Summary WI1402LR: Calculus II

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Based on Calculus 6e (James Stewart) & Lecture notes

Chapter 14: Partial Derivatives

§14.5: The Chain Rule (p. 901)

General version:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Implicit differentiation:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Also holds for other partial derivatives.

§14.6: Directional Derivatives and the Gradient Vector (p. 910)

Directional derivatives make it possible to calculate rates of change in directions other than x and y.

Directional derivative:

$D_u f(x,y) = \nabla f(x,y) \cdot \boldsymbol{u} = f_x(x,y)a + f_y(x,y)b$, with $\boldsymbol{u} = \langle a, b \rangle$ a unit vector in the desired direction.

Gradient vector:

$$\nabla f(x,y) = \langle f_x(x,y,z), f_y(x,y,z) \rangle = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}$$

The dot product of the gradient vector and a unit vector $\boldsymbol{u}$, expresses the directional derivative in the direction of $\boldsymbol{u}$ as the scalar projection of the gradient vector onto $\boldsymbol{u}$.

The maximum value of the directional derivative occurs when $\boldsymbol{u}$ has the same direction as the gradient vector $\nabla f(x,y)$, and is equal to the length of the gradient vector ($|\nabla f(x,y)|$.

Tangent plane and normal line:

The equation for a tangent plane is given by:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

The equation for a normal line is given by:

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

§14.7: Minimum and Maximum Values (p. 922)

Local versus global/absolute and critical point:

A <u>global</u> or <u>absolute</u> minimum or maximum is an extreme value on the full domain of the function, a <u>local</u> minimum or maximum is an extreme value on a designated domain $[a,b]$. The partial derivatives on a <u>critical point</u> are equal to zero. If that equation cannot be solved, the function has no critical point(s).

Stationary/saddle point:

A stationary or saddle point is a local maximum with respect to one curve, and a local minimum with respect to the other. (Think of $y = x^3$ at $x = 0$.)

Second derivatives test:
The second derivatives test gives the nature of a critical point.

First, define $H_f(a,b) = \begin{bmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{bmatrix}$. The determinant of $H_f(a,b)$ and the value of $f_{xx}(a,b)$ show whether the critical point is a local minimum, maximum or a saddle/stationary point.

1. If $\det\left(H_f(a,b)\right) > 0$ and $f_{xx}(a,b) > 0$, (a,b) is a local minimum.
2. If $\det\left(H_f(a,b)\right) > 0$ and $f_{xx}(a,b) < 0$, (a,b) is a local maximum.
3. If $\det\left(H_f(a,b)\right) < 0$, (a,b) is a saddle point.

If $\det\left(H_f(a,b)\right) = 0$, the test is inconclusive.

Global/absolute minimum and maximum:
To find the global/absolute extreme value, there are three steps to take:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest from (1) and (2) is the global/absolute maximum, the smallest the global/absolute minimum.

Chapter 15: Multiple Integrals

§15.1: Double Integrals over Rectangles (p. 951)

Definition:

The double integral is defined in the same way as the single integral, and can be approximated by a (double) Riemann sum.

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

Midpoint Rule:

Rather than taking an arbitrary point $\left(x_{ij}^*, y_{ij}^*\right)$, the Midpoint Rule allows taking the center $\left(\overline{x_\iota}, \overline{y_\jmath}\right)$ of R_{ij} and then computing the Riemann sum.

$$\iint_R f(x,y)dA \approx \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(\overline{x_\iota}, \overline{y_\jmath}\right) \Delta A$$

where $\overline{x_\iota}$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y_\jmath}$ is the midpoint of $\left[y_{j-1}, y_j\right]$.

§15.2: Iterated Integrals (p. 959)

Fubini's theorem:

When evaluating a double (or triple) integral, you are free to choose the order of integration.

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

§15.3: Double Integrals over General Regions (p. 965)

Type I and Type II regions:

Type I: $D = \{(x,y) |\ a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$

Type II: $D = \{(x,y) |\ c \leq y \leq d,\ h_1(x) \leq x \leq h_2(x)\}$

In integration, there is not much of a difference. One only has to pay attention to the order of integration.

