Calculus II

WI1402LR

Alan Hanrahan

Delft University of Technology

TU Delft

Preface

So this is a summary of the core concepts covered in the Calculus 2 module in First year. Up to 2018 you could easily pass the course just by practicing past questions and learning the patterns of how to solve them, but from 2019 on wards you needed to have a more intuitive understanding of the stuff you learned in class. In these notes I go over the basics of what you do in class and try to explain things simply. Obviously I'm not going to every bit of information from the lectures and the book in here, so if you want further clarification I recommend looking back at the slides. These notes go in the same order as the lectures so you should have no problem finding further information. The latest version of these notes is always available on my website (alanrh.com), along with other resources that I find useful. If you find any mistakes and you need anything corrected shoot me an email, but please don't email me if you want further explanation because I can't promise individual help to everyone.

Alan Hanrahan
Delft, April 4, 2021

Contents

1

The Chain Rule

1.1. The Basics

You ought to be familiar with the chain rule, at least when it comes to functions of one variable. You can view simply as being how one function changes with respect to a different functions independent variable. To put it in mathematical terms; if $y = f(x)$ and $x = g(t)$, and f and g are both differentiable functions, then;

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = f'g' \tag{1.1}$$

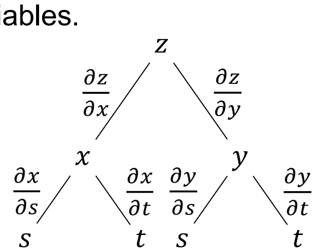
Moving on to functions of 2 variables though, we can find: if $z = f(x, y)$ is a differentiable function of x and y and $x = g(t)$, $y = h(t)$, and g and h are also differentiable functions of t then we can find:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{1.2}$$

This works really well for linear functions. But it will also work for non-linear functions too. Provided you can linearise $f(x, y)$ at some point (x_0, y_0).

1.2. Dependence Charts

A very useful tool for visualising differentials, is a dependence chart, it shows you what variables each term depends on. With the top layer called the dependent variable, the middle section called the intermediate variables, and the bottom section called the independent variables.

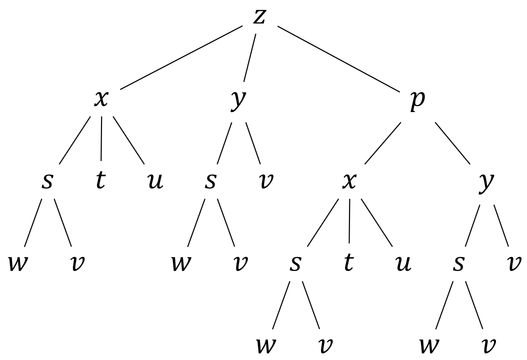


In the example above, we have x and y both being functions of s and t, that's why we use partial derivatives to show how they change with respect to one of these variables. Otherwise if they were only dependent on 1 variable we'd use d instead/

To find the derivative of z with respect to s we need to trace all paths from z to s and add all the possible routes. In this particular case that would be:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

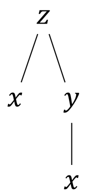
Simple enough! And if you want to find $\frac{\partial z}{\partial t}$ it's the same process. Keep in mind that this also works for much more complicated functions too, and in such cases drawing a tree might prove to be extremely useful for you.



As you can see, the function for z is very complex, but if you want to find $\frac{\partial z}{\partial v}$ for example, you just need to trace all the routes from z to v.

1.3. Implicit Differentiation

One of the applications of the chain rule is implicit differentiation. You've done this before, like in Calculus 1. But here we can do it again. Say we have a function, $z = f(x, y)$ where z is a constant. You can see this in the equation for a circle for instance. What you can do is draw out the dependence chart again and get $\frac{dz}{dx} = 0$.



In such case we find:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

Which as you will be able to recall from Calc 1 translates to:

$$0 = F_x + F_y\frac{dy}{dx}$$

And from here you can easily calculate the derivative.

