Aircraft Stress Analysis and Structural Design

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Chapter 1

Introduction

The primary purpose of this reader is to describe the use of analysis equations and methodologies of structures for design purposes. It is common these days that we hear from industry that the students graduating with engineering degrees do not know how to design whether it is structures, or mechanical systems, or systems from other engineering disciplines. We therefore put the emphasis in this course on design.

Anybody who has some understanding of the design process, however, realises that without a thorough understanding of the use of analysis methods it will not be possible to design at least a reliable system. It is, therefore, important to establish a sound and firm analysis foundation before one can start the design practice. The approach used in this reader, however, is different from the traditional one in which analysis and designs are taught in different portions of the course. Instead we will use an approach in which small portions (sections) of topics from analysis are first introduced immediately followed by their design implementation. A more detailed description of the outline and the contents of the reader is provided in the following.

The reader is divided into two primary sections because of a very important concept, called *statical determinacy*, that has a very strong influence on the way structures are designed. It is of course too early to completely describe the impact of statical indeterminacy on design. It will be sufficient to state at this point that statical determinacy simplifies the design process of a structure made of multiple components by making it possible to design individual components independent from one another. Structural indeterminacy on the other hand causes the internal load distribution in a given structural system to be dependent on the dimensional and material properties of the individual component that are typically being designed. That is, as the design of the individual components change the loads acting on those components for which they are being designed also change. Hence, individidual components cannot be designed independently as the design changes in these components and the other components around them alter the loads that they are being designed for. The resulting process is an iterative one requiring design of all the individual components to be repeated again and again until the internal load redistribution stabilises, reaching an equilibrium state with the prescribed external loading.

For the reason briefly described above, the first section of the reader is dedicated to *statically determinate* structural systems. An important implication of the independence of internal loads to design changes is the fact that stresses in the individual components depend only on the sizing variables, which determine the cross-sectional geometry of the components. That is, since internal stresses are determined by dividing the internal load by a property of the cross section of the component (be it area, or thickness, or area moment of inertia), a designer can make sure that by proper selection of the sizing variables safe levels of stresses compared to stress allowables can be maintained. Furthermore, material selection of the components can be made independent of the rest of the components. The first section therefore concentrates on the use of stress analysis of structural components to design cross sectional properties based on stress considerations only. There are, of course, other considerations that influence the design decision besides stresses even if we limit ourselves to statically determinate systems. For example, structural responses such as buckling and vibration are typically very important for many aircraft and spacecraft design problems. Such considerations will be discussed later in the reader.

The second section of the reader is dedicated to *statically indeterminate* structural systems. An important element of the stress analysis of indeterminate systems is the need to compute displacements and deformations of the members. As stated earlier, internal load distribution in indeterminate systems is influenced by the cross-sectional properties of the individual components as well as their material properties. This dependency is in fact stems from the dependence of the internal loads to the stiffnesses of the individual component, which strongly influence their deformation characteristics. Structural stiffness is a product of the material properties and the dimensions. As such, the design effort requires evaluation of the member stiffnesses as a function of the designed quantities, determining the effect of the member stiffnesses on member deformations, and in turn internal forces. Hence, the second section concentrates on design of indeterminate structures and structures with displacement constraints.

Finally, the material in the reader is organised depending on the type of structural component to be analysed and designed. Most complex structural systems are composed of three different types of components, namely truss and beam members, and plate segments. The primary difference between the three types of components is the type of loads that are carried by them. Truss members are two-load members which can only carry co-linear loads (either tensile or compressive) acting along a line that passes through the two pin joints connecting the member to other truss members or structural components. Furthermore the member is assumed to be straight between the two pins, hence no bending of the member is allowed. The beam members are primarily used for carrying bending loads, which are typically induced by more than two forces, or two force members with non co-linear forces. Finally, plate segments are the more general form of a load carrying component, reacting to in plane and out of plane loads. A more formal definition of these components will be provided in appropriate chapters later on. For the time being we only consider some examples to them in aircraft and spacecraft structures.

Chapter 2

Stress Analysis and Design of Statically Determinate Trusses

A lot of aerospace structures can be idealised as truss structures. This is clearly illustrated in picture 2.1 where it can be seen that the ribs in the wing are built up as a truss structure. Also in space applications, trusses are widely used because of their simplicity and light-weightness. Now the question rises: what is a truss?

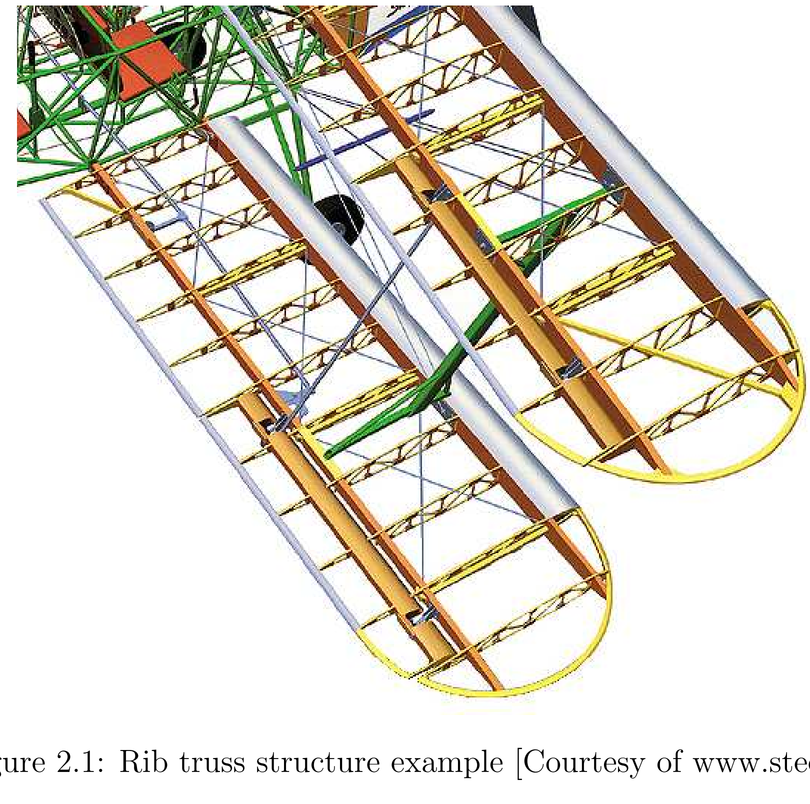


Figure 2.1: Rib truss structure example [Courtesy of www.steenaero.com]

2.1 Truss Properties

A truss is a structure built up from truss members, which are slender bars with a cross-sectional area A and having a Young's modulus E.

A generic picture is given in figure 2.2.

Figure 2.2: Typical simple truss structure [Courtesy of www.compuzone.co.kr]

The following assumptions apply when analysing a truss structure:

- Bar elements can only transfer loads axially. These forces can be either tensile, tending to elongate the bar, or compressive, tending to shorten the bar.
- The bar elements are pin-joined together. This has as a consequence that the joints only transfer forces from one bar element to the other, and no moments. If the bar elements were welded together, or attached with a plate, also moments would have been transferred, which is in contradiction with the earlier mentioning that bar elements are only suited to take axial loads, and no moments.
- Loadings can only be applied at the joints of the truss. This inherently means that the weight of the bar, which would act at the midpoint of a uniform bar, is neglected.

2.2 Static Determinacy of Trusses

A truss is statically determinate if the numbers of unknowns is equal to the numbers of equations (which is twice the number of nodes n because at each node, two forces in x and y direction act) that can be constructed for the problem. The unknowns are the truss member forces and the reaction forces at the truss supports. There are m truss member forces and r reaction forces.

So a necessary condition for static determinacy of a truss structure is

$$m + r = 2n. \tag{2.1}$$

If $m + r > 2n$, the truss structure is indeterminate, but that will be discussed in later chapters

The reaction forces are due to the supports of the truss structure. The two support types that we use for truss structures are the pinned joint and the roller support. Both supports can be inspected in figure 2.3.

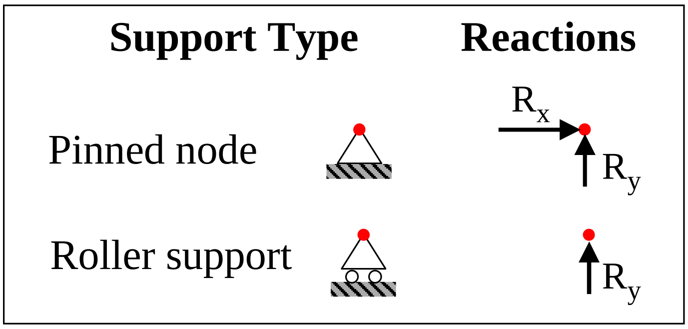


Figure 2.3: Truss support types

Two trusses are given in figure 2.4. One of them is statically determinate and one is indeterminate. Figure out why they are determinate or indeterminate.

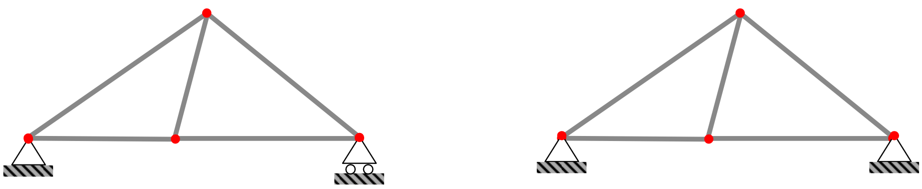


Figure 2.4: Statically determinate and indeterminate trusses

2.3 Truss Analysis

The internal forces created by an external loading to the truss can be solved by using the method of joints. This method implies that the summation of all forces in either of the two directions at each node is equal to zero ($\sum F_x = 0$ and $\sum F_y = 0$). The method of joints is highlighted in figure 2.5.

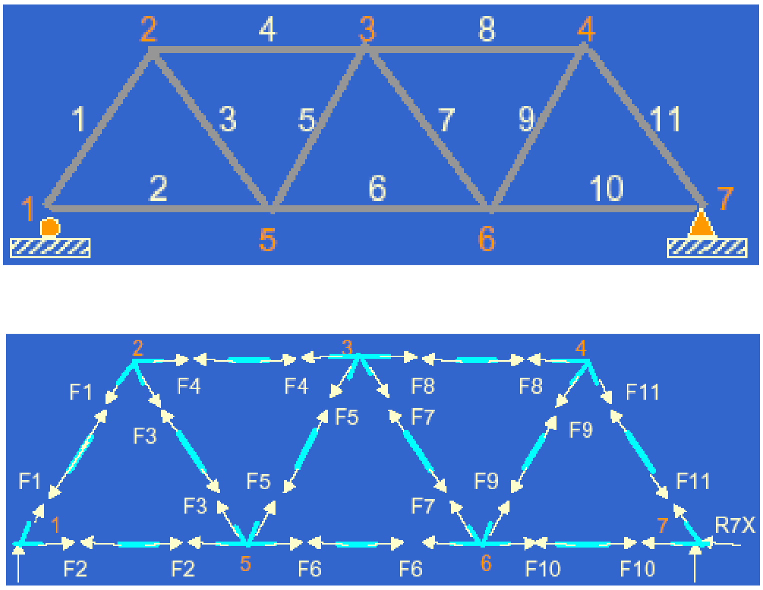


Figure 2.5: Internal force distribution inside a truss

2.3.1 Truss Definition

Defining and analysing a truss structure is not only a matter of stress analysis, but also a matter of good book keeping. Therefore we define the nodal locations, the members along with their begin and end nodes and material properties in an orderly fashion. The i^{th} node has the following location p_i:

$$p_i = \{x_i, y_i\},\ i = 1 \dots n, \tag{2.2}$$

where n is the number of nodes in the truss. The j^{th} member or element e_j has a start point p_s and an end point p_e:

$$e_j = \{p_s, p_e\},\ j = 1 \dots m, \tag{2.3}$$

where m is the number of members. Each member has its Young's modulus E_j and cross-sectional area A_j. The length l_j of the j^{th} member is then defined as:

$$l_j = \sqrt{(x_s - x_e)^2 + (y_s - y_e)^2} \tag{2.4}$$

Finally each node has its own applied force P_j, which can be zero in case of the absence of the force.

2.3.2 Stress Analysis

The procedure for analysing a truss is as follows:

- Define the truss as described in the previous section.
- Draw each node separately including the member forces and if relevant, the applied forces and reaction forces on the node. By convention, we draw the direction of the member forces away from the node (see also figure 2.5). This does not mean that all forces are tensile. Compressive forces will come out negative. The reaction forces should be drawn in the positive axis directions and applied forces should remain in their original direction!
- Sum all the forces per node and per direction and equate them to zero for equilibrium.
- Solve the obtained equations from the previous item to get the unknown forces (notice that in case of a statically indeterminate truss, you would have too few equations!), both member and reaction forces.
- Check global equilibrium to see whether the reaction forces come out right. This gives additional confidence in the solution.

Notice that if the convention of drawing the unknown member forces away from the node is used, the compressive member forces come out negative automatically. Now that the member forces are known, the member stresses σ_j can be retrieved using

$$\sigma_j = \frac{P_j}{A_j}. \tag{2.5}$$

Notice that only the cross-sectional area itself is needed for the calculation of the member stress, so the shape of the cross section, albeit a square or a circle, is irrelevant. Also notice that since a truss member can only take normal forces, only normal stress is calculated, so no shear stress is present in a truss member.

