Calculus I B

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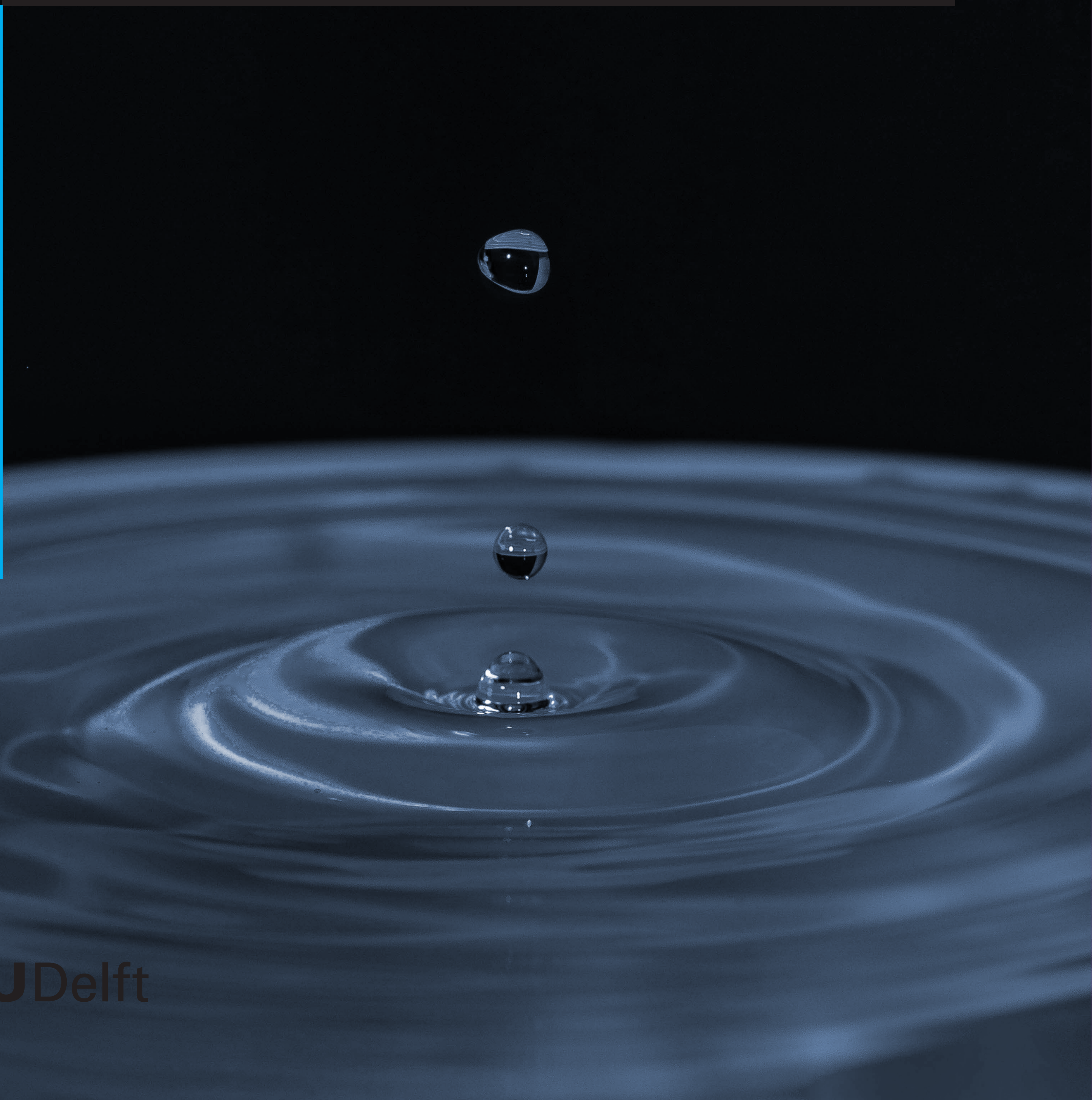

Preface

So this is a summary of the core concepts covered in the Calculus 1 module in First year. Up to 2018 you could easily pass the course just by practicing past questions and learning the patterns of how to solve them, but from 2019 on wards you needed to have a more intuitive understanding of the stuff you learned in class. In these notes I go over the basics of what you do in class and try to explain things simply. Obviously I'm not going to every bit of information from the lectures and the book in here, so if you want further clarification I recommend looking back at the slides. These notes go in the same order as the lectures so you should have no problem finding further information. The latest version of these notes is always available on my website (alanrh.com), along with other resources that I find useful. If you find any mistakes and you need anything corrected shoot me an email, but please don't email me if you want further explanation because I can't promise individual help to everyone.

Alan Hanrahan
Delft, January 27, 2021

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1

Second Order Differential Equations

1.1. The Basics

You should remember from the previous Calculus subject that we dealt with Differential equations of the first order. As in, we only had a single differential of y. Now we're dealing with second order Differential Equations (DEs). more specifically we're dealing with *Linear* second order DEs. We call them Linear if they can be written in the form:

$$P(x)y'' + Q(x)y' + R(x)y = G(x) \tag{1.1}$$

where P,Q,R, and G are all continuous functions of x. We call this form linear, because there are no functions of y so nothing like $y'y''$ for instance. The function G is important. If $G = 0$then we can say that the DE is Homogeneous, otherwise we say that the DE is Non-Homogeneous.

1.2. Homogeneous Differential Equations

1.2.1. Solving a first order Equation

When solving a first order DE you could go through the steps as you recall it from the last quarter, and use the integrating factor, or you could make a guess and say that $y = e^{rx}$ because that's often the case. But remember, this is a guess. If you substitute this into a first order DE:

$$ay' + by = 0$$

$$are^{rx} + be^{rx} = 0$$

$$(ar + b)e^{rx} = 0$$

But exponentials are never equal to zero, so then we can say:

$$ar + b = 0$$

$$r = -\frac{b}{a}$$

Bring this back into our original format for y and we can say that $y = e^{\frac{-b}{a}x}$ is **a** solution for the differential equation. But if we multiply by a constant, our differential equation is unaffected. And so we can say that our answer is in the form:

$$y = Ce^{\frac{-b}{a}x}$$

1.2.2. Solving a Second Order Equation

We can use this same approach of *guessing* the form the differential will be in for second order DEs too. For example, if we have the equation:

$$y'' + 4y' + 3y = 0$$

We can substitute in $y = e^{rx}$ and we find:

$$r^2e^{rx} + 4re^{rx} + 3e^{rx} = 0$$

$$(r^2 + 4r + 3)e^{rx} = 0$$

Once again, an exponential is always non-zero, so we can get rid of that to give us a quadratic equation in r.

$$r^2 + 4r + 3 = 0$$

We are familiar with these, so we can use the quadratic formula to solve for r.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.2}$$

In this case this gives us $r = -3$ or $r = -1$. Thus with these values for r we can say that:

$$y_1 = C_1e^{-3x}$$

and

$$y_2 = C_2e^{-1x}$$

This brings us to an interesting principle. If $y_1(x)$ and $y_2(x)$ are both solutions of the *linear* homogeneous equation $P(x)y'' + Q(x)y' + R(x)y = 0$. And C_1 & C_2 are arbitrary constants, then the function:

$$y(x) = C_1y_1(x) + C_2y_2(x) \tag{1.3}$$

is *also* a solution for y. This is the superposition principle, and this is how we get our general solution.

1.2.3. Different cases

In the quadratic formula (1.2), we say that the value under the square root is the discriminant D. If this value is greater than 0, ($b^2 - 4ac > 0$), then both r_1 and r_2 will be distinct real numbers, and both $y_1 = e^{r_1x}$ and $y_2 = e^{r_2x}$ will be independent functions, and the general solution will work nicely as we expect.

An independent function is one where you can't just multiply one function by a constant to get the other. For example; x^2 and $2x^2$ are not independent because you can just multiply by 2 or $\frac{1}{2}$ to get the other function. But x^2 and x^3 are independent, because the ratio of their values is not constant.

What if the discriminant is zero? In such instance we find that we only get one value for r and so we don't end up with independent equations for $y(x)$.

$$r_1 = r_2 = \frac{-b}{2a}$$

Thus, we don't get a general solution from this because we can simply write it in the form:

$$y(x) = C_1 e^{rx} + C_2 e^{rx} = C_3 e^{rx}$$

No good. Instead we multiply one of our possible solutions of y by x. Why? Because it works. And that's all we need to know for now. But to prove that it works look at the following:

$$ay'' + by' + cy = 0$$

$$y = xe^{rx}$$

$$y' = e^{rx} + rxerx$$

$$y'' = 2re^{rx} + r^2 xe^{rx}$$

$$a2re^{rx} + ar^2 xe^{rx} + be^{rx} + brxerx + cxe^{rx} = 0$$

$$xe^{rx}(ar^2 + br + c) + e^{rx}(2ar + b) = 0$$

We know that exponentials are non-zero, and we know that the quadratic solves to zero. So with the value of $r = \frac{-b}{2a}$, We can check to see if $2ar + b = 0$. And sure enough, the equation works out just right.