2

Directional Derivatives

2.1. The Basics

If f is a function of two variables x and y, then:

$$D_{\vec{u}}f(a,b) = \lim_{h\to 0}\frac{f(a+hu_1,\ b+hu_2) - f(a,b)}{h} \tag{2.1}$$

is called the directional derivative of f at the point (a,b) in the direction of a *unit vector* $\vec{u} = \langle u_1, u_2\rangle$. In Calculus I B last quarter we took partial derivatives of a multi-variable function in the x and y directions. What we're doing now is essentially a more general version of that.

Imagine you're standing in a mountain range, depending on the direction that you walk, the slope you need to climb changes. Essentially that's what we're doing here too.

If f is a differentiable function of x and y then f has directional derivatives in the direction of any unit vector $\vec{u}$ such that:

$$D_{\vec{u}}f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2 \tag{2.2}$$

To prove this we can use (eq 2.1). $x + hu_1$ will be very close to x and the same is true for $y + hu_2$. As such we can linearise the function.

$$D_{\vec{u}}f(x,y) = \lim_{h\to 0}\frac{f(x+u_1h,\ y+u_2h) - f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{L(x+u_1h,\ y+u_2h) - f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{f(x,y) + f_x(x,y)(x+u_1h-x) + f_y(x,y)(y+u_2h-y) - f(x,y)}{h}$$

$$= \lim_{h\to 0}\frac{f_x(x,y)(u_1h) + f_y(x,y)(u_2h)}{h}$$

$$\therefore D_{\vec{u}}f(x,y) = f_x(x,y)u_1 + f_y(x,y)u_2$$

2.2. Gradient Vectors

If f is a function of 2 variables x and y then the gradient of f is the vector function $\vec{\nabla}f$ defined by:

$$\vec{\nabla}f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} \tag{2.3}$$

But if you'll notice, this is looking remarkably similar to (eq 2.2). Well if we multiply with the vector $\vec{u}$ we find the exact same thing. So we can say:

$$D_{\vec{u}}f(x,y) = \vec{\nabla}f(x,y)\cdot\vec{u} = |\vec{\nabla}f(x,y)||\vec{u}|\cos\theta$$

From this we can deduce a few things:

- For a given (x,y) the value of the directional derivative depends only on θ
- The maximum Differential is when $\theta = 0$, ie, when $\vec{u}$ is parallel to $\vec{\nabla}f(x,y)$
- $\vec{\nabla}f(x,y)$ points in the direction of maximal slope.

Essentially that's it, a gradient vector is a vector at a given point, that points in the direction of the greatest slope, and if you think about it, it will always be perpendicular to the level curves. Prove this to yourself.

2.3. Generalised

To bring a gradient vector into higher dimensions you just need to take more partial derivatives. eg:

$$\vec{\nabla}f(x,y,z) = \left\langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\right\rangle \tag{2.4}$$

In 2D, the gradient vector is perpendicular to a level curve, but in 3D the gradient *plane* is perpendicular to the level *surface*. So to find it, Pick a point, find the tangent plane, and perpendicular to that will be $\vec{\nabla}f$.

For example, imagine we take the point (a,b,c), then the tangent plane can be described by vectors along it: $\vec{v} = \langle x-a, y-b, z-c\rangle$. Then take the dot product of this, with a vector on the gradient plane, and you get:

$$(x-a)f_1 + (y-b)f_2 + (z-c)f_3 = 0$$

3

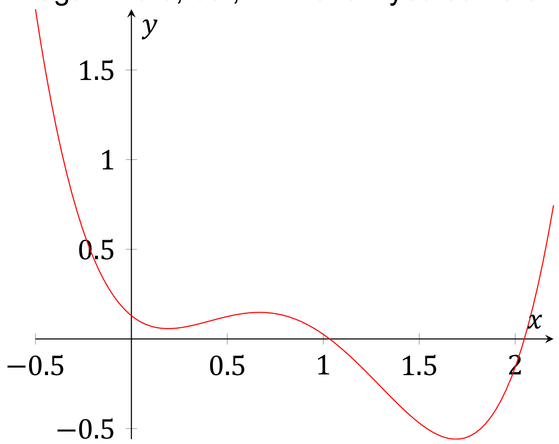
Critical Points

3.1. The Basics

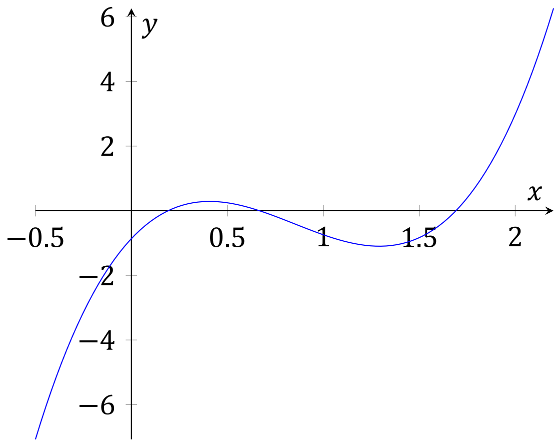
A critical point of a function is a point where the first derivative is 0. This is either a maximum, a minimum, an inflection point, or maybe it's just a horizontal line, in which case all points are critical points. We can distinguish these points (a bit) using the second derivative test.

- $f''(a) < 0 \Rightarrow\ a$ is a local maximum
- $f''(a) > 0 \Rightarrow\ a$ is a local minimum
- $f''(a) = 0 \Rightarrow$ we don't have enough information

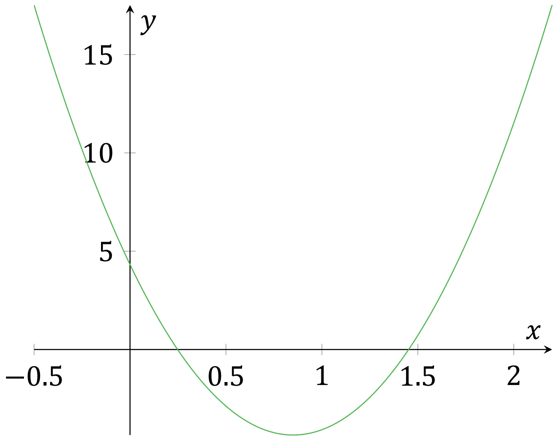
This is all stuff that you should have covered in secondary school, so I'm not going to go over it again here, but, I will show you some examples:



In this function you can see 3 critical points, You can prove this by taking the first derivative and counting the number of times it crosses the x axis.



Then, we want to see at these critical points if the function is going to start increasing (meaning it's a minimum point) or if it will start decreasing (meaning it's a maximum point) so we take the second derivative.



3.2. Higher dimensions

A function f of 2 variables has a local maximum at the point (a, b) if $f(x, y) \leq (a, b)$ when (x, y) is near (a, b), and has a local minimum at the point (a, b) if $f(x, y) \geq (a, b)$ when (x, y) is near (a, b). Note that when I say $f(x, y) \geq (a, b)$, the $\geq$ sign means when $(x, y) = (a, b)$ only.

If $f(x, y) \leq f(a, b)$ hold for all points (x, y) in the whole domain of a function then we have an *absolute maximum*, sometimes called a *global maximum*.

If f has a local maximum or minimum, at (a, b) ans the first order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. This is fact. That's just how it is. If that's *not* the case, then you don't have a local maximum or minimum. You must always have a horizontal tangent plane at a local maximum or minimum in 3D space.

