Summary WI1401LR: Calculus I

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Based on Calculus 6e (James Stewart) & Lecture notes

Chapter 12: Vectors and the geometry of space

§12.1: Three-dimensional coordinate systems (p. 765)

Distances between two points (*Distance formula in three dimensions*):

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of sphere:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

with center $C(h, k, l)$ and radius r.

§12.2: Vectors (p. 770)

Notation:

Vectors are denoted by bold print or an arrow or bar above the sign. Vectors have both a magnitude and a direction.

Scalars are denoted with (normal print) letters. Scalars only have a magnitude.

Vector / scalar properties:

See: page 774.

Vector addition / substraction:

By using Triangle or Parallelogram Law (*kop-aan-staart / paralellogramregel).*

$$\mathbf{a} + \mathbf{b} = \langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Scalar multiplication:

Vector is multiplied by a scalar (which only has magnitude), resulting in a new vector with a different magnitude but equal direction.

$$c\boldsymbol{a} = c\langle a_1, a_2, a_3 \rangle = \langle c \cdot a_1, c \cdot a_2, c \cdot a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Vector between given points:

$$\boldsymbol{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Note the similarity between *Distance formula in three dimensions*, this formula and the formula to calculate the magnitude of a 3D-vector (below).

Length of vector:

Via Pythagoras: $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Unit vectors / standard basic vectors:

Unit vectors have unit length, so $|\boldsymbol{u}| = 1$.

Three standards basic vectors:

$\hat{\boldsymbol{\imath}} = \langle 1,0,0 \rangle$, on x-axis

$\hat{\boldsymbol{\jmath}} = \langle 0,1,0 \rangle$, on y-axis

$\widehat{\boldsymbol{k}} = \langle 0,0,1 \rangle$, on z-axis

Find unit vector of a given vector $\boldsymbol{a}$ with the same direction: divide all terms ($\hat{\boldsymbol{\imath}}, \hat{\jmath}, \widehat{\boldsymbol{k}}$) by $|\boldsymbol{a}|$.

§12.3: The Dot Product (p. 779)

Dot product:

Properties, see: page 779

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ (= value/number)

Angle between (non-zero) vectors:

$cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

Thus, if $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$, the vectors are orthogonal, since then $cos\theta = \frac{0}{|\boldsymbol{a}||\boldsymbol{b}|}$, which gives $cos\theta = 0$, which gives $\theta = \frac{1}{2}\pi = 90°$.

Direction angles and direction cosines:

Direction angles α, β, γ are the angles of a vector between the positive x-, y- and z-axis. Direction cosines are the cosines of those angles.

$cos\alpha = \frac{a_1}{|\boldsymbol{a}|}, cos\beta = \frac{a_2}{|\boldsymbol{a}|}, cos\gamma = \frac{a_3}{|\boldsymbol{a}|}$, or $\frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \langle \cos\alpha, cos\beta, cos\gamma \rangle$

Projections:

Definition / graphical representation, see: page 782/783bpb

Scalar projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $comp_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

Vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $proj_{\boldsymbol{a}}\boldsymbol{b} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}\right)\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|^2}\boldsymbol{a}$

Note: vector projection is scalar projection multiplied by unit vector in the direction of $\boldsymbol{a}$.

§12.4: The Cross Product (p. 786)

Cross product:

Properties, see: page 790

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\boldsymbol{\imath}} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{\boldsymbol{\jmath}} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$$

$(a_2 b_3 - a_3 b_2)\hat{\boldsymbol{\imath}} - (a_1 b_3 - a_3 b_1)\hat{\boldsymbol{\jmath}} + (a_1 b_2 - a_2 b_1)\hat{\boldsymbol{k}}$ (= vector)

Angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (with $0 \leq \theta \leq \pi$):

$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\ sin\theta$

Thus, if $\boldsymbol{a} \times \boldsymbol{b} = 0$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, as then $sin\theta = 0$, which gives $\theta = 0$ or $\theta = \pi$.

$|\boldsymbol{a} \times \boldsymbol{b}|$ gives the area of the parallelogram between $\boldsymbol{a}$ and $\boldsymbol{b}$. $|a \cdot \boldsymbol{b} \times \boldsymbol{c}|$ (*a scalar triple product*) gives the volume of the parallelepiped between $\boldsymbol{b}$ and $\boldsymbol{c}$ that has height a.