2.4 Stress Design

The stresses in the truss members can be calculated according to equation 2.5. This means that if the loading on the member and its geometry are known, the resulting stress can be derived straightly. Now the question occurs whether the resulting stress is larger than the stress the material can take without failure. The latter stress is the allowable stress σ_{all}.

If the occurring stress in a truss member is lower than the allowable stress, it means that there is too much material present. Obviously an aerospace structure should be as light as possible. So therefore in a stress design, one should always strive for equating the occurring stress and the allowable stress. Such a desing philosophy is called a *fully stressed design*. In a statically determinate truss, the member forces are solely dependent on the externally applied loads. As such, one should adapt the cross sectional area to a *new* one as:

$$\sigma_{all} = \frac{F}{A_{new}} \tag{2.6}$$

Keeping in mind that the original expression for the stress is $\sigma = \frac{F}{A_{old}}$, we get as full stressed design update criterion for the cross-sectional area the following expression:

$$A_{new} = \left| \frac{\sigma}{\sigma_{all}} \right| A_{old} \tag{2.7}$$

2.5 Example

Given is a three bar truss, shown in figure 2.6, with all element having equal length. A force P is applied at node B in negative y direction. First free-body diagrams are drawn for each node separately. They can be viewed in figure 2.7.

The equilibrium equations for node A are:

$$\sum F_x^{\rightarrow +} : 0 = A_x + F_{AB}\frac{L_{BC}}{L_{AB}} \tag{2.8}$$

$$\sum F_y^{\uparrow +} : 0 = -F_{AC} - F_{AB}\frac{L_{AC}}{L_{AB}}. \tag{2.9}$$

The equilibrium equations for node B are:

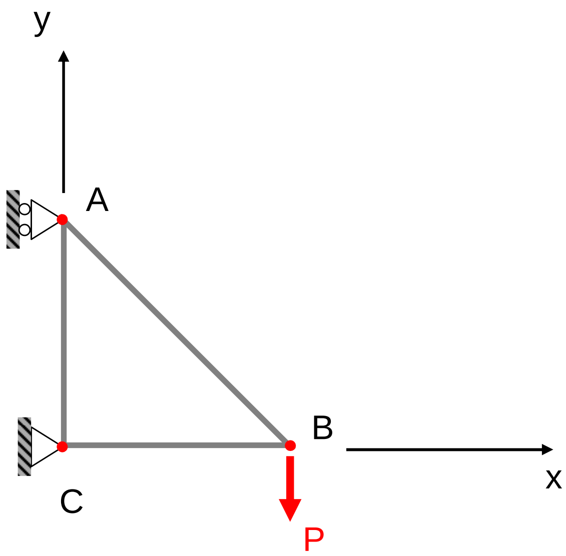


Figure 2.6: Three bar truss example

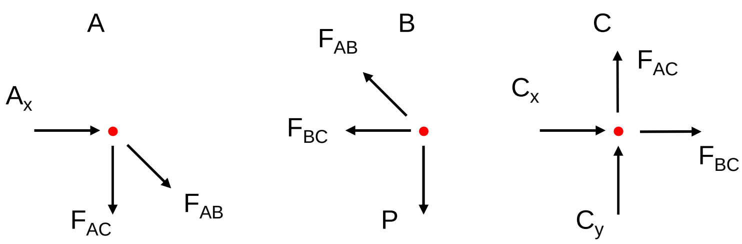


Figure 2.7: Free-body diagram of each node

$$\sum F_x^{\rightarrow +} : 0 = -F_{BC} - F_{AB}\frac{L_{BC}}{L_{AB}} \tag{2.10}$$

$$\sum F_y^{\uparrow +} : 0 = -P + F_{AB}\frac{L_{AC}}{L_{AB}}. \tag{2.11}$$

The equilibrium equations for node C are:

$$\sum F_x^{\rightarrow +} : 0 = C_x + F_{BC} \tag{2.12}$$

$$\sum F_y^{\uparrow +} : 0 = C_y + F_{AC}. \tag{2.13}$$

Solution of the six equations simultaneously yields the following expressions for the unknowns:

$$\left\{ A_x = -P\frac{L_{BC}}{L_{AC}}, C_x = P\frac{L_{BC}}{L_{AC}}, C_y = P, F_{AB} = P\frac{L_{AB}}{L_{AC}}, F_{AC} = -P, F_{BC} = -P\frac{L_{BC}}{L_{AC}} \right\} \tag{2.14}$$

By inspection it is evident that the reaction forces equilibrate each other and the externally applied force P.

Now that the three member forces are known, their cross-sectional areas can be changed such that each of the members becomes fully stressed using equation 2.7. Assume the material has an allowable stress σ_{all}, the new cross sectional areas become:

$$\left\{ A_{AB,new} = \frac{P\frac{L_{AB}}{L_{AC}}}{\sigma_{all}}, A_{AC,new} = \frac{P}{\sigma_{all}}, A_{BC,new} = \frac{P\frac{L_{BC}}{L_{AC}}}{\sigma_{all}} \right\} \tag{2.15}$$

Note that although some of the member forces were negative, their absolute value needs to be taken to calculate the cross-sectional areas because obviously an area cannot be negative.

Chapter 3

Stress Analysis and Design of Statically Determinate Beams

The objective of this chapter is to master the skills of analysing statically determinate structures composed of beams or a combination of beam and truss members using both the continuous and discrete approaches. The discrete approach is based on splitting the structure into short beam segments over which the loading can be assumed constant. In this case, analysing the structure is reduced to well-structured calculation which can be carried out either by hand or using a computer.

3.1 What is a beam?

A beam is any structure or structural member which has one dimension much larger than the other two. This is the simplest possible definition and the one mostly used in engineering practice. The long direction is usually called the beam axis. The beam axis is not necessarily a straight line. The planes normal to the beam axis (in the undeformed configuration) intersect the structure in cross-sections. The point at which the beam axis intersects the cross-section need not be the centroid, although under certain conditions this choice results in considerable simplification.

Beam analysis is common in preliminary design of aerospace structures. As shown in figure 3.1, the complete structure of an aircraft can be idealised as a collection of beams. In other situations, structural members rather than built-up structures are modelled as beam. Examples of such use are tail or wing spars and shafts of a turbojet or turbofan engine as shown in figure 3.2.

Apart from its use in preliminary design, beam models are important for the insight they provide into the behaviour of the structure. Terminology

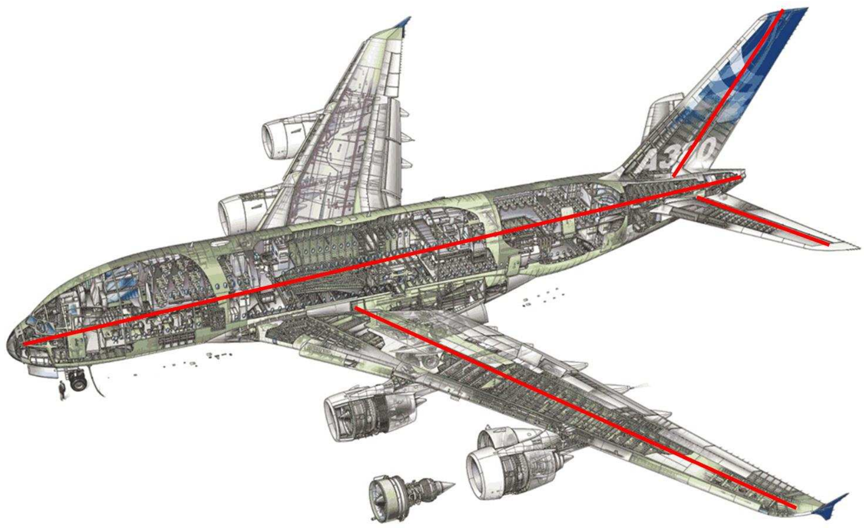


Figure 3.1: Aircraft beam model [Courtesy of www.flightinternational.com]

associated with beam analysis is an essential part of the vocabulary of structural engineers. A thorough understanding of and familiarity with beam theory is one of the cornerstones of the engineering intuition of structural engineers.

3.2 Equilibrium Equations

A beam has cross-section dimensions which are much smaller than the beam length. For this reason a beam is usually sketched as lines representing the beam axis. If we imagine the beam to be separated into two parts at a certain (see figure 3.3) cross-section, the internal forces at that cross-section will have a static resultant force and moment. When resolved along the local beam axes, the static resultant is composed of three force components and three moment components. The force component along the beam axis, x, is termed the normal or axial force. The force components along the two cross-section axes, y and z, are called shear forces. The moment component along the beam axis is called the twisting moment or torque. The moment components along the two cross-section axes are called bending moments.

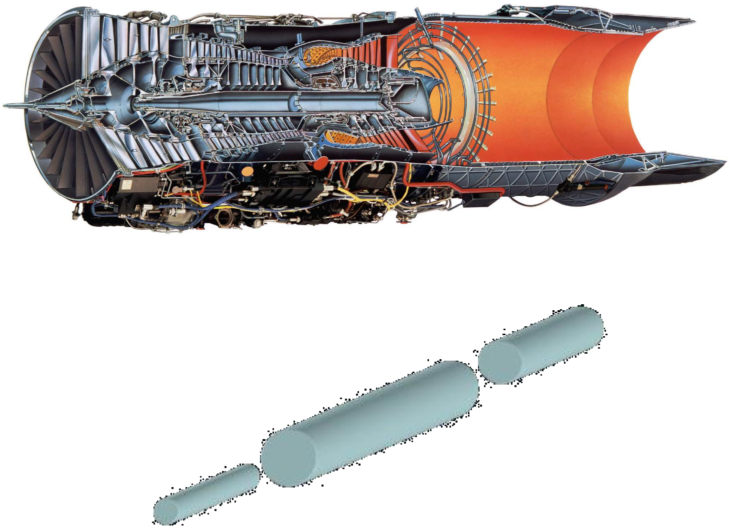


Figure 3.2: Engine beam model [Courtesy of www.milnet.com]

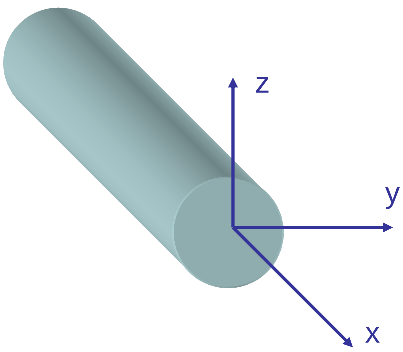


Figure 3.3: Local beam axes

It is important to note that while the names of the different cross-section force and moment components are almost universal, the sign conventions, symbols, and how the variation of these quantities is plotted along the beam axis, can vary between different books and computer software. It is essential to review the conventions adopted by specific books or software packages before using them.

In the following we are going to use a sign convention that is common in aerospace applications. The normal force is denoted by N. The sign convention for the normal force is a positive sign for a tensile force and a negative sign for a compressive force. The torque is denoted by M_x and is positive when rotating counter clockwise out of the cross-section. The bending moments about the two cross-section axes are denoted by M_y and M_z. A bending moment is positive if it causes compression in the first quadrant. Finally, the shear forces are denoted by V_y and V_z. The sign convention of all cross-sectional resultants is shown in figure 3.4.

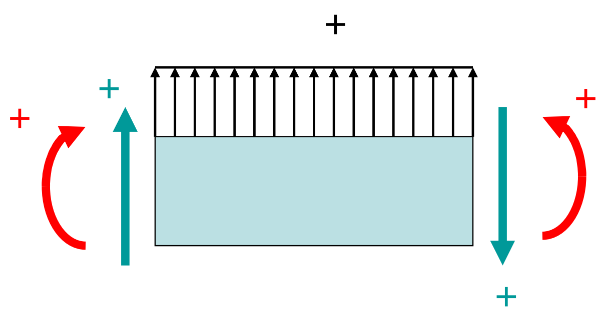


Figure 3.4: Beam sign convection

Consider an arbitrary infinitesimal segment of the beam as shown in figure 3.5. This segment will be in equilibrium under the action of internal and external forces and moments. For simplicity, we derive the vertical equilibrium equations. Similar arguments can be used to derive the rest of the equilibrium equations.

Let $p_z(x)$ define the vertical loading per unit length. If the beam segment is short enough, the load can be assumed uniform. The vertical force equilibrium of the beam segment requires,

$$p_z(x)\Delta x + V_z(x) - V_z(x + \Delta x) = 0. \tag{3.1}$$

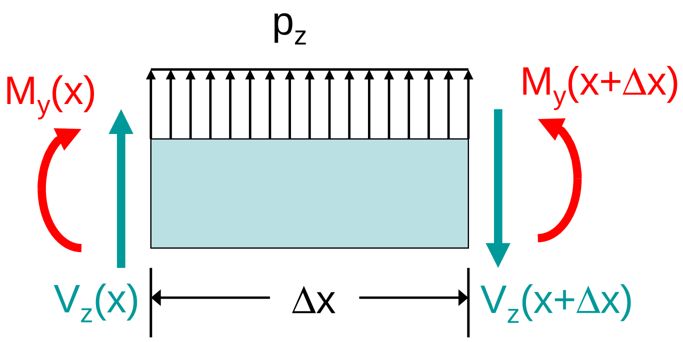


Figure 3.5: Infinitesimal beam segment

Dividing by Δx and taking the limit as $\Delta x \rightarrow 0$, we find the continuous form of the vertical force equilibrium as,

$$\frac{dV_z}{dx} = p_z. \tag{3.2}$$

The student is encouraged to derive the moment equilibrium equation around the y axis. The complete set of equilibrium equations are listed below,

$$\frac{dN}{dx} = -p_x, \quad \frac{dV_y}{dx} = p_y, \quad \frac{dV_z}{dx} = p_z \tag{3.3a}$$

$$\frac{M_x}{dx} = -t_x, \quad \frac{M_y}{dx} = V_z, \quad \frac{dM_z}{dx} = V_y. \tag{3.3b}$$

In these equations, $p_x(x)$ is the distributed thrust load per unit length, p_y and p_z are the lateral distributed loads per unit length in y and z direction respectively, and $t_x(x)$ is the distributid twisting moment per unit length.

