Aerodynamics C Summary

1. Basic Concepts

In this summary we will examine compressible flows. But before we venture into the depths of the aerodynamics, we will examine some basic concepts.

1.1 Basic Concepts of Gases

Usually the atoms in a gas exert forces on each other. If these **intermolecular forces** are negligible, we are dealing with a **perfect gas**. For perfect gases the following **equation of state** is applicable:

$$p = \rho R T, \tag{1.1.1}$$

where p is the **pressure**, ρ is the **density** and T is the **temperature**. R is the **specific gas constant**. Its value is $R = 287 J/kg\,K$ at standard sea-level conditions.

Every molecule in a gas has a certain amount of energy. The sum of all these energies is called the **internal energy** of the gas. The internal energy per unit mass is called the **specific internal energy** e. There also is the **specific enthalpy** h, defined as

$$h = e + pv, \tag{1.1.2}$$

where $v = 1/\rho$ is the **specific volume**. For a perfect gas, both e and h are functions of only the temperature T. In fact, we have

$$de = c_v\,dT \qquad \text{and} \qquad dh = c_p\,dT, \tag{1.1.3}$$

where c_v and c_p are the **specific heat at constant volume** and **specific heat at constant pressure**, respectively. Often c_v and c_p also depend on the temperature T. If they can be assumed constant, then the gas is called a **calorically perfect gas**. We then have

$$e = c_v T \qquad \text{and} \qquad h = c_p T. \tag{1.1.4}$$

Let's take a closer look at the variables c_v, c_p and R. There are relations between them. If we also define $\gamma = c_p/c_v$, then it can be shown that

$$\gamma = \frac{c_p}{c_v}, \qquad R = c_p - c_v, \tag{1.1.5}$$

$$c_p = \frac{\gamma R}{\gamma - 1}, \qquad c_v = \frac{R}{\gamma - 1}. \tag{1.1.6}$$

1.2 The First Law of Thermodynamics

Let's consider a fixed mass of gas, called the **system**. The region outside the system is called the **surroundings**. In between the surroundings and the system is the **boundary**. We can now state the **first law of thermodynamics**, being

$$de = \delta q + \delta w. \tag{1.2.1}$$

Here δq is the amount of heat added and δw is the amount of work done on the system.

Heat can be added and work can be done in many ways. In **adiabatic processes** no heat is added or taken away from the system. In **reversible processes** things like mass diffusion, viscosity and thermal conductivity are absent. Finally **isentropic processes** are both adiabatic and reversible.

1.3 The Second Law of Thermodynamics

It is time to define the **entropy** s of a system. The **second law of thermodynamics** states that

$$ds \geq \frac{\delta q}{T}, \tag{1.3.1}$$

where there is only equality for reversible processes. Furthermore, if the process is adiabatic, then $\delta q = 0$ and thus also

$$ds \geq 0. \tag{1.3.2}$$

If the process is both reversible and adiabatic, then $ds = 0$. The entropy is thus constant for isentropic processes. (This also explains why these processes were named isentropic.)

Now let's try to derive an equation for the entropy. We do this using the first law of thermodynamics. For a reversible process it can be shown that $\delta w = -p\,dv$. Also we have $\delta q = T\,ds$. From this we can find that

$$T\,ds = de + p\,dv = dh - v\,dp. \tag{1.3.3}$$

We can combine the above relations with the equation of state and the relations for de and dh. Doing this will eventually result in

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} = c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1}. \tag{1.3.4}$$

For isentropic processes we have $ds = 0$ and thus $s_2 - s_1 = 0$. Using this fact, we can find that

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}. \tag{1.3.5}$$

1.4 Compressibility

Let's consider some substance. If we increase the pressure on it, its volume will decrease. We can now define the **compressibility** τ as

$$\tau = -\frac{1}{v}\frac{dv}{dp}. \tag{1.4.1}$$

However, when the pressure is increased often also the temperature and the entropy increase. To erase these effects, we define the **isothermal compressibility** τ_T and the **isentropic compressibility** τ_s as the compressibility at isothermal and isentropic processes, respectively. In an equation, this becomes

$$\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \qquad \text{and} \qquad \tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s. \tag{1.4.2}$$

But how can we use this? Using $v = 1/\rho$ we can derive that

$$d\rho = \rho\tau dp. \tag{1.4.3}$$

This equation helps us to judge whether a flow is compressible. A flow is **incompressible** when the density stays (more or less) constant throughout the process. If the density varies, then the flow is compressible. For low-speed flows dp is small, so also $d\rho$ is small. The flow is thus incompressible. For high-speed flows the pressure will change a lot more. Therefore $d\rho$ is not small anymore, and the flow is thus compressible.

1.5 Stagnation Conditions

Let's consider a flow with velocity V. If we move along with the flow, we can measure a certain static pressure p. We can also measure the density ρ, the temperature T, the Mach number M, and so on. All these quantities are static quantities.

Now let's suppose we slow down the flow adiabatically to $V = 0$. The temperature, pressure and density of the flow now change. The new value of the temperature is defined as the **total temperature** T_t. The corresponding **total enthalpy** is $h_t = c_p T_t$.

Using a rather lengthy derivation, it can be shown that the quantity $h + V^2/2$ stays constant along a streamline, in a steady adiabatic inviscid flow. We therefore have

$$h_t = h + \frac{V^2}{2} = \text{constant}. \tag{1.5.1}$$

For a calorically perfect gas (with constant c_p) we also have $T_t = h_t/c_p = \text{constant}$. Keep in mind that this only holds for adiabatic flows.

We can expand this idea even further, if the flow is also reversible, and thus isentropic. In this case, it turns out that the **total pressure** p_t and the **total density** ρ_t also stay constant along a streamline.

2. Normal Shock Waves

Where there are supersonic flows, there are usually also shock waves. A fundamental type of shock wave is the **normal shock wave** – the shock wave normal to the flow direction. We will examine that type of shock wave in this chapter.