Chapter 1: Functions and models

§1.6: Inverse trigonometric functions (p. 67)

Inverse sine:

$f^{-1}(x) = y \Leftrightarrow f(y) = x$ gives $\sin^{-1} x = y \Leftrightarrow \sin y = x$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Inverse cosine:

$f^{-1}(x) = y \Leftrightarrow f(y) = x$ gives $\cos^{-1} x = y \Leftrightarrow \cos y = x$ and $0 \leq x \leq \pi$.

Inverse tangent:

$f^{-1}(x) = y \Leftrightarrow f(y) = x$ gives $\tan^{-1} x = y \Leftrightarrow \tan y = x$ and $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Chapter 3: Differentiation rules

§3.4: The Chain Rule (p. 197)

Chain rule:

$$F'(x) = f'\big(g(x)\big) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Proving the chain rule, see: page (202 and) 203.

§3.5: Implicit differentiation (p. 207)

Definition:

Implicit differentiation consists of differentiation both sides of the equation with respect to x and then solving the resulting equation for y'.

Derivatives of inverse trigonometric functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

§3.10: Linear approximation and differentials (p. 247)

Linear approximation / Tangent line approximation:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Linearization

Linear function whose graph is the tangent line

$$L(x) = f(a) + f'(a)(x - a)$$

Chapter 4: Applications of differentiation

§4.2: The Mean Value Theorem (p. 280)

Mean value theorem:

If f is continuous on $[a, b]$ and differentiable on $\langle a, b\rangle$, then $f'(c) = \frac{f(b)-f(a)}{b-a}$ (or $f(b) - f(a) = f('c)(b-a)$.

Chapter 5: Integrals

§5.2: The definite integral (p. 366)

Comparison properties of the integral:

If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

§5.3: The Fundamental Theorem of Calculus (p. 379)

Part I (FTC1):

If f is continuous on $[a, b]$, then $g(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$ is continuous on $[a, b]$ and differentiable on $\langle a, b\rangle$, and holds $g'(x) = f(x)$.

Part II (FTC2):

If f continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$, where $F' = f$.

§5.5: The substitution rule (p. 400)

The substitution rule:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int f\big(g(x)\big)g'(x)dx = \int f(u)du$.

The substitution rule for definite integrals:

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then $\int_a^b f\big(g(x)\big)g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du$.

Integrals of symmetric functions:

If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ (symmetric in $x = 0$).

If f is odd $[f(-x) = -f(x)]$, the n$\int_{-a}^{a} f(x)dx = 0$ (symmetric in $y = 0$).

Chapter 7: Techniques of integration

§7.1: Integration by parts (p. 453)

Integration by parts:

$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)dx$ or $\int u\, dv = uv - \int v\, du$

§7.5: Strategy for integration (p. 483)

1. Manipulate/simplify integrand
2. Substitution
3. Integration by parts

§7.6: Integration using tables and CAS (p. 489)

Tables:

See: reference pages 7-10.

§7.8: Improper integrals (p. 508)

I: Infinite intervals:

If $\int_a^t f(x)$ exists for every number $t \geq a$, then $\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$.
If $\int_t^b f(x)$ exists for every number $t \leq b$, then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$.
The above integrals are *convergent* if their limit exists and *divergent* if it does not.

$\int_1^\infty \frac{1}{x^P} dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent (see: above), then

$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx.$

II: Discontinuous integrands

The "vertical version" of Type I Improper integrals. Rather than having an infinite interval (and thus a horizontal asymptote), Type II Improper integrals feature a vertical asymptote.

If f is continuous on $[a, b\rangle$ and discontinuous at b, then $\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$.
If f is continuous on $\langle a, b]$ and discontinuous at a, then $\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$.
This integral is *convergent* if the limit exists and *divergent* if it does not.

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent (see: above), then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_b^c f(x)dx$.

Comparing improper integrals

Suppose continuous functions $f(x)$ and $g(x)$ with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is too.
If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is too.