§15.4: Double Integrals in Polar Coordinates (p. 974)

Polar coordinates:

Rather than specifying a point by giving the distances to the origin on multiple axes, polar coordinates specify a point by the (shortest) distance to the origin (radius r) and an angle (θ). Converting can be done with these formula's:

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Jacobian:

When evaluating an integral in polar coordinates, the function to integrate must first be multiplied with a Jacobian (in this case a factor r, printed in bold in the formula below).

$$\iint_r f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, \boldsymbol{r}\, dr\, d\theta$$

§15.5: Applications of Double Integrals (p. 980)

Double integrals can be used to find the center of mass and the moment of inertia of a general volume.

Center of mass:

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\, \rho(x,y)\, dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\, \rho(x,y) dA$$

where the mass m is given by

$$m = \iint_D \rho(x,y) dA$$

Moment of Inertia:

$$I_x = \iint_D y^2 \rho(x,y) dA$$

$$I_y = \iint_D x^2 \rho(x,y) dA$$

$$I_0 = \iint_D (x^2 + y^2) \rho(x,y) dA = I_x + I_y$$

§15.6: Triple Integrals (p. 990)

Definition:

The triple integral of f over the box B is defined as

$$\iiint_B f(x,y,z) dV = \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^8\right) \Delta V$$

Fubini's theorem:

$$\iiint_B f(x,y,z) dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

Applications:

Mind that the result of a triple integration is a hypervolume (since you're integrating *over a volume* (3D already), and by integrating, add an extra dimension), something that can only exist in 4D. Although that is hard to imagine (if possible at all), there are some applications of triple integrals: calculation of moments, centers of mass and moments of inertia, for example. Electrical charge can also be expressed as a triple integral.

§15.7: Triple Integrals in Cylindrical Coordinates (p. 1000)

Just as the polar coordinate system is an alternative to a 2D Cartesian system, a cylindrical coordinate system is an alternative to a 3D Cartesian system. The third dimension is the z-axis and, surprisingly, stays the same when converting to cylindrical coordinates.

$$r^2 = x^2 + y^2$$

$$x = r \cos\theta$$

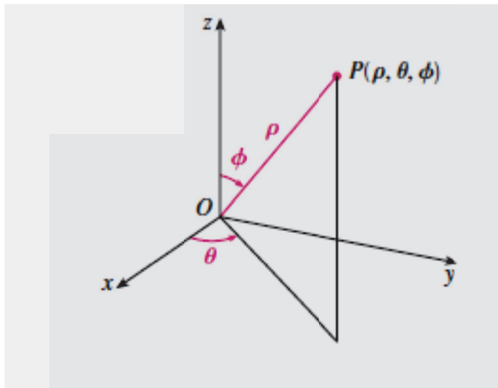
$$y = r \sin\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

§15.8: Triple Integrals in Spherical Coordinates (p. 1005)

Another 3D coordinate system is the spherical coordinate system. Just as the (2D) polar coordinate system, it has a coordinate specifying the distance to the origin (radius, now called ρ) and an angle θ. In addition, the spherical coordinate system introduces another angle (ϕ), which is the angle between the vector of the point and the z-axis.



$\rho > 0$
$0 \le \phi \le \pi$
$0 \le \theta \le 2\pi$

Conversion:

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$
$\rho^2 = x^2 + y^2 + z^2$

Jacobian:

Rather than having to add a Jacobian factor r (as in the polar coordinate system), the Jacobian for a spherical coordinate system is $\rho^2 \sin\phi$.

§15.9: Change of Variables in Multiple Integrals (p. 1012)

Jacobian:

The Jacobian of the (3D) transformation $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = k(u, v, w)$: is given by:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

This determinant can be computed by cofactor expansion. Obviously, in a 2D transformation, the z- and w-terms disappear.

