WI1402-LR Calculus II
Delft University of Technology

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Preface

This summary was written for the course WI1402-LR Calculus II, taught at the Delft University of Technology. All the material treated is taken from [J. Stewart. *Calculus: Early Transcendentals*. Brooks/Cole, 7th edition, 2012.]

Throughout the summary, references to chapters and sections can be found. These are labelled with the aid of the symbol § and can be found in the aforementioned book, where exercises and more explanations are given.

In case of any comments about the content of the summary, please do not hesitate to contact me at `m.facchinelli@yahoo.it`.

"Mathematics is the music of reason."

– James Joseph Sylvester

Changelog

This is version 1.0. Below are listed the changes applied to each version.

Version	Date	Changes
1.0	February 1, 2017	First version

Chapter 14

§14.5

Chain Rule

$$\frac{\mathrm{d}f(g(t),h(t))}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \quad \text{for } \textit{one} \text{ variable}$$

$$\frac{\partial f(g(s,t),h(s,t))}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\partial y}{\mathrm{d}t} \quad \text{for } \textit{two} \text{ variables (same holds for } s)$$

§14.6

Directional Derivative

$$D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z)\cdot\hat{\mathbf{u}}$$

where $\hat{\mathbf{u}}$ is a unit vector.

(i) Another definition is $D_{\mathbf{u}}f(x,y) = f_x(x,y)\ a + f_y(x.y)\ b$, where $\hat{\mathbf{u}} = \begin{bmatrix} a \\ b \end{bmatrix}$.

§14.7

Critical Point at (a, b)

$$\text{if } f_x(a,b) = 0 \text{ and } f_y(a,b) = 0$$

Determinant of Hesse

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

$$\begin{cases} \text{if } D > 0 \text{ and } f_{xx} < 0 & \textit{maximum} \\ \text{if } D > 0 \text{ and } f_{xx} > 0 & \textit{minimum} \\ \text{if } D < 0 & \textit{saddle point} \\ \text{if } D = 0 & \textit{no information} \end{cases}$$

Chapter 15

§15.1, §15.2, §15.3

Double Integrals

$$V = \iint_R f(x,y)\,\mathrm{d}A$$

(i) The average value of a function is found by $f_{avg} = \frac{1}{A(R)}\iint_R f(x,y)\,\mathrm{d}A$;

(ii) If $R = \{(x,y)|\ a \le x \le b,\ c \le y \le d\}$, then $\iint_R f(x,y)\,\mathrm{d}A = \int_a^b\int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_c^d\int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$;

(iii) If R, then $\iint_R g(x)h(y)\,\mathrm{d}A = \int_a^b g(x)\,\mathrm{d}x\int_c^d g(y)\,\mathrm{d}y$;

(iv) If $D = \{(x,y)|\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, then $\iint_R f(x,y)\,\mathrm{d}A = \int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$;

(v) If D, then $A(D) = \iint_D \mathrm{d}A$.

§15.4

Polar Coordinates

$$r^2 = x^2 + y^2 \qquad x = r\cos\vartheta \qquad y = r\sin\vartheta$$

(i) If $R = \{(x,y)|\ 0 \le a \le r \le b,\ \alpha \le \vartheta \le \beta\}$, then $\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta\int_a^b f(r\cos\vartheta, r\sin\vartheta)r\,\mathrm{d}r\,\mathrm{d}\vartheta$;

(ii) Note that $\mathrm{d}A = \mathrm{d}(x,y) = r\,\mathrm{d}(r,\vartheta)$.

§15.5

APPLICATION OF DOUBLE INTEGRALS

$$\begin{cases} \text{mass} & m = \iint_D \rho(x,y)\,\mathrm{d}A \\ \text{centroid} & \bar{x} = \frac{1}{m}\iint_D x\rho(x,y)\,\mathrm{d}A,\ \ \bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,\mathrm{d}A \\ \text{moment of inertia} & I_x = \frac{1}{m}\iint_D y^2\rho(x,y)\,\mathrm{d}A,\ \ I_y = \frac{1}{m}\iint_D x^2\rho(x,y)\,\mathrm{d}A \end{cases}$$

§15.7

TRIPLE INTEGRALS

$$W = \iiint_E f(x,y,z)\,\mathrm{d}V$$

(i) If $B = [a,b] \times [c,d] \times [e,o]$, then $\int_a^b \int_c^d \int_e^o f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$;

(ii) If $E = \{(x,y,z) |\ a \le x \le b,\ c \le y \le d,\ e \le z \le o\}$, then $V(E) = \iiint_E \mathrm{d}V$;

(iii) Mass: $m = \iiint_E \rho(x,y,z)\,\mathrm{d}V$;

(iv) Centroid: $\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\,\mathrm{d}V,\ \ \bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\,\mathrm{d}V,\ \ \bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,\mathrm{d}V$.