Chapter 9: Differential equations

§9.1: Modeling with differential equations (p. 567)

Definition:

In general, a *differential equation* is an equation that contains an unknown function and one or more of its derivatives. The *order* of a differential function is the order of the highest derivative that occurs in the equation.

§9.3: Separable equations (p. 580)

Separable equations:

Can be written as $\frac{dy}{dx} = g(x)f(y)$, which equals to $\frac{dy}{dx} = \frac{g(x)}{h(x)}$, where $h(x) = \frac{1}{f(x)}$ and $f(x) \neq 0$. That gives $dx\, g(x) = dy\, h(y)$, which gives $\int h(y)dy = \int g(x)dx$.

§9.5: Linear equations (p. 602)

Solving linear equations:

To solve $y' + P(x)y = Q(x)$, multiply both sides with $I = e^{\int P(x)dx}$ and integrate both sides.

Appendix H: Complex Numbers (p. A-57)

Axes:

Im(aginary) on y-axis, Re(al) on x-axis.

Conjugates:

$\bar{z}$ is the reflection of z in the Re(al) axis, so $\overline{a + b\imath} = a - bi$.
Properties, see: page A-58.

Modulus / Absolute value:

$|z| = \sqrt{a^2 + b^2}$, denotes distance to origin.

Polar form:

$a + bi$ can be written as $z = r(cos\theta + i\ sin\theta)$, where $r = |z|$ and $tan\theta = \frac{b}{a}$.

De Moivre's Theorem:

$z^n = \left(r(cos\theta + i\ sin\theta)\right)^n = r^n(\cos(n\theta) = i\sin(n\theta))$

Roots:

$w_k = r^{1/n}\left(\cos\left(\frac{\theta + k\cdot 2\pi}{n}\right) + i\sin\left(\frac{\theta + k\cdot 2\pi}{n}\right)\right)$, where $k = 0,1,2, \dots, n-1$.
Angle between roots $\phi = \frac{2\pi}{n}$ is constant for a given number of roots.

Complex exponentials / Euler's formula:

$e^{iy} = \cos y + i\sin y$

Chapter 17: Second-order differential equations

§17.1: Second-order linear equations (p. 1111)

General solution to homogeneous linear equations:

$y(x) = c_1 y_1(x) + c_2 y_2(x)$

Auxiliary /characteristic equation:

$ar^2 + br + c = 0$

If $D > 0$, roots are real and distinct, general solution given by $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.

If $D = 0$, roots are real and equal, general solution given by $y = c_1 e^{rx} + c_2 e^{rx}$.

If $D < 0$, roots are complex numbers $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, general solution given by $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$.

Initial value problem:

Solved by $y(x_0) = y_0$ and $y'(x_0) = y_1$. Solution exists and is unique (if $P(x) \neq 0$).

Boundary value problem:

Solved by $y(x_0) = y_0$ and $y(x_1) = y_1$. Solution does not necessarily exist.

§17.2: Nonhomogeneous linear equations (p. 1117)

General solution to <u>non</u>homogeneous linear equations:

$y(x) = y_p(x) + y_c(x)$, in which $y_p(x) = ay'' + by' + cy = G(x)$ (particular solution) and $y_c(x) = ay'' + by' + cy = 0$ (complementary solution, see: §17.1).

Method of undermined coefficients:

1. Get complementary solution by solving auxiliary equation.
2. Substitute $G(x)$ by another formula, based on the following standard 'guesses':"

$G(x)$	Substitute by
x^p (polynominal)	$a_p x^p + a_{(p-1)} x^{p-1} + \cdots + a_2 x^2 + a_1 x + a_0$
$e^{\mu x}$	$Ae^{\mu x}$
$\sin(\mu x)$ or $\cos(\mu x)$	$A \sin(\mu x) + B \cos(\mu x)$
$x \cos(\mu x)$	$(Ax + B) \cos(\mu x) + (Cx + D) \sin(\mu x)$
$xe^x + \cos(\mu x)$	$(Ax + B)e^x + C \cos(\mu x) + D \sin(\mu x)$

Chapter 11: Infinite sequences and series

§11.1: Sequences (p. 675)

Convergence / divergence:

A sequences $\{a_n\}$ converges if the limit $\lim_{n\to\infty} a_n = L$ exists, and diverges if the limit does not exist.