Change of variables in a double integral:

$$\iint_R f(x, y)\, dA = \iint_S f\big(x(u, v,), y(u, v,)\big) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

Change of variables in a triple integral:

$$\iiint_R f(x, y, z) dV = \iiint_S f\big(x(u, v, w), y(u, v, w), z(u, v, w)\big) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw$$

Chapter 16: Vector Calculus

§16.1: Vector Field (p. 1027)

Definition:

A vector field is a function $\boldsymbol{F}$ which assigns to each point (x, y) a two-dimensional vector $\boldsymbol{F}(x, y)$. Vector fields also exist in 3D: just extrapolate.

The length of a vector is equal to the distance of the point to the origin.

A gradient field is given by:

$$\nabla f(x,y,z) = f_x(x,y,z)\boldsymbol{i} + f_y(x,y,z)\boldsymbol{j}, + f_z(x,y,z)\boldsymbol{k}$$

§16.2: Line Integrals (p. 1034)

Definition:

The line integral of f along a smooth curve C is given by:

$$\int_C f(x,y)ds = \int_a^b f\big(x(t), y(t)\big)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The line integrals with respect to arc length are given by:

$$\int_C f(x,y)dx = \int_a^b f\big(x(t), y(t)\big)\, x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f\big(x(t), y(t)\big)\, y'(t)dt$$

If the smooth curve C is given by a vector function $\boldsymbol{r}(t)$, $a \le t \le b$, the line integral of F along C is:

$$\int_c \boldsymbol{F}\cdot d\boldsymbol{r} = \int_a^b \boldsymbol{F}\big(\boldsymbol{r}(t)\big)\cdot \boldsymbol{r}'(t)dt = \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$$

($\boldsymbol{T}$ is the unit tangent vector, see § 13.2.)`

§16.3: The Fundamental Theorem for Line Integrals (p. 1046)

Definition:

The Fundamental Theorem for Line Integrals:

$$\int_C \nabla f\cdot d\boldsymbol{r} = f\big(\boldsymbol{r}(b)\big) - f\big(\boldsymbol{r}(a)\big)$$

where C is a smooth curve given by $\boldsymbol{r}(t)$, $a \le t \le b$.

Conservative vector fields:

A vector field is conservative if there is a scalar function f such that $\boldsymbol{F} = \nabla f$. Such a function can be found in the following manner:

1. Find the partial derivatives of the function (simply the $\boldsymbol{i}$-, $\boldsymbol{j}$- and possible $\boldsymbol{k}$-terms).
2. Integrate $f_x(x,y,z)$ with respect to x, which gives a formula with a constant (equal to $g(y,z)$).
3. Differentiate this integral and constant to y, in which $g(y,z)$ is replaced by $g_y(y,z)$. When comparing this function to $f_y(x,y,z)$ found in step 1, $g_y(y,z)$ can be determined.
4. Integrate $g_y(y,z)$ and fill it in as the constant in the function found in step 2 ,which then gives a function with another constant, $h(z)$.
5. Differentiate to z and comparing with $f_z(x,y,z)$ determined earlier, $h'(z)$ and thus $h(z)$ can be found.