But what if the Discriminant is negative? In this case, then we find that r_1 and r_2 are complex numbers. In the form $r_1 = \alpha + \beta i$, and $r_2 = \alpha - \beta i$. This isn't useful for us. But if you remember the different ways we can write complex numbers, we can use algebra to our benefit.

$$y_1(x) = e^{\alpha x} e^{\beta i x}$$

$$y_2(x) = e^{\alpha x} e^{-\beta i x}$$

With these forms, we can write the general solution in the form:

$$y(x) = e^{\alpha x}(C_1 e^{\beta i x} + C_2 e^{-\beta i x})$$

Recall $e^{i\theta} = \cos\theta + i\sin\theta$. With this information we can do some algebra (do it yourself to prove it) and find that:

$$\cos\theta = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$$

$$\sin\theta = \frac{1}{2i}e^{i\theta} + \frac{1}{2i}e^{-i\theta}$$

With this knowledge, and the understanding that constants are arbitrary, we can find a general solution of y that gives us real answers:

$$y(x) = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) \tag{1.4}$$

Where $C_1, C_2 \in \mathbb{R}$

1.3. Non-Homogeneous Equations

So we've dealt with homogeneous equations where the independent function $G(x)$ is equal to zero. But to deal with Differential Equations where the function is not zero, we much use those techniques with other techniques too.

1.3.1. Guessing the Form of Y

It should be noted that throughout this course we are assuming that P, Q, R are all constants and not complicated functions of x. As such, we use this information to make a guess at the format of $y(x)$. To do so, we look at the form of $G(x)$. For example:

$$y'' + y' - 2y = 4x$$

The highest power of x in $G(x)$ is x^1. This is a linear equation, so in turn we guess that $y(x)$ is also linear. Each time we differentiate, we reduce the powers of each term, so the highest power term will only appear once, in $y(x)$ and not in any of the derivatives.

The general form of a linear equation is $Ax + B$, with first derivative A, and second derivative 0. Thus we substitute this into our equation and solve for y.

$$0 + A - 2Ax - 2B = 4x$$
$$-2Ax + A - 2B = 4x$$
$$A = -2$$
$$-2 + B = 0$$
$$B = 2$$
$$y = -2x + 2$$

More specifically what we've found is a particular solution of $y(x)$ written as y_p. If we want to find a particular solution of a DE where $G(x)$ is a different format, for instance if $G(x) = \cos x$, then the general solution is $A\cos x + B\sin x$. This is just because of the cyclic nature of trigonometric differentials. Using the same logic as before we can substitute in our guesses for the function of y and work out the coefficients.

1.3.2. Finding a General Solution

We don't just want one particular solution though, we want the general solution. To find this we must add the solution to the complimentary homogeneous equation. This is because the homogeneous equation equals zero, and we can add zero to anything and it will not change the value.

The steps for getting the general solution then, are as follows:

1. Solve the complimentary equation: y_c
   - $ay'' + by' + c = 0$
2. Solve the particular equation: y_p
   - $ay'' + by' + c = G(x)$
3. Sum the solutions (1) and (2)

  - $y(x) = y_p + y_c$

4. Apply the initial conditions and calculate the coefficients C_1 and C_2

Essentially, that it. If your exam is tomorrow and you just need a refresher on the topic you can flick to the next chapter now. But for a more thorough explanation, keep reading.

We saw at the beginning of this section how we solve for the particular solution. If G is an n^{t}h order polynomial then the guess we use is $y_p = Ax^n + bx^{n-1}$ and so on, then we substitute it into the equation and solve for the coefficients

But be warned, there are a few special cases you need to watch out for. Starting with:

- if $G(x) = \cos kx$ or $\sin kx$. In that case, use $y_p = A\cos kx + B\sin kx$
- if $G(x) = \alpha e^{kx}$ then use $y_p = Ae^{kx}$
- if $G(x)$ is a combination of the above, for instance of the form $x^2e^x + \sin x$, then you should find two particular solutions $y_p = y_{p_1} + y_{p_2}$. In this case:
  $y_{p_1} = (Ax^2 + Bx + C)e^x$
  $y_{p_2} = D\cos kx + E\sin kx$
- if the solution y_p works out to be a solution to the complimentary equation, for example: $y'' + y = \cos x$. Then multiply by x.
  $A\cos x + B\sin x \rightarrow Ax\cos x + Bx\sin x$

1.3.3. Practical Applications

These equations are all pretty useful when we want to combine forces to analyse a physical system. Like a spring-mass system that has been pushed from equilibrium. The different forces acting on the mass are:

Mass * accelleration = Spring Force + Friction Force + External force

or in mathematical terms:

$$mx'' = -kx - \gamma x' + F$$

$$= mx'' + \gamma x' + kx = F$$

2

Series

2.1. The Basics

2.1.1. Sequences

Before we start talking about series, we must understand a sequence first. A sequence is an ordered list of numbers. We write it in the form:

$$\{a_n\}_{n=1}^{N} \tag{2.1}$$

$$a_1, a_2, a_3, a_4, \ldots a_N$$

Where we begin with a_n and keep going to a_N. Most of the time in this course we use $N = \infty$. An example of one such sequence would be:

$$\left\{\left(\frac{1}{2}\right)^n + 1\right\}_{n=1}^{\infty}$$

$$= \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \ldots$$

We say that a sequence, $\{a_n\}_{n=1}^{\infty}$ converges to the limit L if:

$$\lim_{n\to\infty} a_n = L$$

If the limit does not exist, the sequence is divergent.

2.1.2. Series

A series is what we get when we sum up all of the terms of a sequence. Given the sequence, $\{a_n\}_{n=1}^{\infty}$, we get the series: $a_1 + a_2 + a_3 \ldots$. This is called an infinite series. We denote it as:

$$\sum_{n=1}^{\infty} a_n \tag{2.2}$$

2.1.3. Rules of Calculation

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series then so are:

- $\sum_{n=1}^{\infty} ca_n$ where c is a constant $\in \mathbb{R}$
- $\sum_{n=1}^{\infty} a_n + bn$
- $\sum_{n=1}^{\infty} a_n - bn$

If we multiply every value in a sequence, then we can factorise it out and write it as: $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

We are able to play around with the starting value so long as we change the form of a_n to fit, For example:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 ...$$

$$= \sum_{n=0}^{\infty} a_{n+1} = a_1 + a_2 + a_3 ...$$

2.1.4. Converging Series

A series is convergent if there is a number (R) such that $\sum_{n=1}^{\infty} a_n = S$. We define a partial sum as $S_N = \sum_{n=1}^{N} a_n$ with this, we can define a sequence of all partial sums.

$$\{S_N\}_{N=1}^{\infty} = a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3, ...$$

Thus, we can say that the series $\{S_N\}_{N=1}^{\infty}$ converges if:

$$\lim_{N\to\infty} S_N = S$$

And because this sequence is a sequence of the partial sums of the series of a_n we can in turn say:

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = \sum_{n=1}^{\infty} a_n = S$$

What we did here is we showed the convergences of the partial sums to show to convergence of a series. If the limit does not exist, then the series is divergent.

2.2. Geometric Series

A geometric series is just a form of a series, written as:

$$\sum_{n=1}^{\infty} ar^{n-1} \tag{2.3}$$

To find the sum of this series, we can use the partial sums again. Start by writing out the series S_N

$$S_N \sum_{n=1}^{N} ar^{n-1} = a + ar + ar^2 + ar^3 ... + ar^{N-1}$$

Then multiply this partial sum by r $rS_N = ar + ar^3 + ar^4 ... + ar^N$

$$S_N - rS_N = a - ar^N$$

$$S_N(1 - r) = a - ar^N$$

$$S_N = \frac{a - ar^N}{1 - r}$$

To show convergence of a geometric series we take the limit as N goes to infinity, and from this we can conclude that the series is only convergent if the magnitude of $|r|$ is less than 1. Thus, if we know that a geometric series is convergent, we can write the sum of the whole series as:

$$S = \frac{a}{1-r} \tag{2.4}$$

2.3. Telescopic Series

A telescopic series is one that can be expanded out into more than one series, kind of like a telescope. Take for example the series $\sum_{n=1}^{\infty} \frac{1}{n(n-1)}$ we can expand this series out into two different series of the form $\sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n-1}$. Knowing that these two fractions multiply to give us our original fraction, we can do some algebra to find:

$$\sum_{n=1}^{\infty} \frac{1}{n(n-1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n-1}$$

But what use is this? Well, if for example you write out the new terms of this series you'll start to notice a pattern:

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n-1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} \ldots$$

Notice how we can cancel pairs of terms? Thus giving us our sum $S = 1$ because nothing cancels with the first term. In my experience the best understanding of telescopic series just comes from examples, so here's another example to wrap your head around:

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} \ldots$$

And what you start to see here is that every second term cancels, as we approach infinity, all terms except two of them at the start will cancel out leaving us with $S = 1 + \frac{1}{2}$.