A sequence $\{r^n\}$ is convergent $-1 < r \leq 1$ and divergent for other r.

Every bounded (see: below) and monotonic (either increasing or decreasing) is convergent.

Limits:

A sequence $\{a_n\}$ has a limit if for every $\epsilon > 0$, there is an integer N such that if $n > N$, then $|a_n - L| < \epsilon$.

Limit laws, see: page 678.

Keep in mind: e is defined as $e = \lim_{\mathrm{n}\to\infty} \left(1 + \frac{1}{n}\right)^n$.

Squeeze Theorem:

If a sequence is 'squeezed' by two other sequences that have a limit $\lim_{n\to\infty} = L$, then the mentioned sequence also has that limit. (If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.)

Bounds:

A sequence is bounded above when it has an 'upper asymptote', i.e. there is an M such that $a_n \leq M$ for all $n \geq 1$. A sequence is bounded below when it has a 'lower asymptote', i.e. there is an m such that $m \leq a_n$ for all $n \geq 1$. A sequence is bounded if it is bounded both below and above.

§11.2: Series (p. 687)

Series:

A series is denoted by $\sum a_n$.

Convergence / divergence:

A series is convergent if there exists a limit $\sum_{n=1}^{\infty} a_n = s$ (limit of the sequence of partial sums), where s is a real number and denotes the sum of the series. If the limit does not exist, the series is divergent.

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$. (NOT reversible!)

If $\lim_{(n\to\infty)} a_n$ does not exist or is unequal to 0, then the series $\sum_{n=1}^{\infty} a_n$is divergent.

Divergence test:

If $\lim_{n\to\infty} a_n$ does not exist or exists and is not equal to zero, $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ might converge.

Geometric series:

$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$, where $a \neq 0$.

Convergent if $|r| < 1$, with $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ (and $|r| < 1$).

Harmonic series:

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Always divergent.

Telescopic series:

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

For $n \to \infty$, the series converges to 1 ($\lim_{n\to\infty} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = 1$).

P-series:

Similar to p-test for integrals.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

§11.6: Absolute convergence and the Ratio and Root Tests (p. 714)

Absolute convergence:

A series is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Ratio test:

Consider $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

If $L < 1$, the series is absolutely convergent.

If $L > 1$ or $L = \infty$, the series is divergent.

If $L = 1$, the ratio test is inconclusive. (So, use another test!)

§11.8: Power series (p. 723)

Power series:

With a power series, it's possible to approximate a function by a (finite) series and has the following form:

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 x + c_2 x^2 + \cdots$, which is a power series centered around a (also known as: a power series about a.)

The power series is quite similar to the geometric series. Note, however, that where a (the coefficient) in the geometric series is a constant, c_n (also the coefficients) in the power series are variables.

Convergence / divergence:

It is impossible to state whether a general power series is convergent or divergent, as there are two dependencies.

In general, there are three possibilities for a certain power series $\sum_{n=0}^{\infty} c_n (x-a)^n$.

1. The series only converges for $x = a$.
2. The series converges for all x.
3. The series converges for some x such that $|x - a| < R$, in which R is the *radius of convergence*. The *interval of convergence* is then $a - R < x < a + R$ (so, depending on bounds, $(a-R, a+R)$, $[a-R, a+R)$, $(a-R, a+R]$ or $[a-R, a+R]$). (Graphical representation on page 725).

§11.9: Representations of functions as power series (p. 728)

As said, power series (or geometric series) can be used to represent functions.

Differentation / integration:

Power series with $R > 0$ can be differentiated and integrated, but one has to do that *term-by-term*.

When a series is integrated, the integration constant C follows from $x = 0$ in the original function.

Radius of convergence:

The radius of convergence of a series is equal to the radius of convergence of the derivative or integral of that series.

