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Aerospace Engineering

Calculus—1st quarter

Summary of formulas

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1 Prerequisits

1.1 Exponents

$$x^m x^n = x^{m+n} \qquad \frac{x^m}{x^n} = x^{m-n}$$
$$(x^m)^n = x^{m\cdot n} \qquad (xy)^n = x^n y^n$$
$$x^{m/n} = \sqrt[n]{m} = \left(\sqrt[n]{x}\right)^m$$
$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$
$$\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$$

1.2 Trigonometric functions

$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c}$$
$$\tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}$$
$$\sin^2 + \cos^2 = 1$$
$$\sin -\theta = -\sin\theta$$
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Functions

$$g_a(x) = a\sin(x)$$

Changes the amplitude of the function.

$$g_b(x) = \sin(bx)$$

Changes the period of the function.

$$g_c(x) = \sin(x+c)$$

Changes the starting value of the function.

Double angle formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos\alpha^2 - \sin\alpha^2$$

Other relations

$$\sin(\pi/2 - \alpha) = \cos\alpha$$
$$\cos(\pi/2 - \alpha) = \sin\alpha$$

Values

θ	rad	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	—

1.3 Logarithmic laws

$$\log_a x = y \quad \Leftrightarrow a^y = x$$
$$\ln x = \log_e x \quad \Leftrightarrow \ln e = 1$$
$$\ln x = y \quad \Leftrightarrow e^y = x$$
$$\log_a(a^x) = x$$
$$a^{log_a x} = x \qquad e^{\ln x} = x$$
$$\log_a(xy) = \log_a x + \log_a y$$
$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$
$$\log_a(x^r) = r\cdot\log_a x$$

1.4 Rationalizing and factoring

Rationalizing square roots

Multiply by conjugate radical

$$\frac{\sqrt{x+4}-2}{x}\cdot\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)$$
$$=\frac{1}{\sqrt{x+4}+2}$$

Factorising quadratic equations

$$x^2+\underbrace{b}_{(r+s)}x+\underbrace{c}_{(r\cdot s)}$$
$$=(x+r)(x+s)$$

Example:

$$x^2+5x-24 \quad r+s=5 \quad r\cdot s=24$$
$$=(x-3)(x+8)$$

1.5 Limits

$$\lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to+\infty} e^x=\infty$$
$$\lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to\infty} \ln x = \infty$$

1.6 Line equations

$$y-y_p=m(x-x_p)$$
$$y=mx+p$$

1.7 Quadratic formulas

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x_{1,2}=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$$

2 Inverse trigonometric functions

Because they are not one-to-one (do the horizontal line check/only one x for y), the domain has to be restricted to make them one-to-one.

Strategy for solving: Take expression in function, equal it to y, solve it so it cancels out the original function.

3 Implicit differentiation

Implicit differentiation can be applied when it is not easily possible, or not possible at all, to solve for y in a complicated function, such as $x^3+y^3=6xy$. The idea is to differentiate both sides with regard to x, then solve for y'.

✔ Example . . .

$$x^2+y^2=25$$
$$\frac{\mathrm{d}}{\mathrm{dx}}x^2+\frac{\mathrm{d}}{\mathrm{dx}}y^2=0$$

y is a function of x, must be differentiated!

$$\frac{\mathrm{d}}{\mathrm{dx}}y^2=\underbrace{2y}_{\text{outer}}\cdot\underbrace{y'}_{\text{inner}}$$
$$2x+2yy'=0$$

Note For expressions like $6xy$ use multiplication rule, too!

$$(6xy)'=6xy'+6y$$

Note If term like

$$x^{-1}+y^{-1}=0\,,$$

derivative is not $(y')^{-1}$ but still y'.

$$-x^{-2}-y^{-2}y'=0$$

4 Linear approximations and differentials

4.1 Linear approximations

Generally, a function resembles its tangent line at a point. This can be used to approximate values.

$$L(x)=f(a)+f'(a)(x-a)$$

4.2 Differentials

Differentials express the change of a function over a certain interval.

$$dy = f'(x)dx$$

dy represents the amount how much the function rises or falls (change in linearization).