§15.8

CYLINDRICAL COORDINATES

$$r^2 = x^2 + y^2 \qquad x = r\cos\vartheta \qquad y = r\sin\vartheta \qquad z = z$$

(i) If $E = \{(x,y,z) |\ 0 \le a \le r \le b,\ \alpha \le \vartheta \le \beta,\ c \le z \le d\}$, then $\iiint_E f(x,y,z)\,\mathrm{d}V = \int_\alpha^\beta \int_a^b \int_c^d f(r\cos\vartheta, r\sin\vartheta, z) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\vartheta$;

(ii) Note that $\mathrm{d}V = \mathrm{d}(x,y,z) = r\,\mathrm{d}(r,\vartheta,z)$.

§15.9

SPHERICAL COORDINATES

$$\rho^2 = x^2 + y^2 + z^2 \qquad x = \rho\sin\varphi\cos\vartheta \qquad y = \rho\sin\varphi\sin\vartheta \qquad z = \rho\cos\varphi$$

(i) If $E = \{(x,y,z) |\ 0 \le a \le \rho \le b,\ \alpha \le \vartheta \le \beta,\ \gamma \le \varphi \le \delta\}$, then $\iiint_E f(x,y,z)\,\mathrm{d}V = \int_\alpha^\beta \int_\gamma^\delta \int_a^b f(\rho\sin\varphi\cos\vartheta,\ \rho\sin\varphi\sin\vartheta,\ \rho\cos\varphi)\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}\vartheta$;

(ii) Note that $\mathrm{d}V = \mathrm{d}(x,y,z) = \rho^2\sin\varphi\,\mathrm{d}(\rho,\varphi,\vartheta)$.

§15.10

JACOBIAN

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{of the transformation } T \text{ given by} \quad \begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

CHANGE OF VARIABLES

$$\iint_D f(x,y)\,\mathrm{d}A = \iint_S f(x(u,v), y(u,v))\frac{\partial(x,y)}{\partial(u,v)}\,\mathrm{d}(u,v)$$

the same holds for three dimentions.

Chapter 16

§16.1

VECTOR FIELDS

$$\text{in } \mathbb{R}^2 \to \mathbf{F}(x, y) = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$$

$$\text{in } \mathbb{R}^3 \to \mathbf{F}(x, y, z) = \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$$

GRADIENT FIELD

$$f(x, y) \to \nabla f(x, y) = \begin{bmatrix} \partial f/\partial x \\ \partial f/\partial y \end{bmatrix} \to \mathbf{F}(x, y)$$

then $\mathbf{F}$ is a conservative vector field and f is a potential function for $\mathbf{F}$.

§16.2

LINE INTEGRAL ALONG A SMOOTH CURVE

$$\int_C f(x, y, z)\,\mathrm{d}s = \int_a^b f(\mathbf{r}(t))\,|\dot{\mathbf{r}}(t)|\,\mathrm{d}t$$

(i) Mass and Centroid: $\int_C \rho(x, y)\,\mathrm{d}s$, $\bar{x} = \frac{1}{m}\int_C x\rho(x, y)\,\mathrm{d}s$, $\bar{y} = \frac{1}{m}\int_C y\rho(x, y)\,\mathrm{d}s$;

(ii) Special cases: $\int_C f(x, y)\,\mathrm{d}x = \int_a^b f(x(t), y(t))\dot{x}(t)\,\mathrm{d}t$, $\int_C f(x, y)\,\mathrm{d}y = \int_a^b f(x(t), y(t))\dot{y}(t)\,\mathrm{d}t$;

(iii) Combination $\int_C P(x, y)\,\mathrm{d}x + \int_C Q(x, y)\,\mathrm{d}y = \int_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$.