2.4. Series Tests

2.4.1. Divergence Test

If we want to take a shortcut to see if a series will converge we can try a test for divergence. If the limit of the sequence $\lim_{n\to\infty} a_n$ is anything other than 0 then the series will diverge. This is because you'll be adding infinite non-zero values so it will obviously diverge. On the other hand, if the limit *does* equal zero, then we can't say anything of significance, the test is inconclusive.

An example of one such series is the harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$. If we try the divergence test we find the limit to be 0, and so we can't say if it's divergent or convergent.

2.4.2. Integral Test

The integral test is another check to see if something converges or diverges. Suppose the function $f(x)$ is a continuous, positive, decreasing function on $[1, \infty]$, and let $a_n = f(n)$. Then we can say that if the integral of $\int_1^\infty f(x)$ converges, then so too must the series $\sum_1^\infty a_n$, and vise versa. The opposite is also true, in that if the integral diverges, so does the series.

To prove this we can imagine a graph where we plot the sequence $\{a_n\}_{n=1}^\infty$ and the function $f(n)$.

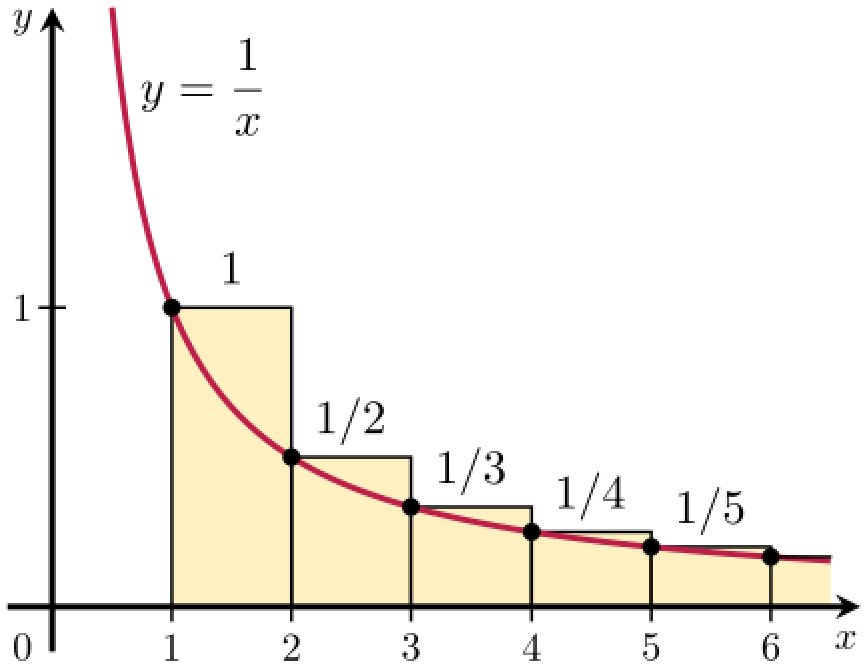


Figure 2.1: *Visualisation of a Divergent Series*

What we can do, is draw the value of the sequence for each value of n then multiply it by $\Delta x = 1$ to get the appropriate area. Notice in figure (2.1) the area of the series is always larger than the area of the integral. And because we know the integral diverges, and the series is always larger than the integral, the series too must diverge.

We can use this same logic to prove convergence too, if instead we shift our vertical bars representing the sequence to the left, so that the area enclosed by the integral is always greater than the area enclosed by the series, then, if the integral converges, and the area of the series is always less than the function, then it follows that the series must also converge.

2.4.3. P Series Test

A power series is one of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{2.5}$$

Where the series is convergent for all all values $p > 1$, and divergent for all $p \leq 1$. When $p = 1$ we get the harmonic series again. Ultimately this test is just useful for making comparisons and relations with other functions, because well, $\frac{1}{n^p}$ is not a super common function.

2.4.4. Comparison Test

In a similar way to how we used comparison tests to see if an integral would converge in Calculus 1A, we can also compare two series to see if they'll converge. Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with **positive terms**.

1. If $\sum_{n=1}^{\infty} b_n$ is convergent, and $a_n \leq b_n$ for all n, then a_n *must* be convergent too.
2. If $\sum_{n=1}^{\infty} b_n$ is divergent, and $a_n \geq b_n$ for all n, then a_n *must* be divergent too.

Think about it. If $\sum b_n$ converges, it must be a finite number greater than 0. And so if $\sum a_n$ is less than this finite number, it is obviously also finite. The same logic applies if $\sum b_n$ diverges, If $\sum a_n$ is greater than an infinite value, it must also be infinite.

For example, $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ is always smaller than $\sum_{n=1}^{\infty} \frac{1}{n^3}$, And due to the P series test, we know that this converges, therefore, so too must $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$.

Similarly, $\sum_{n=1}^{\infty} \frac{1+\sin x^2}{\sqrt{n}}$ is always greater than $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges. Thus, so too must $\sum_{n=1}^{\infty} \frac{1+\sin x^2}{\sqrt{n}}$.

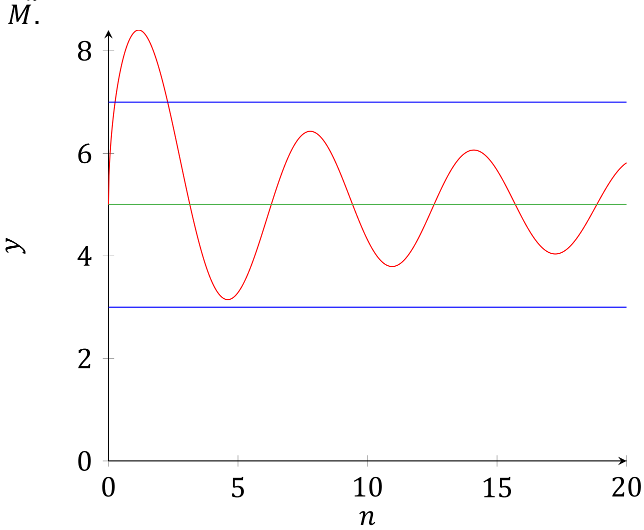
2.4.5. Limit Comparison Test

Suppose $\sum a_n$ and $\sum b_n$ are series with *positive terms*, then, if the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C$$

where C is a finite, positive number, then either; $\sum a_n$ and $\sum b_n$ are both convergent or both divergent.

Seems straightforward enough really, but why is this the case? Let m and M be positive numbers such that $m < C < M$. Because $\lim_{n\to\infty} \frac{a_n}{b_n} = C$, there is a number N such that for $n < \frac{a_n}{b_n} < M$ for all $n \geq N$. Beyond the point of $n = N$ the value of $\frac{a_n}{b_n}$ will be bounded by m and M.



Note how after around $n = 2.5$ the function $\frac{a_n}{b_n}$ is bound between the two blue lines? From this we can say that $N = 2.5$, but that doesn't matter. What matters is that point N exists. With this information at hand, knowing that we're taking a limit at infinity, we can do some algebra:

$$m < \frac{a_n}{b_n} < M$$

$$b_n m < a_n < b_n M$$

.

So from this we can now say, that if $\sum b_n$ is convergent, multiplying it by a constant won't change this. So $\sum b_n M$ is also convergent, and because this is greater than a_n then we can say that $\sum a_n$ is also convergent.

Similarly, if $\sum b_n$ is divergent, so is $\sum b_n m$ and thus $\sum a_n$ is also divergent.

2.5. Alternating Series

2.5.1. The Basics

An alternating series is one where the sequence alternates between positive and negative. Consider the following alternating series:

$$\sum_{n=1}^{\infty} (-1)^n \cdot a_n \tag{2.6}$$

If the series is constantly decreasing, or in other words, if $a_{n+1} \leq a_n$ for all values of n. **And**, if the limit, $\lim_{n\to\infty} a_n = 0$, then we can say that this series is convergent. An interesting concept to introduce now is the idea of Absolute convergence.