The equilibrium equations 3.3 are valid when the applied distributed loads vary smoothly. This, however, is frequently not the case. In many practical situations, load introduction is abrupt and the loading can be idealised as concentrated forces and/or moments. In such cases, the distribution of the cross-sectional forces and moments is not continuous. Special jump conditions have to applied at such locations.

The jump condition when a vertical concentrated force is applied are derived. Referring to figure 3.6, the vertical force equilibrium equation 3.1 is modified to read,

$$F_z + p_z(x)\Delta x + V_z(x) - V_z(x + \Delta x) = 0. \tag{3.4}$$

Taking the limit as $\Delta x \to 0$, we obtain the jump condition,

$$V_z^+ - V_z^- = F_z, \tag{3.5}$$

where V_z^- and V_z^+ are the shear force right before and right after the applied concentrated load F_z.

Similar jump conditions can be derived for all forces and moments. The student should be able to carry out the derivation without difficulty.

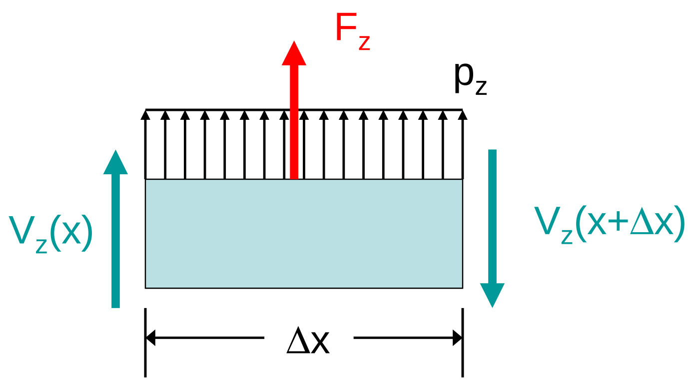


Figure 3.6: Force jump over an infinitesimal beam segment

3.3 Stresses in Beams

Once the distribution of the internal forces and moments is obtained through the integration of the equilibrium equations 3.3, the stress state at any point of the cross section can be calculated. This might at first seem strange since different loading can lead to the same internal force and moment resultant at a given cross-section. The question arises whether the stress distribution due to two statically equivalent loads are the same. The answer to this question is that the stress distributions due to statically equivalent systems of loads are not identically the same but are generally close away from the point of application of load. This is a statement of the de St. Venant principle. Stresses calculated based only on the resultant internal force and moment are usually accurate enough away from points of load introduction, supports, and discontinuity in the cross-sections.

If, for the sake of simplicity, we assume that the beam is loaded in the xz plane only, the formulae for the stress distribution are particularly simple. The distribution of the normal stress σ_x is given by,

$$\sigma_x = \frac{N}{A} - \frac{M_y}{I_y} z, \tag{3.6}$$

where A is the total area of the cross-section, I_y is the second moment of area about the y axis, and the z coordinate is measured from the centroid of the cross section. The distribution of the shear stress is given by,

$$\tau = -\frac{V_z Q}{I_y t}, \tag{3.7}$$

where Q is the first moment of area of the part of the cross section up to the point of calculation.

3.4 Discrete Beam Analysis

While the equations of equilibrium are straightforward to integrate for beams with simple loading, the integration can become very cumbersome for more complex loading (e.g., lift distribution over a wing). For such cases, it is possible to compute approximate distribution of cross-sectional forces and moments by discretising the beam into a finite number of segments, n, and assuming the loading to be constant over each segment.

For simplicity we consider only lateral loads in the xz plane. The technique can be easily extended for the general case. The beam is divided into n segments. The i^{th} segment having a length $\ell^{(i)}$ and uniform applied load $p_z^{(i)}$. It is assumed that any concentrated forces are applied between the segments and not inside the segment. Integration of the equilibrium equations gives the following relations,

$$V_z^{(i)}\big|_L = V_z^{(i-1)}\big|_R + F_z^{(i)}, V_z^{(i)}\big|_R = V_z^{(i)}\big|_L + p_z^{(i)} \ell^{(i)}, \tag{3.8}$$

and

$$M_y^{(i)}\big|_L = M_y^{(i-1)}\big|_R\,,\; M_y^{(i)}\big|_R = M_y^{(i)}\big|_L + \big(V_z^{(i)}\big|_L + p_z^{(i)}\ell^{(i)}/2\big)\ell^{(i)}, \tag{3.9}$$

where the L and R subscripts denote the left end and right end of the segment respectively.

The above equations are very easy to program using Maple, Matlab, or even using a simple excel sheet. The way the equations are used can be best illustrated by an example.

3.5 Example

A representative example will be analysed using both continuous integration and a discrete beam model. Consider an aircraft wing modelled as a cantilever beam and loaded with a vertical load distribution (lift distribution), approximated as,

$$p_z(x) = p_0\Big(3\big(\frac{x}{L}\big) - 3\big(\frac{x}{L}\big)^2 + \big(\frac{x}{L}\big)^3\Big), \tag{3.10}$$

where, for numerical computations, $p_0 = 1000\text{N/m}$ and $L = 5\text{m}$ are used.

The shear force at the leftmost edge is zero since it is a *free* end. This allows us to evaluate the integration constant when integrating equation 3.10. Thus, we can obtain the shear force distribution as,

$$V_z(x) = \frac{p_0 L}{4}\Big(6\big(\frac{x}{L}\big)^2 - 4\big(\frac{x}{L}\big)^3 + \big(\frac{x}{L}\big)^4\Big). \tag{3.11}$$

Similarly, equation 3.11 can be integrated and the integration constant determined by noting that at $x = 0$, the bending moment vanishes. The final expression for the bending moment is,

$$M_y(x) = \frac{p_0 L^2}{20}\Big(10\big(\frac{x}{L}\big)^3 - 5\big(\frac{x}{L}\big)^4 + \big(\frac{x}{L}\big)^5\Big). \tag{3.12}$$

To solve this example using the discrete approach we divide the beam into ten segments. The results of the computation is summarised in Table 3.5. Sample Maple codes for discrete beam analysis are available online.

Table 3.1: Discrete Shear Force and Bending Moment Distribution

Segment	ℓ	p_z	$V_z\|_L$	$V_z\|_R$	$M_y\|_L$	$M_y\|_R$
1	0.5	142.6	0.0	71.3	0.0	17.8
2	0.5	385.9	71.3	264.3	17.8	101.7
3	0.5	578.1	264.3	553.3	101.7	306.1
4	0.5	725.4	553.3	916.0	306.1	673.4
5	0.5	833.6	916.0	1332.8	673.4	1235.6
6	0.5	908.9	1332.8	1787.3	1235.6	2015.7
7	0.5	957.1	1787.3	2265.8	2015.7	3028.9
8	0.5	984.4	2265.8	2758.0	3028.9	4284.9
9	0.5	996.6	2758.0	3256.3	4284.9	5788.5
10	0.5	999.9	3256.3	3756.3	5788.5	7541.6

Chapter 4

Stress Analysis and Design of Statically Determinate Plates

Plates are extensively used structural elements in aerospace constructions. Skin material, ribs, aircraft floors, ... ; their application possibilities are infinite. An example of the use of plate material as aircraft skin is given in figure 4.1. At the very dawn of aviation, aircraft wing and fuselage skins were applied to preserve the required aerodynamic shape, and all the loads acting on the vehicle were carried mainly by truss structures. However, with the introduction of aluminium in the aircraft industry, this changed radiaclly. The so-called *fully stressed skin* was invented. This means that the plates forming the skin take their part of the loads.

A plate can be loaded mainly in two ways, in the plane of the plate and perpendicular to its plane. The stress analysis of bending of plates has proven to be too complicated to deal with in this course, and as such will not be treated. Also the loading in the plane is not exactly trivial either, however, there are some particular cases that will be investigated in this course. These simplified cases already provide initial insight in the essentials of stresses in plates.

4.1 Uniform In-Plane Loading

The loading condition considered is a uniform in-plane loading. This means a constant stress field that is distributed uniformly over the entire plate. Such a stress field is the resultant of an applied stress in one direction of the plate. The plate can also be loaded uniformly in the other direction. This results in a uniform stress field in that direction as well. A combination of two uniform stress fields in either direction of the plate is called a *bi-axial uniform stress*

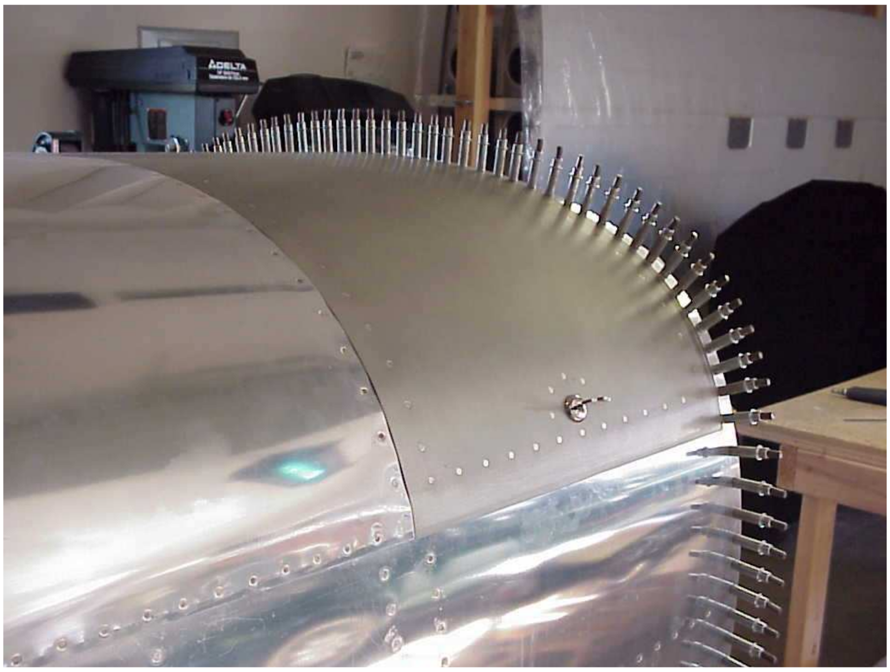


Figure 4.1: Example of an aircraft riveted plate skin [Courtesy of www.romeolima.com]

state. Finally a plate can, in combination with the previously mentioned applied normal stresses, also be loaded in uniform shear. This results in a uniformly distributed shear stress state in the plate. The total combination is shown in figure. Please note the particular directions of the applied shear stress components. The combined load cases can be seen in figure 4.2.

When these three components act on a plate, a stress analysis can be carried out. Even in case of a plate that has a complex stress distribution due to a certain applied load combination, locally the stress distribution is uniform. Such a small local plate element is called a *stress element*.

4.2 Circle of Mohr

If a bi-axial stress state and a uniform shear stress are present, the question is what the occurring normal and shear stress would be in any arbitrary direction of the plate. On top of that, one might wonder what the maximum normal and shear stress would be in the plate, and in which direction that one would act. This issue is solved by looking at the force equilibrium in an

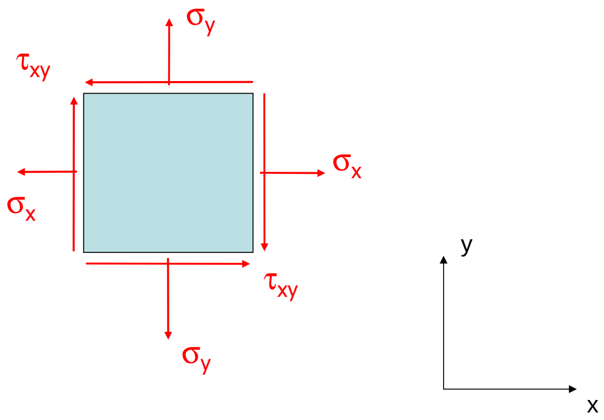


Figure 4.2: Example of fully loaded stress element

arbitrary direction of a stress element, as shown in figure 4.3.

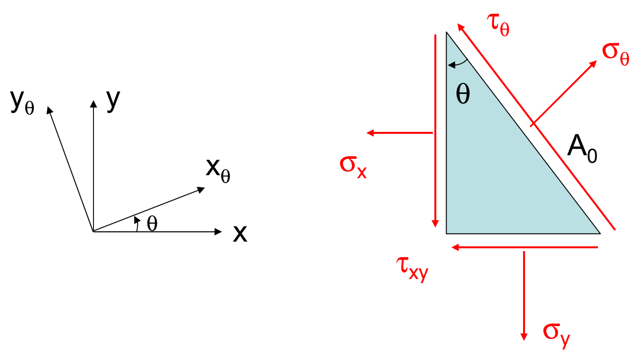


Figure 4.3: Free body diagram for the plate equilibrium equations

Keeping in mind that a force is the applied stress times the area over which the stress is distributed, the force equilibrium in x and y-direction can be written as

$$\begin{cases} \sum F_{x_\theta} : 0 = \sigma_\theta A_0 - \sigma_x (A_0 \cos\theta)\cos\theta - \tau_{xy}(A_0 \sin\theta)\cos\theta - \sigma_y (A_0 \sin\theta)\sin\theta \\ \qquad\qquad -\tau_{xy}(A_0\cos\theta)\sin\theta \\ \sum F_{y_\theta} : 0 = \tau_\theta A_0 + \sigma_x (A_0 \cos\theta)\sin\theta + \tau_{xy}(A_0 \sin\theta)\sin\theta - \sigma_y (A_0 \sin\theta)\cos\theta \\ \qquad\qquad -\tau_{xy}(A_0\cos\theta)\cos\theta \end{cases} \tag{4.1}$$

.

