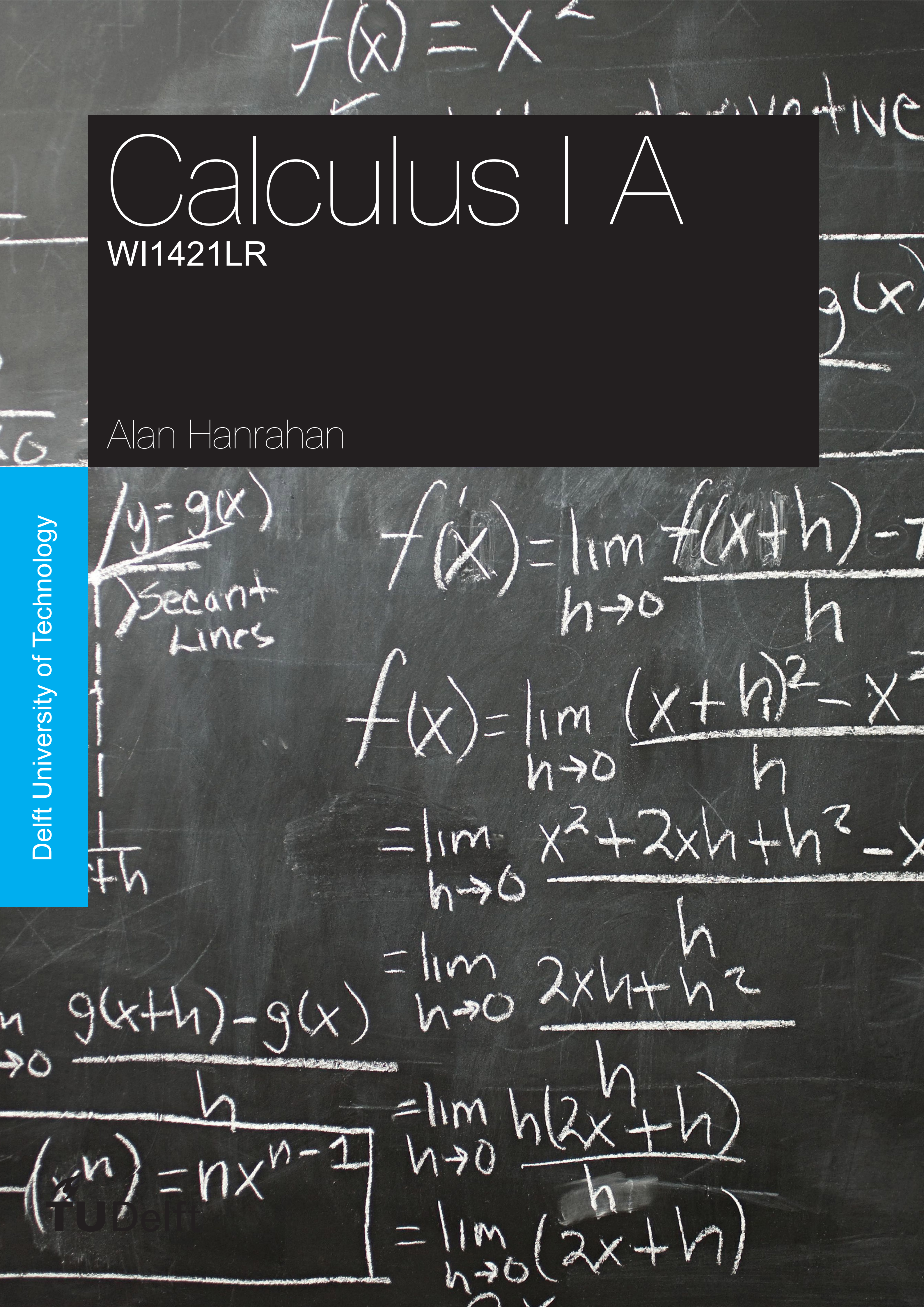

Calculus I A

WI1421LR

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Preface

So this is a summary of the core concepts covered in the Calculus 1 module in First year. Up to 2018 you could easily pass the course just by practicing past questions and learning the patterns of how to solve them, but from 2019 on wards you needed to have a more intuitive understanding of the stuff you learned in class. In these notes I go over the basics of what you do in class and try to explain things simply. Obviously I'm not going to every bit of information from the lectures and the book in here, so if you want further clarification I recommend looking back at the slides. These notes go in the same order as the lectures so you should have no problem finding further information. The latest version of these notes is always available on my website (alanrh.com), along with other resources that I find useful. If you find any mistakes and you need anything corrected shoot me an email, but please don't email me if you want further explanation because I can't promise individual help to everyone.

Alan Hanrahan
Delft, February 4, 2021

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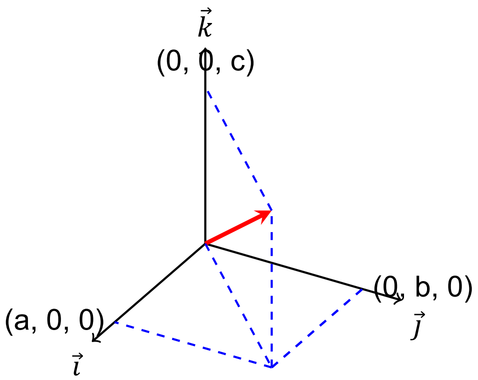
Vectors

1.1. The Basics

I'm sure by now you have a very clear understanding of what a vector is. It's something that has both a scalar magnitude as well as a direction. A vector can point in any direction in space (in any number of dimensions, but in this course we stick to 3 dimensions because we're engineers and why would we ever need to use the 4th dimension), it can be defined by angles or by separating it into its component vectors.

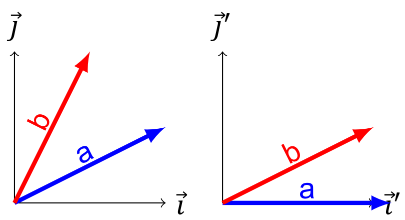
In much the same way that a graph has X, Y, & Z dimensions, a vector has $\vec{\imath}$, $\vec{\jmath}$, & $\vec{k}$ directions. These components can be used to define any vector. Imagine a right-angled triangle. The hypotenuse can be described by using the perpendicular sides, and Pythagoras' theorem. $H = \sqrt{a^2 + b^2}$. Similarly, the magnitude of a vector is found using $M = \sqrt{a\vec{\imath}^2 + b\vec{\jmath}^2 + c\vec{k}^2}$.



1.1.1. Reference Systems

When doing calculations with vectors, you can imagine a coordinate system in any way you want as long as it's consistent, because in all honesty, all of this is just convention, there's no reason for us to use $\vec{\imath}, \vec{\jmath}, \vec{k}$. It's just handy if everyone has the same understanding of how to describe a vector, but if you're only doing some calculations by yourself, it doesn't matter.

Imagine 2 vectors in 2D space. They can be in any orientation you like. To make your calculations easier, you can draw on an auxiliary coordinate system with respect to one of the vectors. That way one of the vectors will only have components in one



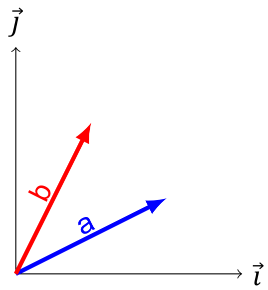
1.1.2. Unit Vectors

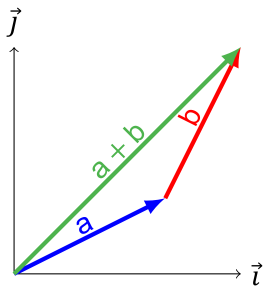
A Unit Vector is really very simple, it's a vector (in some direction) with a magnitude of 1 unit. Wow, how difficult was that? Okay so what's the big deal? Well, unit vectors are actually really useful especially in the Statics and Dynamics courses that you're doing alongside Calculus. You can define a vector (for instance a force vector) as being a scalar, multiplied by the unit vector going in that direction.

For instance, consider the vector $\vec{a} = \langle 3, 4\rangle$. Using Pythagoras you can find the magnitude of this vector is 5. To find the unit vector of $\vec{a}$, just divide the $\vec{\imath}$ and $\vec{\jmath}$ components by the magnitude of the vector. in this case; $\frac{3}{5}, \frac{4}{5}$.We can call this unit vector whatever we like, but it's common to just call it something like; unit vector $\hat{u}$.With this, we can now define $\vec{a} = 5\hat{u}$.

1.2. The Dot Product

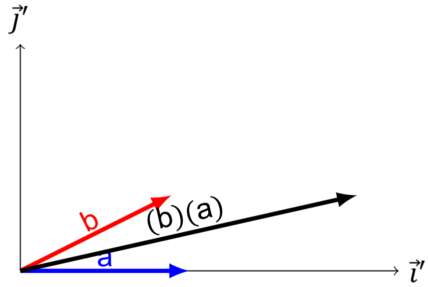
It's very easy to understand basic operations with scalars, or at least I hope it is, you're studying Aerospace Engineering. With vectors understanding addition is very simple. either, graphically or with the components. With components; $\vec{a} + \vec{b} = \langle a1\vec{\imath} + b1\vec{\imath}, a2\vec{\jmath} + b2\vec{\jmath}\rangle$. Or on a plane, just translate one vector from the point of origin to the tip of the other vector, then connect this back to the origin. Subtraction is done in much the same way, except translate the tip of one vector to the tip of the other, and connect the base of the vector back to the origin instead.





But when it comes to multiplying vectors its a different story, it's not so simple. There are actually 2 ways of multiplying vectors, and they give different answers, because of course they do. With scalars there's a bunch of ways of symbolising multiplication, so for vector multiplication we steal two of these symbols; $\cdot$ and $\times$. The dot product (unsurprisingly) is the product you get when you multiply two vectors with the dot symbol($\cdot$).

You can imagine the dot product is a way of saying "multiply $\vec{a}$ in the direction of $\vec{b}$" or vise versa, because the dot product is commutative. Imagine one of the vectors is on an axis, either $\vec{\imath}$ or $\vec{\jmath}$.



As you can see, it's $\vec{b}$ is extended out in the direction of $\vec{a}$, by the magnitude of $\vec{a}$. Remember how you can use a reference system and just make one of the vectors be on an auxiliary axis? Yeah, well this applies here too. You can find the dot product of any two vectors, regardless of if one is on an axis. The dot product, is not actually a vector though, it's a scalar. Why? Because some mathematicians thought it would be fun. So the actual value of $\vec{a} \cdot \vec{b}$ is the magnitude of this new funky vector. That's the idea behind the dot product, but you're not able to use this for any equations yet, so what are some mathematical ways of calculating it?

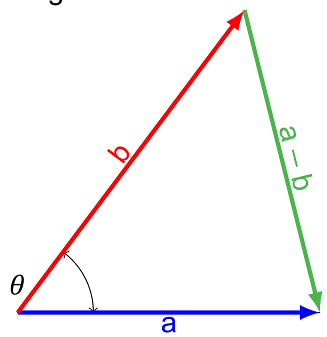
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \tag{1.1}$$

$$\vec{a} \cdot \vec{b} = (a1)(b1) + (a2)(b2) + (a3)(b3) \tag{1.2}$$

To be clear, θ is the angle between the two vectors, and a1, a2, a3 are the components of vector $\vec{a}$.

1.2.1. Geometric Interpretation of the Dot Product

It's useful to relate the dot product to the cosine law. And it's very important to have an understanding of this relation.



If we imagine this as a triangle we can use the cosine law:

$$|a-b|^2 = |a|^2 + |b|^2 - 2|a||b|cos\theta$$

If you remember, the magnitude of a vector squared is the same as the dot product of the vector multiplied by itself.

$$|a-b|^2 = (a-b)\cdot(a-b)$$
$$|a-b|^2 = (a\cdot a) - 2a\cdot b + (b\cdot b)$$
$$|a-b|^2 = |a|^2 + |b|^2 - 2a\cdot b$$

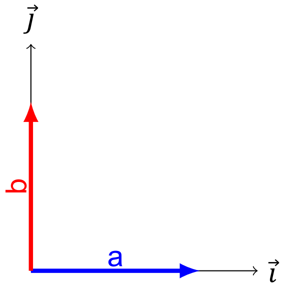
Combining these equations, for the magnitude of a vector due to a dot product, and the side of a triangle due to the cosine law, we find that

$$2a\cdot b = 2|a||b|cos\theta$$

$$a\cdot b = |a||b|cos\theta$$

1.2.2. Uses for the Dot Product

One thing I never really understood when doing all of this, is that it's not really explained as to what the Dot Product is exactly, or why it's useful. As it turns out, it's just another one of those intermediary things that mathematicians made up so that they can get to a useful end point. For us engineers though, it's mainly useful for calculating the angle between vectors, calculating something like work done, or finding out if two vectors are perpendicular to one another. If the dot product of two vectors is zero, that means they are perpendicular to one another. Just imagine, trying to multiply $\vec{b}$ in the direction of $\vec{a}$. It's not gonna work.

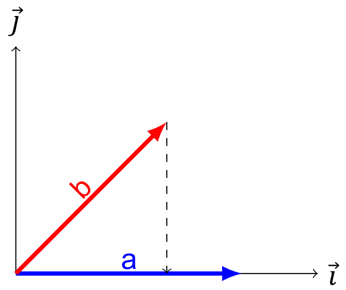


Now, to find the angle between two vectors we're combining equations (1.1) and (1.2) to make;

$$\theta = \arccos \frac{(a1)(b1) + (a2)(b2) + (a3)(b3)}{|\vec{a}||\vec{b}|} \tag{1.3}$$

Projection

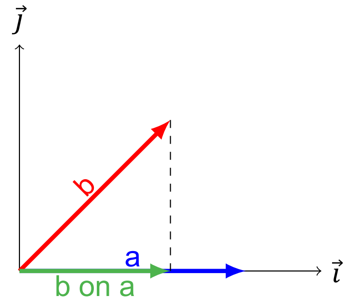
Another use for the dot product is projection. What's projection? Well, is basically like looking perpendicular to vector $\vec{a}$ and seeing what vector $\vec{b}$ looks like from this new viewpoint. Does that make sense? I don't know, look at this diagram:



Basically, we're trying to find how far along vector $\vec{a}$, vector $\vec{b}$ is. In the diagram you can see that there's a right angle triangle formed by $\vec{b}$, the dashed line perpendicular to $\vec{a}$, and $\vec{a}$ where $\vec{b}$ is the hypotenuse. If you remember anything from secondary school trig, you know that finding a side of a triangle with the hypotenuse is as easy as multiplying by the cosine of the angle in between. Or quite simply put: $Proj_a b = |b|cos\theta$ Now recall that $\vec{a} \cdot \vec{b} = |a||b|cos\theta$ and you can substitute this in to the previous equation giving you:

$$Proj_a b = \frac{|a||b|cos\theta}{|a|} = \frac{\vec{a} \cdot \vec{b}}{|a|} \tag{1.4}$$

This tells you the magnitude of $\vec{b}$ that goes in the direction of $\vec{a}$. But what if you want it as a vector and not just a magnitude?