2.5.2. Absolute Convergence

A series $\sum a_n$ is called **absolutely convergent** if the series $\sum |a_n|$ is convergent. And we have a theorem that tells us that if a series is absolutely convergent then it is also normally convergent too. As in if $\sum |a_n|$ is convergent, so too is $\sum a_n$.

A series is called **conditionally convergent** if it is convergent, but not absolutely convergent. As in $\sum a_n$ converges, but $\sum |a_n|$ diverges. For the most part, we don't actually care if something is conditionally convergent because we are able to manipulate terms to make pretty much any series converge or diverge as we please. For example, the alternating harmonic series $\sum_{n=1)}^{\infty} \left(\frac{-1}{n}\right)^n$. This will converge, but only conditionally.

To check for convergence in an alternating series we have, almost a decision tree to follow. First check for absolute convergence, if this is the case then we know the series is convergent. If not, check for conditional convergence, then we see if something is convergent or divergent.

For example, if you want to know if $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges, you can take the absolute value for this series and easily find that it converges absolutely, and thus so too must the original series.

2.5.3. Ratio Test

The ratio test is *yet another* test we can use for series. It will tell us is a series is absolutely convergent, or divergent. That's it. What we do is we take the ratio of two values of a

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \tag{2.7}$$

Using the limit L that we calculate we can make some statements about the series. If $L < 1$ then the series is absolutely convergent, if $L > 1$ then the series is divergent. And if $L = 1$ then the ratio test is inconclusive and you need to do some other test for convergence.

If you think about this for a moment this will make sense. If the terms keep increasing, then of course the series will diverge, and naturally if the terms keep getting smaller, then it will diverge too.

2.6. Power Series

2.6.1. The Basics

A power series is one of the form:

$$\sum_{n=0}^{\infty} C_n x^n \tag{2.8}$$

This will expand out into:

$$C_0 + C_1 x + C_2 x^2....$$

Where the constants, in the sequence $\{C_n\}_{n=0}^{\infty}$ are called the coefficients of the power series. If these are all the same as each other (i.e. $C_0 = C_1 = C_2$) then we have a geometric series on our hands.

To put this in a more general way we can put a constant a into our format.

$$\sum_{n=0}^{\infty} C_n \left(x-a\right)^n \tag{2.9}$$

$$= C_0 + C_1(x-a) + C_2(x-a)^2...$$

2.6.2. Radius of Convergence

We call this a power series centred at $x = a$, or simply, a power series about a. From this we can introduce the concept of *The Radius of Convergence*. To illustrate it; let's do an example. Say we have a series: $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2}$ which is a power series about $x = 0$. But for which values of x does it converge? From equation (2.7) we can just try the ratio test to find out.

$$\lim_{n\to\infty}\left|\frac{x^n}{(n+1)^2}\frac{n^2}{x^{n-1}}\right|$$

$$\lim_{n\to\infty}|x|\cdot\left|\frac{n^2}{n^2+2n+1}\right|$$

$$|x|\cdot\lim_{n\to\infty}\left|\frac{n^2}{n^2+2n+1}\right|$$

$$= |x|$$

What we've deduced here is that the series will absolutely converge when $|x| < 1$ Why? Well, remember the basis of the ratio test, if the limit of the ratio between terms is less than 1, then we have absolute convergence. We found that the limit is $|x|$ so for the series to converge absolutely, $|x|$ *must* be less than 1. As with the ratio test, you must separately analyse the limit when it is equal to 1. The ratio is in conclusive.

But what if we want to make this a bit more general? In that case, take a look at our form (2.9). To check where it converges we first do the ratio test.

$$\lim_{n\to\infty}\left|\frac{C_{n+1}(x-a)^{n+1}}{C_n(x-a)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|\cdot|x-a|$$

The series will absolutely converge when this is less than 1. But we can rearrange this a bit more.

$$\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|\cdot|x-a|<1$$

$$|x-a|<\lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|$$

With this we can see what the radius of convergence is. It is the radius, about point a where the series will converge. We can use shorthand for further analysis, we'll call $\lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|$ the radius of convergence R.

It might make a bit more sense if we draw this out.

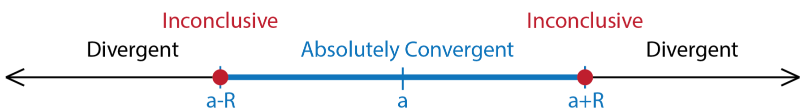


Figure 2.2: *Radius of convergence*

Moving away from inequalities, we can then just write the equation normally. And see the interval for which the series will converge. Remember, this is for a power series about a. So the centre of the interval will be a.

$$-R<x-a<R$$

$$a-R<x<a+R$$

There are 3 possibilities if you do this calculation to find R

- $R=0$ in this case, the series will only converge for $x=a$
- $R=\infty$ in this case, the series will always converge.
- $R>0$ in this case, the series is absolutely convergent for $|x-a|<R$, and you need to check what happens when $x=a-R$ or $x=a+R$

Finding the radius of convergence is a simple task once you work on it enough. You'll need to do some practice but eventually it'll click with you and it will make perfect sense. Try clear this up before we move onto the next topic.

2.7. Functions as a Series

2.7.1. The Basics

So the reason we're dealing with series so much is that they actually become good approximations of functions. In the previous quarter we dealt with linearization to approximate functions, but if we add in more terms (like with a series) we can get more accurate approximations.

When it comes down to it, if we have a converging series we can manipulate it to represent a function. We most commonly work with power series (*I will often call them "P Series", don't worry about it*) and manipulate them along with the functions. This stems from:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{2.10}$$

Valid for $|x| < 1$

2.7.2. Manipulations

This is all well and good but if we want to use this in any meaningful way we can manipulate the functions. What if we want to represent $\frac{1}{5-x}$? Well to do that we'd change it around to get it in a geometric format.

$$\frac{1}{5-x} \to \frac{\frac{1}{5}}{1-\frac{x}{5}} = \frac{1}{1-(x-4)} = \frac{\frac{1}{2}}{1-\frac{x-3}{2}}$$

Any one of these representations will work, and there are many more that you could choose as well. Taking the first one, if you replace x with $\frac{x}{5}$ then, it is then valid for $\left|\frac{x}{5}\right| < 1$ or in other terms, $-5 < x < 5$.

Essentially we can represent a function as a P series about almost any value of x, which gives convergence with different radii of convergence (R). So lets make a power series for $\frac{1}{5-x}$ about a.

$$\frac{1}{5-x} = \frac{1}{(5-a)-(x-a)} = \frac{\frac{1}{5-a}}{1-\frac{x-a}{5-a}}$$

And this can then easily be written as a geometric series:

$$\frac{1}{5-x} \sum_{n=0}^{\infty} \frac{1}{5-a} \left(\frac{x-a}{5-a}\right)^n$$

The centre of convergence is, obviously, a. Because that what we chose it to be. but to find the radius of convergence it's:

$$\left|\frac{x-a}{5-a}\right| < 1$$

$$a-5<x-a<5-a$$
$$2a-5<x<5$$

In general, to use this for any function you need to:

1. Start with a function that has a known power series equivalent. (eg: $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$)
2. Find a series of manipulations that transform your standard function (eg: $\frac{1}{1-x}$) into the desired function.
3. Apply these same manipulations to the P series to get a representation of the function as a series.

There are a whole load of different ways to manipulate functions, but for the purposes of generalisation, I'm going to define two functions as: $f(x)=\sum_{n=0}^{\infty}a_nx^n$ for $|x|<R_1$ and $g(x)=\sum_{n=0}^{\infty}b_nx^n$ for $|x|<R_2$

- **Substitution**: $f(cx^k)=\sum_{n=0}^{\infty}a_nc^nx^{kn}$ for $|cx^k|<R_1$
- **Multiplication**: $cx^k\cdot f(x)=\sum_{n=0}^{\infty}ca_nx^{n+k}$ $|x|<R_1$
- **Translation**: $f(x-c)=\sum_{n=0}^{\infty}a_n(x-c)^n$ for $|x-c|<R_1$
- **Addition**: $f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+bn)x^n$ for $|x|<\{R_1,R_2\}_{min}$
- **Differentiation**: $f'(x)=\sum_{n=0}^{\infty}na_nx^{n-1}$ for $|x|<R_1$
- **Integration**: $\int f(x)dx=\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ for $|x|<R_1$

Again, these all rely on you beginning with a known power series. For example:

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

For $|x|<1$

$$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

For all $x\in\mathbb{R}$

2.8. Taylor Series

2.8.1. The Basics

"The Taylor series of a function is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point"-*(from Wikipedia)*. Essentially, the Taylor series is another was of representing functions as a series. But it comes from taking lots and lots of derivatives. It's used all the time in computing to calculate complex functions, and in engineering, we use it all the time to simplify our calculations.

