Chapter 1

Some Basic Concepts

1.1 Newtons law and inertial frames

The three laws of newton

First law:	Every particle continues in its state of rest or uniform motion in a straight line relative to an inertial reference frame, unless it is comepelled to change that state by forces acting upon it
Second law:	The time rate of change of linear momentum of a particle relative to an inertial reference frame is proportional to the resultant of all forces acting upon that particle and is collinear with and in the direction of the resultant force
Third law:	If two particles exert forces on each other, these forces are equal in magnitude and opposite in direction (action - reaction)

The first law defines an inertial reference frame. If one inertial reference frame is know a whole class of inertial reference frames is known. These are non rotating frames which are in rest or in uniform motion in a straight line with respect to the first inertial reference frame. This can be shown with the *Galilei transformations* eq [1.1]. Assume that the first frame moves with a constant speed of V and that the second frame moves with a constant velocity W with respect to the first frame. The time in frame X'Y'Z' differs with a constant T and for simplicity sakes it is assumed that at t_0 the origins coincided. The following holds: (also see fig 1.1)

$$\bar{r}' = \bar{r} - \bar{W}(t - t_0) \quad ; \quad t' = t + T \tag{1.1}$$

$$\bar{V} = \frac{d\bar{r}}{dt} \quad ; \quad \bar{V}' = \frac{d\bar{r}'}{dt'} \tag{1.2}$$

differentiating eq [1.1] with respect to dt and substituting eq: [1.2]

$$\frac{d\bar{r}'}{dt} = \frac{d\bar{r}}{dt} - \bar{W} \quad ; \quad \frac{dt'}{dt} = 1 \tag{1.3}$$

$$\frac{d\bar{r}'}{dt}\frac{dt}{dt'} = \bar{V} - \bar{W} \tag{1.4}$$

$$\frac{d\bar{r}'}{dt'} = \bar{V}' = \bar{V} - \bar{W} \tag{1.5}$$

We defined that $\bar{V}$ and $\bar{W}$ were constant so also $\bar{V}'$ has to be constant so X'Y'Z' has to be a inertial reference frame.
Newtons second law is:

$$\bar{F} = \frac{d}{dt}(m\bar{V}) \tag{1.6}$$

From wikipedia: *In mathematics and theoretical physics, an invariant is a property of a system which remains unchanged under some transformation.* It would be handy if this was also the case for Newtons laws. We examine newtons second law in reference frame X'Y'Z' and substitute eq: [1.5]

$$\bar{F} = \frac{d}{dt}(m\bar{V}') = \frac{d}{dt}(m(\bar{V} - \bar{W})) = \bar{F} = \frac{d}{dt}(m\bar{V}) - \bar{W}\frac{dm}{dt} \tag{1.7}$$

So Newtons second law is only invariant if $\frac{dm}{dt} = 0$ i.e. for a body of constant mass. For a rocket, which has a varying mass m is considered to be the *instantaneous* mass and extra *apparent forces* needs to be added to compensate. The *solidification principle* can be used to describe the rocket as a body with an instantaneous mass m and the thrust as an external force.

1.2 Gravity force and potential

The gravity force of body m_2 can be described as follows

$$\bar{F}_2 = m_2 g_2 = -G\frac{m_1 m_2}{r_{12}^3}\bar{r_{12}} \tag{1.8}$$

$$g_2 = -G\frac{m_1}{r_{12}^3}\bar{r_{12}} \tag{1.9}$$

$\bar{g}_2$ is called the *field strength* and can simply be found by dividing both sides of eq [1.8] with m_2. The field strength can also be found by taking the (negative) gradient of the potential field. (In astrophysics it is assumed that the potential at infinity is zero so that $U_{2_0} = 0$). Furthermore we assume for the scaler U_2 eq [1.10]

$$U_2 = -G\frac{m_1}{r_{12}} + U_{2_0} = -G\frac{m_1}{r_{12}} + 0 \tag{1.10}$$

$$\bar{g}_2 = -\bar{\nabla}U_2 = Gm_1\bar{\nabla}\frac{1}{r_{12}} = Gm_1\left(\frac{d}{dx}\frac{1}{r_{12}}, \frac{d}{dy}\frac{1}{r_{12}}, \frac{d}{dz}\frac{1}{r_{12}}\right) = -Gm_1\left(\frac{1}{X^2}, \frac{1}{Y^2}, \frac{1}{Z^2}\right) = -G\frac{m_1}{\bar{r}_{12}^2} = -G\frac{m_1}{r_{12}^3}\bar{r}_{12} \tag{1.11}$$

The end of equation [1.11] is the same as equation [1.9] so apparently the potential at an arbitrary distance r can be written as:

$$U = -G\frac{m_1}{r} \tag{1.12}$$

eq [1.12] can be differentiated wrt the mass. To find the gravity field of a body this equation needs to be integrated over the body. Only for spherical shells of constant mass density and spheres with a radial symmetric mass density distribution there is a closed form analytical solution.

$$dU = -G\frac{dm}{r} \tag{1.13}$$

For a spherical shell the following holds (see also fig 1.3)

$$dm = (2\pi R\sin\theta)(Rd\theta)t\rho \tag{1.14}$$

$$dU_P = -G\frac{2\pi R^2 t\rho\sin\theta d\theta}{r} \tag{1.15}$$

$$M = 4\pi R^2 t\rho \tag{1.16}$$

$$\begin{aligned} r^2 &= (R\sin\theta)^2 + (l - R\cos\theta)^2 \\ &= R^2\sin^2\theta + l^2 - 2Rl\cos\theta + R^2\cos^2\theta \\ &= R^2(\sin^2\theta + \cos^2\theta) + l^2 - 2Rl\cos\theta \\ &= R^2 + l^2 - 2Rl\cos\theta \end{aligned} \tag{1.17}$$

$$2rdr = 2RL\sin\theta d\theta \tag{1.18}$$

$$\frac{dr}{d\theta} = \frac{Rl\sin\theta}{r} = \frac{Rl\sin\theta}{\sqrt{R^2 + l^2 - 2Rl\cos\theta}} \tag{1.19}$$

$$d\theta = \frac{\sqrt{R^2 + l^2 - 2Rl\cos\theta}}{Rl\sin\theta}dr \tag{1.20}$$

substituting [1.14] into [1.13] gives [1.15]. Substituting [1.16] and [1.17] into [1.15] gives [1.21]. Substituting [1.17] into [1.18] gives [1.19].