2.1 Basic Relations

Let's consider a rectangular piece of air (the system) around a normal shock wave, as is shown in figure 2.1. To the left of this shock wave are the initial properties of the flow (denoted by the subscript 1). To the right are the conditions behind the wave.

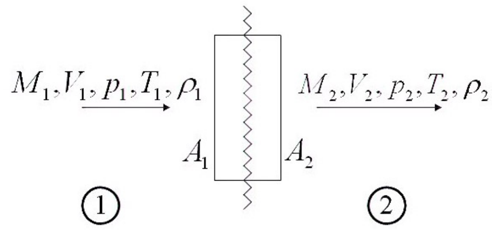

Figure 2.1: A normal shock wave.

We can already note a few things about the flow. It is a steady flow (the properties stay constant in time). It is also adiabatic, since no heat is added. No viscous effects are present between the system and its boundaries. Finally, there are no body forces.

Now what can we derive? Using the continuity equation, we can find that the mass flow that enters the system on the left is $\rho_1 u_1 A_1$, with u the velocity of the flow in x-direction. The mass flow that leaves the system on the right is $\rho_2 u_2 A_2$. However, since the system is rectangular, we have $A_1 = A_2$. So we find that

$$\rho_1 u_1 = \rho_2 u_2. \tag{2.1.1}$$

We can also use the momentum equation. The momentum entering the system every second is given by $(\rho_1 u_1 A_1) u_1$. The momentum flow leaving the system is identically $(\rho_2 u_2 A_2) u_2$. The net force acting on the system is given by $p_1 A_1 - p_2 A_2$. Combining everything, we can find that

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2. \tag{2.1.2}$$

Finally let's look at the energy. The energy entering the system every second is $(\rho_1 u_1 A_1)\left(e_1 + u_1^2/2\right)$. Identically, the energy leaving the system is $(\rho_2 u_2 A_2)\left(e_2 + u_2^2/2\right)$. No heat is added to the system (the flow is adiabatic). There is work done on the system though. The amount of work done every second is $p_1 A_1 u_1 - p_2 A_2 u_2$. Once more, we can combine everything to get

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{2.1.3}$$

This equation states that the total enthalpy is the same on both sides of the shock wave. Since the shock wave was adiabatic, we actually already knew that. So this was no surprising result.

The three equations we have just derived hold for all one-dimensional, steady, adiabatic, inviscid flows. But let's take a closer look at them. Let's suppose that all upstream conditions ρ_1, u_1, p_1, h_1 and T_1 are known. We can't solve for all the downstream conditions just yet. We have only three equations, while we have four unknowns. We need a few more equations. These equations are

$$h = c_p T, \tag{2.1.4}$$

$$p = \rho R T. \tag{2.1.5}$$

That wasn't much new, was it? We now have 5 unknowns and 5 equations. So we can solve everything.

2.2 The Speed of Sound

A special kind of normal shock wave is a sound wave. In fact, it is an infinitesimally weak normal shock wave. Because of this, dissipative phenomena (like viscosity and thermal conduction) can be neglected, making it an isentropic flow.

At what velocity does this shock wave travel? Let's call this velocity the **speed of sound** a. Note that $a = u_1$. Because the shock wave is very weak, we can also state that $p_2 = p_1 + dp$, $\rho_2 = \rho_1 + d\rho$ and $a_2 = a_1 + da$. If we combine these facts with the three equations we derived in the previous paragraph, we eventually find that

$$a^2 = \frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s. \tag{2.2.1}$$

The last part in the above equation is to indicate that the changes in p and ρ occur isentropically. For isentropic processes we have

$$p = c\rho^\gamma \qquad \Rightarrow \qquad \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho}. \tag{2.2.2}$$

This results in

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma R T}, \tag{2.2.3}$$

where we used the equation of state in the last part. So apparently, for a given medium, the speed of sound only depends on the temperature.

Do you still remember the compressibility we introduced in the previous chapter? From the equation $d\rho = \rho \tau dp$, we can also derive that

$$a = \sqrt{\frac{1}{\rho \tau_s}}. \tag{2.2.4}$$

Note that we have used the isentropic compressibility because the process is isentropic. So we see that the lower the compressibility of a substance, the faster sound travels in it.

2.3 The Mach Number

The Mach number M is defined as

$$M = \frac{u}{a}. \tag{2.3.1}$$

A lot of properties can be derived from the Mach number. Let's recall the total temperature T_0. This can be found using

$$c_p T_0 = c_p T + \frac{u^2}{2}. \tag{2.3.2}$$

From this we can derive that

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2. \tag{2.3.3}$$

Using the isentropic flow relations, we can also find that

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}, \tag{2.3.4}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}. \tag{2.3.5}$$

From equation (2.3.2) we can also derive that

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma - 1} = \text{constant}. \tag{2.3.6}$$

2.4 Sonic Conditions

When you slow an airflow down adiabatically to $u = 0$ (and thus $M = 0$) you find the total temperature T_t, total pressure p_t, total density ρ_t, and so on. Similarly, we can change the velocity of a flow adiabatically such that $M = 1$. The corresponding **temperature at sonic conditions** is denoted by T^*. The **characteristic speed of sound** a^* can now be found using $a^* = \sqrt{\gamma R T^*}$. However, we can also determine that

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma - 1} + \frac{a^{*2}}{2} = \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} = \text{constant}. \tag{2.4.1}$$

Just like we can examine the speed of sound at sonic conditions, we can also look at the temperature T^*, pressure p^* and density ρ^* at such conditions. By inserting $M = 1$ in equations (2.3.3) to (2.3.5) we find that

$$\frac{T_0}{T^*} = \frac{\gamma + 1}{2}, \qquad \frac{p_0}{p^*} = \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma - 1}} \qquad \text{and} \qquad \frac{\rho_0}{\rho^*} = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma - 1}}. \tag{2.4.2}$$

Finally we can define the **characteristic Mach number** M^* as

$$M^* = \frac{u}{a^*}. \tag{2.4.3}$$

We can find that M and M^* are related, according to

$$M^2 = \frac{2M^{*2}}{(\gamma + 1) - (\gamma - 1) M^{*2}} \qquad \Leftrightarrow \qquad M^{*2} = \frac{(\gamma + 1) M^2}{2 + (\gamma - 1) M^2}. \tag{2.4.4}$$

The parameters M and M^* are quite similar. If one is bigger than 1, so is the other, and vice verse.