Instead of calculating a really long function to 100% accuracy, instead what we can do is just take the first 4 terms of the Taylor series, and we'll get a result that's close enough. Before we get started though there are a few things we need to cover.

Derivatives should make sense to you by now, but all we need to do now is introduce some new notation. The first, second, and third derivatives are written like this:

$$f', f''. f'''$$

This is fair enough, but what if you want to take the tenth derivative?

$$f''''''''''$$

You can see where this is going. With the Taylor series we take (in theory) infinitely many derivatives so the notation gets a bit out of hand. So to combat this we just write the number of dashes that should be there:

$$f''' = f^{(3)}; \quad f'''''''''' = f^{(10)}$$

So, let's condense this down a bit into an actual mathematical sentence. If $f(x)$ has a power series expansion at $x = a$, that is, for some Radius of convergence R which is greater than 0;

$$f(x) = \sum_{n=0}^{\infty} C_n \, (x-a)^n \; : \; |x-a| < R$$

then its coefficients are given by:

$$C_n = \frac{f^{(n)}(a)}{n!} \tag{2.11}$$

To understand this for yourself you can write out the different derivatives of the power series and see the pattern that emerges.

2.8.2. Taylor Polynomials

The Taylor series of $f(x) = f(a)$ is what we call the following series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \, (x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 ...$$

There is a special name for Taylor series about the origin (or when $a = 0$), it's called a Maclaurin series, after the Colin Maclaurin.

A Taylor Polynomial is a truncated Taylor series. So instead of using infinite terms to represent the function, we cut it off at a point and use that as an approximation of the function. This is the most common use of a Taylor series.

$$T_N(x) \; = \; \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} \, (x-a)^n \tag{2.12}$$

With this you can start to recognise a few things:

- $T_0(x) = f(a)$; Which is a constant.
- $T_1(x) = f(a) + f'(a)(x-a)$; Which is a linearisation of a function. We covered this in much more detail last quarter.
- $T_2(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2$ Notice how with each term our polynomial gets closer to the real function?

A very nice visualisation of this is the Taylor polynomial of $\sin x$ about $a = 0$. You can see that with increasing terms we get a polynomial that's closer and closer to $\sin x$, and that close to $x = 0$ the taylor functions are extremely close to the original. This shows us how we can approximate and still get a useful answer.

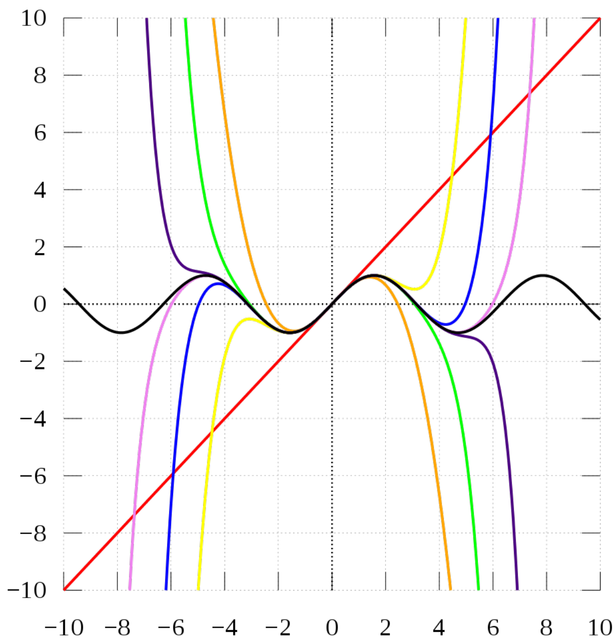


Figure 2.3: *Taylor polynomials of increasing degrees.*

2.8.3. Applications

Taylor polynomials are great for approximating a lot of functions, but in particular, we're going to look at how it helps us with integrals and with limits.

Starting with limits, for the most part we only really do this if we encounter a limit of indeterminate form (ie; $\frac{0}{0}$). Let's walk through an example.

$\lim_{x\to 0} \frac{\sin x}{x}$

Well, if you write out the Taylor series of $\sin x$ about $a = 0$ you will find that it is: $x - \frac{x^3}{6} + \frac{x^5}{120}$...., so with this at hand, we can rewrite our fraction as: $1 - \frac{x^2}{6} + \frac{x^4}{120}$..... Getting the limit of this is easy, it's 1.

Another example of finding a limit would be using the Taylor series of e^x:

$$lim_{x\to 0}\frac{e^x - 1 - x}{x^2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3....$$
$$\Rightarrow lim_{x \to 0} \frac{\frac{1}{2}x^2 + \frac{1}{6}x^3....}{x^2}$$
$$= \frac{1}{2}$$

Integrals are a little bit trickier,

$$\int_0^{\frac{1}{2}} xe^{-4x}dx$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This is a series around $x = a = 0$ which not ideal but it's easy. Ideally we'd select a to be the midpoint of the limits, but it doesn't make too much of a difference.

$$\Rightarrow xe^{-4} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{4n+1}$$

2.9. The Binomial Series

2.9.1. The Basics

The binomial series is kind of a shortcut we can use to express $(x + y)^n$. This is something you're probably familiar with from secondary school but for example, $(x+y)^2$ absolutely cannot be written as just $x^2 + y^2$ that's just, completely wrong, instead you need to write it out as $(x + y)(x + y)$ and multiply it out. Now, this is all well and good for small powers, but what if you want to raise it to a higher power? Not so easy then. That's where the Binomial series comes into play.

2.9.2. Combinations

If you have a collection of n things and you want to know how many combinations of m things you can choose from this collection how would you work that out? Well, when you select your first item, you have n options. Then for your second item you have $n - 1$ choices, and so on until you get to your mth item, when you have $(n + 1) - m$ options.

But then we must deal with the order of items picked. Because a combination of ABC is the same as CBA, and so on. So if you recall from secondary school; if you have m items then you can arrange it in $m!$ ways. If you don't understand this go to khan academy and revise it.

Because this is such a common thing to want to calculate we have a specific notation for it: $\binom{n}{m}$, you read this is "n choose m", meaning choose m items from a collection of n. Your calculator probably has a function to do this too, probably written as nCr

Putting this all together then, how do we write it mathematically?

$$\binom{n}{m} = \frac{n(n-1)(n-2)...(n+1-m)}{m!} = \frac{n!}{(n-m)!m!} \tag{2.13}$$

2.9.3. Using The Binomial Series

The binomial series for $(x+y)^n$ is as follows:

$$(x+y)^n = \sum_{n=0}^{m} \binom{n}{m} x^n y^{m-n} \tag{2.14}$$

The combinations bit gives us the *binomial coefficients* of the series. If you do out a whole load you'll notice a pattern emerging in the coefficients, and that's with an increasing m. You'll see Pascal's triangle, where each new layer is a for an increased m.

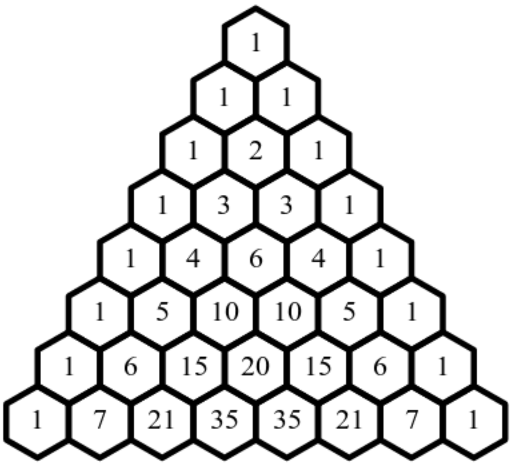


Figure 2.4: *The First Seven Rows of Pascal's Triangle*

2.9.4. Non-Integers

What we've covered so far is basically everything that you would have covered in secondary school, depending on where you went to school that is. But what about if you don't want to use an integer? Like what about $(1+x)^{\frac{1}{2}}$? In such cases, we have a very similar process.

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \tag{2.15}$$

This holds for $|\alpha| < 1$ and $|x| < 1$. What we see here is that this is actually another Taylor series. if you do out the differentials you'll see that we're essentially just creating the binomial coefficients again:

$$f(x) = (1+x)^\alpha$$

$$f'(0) = \frac{1}{0!}$$

$$f''(0) = \frac{\alpha}{1!}$$

$$f^3(0) = \frac{\alpha(\alpha-1)}{2!}$$

And so on.

As an example let's find $\sqrt{1+x}$ using this method. Because this is just another way of getting Taylor polynomials we can phrase it that way too. Let's find T_4 of this function.