How to find approximate relative errors

- Express unknown, e.g. area, as a function
- Calculate the differential
- Plug in the values, also for the change in x (dx)

5 Substitution rule

5.1 For indefinite integrals

If $u = g(x)$ is differentiable, then $du = g'(x)dx$. Thus, plugging in, the integral $\int f(x)dx$ turns into

$$\int \underbrace{f(g(x))}_{=f(u)} \underbrace{g'(x)dx}_{du}$$

Notice that $du = g'(x)dx$ is a *differential* that we plug in.

It is permissible to operate with dx and du after integral signs as if they were differentials!

Always remember that the idea behind the substitution rule is to replace a relatively complicated integral by a simpler one. The strategy is to find a substitution whose differential occurs in the integral. Also consider to substitute the dx after solving for it.

5.2 For definite integrals

This rule is more important than you think and *very much preferable* to solving the indefinite integral.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

In words: Replace the integrating boundaries by the value of the function of the substitution.

✔ Example . . .

$$\int_0^4 \sqrt{2x+1}dx$$
$$= \int_1^9 \frac{1}{2}\sqrt{u}du$$
$$= \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right]_1^9 = \frac{26}{3}$$

So note that we have not substituted back, but have indeed plugged in the values into the function $u = g(x)$ and calculated the integral, which saved us time!

What can also save us some time is the symmetry of integrals.

1. If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$.
2. If f is odd $[f(x) = -f(-x)]$, then $\int_{-a}^{a} f(x)dx = 0$.

6 Integration by parts

You are writing an exam and have suddenly forgotten the rule. Don't worry, you can easily derive it:

$$(uv)' = u'v + uv'$$
$$\int (uv)' = \int (u'v) + \int (uv')$$
$$\Rightarrow \int u dv = uv - \int v du$$

A little "trick", that is possible:

$$\int \ln x dx$$

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}dx \qquad v = x$$

Now you can easily solve this integral.

7 Stuck on integrating?

Sometimes you can't find a way to solve an integral. Try out the following strategies.

1. Simplify the integrand (if possible)
2. Look for obvious substitution
3. Look for a not-so-obvious substitution
4. Look for an even less obvious substitution
5. Try to integrate by parts

Doesn't work at all? That sucks, but don't give up! Here are some tricks:

$$\tan = \sin / \cos, e^{\sqrt{x}}, \quad u = \sqrt{x}, \quad u^2 = x$$

Always remember that you can still play around with the substitution! Maybe if you square the substitution or take the root, you can substitute it into the integral.

8 Discontinous Integrals

Integrals that have a discontinuity in $[a,b]$ are *improper* integrals, for instance $\int_{-1}^{1} 1/x^2 \mathrm{d}x$.

If an integral is continous at $[a,b)$ and discontinous at b), then

$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to b^+} \int_a^t f(x)\mathrm{d}x\ .$$

If f is continous at $(a,b]$ and discontinous at a, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to b^+} \int_t^b f(x)\mathrm{d}x$$

If $a < c < b$ and f is discontinous at c, then

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

When coming from the right to the left side: a^+ (positive side).
When coming from the left to the right side: a^- (negative side).

If the corresponding limit exists, the integral is *convergent*. If it does not exist (the integral turns out to be an inifite value), it is *divergent*.

Stuff to remember:

$$\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{t \to -\infty} \int_t^b f(x)\mathrm{d}x$$

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x) = \int_{-\infty}^{a} f(x)\mathrm{d}x + \int_a^{\infty} f(x)\mathrm{d}x$$

✔ Example . . .

$$\int_2^5 \frac{\mathrm{d}x}{\sqrt{x-2}} = \lim_{t \to 2^+} \int_2^5 \frac{\mathrm{d}x}{\sqrt{x-2}}$$

$$\lim_{t \to 2^+} 2(\sqrt{3} - \sqrt{t-2}) = 2\sqrt{3}$$

$\Rightarrow$ Integral is convergent.

Short check

A short check for the con- or divergency of a special integral:

$$\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x$$

is convergent if $p > 1$, and divergent if $p \leq 1$,

Note While $1/x^2$ is convergent, $1/x$ is not, since $1/x$ does not reach the x-axis fast enough.

Comparison Theorem

It is impossible to find the exact value of an improper integral, yet you want to find out whether it is improper or not? Use the comparison theorem:

$$f(x) \geq g(x) \geq 0 \quad \text{for} \quad x \geq a$$

Now, if $g(x)$ is con/divergent, then $f(x)$ is, too! Just take a comparable, easier function $g(x)$ and evalute it, it will be the same result as $f(x)$.

