\infty)$. So

by the Intermediate Value Theorem, there is a number c in $(5,6)$ such that $f(c)=0$. This implies that $\frac{1}{c+3}=\sqrt{c-5}$.

(b) Using the intersect feature of the graphing device, we find that the root of the equation is $x=5.016$, correct to three decimal places.



55. ($\Rightarrow$) If f is continuous at a , then by Theorem 8 with $g(h)=a+h$, we have $\lim_{h\to 0} f(a+h)=f\left(\lim_{h\to 0}(a+h)\right)=f(a)$.

($\Leftarrow$) Let $\varepsilon>0$. Since $\lim_{h\to 0} f(a+h)=f(a)$, there exists $\delta>0$ such that $0<|h|<\delta \Rightarrow |f(a+h)-f(a)|<\varepsilon$. So if $0<|x-a|<\delta$, then $|f(x)-f(a)|=|f(a+(x-a))-f(a)|<\varepsilon$. Thus, $\lim_{x\to a} f(x)=f(a)$ and so f is continuous at a .

56.

$$\begin{aligned}\lim_{h\to 0}\sin(a+h) &= \lim_{h\to 0}(\sin a\cos h+\cos a\sin h)=\lim_{h\to 0}(\sin a\cos h)+\lim_{h\to 0}(\cos a\sin h)\\ &=\left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\cos h\right)+\left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\sin h\right)\\ &=(\sin a)(1)+(\cos a)(0)=\sin a\end{aligned}$$

57. As in the previous exercise, we must show that $\lim_{h\to 0}\cos(a+h)=\cos a$ to prove that the cosine function is continuous.

$$\begin{aligned}\lim_{h\to 0}\cos(a+h) &= \lim_{h\to 0}(\cos a\cos h-\sin a\sin h)\\ &=\lim_{h\to 0}(\cos a\cos h)-\lim_{h\to 0}(\sin a\sin h)\\ &=\left(\lim_{h\to 0}\cos a\right)\left(\lim_{h\to 0}\cos h\right)-\left(\lim_{h\to 0}\sin a\right)\left(\lim_{h\to 0}\sin h\right)\\ &=(\cos a)(1)-(\sin a)(0)=\cos a\end{aligned}$$

58. **(a)** Since f is continuous at a , $\lim_{x\to a} f(x)=f(a)$ .Thus, using the Constant Multiple Law of Limits,

we have $\lim_{x \to a} (cf)(x) = \lim_{x \to a} cf(x) = c\lim_{x \to a} f(x) = cf(a) = (cf)(a)$. Therefore, cf is continuous at a .
(b) Since f and g are continuous at a , $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$. Since $g(a) \neq 0$, we can use the Quotient Law of Limits: $\lim_{x \to a} \left(\frac{f}{g} \right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g} \right)(a)$. Thus, $\frac{f}{g}$ is continuous at a .

59. $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere. For, given any number a and any $\delta > 0$, the interval $(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $f(a) = 0$ or 1 , there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|f(x) - f(a)| = 1$. Thus, $\lim_{x \to a} f(x) \neq f(a)$. [In fact $\lim_{x \to a} f(x)$ does not even exist.]

60. $g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$ is continuous at 0 . To see why, note that $-|x| \leq g(x) \leq |x|$, so by the Squeeze Theorem $\lim_{x \to 0} g(x) = 0 = g(0)$. But g is continuous nowhere else. For if $a \neq 0$ and $\delta > 0$, the interval $(a - \delta, a + \delta)$ contains both infinitely many rational and infinitely many irrational numbers. Since $g(a) = 0$ or a , there are infinitely many numbers x with $0 < |x - a| < \delta$ and $|g(x) - g(a)| > |a|/2$. Thus, $\lim_{x \to a} g(x) \neq g(a)$.

61. If there is such a number, it satisfies the equation $x^3 + 1 = x \Leftrightarrow x^3 - x + 1 = 0$. Let the left–hand side of this equation be called $f(x)$. Now $f(-2) = -5 < 0$, and $f(-1) = 1 > 0$. Note also that $f(x)$ is a polynomial, and thus continuous. So by the Intermediate Value Theorem, there is a number c between -2 and -1 such that $f(c) = 0$, so that $c = c^3 + 1$.

62. **(a)** $\lim_{x \to 0^+} F(x) = 0$ and $\lim_{x \to 0^-} F(x) = 0$, so $\lim_{x \to 0} F(x) = 0$, which is $F(0)$, and hence F is continuous at $x = a$ if $a = 0$. For $a > 0$, $\lim_{x \to a} F(x) = \lim_{x \to a} x = a = F(a)$. For $a < 0$, $\lim_{x \to a} F(x) = \lim_{x \to a} (-x) = -a = F(a)$. Thus, F is continuous at $x = a$; that is, continuous everywhere.
(b) Assume that f is continuous on the interval I . Then for $a \in I$, $\lim_{x \to a} |f(x)| = \left| \lim_{x \to a} f(x) \right| = |f(a)|$ by Theorem 8. (If a is an endpoint of I , use the appropriate one–sided limit.) So $|f|$ is continuous on I .
(c) No, the converse is false. For example, the function $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ is not continuous at $x = 0$, but $|f(x)| = 1$ is continuous on R .

63. Define $u(t)$ to be the monk's distance from the monastery, as a function of time, on the first day, and define $d(t)$ to be his distance from the monastery, as a function of time, on the second day. Let D be the distance from the monastery to the top of the mountain. From the given information we know that $u(0)=0$, $u(12)=D$, $d(0)=D$ and $d(12)=0$. Now consider the function $u-d$, which is clearly continuous. We calculate that $(u-d)(0)=-D$ and $(u-d)(12)=D$. So by the Intermediate Value Theorem, there must be some time t_0 between 0 and 12 such that $(u-d)(t_0)=0 \Leftrightarrow u(t_0)=d(t_0)$. So at time t_0 after 7:00 A.M., the monk will be at the same place on both days.