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$\sum_{n=0}^{4} \binom{\frac{1}{2}}{n} x^n$$

$$\binom{\frac{1}{2}}{0} = 1$$

$$\binom{\frac{1}{2}}{1} = \frac{1}{2}$$

$$\binom{\frac{1}{2}}{2} = \frac{-1}{8}$$

$$\binom{\frac{1}{2}}{3} = \frac{1}{16}$$

$$\binom{\frac{1}{2}}{4} = \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdot \frac{-5}{2}}{4!} = \frac{-5}{128}$$

$$T_4 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$$

2.10. Series for Differential Equations

That's right! Once again we're taking a look at differential equations. They're quite common in Calculus apparently. Recalling the standard formats for a linear second order differential (eq:1.1) and a power series (eq:2.8):

$$P(x)y'' + Q(x)y' + R(x)y = G(x)$$

$$y(x) = \sum_{n=0}^{\infty} C_n x^n$$

1. Substitute the format for the power series into the Differential Equation
2. Rewrite the sums in order to match the powers of x
3. Combine the the sums into one sum
4. Find a recursive formula for the coefficients
5. Split the solutions into independent solutions.

This sounds a bit confusing, and rightfully so. So let's just work our way through an example and think about each step.

$$y'' + y = 0 :^{y(0)=1}_{y'(0)=0}$$

Recalling **step 1**, substitute in $y = \sum_{n=0}^{\infty} C_n x^n$ for $|x| < R$. To do this though, we need to calculate the differentials of a series. That's simple enough;

$$y' = \sum_{n=0}^{\infty} nC_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)C_n x^{n-2}$$

Now you can substitute this into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^n = 0$$

For **step 2** you need to rewrite the sums in the same powers of x. Because as you'll notice $x^n \neq x^{n-2}$. But, what happens if we write out the terms of the series $\sum_{n=0}^{\infty} n(n-1)C_n x^{n-2}$? due to that term of n in there we're multiplying by 0, meaning our terms look like:

$$0 + 0 + 2C_2 + 6C_3 x\ldots$$

So we can start at $n = 2$ and it won't actually make any difference to the sum of the series. We can then just, *shift* the series back to $n = 0$ as long as we manipulate the terms accordingly

$$\sum_{n=0}^{\infty} n(n-1)C_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} x^n + \sum_{n=0}^{\infty} C_n x^n = 0$$

Now that we've put the terms of x to the same power, for **step 3** we can easily combine the sums into:

$$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2} + C_n x^n = 0$$

For **step 4** we need to find a recursion formula for the coefficients. How do we do that? Well, we know that $(n+2)(n+1)C_{n+2} + C_n = 0$, and from here we can extrapolate. Rearranging our equation we can get a definition of C_{n+2}

$$C_{n+2} = \frac{-C_n}{(n+2)(n+1)}$$

If we have say, C_0 then we can find C_2 and if we have C_2 we can find C_4 and so on. Let's write out some terms to see if we notice a pattern emerging.

$$C_2 = \frac{-C_0}{(2)(1)}$$

$$C_3 = \frac{-C_1}{(3)(2)}$$

$$C_4 = \frac{-C_2}{(4)(3)} = C_0(4)(3)(2)(1)$$

$$C_5 = \frac{-C_3}{(5)(4)} = \frac{C_1}{(5)(4)(3)(2)}$$

You might see where this is going, for even values of n you have one pattern based off of C_0 and for odd values you have a pattern based off of C_0. This brings us to **step 5** where we split out solution into 2 different solutions, independent of one another.

$$C_{2n} = \frac{-1^n}{(2n)!}C_0 \ : \ C_{2n+1} = \frac{-1^n}{(2n+1)!}C_1$$

$$\Rightarrow y(x) = \sum_{n=0}^{\infty} C_{2n}x^{2n} + \sum_{n=0}^{\infty} C_{2n+1}x^{2n+1}$$

$$y(x) = C_0 \sum_{n=0}^{\infty} \frac{-1^n}{(2n)!}x^{2n} + C_1 \sum_{n=0}^{\infty} \frac{-1^n}{(2n+1)!}x^{2n+1}$$

From this point, you can then use the initial conditions to find the values of C_0 and C_1.

$$y(0) = \sum_{n=0}^{\infty} C_n x^n = C_0 + 0 + 0 + 0....$$

$$y'(0) = \sum_{n=0}^{\infty} nC_n x^{n-1} = 0 + C_1 + 0 + 0 + 0....$$

3

Multi-variable Functions

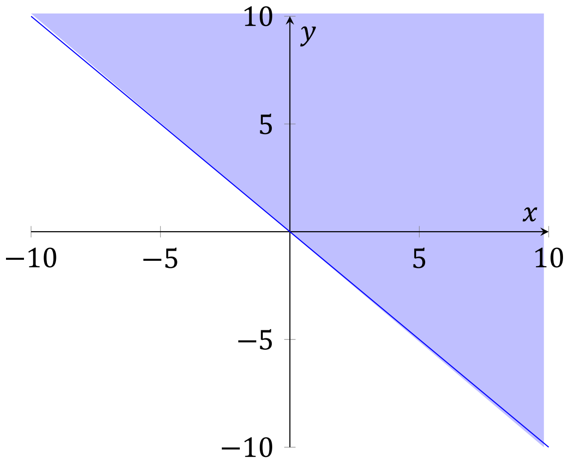
3.1. The Basics

Up to this point we have dealt with functions of just one variable ($f(x)$). But here we deal with functions of more than one, but usually two ($f(x, y)$). With this we have to rethink some of the definitions we used for terms in the last quarter.

Maximal Domain, or the Natural Domain of $f(x, y)$ is the set of all points (x, y) for which a function value exists. You can test this mathematically, by looking at the function at hand, and then drawing it on a plane.

$$f(x, y) = \sqrt{x + y}$$

$$\Rightarrow x + y \geq 0$$



$$D_f = \{(x, y)|y \geq -x\} \tag{3.1}$$

To describe the domain we can write it in a format similar to (eq 3.1). Dissecting that to see what it means: D_f means the domain of f. The curly brackets {} means a set of points, and the | means 'with the property of'.

The Range is the set of all possible values $f(x, y)$ with $(x, y) \in D_f$. Essentially, it's just all possible outputs of the function. Not very different from what we did last quarter.

A Graph of a multi-variable Function is a set of all points (x, y, z) where $z = f(x, y)$. This is a two variable function, and as such we can represent it in 3D. Going to higher dimensions is difficult due to the physical limitations of the universe, though it can in theory be done.

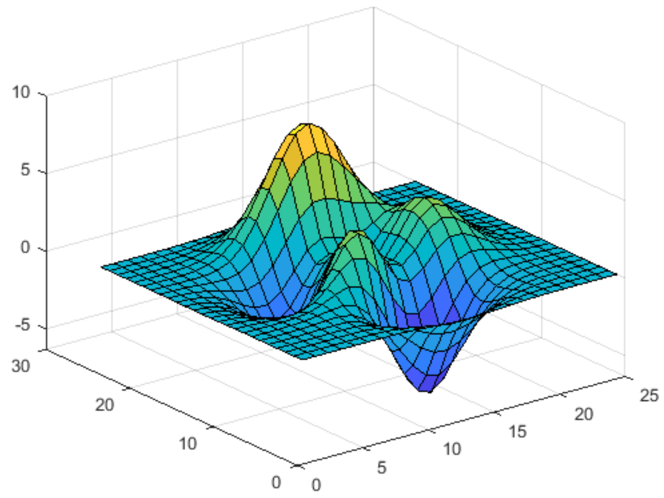


Figure 3.1: *A 3D Graph of a 2 Variable Function*

A Level Curve is a curve in $\mathbb{R}^2$ with a constant value of $f(x, y)$. It makes more sense if you look at a contour plot, or map. Whatever you want to call it.

A Contour Map is a way of representing a 3D graph in a 2D picture. It's made up of many level curves, each of them a section of the graph taken at a different height. These are often used in maps of terrain to show altitude differences. They're also used to show pressure differences in the atmosphere with isobars. They're useful.