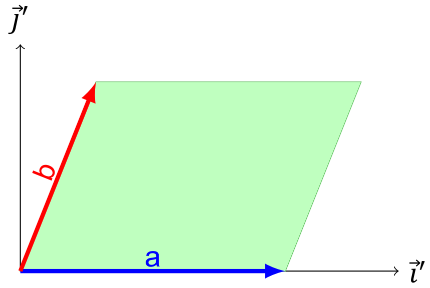
In that case you need to multiply the magnitude (which we already calculated in equation (1.4)), by a unit vector in the direction of $\vec{a}$. Which is just $\frac{\vec{a}}{|a|}$. This then gives us the orthogonal projection of $\vec{b}$ onto $\vec{a}$ or in mathematical terms:

$$Proj_a b = \frac{\vec{a} \cdot \vec{b}}{|a|} \frac{\vec{a}}{|a|} = \frac{\vec{a} \cdot \vec{b}}{|a|^2} \vec{a} \tag{1.5}$$

1.3. The Cross Product

The cross product is the other way of multiplying vectors. Unlike the dot product this actually does result in a vector, but not in a way that is intuitive at all, oh no, it makes a vector that is perpendicular to both vectors. Why? Because convention. To find the magnitude of the cross product, think of it the same way you find the area of a parallelogram. Multiply the base, by the perpendicular height, easy! Well, kinda, the two vectors don't define the perpendicular height and base, instead they define two sides of a parallelogram.

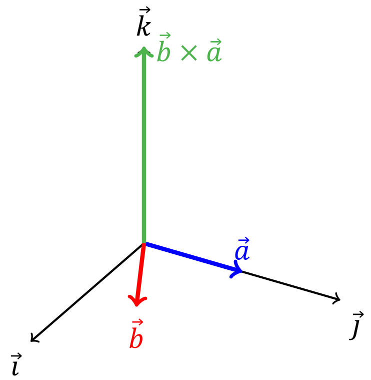
But we can deal with this, set one of the vectors to be base (remember, reference systems, we can manipulate the vectors however we want). Now to find the perpendicular height of the parallelogram we take the other vector, and find its $\vec{\jmath}'$ component. Which, if you imagine it as the hypotenuse of a right angle triangle is just the $\sin$ of the angle between the vectors (θ).



The area of this parallelogram is the just magnitude of the cross product, or if you'd like just remember this equation:

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \tag{1.6}$$

Remember, there's more to the cross product than just the magnitude, so now we need to find the direction of the resulting vector. It's perpendicular to the two vectors $\vec{a}$ and $\vec{b}$ but in which direction? Once again it comes back to convention. Going by the right hand rule - I hope for your own sake that you already understand that - imagine the three positive axes, assume $\vec{a}$ and $\vec{b}$ are on the X,Y plane, then the cross product will be in the Z direction. If the cross product is positive, it'll be in the positive direction, if it's negative, it'll be in the negative direction.

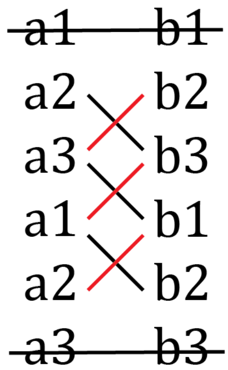


But now hold on here's something fun to keep in mind, the cross product is NOT commutative, so while the magnitude won't change, the direction will. Or in mathematical terms:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \tag{1.7}$$

The Amsterdam Method

There are a lot of ways of calculating the actual vector result of a cross product, but you only really need to know one. I use the Amsterdam method because it's wildly easier than anything else I've seen so far, but it goes like this: write out the components of the vectors, vertically, twice. Strike out the top and bottom lines. Draw in 3 crosses (these will show you what to multiply to get the new components).



Start with the first cross. Multiply the numbers, as shown by the cross, then subtract them as shown. This gives you the $\vec{\imath}$ component, then do the same for the other crosses to get the other components, giving you the result below. It looks complicated, but it's actually very simple, give it a go with some sample numbers yourself.

$$\vec{a} \times \vec{b} = \begin{bmatrix} (a2b3) - (a3b2) \\ (a3b1) - (a1b3) \\ (a1b2) - (a2b1) \end{bmatrix}$$

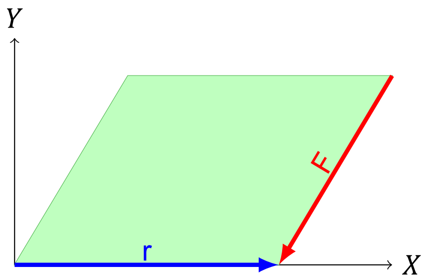
(1.8)

$$\langle 1, 2, 3 \rangle \times \langle 3, 2, 1 \rangle = \begin{bmatrix} (2 * 1) - (3 * 2) \\ (3 * 3) - (1 * 1) \\ (1 * 2) - (2 * 3) \end{bmatrix} = \begin{bmatrix} 2 - 6 \\ 9 - 1 \\ 2 - 6 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -4 \end{bmatrix}$$

1.3.1. Uses For the Cross Product

So, the cross product is also a lot like the dot product, in that it's completely made up. But unlike the dot product there are some more useful, and obvious uses for the cross product. For instance, calculating torque. As you might remember from secondary school, torque on something depends on the magnitude of force applied, and the distance between the point of application and the axis of rotation.

If you went to a school similar to mine, you were told that torque is equal to the force applied times the perpendicular distance to the axis of rotation. This is correct, but oftentimes you have to do trigonometry to find out what the perpendicular distance is. And no-one wants to do unnecessary trig. The simple solution is to use the cross product.



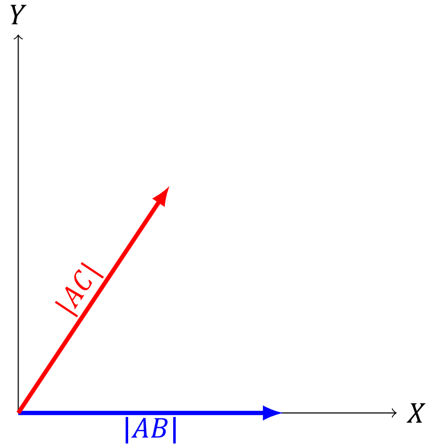
In this diagram ”r” is the radius of rotation and ”F” is the force applied. This is the convention used everywhere so get used to it. Imagine you're pushing a door open by the handle; ”r” is the distance from the handle to the hinges. The angle θ is the angle between your arm and the door, and in this case, is represented by ”F”. Intuitively, you know that it's easier to open a door by the handle than near the hinges. This clearly shows that more torque is made when you're farther from the axis of rotation, even with the same applied force.

To get a feel for how the angle impacts the torque generated, imagine the extreme cases. Pushing perpendicular to the door will be easy and will generate the most torque possible, and pushing in line with the door will have you looking like an idiot, as the door won't budge. Here you can see how the cross product plays into this, the magnitude of the cross product is the parallelogram show in the diagram above, when the vectors are perpendicular, the torque is maximised, and when they're aligned the torque is zero.

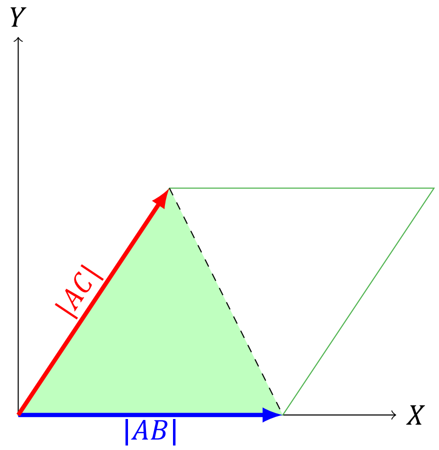
$$T = r \times F \tag{1.9}$$

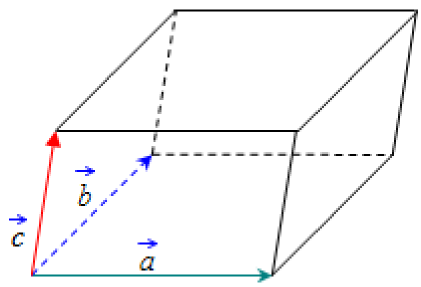
This shows us the magnitude of the torque, but isn't the cross product a vector? It is! As you remember, the cross product gives us a vector in either the positive or negative direction. If it's positive, the torque is anticlockwise, and if it's negative, then it's clockwise. Because of this, it's **very** important that you don't mix up the order of $r \times F$, if you write it the wrong way around, you'll have it rotating in the wrong direction. Consequently, torque is frequently represented by a vector in diagrams, rather than the usual curved arrow you're probably used to from secondary school.

Finding area of a Triangle:
See, sometimes you want to find the area of a triangle when given coordinates in 3D space. Why? well think of how computer models work, they're all just lots and lots of tiny triangles, building up a polyhedron. Anyway, if you have the coordinates of the 3 vertices the first step is to make two position vectors from them.



These vectors go from point A to points B and C. Obviously. If you get the magnitude of the cross product of these vectors you will get the area of a parallelogram spanned by these vectors. But if you have a basic understanding of geometry, you'll know that a parallelogram is just two triangles. So then you can just divide the area of the parallelogram in half to get the area of the given triangle.





Scalar Triple Product:

The Scalar Triple Product is a fun one. This describes the volume of a parallelepiped spanned by 3 vectors. What is a parallelepiped (aside from being hard to spell)? It's basically just a cuboid that has been squished a bit. The Cross product gives you the area of a parallelogram, and then the dot product gives you the area of the parallelogram multiplied up, in the direction of the other vector.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \tag{1.10}$$

You can also use the scalar triple product to find if 3 vectors are co-planar. Any 2 vectors will define a plane, but the question is if all 3 lie on the same plane. If the scalar triple product is equal to 0, then they are co planar. Imagine the volume of a parallelepiped where all edges were in the same plane. It would have a volume of 0.

If you want a more mathematical interpretation of it, recall that the cross product of two vectors is perpendicular to both of them, so $(b \times c)$ would be perpendicular to the plane defined by $\vec{b}$ and $\vec{c}$. And also recall that the dot product of two perpendicular vectors is 0. So if $\vec{a}$ is on this plane, it must be perpendicular to $(b \times c)$. And Thus; $a \cdot (b \times c) = 0$

1.4. Useful Things for the Exam

So I've gone over the fundamentals of vectors and the dot product, and the cross product, but there's a few more things you need to keep in mind before the exam. These are all fairly self explanatory but you mightn't think of them on the day of the exam.

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} \tag{1.11}$$

$$\vec{a} \cdot (\lambda \vec{b}) = (\lambda \vec{a}) \cdot \vec{b} = \lambda (\vec{a} \cdot \vec{b}) \tag{1.12}$$

Where λ is a scalar.

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2 \tag{1.13}$$

$$\lambda \vec{a} \times \vec{b} = \lambda (\vec{a} \times \vec{b}) = \vec{a} \times \lambda \vec{b} \tag{1.14}$$

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) \tag{1.15}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} \tag{1.16}$$

2

Functions

2.1. The Basics

So you may be asking, ”why the hell do we need to go over the basics of functions in Uni?”. Well it’s because you probably have a grasp and understanding of what a function is, but it’s not clear and precise enough for what we’ll be doing. A function is something that maps something onto one other thing. That’s a very basic sentence, but it’s important not to get locked into thinking that a function is just a mathematical expression with a variable in it. That’s a function for sure, and it’s what we encounter most often, but that’s not all functions.

You can have functions that exist in multiple segments, for instance:

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 3 \\ x^{1.5} & \text{for } x < 3 \end{cases}$$

This is very clearly a function. Because it fulfils the requirements of mapping values onto new values. Note how it only maps **onto one value** for each input. A function only has one output for each input. So $y = \pm x^2$ is absolutely not a function. Because it maps from one input to two outputs. However, this doesn’t mean that two inputs can’t map to the same output. Like $y = x^2$ here we see that $-x$ and x both result in the same output.

It’s important to know the terminology for talking about functions because you’ll hear people talk about functions and not understand what they’re telling you, and you’ll look stupid in front of all of your friends. Which would be awful.

-Domain: This is the set of values that are your inputs. In a mathematical function, you can set this to be whatever you want, Reals, Integers, whatever.
-Co-Domain: This is the set of all *possible* outcomes.
-Range: This however, is the set of all of the *actual* outcomes of a function.