9 L'Hospitals rule

This rule is not on the schedule, but "it can make your life much easier", so I give it here! If

$$\lim_{x\to a} f(x) = 0 \quad \text{or} \quad \lim_{x\to a} f(x) = \pm\infty$$

and the same is true for $g(x)$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

So this means, that the limit of $f(x)/g(x)$ is the same as the fraction of their derivatives. In many cases, this is much easier to solve!

✔ Example . . .

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t}$$
$$\lim_{t\to 0^+} \frac{1/t}{-1/t^2} = \lim_{t\to 0^+} -t = 0$$

10 Differential Equations

10.1 Separable equations

A special type of first order differential equations. In such functions, $\mathrm{d}y/\mathrm{d}x$ can always be factored as a function of x times a function of y.

$$h(y)\mathrm{d}y = g(x)\mathrm{d}x$$

Now simply integrate both sides and solve for y:

$$\int h(y)\mathrm{d}y = \int g(x)\mathrm{d}x$$

Note that if the integral yields something like $\ln y$ you should still solve for y by raising it to the power of e.

10.2 Linear differential equations

Always have the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

For instance $xy' + y = 2x$, because $y' + y/x = 2$. To solve, multiply by the *integrating factor*

$$I(x) = e^{\int P(x)\mathrm{d}x}$$

and then integrate both sides. Note that the left side will always be an expanded form of the multiplication rule and can be simplfied,

$$\begin{aligned} y' + P(x)y &= Q(x) \quad |\cdot I(x) = e^{\int P(x)\mathrm{d}x} \\ (y\cdot I(x))' &= Q(x)\cdot I(x) \end{aligned}$$

which makes it a child's play to integrate it.

10.3 Second order differential equations

10.3.1 Homogenous second order . . .

$$ay'' + by' + cy = 0$$

This kind of equation is solvable with

$$y = e^{rx} \to y' = re^{rx} \to y'' = r^2e^{rx} .$$

Plugging this in, you get the auxilliary equation, with which you have to solve.

$$ar^2 + br + c = 0$$

Often, a quck way to solve it is to factor it (see section 1.4). Otherwise, use the quadratic formulas to solve it.

Possible solutions:

Roots of aux. eqn.	General solution
r_1, r_2 real and distinct	$y = c_1e^{r_1x} + c_2e^{r_2x}$
$r_1 = r_2 = r$	$y = c_1e^{rx} + c_2xe^{rx}$
r_1, r_2 complex: $\alpha \pm i\beta$	$y = e^{\alpha x}(c_1\cos\beta x + c_2x\sin\beta x)$

10.3.2 Nonhomogenous equations

To solve such equations, solve the complementary equation, then solve the particular equation, add them.

$$ay'' + by' + cy = G(x)$$

Complementary equation:

$$ay'' + by' + cy = 0$$

The solution $y(x)$ of a nonhomogenous equation can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is the particular solution and y_c the complementary.

✔ Example . . .

$$y'' + y' - 2y = x^2$$

$$y_c = c_1 e^x + c_2 e^{-2x}$$

Now guess what y_p could look like:

$$y_p(x) = Ax^2 + Bx + C$$

Differentiate this equation and substitute it into the original equation:

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$[\ldots]$$
$$-2A = 1 \quad 2A - 2B = 0 \quad 2A + B - 2C = 0$$
$$\Rightarrow A = -\frac{1}{2} \quad B = -\frac{1}{2} \quad C = -\frac{3}{4}$$
$$y(x) = y_c(x) + y_p(x) = \ldots$$

Note The solution y_p must never equal the solution y_c. In that case, a x has to be added to the solution. For example:

$$\begin{aligned} y'' - 3y' + 2y = 2x + e^{2x} + e^{-2x} \\ y_c = c_1 e^x + c_2 e^{2x} \\ y_p = ax + b + cxe^{2x} + de^{2x} \end{aligned}$$

The particular solution y_p must be "linearly independent" of y_c.