$$U_P = -\frac{1}{2} GM \int_{\theta=0}^{\pi} \frac{\sin\theta d\theta}{\sqrt{R^2 + l^2 - 2Rl\cos\theta}} \tag{1.21}$$

Substituting [??] into [1.21] gives [1.22] for l< R (point inside sphere) and [1.23] for $l > R$ (point outside sphere).

$$U_{P_{inside}} = -\frac{1}{2}\frac{GM}{Rl}\int_{r=R-l}^{r=l+R} dr = -\frac{1}{2}\frac{GM}{Rl}(R + l - (R - l)) = -\frac{GM}{R} \tag{1.22}$$

$$U_{P_{outside}} = -\frac{1}{2}\frac{GM}{Rl}\int_{r=R-l}^{r=l-R} dr = -\frac{1}{2}\frac{GM}{Rl}(R + l - (l - R)) = -\frac{GM}{l} \tag{1.23}$$

If a body has a radially symmetric density distribution it can be modeled as a series of spherical shells with the same center. Giving each shell an index i with a mass M_i the following is valid

$$U_{P_{outside}} = -\frac{G}{l}\sum_i M_i = -\frac{GM_T}{l} \tag{1.24}$$

Looking back one can say that a force in an arbitrary direction, lets say x can be expressed as [1.25]

$$F_{P_x} = -m_p \frac{\delta U_p}{\delta x} \tag{1.25}$$

$$F_{P_{x_{inside}}} = m_p \frac{\delta}{\delta x}\frac{GM}{R} = 0 \tag{1.26}$$

$$F_{P_{x_{outside}}} = m_p \frac{\delta}{\delta x}\frac{GM}{l} = -\frac{GM_T m_p}{l^2} \tag{1.27}$$

So if the point lies inside the sphere the neto force is 0 and if it lies outside the sphere the force is directed along l and can be calculated with [1.27].

It is of course also possible to write out [1.13] for arbitrary bodies. To come to a solution one can do a series expension and truncated all higher order terms. Rearranging and simplifying gives [1.28]

$$U = -\frac{GM}{l} - \frac{1}{2}\frac{G}{l^3}(A + B + C - 3D) \tag{1.28}$$

$$A = \int (y^2 + z^2) dm \tag{1.29}$$

$$B = \int (x^2 + z^2) dm \tag{1.30}$$

$$C = \int (x^2 + y^2) dm \tag{1.31}$$

$$D = \int r^2 \sin^2\theta dm \tag{1.32}$$

These terms might look familiar because A,B and C are the moment of inertia around the X,Y and Z axis and D is the moment of inertia around the line OP (the line between the Origin of the reference frame of the body and the particle P). For a sphere all the inertias are equal which gives equation [1.27] again.

One can also simplify equation [1.28] for rotational symmetric bodies. For such bodies A=B and $D = A\cos^2\phi + C\sin^2\phi$. This results in [1.34]

$$\left(\frac{C-A}{Ml^2}\right)_{earth} = J_2 \approx 1.086 \cdot 10^{-3} \quad (1.33)$$

$$U = -\frac{GM}{l}[1 - \frac{C-A}{Ml^2}(3\sin^2\phi - 1)] = -\frac{GM}{l}[1 - J_2(3\sin^2\phi - 1)] \quad (1.34)$$

This is solved by me, and not in the reader. Please confirm yourself
Per definition the force in the direction of l, so the find it, you only have to differentiate with respect to l

$$F_l = -m_2\frac{dU}{dl} = m_2\frac{d}{dl}GMl^{-2} + \frac{1}{2}J_2R^2(3\sin^2\phi - 1)l^{-3} \quad (1.35)$$

$$= m_2\left(-\frac{GM}{l^2} - \frac{3}{2}J_2R^2\frac{GM}{l^4}(3\sin^2\phi - 1)\right) \quad (1.36)$$

$$= \frac{GMm_2}{l^2}\left[1 + J_2\frac{3}{2}\left(\frac{R}{l}\right)^2(3\sin^2\phi - 1)\right] \quad (1.37)$$

hmm This is incorrect because the final answer should be:

$$F_l = \frac{GMm_2}{l^2}\left[1 + J_2\frac{3}{2}\left(\frac{R}{l}\right)^2(3\cos^2\phi - 1)\right] \quad (1.38)$$

1.3 rocket maneuvering

When using an impulsive shot the following is valid:

$$\Delta\bar{H} = \bar{r_0} \times \bar{V_1} - \bar{r_0} \times \bar{V_0} = \bar{r_0} \times \Delta\bar{V} \quad (1.39)$$

$$\Delta\mathscr{E} = \frac{1}{2}(V_1^2 - V_0^2) = \frac{1}{2}\left((\bar{V_0} + \Delta\bar{v})\cdot(\bar{V_0} + \Delta\bar{v}) - V_0^2\right) = \frac{1}{2}(\Delta V)^2 + \bar{V_0}\cdot\Delta\bar{V} \quad (1.40)$$

From these equation the following can be determined:

- For a given ΔV the maximum change in orbital angular momentum (H) is achieved if the impulsive shot is executed when the spacecraft is farthest away from Earth and of $\Delta\bar{V}$ is perpendicular to $\bar{r_0}$.
- If the direction of the orbital angular momentum vector should bot be changed, $\Delta\bar{V}$ should be directed in the initial orbital plane and vice versa
- The maximum change in (total) orbital energy is achieved if the impulsive shot is executed at the point in the orbit where the velocity reaches a maximum value, and if $\Delta\bar{V}$ is directed along the velocity vector$\bar{V_0}$