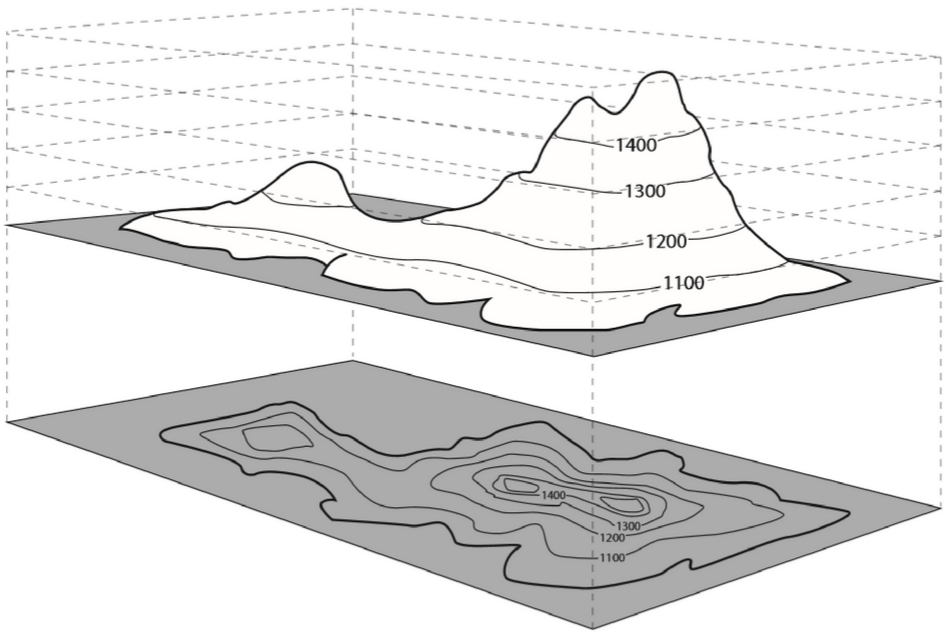


Figure 3.2: *How a Contour Map is Drawn*

3.2. Limits

3.2.1. Non-existant Limits

Oh joy, how fun it is to be working with limits again. I know, we've dealt with them an awful lot already, but bear with me on this. Let's just go over limits of 1 variable again to clear up some things.

$\lim_{x\to a} f(x) = L$

When we write out this equation what we're saying is "*if x approaches a, the $f(x)$ approaches L*". With two variables it's much the same. "*if (x, y) approaches (a, b), the $f(x, y)$ approaches L*"

$$\lim_{(x,y)\to(a,b)} f(x) = L \tag{3.2}$$

So while a limit of 1 variable exists if you approach it from the left or the right, a limit of two variables exists if you can approach from any direction and still get the same answer. If you take a top down view of the XY plane you can approach (a, b) from the north, south, east, west, or anywhere in between. There are infinitely many ways to approach a point so it's much harder to prove that a limit does exist.

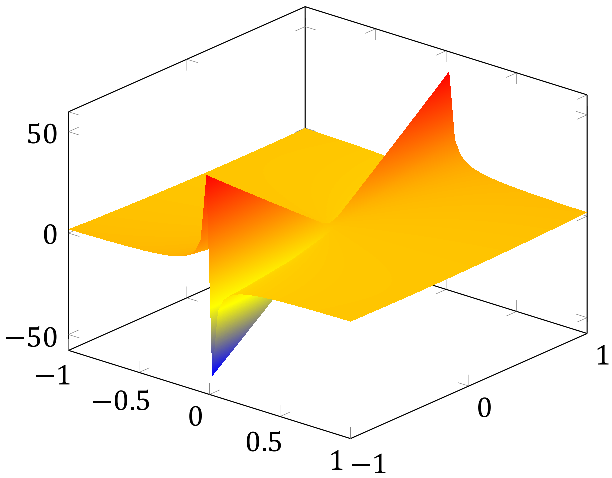
To prove that a limit does not exist though, all you need to do i show that any two paths approaching (a, b) will give a different result. In mathematical terms:

$$\text{if } \lim_{(x,y)\to(a,b)} f(x, y) = L_1 \text{ along } C_2$$

$$\text{and } \lim_{(x,y)\to(a,b)} f(x, y) = L_3 \text{ along } C_2$$

$$\text{and } L_1 \neq L_2 \text{, then } \lim_{(x,y)\to(a,b)} f(x, y) \text{ does not exist}$$

To give you an example, take the function $f(x, y) = \frac{xy}{x^2+y^2}$



If we want to find the limit as the function approaches $(0, 0)$ we can pick any path we like. Taking the path of $y = x$ we find that the limit is:

$$f(x, x) = \frac{x \cdot x}{x^2 + x^2} = \frac{1}{2}$$

This is one limit given by one path. And we can find this extremely easily because a 3D graph is essentially just infinite 2D graphs, and we already know how to deal with 2D limits. Taking another path of $y = 0$ we find:

$$f(x,0) = \frac{x \cdot 0}{x^2 + 0} = 0$$

The limit clearly doesn't exist.

3.2.2. Limits That do Exist

So it's impossible to show that a limit exists, but we can show that a limit *probably* exists. So we can use the **epsilon-delta** definition of a limit: Say we want $|f(x) - L|$ to be very small; smaller than a given value ϵ. Then there is a value $\delta > 0$ such that for all x; with $|x - a| < \delta$ we have $|f(x) - L < \epsilon|$

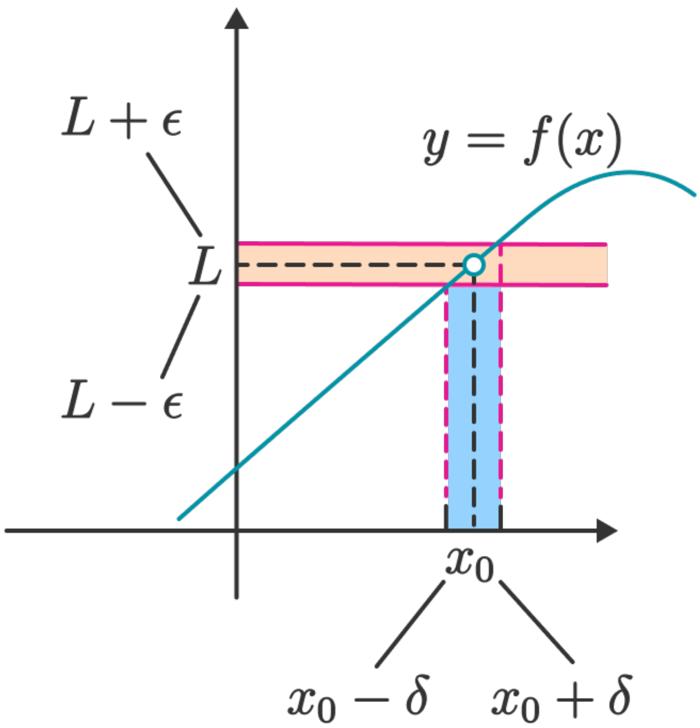


Figure 3.3: *Proving the existence of a limit*

If a limit doesn't exist, you won't be able to draw an infinitely small "*box*" around the whole function, ie, at a discontinuation. To bring this into 3D we instead try to draw a cylender around the function at the point.

Let f be a function of 2 variables with the domain D that includes points that are arbitrarily close to (a,b). Then the limit exists, if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $(x,y) \in D$ with $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ we have $|f(x,y) - L| < \epsilon$.

Again, this means that for a limit to exist, you need to be able to enclose the entire function at the given point. think this over a bit and recall that $\sqrt{(x-a)^2 + (y-b)^2}$ is the formula for a circle.

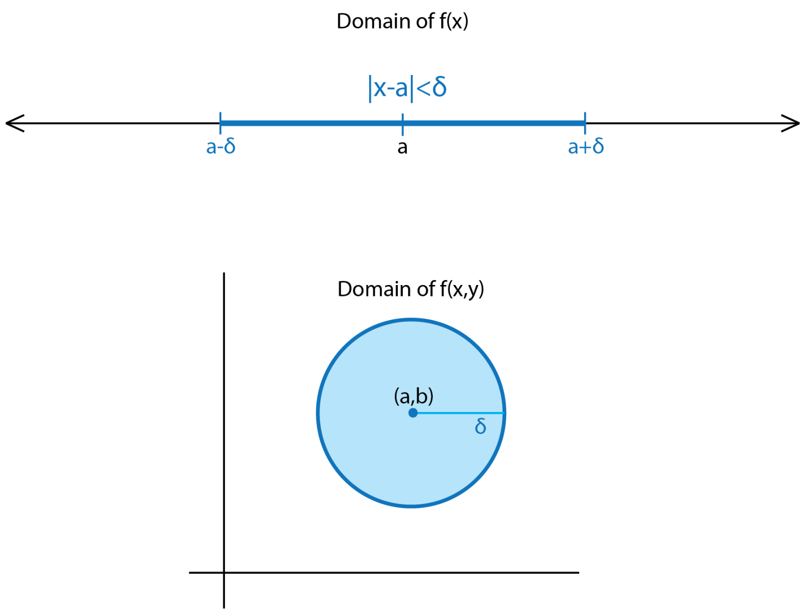


Figure 3.4: *The area around point a in the Domain in 2D and 3D*

3.2.3. Continuity

Continuity is the same again as in 2D. A function $f(x,y)$ is continuous at (a,b) if the limit $\lim_{(x,y)\to(a,b)} = f(a,b)$.