§11.10: Taylor & Maclaurin series (p. 734)

As said, power series can be used to represent functions. In §11.9, geometric series (constant coefficient) were used: the difficulty with power series are the variable coefficients. They can be found by putting $x = a$ ($f(a) = c_0$) for the first coefficient, and differentiation for the succeeding coefficients ($f'(a) = c_1$). In general:

$c_n = \frac{f^{(n)}(a)}{n!}$, which gives $\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

Thus, íf a function can be represented by a power series, that series is of the form given above. It is called a Taylor series centered around a. If $a = 0$, the series is a Maclaurin series.

Taylor series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Maclaurin series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Finite $\boldsymbol{n}$:

$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the n^{th}-order Taylor polynomial, with $\lim_{n\to\infty} T_n(x) = f(x)$.

The remainder is defined as $R_n(x) = f(x) - T_n(x)$, with $\lim_{n\to\infty} R_n(x) = 0$.

Taylor's inequality:

If $\left|f^{(n+1)}(x)\right| \leq M$ (i.e. bounded) for $|x-a| < d$, then it is possible to bound $R_n(x)$ of the Taylor Series:

$|R_n(x)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$ for $|x-a| < d$.

Binomial coefficient and series:

The binomial coefficient is $\frac{k!}{n!(k-n)!}$ and is (also) denoted by $\binom{k}{n}$.

Binomial series:

$\sum_{n=0}^{\infty} \binom{k}{n} x^n = \sum_{n=0}^{\infty} \frac{k!}{n!(k-n)!} x^n = (1+x)^k$ for $k \ni \mathbb{R}$ and $|x| < 1$

Important Maclaurin series and their R:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, R = \infty$$

$$\sin(x) = \sum_{(n=0)}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, R = \infty$$

$$\cos(x) = \sum_{(n=0)}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, R = \infty$$

$$\tan^{-1}(x) = \sum_{(n=0)}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = \sum_{n=0}^{\infty} \frac{k!}{n!\,(k-n)!} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots, R = 1$$

Chapter 13: Vector functions

§13.1: Vector functions and space curves (p. 817)

Vector(-valued) functions:

Domain: set of real numbers.

Range: set of vectors.

Component functions:

The component functions are the (three, in this chapter) functions from which the vector is built up:

$$\boldsymbol{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$$

Limits / continuity:

By evaluating the limits of the component functions, the limit of the vector function can be found. If $\lim_{t\to a} \boldsymbol{r}(t) = \boldsymbol{r}(a)$, the vector function is continuous at a.

Space curves:

The set of all points described by the parametric equations of that set (i.e., the set of points described by $x = f(t)$, $y = g(t)$ and $z = h(t)$) through a certain interval is a space curve**.**

The vector function of the parametric equations (i.e. $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$) gives the position along the space curve at a certain t. (Remember (2D) Lissajous-curves from high school.)

§13.2: Derivatives and integrals of vector functions (p. 824)

Derivative:

The derivative of a vector function is given by the vector function of the derivatives of the component functions of the original vector function:

$$\boldsymbol{r}'(t) = f'(t)\boldsymbol{i} + g'(t)\boldsymbol{j} + h'(t)\boldsymbol{k}$$

Dividing this *tangent vector* $\boldsymbol{r}'(t)$ by its length gives the unit tangent vector:

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

The angle between two space curves at a certain point is equal to the angle between the two tangents of these space curves at that point.

Differentiation rules:

1. $\frac{d}{dt}[\boldsymbol{u}(t) + \boldsymbol{v}(t)] = \boldsymbol{u}'(t) + \boldsymbol{v}'(t)$ (addition)
2. $\frac{d}{dt}[c\boldsymbol{u}(t)] = c\boldsymbol{u}'(t)$ (scalar multiplication)
3. $\frac{d}{dt}[f(t)\boldsymbol{u}(t)] = f'(t)\boldsymbol{u}(t) + f(t)\boldsymbol{u}'(t)$ (product rule)
4. $\frac{d}{dt}[\boldsymbol{u}(t) \cdot \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \cdot \boldsymbol{v}(t) + \boldsymbol{u}(t) \cdot \boldsymbol{v}'(t)$ (product rule, dot product)
5. $\frac{d}{dt}[\boldsymbol{u}(t) \times \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \times \boldsymbol{v}(t) + \boldsymbol{u}(t) \times \boldsymbol{v}'(t)$ (product rule, cross product)
6. $\frac{d}{dt}\left[\boldsymbol{u}\big(f(t)\big) = f'(t)\boldsymbol{u}'\big(f(t)\big)\right]$ (chain rule)