Assembling corresponding terms of the above equation yields

$$\begin{cases} \sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\ \tau_\theta = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \end{cases} \tag{4.2}$$

.

Using standard trigonometric relations, the above expressions can be simplified even further:

$$\begin{cases} \sigma_\theta = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_\theta = -\frac{\sigma_x-\sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{cases} \tag{4.3}$$

.

If the second of the above equation is substituted into the first one, one obtains an expression of a circle:

$$\left(\sigma_\theta - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_\theta^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \tag{4.4}$$

This circle is called *Mohr's circle*. It is obvious that this is a circle in the $\sigma_\theta - \tau_\theta$ - plane and has as centre point $\frac{\sigma_x+\sigma_y}{2}$, which is the average normal stress σ_{av} and radius $R = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$. Assume the applied stresses σ_x, σ_y and τ_{xy} are known, the stresses σ_θ and τ_θ can be determined using the equation of the circle of Mohr. The layout of the circle is given in figure 4.4.

It is clear that the maximum (σ_1) and minimum (σ_2) normal stresses can be read off Mohr's circle directly (where the circle intersects with the σ_θ-axis), as well as the maximum shear stress. In some cases it can also occur that σ_2 becomes larger than σ_1 in absolute value. Obviously σ_2 will be the maximum normal stress in that case. Also the double of the direction ϑ in which the normal stresses act can be read from the circle. It is the angle between the σ_θ-axis and the line connecting the circle's centre point and the (σ_x, τ_{xy}) point. The reason why it is a double angle can be found in equation 4.3. The equation for the direction angle ϑ is

$$\vartheta = \frac{1}{2}\arctan\left(\frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}\right) \tag{4.5}$$

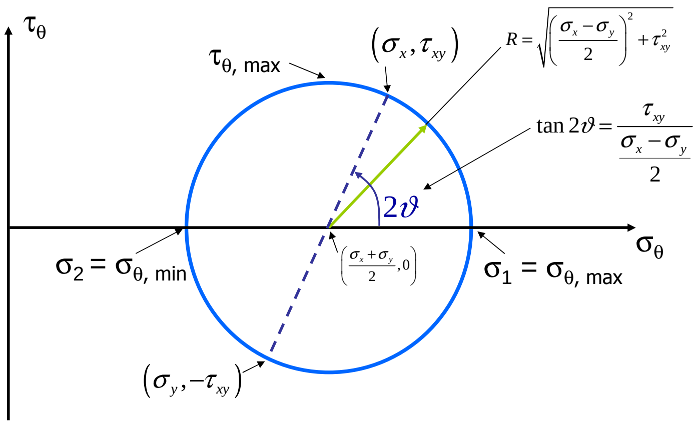


Figure 4.4: Layout and definition of Mohr's circle

4.3 Stress Design

The stress design of a statically determinate plate is very similar to that of a statically determinate truss or beam. In this case, also the *fully stressed design* philosophy applies, as already explained in section 2.4. The major difference is that in the case of a plate, we are not looking to update the cross sectional area, but the thickness, since the other geometric quantities are usually prescribed. The plate thickness can be updated according to two design criteria, maximum normal stress or maximum shear stress. The maximum normal stress is the maximum value of the two stresses σ_1 and σ_2 - in absolute value - obtained using Mohr's circle. The maximum shear stress is obtained directly from using the circle of Mohr. The *new* plate thickness is calculated according to the following equations, which are expressions for maximum normal and shear stress, respectively:

$$t_{new} = \frac{\sigma}{\sigma_{all}} t_{old} \tag{4.6}$$

$$t_{new} = \frac{\tau}{\tau_{all}} t_{old} \tag{4.7}$$

4.4 Example

Special cases of Mohr's circle

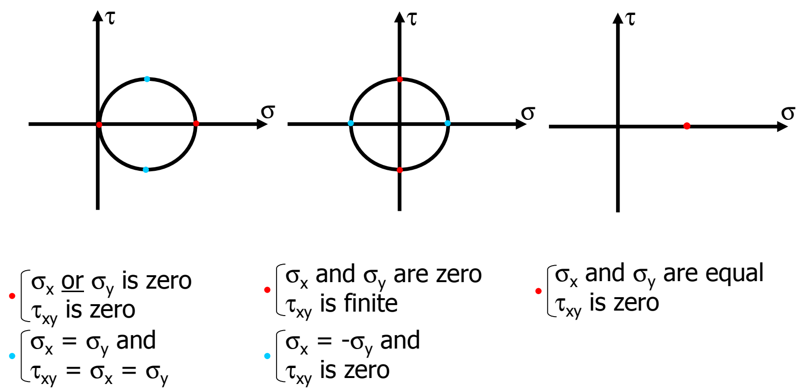


Figure 4.5: Special cases of Mohr's circle

Bi-axially Loaded Stress Element

A stress element is loaded with two normal stresses and one shear stress as indicated in figure 4.6.

In orde to construct Mohr's circle, we need the circle midpoint and the circle radius. The midpoint, located on the σ_θ-axis is the following:

$$\sigma_{av} = \frac{\sigma_x + \sigma_y}{2} = \frac{100 + 50}{2} = 75MPa \tag{4.8}$$

The circle radius is:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{100 - 50}{2}\right)^2 + 25^2} = 35.36MPa \tag{4.9}$$

Using this information, the maximum normal and shear stress can be found:

$$\sigma_1 = \sigma_{av} + R = 110.36MPa \tag{4.10}$$

$$\tau_{max} = R = 35.36MPa \tag{4.11}$$

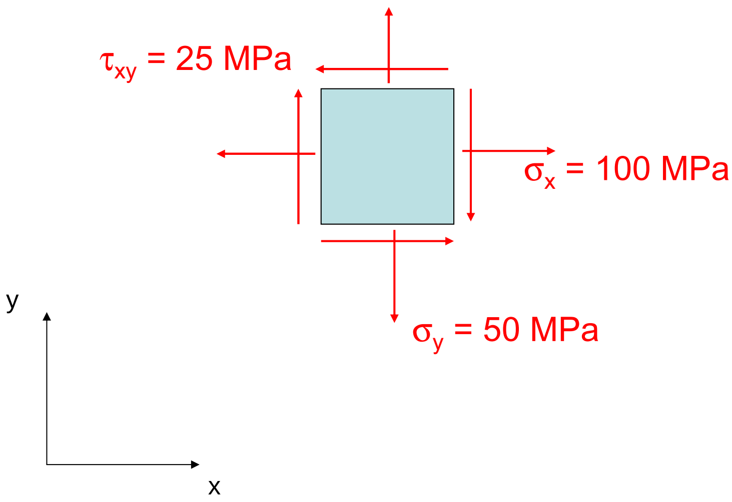


Figure 4.6: Example load case on a stress element

The value for σ_2 in this case is 39.64 MPa. The direction ϑ in which the maximum normal stress acts, is calculated as follows:

$$\vartheta = \frac{1}{2}\arctan\left(\frac{25}{(100-50)/2}\right) = 22.5^\circ \tag{4.12}$$

Assuming the initial thickness of the plate is 2 mm, the allowable normal stress is 100 MPa and the allowable shear stress is 50 MPa, the fully stressed thickness both for maximum normal and shear stress becomes, respectively:

$$t_{new} = \frac{110.36}{100}2 = 2.2\text{mm} \tag{4.13}$$

$$t_{new} = \frac{35.36}{50}2 = 1.4\text{mm} \tag{4.14}$$

It is evident that one can get totally different answers based on the stress criterion one uses.

Chapter 5

Stress Concentrations and Multiple Loads

5.1 Stress Concentrations

We have seen in the stress analysis methods of trusses, beams and plates that one deals with average stresses uniformly distributed over the cross-section of the structure. In most cases that is a fair assumption and leads to good analysis results. However particular load introductions, holes in a structure or sudden changes in cross-section can cause stress concentrations to occur. These stresses are larger than the average stresses calculated far from those critical locations, and as such can cause the occurring stress to become larger than the allowable stress at certain locations in the structure. Therefore it is important to take these stress concentrations into account and involve them into the stress analysis.

Stress concentrations happen at places where there are discontinuities in the structure, such as holes, cracks, cross-section changes. They are also present at load introduction locations. It can be stated generally that brittle materials are more susceptible to stress concentrations than ductile ones. The reason for this is twofold. First of all cracks and cognates influence the fatigue life of a structure, and it is generally known that brittle materials are more sensitive to fatigue. Secondly the elevated stresses might cause plasticity to emerge in the material. Also in this case, ductile materials are more favourable than brittle ones.

The stress level at the stress concentration location, σ_{max}, is higher than the average stress in the structure, σ_{av}. The ratio between these two stresses is a measure for the intensity of the stress concentration and indicated with the stress concentration factor K:

$$K = \frac{\sigma_{max}}{\sigma_{av}}. \tag{5.1}$$

De Saint-Venant's Principle

The question is now how big the influence of such local phenomena is on the remainder of the structure. De Saint-Venant postulated a theorem on this matter, which is still accepted today. He claims that such local phenomena have only a local influence. Far from the influence, the occurring stresses are independent of how the loads are applied, or whether there are holes present or not. This is depicted in figure 5.1. As a rule of thumb one may assume for load introduction problems that the effect of the applied axial load is relevant up to a distance equal to the width of the plate.

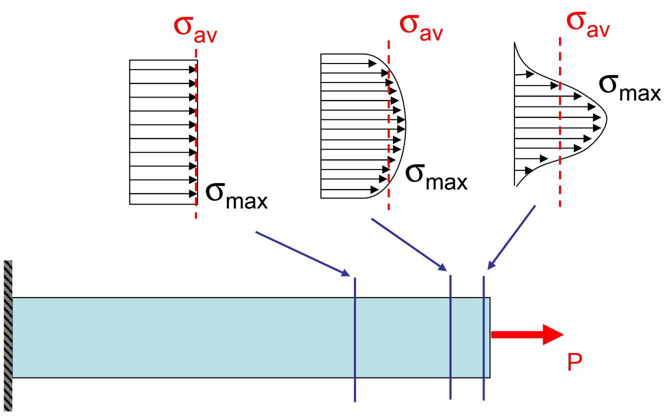


Figure 5.1: Illustration of the principle of De Saint-Venant

Circular holes and Cross-Section Changes

As mentioned earlier, stress concentrations can occur near circular holes or cross-section changes. The question is how to determine the stress concentration factor K based on the geometric properties of the holes or area change. The following geometric quantities are important when calculating K:

- Circular hole: the ratio of hole radius r and the distance from the hole to the plate edge d.

- Cross-section change:
    1. the ratio of the fillet radius r and the length of the smallest cross-section d.
    2. the ratio of the smallest cross-section length to the largest cross-section length.

Symbols r and d are used multiple times, however, these are the conventions for hole and cross-section change stress concentration factors, so be always mindful of what K factor you are dealing with.

The geometric definitions and K factor relations for the hole are given in figure 5.2 while those for the cross-section change are given in figure 5.3.

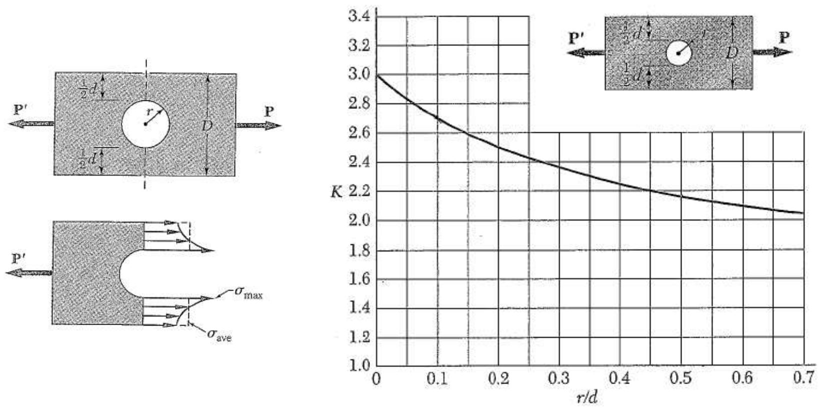


Figure 5.2: Geometric definitions and K factor for a hole [Courtesy of Beer, P.F., *Mechanics of Materials*]

5.2 Design for Multiple Loads

The terms *combined load* and *multiple loads* are often used in design. In the first case, a structure is submitted to a combination of loads simultaneously. A good example is an aircraft wing; this structure is submitted to a lift force, a drag force and an aerodynamic moment at the same time during flight (see figure 5.4).