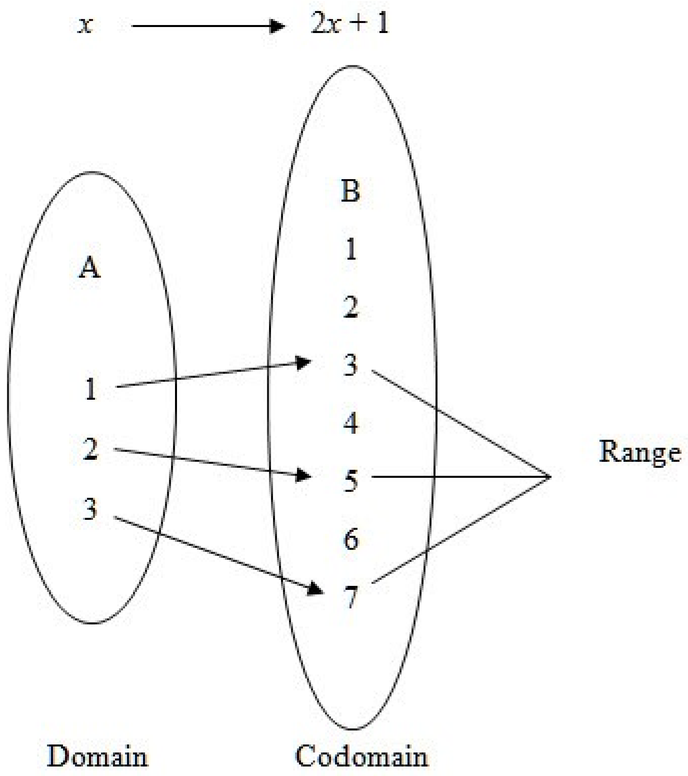


Figure 2.1: *Domain, Range, Codomain*

2.2. Bijective Functions

2.2.1. Injective Functions

Here we're going to look at a specific type of function, known as an injective function. In this case, all of the values in the range only have one value mapped **to** it. This is the case for the function seen above, and for all linear functions, but it's not true for $y = x^2$. Because take for example 4; this is in the range, but it is mapped to by both -2 & 2. You can visualise these by imagining a graph of the full function, and drawing a horizontal line at a random point. if you can draw a horizontal line somewhere, *anywhere* on the graph that cuts the function at two points, then it's not injective.

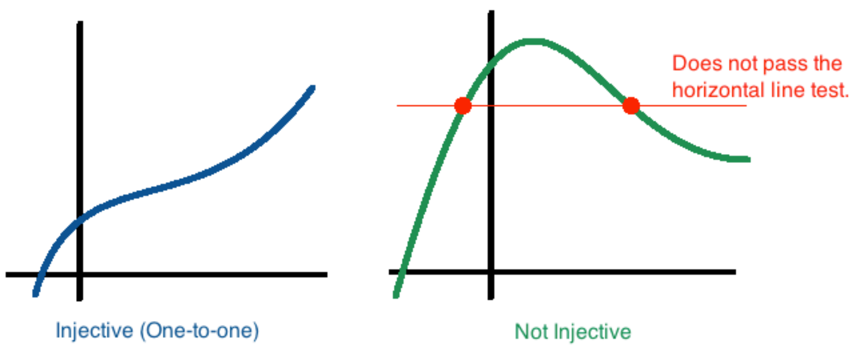


Figure 2.2: *The Horizontal test of Injectivity*

2.2.2. Surjective Functions

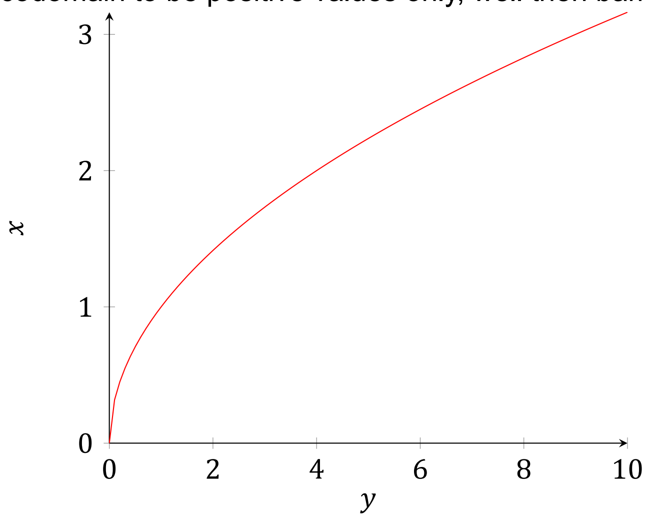
Surjective and injective functions are not mutually exclusive. Finding out if a function is injective or not tells you nothing about its surjectivity. A surjective function is one where the range and the co-domain are the same. That is, every possible outcome, has an input mapped to it. To do the Horizontal test for surjectivity, make sure that you can draw a horizontal line anywhere on the graph and have it cut the function *at least* once.

A Bijective function is both injective and surjective.

2.3. Inverse Functions

Inverse functions are pretty self explanatory, instead of having y in terms of x you just flip the equation and have x in therms of y. Really quite easy to understand. You can even imagine an inverse function as being a mirror around the line $y = x$ The only thing you need to keep in mind is that you can only make an inverse function of a bijective function.

Think about it. Imagine trying to get the inverse of a function that isn't injective. You end up with something with that gives two outputs from the same input. And that wouldn't be a function, remember, unique outputs only. You can't have an inverse function of $y = x^2$ because the inverse of that would be $x = \pm\sqrt{y}$. You can however apply a window or a frame to the graph (I don't actually know what the correct terminology is but, shush) If you limit the codomain to be positive values only, well then bam! You've got $x = \sqrt{y}$ and you're all good.



Now imagine you're trying to make an inverse function of something that isn't surjective. This one is very easy to understand. if the codomain has something that isn't mapped to it, then the inverse of the function will have a value in the domain, that isn't mapped to anywhere. And a functions *only* job is to map all the values in the domain to other values.

2.4. Useful Things for the Exam

It's important to know that for inverse trigonometric functions, we also apply a domain to the functions. That's because if you look at the functions over the domain (-∞,∞) they won't be bijective. To combat this we have the domains:

$$y = \arcsin(x) \Leftrightarrow \sin(y) = x$$
with $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

$$y = \arccos(x) \Leftrightarrow \cos(y) = x$$
with $0 \leq y \leq \pi$

$$y = \arctan(x) \Leftrightarrow \tan(y) = x$$
with $\frac{-\pi}{2} < y < \frac{\pi}{2}$

Note: Pay attention to the signs. tan is undefined for $\frac{\pi}{2}$ etc.

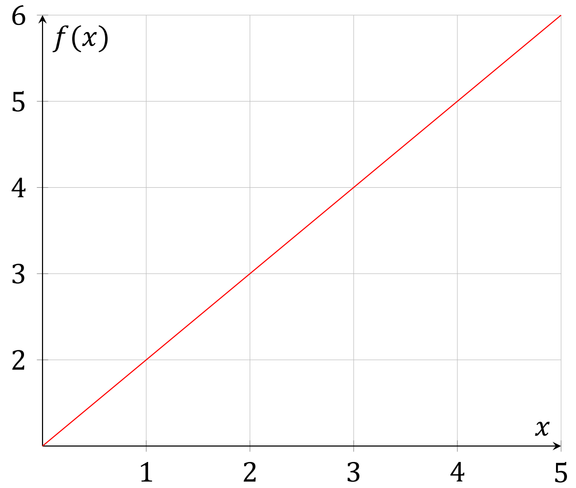
3
Limits

3.1. The Basics

3.1.1. What is a Limit?

Good question. It's basically trying to figure out what $f(x)$ is at a based on the trend of the function as you get close to a. Why? It's because sometimes you can't actually find a defined value for a. This is what we call "undefined", but limits can be applied to any function, take for example $f(x) = x + 1$ This is a really simple linear function, for which no standard input gives an undefined output.

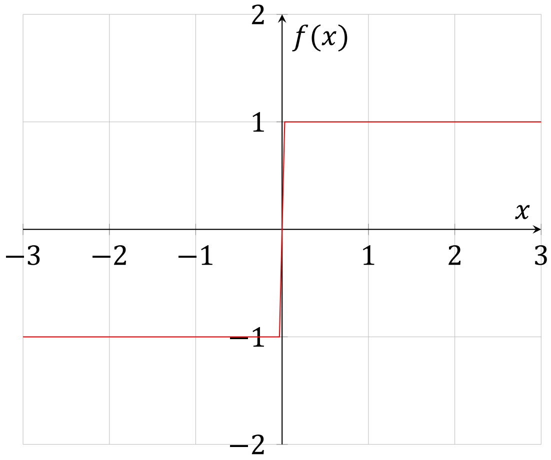
What if we try to answer the question: "What is the limit of $f(x)$ as x approaches 1". A good way to get an intuitive understanding of limits is with the use of a graph. Looking at the graph below, You can clearly see, that as x nears 1, $f(x)$ is getting closer and closer to 2. This is true from both directions. Thus we can say $\lim_{x\to 1} f(x) = 2$



3.1.2. Directional Limits

In the previous section we saw how we find a limit as x *approaches* a. But what do we mean by "approaches" anyway? Does that mean we get close to a from the negative side or the positive side? It's actually both. See, not all functions are continuous, so a function only has a limit if $f(x)$ approaches the same value from both sides of a.

Take for example the function $f(x) = \frac{x}{|x|}$. This is a perfect opportunity for a limit, because at $x = 0$, $f(x)$ is undefined.



If we start approaching $x = 0$ from the negative side (the left side), it seems pretty obvious that the limit is going to be $\lim_{x\to 0} f(x) = -1$. What about when we approach from the positive side (the right side)? Well in that case we end up with the obvious case that $\lim_{x\to 0} f(x) = 1$. I hope I don't need to explain that doesn't make sense because $-1 \neq 1$. Clearly in this case the limit does not exist.

In cases like this we can specify limits of a function from a specific direction. For example, if we want the limit as we approach $x = 0$ from the negative side we'd write: $\lim_{x\to 0^-} f(x) = -1$ and similarly for the positive side: $\lim_{x\to 0^+} f(x) = 1$.

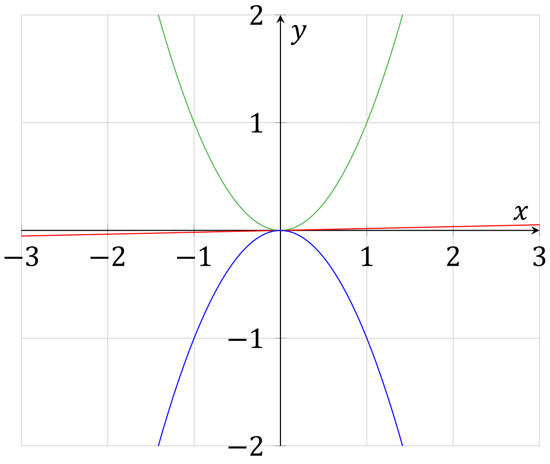
Only when these two limits are equal can we say that $\lim_{x\to a}$ exists.

3.2. The Squeeze Theorem

Don't you love it when mathematicians name things something funny? In the squeeze theorem we imagine that $f(x)$ -the function we're trying to find the limit of- is being squeezed between two limits $g(x), h(x)$. We need to make certain that $g(x)$ is **always** less than or equal to the value of $f(x)$, and that $h(x)$ is **always** greater than or equal to the value of $f(x)$. Such that:

$$g(x) \leq f(x) \leq h(x)$$

for the relevant interval.



This is an extremely simple example, and there's no way you'd use the squeeze theorem for this in real life, but it explains the concepts simply. The limits of the two polynomial functions are the same at $x = 0$, and because $f(x)$ is always between these two functions (over the relevant interval), the limit of $f(x)$ must also be the same.

$$\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$$

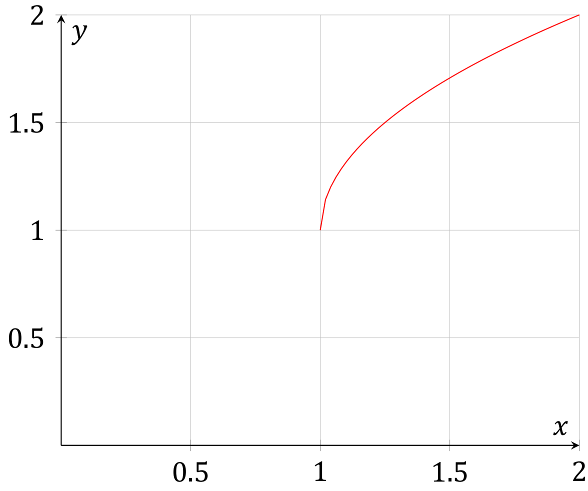
$$\therefore \lim_{x \to a} f(x) = L$$

This also applies for limits at infinity! Just find two functions that trend towards a single value for $x \to \infty$, and then you'll get a limit (if there is one) for the function you're analysing.

3.3. Continuity

It's very important that you understand continuity. We say that a function is continuous at $x = a$ if the limit is equal to $f(a)$. Think about that. This lets us know that there is no "break" in the function at a. A limit is what we expect the output of $f(a)$ to be, and if these are equal, then the function acts as we expect. This is continuity at a point. We can also say that a function is continuous from left, if $\lim_{x \to a^-} f(x) = f(a)$. The same can be said for continuity from the right. Continuity over an interval is just that, the function is continuous at every point over a specified interval.

Something that was covered in class but not fully explain was the idea that a function can be continuous even if it's undefined for a part of the domain. That's seems counter intuitive, but it makes sense when you look at a graph.



In the graph you can see that, over the domain of the graph, it's got a definite output. just once you go below $x = 1$ the function doesn't make sense. So it's a continuous function, it doesn't't *break* anywhere along the graph, it just has a lower bound.

3.4. L'Hospital

L'Hospital's rule is a simple way of getting a limit of a function that is composed of two other functions. Or written in mathematical terms:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{3.1}$$

This is useful when the fraction is undefined, be that because it goes to $\frac{0}{0}$ or . It's just another tool to add to your toolbox. The reason it works is that as you zoom in to a point, you can linearise the function and it will be close enough to the real thing. And if you recall, a differentiation is taking a tangent at a point of a function. But what about the constants? Don't you lose them when you differentiate? Yes, yes you do, however l'Hospital's rule only works when $f(x)$ and $g(x)$ are converging at the same point, so the ratio of the slopes is all that matters. There are a number of proofs for this, but all you really need to know is that we differentiate to get an approximation of a function at a point. That's basically all there is to it, at least that's all there is to it in this course.