Attempts for solving nonhomogenous equations

Note If you have an equation like

$$ay'' + by' + cy = G(x) + H(x)$$

the solution may be acquired by

$$ay'' + by' + cy = G(x)$$
$$ay'' + by' + cy = H(x)$$

then $y(x) = y_{p1} + y_{p2}$.

$$ay'' + by' + cy = G(x)$$

$G(x)$	$y_p(x)$
const	A
x^n	$Ax^n + Bx^{n-1} + \ldots + C = y_p(x)$
e^{nx}	Ae^{nx}
$\sin x$	$A\cos x + B\sin x$
$x\cos nx$	$(Ax + B)\cos nx + (Cx + D)\sin nx$

As is the case with the last row, the solutions should be combined in order to find a solution.

10.3.3 Initial value problems

Those are especially easy, all you have to do is plug in the given values into your solution and solve for the c_1 and c_2.

Just to be sure. . .

$$y(\underbrace{0}_{x}) = \underbrace{1}_{y}$$

11 Complex numbers

Complex numbers suck, but you have to know them.

$$z = \underbrace{\alpha}_{\text{real}} + \underbrace{\beta i}_{\text{complex}}$$

The main property of i is $i^2 = -1$. Therefore: $\sqrt{i} = \sqrt{-1}$ But also: $(-1)^2 = -1 \Rightarrow \sqrt{-i} = \sqrt{-1}$ In general: $\sqrt{-c} = \sqrt{c}i$

Fundamental Theorem of Algebra

If you want to show off, this was proofed by Gauss!

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

does always have a solution among the complex numbers.

Basic rules

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
$$(a + bi)(c + di) = a(c + di) + (bi)(c + di)$$

Conjugates

Complex number: $z = a + bi$

Conjugate: $\overline{z} = a - bi$

$$\overline{z + w} = \overline{z} + \overline{w} \quad \overline{zw} = \overline{z} \cdot \overline{w} \quad \overline{z^n} = \overline{z}^n$$

Division of complex numberx

To divide complex numbers, multiply them with their conjugate denominator.

$$\frac{-1+3i}{\underbrace{2+5i}_{z}} = \frac{-1+3i}{2+5i} \cdot \frac{\overbrace{2-5i}^{\bar{z}}}{\underbrace{2-5i}_{\bar{z}}}$$
$$= \frac{13+11i}{2^2+5^2} = \frac{13}{29} + \frac{11}{29}i$$

Modulus

Distance from the origin in the complex number diagram.

$$|z| = \sqrt{a^2+b^2}$$
$$z\bar{z} = |z|^2$$
$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2}$$

Polar form

Expresses a complex number by means of the complex number diagram.

$$a = r\cos\theta, \quad b = r\sin\theta$$
$$z = a+bi$$
$$z = r(cos\theta + i\sin\theta)$$

Where $r = |z| = \sqrt{a^2+b^2}$ and θ is the "argument" of z, $\theta = \arg(z) = \arctan(b/a)$.

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right], z_2 \neq 0$$
$$z^{-1} = \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta)$$

De Moivre's Theorem

$$z^n = [r(\cos\theta + i\sin\theta)]^n$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

Roots of complex numbers

Roots of complex numbers suck especially. For the n^{th} root, you need n iterations…

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right],$$

where $k = 0, 1, \dots, n-1$.

Complex exponentials

$$e^z = e^{a+bi} = e^a e^{bi} = e^a(\cos b + i\sin b)$$

Show off note:

$$e^{i\pi} + 1 = 0$$

This formula comprises the very most important mathematical numbers. Also note:

$$[r(\cos\theta + i\sin\theta)]^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

Trick for complex numbers with high exponentials:

$$z^{10} = (z^2)^5$$

A Recurring Derivatives

And hence integrals as well…

$$\frac{\mathrm{d}}{\mathrm{dx}}a^x = a^x \ln a$$
$$\frac{\mathrm{d}}{\mathrm{dx}}(log_a x) = \frac{1}{x\ln a}$$
$$\frac{\mathrm{d}}{\mathrm{dx}}\tan x = \frac{1}{\cos^2 x}$$
$$\frac{\mathrm{d}}{\mathrm{dx}}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{dx}}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{\mathrm{d}}{\mathrm{dx}}\arctan x = \frac{1}{1+x^2}$$
$$\frac{\mathrm{d}}{\mathrm{dx}}x\ln|\cos x| = \tan x$$
$$\frac{\mathrm{d}}{\mathrm{dx}}\frac{1}{2}\ln(1+x^2) = \ln(1+x^2)$$