Calculating the stresses due to such a combined load can be approached by calculating the internal normal and shear forces, and the bending and torsional moments resulting from the combined load at the location where

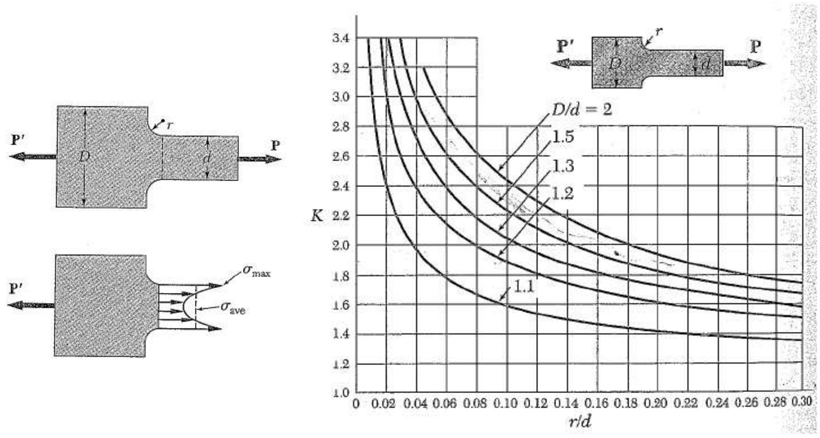


Figure 5.3: Geometric definitions and K factor for a cross-section change [Courtesy of Beer, P.F., *Mechanics of Materials*]

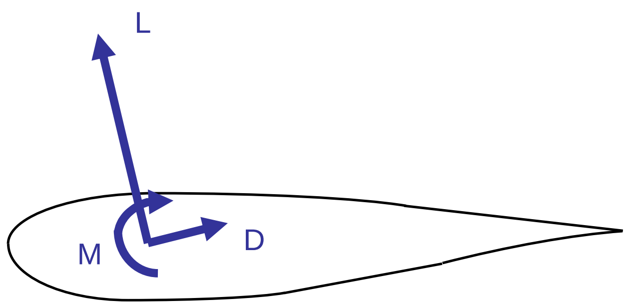


Figure 5.4: Combined airfoil load

one desires to know the stresses. The stresses resulting from each of these internal forces can added together since we are dealing with linear structures throughout the entire course. As such, desinging for combined loads is not fundamentally different from designing for a single load only.

When one designs for multiple loads on the other hand, the objective is to make sure that the structure is able to withstand several load cases which do not occur simultaneously. To clarify the difference between multiple load and combined load, each of the individual load cases of a multiple loading condition can be a combined load. As an example for a multiple load case serves a flight envelope (see figure 5.5). An aircraft wing needs to be designed for each of the points inside the boundaries of the flight envelope.

The challenge is that one is often confronted with completely different

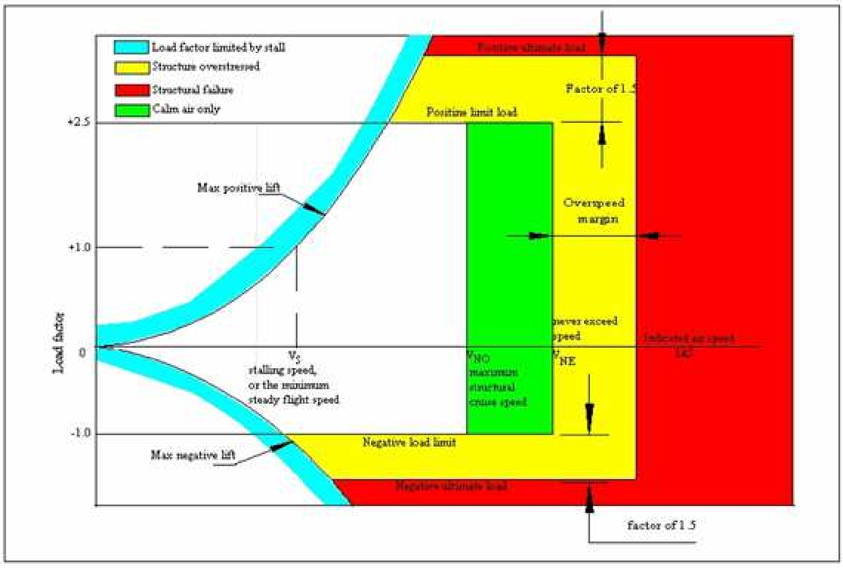


Figure 5.5: Flight envelope containing multiple load cases

requirements. Take as an example a simply supported beam which needs to be designed for tension and compression simultaneously. In the tensile case, the only important factor is the total cross-section area of the beam. On the other hand, in the compressive case, the beam is prone to buckling. For a simply supported beam, the buckling load P_{cr} is equal to $\frac{\pi^2 EI}{L^2}$, where I is the second moment of area of the beam, and as such not only the area of the beam becomes important, but also the shape. Let us assume a rectangular thin-walled shape with wall thickness 1 mm, width b and height h. The applied tensile/compressive load is 10,000 N and the material Young's modulus is 70 GPa and its allowable stress 310 MPa. This leads to two requirements:

$$A_{min} \geq \frac{P}{\sigma_{all}} \tag{5.2}$$

$$I_{min} \geq \frac{PL^2}{\pi^2 E}. \tag{5.3}$$

If we keep the formulas for A and I in mind ($A = 2t(b+h)$ and $I = th^2(\frac{1}{6}h + \frac{1}{2}b)$), and plugging in the values given above, we obtain the following inequalities:

$$32.3 \leq 2(b+h) \tag{5.4}$$

$$145.0 \leq h^2(\frac{1}{6}h + \frac{1}{2}b). \tag{5.5}$$

Plotting both curves gives a graph as depicted in figure 5.6.

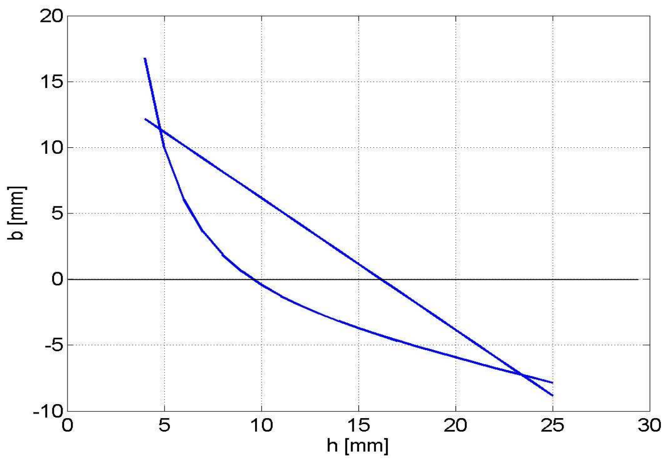


Figure 5.6: Plotted inequalities for the multiple load case

The inequalities given above indicate that all combinations of b and h that are above *both* curve are good candidates for the design problem. Obviously both values should be positive, so $b = 0$-line forms a boundary as well. If both tensile failure and buckling should occur simultaneously, then the intersection of both graphs in the positive b-plane is the only option.

Chapter 6

Displacement Analysis and Design of Statically Determinate Trusses

No structure is rigid. The floor you were just walking on deformed under your feet, the chair you are sitting on shortened just a little bit. However not evident from every day life practice, everything deforms under a certain loading. Especially in the aerospace industry, deformations are important as a consequence of the design philosophy. Weight is the driving factor in every aerospace design, and this constant striving for lightness renders structures to become flexible. The danger of such structures is that the occurring displacements might jeopardise the structural functionality. Imagine that because of weight savings, the torsional rigidity of a wing would be reduced. This would result in large local angles-of-attack altering the aerodynamic performance considerably.

Therefore a structure should always be designed such that it is able to carry the applied loads without violating certain displacement constraints. Because of that, displacement analysis of a structure is very important. Moreover, as we will see later, the internal stress distribution in a statically indeterminate structure depends entirely on the displacement field. This is a second example the importance of displacement calculation.

In this chapter we will deal with the displacement of trusses. But first we introduce a tool with which we can calculate displacements of trusses and beams using energy principles. This tool is called the *Dummy Load Method*.

6.1 Dummy Load Method

Energy Principles

Work W exerted on a structure is equal to a force F times the displacement δ the force creates:

$$W = F\delta \tag{6.1}$$

Doing work requires energy U, which is stored in the structure. If all energy is released when the applied forces are removed from the structure, such a structure is called *conservative* or *elastic*. If some energy remains in the structure, it is called *nonconservative* or *plastic*. The externally applied work, resulting in an external energy U_e needs to be in equilibrium with the internal structural strain energy U_i. This principle is called *conservation of energy*.

Another important concept is the principle of *virtual work*. This type of work can be interpreted as an applied force effectuating an infinitessimal small displacement in the direction of the force. It can be explained by looking at figure 6.1.

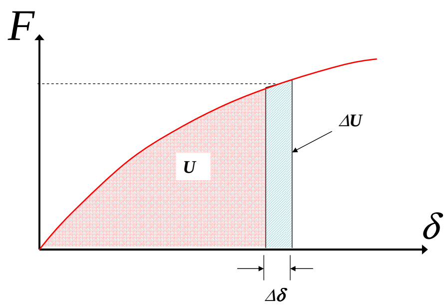


Figure 6.1: Principle of virtual work

It is clear that the energy stored in the structure due to a finite displacement $\Delta\delta$ is equal to $\Delta U = F\Delta\delta$. If the latter displacement is going to zero in the limit - in order to obtain the infinitessimally small displacement - the principle of virtual work becomes

$$dU = Fd\delta. \tag{6.2}$$

This means that the rate of change of the internal strain energy with the displacement is equal to the applied force. Associated with virtual work is the principle of *complementary energy* U^*. This can be inspected in figure 6.2.

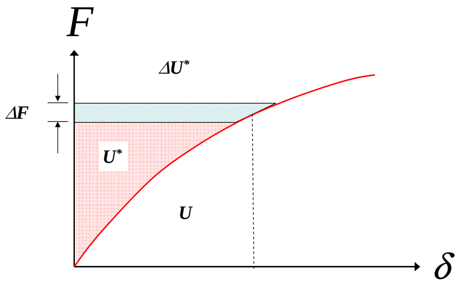


Figure 6.2: Principle of complementary energy

Instead of an infinitessimal displacement, we are now looking at a small force ΔF. Analoguous to the method described above for the virtual work, we obtain

$$dU^* = dF\ \delta \tag{6.3}$$

This formula indicates that the rate of change of the complementary energy with the force is equal to the resulting displacement.

Castigliano's Theorem

For linear structures, which we use thorugout this course, the relation between an applied force and the resulting displacement is linear $F = k\delta$, where k is the structural stiffness. It is evident from figure 6.3 that in the linear case the internal energy is equal to the complementary energy $U = U^*$.

Therefore the following important relation can be derived from equations 6.2 and 6.3:

$$\frac{dU}{dF} = \delta. \tag{6.4}$$

This means that we can find the displacement δ in the direction of a force F by differentiating the strain energy with respect to that force. This equation is called the *theorem of Castigliano*. If multiple forces act on the structure, each displacement in the direction of a particular force F_i is then expressed as:

$$\frac{dU}{dF_i} = \delta_i. \tag{6.5}$$

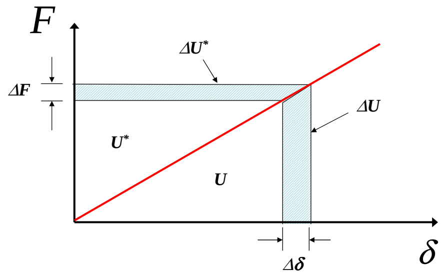


Figure 6.3: Linear force-displacement diagram

Note that the strain energy U is a result from all applied forces F, while the displacement in the direction of a particular force F_i is the derivative with that force only.

If we apply the above explanation to a truss member in particular, we end up with the following expression for the energy:

$$U = \int_0^F \delta dF = \frac{F^2 L}{2EA}, \tag{6.6}$$

keeping in mind that $F = k\delta$ and that the stiffness k of an axial truss member is equal to $k = \frac{EA}{L}$. For an entire truss structure, the energy becomes the summation over the individual contributions of each member:

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2E_i A_i} \tag{6.7}$$

where n is the number of truss members in the truss structure. This equation assumes the properties like cross-section, length and Young's modulus to be constant over the truss member.

If we apply Castigliano's theorem to the above expressions, we obtain an expression for the displacement in the direction of a certain applied force P (do not confuse with the internal forces F):

$$\delta = \frac{dU}{dP} = \sum_{i=1}^{n} \frac{F_i \frac{dF_i}{dP} L_i}{E_i A_i}. \tag{6.8}$$

The quantity $\frac{dF_i}{dP}$ is how the internal force in truss member i changes if the external loading P changes. This is also denoted with the symbol f_i:

$$\delta = \frac{dU}{dP} = \sum_{i=1}^{n} \frac{F_i f_i L_i}{E_i A_i}. \tag{6.9}$$

Notice that f_i is dimensionless.

Dummy Load Method

In practice, the above derived theory works as follows. The most difficult and important part of the expression is to determine the unity force distribution f_i. This quantity indicates how the internal forces change with changing external force. Keeping this in mind, and also thinking of the fact that Castigliano's theorem says that the displacement is obtained by deriving the total strain energy U by a force in that direction, one can conclude the following. If one wants to know the displacement of a structure in a certain direction, place a unit load (remind that f_i is dimensionless) at that particular point in the direction in which one desires to know the displacement. Do this regardless whether an external load P is already applied or not. Using this applied unit load, one can calculate f_i. Use the following approach to calculate the displacements of a truss:

- Calculate the internal force F_i in each truss member due to the externally applied forces P_i.
- Remove all externally applied forces.
- Apply a unit load at the point where you want to know the displacement in the direction you want to calculate the displacement.
- Calculate the reaction forces at the supports due to the dummy load.
- Calculate the internal unity forces f_i due to the dummy load.
- Apply equation 6.8 to obtain the magintude of the displacement in the direction of the dummy load.