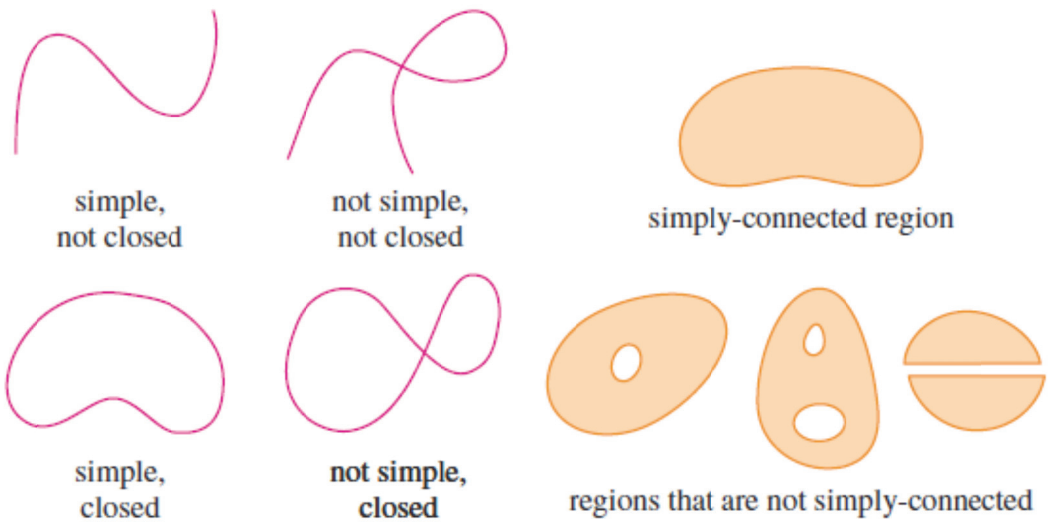
Line integrals of a conservative vector field are <u>independent of path</u>. This means that $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{r}$, as long as the starting and ending points of C_1 and C_2 are equal.

If $\boldsymbol{F}(x,y) = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$ is a conservative vector field, $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ (by Clairaut's Theorem, which states $f_{xy} = f_{yx}$). The statement is not always reversible (although it can be in some cases, namely, when the region is not *closed* and *simply connected*).

Closed and simple paths and simply connected regions:
If the starting and ending points of a path are the same (if $\boldsymbol{r}(b) = \boldsymbol{r}(a)$) the path C is <u>closed</u>, and $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$. Closed paths thus need to be split up into two (or more) separate, open paths.

A <u>simple path</u> is a path that does not intersect itself.

A <u>simply connected region</u> is a region without any holes in it.



§16.4: Green's Theorem (p. 1055)

Green's Theorem gives the <u>relationship between a line integral around a simple closed curve C and a double integral over the plan region D bounded by C</u>. A positive orientation of C relates to a counterclockwise traversal of C.

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

provided that P and Q have continuous partial derivatives on D.

§16.5: Curl and Divergence (p. 1061)

Curl:
If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the curl of $\boldsymbol{F}$ is the vector field on $\mathbb{R}^3$ defined by:

$$\text{curl}\,\boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k} = \nabla \times \boldsymbol{F} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

If f is a function of three variables and has continuous second order partial derivatives, then $\text{curl}(\nabla f) = 0$.

If $\text{curl } \boldsymbol{F} = 0$ and $\boldsymbol{F}$ has continuous partial derivatives, then $\boldsymbol{F}$ is a conservative vector field.

Divergence:
If $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of $\partial P/\partial x$, $\partial Q/\partial y$ and $\partial R/\partial z$ exist, then the divergence of $\boldsymbol{F}$ is the function of three variables defined by:

$$\text{div } \boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \boldsymbol{F}$$

If the vector field $\boldsymbol{F}$ has continuous second-order partial derivatives, then $\text{div curl } \boldsymbol{F} = 0$.

Green's Theorem:
Using curl and div, it is possible to rewrite Green's Theorem in the following vector forms:

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_D (\text{curl } \boldsymbol{F}) \cdot \boldsymbol{k} \, dA$$

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{n} \, ds = \iint_D \text{div } \boldsymbol{F}(x, y) \, dA$$

§16.6: Parametric Surfaces and their Areas (p. 1070)

Definition:
A parametric surface is a surface described by the tip of the position vector $\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}$ as (u, v) moves through the region D. The equations $x = x(u, v)$, $y = y(u, v)$ and $z = z(u, v)$ are parametric equations.