Most rockets burns are not impulse shot so *Tsiolkovski's law* has an extra term for the gravity losses (second right hand term)

$$\Delta V = v_e \ln\frac{M_0}{M_e} - \int_{t_0}^{t_e} g\sin\gamma dt \quad (1.41)$$

Chapter 2

Many-body problem

$$\text{Second law of Newton}: \qquad \sum \bar{F} = m \cdot \frac{d^2 r}{dt^2} \tag{2.1}$$

$$\text{Gravity law of Newton}: \qquad F_{ij} = G\frac{m_i m_j}{r_{ij}^3}\bar{r_{ij}} \tag{2.2}$$

$$\text{EoM of body } i: \qquad m_i \frac{d^2 r_i}{dt^2} = G\sum_{j\neq i}\frac{m_i m_j}{r_{ij}^3}\bar{r_{ij}} \tag{2.3}$$

Potential field

$$g = -\nabla U = -\nabla\left(\sum_{j\neq i} G\frac{m_i m_j}{r_{ij}}\right) + \nabla U_{i_0} \tag{2.4}$$

$$= -G\sum_{j\neq i} m_j \nabla \sum_{j\neq i} r_{ij}^{-1} = G\sum_{j\neq i} m_j \sum_{j\neq i} \bar{r_{ij}}^{-2} = G\sum_{j\neq i}\frac{m_j}{r_{ij}^3}\bar{r_{ij}} \tag{2.5}$$

This forcefield is not conservative because when looking to a point fixed in an inertial frame the potential field will differ because the bodies j move in time. The sum of the potential and kinetic and potential energy of body m_i is not constant. (Part its energy will flow into the bodies j and vice versa)

With the use of a auxiliary variable and differentiation the next equation can be found:

$$\sum_i \frac{1}{2} m_i V_i^2 - \frac{1}{2} G \sum_i \sum_{j\neq i} \frac{m_i m_j}{r_{ij}} = C \tag{2.6}$$

Or in short (watch the sign change and definition of $\mathscr{E}_p$)

$$\mathscr{E}_k + \mathscr{E}_P = C \tag{2.7}$$

When taking a closer look one can see that as two bodies approach each other very closely ,$r_{ij} \rightarrow \infty$, the potential energy goes to infinity and the only solution is that the speed of one of the bodies also goes to infinity ($V_i \rightarrow \infty$). Or in other words, at least one body reaches escape velocity ($r_i \rightarrow \infty$).

By using steiners rule for the moment of inertia, differentiating twice using an auxiliary variable and substituting previous equations the following can be found:

$$\frac{dI}{dt} = 2\sum_i m_i \bar{r}_i \cdot \bar{V}_i \tag{2.8}$$

$$\frac{d^2 I}{dt^2} = 4\mathscr{E}_k + 2\mathscr{E}_P \tag{2.9}$$

To have a stable system it is necessary that $\frac{d^2I}{dt^2} < 0$ or else the moment of inertia at the end will grow to infity, meaning that at least one body is a infinity distance which is not of course not a stable system. Substituting 2.7 in 2.9 gives:

$$\frac{d^2I}{dt^2} = 4\mathscr{E}_k + 2(C - \mathscr{E}_k) = 2C + 2\mathscr{E}_k \tag{2.10}$$

So to have a stable system $C < 0$ but one also has to look further. To determine if a system is really stable we integrate 2.9 over a long period of time, t_e, and take the average.

$$\frac{1}{t_e}\int_0^{t_e} \frac{d^2I}{dt^2}dt = \frac{4}{t_e}\int_0^{t_e} \mathscr{E}_k dt + \frac{2}{t_e}\int_0^{t_e} \mathscr{E}_P dt \tag{2.11}$$

$$\frac{1}{t_e}\left(\frac{dI}{dt}\right)_0^{t_e} = 4\bar{\mathscr{E}}_k + 2\bar{\mathscr{E}}_P \tag{2.12}$$

substitute 2.8 gives:

$$\frac{2}{t_e}[\sum_i m_i \bar{r}_i \cdot \bar{V}_i]_0^{t_e} = 4\bar{\mathscr{E}}_k + 2\bar{\mathscr{E}}_P \tag{2.13}$$

$$\frac{1}{t_e}[\sum_i m_i \bar{r}_i \cdot \bar{V}_i]_0^{t_e} = 2\bar{\mathscr{E}}_k + \bar{\mathscr{E}}_P \tag{2.14}$$

Because in a stable system no collision or escapes happen the term between square brackets stays finite. In the limit case ($\lim_{t_e \to 0}$) the left hand side of the equation will become zero. Giving:

$$2\bar{\mathscr{E}}_k + \bar{\mathscr{E}}_P = 0 \tag{2.15}$$

$$\bar{\mathscr{E}}_k = -\frac{1}{2}\bar{\mathscr{E}}_P \tag{2.16}$$

Also the following holds when substituting 2.7 in 2.15

$$2\bar{\mathscr{E}}_k + (C - \mathscr{E}_k) = 0 \tag{2.17}$$

$$\mathscr{E}_k = -C \tag{2.18}$$

Chapter 3

Three body problem

3.1 Equations of motion

When describing the motion of a body while there are only two other bodies the EoM reduces to:

$$\frac{d^2\bar{r_i}}{dt^2} = G\frac{m_j}{r_{ij}}\bar{r_{ij}} + G\frac{m_k}{r_{ik}}\bar{r_{ik}} \tag{3.1}$$