2.5 Normal Shock Wave Relations

There are several other relations that hold for normal shock waves. We will discuss some of them. We start with the **Prandtl relation**, stating that

$$a^{*2} = u_1 u_2 \qquad \Leftrightarrow \qquad 1 = M_1^* M_2^*. \tag{2.5.1}$$

From this follows that

$$M_2^2 = \frac{2 + (\gamma - 1) M_1^2}{2\gamma M_1^2 - (\gamma - 1)}. \tag{2.5.2}$$

This is an important relation. If $M_1 > 1$ we have $M_2 < 1$. If $M_1 = 1$, then also $M_2 = 1$. (If this is the case we are dealing with an infinitely weak shock wave, called a **Mach wave**.) However, if $M_1 < 1$ it would seem that $M_2 > 1$. But this seems rather odd. Suddenly a subsonic flow becomes supersonic! A more detailed look would show that in this case also the entropy s would decrease. But the second law of thermodynamics states that the entropy can only increase. What can we conclude from this? It means that in subsonic flows no shock waves can appear. Shock waves are thus only present in supersonic flows.

Now we know how to find M_2. But can we also find the other properties behind the shock wave? It turns out that we can, using

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)\, M_1^2}{2 + (\gamma-1)\, M_1^2}, \tag{2.5.3}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right), \tag{2.5.4}$$

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \left(1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)\right)\left(\frac{2 + (\gamma-1)\, M_1^2}{(\gamma+1)\, M_1^2}\right). \tag{2.5.5}$$

It would also be interesting to know how the total temperature T_t and the total pressure p_t change across the shock wave. Since a shockwave is an adiabatic process we know that $h_1 = h_2$ and thus also $T_{t,1} = T_{t,2}$. Finally, using the relation for entropy we can find that

$$\frac{p_{t,2}}{p_{t,1}} = e^{-\frac{s_2 - s_1}{R}}. \tag{2.5.6}$$

So what can we derive from all the above equations? When passing through a shock wave, the properties of the flow change drastically. The pressure, temperature and density increase, while the total pressure and the Mach number decrease. The total temperature and the enthalpy stay constant.

2.6 Measuring the Velocity

When an aircraft is flying, it would be nice to know how fast it is going. To find this out, a Pitot tube is used, measuring the total pressure p_t. We also assume that the static pressure p is known.

To find the velocity during a subsonic flight, we can simply use the relation

$$\frac{p_t}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}. \tag{2.6.1}$$

Solving for M^2 and using $u^2 = M^2 a^2$ we find that

$$u^2 = \frac{2a^2}{\gamma-1}\left(\left(\frac{p_t}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right). \tag{2.6.2}$$

So to find the velocity, we also need to know the speed of sound. But if we know that, it's easy to find the velocity.

To find the velocity during a supersonic flight is a bit more difficult, since there is a shock wave. This time the Pitot tube measures the total pressure behind the shock wave $p_{t,2}$. The static pressure that was known is now called p_1. This time we need to use the relation

$$\frac{p_{t,2}}{p_1} = \frac{p_{t,2}}{p_2}\frac{p_2}{p_1} = \left(\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right)^{\frac{\gamma}{\gamma-1}} \frac{(1-\gamma) + 2\gamma M_1^2}{\gamma+1}. \tag{2.6.3}$$

This equation is called the **Rayleigh Pitot tube formula**. In its derivation we used the normal shock wave relations for the ratio p_2/p_1. We used the relation for total pressure in an isentropic flow for the ratio $p_{t,2}/p_2$. From this equation the Mach number can be solved. Then only the speed of sound is still needed to find the flight velocity u.

3. Oblique Shock Waves

In reality normal shock waves don't often occur. Oblique shock waves are more common. We would like to know what causes them, and how we can calculate flow properties around them.

3.1 Shock Wave Angles

When an aircraft is flying, it creates disturbances in the flow. These disturbance spread around with the speed of sound a. Figure 3.1 visualizes these disturbances for an airplane traveling from point A to point B.

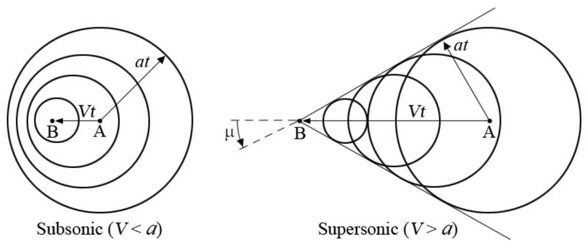

Figure 3.1: Visualization of the disturbances in a flow.

When the airplane flies at a subsonic velocity ($V < a$), the disturbances can move upstream. If the airplane, however, flies at a supersonic speed ($V > a$), the disturbances can not. In fact, they all stay within a cone and stack up at the edge, forming a so-called **Mach wave**. This cone has an angle μ, where μ is called the **Mach angle**. From figure 3.1 it can be derived that

$$\sin \mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}. \tag{3.1.1}$$

The above relation is, however, only theoretical. In practice the shock wave doesn't have an angle μ but an angle β, called the **wave angle**. For shock waves we always have $\beta > \mu$. Finally there is the special case with $\beta = 90°$, at which we once more have a normal shock wave. So a normal shock wave is just a special case of the oblique shock wave.