6.2 Displacement Design

If the geometry of a structure is given, including the loading conditions, the construction displaces according to the given parameters. However, it might be desirable to alter the geometry of the construction such that a certain point of the truss meets a prescribed displacement δ_0. Usually the truss layout in terms of member lenghts is fixed as well as the material properties. This leaves the cross-sectional areas as variables. Assume that only the cross-sectional area of member n, A_n is variable, we get the following equation for the displacement (analogue to equation 6.8):

$$\sum_{i=1}^{n-1} \frac{F_i f_i L_i}{E_i A_i} + \frac{F_n f_n L_n}{E_n A_n} = \delta_0. \tag{6.10}$$

The only unknown in this equation is the unknown cross-section area A_n which can be determined easily. The same procedure holds if a certain length L_i or Young's modulus E_i is unknown.

6.3 Example

As an example, we take the same truss as in the example of chapter 2, see figure 2.6. We already know the internal force distribution due to the applied load P, which is given in section 2.5, so we know all the unknown quantities F_i. Now it we want to know the displacement of node B in positive y-direction, we first remove the load P and apply a dummy load as given in figure 6.4.

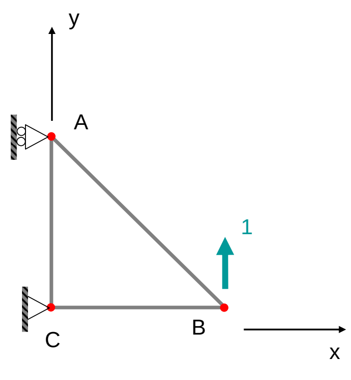


Figure 6.4: Truss structure with applied dummy load

By inspection it is evident that the reaction dummy force c_y is equal to 1. Furthermore $a_x = \frac{L_{BC}}{L_{AC}}$ and as such, $c_x = -\frac{L_{BC}}{L_{AC}}$. Note that each of these dummy reaction forces are dimensionless as well. The internal dummy forces f_i can be calculated by looking at nodal equilibrium as given in figure 6.5.

The equilibrium equations for node A are:

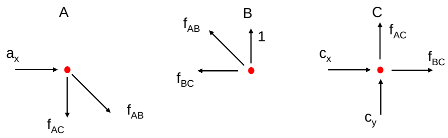


Figure 6.5: Nodal internal dummy loads

$$\sum F_x^{\rightarrow +} : 0 = a_x + f_{AB}\frac{L_{BC}}{L_{AB}} \tag{6.11}$$

$$\sum F_y^{\uparrow +} : 0 = -f_{AC} - f_{AB}\frac{L_{AC}}{L_{AB}}. \tag{6.12}$$

The equilibrium equations for node B are:

$$\sum F_x^{\rightarrow +} : 0 = -f_{BC} - f_{AB}\frac{L_{BC}}{L_{AB}} \tag{6.13}$$

$$\sum F_y^{\uparrow +} : 0 = 1 + f_{AB}\frac{L_{AC}}{L_{AB}}. \tag{6.14}$$

The equilibrium equations for node C are:

$$\sum F_x^{\rightarrow +} : 0 = c_x + f_{BC} \tag{6.15}$$

$$\sum F_y^{\uparrow +} : 0 = c_y + f_{AC}. \tag{6.16}$$

Solution of the six equations simultaneously yields the following expressions for the unknowns:

$$\left\{ a_x = \frac{L_{BC}}{L_{AC}}, c_x = -\frac{L_{BC}}{L_{AC}}, c_y = -1, f_{AB} = -\frac{L_{AB}}{L_{AC}}, f_{AC} = 1, f_{BC} = \frac{L_{BC}}{L_{AC}} \right\} \tag{6.17}$$

Note that the reaction forces are indeed the ones we predicted, which gives confidence that our solution is correct indeed. Note furthermore that the dummy forces f_i are $-\frac{dF_i}{dP}$ indeed. The minus sign comes from the opposite signs of P and the dummy load.

If then equation 6.8 is applied, and assuming lengths L_{AC} and L_{BC} are equal to L (to simplify the resulting expression), we obtain for the vertical displacement:

$$\delta = -\frac{P2(1+\sqrt{2})L}{EA}. \tag{6.18}$$

Since we chose the dummy load to be positive upwards, the resulting displacement is negative, since node B will displace downwards under the applied load P. Verify this by inspection.

Assume we prescribe the vertical displacement, positive upwards according to the dummy load direction, to be $-\delta_0$. All geometric values are known, except the cross-sectional area A_{AC}. The other cross-sectional areas are all equal to A. Determine that value in order the displacement to be δ_0.

$$\delta_0 = -\left(\frac{F_{AC}L}{EA_{AC}} + \frac{F_{BC}L}{EA} + \frac{F_{AB}\sqrt{2}L}{EA}\right). \tag{6.19}$$

The unknown can now be solved as:

$$A_{AC} = \frac{P\ L\ A}{A\ E\ \delta_0 - (2\sqrt{2}+1)PL} \tag{6.20}$$

Chapter 7

Displacement Analysis and Design of Statically Determinate Beams

In this chapter, we consider the calculation of beam displacements and rotations *at selected points*. This is done, as explained before for trusses, by applying the unit load method. Again, we restrict ourselves to beams in the x-z plane subject to lateral loading.

If it is desired to calculate the displacement at a certain point p, we apply a unit later force at p. The direction of the unit load is arbitrary (it can be pointing up or down). It must be kept in mind that the displacement calculated will be *in the direction of the load*. Thus, a negative displacement means that the actual sense of the displacement is opposite to that of the unit load. Similarly, if it is desired to calculate the beam *rotation* at a certain point, we apply a unit moment at this point.

The bending moment distribution due to the application of the unit load (force or moment) is denoted by $m_y(x)$. Then, the displacement (or rotation) at the given point in the direction of the unit load is given by,

$$\delta = \int\limits_{beam} \frac{m_y(x) M_y(x)}{EI(x)} dx, \tag{7.1}$$

where $M(x)$ is the bending moment distribution due to *actual* applied loads.

The integration in 7.1 can be evaluated analytically only in special cases. For general load distribution and variable cross-sectional geometry along the beam (e.g., due to tapering), it is best to evaluate the integral 7.1 in conjunction with the discrete method described in the previous chapter. The discrete version of the equation reads,

$$\delta = \sum_{i=1}^{n} \frac{\bar{m}_y^{(i)} \bar{M}_y^{(i)}}{EI^{(i)}} dx, \tag{7.2}$$

where,

$$\bar{M}^{(i)} = \left(M_y^{(i)}\big|_L + M_y^{(i)}\big|_R \right)/2, \tag{7.3}$$

and similarly for $\bar{m}_y^{(i)}$.

7.1 Example

We take a beam with length L, Young's modulus E and moment of inertia I, which is loaded with a constant distributed load q_0. The objective is to calculate the tip rotation of the beam. For that purpose, we are going to use the integral equation as given in equation 7.1. We first calculate the moment distribution due to the distributed loading q_0:

$$M(x) = -\frac{q_0}{2}(L-x)^2 \tag{7.4}$$

Since we are interested in the tip rotation, we apply a unit moment at the tip of the beam, as depicted in figure 7.1. The resulting moment distribution is hence:

$$m(x) = 1 \tag{7.5}$$

Applying the integral equation, you end up with a tip rotation θ:

$$\theta = -\frac{q_0}{2EI} \int_0^L (L - x^2) dx = -\frac{q_0 L^3}{6EI} \tag{7.6}$$

Check this expression with the basic mechanics expression for a tip rotation of a beam loaded under a distributed load.

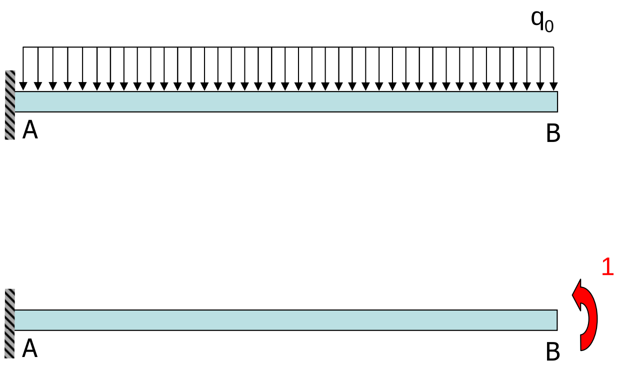


Figure 7.1: Statically determinate beam example

Chapter 8

Analysis of Statically Indeterminate Trusses

In previous chapters, statically determinate trusses have been discussed. Also the way to determine whether a structure is determinate or indeterminate and the degree of indeterminacy is explained. This basically breaks down to checking whether the number of available equilibrium equations is equal to the number of unknown forces in the system. If that is the case, then the structure is statically determinate. If not, the number of *missing* equations indicates the degree of indeterminacy. A structure can be indeterminate for several reasons. Trusses are often indeterminate for redundancy reasons, meaning that additional truss member are added to the structure in case another member would fail. This makes the stress analysis of trusses more complex since additional equations need to be formulated in order to solve for all unknowns. Such equations can be formulated as displacement compatibility equations, as will be explained next.

8.1 Displacement Compatibility

Essential in the technique of displacement compatibility is the fact that the structural response is linear. This allows the application of the principle of superposition. Resultant displacements of a set of loadings acting simultaneously is equal to the sum of the effect of individual loadings acting alone. This is illustrated in figure 8.1.

So if this is applied to a redundant or statically indeterminate truss, one can split up the indeterminate structure into a statically determinate part, and a redundant force, as is shown in the following figure.

Figure 8.2 shows that the internal forces can be calculated as function of

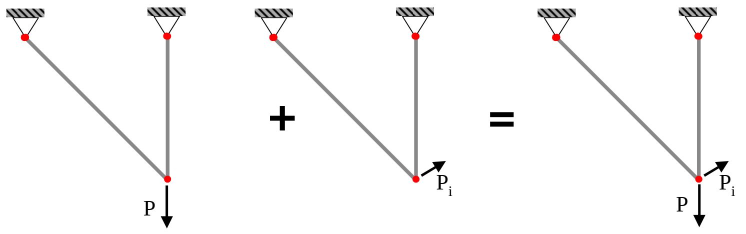


Figure 8.1: Force superposition principle

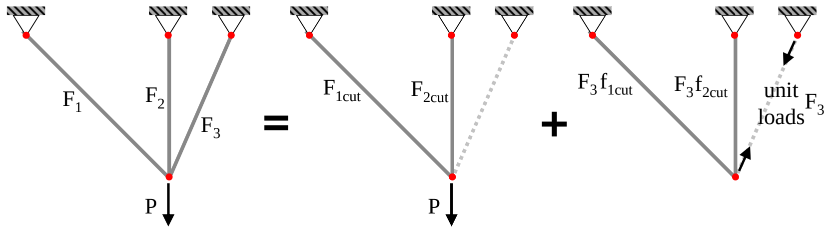


Figure 8.2: Split-up of an indeterminate truss

the statically indeterminate force F_3. So this particular force remains to be solved. This can be done by introducing a displacement compatibility at the location where the structure has been cut to create the statically determinate version. The relative displacement of the point where the structure is cut needs to be compatible for both superposed configurations. This can be illustrated easily if we take figure 8.2 as an example. The relative displacement of the point where force P is applied needs to be zero for the two superposed structures. First of all, the displacement of the aforementioned point of the statically determinate part is as follows:

$$\delta^{det} = \sum \frac{F_i^{cut} f_i^{cut} L_i}{E_i A_i}. \tag{8.1}$$

The displacement due to the indeterminate force is now calculated as follows:

$$\delta^{F_3} = \sum \frac{\left(F_3 f_i^{cut}\right) f_i^{cut} L_i}{E_i A_i}. \tag{8.2}$$

Now the relative displacement needs to be zero, enforced by the following displacement compatibility equation:

$$\delta^{det} + \delta^{F_3} = 0. \tag{8.3}$$

Note that in the above equation, the only unknown is the indeterminate force F_3. If the value for that force is known, the internal force distribution is defined. It can be seen from equation 8.3 that the force F_3 is a function of the geometric parameters such as cross-sectional areas and Young's moduli. This means that if the geometry changes, the internal force distribution is altered as well. It is important to keep this in mind when designing an indeterminate structure.

8.2 Action Item List for Analysis of Indeterminate Trusses

This following list of actions should be followed when analysing a statically indeterminate truss:

- Make the structure statically determinate: remove truss members or supports that render the truss to be indeterminate. Often this can be done in multiple ways, though the different individual approaches should yield the same result. This can be a convenient way of checking the correctness of the solution.

- Identify the displacement compatibility at the locations where the statically indeterminate forces have been removed. At those locations, calculate the displacements in the direction of the released forces using the dummy load method.

- Next calculate the displacements, again in the aforementioned directions, due to the released forces.

- Sum up all individual displacements and equate them to zero. Obtain an equation for each displacement that needs to be compatible. As such, you will get as many equations as unknowns, which should be a solvable system of equations.