3.4.1. Using e as a limit

Euler's number e is defined as: $y = e^x$ having a slope of 1 at (0,1), thus:

$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} \tag{3.2}$$

This is handy and we can substitute this into a lot of equations to find the limits in terms of e. Also recall that $w = \frac{1}{x}$ for $x \to \Leftrightarrow w \to 0$, for instance:

$$e = \lim_{w\to}(1+\frac{1}{w})^w$$

or

$$e^2 = \lim_{x\to 0}(1+2x)^{\frac{1}{x}}$$

3.5. Useful Things for the Exam

There are a hand full of ways to find the limit of a function but some useful techniques include:

Factorising, eg:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

$$\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}$$

$$\lim_{x\to 2}\frac{x+2}{1}$$

Rationalising, eg:

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{x}$$

$$\lim_{x\to 0}\frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2}{)}x(\sqrt{4+x}+2)$$

$$\lim_{x\to 0}\frac{4+x-4}{x(\sqrt{4+x}+2)}$$

$$\lim_{x\to 0}\frac{x}{x(\sqrt{4+x}+2)}$$

$$\lim_{x\to 0}\frac{1}{\sqrt{4+x}+2}$$

4

Differentiation

4.1. The Basics

For the purposes of this summary, I'm going to assume you have a reasonable understanding of secondary school calculus. When we differentiate we're trying to find the rate of change of something, in a given instant. Which can be represented as the slope of a function at any given point. Which is the slope of a line tangential to the function at a given point. there are many ways to visualise it, but keep the core ideas in your head. We differentiate to find the rate of change of something in a given instant.

Over the course of this subject we'll look at a few funky ways of differentiating. The chain rule, implicit differentiation, and differentials of inverse trigonometric functions. Let's start at the beginning with the chain rule.

4.2. The Chain Rule

Take the function $F(x)$ which is a composition of both $f(x)$ and $g(x)$ such that $F(x) = f(g(x))$. If both f and g are differentiable at x then we can say F is also differentiable at x.

$$F'(x) = f'(g(x)) * g'(x) \tag{4.1}$$

We can also look at a different notation to get a different understanding of how this works. Good oul' Leibniz's got our back here. If we call $y = f(u)$ and we call $u = g(x)$. Then we can show this:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Think of how we're substituting in a different variable (u) to make differentiating the function f a standard differentiation. Then we just multiply by how u changes with respect to x. This will come back to be useful when we get to integration.

What does it mean to be differentiable? A function is differentiable when you can actually find a single value for an input when you differentiate. Think back to our limits. A function only has a limit when the limits from both sides of a point are equal. So, When we differentiate, we are finding the slope of the function of a distance Δx, With this value being as small as possible. So, for a function to be differentiable:

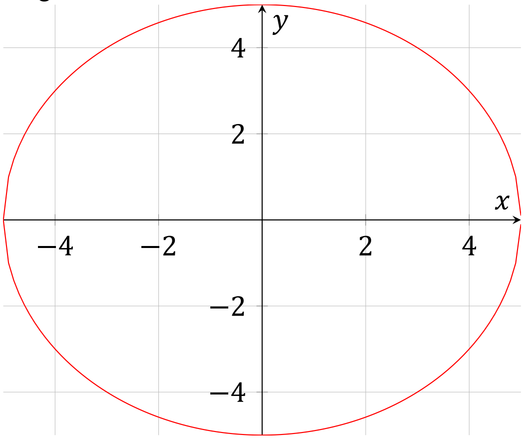
$$\lim_{\Delta x \to 0^-} = \lim_{\Delta x \to 0^+}$$

Thus:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x\Delta x) - f(x)}{\Delta x} \tag{4.2}$$

4.3. Implicit Differentiation

The next topic we look at in this course is Implicit differentiation. Which is when we differentiate an implicit function. But what *is* an implicit function. An explicit function defines a strict relationship between x and y (or whatever other variables you're using), whereas an implicit function just *implies* a relation between the two. The most obvious example that you probably know of is a circle. It's defined as $x^2 + y^2 = r^2$ you can also put in the coordinates for the location of the circle but that doesn't matter here. We're only trying to find the rate of change, nothing more.



When we differentiate we're seeing how y changes with respect to x, hence the notation $\frac{dy}{dx}$. So using the logic we just got from the chain rule, we can pretend that y is a function of x, and the use the chain rule, to allow us to differentiate with respect to y, and then multiply by $\frac{dy}{dx}$. For example:

$$\frac{d}{dx}\, x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Essentially when we differentiate implicitly we have to keep in mind that y changes with respect to x, and then we apply the chain rule. Beyond that, it's as simple as the example above. But, it's not like a normal function where the slope of the tangent line is given as a function of just x, it's usually given as a function of x and y. So you need to find the tangent at a specific coordinate.

4.4. Inverse Trigonometric Differentials

In this section we're going to take a look at the differentials of inverse trig functions. There aren't any new concepts to understand here, it's just good to see how they're derived.

Differentiating arcsin(x) Let's start by assuming $y = \arcsin x$ Therefore $x = \sin y$. We can use the chain rule to get an expression for this:

$$\frac{d}{dx}x = \frac{d}{dx}sin(y)$$

On the left hand side, we're differentiating x which is a simple operation, but on the right, we need to employ some implicit differentiation.

$$1 = \cos y \frac{dy}{dx}$$

$$\frac{1}{\cos y} = \frac{dy}{dx}$$

This isn't very useful though, because we want the derivative in terms of x not y. If we can get $\cos y$ in terms of $\sin y$ then we can just substitute x back in. Looking at our trigonometric identities we know that $\cos^2 y + \sin^2 y = 1$. Thus:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Differentiating arccos(x) This is done in much the same way:

$$y = \arccos x \therefore x = \cos y$$

$$\frac{d}{dx}x = \frac{d}{dx}\cos y$$

$$1 = -\sin y \frac{dy}{dx}$$

$$\frac{-1}{\sin y} = \frac{dy}{dx}$$

$$\frac{-1}{\sqrt{1 - \cos^2 y}}$$

$$\frac{-1}{\sqrt{1 - x^2}}$$

Differentiating arctan(x) We begin, as usual, by defining $x = \tan y$

$$\frac{d}{dx}x = \frac{d}{dx}\tan y$$

$$1 = \frac{1}{\cos^2 y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \cos^2 y$$

Defining $\cos^2 y$ in terms of $\tan y$ is a bit tricky, But lets divide by $1 = \cos^2 y + \sin^2 y$

$$\frac{dy}{dx} = \frac{\cos^2 y}{\cos^2 y + \sin^2 y}$$

Cancel out $\cos^2 y$

$$\frac{dy}{dx} = \frac{1}{1 + \frac{\sin^2 y}{\cos^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

4.4.1. Useful things to remember

The derivations above are important to understand, but you don't need to remember them line for line in the exam. Instead you should be familiar with them, and instead memorise these relations below:

$$\frac{d}{dx}\arcsin\frac{x}{a} = \frac{x'}{\sqrt{a^2 - x^2}} \tag{4.3}$$

$$\frac{d}{dx}\arccos\frac{x}{a} = \frac{-x'}{\sqrt{a^2 - x^2}} \tag{4.4}$$

$$\frac{d}{dx}\arctan\frac{x}{a} = \frac{ax'}{a^2 + x^2} \tag{4.5}$$

4.5. Linear Approximations

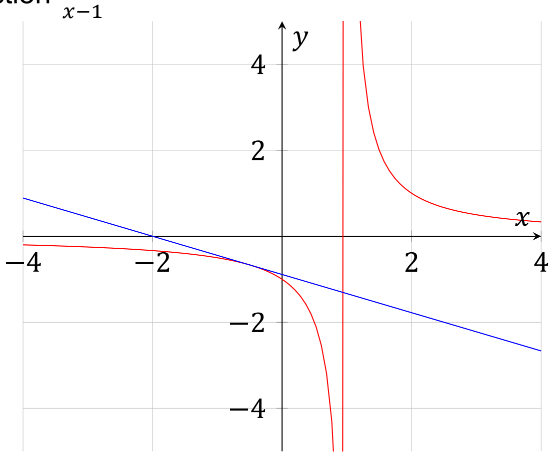
In any function you can pick a point and draw a tangent line at that point. This line will have a simple linear function in the form $y = mx + c$ If you're dealing with values close to this point, the linear function will result in a value close enough to what the real function is. Thus we can approximate a function with a "Linear Approximation"

$$L(x) = f(a) + f'(a)(x - a) \tag{4.6}$$

Here we can see the components and their roles;

- $f(a)$ gives us the point where we draw the tangent
- $f'(a)$ gives us the slope of the tangent
- $(x - a)$ gives us the distance from the tangent point.

We combine these as shown because the at a we have the correct point, then as we move along the line we just multiply by the slope, as with any linear function. Take for example the function $\frac{1}{x-1}$



See in the above graph I've drawn a line tangential to the function. This is The tangent line $L(x)$, at point $x = -0.5$. We can use this to approximate the value of $f(x)$ in a region close to $x = -0.5$.

4.5.1. Applications

You'll find many uses for linearisation over the course of your career, but for this course in particular you should know how to use linearisation to find the approximate square root of a number, and the approximate decimalisation of a fraction. For the sake of simplicity, I'm just going to use two examples:

Square roots For example, if we want to find the approximate value of $\sqrt{63.6}$. Firstly we define the function $f(x) = \sqrt{x}$, then we can find recall equation 4.6 to get a linearisation at $x = 64$

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = \sqrt{a} + \frac{1}{2\sqrt{a}}(x - a)$$

$$L(x) = \sqrt{64} + \frac{1}{2\sqrt{64}}(x - 64)$$

And because we're trying to approximate $x = 63.6$, we can just substitute that in here to get:

$$L(x) = \sqrt{64} + \frac{1}{2\sqrt{64}}(63.6 - 64) = 7.974$$

Decimals Getting decimals for weird fractions is difficult to do without a calculator but if we turn it into a function we can get there. Say we want to write $\frac{1}{101}$ in decimals, well, we can do that in a similar process. I'm going to define $f(x) = 1/x$ and then work from there:

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(100) + f'(100)(x - 100)$$

$$L(x) = 0.01 + \frac{-1}{100^2}(x - 100)$$

And we're finding the approximation for $x = 101$ so we can substitute in x.

$$L(x) = 0.01 - 0.0001(101 - 100) = 0.0099$$

4.6. Differentials

A differential is, the change of something, and the rate of change of one thing, with respect to another is what we find when we differentiate. We find, for instance, $\frac{dy}{dx}$, dy or dx in this case, is a differential. For example, Find the differential of y where:

$$y = x^2$$

A simple equation yes, and differentiating it is also simple. But we don't want the slope of the tangent, or the rate of change, we want the differential of y.

$$\frac{dy}{dx} = 2x$$

$$dy = 2x * dx$$

We can use this for finding a discrete change in something. For example, (this example taken from the lectures) what if we want to find the volume increase in a sphere, needed to make the radius change by 0.1cm. When the starting radius is 10cm. Well, we know that $V = \frac{4}{3}\pi r^3$, and we know $dr = 0.1$.We can differentiate to get:

$$\frac{dV}{dr} = 4\pi r^2$$

$$dV = 4\pi r^2 dr$$

Thus, we can now substitute in the known values to find the change in volume (dV) needed.

$$dV = 4\pi(10)^2(0.1) = 40\pi$$

. This same logic can be applied across a number of problems, such as finding the error in certain things. It's good to remember that the relative error in something is $\frac{dx}{x}$, or whatever variable you're using.

5

Hyperbolic Functions

5.1. The Basics

Hyperbolic functions are similar to trigonometric functions in that you can use them to trace out a shape. With trigonometric functions you can trace out a circle with a Pythagorean relationship:

$$\cos^2 x + \sin^2 x = 1 \tag{5.1}$$

This defines the unit circle. They hyperbolic functions trace out the shape of a hyperbola with the following relationship:

$$\cosh^2 1 - \sinh^2 x = 1 \tag{5.2}$$

They are defined as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{5.3}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{5.4}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \tag{5.5}$$

Useful rules to keep in mind:

- $\sinh x = \sinh -x$
- $\cosh x = \cosh -x$
- $\cosh^2 1 - \sinh^2 x = 1$
- $\sinh x + y = \sinh x \cosh x + \sinh y \cosh y$
- $\cosh x + y = \sinh x \sinh y + \cosh x \cosh y$
- $\frac{d}{dx} \cosh x = \sinh x$

You can prove these relations by writing them out the long way and doing some algebra, but just. Keep this in mind. Also pay attention to the inverse hyperbolic equations, and their derivatives.

- $arc \sinh x = \ln (x + \sqrt{x^2 + 1})$
- $arc \cosh x = \ln (x + \sqrt{x^2 - 1})$
- $arc \tanh x = \frac{1}{2} \ln \frac{1+x}{1-x}$

6 Integration

6.1. The Basics

6.1.1. Riemann Sum

There are multiple ways of interpreting integration, but in this course we will go with the geometric interpretation. That is, that the integration of a function is equal to the area enclosed by the graph of the function and the x axis. Let's calculate this with what we call a **Riemann Sum**.

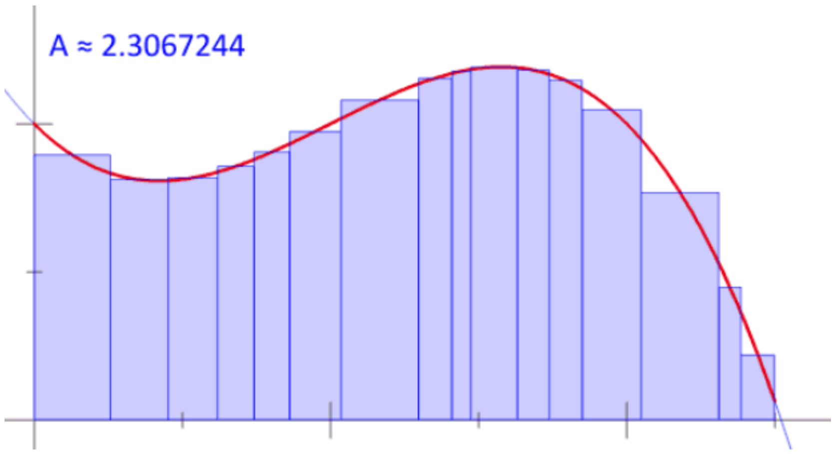


Figure 6.1: *A Riemann Sum of a Function*

What we see in the figure above, is the approximate area "under the graph" we find this by setting boundaries on the domain of the graph $[a, b]$. We then divide this into small sub elements. In each sub interval we pick a point x_i^*. We use *these* points to evaluate $y = f(x_i^*)$, this gives us the height of the rectangles. The area of which is equal to the width by the height; $(x_{i+1}^* - x_i^*)(f(x_i^*))$. And so the integral is roughly:

$$\Sigma(f(x_i^*))(x_{i+1}^* - x_i^*)$$

This isn't totally accurate, of course, so to make it *more* accurate we reduce the with of the rectangles and "increase the resolution", ∴ make $(x_{i+1}^* - x_i^*)$ smaller.

$$(x_{i+1}^* - x_i^*) = \Delta x$$

If we say that we have n sub intervals, we can say:

$$\frac{b-a}{n} = \Delta x$$

$$\therefore A = \sum_{i=0}^{n} f(x_i^*)\Delta x$$

If we want to get as precise as we possible can, we must take a limit of Δx going to 0.

$$\lim_{\Delta x \to 0} \sum_{i=0}^{n} f(x_i^*)\Delta x$$

$$= \int_a^b f(x)dx$$

We say that an integral is the ”area under the graph” but that’s not totally correct. It’s the area enclosed by the function and the x axis. If the function becomes negative, this then results in a ”negative area”, so the integral the positive area enclosed by the graph, plus the negative area. but the area enclosed, is the positive area, minus the negative area.