3.2 Oblique Shock Wave Relations

We will try to derive some relations for oblique shock waves. But before we can do that, we need to make some definitions.

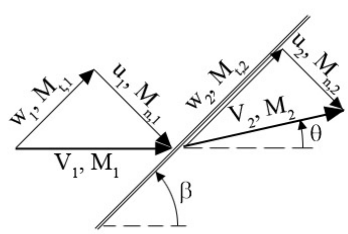

Figure 3.2: Properties of the oblique shock wave.

We know that the velocity V_1 before the shock wave is directed horizontally. We examine two components of this velocity: The component normal to the shock wave u_1 and the component tangential to the shock wave w_1. Corresponding are the Mach number normal to the shock wave $M_{n,1}$ and the Mach number tangential to the shock wave $M_{t,1}$. We can do the same for the velocities after the shock wave (but now with subscript 2). All the properties have been visualized in figure 3.2. Also note the **deflection angle** θ.

Using the variables described above, we can derive some relations. It turns out that these relations are virtually the same as for a normal shock wave. There's only one fundamental difference. Instead of using the total velocity, we only need to consider the component of the velocity normal to the shock wave (being u). We then get

$$\rho_1 u_1 = \rho_2 u_2, \tag{3.2.1}$$

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2, \tag{3.2.2}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{3.2.3}$$

But what about the tangential component of the velocity w? Well, using the momentum equation, we can derive the simple relation

$$w_1 = w_2. \tag{3.2.4}$$

So now we have used the continuity equation, the momentum equation and the energy equation. In the previous chapter we now continued to express ratios like p_2/p_1 as a function of the Mach number. We can do the same again. However, this time we express everything in the component of the Mach number normal to the flow, being

$$M_{n,1} = M_1 \sin\beta. \tag{3.2.5}$$

Going through a lot of derivations, we can find that

$$M_{n,2}^2 = \frac{2 + (\gamma - 1) M_{n,1}^2}{2\gamma M_{n,1}^2 - (\gamma - 1)} \qquad \text{with} \qquad M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)}, \tag{3.2.6}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma + 1) M_{n,1}^2}{2 + (\gamma - 1) M_{n,1}^2}, \tag{3.2.7}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_{n,1}^2 - 1\right), \tag{3.2.8}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \left(1 + \frac{2\gamma}{\gamma + 1}\left(M_{n,1}^2 - 1\right)\right)\left(\frac{2 + (\gamma - 1) M_{n,1}^2}{(\gamma + 1) M_{n,1}^2}\right). \tag{3.2.9}$$

We can once more see that these equations are virtually the same as for a normal shock wave. The only difference is that we now need to take the component of the Mach number normal to the flow.

3.3 The Deflection Angle

There is one last variable for which we can derive an equation. That variable is the deflection angle θ. This angle is usually determined by the shape of the object causing the shock waves. We can find that

$$\tan\theta = \frac{2}{\tan\beta}\frac{M_1^2 \sin^2\beta - 1}{M_1^2(\gamma + \cos(2\beta)) + 2}. \tag{3.3.1}$$

This equation is called the **θ-β-M relation**. Many important things can be derived from it.

Let's suppose we know θ and M_1. We can then find the corresponding values of the wave angle β. For relatively low values of θ you will find two solutions for β. There are thus two possible shock waves. The shock wave with the higher angle of β is called the strong shock wave, while the one with the lower angle is called the weak shock wave. In nature, the weak shock wave is almost always present. So usually the smallest of the two solutions can simply be used.

But what happens if θ gets bigger? Soon θ will reach a maximum value θ_{max}. At this point there is only one solution for β. If θ gets even bigger, no solutions exist anymore. So an oblique shock wave is not possible then. Instead, the shock wave will detach and get a curved shape. We will briefly examine the detached shock wave later in this chapter.

3.4 Multiple Shock Waves

There are many cases in which multiple shock waves occur. We will examine a few. First let's consider a single shock wave with wave angle β_1, colliding with a wall parallel to the free stream. What happens to this shock wave?

To answer this question, we look at the flow after the shock wave. This flow has been deflected towards the wall by an angle θ. Since the flow can't go through the wall, it needs to be deflected the other way, by the same angle θ. To accomplish this, there will be a new shock wave.

You may initially think that this new shock wave has the same wave angle β_1. This is, however, not true. To see why, we need to look at the Mach numbers. Before the first shock wave, the flow had a Mach number M_1. After the first shock wave (and before the second), the flow has a lower Mach number M_2. By combining this new Mach number with the deflection angle θ, the new wave angle β_2 can be found. So the second shock wave will have a wave angle β_2.

Now let's look at another situation: the case where two shock waves A and B intersect each other. At the point of intersection, two new shock waves C and D will appear, each with different wave angles β_C and β_D. What information can we use to determine these wave angles?

Experiments have shown that, after the two new waves, the flows from both waves travel in the same direction. In between these two flows is the so-called **slip line**. This line is called the slip line, because the two flows "slip" with respect to each other – they usually have a different velocity.

So the final directions of both flows are the same. However, because there is a straight line between the two flows, their pressures must be equal as well. So $p_{C,2} = p_{D,2}$. Using these two boundary conditions the wave angles β_C and β_D can be determined.

3.5 The Detached Shock Wave

If we put a rather blunt body in a supersonic flow, we won't get an (attached) oblique shock wave. Instead, we will get a **detached shock wave**. The properties of this shock wave vary along the shock wave. At the front of the shock wave, the wave angle β is $90°$. So we have a normal shock wave there. Behind this shock wave, the flow is subsonic.

As we go further from the shock wave, the wave angle β decreases. As β decreases, the deflection angle θ initially increases. It soon reaches its maximum, after which it once more starts to decrease.

Not much after we reached θ_{max}, we find the **sonic line**. At this line the Mach number of the flow behind the shock wave is $M_2 = 1$. As we continue our travel along the shock wave, the shock wave loses strength. It's not longer able to slow down the flow to subsonic velocities. So the Mach number behind the shock wave M_2 will be above 1.