When this set is written as an set of first order differential equations we have three bodies with each six differential equations (dx dy and dz with respect to two other bodies) giving us a total of 18 unknowns, or in other words of the order 18. By using the ten integrals of motion, the invariable plane of Laplace as reference plane and a angular coordinate to replace time, the order can reduced with a factor 12. This still leaves an order six differential equation to be solved. An other approach is by using center-of-mass integrals, rewriting the equations in such a way that they are with respect to the center of mass of the system and with respect to the center of mass of body P_1 and P_2 combined, Jacobi reduced this set to an order 12. (which can be solved with the above mentioned tricks) The Jacobi set is:

$$\frac{d^2\bar{R}}{dt^2} = -GM\left[\alpha\frac{\bar{r_{13}}}{r_{13}^3} + (1-\alpha)\frac{\bar{r_{23}}}{r_{23}^3}\right] \tag{3.2}$$

$$\frac{d^2\bar{r_{12}}}{dt^2} = -G\left[(m_1+m_2)\frac{\bar{r_{12}}}{r_{12}^3} + m_3\left(\frac{\bar{r_{13}}}{r_{13}^3} - \frac{\bar{r_{23}}}{r_{23}^3}\right)\right] \tag{3.3}$$

$$\alpha = \frac{m_1}{m_1+m_2} \tag{3.4}$$

R is the vector between the CoM of body P_1 and P_2 combined and P_3. For the *Lunar case* were P_1 is the Earth, P_2 the Moon and P_3 is the sun. And for the *planetary case* were P_1 is the Sun, P_2 is the Earth and P_3 and planet, the following holds:

$$\text{lunar case:} \quad \alpha \approx 1 \quad ; \quad \bar{r_{13}} \approx \bar{r_{23}} \approx \bar{R} \tag{3.5}$$

$$\text{planetary case:} \quad \alpha \approx 1 \quad ; \quad \frac{m_3}{m_1+m_2} \ll 1 \quad ; \quad \bar{r_{13}} \approx \bar{R} \tag{3.6}$$

Both set of approximations give the same simplified results:

$$\frac{d^2\bar{R}}{dt^2} = -GM\frac{\bar{R}}{R^3} \tag{3.7}$$

$$\frac{d^2\bar{r_{12}}}{dt^2} = -G(m_1+m_2)\frac{\bar{r_{12}}}{r_{12}^3} \tag{3.8}$$

The first equation describes the motion of CoM of body P_1 and P_2 with respect to P_3 and the second equation describes the relative motion of body P_1 with respect to P_2. In these special cases the motion of the bodies may be approximated by a superposition of two two-body trajectories. (The earth moon system circles around the sun, and the moon circles around the earth)

3.2 Circular restricted three-body problem

By assuming that the mass of body P_1 and P_2 are much larger than the mass of P_3 and by assuming that the two massive bodies move in circular orbits around the CoM of the system equation 3.1 reduces. When also a new (pseudo-)inertial reference frame is used this gives: (see also fig 3.4)

$$\frac{d^2\bar{r}}{dt^2} = G\frac{m_1}{r_1}\bar{r_1} + G\frac{m_2}{r_2}\bar{r_2} \tag{3.9}$$

By rewriting this equation for the movement of body P_3 in a rotating reference frame with the same origin and its X-axis alligned with the line r_{12}, assuming that the velocity body P_3 in this reference frame is 0 gives the equation for the *surface of Hill.* In this equation x and y are the coordinates in the rotating reference frame and r_1 and r_2 are the distances between body P_1 and P_3, and between P_2 and P_3. μ is a mass ratio and C is an intergration constant which is determined by the position and velocity of body P_3 at time t =0.

$$x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = C \tag{3.10}$$

$$\mu = \frac{m_2}{m_1 + m_2} \tag{3.11}$$

By looking at cross section in the XY plane for z=0 simplifies this equation to:

$$r^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} = C \tag{3.12}$$

If we look for solutions when C is very large there are three possibilities: r^2 is very large (so body P_3 very far away from the CoM of the whole system), r_1 is very small (Body P_3 is very close to Body P_1) and r_2 is very small (Body P_3 is very close to Body P_2). If C becomes smaller there will be more possible solutions, It is possible for body P_3 to be located further from P_1 and P_2. So the areas will grow and the places where to areas meet for the first time a Lagrange point is located. (see fig 3.6)

A Lagrange libration point is place where centripetal forces on body P_3 have the same magnitude and opposite direction as the combined gravitational force of body P_1 and P_2 of body P_3. (Remember the whole reference frame is rotating, Body P_1 and P_2 are still rotating around the CoM of the whole system, and with them body P_3. Only the relative position of body P_3 does not change). The equilibrium in point L_1, L_2 and L_3 are not stable, point L_4 and L_5 are stable for certain values of μ and eccentricity. Sidenote, a lissajous orbit is a quasi-periodic orbital trajectory and does not have to be round or elliptical (think figure of eight and even more complex figures). However for realistic cases they can be considered as slowly-changing elliptical paths. It can be shown that for the collinear libration points (L_1, L_2 and L_3) the major axis is aligned with the Y axis (perpendicular to the line between P_1 and P_2). The orbit direction is clockwise looking top down. For the equilateral libration points(L_4 and L_5) it can be shown that major axis is rotated 30 degrees with respect to the X axis. Halo orbits are special lissajous orbits around collinear libration points which are very long elongated but have fixed geometry and size and are closed loop. (They are "real" orbits).

Halo orbits are very handy if you want the have a fixed position with respect to the other body. The Earth-Sun L1 point is multiple times used to study the sun. The Sun-Earth L2 point can be used if you always want to be in the dark (cold), to study deep space. Furthermore the gravity gradient is very low (gravity field is weak) so gravity distorsions are lower and higer pointing accuracy can be obtained (deep space telescopes)

Chapter 4

Relative motion in the many-body problem

In real life there are no real two body problems. There are always other bodies like, the sun, other planets and even galaxies which have an influence on the trajectory. These forces are small compared to the forces between the two main bodies and can be seen as disturbance forces. These forces are small because the distance is very large or because the mass is relatively low. To derive a usefull equation we start again with

$$m_i \frac{d^2 r_i}{dt^2} = G \sum_{j \neq i} \frac{m_i m_j}{r_{ij}^3} \bar{r_{ij}} \tag{4.1}$$