6.1.2. The Fundamental Theorem of Calculus

The fundamental theorem of calculus comes in two parts. The first part is:

$$\frac{d}{dx}\int_a^x f(x)dx = f(x) \tag{6.1}$$

The second part is:

$$\int_a^b f(x)dx = F(b) - F(a) \tag{6.2}$$

Let’s look at these in more detail, The first part (6.1) says that the integral of a function, and the differential of a function are essentially opposite. Keep in mind that when you integrate you get a constant of integration C, but differentiating a constant gives you 0 so you’re fine. Note this only holds true for continuous functions over an interval.

The second part requires us to define $F(x)$, this is what we call the *primitive function* or more commonly, the *Anti-derivative*. Where

$$\int f(x)dx = F(x) + C$$

, and

$$\frac{d}{dx}F(x) = f(x)$$

. $F(a)$ then, is the area under the graph, up to the point $x = a$, and so on. It only makes sense then that the area between point a and b is the difference between $F(b)$ and $F(a)$. If we have an indefinite integral, where we don’t have boundaries, we then just add a constant of integration ”C” to account for the unknown area.

6.2. U Substitution

There are a number of tricks to help integrate, usually based on common differentiation rules. First and foremost is the *substitution rule*, or as I have been calling it for years; *U substitution*. This comes from the the chain rule, but backwards. Recall the chain rule:

$$\frac{d}{dx}F(g(x))$$
$$= F'(g(x))g'(x)$$
$$= f(g(x))g'(x)$$

What you can do if you find a function like this is write it out, identify a function of x and it's derivative, and substitute for u. For example:

$$\int f(g(x))g'(x)dx$$
$$u = g(x) \qquad \frac{du}{dx} = g'(x)$$
$$= \int f(u)\frac{du}{dx}dx$$
$$= \int f(u)du$$
$$= F(u) = F(g(x))$$

Essentially, that's it. That's all there is to U substitution. but there are few things to deal with if you want to use this often. You can manipulate the equation to put $\frac{du}{dx}$ into it, that could include things like, multiplying by 1, in the form of $\frac{z}{z}$, just so long as you don't change the value of anything.

Note that when you're integrating with limits, you must change the values as well. This is because the limits are "integrate between $x = a$ and $x = b$", so now that we're integrating with respect to u, we must change the limits to be when u is c and d. As u is a function of x, just change the limits to $u(a)$ and $u(b)$. Then integrate as normal.

This is all simple in theory, but you'll need to do a lot of practice to be able to recognise when to use it. so to give you an example I'll do an integration:

$$\int x^2(x+1)^{20}dx$$
$$u = (x+1) \qquad \frac{du}{dx} = 1 \qquad x^2 = (u-1)^2$$
$$= \int (u-1)^2u^{20}du$$
$$= \int (u^2+2u+1)u^{20}du$$
$$= \int u^{22} + 2u^{21} + u^{20}du$$
$$= \frac{1}{23}u^{23} + \frac{1}{22}2u^{22} + \frac{1}{21}u^{21} + C$$
$$= \frac{1}{23}(x+1)^{23} + \frac{1}{11}(x+1)^{22} + \frac{1}{21}(x+1)^{21} + C$$

6.3. Integration by Parts

If U substitution is just the chain rule, but backwards, then integration by parts is just the product rule backwards. It's just yet another trick that we can use to make out equations easier to work with. Recall the product rule:

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x) \tag{6.3}$$

As we know, integration is directly related to integration, so we want to use this to help us integrate an equation. The first step is to integrate both sides

$$\int (\frac{d}{dx}f(x)g(x))dx = \int f'(x)g(x) + f(x)g'(x)dx$$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\int f(x)g'(x)dx \ = \ f(x)g(x) - \int f'(x)g(x)dx \tag{6.4}$$

This is useful in cases where $f'(x)g(x)$ is an easier expression to integrate, for example:

$$\int xe^{4x}dx$$

Note, that sometimes the notation is slightly different, and it's written like $\int fdg = fg - \int gdf$ but that's just the same, it's just a shorthand for $\frac{dg}{dx} = g' \Rightarrow dg = g'dx$

$$\begin{array}{ll} f(x) = x & f'(x) = 1 \\ g'(x) = e^{4x} & g(x) = \frac{e^{4x}}{4} \end{array}$$

$$\therefore \int xe^{4x}dx = \frac{xe^{4x}}{4} - \int \frac{e^{4x}}{4}dx$$

That's essentially it, that's the basic principles behind this, but when is it useful? Quite often actually, because you can use integration by parts multiple times per equation. For example:
1) Reducing the order of polynomials:

$$\int x^2e^xdx$$

$$\begin{array}{ll} f(x) = x^2 & f'(x) = 2x \\ g'(x) = e^x & g(x) = e^x \end{array}$$

$$= x^2e^x - \int 2xe^xdx$$

$$= x^2e^x - (2xe^xdx - \int 2e^xdx)$$

2) We can also take advantage of logarithms

$$\int x^4 \ln(x) dx$$

$$f(x) = \ln(x) \quad f'(x) = \frac{1}{x}$$
$$g'(x) = x^4 \quad g(x) = \frac{x^5}{5}$$

$$= \frac{x^5 \ln(x)}{5} - \int \frac{x^5}{5x} dx$$

3) Trig functions and their cyclic nature can be useful too:

$$(\cos x)'' = (-\sin x)' = -\cos x$$

$$\int \cos^3 x dx$$
$$= \int \cos^2 x \cos x dx$$

$$f(x) = \cos^2 x \quad f'(x) = -2 \sin x \cos x$$
$$g'(x) = \cos x \quad g(x) = \sin x$$

$$= \cos^2 x \sin x - \int -2 \cos x \sin x \sin x dx$$
$$= \cos^2 x \sin x + 2 \int \cos x (1 - \cos^2 x) dx$$
$$= \cos^2 x \sin x + 2 \int \cos x dx - 2 \int \cos^3 x dx$$
$$\int \cos^3 x = \cos^2 x \sin x + 2 \int \cos x dx - 2 \int \cos^3 x dx$$
$$3 \int \cos^3 x = \cos^2 x \sin x + 2 \int \cos x dx$$
$$\int \cos^3 x = \frac{1}{3} (\cos^2 x \sin x + 2 \int \cos x dx)$$

4) We're also able to make our own composite function to make our lives easier, just by multiplying by 1. Because $\frac{d}{dx}x = 1$.

$$\int \arctan x dx$$

$$\int 1 \cdot \arctan x dx$$

$$\begin{array}{ll} f(x) = \arctan x & f'(x) = \frac{1}{1+x^2} \\ g'(x) = 1 & g(x) = x \end{array}$$

$$= x \arctan x - \int x \frac{1}{1+x^2} dx$$

$$= x \arctan x - \int x \frac{1}{1+x^2} dx$$

Notably with integrals, you can manipulate the equation with constants, as long as you divide by the same constant outside of the integral, so the net impact is just multiplying by 1.

$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$

$$= x \arctan x - \frac{1}{2} \ln(1+x^2)$$

$$= x \arctan x - \ln(1+x^2)^{\frac{1}{2}}$$

$$= x \arctan x - \ln \sqrt{1+x^2}$$

6.4. Improper Integrals

An improper integral is not to be confused with an indefinite integral. It is instead an integral with at least one undefined boundary, or it has one point on the function that is undefined. The distinction here is why we have two types of improper integrals.

Type 1 integrals have at least one undefined boundary on the domain:

$$\int_a^b f(x)dx$$

Where either, $a = -\infty$, $b = \infty$ or both!

Type 2 integrals are integrals that are not continuous over the specified interval $[a, b]$. For example:

$$\int_{-1}^{1} \frac{1}{x^2} dx$$

6.4.1. Type 1

Type 1 integrals are generally easier to work with compared to Type 2. Essentially what we're doing is integrating between point a and ∞ (or we could go from $-\infty$, but let's not worry about that for now). Because ∞ is not a number, we can't just put it in and integrate to it. Instead we substitute it for a variable b and integrate to there instead. But because we want to get as close as possibly to $b = \infty$ we use limits.

Take for example the improper integral :

$$\int_1^{\infty} \frac{1}{x^2} dx$$

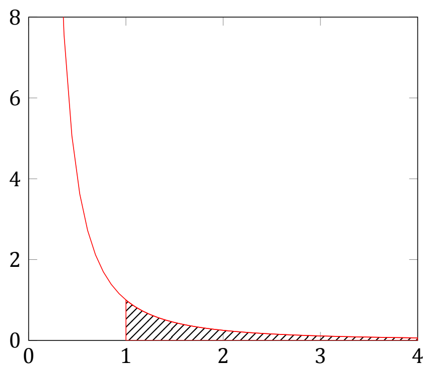
if we integrate this we get:

$$[\frac{-1}{x}]_1^{\infty}$$

But this isn't helpful, so the next step is to substitute in the variable b and find the limit.

$$\lim_{b\to\infty}(\frac{-1}{b} - \frac{-1}{1})$$

$$= 0 + 1$$



The same logic applies if you're dealing with a lower boundary of $-\infty$. The slight twist comes when both boundaries are undefined, so integrating over the interval $[-\infty, \infty]$. In this case what you have to do is pick a value c where $a < c < b$ and then separate the integral into these two sections:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

From here we apply the same process of replacing ∞ with a variable a or b, then taking a limit as they approach ∞.

Convergent and Divergent Type 1

A one sided improper type 1 integral is called **convergent** if the corresponding limit exists, and if it doesn't, we call it **Divergent**. For example:

$$\int_{0}^{\infty} \sin x dx$$

$$= \lim_{b\to\infty}[-\cos x]_0^b$$

This will **diverge**, because as you keep increasing x you wont get any closer to a specific value of y, the cyclic nature prevents us from getting a proper limit.

$$\int_{1}^{\infty} \frac{1}{x} dx$$

$$\lim_{b\to\infty}[\ln x]_1^b$$

$$= \lim_{b\to\infty} \ln b = \infty$$

$\ln(x)$ is unbounded, so the integral is divergent. And in fact, for any function in the form $\frac{1}{x^n}$ if n is greater than 1 the function will converge, and if it's less than or equal to 1 it will diverge,

6.4.2. Type 2

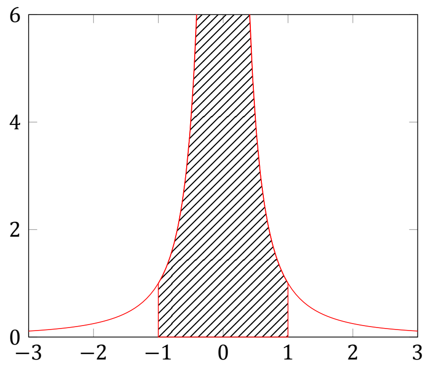
Type 2 integrals are where we integrate over a non-continuous section of a function. And you must be careful when dealing with these because they're not always obvious. It's clear when you're integrating to infinity because the limits will say so, but you need to check and make sure a function is continuous for example, what about integrating:

$$\int_{-1}^{1} \frac{1}{x^2} dx$$

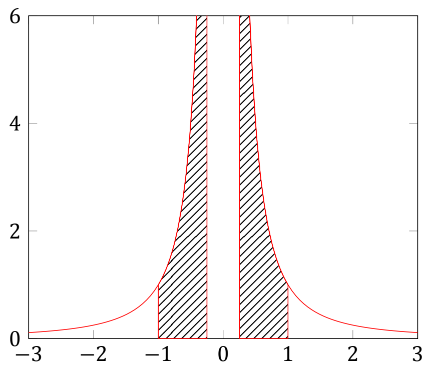
Seems innocent enough right? just integrate and substitute the limits:

$$[\frac{-1}{x}]_{-1}^{1} = (\frac{-1}{1} - \frac{-1}{-1}) = -2$$

That's not right. That's not right at all, How do I know? Well look at the graph of the function:



That's not a negative area! It's not possible for that to be -2. When it comes to integrating a Type 2 what we do is we take find the of discontinuity, and give it a label, c. In the example above this point is $c = 0$. Then we divide this integral in two, at point c. And we introduce two more variables s and t to help us integrate.



What we do is we find the limit as s and t approach c. So, in this specific example we'd have:

$$\int_{-1}^{s} \frac{1}{x^2} dx + \int_{t}^{1} \frac{1}{x^2} dx$$

$$= \lim_{s \to c^-} \left(\frac{-1}{s} - \frac{-1}{-1}\right) + \lim_{t \to c^+} \left(\frac{-1}{1} - \frac{-1}{t}\right)$$

This diverges. How?? Well, that confused me too, because for the rest of the course we think like an engineer, but in calculus, we think like mathematicians, so, while intuitively you may think that because s and t are approaching the same value, we can just say that $\frac{1}{s} = \frac{1}{t}$, well, no, we can't. Because the two values could be approaching c at a different rate. For example, s could be defined as $-2t$ and the distance between t and c would always be twice as large as s to c. It's weird, I know, but because we've got these two limits, we have to say this is divergent. But also because $\lim_{s \to 0^-} \frac{1}{s}$ is ∞ which, yeah that's obviously divergent, the area enclosed is not a proper value.