As we go even further away from our blunt body, the shock wave will continue to lose strength. Eventually, when $\theta = 0$ again, its strength will have disappeared entirely.

Performing calculations on a blunt shock wave is very difficult. It is therefore not part of this course.

3.6 Expansion Waves

Suppose we have an airflow moving along a wall, which suddenly makes an angle θ away from the flow. We then get an **expansion wave**. In this expansion wave, the airflow "bends" around the wall edge. While the airflow changes direction, its velocity also changes. This happens according to

$$d\theta = \sqrt{M^2 - 1}\frac{dV}{V}. \tag{3.6.1}$$

Now how can we find the Mach number after the expansion wave? For that, we first have to rewrite dV/V to

$$\frac{dV}{V} = \frac{2}{2 + (\gamma - 1) M^2}\frac{dM}{M}. \tag{3.6.2}$$

Using this, we can find that θ is equal to

$$\theta = \int_{M_1}^{M_2} \frac{2\sqrt{M^2 - 1}}{2 + (\gamma - 1) M^2}\frac{dM}{M}. \tag{3.6.3}$$

The integral is kind of complex, but it can be solved. Because of its importance, it has gotten its own symbol and name. This integral is named the **Prandtl-Meyer function** $\nu(M)$, defined as

$$\nu(M) = \int \frac{2\sqrt{M^2 - 1}}{2 + (\gamma - 1) M^2}\frac{dM}{M} = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \arctan\left(\sqrt{\frac{\gamma - 1}{\gamma + 1}\left(M^2 - 1\right)}\right) - \arctan\left(\sqrt{M^2 - 1}\right). \tag{3.6.4}$$

Using this function, we can derive an expression for θ, being

$$\theta = \nu(M_2) - \nu(M_1). \tag{3.6.5}$$

However, we usually don't need to calculate θ. Usually we know θ and M_1 and we need to know M_2. How do we find M_2 then? Well, we first use M_1 to find $\nu(M_1)$. We then add this result up to θ to find $\nu(M_2)$. From this we can derive M_2 (often using tables). In general we can say that $M_2 > M_1$.

How do the various flow properties behave during expansion waves? It can be shown that the flow is isentropic, so the entropy s stays constant. Therefore also T_t and p_t stay constant. From this we can derive that

$$\frac{T_2}{T_1} = \frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M_2^2}, \tag{3.6.6}$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma - 1}} = \left(\frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M_2^2}\right)^{\frac{1}{\gamma - 1}}, \tag{3.6.7}$$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{2 + (\gamma - 1) M_1^2}{2 + (\gamma - 1) M_2^2}\right)^{\frac{\gamma}{\gamma - 1}}. \tag{3.6.8}$$

In a shock wave the pressure, density and temperature increase. In an expansion wave it is exactly opposite: they all decrease.

4. Flow Through Wind Tunnels

To be able to test with supersonic flows, wind tunnels are used. To reach a supersonic flow, they must have a characteristic shape. Why is this? And how does this shape effect the flow? We will try to find that out.

4.1 Basic Equations

Let's consider a wind tunnel. The flow in it is not entirely one-dimensional. As the cross-section changes, the flow also goes in the y and z-direction. However, if we assume that the cross-section changes only very gradually, then these components are small with respect to the x-direction. We would then approximately have a one-dimensional flow: a so-called **quasi-one-dimensional flow**. In this flow all parameters p, ρ, u and also A only depend on x.

What equations hold for such a flow? From the general continuity, momentum and energy equation we can derive that

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2, \tag{4.1.1}$$

$$p_1 A_1 + \rho_1 u_1^2 A_1 + \int_{A_1}^{A_2} p\, dA = p_2 A_2 + \rho_2 u_2^2 A_2, \tag{4.1.2}$$

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{4.1.3}$$

There aren't many surprises in the first and third of these equations. But in the middle one is an integral! This is because the walls of the wind tunnel aren't horizontal. They can thus also exert a pressure force in x-direction on the flow.

Of course having an integral in an equation isn't convenient. To prevent that, we simply consider two points, with an infinitely small distance dx between them. So we would then have $p_2 = p_1 + dp$, $\rho_2 = \rho_1 + d\rho$, and so on. Filling this in, and working it all out, we would get

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0, \tag{4.1.4}$$

$$dp = -\rho u\, du, \tag{4.1.5}$$

$$dh + u\, du = 0. \tag{4.1.6}$$

The middle one of these three equations (the one derived from the momentum equation) is called **Euler's equation**. Using the above equations, we can derive new equations for the flow through wind tunnels, as we will see in the coming paragraph.

4.2 Area, Velocity and Mach Number

We can extensively rewrite and combine the equations we just found. By doing so, we can derive another important relation, called the **area-velocity relation**. It states that

$$\frac{dA}{A} = \left(M^2 - 1\right) \frac{du}{u}. \tag{4.2.1}$$

Now what does this equation tell us? First let's suppose that $M < 1$. If the cross-sectional area gets bigger ($dA > 0$), then the velocity decreases ($du < 0$). Also, if the area gets smaller, the velocity increases. This is rather intuitive. However, the counterintuitive part comes when $M > 1$. Now things are exactly opposite. If the area gets bigger, then the velocity also increases. Similarly, if the wind tunnel decreases in size, then the flow also reduces its velocity.

A special case occurs if $M = 1$. If this is true, then we must have $dA = 0$. So a sonic flow can only occur when the cross-section is at a minimum (at a so-called **throat**). Note that the flow properties at this point are the flow properties at sonic conditions, which we denoted with a star (*). So we would have a pressure p^*, a density ρ^* and a flow velocity $u^* = a^*$.