Integrals:

The integral of a vector function is given by the vector function of the integrals of the component functions of the original vector function:

$$\int_a^b \boldsymbol{r}(t)dt = \left(\int_a^b f(t)dt\right)\boldsymbol{i} + \left(\int_a^b g(t)dt\right)\boldsymbol{j} + \left(\int_a^b h(t)dt\right)\boldsymbol{k}$$

which gives:

$$\int_a^b \boldsymbol{r}(t)dt = \boldsymbol{R}(t)\Big]_a^b = \boldsymbol{R}(b) - \boldsymbol{R}(a)$$

§13.3: Arc length (p. 830)

Arc length:

The arc length of a plane or space curve is defined as the summation of the lengths of the inscribed polygons (see: graphic §13.3, figure 1):

$$L = \int_a^b |\boldsymbol{r}'(t)|\ dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt$$

Parametrizations:

Different formulas that represent the same curve are called parametrizations (of the curve they represent).

When (re)parametrizating a curve with respect to arc length (i.e., instead of having a time-variable, there is a length-variable), the *arc length function* s comes into play:

$$s = s(t) = \int_a^t |\boldsymbol{r}'(u)|\ du$$

Furthermore, $\frac{ds}{dt} = |\boldsymbol{r}'(t)|$.

Normal and binormal vectors:

Principal unit normal / unit normal:

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} \text{(with } \boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}\text{, §13.1)}$$

Binormal vector, which is perpendicular to both $\boldsymbol{T}$ and $\boldsymbol{N}$ and is a unit vector:

$$\boldsymbol{B}(t) = \boldsymbol{N}(t) \times \boldsymbol{T}(t)$$

Normal and osculating plane:

The plane determined by $\boldsymbol{B}$ and $\boldsymbol{N}$ at a point P on a curve C is called the normal plane of C at P. The plane determined by $\boldsymbol{T}$ and $\boldsymbol{N}$ is the osculating plane of C at P.

Chapter 14: Partial derivatives

§14.1: Functions of several variables (p. 855)

Functions of two variables:

A function f of two variables assigns a unique number ($f(x,y)$) to each ordered pair of real numbers (x,y) in a set D. D is the domain of f, and $\{f(x,y)|(x,y) \in D\}$.

Graphs:

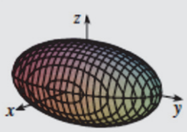
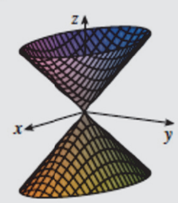
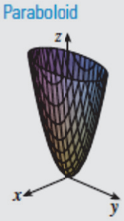
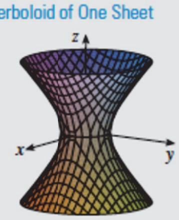
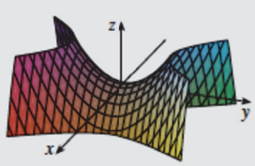
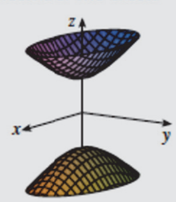
If f is a function of two variables with domain D, then the graph of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and $(x,y) \in D$.

Level curves:

The level curves:

- are contours of the function $z = f(x,y)$;
- are the lines where z is held constant ($k = f(x,y)$);
- shows the domain D of the graph, when the height is k.

The closer to each other the level curves are, the larger $\frac{dz}{dt}$ is.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

(Calculus 6e, §12.6, page 808.)

Functions of three variables:

When a function has three variables, some things change:

- $D \in \mathbb{R}^3$ (instead of $D \in \mathbb{R}^2$ for two-variable-functions);
- There are <u>level surfaces</u> rather than level curves.

§14.2: Limits and continuity (p 870)

Limits:

Let $f(x, y)$ be a function defined on a domain D that includes points arbitrarily close to a point (a, b). Then the limit of $f(x, y)$ as a point (x, y) approaches (a, b) is L:

$\lim_{(x,y)\to(a,b)} f(x, y) = L$

The limit L exists if for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ when $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ for $(x, y) \in D$. (See: pp. 871-872.)