Lets assume that there is one main body k, a body i and a disturbing body j. First step is the write the EoM of body i and k with respect to an inertial reference frame. By substrating the EoM of body k from body i, the equation of motion for body i in a non rotating (non inertial) with as center body k can be found. (The EoM of body i with respect to body k)

$$m_i \bar{\ddot{r}}_i = G \frac{m_i m_k}{r_{ik}^3} \bar{r_{ik}} + G \sum_{j \neq i \neq k} \frac{m_i m_j}{r_{ij}} r\bar{i}j \tag{4.2}$$

$$m_k \bar{\ddot{r}}_k = G \frac{m_k m_i}{r_{ki}^3} \bar{r_{ki}} + G \sum_{j \neq i \neq k} \frac{m_k m_j}{r_{kj}} r\bar{k}j \tag{4.3}$$

$$\bar{\ddot{r}}_i = G \frac{m_k}{r_{ik}^3} \bar{r_{ik}} + G \sum_{j \neq i \neq k} \frac{m_j}{r_{ij}} r\bar{i}j \tag{4.4}$$

$$\bar{\ddot{r}}_k = G \frac{m_i}{r_{ki}^3} \bar{r_{ki}} + G \sum_{j \neq i \neq k} \frac{m_j}{r_{kj}} r\bar{k}j \tag{4.5}$$

using the following vector rules

$$\bar{r_{ik}} = -\bar{r_{ki}} \quad ; \quad \bar{r_{ki}} = \bar{r_i} - \bar{r_k} \quad ; \quad \bar{r_{ij}} = \bar{r_j} - \bar{r_i} = \bar{r_{kj}} \bar{r_{ki}} \tag{4.6}$$

It can be simplified to

$$\begin{aligned} \bar{\ddot{r}}_{ki} = \bar{\ddot{r}}_i - \bar{\ddot{r}}_k &= G \frac{m_k}{r_{ik}^3} \bar{r_{ik}} + G \sum_{j \neq i \neq k} \frac{m_j}{r_{ij}} r\bar{i}j - G \frac{m_i}{r_{ki}^3} \bar{r_{ki}} - G \sum_{j \neq i \neq k} \frac{m_j}{r_{kj}} r\bar{k}j \\ &= -G \frac{m_i + m_k}{r_{ki}^3} \bar{r_{ki}} + G \sum_{j \neq i \neq k} m_j \frac{\bar{r_{kj}} - \bar{r_{ki}}}{r_{ij}^3} - \frac{\bar{r_{kj}}}{r_{kj}} \end{aligned} \tag{4.7}$$

The subscript k can now be dropped because everything is described with respect to k, and the above mention reference frame is used.

$$\bar{\ddot{r}}_{ki} = -G\frac{m_i + m_k}{r_i^3}\bar{r}_i + G\sum_{j\neq i\neq k} m_j \frac{\bar{r}_j - \bar{r}_i}{r_{ij}^3} - \frac{\bar{r}_j}{r_j} \tag{4.8}$$

Assuming a satellite orbiting the earth the equation can be simplified because $m_s \ll m_E$. The **magnitude** of the main force and the disturbing force can now be written as.

$$a_m = G\frac{m_E}{r_i^2} \tag{4.9}$$

$$a_d = Gm_d\left|\frac{\bar{id}}{r_{id}^3} - \frac{\bar{r}_j}{r_j}\right| \tag{4.10}$$

$$= Gm_d\sqrt{\left(\frac{\bar{id}}{r_{id}^3} - \frac{\bar{r}_j}{r_j}\right)\cdot\left(\frac{\bar{id}}{r_{id}^3} - \frac{\bar{r}_j}{r_j}\right)} = Gm_d\sqrt{\frac{\bar{id}}{r_{id}^3}\cdot\frac{\bar{id}}{r_{id}^3} - 2\frac{\bar{id}}{r_{id}^3}\cdot\frac{\bar{d}}{r_d^3}\frac{\bar{d}}{r_d^3}\cdot\frac{\bar{d}}{r_d^3}}$$

$$= Gm_d\sqrt{\frac{r_{id}^2}{r_{id}^6} + \frac{r_d^2}{r_d^6} - \frac{2r_d r_{id}\cos\beta}{r_d^3 r_{id}^3}} = Gm_d\sqrt{\frac{1}{r_{id}^4} + \frac{1}{r_d^4} - \frac{2\cos\beta}{r_d^2 r_{id}^2}} \tag{4.11}$$

used geometry rules

$$\bar{a}\cdot\bar{b} = |a||b|\cos\beta \qquad ; \qquad \bar{a}\cdot\bar{a} = ||a||^2 = |a^2| \qquad ; \qquad c^2 = a^2 + b^2 - 2ab\cos\alpha \tag{4.12}$$

Lets introduce $\gamma = r_i/r_d$ and α which is the angle between $\bar{r}_i$ and $\bar{r}_d$ (β is the angle between $\bar{r}_d$ and $\bar{r_i}d$)

$$r_{id}^2 = r_d^2 + r_i^2 - 2r_i r_d\cos\alpha \tag{4.13}$$

$$\cos\beta = \frac{A}{S} = \frac{r_d - r_i\cos\alpha}{r_{id}} \tag{4.14}$$

$$\gamma = \frac{r_i}{r_d} \tag{4.15}$$

$$a_d = Gm_d\sqrt{\frac{1}{(r_d^2 + r_i^2 - 2r_i r_d\cos\alpha)^2} + \frac{1}{r_d^4} - \frac{2(r_d - r_i\cos\alpha)}{r_d^2 r_{id}^3}}$$

$$= G\frac{m_d}{r_d^2}\sqrt{\frac{1}{(1+\gamma^2 - 2\gamma\cos\alpha)^2} + 1 - \frac{2(1-\gamma\cos\alpha)}{r_d^5(r_d^2 + r_i^2 + r_d r_i\cos\alpha)^{3/2}}}$$

$$= G\frac{m_d}{r_d^2}\sqrt{1 + \frac{1}{(1 - 2\gamma\cos\alpha + \gamma^2)^2} - \frac{2(1-\gamma\cos\alpha)}{(1 - 2\gamma\cos\alpha + \gamma^2)^{3/2}}} \tag{4.16}$$

THERE IS AN ERROR SOMEWHERE r_d^5 SHOULD BE r_d^6

In this equation k is the main body with mass $m_k \gg m_i$. α is the angle between the vector r_{ki} and r_{kd} and β is the angle between the vector r_{kd} and r_{di}. (see fig 4.2). With this equation it can be shown that bodies outside the solar system have a negligible influence on the trajectories around the sun, and that the planets have a negligible influence on satellites circling the earth.