So we have found that the cross-sectional area A and the Mach number M are linked. But how? To find that out, we can derive that

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left(\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right)^{\frac{\gamma+1}{\gamma-1}}, \tag{4.2.2}$$

where A^* is the cross-sectional area at the throat. (Note that if we fill in $M = 1$ we would get $A = A^*$.) This important equation is called the **area-Mach number relation**. It shows that the Mach number only depends on the ratio A/A^*.

4.3 Flow in a Nozzle

Let's examine a nozzle. We can consider three parts in it. To the left is a reservoir of air. At this point the cross-sectional area A is very big. The velocity is therefore very low. The pressure and temperature at this point are thus equal to the total pressure p_t and the total temperature T_t.

In the middle of the nozzle is the throat. To the right of that, the tunnel gets wider again. Eventually there is an exit, with exit pressure p_e and exit temperature T_e.

Flow doesn't go through the nozzle spontaneously. It flows because $p_e < p_t$. This pressure difference causes the air to move. However, the flow doesn't always reach supersonic velocities. To check how the flow behaves, we need to examine the ratio p_e/p_t. While doing that, we can consider 6 stages.

In the first stage, the ratio $p_{e,1}/p_t \approx 1$. This causes the flow to move, but only slowly. Not much special is going on. In the second stage, the ratio $p_{e,2}/p_t$ becomes smaller. However, the flow remains subsonic. In stage three, the ratio $p_{e,3}/p_t$ is sufficiently small to cause a sonic flow in the throat. So at the throat finally $M = 1$. However, after the throat the flow becomes subsonic again.

Now what happens if we decrease the exit pressure even further? We then reach stage four. In this stage, the flow becomes supersonic after the throat. However, the pressure difference isn't big enough to continue this supersonic flow. So a normal shock wave appears, slowing the flow down to subsonic velocities. To know where the shock wave appears, you have to look at the pressure. The pressure drop in the normal shock wave should be such that, at the exit, the exit pressure $p_{e,4}$ is reached.

If we decrease the exit pressure further, the position of the normal shock wave changes. In fact, it moves to the right. This continues until we reach stage 5. In stage 5 the normal shock wave is at the exit of the nozzle.

If we decrease the exit pressure just a little bit further, we reach stage 6. A supersonic flow now exits the nozzle. In this case the exit pressure is always a fixed value $p_{e,6}$. However, now the **back pressure** p_B is also important. This is the pressure behind the nozzle. (Previously the back pressure p_B was equal to the exit pressure p_e. Now this is not the case.) We can now consider three cases:

- If $p_B > p_{e,6}$, the flow has expanded too much (it is **overexpanded**) and will decrease in size once it exits the nozzle. This causes oblique shock waves.
- If $p_B < p_{e,6}$, the flow hasn't expanded enough (it is **underexpanded**) and will increase in size once it exits the nozzle. Due to this, expansion waves will occur.
- If $p_B = p_{e,6}$, the flow will just exit the nozzle without any waves.

During stages 3 to 6, an important phenomenon occurs. In all these stages, we have $M = 1$ at the throat. From this follows that the pressure in the throat is always

$$p^* = \frac{p_t}{\left(1 + \frac{\gamma-1}{2}\right)^{\frac{\gamma}{\gamma-1}}} = 0.528 p_t. \tag{4.3.1}$$

From this we can derive that the mass flow in the throat is always the same during stages 3 to 6. So although the exit pressure (or back pressure) decreases, the mass flow through the nozzle stays the same. This effect is called **choked flow**.

4.4 Wind Tunnels

Suppose we have a model of an airplane, which we want to test at supersonic velocities. What kind of wind tunnel do we need? We can simply take a nozzle, having only one throat. If we do this, we can get supersonic velocities. However, a huge pressure ratio p_t/p_e will be needed. This means expensive equipment, which is of course undesirable.

The solution lies in a **diffuser**. A diffuser slows the flow down, back to subsonic velocities. During this process, the pressure increases. So in a wind tunnel we would first have a nozzle, then our test model, and finally a diffuser. To the left of the nozzle is the high pressure p_t. At the test model is a low pressure, but a high velocity. Finally, after the diffuser, there is a low velocity, but a more or less high pressure p_e. Although still $p_e < p_t$, the ratio p_t/p_e is much smaller than normal. This therefore makes supersonic wind tunnels feasible.

Let's take a closer look at this diffuser. A diffuser has a similar shape as a nozzle: it has a throat. However, this time there is a supersonic flow ($M > 1$) to the left of the throat, and a subsonic flow to the right. Once more, we have $M = 1$ at the throat. After the throat will be a subsonic flow ($M < 1$). Ideally, this would occur isentropically, without any shock waves. In reality, there are viscous effects near the edges of the diffuser. These viscous effects eventually cause shock waves.

When designing a wind tunnel, we would like to know how big the cross-sectional area of the diffuser throat should be. In the wind tunnel we will be having two throats: one in the nozzle (with cross-sectional area $A_{t,1}$) and one in the diffuser (with area $A_{t,2}$). (Note that the subscript t now stands for throat; not total.) These areas relate to each other, according to

$$\frac{A_{t,2}}{A_{t,1}} = \frac{p_{t,1}}{p_{t,2}}. \tag{4.4.1}$$

In an ideal (isentropic) situation we have $p_{t,1} = p_{t,2}$, so also $A_{t,1} = A_{t,2}$. In reality, however, there are viscous effects. Due to this we have $p_{t,1} > p_{t,2}$ and thus also $A_{t,2} > A_{t,1}$. So the diffuser throat should always be bigger than the nozzle throat.

Now what happens if $A_{t,2}$ is too small? In this case the diffuser will **choke**. It can't handle the mass flow. This causes shock waves in the test section. This can ultimately lead to an entirely subsonic test section. In such a case, the wind tunnel is said to be **unstarted**.

5. Subsonic Compressible Flow over Airfoils

It is time to turn theory into practice. What can we say about flow over airfoils? In this chapter we consider compressible subsonic flow over airfoils. The next chapter focusses on supersonic flow.