Convergent and Divergent Type 2

To be honest, the most important thing you get from this chapter is being able to identify if an integral is convergent of divergent. This applies to both Type 1 and 2. The key thing to remember is that if the area enclosed is infinite, it diverges. If you can't get a proper limit of the integral, it diverges. Only when you can find an actual limit of the area under the graph can you say the function converges.

6.4.3. Comparison Test

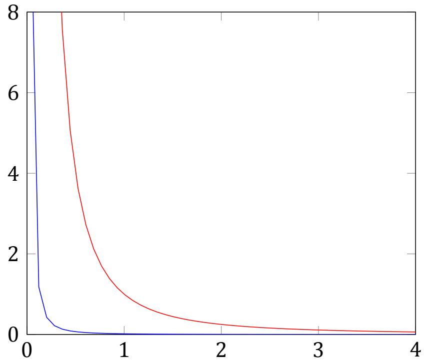
When it comes to finding out if a function is convergent or divergent by comparing it to a differnt function of known characteristics. It states: if f and g are continuous, with $0 \leq g \leq f$ for $x \geq a$, if

$$\int_a^\infty fxdx$$

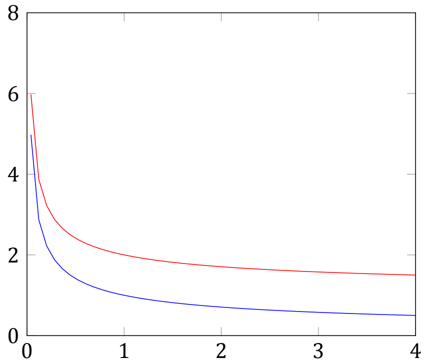
is convergent, then

$$\int_a^\infty gxdx$$

must *also* be convergent. This makes perfect sense if you think about it graphically, if f converges to a point, and g is always less than f, it is bound within a finite area, and must also converge.



Similarly we can say that if g diverges, so must f. Because if the area under g is infinite, and f is always greater than g then the area under f must also be infinite.



7

Differentials

7.1. The Basics

Yes, we have covered differentiation before, but a differential equation is slightly different. A Differential Equation (henceforth abbreviated to DE) is basically an equation involving an unknown function $y(x)$ and one or more of its derivatives. We call y the *dependent variable* and we call x the *independent variable*. The order of the DE is equal to the order of the highest derivatives in the equation. For example:

$$y'' + 2y' + y = xe^x$$

This is a second order DE because of the y''

$$y^2y'''' + x^2y^2y'' = (y')^3$$

This is a fourth order DE because of the y''''

If a DE has just one independent variable (y) then we call it an Ordinary Differential Equation (ODE), but if it has more than one, (y, z, w, etc) we call it a Partial Differential Equation. Which is not necessary knowledge for this course. We only deal with ODEs of the form $\frac{dy}{dx} = F(x, y)$ (ignore the capital F, I'm not talking about anti derivatives for now) where we solve for $y(x)$.

When it comes to DEs we can call them separable differentials if they can be described as:

$$\frac{dy}{dx} = g(x)f(y)$$

Where $f(y) \neq 0$. This allows us to **separate** the terms and write it as:

$$\frac{1}{f(y)}dy = g(x)dx$$

This is why $f(y)$ cannot be 0. Note that we must have a product of f and g, so something like $f(y) + g(x)$ is not separable.

7.2. Solving Separable Differentials

The way we solve these is a method called "the separation of variables" maybe, I don't know, it doesn't matter what it's called. I'll just do an example here:

$$y' = xy$$

Seems a bit tricky at first, but if we write y' slightly differently, as $\frac{dy}{dx}$. we can do a few tricks to help us calculate this.

$$\frac{dy}{dx} = xy$$

What we want to do from here is take all x elements and put them on one side, and put all the y elements and put them on the other side, hence the name.

$$dy = xydx$$

$$\frac{1}{y}dy = xdx$$

This is starting to look promising here. And remember we're working towards getting $y(x)$ so the next step here is to integrate both sides.

$$\int \frac{1}{y}dy = \int xdx$$

$$\ln|y| + C = \frac{1}{2}x^2 + C$$

Now, notice how there's an unknown constant on both sides of the equation. What we can do, is just, combine these into one unknown constant and leave it on one side of the equation.

$$\ln|y| = \frac{1}{2}x^2 + C$$

From here we want to move closer to finding $y(x)$

$$y = e^{\frac{1}{2}x^2+C}$$

This is a bit confusing, so let's try writing it in a simpler way with

$$y = e^C \cdot e^{\frac{1}{2}x^2}$$

But, if you remember, C is just an unknown constant, and e^C is *also* an unknown constant, so we can just merge these down and call it C. This gives us out final answer of

$$y = C \cdot e^{\frac{1}{2}x^2}$$

7.3. Applications

So now that we've covered the technique behind solving one of these equations, how about we look at some examples of where to use it. Most obvious is population growth. Say we have a population P and a change in population with respect to time $\frac{dP}{dt}$. What we can say (in this case at least) is that the growth in population, is directly related to the population at a given time; thus $\frac{dP}{dt} = kP$. And what we want to do is find P as a function of t, given $P(o)$. That's what we call the initial conditions, they help us solve these equations in more than just a general way.

$$\frac{dP}{dt} = kP$$

$$\frac{1}{P}dP = kdt$$

$$\int \frac{1}{P}dP = k\int dt$$

$$\ln|P| = kt + C$$

$$P = Ce^{kt}$$

From here we can't do much without k or $P(0)$, this is why we need those initial conditions to solve the equation. I'm going to arbitrarily make $k = 0.1$ and $P(0) = 100$. Subbing these into the equation we find:

$$100 = Ce^{0.1 \cdot 0}$$

$$100 = C$$

Therefore we find the specific equation for the population of bacteria at any given time.

$$P = 100e^{0.1t}$$

But this isn't a completely accurate way of modelling population growth, it implies that there is no maximum population possible. Which is wrong. There are only finite resources in the universe. Instead we use a different model called **Logistic Growth** as opposed to exponential growth.

$$\frac{dP}{dt} = kP(1 - \frac{P}{M}) \tag{7.1}$$

Where M is the natural maximum of the population. If P is much smaller than M then $1 - \frac{P}{M}$ will roughly be 1 so $\frac{dP}{dt}$ will roughly be kP, and if the current population is near the natural maximum, then the rate of change $\frac{dP}{dt}$ will roughly equal 0.

Now, this also applies if for some reason the population goes above M, say a scientist introduces more bacteria into the environment. If P is significantly higher than M, then $\frac{dP}{dt}$ will be very negative.

If we go through our usual route of separating then integrating We get the equation:

$$\int \frac{1}{P} + \frac{1}{M - P} dp = k \int dt$$

You can prove to yourself that $\frac{1}{P(1-\frac{P}{M})} = \frac{1}{P} + \frac{1}{M-P}$ by doing the sums out by hand. but for now just go with it.

$$= \ln|P| - \ln|M - P| = kt + C$$

$$= \ln\frac{|P|}{|M - P|} = kt + C$$

$$= \frac{P}{M - P} = Ce^{kt}$$

And this is our general solution!

7.3.1. Salt Solutions

One very important application to wrap your head around is the salt solution problem. The one where you have some water in a container, with a certain concentration of salt in it, with other water, with a different concentration of salt pouring in, and sometimes there will be a hole at the base of the container letting water flow out.

For these problems we assume the solution is always thoroughly mixed, obviously that's not how fluid dynamics works in real life, but we imagine that as soon as salt enters the vessel, the concentration of the whole volume of water changes.

Figure 7.1: *A Water tank with water flowing in, and out. Both with different salt concentrations.*

What we're aiming to find in these problems is how the mass of salt (y) in the container changes as a function of time (t). You can call these whatever you want, but I'll be using y and t. What we want to do is write a function of the rate change in salt in the container. This is expressed as the rate of salt coming in, minus the salt exiting the container.

$$\frac{dy}{dt} = \frac{dy}{dt}_{in} - \frac{dy}{dt}_{out}$$

The incoming salt is usually given as a constant (but sometimes it isn't!!!), where you have a rate of water coming in, and a concentration of salt. Say, 2L per minute, with 5g of salt per L. (that means you have 10g of salt coming in per minute). Easy!!

The out flowing salt is more difficult, because we may have a constant rate of water being pumped out, but the concentration of salt in the tank is not constant. Remember how we called the total amount of salt in the container y? That comes into play here: $\frac{y}{V}$, is the concentration of salt at any one moment. The amount of out-flowing salt is the concentration of the water, multiplied by the volume flowing out.

Let's do an example to try clear things up a bit. Imagine a Drum of volume $100L$, with $5L$ of $2g/L$ salt water flowing in, and $5L$ of water flowing out. Notice how the volume remains constant? That's handy! Let's draw up our balance equation.

$$\frac{dy}{dt} = \frac{dy}{dt}_{in} - \frac{dy}{dt}_{out}$$

$$\frac{dy}{dt} = 5L \cdot 2\frac{g}{L} - \frac{y}{100} \cdot 5L$$

$$\frac{dy}{dt} = 10 - \frac{y}{20}$$

$$\frac{dy}{dt} = \frac{200 - y}{20}$$

$$\frac{20}{200 - y} dy = dt$$

$$\frac{-20}{y - 200} dy = dt$$

$$-20 \int \frac{1}{y - 200} dy = \int dt$$

$$-20 \ln|y - 200| = t + C$$

$$\ln|y - 200| = \frac{-t}{20} + C$$

$$y - 200 = Ce^{\frac{-t}{20}}$$

$$y = Ce^{\frac{-t}{20}} + 200$$

This gives is the general equation for finding the amount of salt (y) in the container. But without a specific data point, we cant get rid of that constant C. If we're told that, $y(0)$ is 50 for example, we can then use this information to get rid of C

$$50 = Ce^0 + 200$$

$$50 = C + 200$$

$$C = -150$$

This is the most basic of examples, sometimes the volume isn't constant. Sometimes the incoming salt ins't constant. but as long as you apply basic principles to the problem, and you don't try to rush ahead and skip some steps you have a good chance of avoiding mistakes.

7.4. Direction Fields

The differential of a function gives you the tangent line at any specific point. But it can also be described implicitly. For instance we can say that $\frac{dy}{dx} = y - x$ this gives us the slope of a function at *any* point. See, we can say, $x = 1$ and that tells us nothing about y, so for each value of y there is a different differential. And we can draw these all on a graph, drawing $\frac{dy}{dx}$ at every point, giving us something like this:

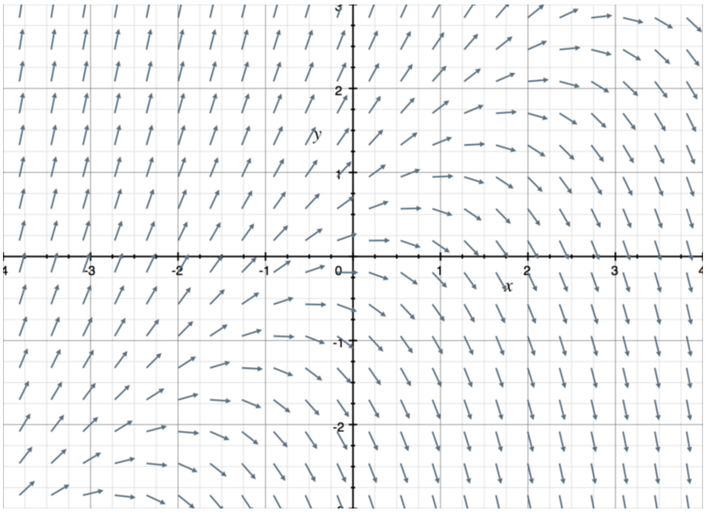


Figure 7.2: *A Directional Field*

This is what we get when we solve a DE generally, but when we're given initial conditions, we are given one specific point on the curve. If we go to those coordinates on the directional fields, you can follow the arrows at that point and then plot the curve of the specific function. For example, if we go to the coordinates $(-2, -1)$ and follow the directions we get the following curve:

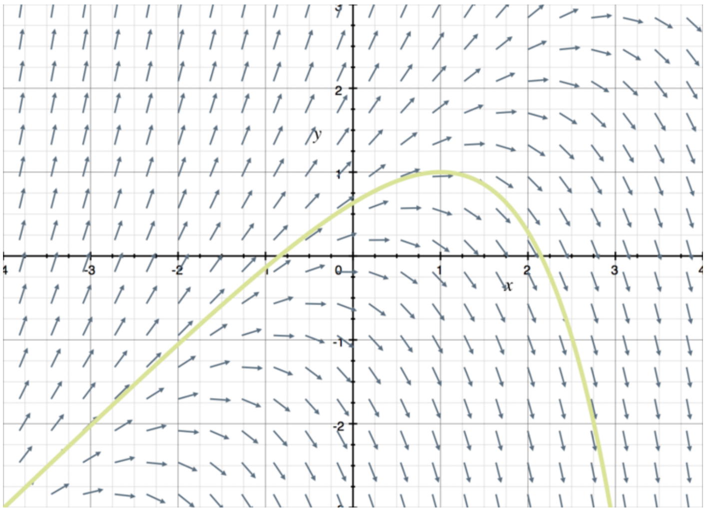


Figure 7.3: *A Specific Function in a Directional Field*

7.5. Linear First-Order Differentials

Another way of looking at DEs is A Linear First Order Differential. They're called this because they feature y and no higher powers of y. They can all be written in the form:

$$\frac{dy}{dx} + P(x)y(x) = Q(x) \tag{7.2}$$

Where both P and Q are continuous functions of x over a given interval. A DE can be both linear or separable, and in such circumstances, you can choose how to deal with them.

7.5.1. The Integration Factor

You may get lucky and see a simple way of relating these terms together, but often that's not the case. Instead we introduce an integrating factor $I(x)$ to make our lives easier.

$$I(x)\frac{dy}{dx} + I(x)P(x)y(x) = I(x)Q(x)$$

This isn't much good to us as is. We want to move to the form:

$$(I(x)y(x))' = I(x)Q(x)$$

Because this will allow us to find $y(x)$ much more easily, and hence *solve* the equation. From the product rule we know that:

$$(I(x)y(x))' = I(x)y' + I'y(x)$$

If we know this, then we can relate it to the previous expression, and state that:

$$I'y(x) = I(x)P(x)y(x)$$

This is handily enough a separable differential equation. I'll skip over the calculations, but from here we find that $I(x) = e^{\int P(x)dx}$. Because we multiplied both sides of the equation by $I(x)$ we can ignore the integration constant for I **only**. As we can divide it out later.