3.3. Partial Derivatives

The Basics So a partial derivative is very closely related to a derivative of a single variable. If you can recall that those are given by:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \tag{3.3}$$

In a similar manner, partial derivatives with respect to x and y at (a,b) are given by:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} \tag{3.4}$$

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h} \tag{3.5}$$

The notation involved is, for some reason, extensive. The subscript f_x means that you take a partial derivative with respect to x, i.e. with a constant y. This can also be written as $\frac{\partial f}{\partial x}$, or $\frac{\partial}{\partial x} f(x,y)$, or $\frac{\partial f}{\partial x} |_y$. It makes more sense if you look at a diagram. We take a plane, at a constant point of $y = b$, or if we're differentiating with respect to y, we take a constant $x = a$.

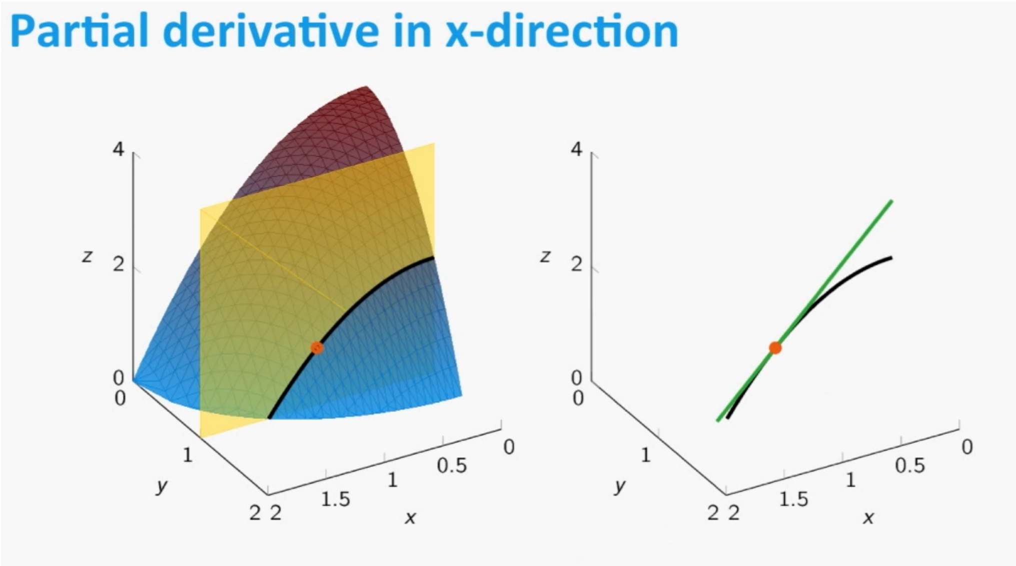


Figure 3.5: *Partial Derivative With a Constant Y*

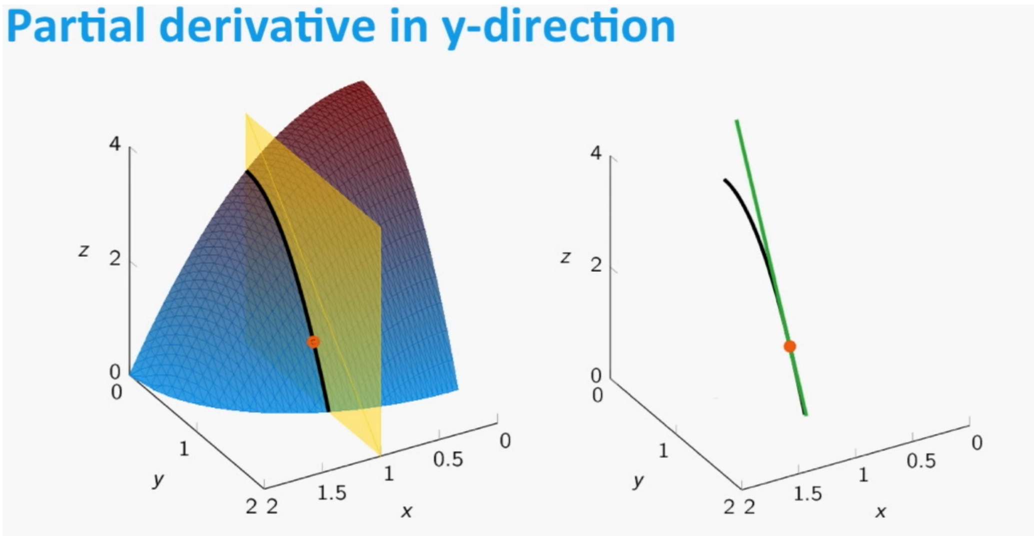


Figure 3.6: *Partial Derivative With a Constant X*

3.3.1. Clairaut's Theorem

When it comes to taking higher partial derivatives the notation matters. For example f_{xx} means you differentiate with respect to x, then again with respect to x. f_{xy} means differentiate with respect to x and then with respect to y. You get the picture.

Clairaut's theorem is; If f_{xy} and f_{yx} are both continuous in a neighbourhood around (a, b), then $f_{xy} = f_{yx}$. That's the theorem, but what's the proof?

$$f_x(x_0, y_0) \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$f_{xy}(x_0, y_0) \lim_{y \to y_0} \frac{f_x(x_0, y) - f_x(x_0, y_0)}{y - y_0}$$

$$= \lim_{y \to y_0} \lim_{x \to x_0} \frac{f(x,y) - f(x_0,y) - f(x,y_0) + f(x_0,y_0)}{(x-x_0)(y-y_0)}$$

If the derivatives are both continuous, then all limits are OK.

3.4. Tangent Planes

3.4.1. The Basics

The tangent plan to the graph of f, at the point P is the plane through the tangent lines T_1 and T_2, which are tangent to the curves C_1 and C_2 at point P, on the surface $x = f(x,y)$. Noting that C_1 and C_2 should not be parallel neat point P.

Finding a tangent plan involves a simple step-by-step procedure:

1. Find T_1 and T_2
2. Construct a vector normal to the surface of the graph (and the tangent plane)
3. Use the normal vector to find a formula for the tangent plane

You can find the normal vector $\vec{n}$ by getting the cross product of two vectors along the tangent lines T_1 and T_2. We'll call these $\vec{v}_1$ and $\vec{v}_2$ respectively.

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}, \ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$$

Notice that we're using the partial derivatives to define our vectors. If we take a cross product of these (the order we take them doesn't matter, as the vector will still be normal to the tangent plane), we get a resultant vector of:

$$\vec{v}_2 \times \vec{v}_1 = \vec{n} = \begin{pmatrix} f_x \\ f_y \\ -1 \end{pmatrix}$$

Note the dot product of $\vec{n}$ with any vector in the tangent plan must be 0. Therefore:

- Construct a vector in the tangent plane
- Choose point $A = (x,y,z)$ in the plane
- Draw the vector $\vec{PA}$

$$\vec{PA} = \begin{pmatrix} x-a \\ y-b \\ z-f(a,b) \end{pmatrix}$$

$$\vec{n} \cdot \vec{PA} = 0$$

Writing this out the long way yields:

$$z = f_x(x-a) + f_y(y-b) + f(a,b)$$

Does this look familiar to you? It should, it's the 3D version of the linearisation function we saw in the last quarter. $L(x) = y = f(a) + f'(a)(x - a)$. If we want to bring it into 4D we get: $w = f(a,b,c) + f_x(x - a) + f_y(y - b) + f_z(z - c)$.

Linearisation is good when you have a reasonably smooth surface, i.e. if we're looking for the linearisation of $f(x,y)$ near point (a,b) then the linearisation is a good approximation, only if the function is differentiable. If the partial derivatives of f_x and f_y exist in a neighbourhood around (a,b), and are continuous at (a,b), then $f(x,y)$ is differentiable at (a,b)

3.4.2. Differentials

For a differentiable function of 2 variables, $z = f(x,y)$, we define the differentials dx and dy to be independent variables. Then the total differential dz is defined by:

$$dz = f_x(x,y)dx + f_y(x,y)dy \tag{3.6}$$

If we merge this with our tangent plane formula we get:

$$z - f(a,b) = f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

Where

- $dz = z - f(ab)$
- $dx = (x - a)$
- $dy = (y - b)$

We can call the differentials "Errors" or "Deviations"or "changes", if we like, it depends on the context. dz being the resulting error etc.