5.1 The Velocity Potential Equation

In a previous aerodynamics course we have seen the velocity potential ϕ. It was defined such that

$$\mathbf{V} = \nabla\phi. \tag{5.1.1}$$

From the velocity potential we can find the velocity components

$$u = \frac{\partial\phi}{\partial x} \qquad \text{and} \qquad v = \frac{\partial\phi}{\partial y}. \tag{5.1.2}$$

For incompressible flows (ρ is constant) we would get **Laplace's equation**

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{5.1.3}$$

This equation is a linear differential equation. There exist solutions for it. If ρ is not constant, things are a lot more difficult. Using (among others) the continuity equation and Euler's equation, we can eventually derive that

$$\left(1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right)\frac{\partial^2\phi}{\partial x^2} + \left(1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right)\frac{\partial^2\phi}{\partial y^2} - \frac{2}{a^2}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y} = 0. \tag{5.1.4}$$

This important equation is called the **velocity potential equation**. Note that for incompressible flows we would have ρ constant and thus $a = \infty$. The above equation then reduces back to Laplace's equation. However, a is not infinite. It also depends on the velocity potential. This is according to

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right), \tag{5.1.5}$$

where a_0 is constant for the entire flow.

5.2 The Linearized Velocity Potential Equation

The velocity potential is a nonlinear equation. It is therefore very hard to solve. To solve it, we have to use assumptions, through which we can turn the above equation into a linear equation.

But before we do that, we have to introduce the **perturbation velocities** $\hat{u}$ and $\hat{v}$. Let's suppose we are flying with a **free stream velocity** V_∞ in x-direction. The velocity perturbations are now defined as the change in velocity, with respect to the free stream velocity. So

$$\hat{u} = u - V_\infty \qquad \text{and} \qquad \hat{v} = v. \tag{5.2.1}$$

Identically, we can define the **perturbation velocity potential** $\hat{\phi}$ such that

$$\hat{u} = \frac{\partial\hat{\phi}}{\partial x} \qquad \text{and} \qquad \hat{v} = \frac{\partial\hat{\phi}}{\partial y}. \tag{5.2.2}$$

Using this perturbation velocity, we can derive a very complicated equation, similar to the velocity potential equation. However, for certain **free stream Mach numbers** M_∞, this equation can be simplified. If $0 \leq M_\infty \leq 0.8$ or $M_\infty \geq 1.2$, certain parts can be neglected. If also $M_\infty < 5$ even more parts can be neglected. We also have to make the assumption that the velocity perturbations $\hat{u}$ and $\hat{v}$ are small. This is usually the case for thin bodies at small angles of attack. Based on all these assumptions, the complicated equation reduces to

$$\left(1 - M_\infty^2\right) \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0. \tag{5.2.3}$$

This is the **linearized perturbation velocity potential equation**. It is a linear partial differential equation. With it, the perturbation velocity potential can be found. However, when doing that, we also need boundary conditions. There are two boundary conditions that can be used. First of all, at $x = \infty$, we have $\hat{u} = \hat{v} = 0$ and thus $\hat{\phi} =$ constant. Second, we know that if the airfoil is at an angle θ with respect to the free stream flow, then also

$$\frac{\partial \hat{\phi}}{\partial y} = \hat{v} = (V_\infty + \hat{u}) \tan\theta \approx V_\infty \tan\theta. \tag{5.2.4}$$

So now we know how to find $\hat{\phi}$. What can we do with it? Well, with it we can find the pressure coefficient. The **pressure coefficient** could normally be found using

$$C_p = \frac{p - p_\infty}{q_\infty} = \frac{2}{\gamma M_\infty^2} \left(\frac{p}{p_\infty} - 1 \right). \tag{5.2.5}$$

For small velocity perturbations the ratio p/p_∞ can be approximated by

$$\frac{p}{p_\infty} = 1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2} \right). \tag{5.2.6}$$

Using this, the relation for the pressure coefficient can be simplified to

$$C_p = -\frac{2\hat{u}}{V_\infty}. \tag{5.2.7}$$

5.3 Compressibility Corrections

Instead of deriving entirely new equations for compressible flows, we can also slightly change existing equations for incompressible flows, such that they approximate compressible flows. Such adjustments are called **compressibility corrections**.

The first compressibility correction is the **Prandtl-Glauert correction**. It stated that the pressure coefficient C_p in a compressible flow can be derived from the pressure coefficient $C_{p,0}$ in an incompressible flow, according to

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2}}. \tag{5.3.1}$$

The lift coefficient and moment coefficient for compressible flow can be derived similarly, using

$$c_l = \frac{c_{l,0}}{\sqrt{1 - M_\infty^2}} \qquad \text{and} \qquad c_m = \frac{c_{m,0}}{\sqrt{1 - M_\infty^2}}. \tag{5.3.2}$$

The Prandtl-Glauert rule is based on the linearized velocity potential equation. Other compressibility corrections do take the nonlinear terms into account. Examples are the **Karman-Tsien** rule, which states that

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2} + \frac{M_\infty^2}{1 + \sqrt{1 - M_\infty^2}} \frac{C_{p,0}}{2}}, \tag{5.3.3}$$

and Laitone's rule, stating that

$$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2} + M_\infty^2 \left(1 + \frac{\gamma-1}{2} M_\infty^2\right) \sqrt{1-M_\infty^2} \frac{C_{p,0}}{2}}. \tag{5.3.4}$$

5.4 The Critical Mach Number

The flow velocity is different on different positions on the wing. Let's say we know the Mach number M_A of the flow over our wing at a given point A. The corresponding pressure coefficient can then be found using

$$C_{p,A} = \frac{2}{\gamma M_\infty^2} \left(\left(\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_A^2} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right). \tag{5.4.1}$$