When flying further away from a planet the sun will become more dominant in play of forces. At some point it can not be regarded any more as a disturbing force. The region were it can be regarded as an disturbing force is called the *sphere of influence* of a planet. Beyond this sphere the planet will become a disturbing force. These sphere is are relatively small. An interplanetary trajectory can be modeled in three separate parts, making the analysis easier.

Chapter 5

Two-body problem

Remember these formulas:

$$r = \frac{p}{(1 + e\cos\theta)} \tag{5.1}$$

$$p = \frac{H^2}{\mu} \tag{5.2}$$

$$H = \dot{\varphi}r^2 = \dot{\theta}r^2 \tag{5.3}$$

In the pericenter $\theta = 0$ and in the apocenter $\theta = \pi$. By differentiating 5.5 en substitute 5.5 back into its derivative and substituting 5.3 a simple function for $\dot{r}$ can be found

$$2a = r_{pericenter} + r_{apocenter} = r_{\theta=0} + r_{\theta=\pi} = \frac{p}{(1 + e\cos 0)} + \frac{p}{(1 + e\cos\pi)}$$

$$= \frac{p}{(1+e)} + \frac{p}{(1-e)} = \frac{p + ep + p - ep}{1^2 + ep - ep - e^2} = \frac{2p}{1 - e^2}$$

$$p = a(1 - e^2) \tag{5.4}$$

$$r = \frac{p}{(1 + e\cos\theta)} = \frac{a(1 - e^2)}{(1 + e\cos\theta)} = \frac{\frac{H^2}{\mu}}{(1 + e\cos\theta)} \tag{5.5}$$

$$\dot{r} = \frac{dr}{dt} = \frac{H^2}{\mu}(-(1 + e\cos\theta)^{-2})(-e\sin\theta)\dot{\theta} = \frac{r^2}{\frac{H^2}{\mu}}e\dot{\theta}\sin\theta = r^2\dot{\theta}\frac{\mu}{H^2}e\sin\theta = \frac{\mu}{H}e\sin\theta \tag{5.6}$$

Combining 5.5 and 5.3 gives:

$$r = \frac{\frac{H^2}{\mu}}{(1 + e\cos\theta)} = \frac{r^2\dot{\theta}\frac{H}{\mu}}{(1 + e\cos\theta)} \tag{5.7}$$

$$r\dot{\theta} = \frac{\mu}{H}(1 + e\cos\theta) \tag{5.8}$$

One can also write the speed in a component which is perpendicular to the radius vector V_n and a component which is perpendicular to the axis of symmetry of the conic section. V_l. (see fig 5.8)

$$V_n = r\dot{\theta} + \frac{\dot{r}}{tan(\frac{\pi}{2} - \alpha)} = r\dot{\theta} + \frac{\dot{r}}{tan(\frac{\pi}{2} - (\theta - \frac{\pi}{2}))} = r\dot{\theta} + \frac{\dot{r}}{tan(\theta - \pi)} = r\dot{\theta} + \frac{\dot{r}}{tan\theta} \tag{5.9}$$

$$= \frac{\mu}{H}(1 + e\cos\theta) - \frac{\mu}{H}e\sin\theta\frac{\cos\theta}{\sin\theta} = \frac{\mu}{H} + \frac{\mu}{H}e\cos\theta - \frac{\mu}{H}e\cos\theta = \frac{\mu}{H} \tag{5.10}$$

$$V_l = \frac{\dot{r}}{\cos(\theta - \frac{\pi}{2})} = \frac{\dot{r}}{\sin\theta} = \frac{\mu}{H}e\sin\theta\frac{1}{\sin\theta} = e\frac{\mu}{H} \tag{5.11}$$

Formula 5.6 and 5.8 can be used to make velocity hedographs. For elliptical (e<1), parabolic (e=1) and hyperbolic (e>1) trajectories different speeds are calculated. The pericenter is located at the top, where the tangential velocity is the largest, the apogee at the bottom. See also fig 5.7

	Ellipse		parabola		Hyperbola	
θ	$\dot{r}$	$r^2\dot{\theta}$	$\dot{r}$	$r^2\dot{\theta}$	$\dot{r}$	$r^2\dot{\theta}$
0	0	$\frac{\mu}{H}(1+e)$	0	$2\frac{\mu}{H}$	0	$\frac{\mu}{H}(1+e)$
$\frac{\pi}{2}$	$e\frac{\mu}{H}$	$\frac{\mu}{H}$	$\frac{\mu}{H}$	$\frac{\mu}{H}$	$e\frac{\mu}{H}$	$\frac{\mu}{H}$
π	0	$\frac{\mu}{H}(1-e)$	0	0	0	$\frac{\mu}{H}(1-e)<0$
$\frac{3\pi}{2}$	$-e\frac{\mu}{H}$	$\frac{\mu}{H}$	$-\frac{\mu}{H}$	$\frac{\mu}{H}$	$-e\frac{\mu}{H}$	$\frac{\mu}{H}$

The Theorem of Whittaker states that at the pericenter V_n and V_l have to same direction and can therefore be added to find the maximum speed. In the apocenter V_n is directed in the the opposite way, and the minimum speed can be obtained by subtracting V_n from V_l.

$$V_p = V_l + V_n = e\frac{\mu}{H} + \frac{\mu}{H} = \frac{\mu}{H}(1+e) \tag{5.12}$$

$$V_a = V_l - V_n = e\frac{\mu}{H} - \frac{\mu}{H} = \frac{\mu}{H}(1-e) \tag{5.13}$$

5.1 Relativistic effects

With the use of the auxiliary variable u =1/r and some rearranging equation 5.14 can be writtin as eq 5.16

$$\ddot{r} - r\dot{\varphi}^2 = \frac{\mu}{r^2} \tag{5.14}$$

$$u = \frac{1}{r} \tag{5.15}$$

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{H^2} \tag{5.16}$$

According to general theory of relativity a good approximation for the relativistic motion of a body can be given by:

$$\frac{d^2u}{d\varphi^2} + u = \frac{\mu}{H^2} + 3\frac{\mu}{c^2}u^2 \tag{5.17}$$