8.3 Example

Consider the six bar truss shown in figure 8.3. This statically indeterminate square truss with length $L = 1000\ mm$ is loaded by a force $P = 1500\ N$ which is applied vertically in downward direction at node C. The cross-sectional areas of the truss members are the same and equal to 15 mm^2. The Young's modulus is 200 GPa.

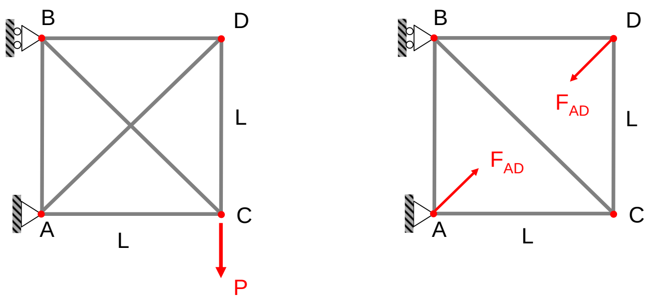


Figure 8.3: Statically indeterminate six bar truss

Member AD is removed to make the structure statically determinate, and the indeterminate force F_{AD} is added as the redundant force. We calculate all the necessary ingredients for the displacement compatibility equation, used to calculate the redundant force F_{AD} under the loading P.

member	E	A	L	F_{sd}	f	ϵ_{sd}	$f\epsilon L$	f^2L/EA
AB	200×10^3	15	1000	-1500	-0.7071	-5.0000×10^{-4}	0.354	1.667×10^{-4}
AC	200×10^3	15	1000	-1500	-0.7071	-5.0000×10^{-4}	0.354	1.667×10^{-4}
AD	200×10^3	15	1414	0	1.0	0	0.0	4.714×10^{-4}
BC	200×10^3	15	1414	2121	1.0	7.0710×10^{-4}	1	4.714×10^{-4}
BD	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
CD	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
$\sum$							1.707	1.609×10^{-3}

Table 8.1: Calculation of redundant force under mechanical load

The value of the redundant reaction in member AD is calculated from the condition,

$$1.707 + 1.609 \times 10^{-3}\, R_{AD} = 0, \Longrightarrow R_{AD} = -1061\text{N}. \qquad (8.4)$$

For the calculation of displacements we need the total member forces F which is obtained by superposing the statically determinate values with the values due to the redundant reaction. The unit load at point C needed to calculate the displacement is applied to *a* statically determinate structure. For simplicity, we use the same statically determinate structure as was used for the calculation of the redundant.

member	E	A	L	F	f	ϵ	$f\epsilon L$
AB	200×10^3	15	1000	-750	-1	-2.500×10^{-4}	0.250
AC	200×10^3	15	1000	-750	-1	-2.500×10^{-4}	0.250
AD	200×10^3	15	1414	-1061	0	-3.536×10^{-4}	0.0
BC	200×10^3	15	1414	1061	1.414	3.536×10^{-4}	0.707
BD	200×10^3	15	1000	750	0	2.500×10^{-4}	0
CD	200×10^3	15	1000	750	0	2.500×10^{-4}	0
$\sum$							1.207

Table 8.2: Calculation of displacement under mechanical load

Chapter 9

Analysis of Statically Indeterminate Beams

In the previous chapter, statically indeterminate trusses have been discussed. Obviously, beams can be indeterminate as well. The reason why beams are indeterminate is because of redundancy reasons as well, or often supporting truss members are connected to beams to restrict displacements or redistribute stresses yielding lighter designs for the beam. Just as for trusses, this makes the stress analysis of the beam more complex, since the number of available equilibrium equations is smaller than the number of internal forces and moments that need to be solved for. Therefore displacement compatibility equations need to be formulated, which will make the internal force and moment distribution dependent on the structure's geometry.

The principle of static indeterminacy and analysis is basically the same as for trusses. This is discussed in the previous chapter. However, there are certain differences, as already highlighted in the chapters on statically determinate trusses and beams. The main issue is that beams, apart from axial forces, can also take bending moments. This extends the principle of displacement compatibility of beams to displacement and rotation compatibility.

9.1 Force Method of Analysis

This following list of actions should be followed when analysing a statically indeterminate beam:

- Make the structure statically determinate: remove additional forces and moments that render the beam to be indeterminate. Often this can be done in multiple ways, though the different individual approaches should yield the same result. This can be a convenient way of checking the correctness of the solution.

- Identify the displacement and rotation compatibility at the locations where the statically indeterminate forces and moments have been removed. At those locations, calculate the displacements and rotations in the direction of the released forces using the dummy load method.

- Next calculate the displacements and rotations, again in the aforementioned directions, due to the released forces and moments.

- Sum up all individual displacements and rotation and equate them to zero. Obtain an equation for each displacement and rotation that needs to be compatible. As such, you will get as many equations as unknowns, which should be a solvable system of equations.

In case of beams, it is convenient to use the principle of flexibility coefficients. This technique is highlighted in the following section.

9.2 Flexibility Coefficients

The flexibility coefficient f_{BA} is defined as the displacement in point B due to a unit force in point A. This definition is illustrated in figure 9.1.

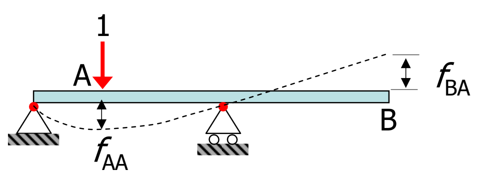


Figure 9.1: Illustration of the flexibility coefficient

The use of such coefficients is very effective since if one wants to know the displacement in let's say point X due to a force in point Y, one takes the flexibility coefficient f_{XY} and one multiplies that coefficient with the force in point Y and you obtain the displacement in point X. Moreover the flexibility coefficient f_{XY} is equal to f_{YX} according to Maxwell's theorem. The convenience of flexibility coefficients becomes clear when looking at the following figure:

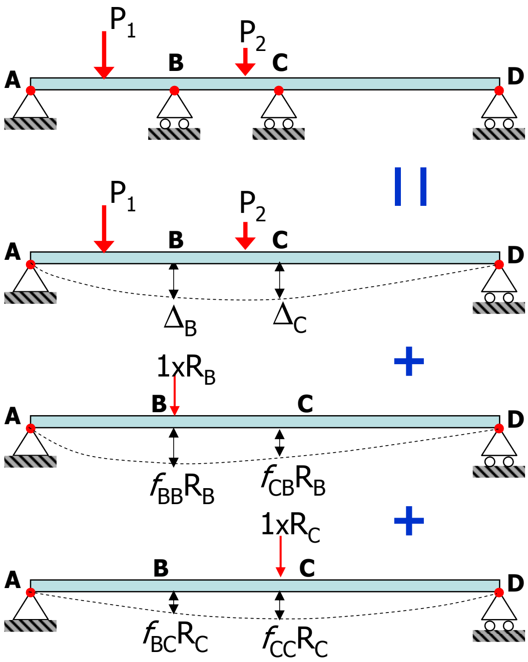


Figure 9.2: Solution of a statically indeterminate beam using flexibility coefficients

It is easy to see that this beam is statically indeterminate to the second degree. This structure can be solved by enforcing the displacement compatibility at points B and C. The displacement at point B is calculated by adding the displacements due to the external loads P_1 and P_2, and due to the statically indeterminate forces R_B and R_C. As such, the displacement compatibility equations for points B and C are calculated as:

$$0 = \Delta B + f_{BB}R_B + f_{BC}R_C \tag{9.1}$$

$$0 = \Delta C + f_{BC}R_B + f_{CC}R_C \tag{9.2}$$

The flexibility coefficients can be calculated easily by using the dummy load method.

9.3 Example

Let us consider a beam of length L which is cantilevered on the left hand side in point A and supported by a roller on the right hand side in point

B. The beam has Young's modulus E and a moment of inertia I. It can be seen easily that the beam is statically indeterminate. In order to analyse the structure, the first step is to make it statically determinate and superpose the statically indeterminate force. This is shown in figure 9.3.

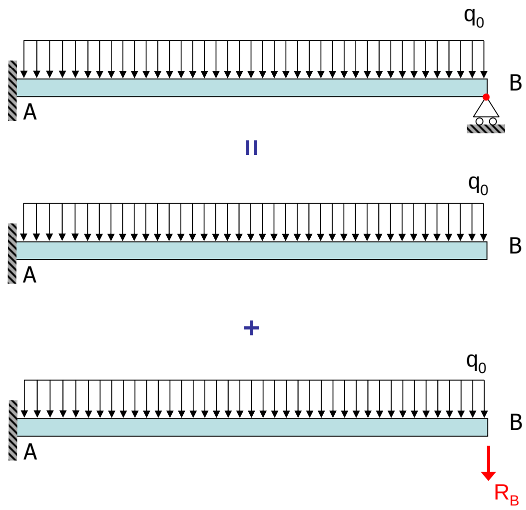


Figure 9.3: Analysis example of a statically indeterminate beam

In this example, the use of flexibility coefficients comes in handy. In this case, the flexibility coefficient f_{BB} is needed. The reason for this is that we are going to apply compatibility at the beam tip. It can be seen easily that the tip deflection in point B needs to be equal to zero. For this particular beam, this influence coefficient is equal to:

$$f_{BB} = \frac{L^3}{3EI}. \tag{9.3}$$

Try to figure out by using the dummy load method. Furthermore, from standard mechanics it is evident that the tip deflection due to a distributed load is equal to:

$$\delta_B = \frac{q_0 L^4}{8EI}. \tag{9.4}$$

The compatibility equation now reads:

$$0 = \frac{q_0 L^4}{8EI} + f_{BB} R_B. \tag{9.5}$$

By adding the sign, we assume the force R_B to be acting in the downward direction. Try to figure out yourself why. The solution of this equation gives the expression for the unknown force R_B:

$$R_B = -\frac{3}{8} q_0 L. \tag{9.6}$$

Note the minus sign, indicating that the force R_B acts in the opposite than defined, meaning the force points upwards, which is consistent with the physical problem.

Chapter 10

Analysis of Statically Indeterminate Plates

The analysis of plates is rather intricate, as already indicated in the chapter on statically determinate plates. Only in exceptional cases of uniaxial or biaxial stress states, analytical methods can offer a way out. In this chapter, it is explained how internal stresses in plates can be calculated for indeterminate plates. Again, just as in the case of trusses and beams, displacement compatibility is the key to solving the stress state.

As a start, we consider a statically determinate plate. If a square plate of length L and thickness t is loaded in x-direction with a force P, the resulting stress in that direction is $\sigma_x = \frac{P}{Lt}$, and according to Hooke's law, the strain is $\epsilon_x = \frac{P}{ELt}$. Due to the Poisson effect, which accounts for the fact that a plate shrinks in a direction perpendicular to the loading direction, the strain in the transverse direction is $\epsilon_y = \nu\epsilon_x$. ν is the Poisson ratio, with has a typical value of around 0.3 for metals. Note that the stress in the transverse direction remains zero.

10.1 Biaxial Stress State

Let us first consider a simple biaxial stress state plate, which is given in figure 10.1.

Obviously there is a stress in x-direction due to the applied force P, but because of both side restraints on the plate, there is normal stress in y-direction as well due to the constraint force R_y. The displacement in x-direction is defined as:

$$\delta_x = \frac{PL}{Ewt} - \nu\frac{R_y}{Et}, \tag{10.1}$$

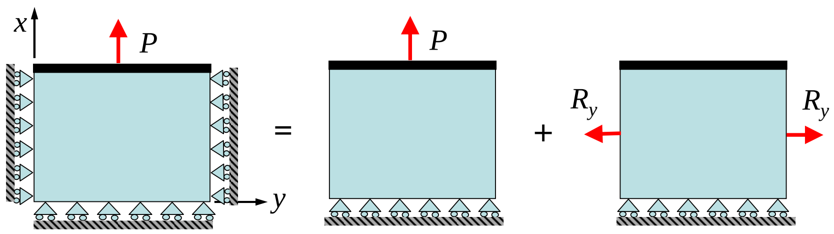


Figure 10.1: Biaxial stress state plate

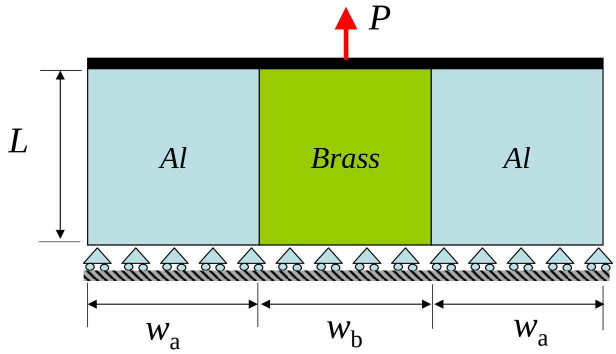
while the transverse displacement is equal to:

$$\delta_y = \frac{R_y w}{ELt} - \nu \frac{P}{Et}. \tag{10.2}$$

Now the unknown force R_y is calculated by requiring the displacement in y-direction is equal to zero (due to the restraints).

10.2 Multimaterial Plates

Plates consisting of multiple materials are inherently indeterminate, even if the geometric boundary conditions do not seem to indicate so. For instance the left plate of figure 10.2 contains two different materials, each having their own internal force. In this particular case, there are two unknown forces and one force equilibrium equation. Therefore a displacement compatibility equation needs to be formulated. If we take a closer look at the problem itself, it is evident that the vertical displacements of the three parts of the plate need to extend equally under the given load. Therefore an additional equation can be formulated requiring the displacement of the Aluminium being equal to the displacement of the Brass. Note that this condition only applies because of the fact that the two Aluminium part have exactly the same dimensions, otherwise the force in both Aluminium plates would be different.