When we want to move on from here we can rewrite: $Iy' + I'y = IQ$ as $(Iy)' = IQ$, because of the product rule. From the fundamental theorem of calculus you can make a quick move from here by integrating both sides.

$$Iy = \int IQdx$$

And the final step to then divide by $I(x)$, to find the function of $y(x)$.

$$y = \frac{1}{I}\int IQdx$$

7.5.2. Solving Linear Differentials

So we've looked at the concept of an integration factor, but how do we actually use this to solve a problem. Well because of how this subject is examined you don't actually need to derive anything written above, instead you just need to know the following few steps:

- Write the formula in the form $\frac{dy}{dx} + P(x)y(x) = Q(x)$
- Multiply by $I(x)$
- Calculate $I = e^{\int Pdx}$
- Substitute in the new value of I and solve for y

That's pretty much it. That's all of the new concepts introduced in this chapter, but this does rely heavily on past knowledge with respect to differentiation and integration. The Chain rule, integration by parts, U substitution, are all important things for solving these questions.

7.6. Applications

7.6.1. Simple Example

In my experience you get the best understanding of something when you see it put to use. So let's work through some examples to see how this is actually used.

$$y' = 6 - 2y$$

First of all we want to rewrite this into the standard format:

$$y' + 2y = 6$$

In this equation we can see that $P = 2$ and $Q = 6$. Not particularly exciting functions of x, but they are nonetheless. From this we can work out what I, the integrating factor is.

$$I = e^{\int 2dx} = e^{2x}$$

Remember, the integrating constant doesn't matter because we can just divide it out on both sides of the equation. Substitute this into the format $(Iy)' = IQ$ to get:

$$(ye^{2x})' = 6e^{2x}$$

$$= ye^{2x} = \int 6e^{2x}$$

$$= ye^{2x} = 3e^{2x} + C$$

$$= y = 3 + Ce^{-2x}$$

And this is our final solution. We can't go any further without any extra information. If we had some initial conditions then we could substitute in some values for C, but we don't, so we can't.

7.6.2. More Difficult Salt Solutions

We saw the salt solution problem from earlier, but that was a reasonably simple one to solve. The volume remained constant, so we could easily set up a separable differential equation. But in this next example that's not the case.

Let's examine this problem with these conditions:

- A tank containing 800L of Water and 240g of salt
- 50L/min flowing in, with a concentration of 0.6g/L
- 20L/min flowing out

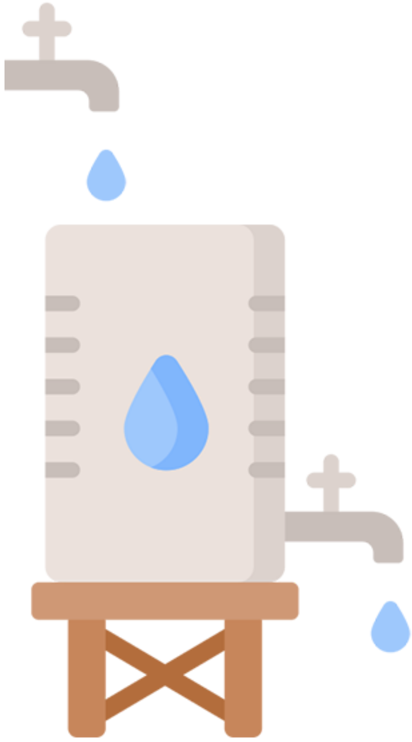


Figure 7.4: *A New Water tank with 50L/min flowing in and 20L/min flowing out*

If we want to find the volume of the container after a given amount of time, we can start with our balance equation and work from there.

$$\frac{dV}{dt} = \frac{dV}{dt}_{in} - \frac{dV}{dt}_{out}$$
$$= \frac{dV}{dt} = 50 - 20$$
$$= dV = 30dt$$
$$= \int dV = \int 30dt$$
$$= V = 30t + C$$

But, because we have those initial conditions of $V(0) = 800$ we can find $C = 800$. This is obviously one way of finding the equation for volume at a given time, but you could probably have done this intuitively to get $V = 800 + 30t$, but sometimes the problems won't be that simple, so this is just a backup plan that works every time.

The next step is to set up the balance equation for the amount of salt in the container. I'm again going to call salt y for the purposes of the equation, but you can use whatever you want.

$$\frac{dy}{dt} = \frac{dy}{dt}_{in} - \frac{dy}{dt}_{out}$$
$$= \frac{dy}{dt} = (50 \cdot 0.6) - (20 \cdot \frac{y}{800 + 30t})$$
$$= \frac{dy}{dt} = 30 - \frac{20y}{800 + 30t}$$

This is our balance equation. It's not visibly separable, but we can rearrange it to be a linear differential.

$$= \frac{dy}{dt} + \frac{2y}{80 + 3t} = 30$$

Notice anything familiar? It's in the form of Eq(7.2) where $P = \frac{2}{80+3t}$ and $Q = 30$. Remember the next step? Calculate I.

$$I = e^{\int P dx}$$

$$I = e^{\int \frac{2}{80+3t} dx}$$

$$I = e^{\ln (80+3t)^{\frac{2}{3}}}$$

$$I = (80 + 3t)^{\frac{2}{3}}$$

Now that we have I we can jump ahead and substitute it into the format of $(Iy)' = IQ$.

$$((80 + 3t)^{\frac{2}{3}} y)' = 30(80 + 3t)^{\frac{2}{3}}$$

$$(80 + 3t)^{\frac{2}{3}} y = \int 30(80 + 3t)^{\frac{2}{3}}$$

$$(80 + 3t)^{\frac{2}{3}} y = 30(\frac{3}{5})(\frac{1}{3})(80 + 3t)^{\frac{5}{3}} + C$$

Why are we dividing by 3 in this step? It's because of the chain rule. If we were to differentiate something, we multiply by the differential of what's in the brackets. When we integrate, we do the opposite.

$$(80 + 3t)^{\frac{2}{3}} y = 6(80 + 3t)^{\frac{5}{3}} + C$$

$$y = 6(80 + 3t) + C(80 + 3t)^{\frac{-2}{3}}$$

$$y = 480 + 18t + C(80 + 3t)^{\frac{-2}{3}}$$

We've arrived at a familiar spot here, using our initial conditions of $y(0) = 240$ we can calculate for C.

$$240 = 480 + C(80)^{\frac{-2}{3}}$$

$$-240 = C(80)^{\frac{-2}{3}}$$

$$C = -4456$$

Substitute this back into the original our general solution and we now have a specific solution for the amount of salt in the drum:

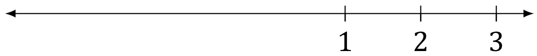
$$y = 480 + 18t - 4456(80 + 3t)^{\frac{-2}{3}}$$

8

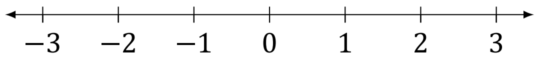
Complex Numbers

8.1. The Basics

Let's begin our discussion of what complex numbers are by first reminding ourselves what "Normal numbers" are. Starting from the simple ways of counting that we learned in primary school, we begin with Natural Numbers $\mathbb{N}$. If we plot these on a number line, we get a line with a whole bunch of dots on it from 1 to infinity.



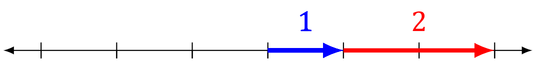
What about below that? Then we introduce the concept of Integers $\mathbb{Z}$, These numbers include all "whole numbers" less than 1. And on a number line that would look something like this:



From here we can start to fill in the gaps between all of the dots along this line, and we can start bu filling in all of the fractions. These are all Rational Numbers $\mathbb{Q}$, or *Quotients* if you will. Then with all of those filled in we can finally complete the number line with the Real Numbers $\mathbb{R}$. This collection of numbers is everything on the number line. So, even including numbers that cannot be expressed as fractions like π, or $\sqrt{2}$. But there are more numbers to be discussed, and nowhere to put them on the line.

Let's take a step to the side and look at how we operate with numbers. For the purposes of this we can imagine that numbers are represented by vectors on the number line. When we add two numbers we do the "head to tail" operation of vectors, and find our end position.

$1 + 2 =$

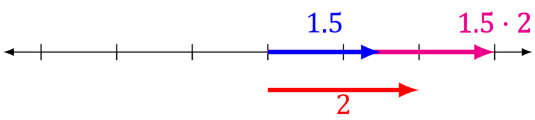


$= 3$

You can extrapolate out from here how subtracting a number works.

Multiplication is a little bit more tricky, In this case we are stretching the vector by a factor of the other vector to get our final answer.

$1.5 \cdot 2 =$



$= 3$

We can also imagine that multiplication is when we add up the angles of the vectors as well. If we take 0 to be the origin that is. So, positive numbers have an argument of $0rad$, and negative numbers have an argument of πrad. Let's take some very simple examples here:

$$1 \cdot 1 = 1$$

And the arguments?

$$0 + 0 = 0$$

So this shows us we have a vector with the argument of 0.

$$-1 \cdot 1 = -1$$

$$\pi + 0 = \pi$$

This shows us we have a vector with an argument of π so it will be pointing in the negative direction.So now what if we have something a bit trickier than this?

$$x^2 = -1$$

Well, we can say that x has an argument of α, and -1 has an argument of either π, or 3π because 2π is a full circle. Thus:

$$x \cdot x = -1$$

$$\alpha + \alpha = \pi, 3\pi$$

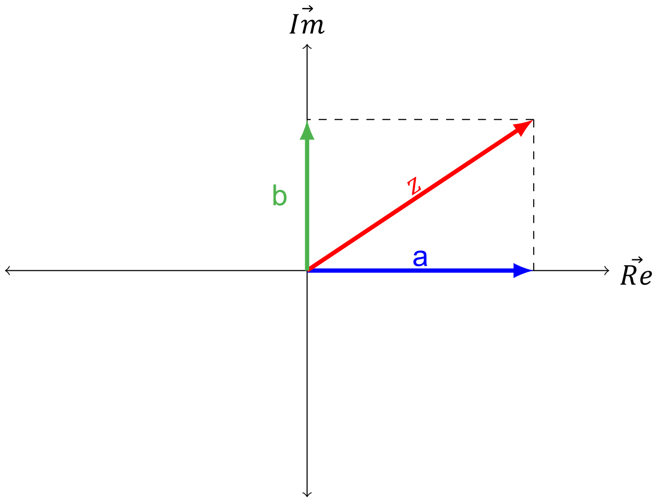
$$\alpha + = \frac{\pi}{2}, \frac{3\pi}{2}$$

But now wait a minute, that means that the square root for -1 is perpendicular to the number line? Exactly, and we use this new perpendicular line to define the "Complex Plane" or "Argand Diagram" if you want to be fancy. The two axes are the *Real Axis*, which is horizontal, and the *Imaginary Axis* is the vertical one. Where the real axis uses units of 1, the imaginary axis uses units of i where i is the imaginary unit, defined as $i = \sqrt{-1}$.

8.2. The Complex Plane

Remember everything we did with vectors back at the start? Well this is pretty similar. We can treat the complex plane with similar techniques. so when we have a complex number, we can treat it's real and imaginary components as coordinates.

We often write imaginary numbers as z or w but honestly it doesn't matter what you call them. A complex number is defined as $z = a + ib$ where a and b are both real numbers. But i is still the imaginary unit. If we plot this on the complex plane we see the following:



8.3. Adding Complex Numbers

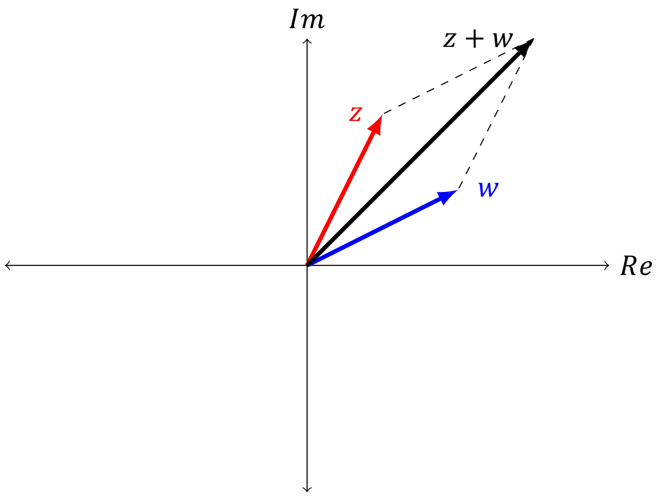
Now, the comparison to vectors goes even further, because say you have two complex numbers, that you want to add, you'd add them up the same way you'd add two vectors. Combine the $\vec{i}$ components and then combine the $\vec{j}$ components. For example:

$$z + a + ib$$

$$w = c + id$$

$$z + w = a + c + i(d + b)$$

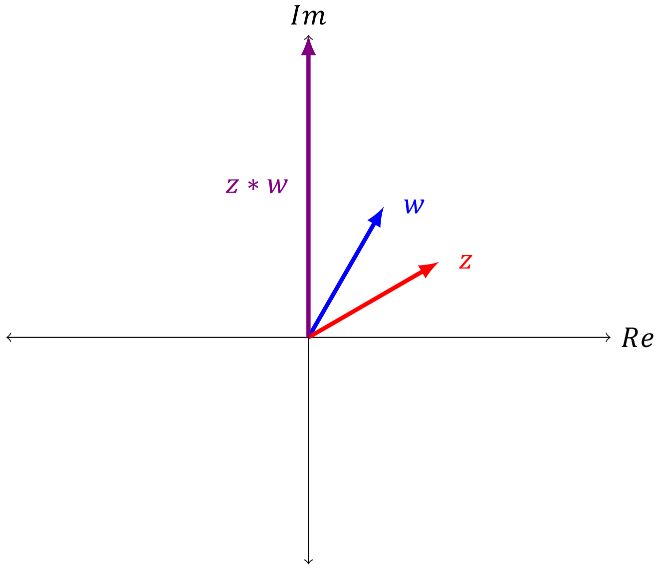
Thus we can say the real component of $(z + w)$ is $Re(z + w) = (a + c)$ and the imaginary component of $(z + w)$ is $Im(z + w) = (d + b)$



In a vector plane we'd call the length of the vectors the "magnitude" of the vectors, in the Complex plane we call it the "modulus" of the number, and we calculate it in pretty much the exact same way. With Pythagoras.