In these equation μ is the *gravitation parameter* of the large body, c is the speed of light, u = 1/r and H is the classical angular momentum. Comparing the two equations it becomes clear that the most right part of the right hand side of the equations is due to relativity. The right hand side can be rewritten

$$H = \dot{\varphi}r^2 = \dot{\varphi}\frac{1}{u^2} \tag{5.18}$$

$$\frac{\mu}{H^2} + 3\frac{\mu}{c^2}u^2 = \frac{\mu}{H^2}\left(1 + 3\frac{H^2u^2}{c^2}\right) = \frac{\mu}{H^2}\left(1 + 3\frac{\dot{\varphi}^2}{u^2c^2}\right) = \frac{\mu}{H^2}\left(1 + 3\frac{(\dot{\varphi}r)^2}{c^2}\right) = \frac{\mu}{H^2}\left(1 + 3\frac{V_\varphi^2}{c^2}\right) \tag{5.19}$$

The relativistic part depends on the ration between the velocity of the body and the speed of light. In celestical mechanics the speed of the body is much smaller than the speed of light ($V_\varphi \ll c$). Therefore the ration will be small and therefore the right term will be much smaller than the left term. A first order approximation is:

$$u = \frac{\mu}{H^2}\left[1 + e\cos(\varphi - \omega)\right] + \alpha\frac{\mu^2}{H^4}\left[1 + \frac{1}{2}e^2 + e\varphi\sin(\varphi - \omega) - \frac{1}{6}e^2\cos 2(\varphi - \omega)\right] \tag{5.20}$$

$$\alpha = 3\frac{\mu}{c^2} \tag{5.21}$$

The relativistic part of this equation consist of three parts:

- The constant part $(1 + 1/2e^2)$ which increase the value of u with the constant value $\alpha\mu^2(1 + e^2/2)/H^2$. Time has no influence on this part.
- $e\varphi\sin\varphi - \omega$ is a fluctuating term, of which the amplitude continuously increases with increasing values of φ. Because φ is the total angle traveled, φ and this term will become larger in time.

- $1/6e^2 \cos 2(\varphi - \omega)$ is a pure oscillation with a constant amplitude.

So in the long run only the second term has influence and the equation can be simplified:

$$u = \frac{\mu}{H^2} \left[1 + e \cos(\varphi - \omega) + \beta e \varphi \sin \varphi - \omega\right] \tag{5.22}$$

$$\beta = \alpha \frac{\mu}{H^2} = 3 \frac{mu^2}{c^2 H^2} \tag{5.23}$$

In celestial mechanics the value of β is very small. It is so small that even for large values of φ the combination of the two stays small. Therefore the following is valid:

$$\cos \beta\varphi \approx 1 \quad : \quad \sin \beta\varphi \approx \beta\varphi \tag{5.24}$$

With this known equation 5.22 can be even further simplyfied:

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \tag{5.25}$$

$$\begin{aligned} u &= \frac{\mu}{H^2} \left[1 + e \cdot 1 \cdot \cos(\varphi - \omega) + e \cdot \varphi\beta \cdot \sin(\varphi - \omega)\right] \\ &= \frac{\mu}{H^2} \left[1 + e \cos \beta\varpi \cos(\varphi - \omega) + e \sin \varphi\beta \sin(\varphi - \omega)\right] \\ &= \frac{\mu}{H^2} \left[1 + e \cos(\varphi - \omega - \beta\varphi)\right] \end{aligned} \tag{5.26}$$

In the cosine the instantaneous argument of the pericenter is given by $\omega + \omega\beta$. Because this term depends on φ the location of the pericenter will change over time. The amount of change after one full rotation, $\Delta\omega$, can be found by taking 2π for φ (one full rotation). This gives:

$$\Delta\omega = \beta 2\pi = \alpha \frac{\mu}{H^2} \cdot 2\pi = 6\pi \frac{mu^2}{c^2 H^2} \tag{5.27}$$

When looking to this equation the only term which differs for planets in our solar system is H. Assuming that the orbits for the planets are more or less circular ($P \approx r$) one can say that $H^2 \approx \mu r$ which shows that the relativistic effects is the largest when the planet is closest to the sun, in other words Mercuries.

5.2 Solar radiation pressure

Photons emitted by the sun reflect on satellites, losing kinetic energy/ apply a force on the satellite. This can be modeled as follows:

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\mu}{r^2} + \frac{F}{m} \sin \delta \tag{5.28}$$

$$\frac{d}{dt}(r^2 \dot{\varphi} = \frac{F}{m} r \cos \delta \tag{5.29}$$

The solar radiation force can be written as

$$F = C_R \frac{WS}{c} \tag{5.30}$$

C_R is the reflectivity, c is the speed of light, W is the power density of the light falling on the object and S is the effective surface. Because the surface area of a sphere is $4\pi r^2$ the power density dimmished with a factor r^2. When the power density at the sun is know (W_S) the local power density can be found by multiplying it with R_S/r where R_S is the radius of the sun and r is the distance from the center of the sun to the object. Assuming the object is a perfect sphere, the effective area is a circle with a surface of πR^2 and a mass of $4/3\pi R^3 \rho$. This gives:

$$\frac{F}{m} = C_R W_S (\frac{R_S}{r})^2 \frac{1}{c} \pi R^2 \frac{1}{4/3\pi R^3 \rho} = \frac{3}{4} \frac{C_R W_S R_S^2}{c\rho R} \frac{1}{r^2} = \frac{\alpha}{r^2} \tag{5.31}$$

Because the body has a radial velocity component with respect to the sun, the frequency of the light is shifted because of the Doppler effect. The power density is linear with the frequency (a lower frequency means that the photons have less energy, so produce a smaller force) Therefore:

$$v' = v(1 - \frac{\dot{r}}{c}) \tag{5.32}$$

$$W' = W(1 - \frac{\dot{r}}{c}) \tag{5.33}$$

Furthermore because also light needs time to travel the distance between the sun and the object, the object already has rotated and it appears that the light comes more from the front.