Surface area:
If a smooth parametric surface is given by $\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}$, $(u, v) \in D$ and S is covered once (as (u, v) moves through D), the surface area of S is:

$$A(S) = \iint_D |\boldsymbol{r}_u \times \boldsymbol{r}_v| \, dA$$

where $\boldsymbol{r}_u = \frac{\partial x}{\partial u}\boldsymbol{i} + \frac{\partial y}{\partial u}\boldsymbol{j} + \frac{\partial z}{\partial u}\boldsymbol{k}$ and $\boldsymbol{r}_v = \frac{\partial x}{\partial v}\boldsymbol{i} + \frac{\partial y}{\partial v}\boldsymbol{j} + \frac{\partial z}{\partial v}\boldsymbol{k}$.

When S is given by $z = f(x, y)$ and $f(x, y)$ has continuous partial derivatives, S is given by:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

Tangent plane:

1. Find the tangent vectors ($\boldsymbol{r}_u$ and $\boldsymbol{r}_v$);
2. Find the normal vector to the tangent plane ($\boldsymbol{r}_u \times \boldsymbol{r}_v$);
3. Find a parameterization of (x, y, z) in (u, v);
4. Compute the tangent plane by replacing the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ unit vectors from step 2 by $x - x_0$, $y - y_0$ and $z - z_0$ and (obviously) equating to 0.

§16.7: Surface Integrals (p. 1081)

A surface integral is to a surface area what a line integral ($\int_C f(x, y, z) \, ds = \int_a^b f(\boldsymbol{r}(t))|\boldsymbol{r}'(t)| \, dt$ is to arc length:

$$\iint_S f(x, y, z) \, dS = \iint_D f(\boldsymbol{r}(u, v)) \, |\boldsymbol{r}_u \times \boldsymbol{r}_v| \, dA$$

Also note that the value of the surface integral is equal to the surface area for $f(x,y,z)=1$:

$$\iint_S 1\, dS = \iint_D |\boldsymbol{r}_u \times \boldsymbol{r}_v|\, dA = A(S)$$

Graphs:
For graphs with parametric equations $x = x$, $y = y$ and $z = g(x,y)$, the surface integral is given by:

$$\iint_S f(x,y,g(x,y))\, dS = \iint_D f(x,y,g(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Formulas similar to the one above (with a projection on the xy-plane) can be derived for projections on the xz- ($x = x$, $y = h(x,z)$ and $z = z$) and yz-planes `($x = k(y,z)$, $y = y$ and $z = z$).

Oriented surfaces:
Surface integrals can only be applied to oriented surfaces. Thus, these need to be defined. The unit normal vector gives the upward orientation of the surface by:

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}$$

Vector fields:
If $\boldsymbol{F}$ is a continuous vector field defined over an oriented surface S with unit normal vector $\boldsymbol{n}$, the the surface integral (or *flux integral*) of $\boldsymbol{F}$ over S is

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$$

If S is given by $\boldsymbol{r}(u,v)$, then the equation above simplifies to $\iint_S \boldsymbol{F} \cdot (\boldsymbol{r}_u \times \boldsymbol{r}_v)\, dA$.

Furthermore, if S is given by $z = g(x,y)$, x and y can be thought of parameters and $\boldsymbol{F} \cdot (\boldsymbol{r}_x \times \boldsymbol{r}_y)$ can be rewritten as $(P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}) \cdot \left(-\frac{\partial g}{\partial x}\boldsymbol{i} - \frac{\partial g}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right)$, with finally results into a new equation for the surface integral (assuming the upward orientation for S):

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

Similar formulas exist for the case where $x = k(y,z)$ or $y = h(x,z)$.

§16.8: Stokes' Theorem (p. 1092)

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem (§16.4).

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\boldsymbol{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S \mathrm{curl}\boldsymbol{F} \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r}$$

§16.9: The Divergence Theorem

When Green's Theorem ($\int_C \boldsymbol{F} \cdot \boldsymbol{n}\, ds = \iint_D \mathrm{div}\boldsymbol{F}(x,y)\, dA$) is extended to vector fields on $\mathbb{R}^3$, this yields the Divergence Theorem

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iiint_E \mathrm{div}\boldsymbol{F}(x,y,z)\, dV$$

where S is the boundary surface of the solid region E.