8.4. Multiplying Complex Numbers

Multiplying complex numbers is again, really simple if you can recall what we defined earlier. We stretch out a vector by the length of the other one, and then we add the arguments of the two vectors together to find the argument of the resultant.

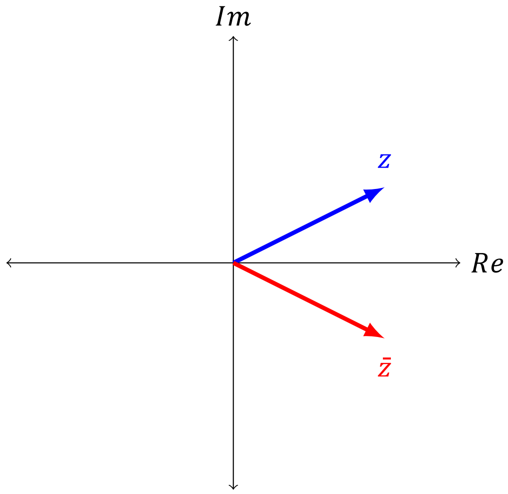


You can also work this out algebraically, remembering that i^2 is just -1

$$(a + ib)(c + id) = ac + iad + ibc + i^2 bd$$

8.5. Complex Conjugate

So if z is $x+iy$ then the conjugate of z is $x-iy$. We mirror it about the Real Axis in the complex plane.



If we multiply a complex number by it's conjugate we get:

$$z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

Which just so happens to be the modulus of z, squared!

Let's talk about the arguments for a second again. that's the angle between the vector and the x axis, in the same was the unit circle works. Thus, we can use our friends from the world of trigonometry again here. So, to find the argument of z we just need the $\arctan \frac{y}{x}$, or if x happens to be negative, then we say the argument is the $\arctan \frac{y}{x} + \pi$. We're using radians here still.

8.6. Applications of Complex numbers

8.6.1. Division

What if we want to divide a complex number by a real number? That's easy! On the complex plane, just scale the vector, and add up the argument. But what about algebraically? Let's say we want to find; $\frac{z}{x}$ in that case we'd just get: $\frac{z}{x} = \frac{a}{x} + i\frac{b}{x}$. This is easy for us to work with because the denominator is a real number. But what if we want to divide a complex number by another complex number $\frac{z}{w}$? in such a case we first multiply by 1, in the form of $\frac{\overline{w}}{\overline{w}}$. This is the conjugate of w and if you remember from earlier, a complex number multiplied its conjugate gives us the modulus squared; which is a real number! So this is nice to work with.

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\frac{c-id}{c-id} = \frac{(a+ib)(c+id)}{c^2+d^2}$$

8.6.2. Quadratic equations

Quadratic equations. You often want to find the roots of them, as in; when the equation is equal to zero. Not always possible though is it? Well it is if you think of it on the complex plane! There will always be 2 solutions to a quadratic equation ($ax^2 + bx + c = 0$) on the complex plane, and they will be complex conjugates of each other! How cool is that!? You can think about this using the quadratic formula (or the minus b formula, or the abc formula, whatever you call it).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What if $b^2 - 4ac$ is a negative number? Then the solutions aren't real, they're complex. And with the knowledge of the imaginary unit i, you can use this to help you out whenever you need it. For instance:

$$\sqrt{-a} = i\sqrt{|a|}$$

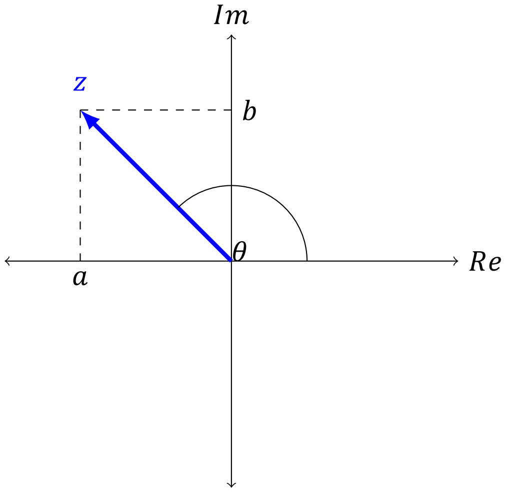
8.7. Other Ways of Representing Complex Numbers

we're all familiar with Cartesian coordinates (x, y), but there are more ways of describing the location of something in 2D space, and for that matter we have more than one way of representing a complex number. Because after all, a complex number is made of a real and an imaginary component.

- Cartesian Coordinates:
$$z = a + ib \qquad a, b \in \mathbb{R}$$
- Polar Form:
$$z = r(\cos\theta + i\sin\theta) \qquad r \in \mathbb{R}$$
- Exponential Form:
$$z = re^{i\theta} \qquad r \in \mathbb{R}$$

These are the three ways of looking at complex numbers that we'll deal with in this course, but to understand them a bit clearer let's start by graphing them out visually.

8.7.1. The Polar Form



Above you can see a complex number z plotted on the complex plane. You can define it with the components a and b if you like, but sometimes that makes doing calculations difficult. As such we start defining new things to make calculations easier. You'll remember the **modulus** from earlier. This is the length of the vector point to z it's defined as $\sqrt{z \cdot \overline{z}}$, or for our usecase right here; $\sqrt{a^2 \cdot b^2}$ where a is the real part of the number, and b is the imaginary part.

The Argument is the angle of the vector with respect to the positive real axis. Labeled as θ in the graph above. We already discussed the argument earlier, but just to make sure it's clear as day, the argument can be calculated as:

$$Arg(z) = \begin{cases} \arctan \frac{b}{a} & \text{if } a > 0 \\ \\ \arctan \frac{b}{a} + \pi & \text{if } a < 0 \end{cases} \tag{8.1}$$

If a just so happens to be equal to one, then the number is on the imaginary axis, so θ will obviously be either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ depending on if b is positive or negative. Thus, to define somewhere in polar form we need polar coordinates (r,θ). You can think of these as if r is the radius of a circle, and θ is the position along the circumference.

If you remember how triangles work, you can see that we can define the horizontal and vertical components (a and b) by using trigonometric functions:

$$a = r\cos\theta$$

$$b = r\sin\theta$$

and as such we finally get to the polar representation of a complex number:

$$z = r(\cos\theta + i\sin\theta) \tag{8.2}$$

8.7.2. The Exponential Form

You're just going to have to trust me that this works because the background information you need to derive it come up next quarter. Anyway, the exponential form comes from a combination of the polar format and Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{8.3}$$

Just, trust me. This works. This will make your life easier, but we don't have time to go into it now. If we recall our way of describing the polar format (8.2) you can see how we get our exponential form.

$$z = re^{i\theta} \tag{8.4}$$

With this form we can make our lives easier, because multiplication, division, and exponents suddenly become miles easier.

Let's first start by defining two complex numbers so that we can demonstrate how these operations work.

$$z_1 = r_1 e^{i\theta_1}$$

$$z_2 = r_2 e^{i\theta_2}$$

If we multiply the two numbers, we get the following:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \tag{8.5}$$

This is logical if you think about it in terms of normal algebra. But it's even smarter when you think about it, because $r_1 r_2$ is where we stretch the magnitude of the vectors, and $\theta_1 + \theta_2$ is when we add the arguments of the two numbers together. Just like from the very beginning of this chapter! How cool is that!?

Division is very similar to multiplication. Just treat the numbers as you would any exponents.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \tag{8.6}$$

The rules of exponents come in *really* handy in this chapter. If you want to raise a complex number by a power then it becomes really simple in the exponential form.

$$z_1^n = (r_1 e^{i\theta_1})^n = r_1^n e^{in\theta_1} \tag{8.7}$$

8.8. De Moivre's Theorem

Jumping straight from eq(8.7) we can arrive at one of the most useful theorems in maths. If $\theta \in \mathbb{R}$ and n is a positive integer, then:

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta) \tag{8.8}$$

Why does this work? well think of the polar form.

$$(e^{i\theta})^n = e^{in\theta}$$

we can see that the new argument is now $n\theta$. With our knowledge of Euler's formula (8.3) we can see why this works.

8.9. Trigonometric Identities

In secondary school you probably just took trigonometric identities for granted, they're just something that exist out there. But with De Moivre's theorem, we can actually derive them.

8.9.1. Double Angle Formulae

The double angle formulae are as follows:

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{8.9}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{8.10}$$

But how does De Moivre fit into this? Well:

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

But if we multiply out the left hand side we find it is equal to:

$$\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta$$

And because we have real and imaginary components we can separate them out and equate them:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$i\sin 2\theta = 2i\sin\theta\cos\theta$$

(We can just divide i out of both sides to get the original equation(8.9))

But what if we have different angles?

$$\sin(\theta + \Phi) = \sin\theta\cos\Phi + \sin\Phi\cos\theta \tag{8.11}$$

$$\cos(\theta + \Phi) = \cos\theta\cos\Phi + \sin\theta\sin\Phi \tag{8.12}$$

To deal with this we should look back to our good friend; the exponential form:

$$\cos\theta + \Phi + i\sin\theta + \Phi = e^{i\theta+\Phi}$$

$$= e^{i\theta} \cdot e^{i\Phi}$$

$$= (\cos\theta + i\sin\theta)(\cos\Phi + i\sin\Phi)$$

$$= (\cos\theta\cos\Phi - \sin\theta\sin\Phi) + i(\sin\theta\cos\Phi + \sin\Phi\cos\theta)$$

And once again, we have both a real and imaginary part, so we can separate them and get our formulae. By now you're probably starting to see the pattern emerging. You can use the exponential form and De Moivres theorem to find how angles relate to each other. Try this out for yourself. Try to define $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$. It starts off from $e^{i3\theta}$

8.10. Roots of Unity

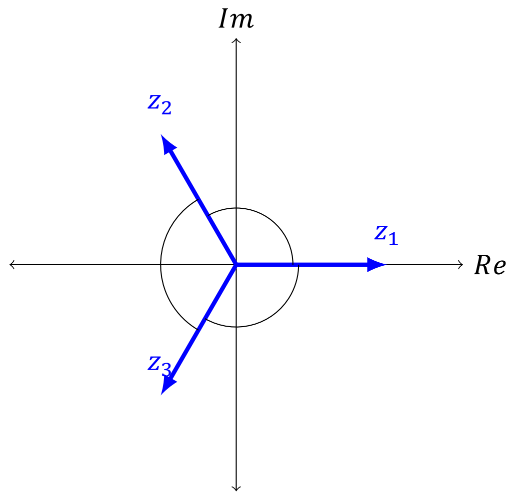
In my opinion this is one of the most fun and cool parts of complex numbers, and it all relates back to the fundamental theorem of algebra (no we didn't cover it, no you don't need to know it). Essentially what we are saying is that in circumstances as: $x^n = a$ solve for x. there are n solutions, and when you plot them on the complex plane, they arguments will all differ by $\frac{2\pi}{n}$. Honestly, I think this is fascinating! we can prove this, again, with the exponential form.

$$e^{i\alpha+i2\pi k} = e^{ni\theta}$$

$$i\alpha + i2\pi k = ni\theta$$

$$\frac{\alpha}{n} + \frac{2k\pi}{n} = \theta$$

With k being a an integer, we can see how this cycles around. Below we can see the three cubic roots of 1:



A

Appendix

A.1. Extra things to know

This Calculus exam will likely be very different to exams that you're used to. You're not allowed to used a calculator, and you're not allowed to have any formulae sheets with you, so you need to be familiar with trig functions so you can calculate them without using a calculator. Hence: this appendix exists.

A.2. Angles

Degrees	0°	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	

A.3. Co-functions

$$\sin(\frac{\pi}{2} - x) = \cos x$$
$$\cos(\frac{\pi}{2} - x) = \sin x$$
$$\tan(\frac{\pi}{2} - x) = \cot x$$
$$\cot(\frac{\pi}{2} - x) = \tan x$$
$$\sec(\frac{\pi}{2} - x) = \csc x$$
$$\csc(\frac{\pi}{2} - x) = \sec x$$

A.4. Double Angles

$$
\begin{aligned}
\sin(2x) &= 2\sin x\cos x \\
\cos(2x) &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \\
&= 1 - 2\sin^2 x \\
\tan(2x) &= \frac{2\tan x}{1-\tan^2 x}
\end{aligned}
$$

A.5. Half angles

$$
\begin{aligned}
\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\
\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\
\tan\frac{x}{2} &= \frac{1-\cos x}{\sin x} \\
&= \frac{\sin x}{1+\cos x}
\end{aligned}
$$

A.6. Power reducing formulas

$$
\begin{aligned}
\sin^2 x &= \frac{1-\cos 2x}{2} \\
\cos^2 x &= \frac{1+\cos 2x}{2} \\
\tan^2 x &= \frac{1-\cos 2x}{1+\cos 2x}
\end{aligned}
$$

A.7. Product to sum

$$
\begin{aligned}
\sin x\sin y &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \\
\cos x\cos y &= \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \\
\sin x\cos y &= \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] \\
\tan x\tan y &= \frac{\tan x + \tan y}{\cot x + \cot y} \\
\tan x\cot y &= \frac{\tan x + \cot y}{\cot x + \tan y}
\end{aligned}
$$

A.8. Pythagorean identities

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x
\end{aligned}
$$

A.9. Sum and difference of angles

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

A.10. Sum to product

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$
$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}$$
$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x \cos y}$$

A.11. Source

To be clear, I didn't type all of this out, instead I took it from here: Which saved me an incredible amount of time.

http://evgenii.com/blog/basic-trigonometric-identities

A.12. Rules of Integration

Remember that integration is just differentiation backwards.

$$\int \frac{x'}{x} dx = \ln x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \tan x dx = -\ln \cos x + C$$

$$\int \frac{x'}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \frac{x'}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{x'}{\cos^2 x} dx = \tan x + C$$

$$\int \ln x dx = x \ln x - x + C$$