$$\Delta t = \frac{r}{c} \tag{5.34}$$

$$\gamma = \dot{\varphi}\Delta t = \frac{\dot{\varphi} r}{c} \tag{5.35}$$

A bit of geometry and because γ is very small (small angle approach)

$$\sin\delta = \sin(\frac{\pi}{2} + \gamma) = \cos\gamma \approx 1 \qquad ; \qquad \cos\delta = \cos(\frac{\pi}{2} + \gamma) = -\sin\gamma \approx= -\gamma \tag{5.36}$$

Substituting these answers back into 5.28 gives:

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\mu}{r^2} + \frac{\alpha}{r^2}\left(1 - \frac{\dot{r}}{c}\right) \tag{5.37}$$

$$\frac{d}{dt}(r^2\dot{\varphi} = -r\frac{\alpha}{r^2}\left(1 - \frac{\dot{r}}{c}\right)\frac{r\dot{\varphi}}{c} \tag{5.38}$$

Rewriting and neglecting second order terms (c^2)

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\mu - \alpha}{r^2} - \frac{\alpha\dot{r}}{cr^2} \tag{5.39}$$

$$\frac{d}{dt}(r^2\dot{\varphi} = -\frac{\alpha\varpi}{c} \tag{5.40}$$

A couple of things can be noticed here. All particle will be slowed down because of equation 5.40. Due to this drag force the object will spiral down in the direction of the sun. Equation 5.39 can be even further simplified by neglected the second term.

$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\mu}{r^2}\left(1 - \frac{\alpha}{\mu}\right) = -\frac{\mu}{r^2}(1 - \beta). \tag{5.41}$$

For the right values of α/μ ($\beta > 1$) particles will not spiral inwards but will be blown out of the solar system. When taking a closer look to definition of α one can see that small, low density, high reflectivity particles will have larger forces applied to them. This is why our sun is bright and not hazy because all dust is blown away.

Chapter 6

Parabolic and hyperbolic orbits

The motion of a satellite in an elliptical orbit can be described with:

$$r = a(1 - e\cos E) \quad ; \quad \tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \quad ; \quad E - e\sin E = n(t-\tau) = M \tag{6.1}$$

For a hyperbolic orbit:

$$r = a(1 - e\cosh F) \quad ; \quad \tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tanh\frac{F}{2} \quad ; \quad e\sinh F - F = \bar{n}(t-\tau) = \bar{M} \tag{6.2}$$

In these equations r is the radius, a is the semi-major axis, e is the eccentricity, θ is the *true anomaly*, E is the *eccentric anomaly*, F is the *hyperbolic anomaly*, n is the *mean angular motion*, $\bar{n}$ is an angular velocity, t is the time, τ is the time of (last) pericenter passage, M is the *mean anomaly* and $\bar{M}$ is also a kind of mean anomaly. see fig 6.7 and 8.1

$$n = \frac{T}{2\pi} = \frac{\mu}{a^3} \quad ; \quad \bar{n} = \frac{T}{2\pi} = \frac{\mu}{-a^3} \tag{6.3}$$

To demonstrate how these equations work, the calculation of the orbital period T and the position (θ and r) are calculated for a satellite orbiting the Earth in an orbit with e=0.05 and a perigee altitude of $h_p = 500$ km after 60 minutes. Furthermore given is that $\mu = 398600.4\ km^3/s^2$ and R = 6378.14 km.

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{6.4}$$

$$a = \frac{r(1+e\cos\theta)}{1-e^2} = \frac{(500+6378.14)(1+0.05\cos 0)}{1+0.05^2} = 7204.036908 \text{ Km} \tag{6.5}$$

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} = 2\pi\sqrt{\frac{7204^3}{398600.4}} = 6085\ s \tag{6.6}$$

$$n = \sqrt{\frac{\mu}{a^3}} = \sqrt{\frac{398600.4}{7204^3}} = 0.0010325355\ s^{-1} \tag{6.7}$$

$$t = 60 \cdot 60 = 3600 < 6085 \rightarrow \tau = 0 \tag{6.8}$$

$$M = n(t-\tau) = 0.0010325355(3600-0) = 3.717127 \tag{6.9}$$

$$E_{k=0} = M = 3.68935344 \tag{6.10}$$

$$E_{k+1} = M + e\sin E_k \tag{6.11}$$

$$E_1 = 3.717127 + 0.05\sin 3.717127 = 3.689913439 \tag{6.12}$$

$$E_2 = 3.717127 + 0.05\sin 3.689913439 = 3.691064876 \tag{6.13}$$

$$E_3 = 3.717127 + 0.05\sin 3.691064876 = 3.691015761 \tag{6.14}$$

$$E_4 = 3.717127 + 0.05\sin 3.691015761 = 3.691017855 \tag{6.15}$$

$$r = a(1-e\cos E) = 7204.036908(1-0.05\cos 3.691017855) = 7511.225Km \tag{6.16}$$

$$\theta = 2 \cdot \arctan\left(\sqrt{\frac{1+0.05}{1-0.05}}\tan\frac{3.691017855}{2}\right) = -2.62 \tag{6.17}$$

$$\theta = \frac{\pi + 2.64}{\pi} \cdot 180 = 330^\circ \tag{6.18}$$

As can be seen in the above example, the eccentric anomaly is found with an iterative process (you van also plot two lines with your GRM and find the intersection). For elliptical trajectories is possible to find an analytical approximation by using a series expension. For the hyperbolic case this can not be done because these series do not converge because $1 > e > \infty$ and because the function sinhF and $\bar{M} - F$ are non-periodic.

Hyperbolic problems can be solved numerical or graphical. A graphical method is to plot the function sinhF and the linear function $(F + \bar{M})/e$. The intersection of the two is the solution. The linear function can be plotted by drawing a line through ($-\bar{M}$,0) and ($-\bar{M} + e$),1), or use a GRM.