The velocity of the flow on top of our wing is generally bigger than the free stream velocity V_∞. So we may have sonic flow ($M = 1$) over our wing, while we are still flying at $M_\infty < 1$. The **critical Mach number** M_{cr} is defined as the free stream Mach number M_∞ at which sonic flow ($M = 1$) is first achieved on the airfoil surface. It is a very important value. To find it, we use the **critical pressure coefficient** $C_{p,cr}$. The relation between M_{cr} and $C_{p,cr}$ can be found from the above equation. This relation is

$$C_{p,cr} = \frac{2}{\gamma M_{cr}^2} \left(\left(\frac{1 + \frac{\gamma-1}{2} M_{cr}^2}{1 + \frac{\gamma-1}{2}} \right)^{\frac{\gamma}{\gamma-1}} - 1 \right). \tag{5.4.2}$$

However, the above equation has two unknowns. So we need an additional equation. We can use any of the compressibility corrections for that. But, to do that, we first need to know $C_{p,0}$. This can be found using low-speed wind tunnel tests. Just measure the minimal pressure coefficient over the entire wing. This is the position of minimum pressure and thus maximum velocity. So once $C_{p,0}$ is known, we have two equations with two unknowns. It can be solved.

5.5 The Increase in Drag

You may wonder, why is the critical Mach number so important? We can see that if we plot the drag coefficient c_d with respect to the free stream Mach number M_∞. Initially c_d has the constant value of $c_{d,0}$. However, when M_∞ gets bigger than M_{cr}, shock waves will appear. This causes additional drag. So the critical Mach number relates to the velocity at which the drag increases.

At a certain free stream Mach number the drag coefficient suddenly starts to increase enormously. The Mach number at which this occurs (which is often slightly bigger than M_{cr}) is called the **drag-divergence Mach number**. However, once we have passed $M_\infty = 1$, the drag coefficient c_d decreases. We have then passed the so-called **sound barrier**.

Normally, the drag coefficient can get as big as ten times $c_{d,0}$. There are, however, ways to prevent this. One way is the so-called **area rule**. It seems that sudden changes in the cross-sectional area of a wing cause a high drag coefficient at the sonic region. So, the cross-sectional area of an airplane should be as constant as possible. At the positions of the wings, the cross-section of the aircraft usually increases. To prevent this, the cross-section of the fuselage at those points should decrease. This can reduce the drag coefficient at $M = 1$ by an entire factor 2.

Another way of reducing the drag around $M = 1$ is by using **supercritical airfoils**. The idea behind this is not to increase the critical Mach number M_{cr}. Instead, it is to increase the drag-divergence Mach number. This is done by making the top of the airfoil as flat as possible. By doing this, airplanes can fly at higher velocities, without experiencing the massive increase in drag just yet.

6. Supersonic Flow over Airfoils

In the previous chapter we treated subsonic flow over airfoils. In this final chapter we will take a look at supersonic flow. How do airfoils behave at $M > 1$?

6.1 The Linearized Supersonic Pressure Coefficient Equation

In the previous chapter, we derived the linear perturbation velocity potential equation. If we define $\lambda = \sqrt{M_\infty^2 - 1}$, we can rewrite it to

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = 0. \tag{6.1.1}$$

Any function $\hat{\phi} = f(x - \lambda y)$ satisfies this equation. So it initially may not seem helpful. However, we do know that if $x - \lambda y = \text{constant}$, also $\hat{\phi}$ stays constant. Also, $x - \lambda y$ is constant, if

$$\frac{dy}{dx} = \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}} = \tan \mu, \tag{6.1.2}$$

where μ is the **Mach angle**, which was introduced in the chapter on oblique shock waves. So we find that $\hat{\phi}$ is constant along a **Mach line**.

From the fact that $\hat{\phi} = f(x - \lambda y)$, we can also derive another important relation. From this follows that, for a certain position on the wing with angle θ, we have

$$\hat{u} = -\frac{V_\infty \theta}{\lambda}. \tag{6.1.3}$$

The pressure coefficient can now be found using

$$C_p = -\frac{2\hat{u}}{V_\infty} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}. \tag{6.1.4}$$

This important equation is called the **linearized supersonic pressure coefficient equation**. It is a rather simple way to find C_p. The sign of θ, and thus also of C_p can, however, be rather tricky. Luckily you only have to remember one important thing. If the surface of the airfoil is inclined into the free stream, there is a relatively high pressure, and C_p is thus positive. On the other hand, if the surface is inclined away from the free stream, the pressure is relatively low, and C_p is thus negative.

6.2 Lift and Drag Coefficients of a Flat Plate

Let's give an example of how to use the relation that was just derived. Let's calculate the lift and drag coefficient of a flat plate at an angle of attack α in a supersonic flow. The pressure coefficients at the lower and upper side of the plate, $C_{p,l}$ and $C_{p,u}$, respectively, are given by

$$C_{p,l} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}} \qquad \text{and} \qquad C_{p,u} = -\frac{2\alpha}{\sqrt{M_\infty^2 - 1}}. \tag{6.2.1}$$

The component of the force acting normal to the plate c_n can now be found using

$$c_n = \frac{1}{c} \int_0^c (C_{p,l} - C_{p,u})\, dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}. \tag{6.2.2}$$

Since the plate has no thickness, there is no component of the force acting parallel to the plate. So we have

$$c_l = c_n \cos\alpha \qquad \text{and} \qquad c_d = c_n \sin\alpha. \tag{6.2.3}$$

Using $\cos\alpha \approx 1$ and $\sin\alpha \approx \alpha$ we eventually get

$$c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \qquad \text{and} \qquad c_n = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}. \tag{6.2.4}$$

These equations are only valid for flat plates at small angles of attack. Supersonic airplanes, however, usually have relatively flat wings, and also fly at low angles of attack. So the above equations can often also be applied for the wings of supersonic aircrafts. Isn't it surprising that such simple equations can say so much about such complicated aircrafts?