$|f(x, y) - L|$ is the distance between z and $L = f(a, b)$.

$\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the points (x, y) and (a, b).

The above definition holds for any way (a, b) is approached (see: p. 871, figure 3). If the limits found for two different paths are not equal, the limit does not exist.

To check all paths, look at the numerator $x^\alpha y^\beta$. When $\alpha = 1$ and $\beta > 1$, substitute $x = y^\beta$ and $y = mx$ (and vice versa) and find the limit. If these are equal, the limit *might exist*. If not, it *does not exist*. If it might exist, let $\epsilon > 0$ and find $\delta > 0$ such that $|f(x, y) - L| < \epsilon$. Then express δ in ϵ, and check.

Continuity:

$z = f(x, y)$ is continuous at a point (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) = L$.
$z = f(x, y)$ is continuous on domain D if it is continuous at every point $(a, b) \in D$.
In normal words, this means that if (a, b) changes by a small amount, (x, y) has to change by the same small amount (i.e. there are no holes/jumps/gaps in the graph).

All polynomials are continuous on $\mathbb{R}^2$.

§14.3: Partial derivatives (p. 878)

Partial derivatives:

When the change in one variable of a multi-variable function is calculated, when keeping the other(s) constant, it is a partial derivative:

$f_x(a, b) = g'(a)$ where $g(x) = f(x, b)$ with b constant.

$f_y(a, b) = g'(b)$ where $g(x) = f(a, y)$ with a constant.
(Other notations, see: page 880.)

Following from the definition above, partial derivatives are calculated by taking the derivative with respect to the indicated variable, and treating the other variable(s) as a constant. Mind: when taking a variable as a constant, its derivative is zero.

<u>Geometrically</u>, partial derivatives can be interpreted as the slopes of the tangents to C_1 and C_2, which are planes through the surface S described by $f(x, y)$ when $y = b$ or $x = a$, respectively. (See: page 881, figure 1 and page 882, figures 4 and 5.)

Partial derivatives are defined by <u>limits</u> (as are normal derivatives):

$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$

$f_y(x, y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$

Higher-order partial derivatives and Clairaut's theorem:

Second partial derivatives are calculated by taking the derivative of a partial derivative:

$(f_x)_x$.

The second partial derivative of f_x with respect to y is equal to the second partial derivative of f_y with respect to x (i.e. $f_{xy}(a,b) = f_{yx}(a,b)$) as long as both functions are continuous on the disk containing the point (a,b). The same goes for higher-order partial derivatives (e.g. $f_{xyz} = f_{yzx} = f_{zxy}$). Proven by Clairaut's theorem.

Partial differential equations:

Laplace's equation:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, produces harmonic functions (fluid flow, heat conduction, etc.).

Wave equation:

$\frac{\partial^2 u}{\partial^2 t} = a^2 \frac{\partial^2 u}{\partial x^2}$, describes waveforms.

§14.4: Tangent planes and linear approximations (p. 892)

Linear approximations:

In 1D, the tangent (line) of $y = f(x)$ is given by $y = f(a) + f'(a)(x - a)$, which is known as the *linearization* $L(x)$.

In 2D, the tangent (plane) of $z = f(x,y)$ is given by $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$, which is also known as the *linearization* $L(x,y)$.

Note that the *linearizations* comprise the first few terms of the Taylor Series (§11.10).

Increments and differentials:

In 1D, the increment Δy is given by $\Delta y = f(a + \Delta x) - f$ (change along the curve). The differential dy is given by $dy = f'(x)dx$ (change along tangent line of the curve).

In 2D, the increment Δz is given by $\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$ (change along the surface). The differential dz is given by $dz = f_x(a,b)dx + f_y(a,b)dy$ (change along tangent plane of the surface).

Differentiability:

A function $f(x,y)$ is differentiable at (a,b) if Δz can be expressed as $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where $\epsilon_1, \epsilon_2 \to 0$ as $(a,b) \to (0,0)$.

If the partial derivatives f_x, f_y exist near (a,b) and are continuous at (a,b), then $f(x,y)$ is differentiable (at (a,b)).