LINE INTEGRAL OF A VECTOR FIELD

$$\int_C \mathbf{F}\,\mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t)\,\mathrm{d}t$$

or $\int_C \mathbf{F}\,\mathrm{d}\mathbf{r} = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}x$, where $\mathbf{F}(x, y, z) = \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$.

§16.3

FUNDAMENTAL THEOREM OF LINE INTEGRALS

$$\int_C \nabla f\,\mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

(i) $\int_C \mathbf{F}\,\mathrm{d}\mathbf{r}$ is called independent of path if $\int_{C_1} \mathbf{F}\,\mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F}\,\mathrm{d}\mathbf{r}$, where C_1 and C_2 have same initial and starting points;

(ii) $\int_C \mathbf{F}\,\mathrm{d}\mathbf{r}$ is independent of path if and only if $\oint_C \mathbf{F}\,\mathrm{d}\mathbf{r} = 0$.

(iii) If $\mathbf{F}$ is continuous in an open and connected regionand $\int_C \mathbf{F}\,\mathrm{d}\mathbf{r}$ is independent of path, then $\mathbf{F}$ is conservative (hence $\mathbf{F} = \nabla f$);

(iv) Let $\mathbf{F} = \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$ on an open simply-connected region and suppose $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then $\mathbf{F}$ is conservative.

§16.4

GREEN'S THEOREM

$$\oint_C \mathbf{F}\, d\mathbf{r} = \oint_{\partial D} P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

AREA OF A REGION D

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = x \end{cases} \rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \rightarrow \oint_C x\, dy$$

$$\begin{cases} P(x,y) = -y \\ Q(x,y) = 0 \end{cases} \rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \rightarrow \oint_C y\, dx$$

$$\begin{cases} P(x,y) = -\frac{1}{2}y \\ Q(x,y) = \frac{1}{2}x \end{cases} \rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \rightarrow \frac{1}{2}\oint_C x\, dy - y\, dx$$

§16.5

CURL

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

(i) If $\mathbf{F}$ is defined in $\mathbb{R}^3$ and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is *conservative*;

(ii) Green's Theorem can be rewritten as $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$ in $\mathbb{R}^2$.

DIVERGENCE

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

(i) If $\mathbf{F}$ is defined in $\mathbb{R}^3$, div (curl $\mathbf{F}$) $= 0$;

(ii) Green's Theorem can be rewritten as $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \cdot \mathbf{F})\, dA$ in $\mathbb{R}^2$.

§16.6

SURFACE AREA

$$A(S) = \iint_S |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

which is the area of the smooth parametric surface $\mathbf{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ in the domain D, and with $\mathbf{r}_u = \begin{bmatrix} \partial x/\partial u \\ \partial y/\partial u \\ \partial z/\partial u \end{bmatrix}$ and $\mathbf{r}_v = \begin{bmatrix} \partial x/\partial v \\ \partial y/\partial v \\ \partial z/\partial v \end{bmatrix}$.

(i) Tangent Plane $S : \mathbf{r}(u,v) = \mathbf{r}_0 + u\mathbf{r}_u + v\mathbf{r}_v$.

§16.7

SURFACE INTEGRAL

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

(i) Mass: $m = \iint_S \rho(x,y,z)\, dS$;

(ii) Centroid: $\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\, dS$, $\bar{y} = \frac{1}{m}\iint_S y\rho(x,y,z)\, dS$, $\bar{z} = \frac{1}{m}\iint_S z\rho(x,y,z)\, dS$.

FLUX INTEGRAL

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

where $\mathbf{n}$ is the normal vector.

§16.8

Stokes' Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \iint_D (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

where $C = \partial S$.

§16.9

Divergence (or Gauss) Theorem

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F}) \, dV$$

(i) If the survace is not closed: $\iint_S \mathbf{F} \cdot d\mathbf{S} = \oiint_{S \cup S'} \mathbf{F} \cdot d\mathbf{S} - \iint_{S'} \mathbf{F} \cdot d\mathbf{S}$

General Relations

Clairaut's Theorem

If f has continuous partial derivatives, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Gradient Theorem

If f represents a scalar field, then

$$\oiint_S f \, d\mathbf{S} = \iiint_E \nabla f \, dV$$

This was:

WI1402-LR Calculus II

Version 1.0

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