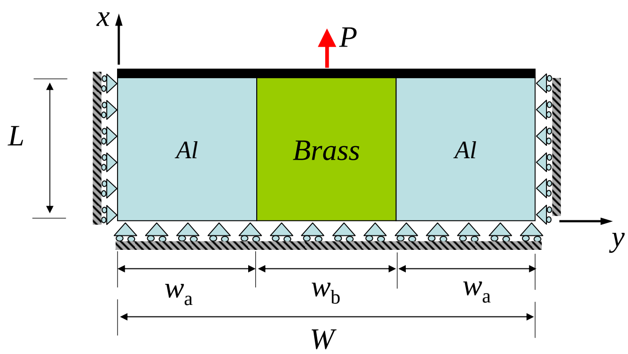


Figure 10.2: Indeterminate multimaterial plates (left: one statically indeterminate force, right: two statically indeterminate forces)

Chapter 11

Actuated Structures

Linear actuation can be achieved through various means. A simple example is the change in length of bars under thermal actuation (sometimes referred to as thermal load). Hydraulic and pneumatic cylinders, which work with pressurised liquids and gases respectively, are the most common commercially available linear actuators. A more modern version of similar devices are the electrically actuated active material systems such as piezoelectric devices.

While we consider temperature as an *actuation*, the methods presented here are also applicable for the practically important case of thermal *loading*. For many aerospace structures such as launchers, reentry vehicles and engines, structural components are subject to elevated temperatures that lead to considerable stresses and deformations. The stress and deflection analysis of thermally loaded structures follows the same methodology as thermal actuation.

11.1 Thermal Actuation

A bar with a uniform cross section A will change its length due to a uniform applied change in the temperature $\Delta\ T$. The change in length is linearly proportional to the temperature change, as shown in figure 11.1. Typically this relation is represented in terms of thermal strain, and is given by the relation,

$$\epsilon_T = \alpha \Delta\ T. \tag{11.1}$$

where α is the coefficient of thermal expansion (CTE). This is a special one dimensional case of a more general three dimensional description of thermal expansion, and is sometimes called linear coefficient of thermal expansion. In general, application of the thermal field would also generate strain in other directions (like in the cross-sectional expansion of a bar), but for the time being we will ignore that for one-dimensional bars.

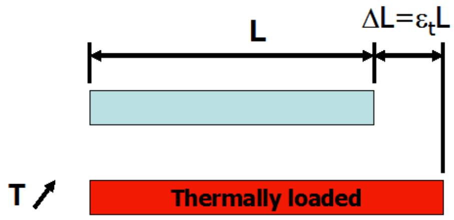


Figure 11.1: Thermal extension of a bar

Following are the CTEs for three popular materials, namely copper, aluminium, and steel, respectively, all in the units of cm/cm/°C (strain/°C): $\alpha_{cu} = 1.8 \times 10^{-5}$, $\alpha_{Al} = 2.5 \times 10^{-5}$, and $\alpha_{st} = 1.9 \times 10^{-5}$.

Note that there is a limit on the maximum thermal strain that can be achieved for a given material dictated by the maximum temperature that the material is able to withstand without loosing its elastic properties. We will call this strain $\epsilon_a^{\max}$. Also the temperature difference may be either positive or negative with respect to the room (or operating) temperature.

11.2 Piezoelectric Actuation

For a piezoelectric material, an electric field E_3 produces strains represented by S_33, as shown in figure 11.2,

$$S_{33} = d_{33}E_3. \tag{11.2}$$

In this representation the subscript 33 refers to the thickness direction in the piezoelectric material. The use of such subscripts should not be confusing, as the basic equation above 11.2 is same as thermal actuation of a bar presented earlier in equation 11.1, it is just different notation.

In an actual piezoelectric material (like in the case of thermal loads) the same electric field across the thickness also produces strains in the transverse directions, but these will not be needed until we study bending actuators constructed using piezoelectric material.

For a typical piezoelectric material $d_{33} = 500 \times 10^{-12} m/V$.

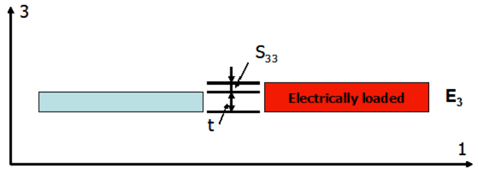


Figure 11.2: Piezoelectric thickness change of a bar

In contrast with the thermal case, we rarely use the applied electric field directly. The applied actuation is customarily specified as applied voltage. The electric field E_3 generated by applying voltage difference V across a piezoelectric substrate of thickness t is,

$$E_3 = \frac{V}{t}. \tag{11.3}$$

Similar to the thermal case in which there is a maximum temperature that can be applied to the material, there is a maximum electric field that can be applied, called the coercive electric field, E_c . Strain corresponding to this coercive field is,

$$S_{33}^{\max} = d_{33}E_3. \tag{11.4}$$

For a typical value of the coercive field $E_c = 0.4 \times 10^6 V/m$, the maximum possible strain is $S_{33}^{\max} = 0.2 \times 10^{-3}$.

Typically, in order to increase the stroke (displacement) capability, the piezoceramic material is arranged in a stacks. Lets say we put 200 layers on top of one another. In this way, the displacement output is amplified 200 times. Piezoelectric material arranged in a stack is rather stiff, on the other hand. They produce small deformations, but they can produce very large actuation forces. This will be demonstrated in the next section. However, we first learn how to handle strains/deformations created under the combined action of applied forces and applied actuation.

11.3 Response of an Actuated Member

More than often, the actuators described above will have to operate under mechanical loads acting on the members. In this section we will assume that;

the entire bar is made of a material that will be actuated, the cross sectional area of the bar is uniform A, the modulus is E , and the bar is under the action of an axial load F. Also, for simplicity, we will consider the thermal case only; the piezoelectric case being similar.

Under the action of combined thermal and mechanical *load*, the strain that the active member will experience is the superposition of the strains due to the thermal load and the mechanical loads,

$$\epsilon = \epsilon_T + \epsilon_M \tag{11.5}$$

The mechanical part of the strain, ϵ_M, is the one responsible for the generation of mechanical stress, thus,

$$\sigma = E\epsilon_M, \tag{11.6}$$

where E is Young's modulus of the material. The stress in the member can be expressed as,

$$\sigma = \frac{F}{A}. \tag{11.7}$$

The strain in the member is,

$$\epsilon = \alpha\Delta\, T + \frac{F}{EA}. \tag{11.8}$$

Clearly, when the applied external force is zero then the total strain of the actuator is *force free* ($\epsilon_M = 0$) and is equal to the ϵ_T.

The other extreme case is when the expansion of the bar is completely restrained ($\epsilon = 0$). In this case, the corresponding force is called the *blocked force*, F_b, which can be computed by setting $\epsilon = 0$ in 11.8 to obtain,

$$F_b = \alpha E\Delta\, TA, \tag{11.9}$$

with a corresponding (compressive) stress of,

$$\sigma_b = -\alpha E\Delta\, T. \tag{11.10}$$

In realistic applications, when an actuated member is part of a structure that is performing a function, the members is neither fully blocked nor completely free to extend. It will be under the action of combined mechanical and thermal (actuation) loads.

In the following section, we will use the dummy-load method to determine the actuation temperature of an axial member.

11.4 Dummy Load Method for Structures with Actuated Members

We certainly do not need the dummy-load method to determine the response of a single member under a mechanical load and actuation. The equation derived in the previous section, based on superposition, would have been enough to compute anything/everything we need. However, as we will see later with examples, the dummy-load method provides a powerful tool to compute response efficiently in more complicated structural geometries, especially the ones that are statically indeterminate. Therefore, although we will still use the method of superposition, the argument provided for the calculation of the displacement under the thermal load may be useful (even though it is implementation is ridiculously simple).

The basic equation for the unit load method is not changed. The displacement at a given point and in a given direction is calculated by applying the corresponding unit load and calculating,

$$\delta = \sum f_i \epsilon_i L_i \tag{11.11}$$

where f_i are the member forces under the unit load. The only additional consideration is that the strain in the i-th member is calculated according to 11.8.

11.5 Example

The same truss is considered as in chapter 8. The picture is shown again in figure 11.3. The length of the truss is 1000 mm, the cross-sectional areas of the members are 15 mm^2, and the applied force P is 1500 N. The Young's modulus is 200 GPa.

As usual we use superposition. Moreover, we will compute the displacement due to mechanical and thermal loads separately. One important point to remember is that for statically indeterminate structures under thermal loads, member forces are *not* zero.

First we calculate all the necessary ingredients for the displacement compatibility equation, used to calculate the redundant force F_{AD} under the loading P.

The value of the redundant reaction in member AD is calculated from the condition,

$$1.707 + 1.609 \times 10^{-3}\, R_{AD} = 0, \Longrightarrow R_{AD} = -1061\text{N}. \tag{11.12}$$

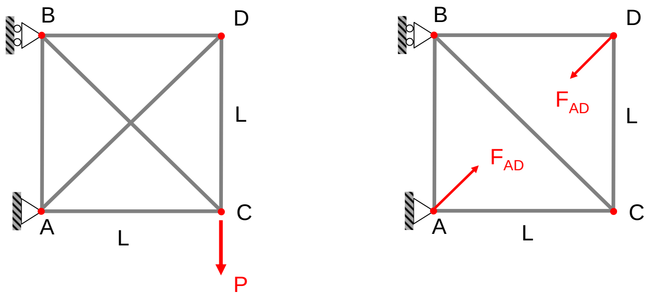


Figure 11.3: Statically indeterminate six bar truss

member	E	A	L	F_{sd}	f	ϵ_{sd}	$f\epsilon L$	f^2L/EA
AB	200×10^3	15	1000	-1500	-0.7071	-5.0000×10^{-4}	0.354	1.667×10^{-4}
AC	200×10^3	15	1000	-1500	-0.7071	-5.0000×10^{-4}	0.354	1.667×10^{-4}
AD	200×10^3	15	1414	0	1.0	0	0.0	4.714×10^{-4}
BC	200×10^3	15	1414	2121	1.0	7.0710×10^{-4}	1	4.714×10^{-4}
BD	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
CD	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
$\sum$							1.707	1.609×10^{-3}

Table 11.1: Calculation of redundant force under mechanical load

For the calculation of displacements we need the total member forces F which is obtained by superposing the statically determinate values with the values due to the redundant reaction. The unit load at point C needed to calculate the displacement is applied to *a* statically determinate structure. For simplicity, we use the same statically determinate structure as was used for the calculation of the redundant.

member	E	A	L	F	f	ϵ	$f\epsilon L$
AB	200×10^3	15	1000	-750	-1	-2.500×10^{-4}	0.250
AC	200×10^3	15	1000	-750	-1	-2.500×10^{-4}	0.250
AD	200×10^3	15	1414	-1061	0	-3.536×10^{-4}	0.0
BC	200×10^3	15	1414	1061	1.414	3.536×10^{-4}	0.707
BD	200×10^3	15	1000	750	0	2.500×10^{-4}	0
CD	200×10^3	15	1000	750	0	2.500×10^{-4}	0
$\sum$							1.207

Table 11.2: Calculation of displacement under mechanical load

Now, we repeat the exercise for the thermal load.

member	α	E	A	L	$\Delta\ T$	f	ϵ_{sd}	$f\epsilon L$	f^2L/EA
AB	1.9×10^{-5}	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
AC	1.9×10^{-5}	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
AD	1.9×10^{-5}	200×10^3	15	1414	0	1.0	0	0	4.714×10^{-4}
BC	1.9×10^{-5}	200×10^3	15	1414	1	1.0	1.9×10^{-5}	0.0269	4.714×10^{-4}
BD	1.9×10^{-5}	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
CD	1.9×10^{-5}	200×10^3	15	1000	0	-0.7071	0	0	1.667×10^{-4}
$\sum$								0.0269	1.609×10^{-3}

Table 11.3: Calculation of redundant force under thermal load

The value of the redundant reaction in member AD is calculated from the condition,

$$0.0269 + 1.609\times10^{-3}\,R_{AD} = 0, \Longrightarrow R_{AD} = -16.7\text{N}. \tag{11.13}$$

To calculate the vertical displacement at point C, we follow the same procedure as for mechanical loading except that now the strain is composed of mechanical and thermal parts.

member	E	A	L	F	f	ϵ	$f\epsilon L$
AB	200×10^3	15	1000	11.81	-1	3.935×10^{-6}	-3.935×10^{-3}
AC	200×10^3	15	1000	11.81	-1	3.935×10^{-6}	-3.935×10^{-3}
AD	200×10^3	15	1414	-16.7	0	-5.565×10^{-6}	0.0
BC	200×10^3	15	1414	-16.7	1.414	1.344×10^{-6}	2.687×10^{-2}
BD	200×10^3	15	1000	11.81	0	3.935×10^{-6}	0
CD	200×10^3	15	1000	11.81	0	3.935×10^{-6}	0
$\sum$							0.019

Table 11.4: Calculation of displacement under thermal load

With the vertical displacement at C under both mechanical and thermal loads known, we can calculate the required actuation temperature to nullify the displacement at C.

$$\delta_C = 1.2071 + 0.019\Delta\ T = 0, \Longrightarrow \Delta\ T = -63.5°\text{C}. \tag{11.14}$$