A point (a, b) is a critical point, or stationary point of f if:

$$f_x(a, b) = 0 \;\&\; f_y(a, b) = 0$$

There are 3 notable critical points that we'll pay attention to throughout this course. Maxima, minima, and saddle points. They look like this:

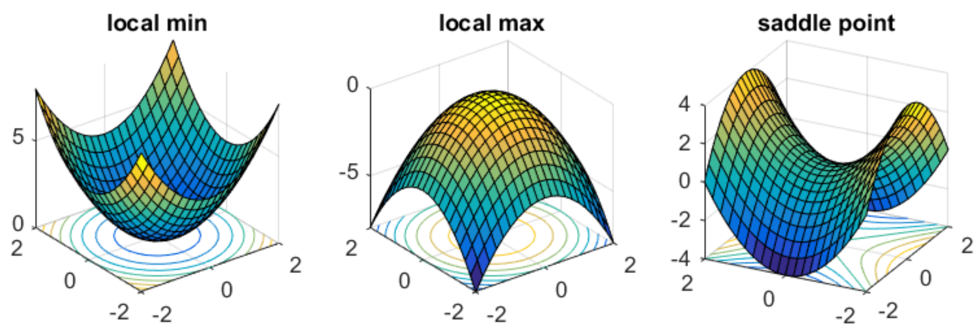


Figure 3.1: *Different 3D Critical Points*

Before we go any further we need to define a discriminant. We say that it is:

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 \tag{3.1}$$

These are the second partial derivatives. So it's much the same as when we were dealing with a function of 1 variable.

Suppose the second partial derivatives are continuous on a disk with centre (a,b) and suppose $f_x(a,b) = 0$ and $f_y(a,b) = 0$. If:

- $D > 0$ and $f_{xx}(a,b) > 0$ we have a local minimum
- $D > 0$ and $f_{xx}(a,b) < 0$ we have a local maximum
- $D < 0$ Then we have a saddle point

There are all sorts of different conditions that are possible, but really, they're not important so we just focus on these 3 possibilities for the most part.

4

Extreme values

4.1. The Basics

In the last chapter we dealt with local maxima and minima, but what we're doing now s dealing with the absolute maxima and minima of functions. Before we get there though we need to define a few things first.

A Closed Set is a set in $\mathbb{R}^2$ that contains all of its *boundary points*. That is, all of the points on the border that encloses it.

A Bounded Set is a set in $\mathbb{R}^2$ is on that is contained within some disk of finite radius. Essentially if you can draw around it on a graph then you have a bounded set. Note that it is possible to have a closed, but non-bounded set, provided the boundaries are included in the set, but extend to infinity. eg: $\{(x, y) \in \mathbb{R}^2 \mid x^2 \leq 1\}$.

The Extreme Value Theorem: if f is continuous on a closed and bounded set D in $\mathbb{R}^2$ then, f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at the points (x_1, y_1) and (x_2, y_2) in D.

Note, that the extreme values aren't necessarily critical points. And also, if the domain D is unbounded, ther *may* still be an absolute maximum or minimum present.

4.2. Application

So what is our goal with all of this anyway? Why do we need these definitions? We want to find the absolute maxima, and minima of functions with closed and bounded domains D, and the corresponding coordinates. So let's think about this for a bit. Where would we find these extreme points in a domain?

- Critical points on D
- Along the boundary lines
- At the vertices

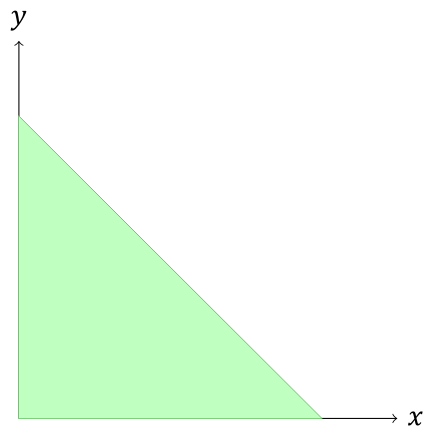
From here we can come up with a simple step by step solution to finding extreme values in a domain.

1. Find the values of f at the critical points *in* D
2. Find the extreme values of f on the boundary of D

3. Make sure to calculate f at the vertices of D
4. The largest of these values will give the absolute maximum and the smallest will give the absolute minimum

Note that there is no need to apply the second derivative test.

Let's try an example. Take the function $4y - 2x - y^2 + x^2$ with the domain D which is a triangle with vertices $(0,0), (0,4), (4,0)$.



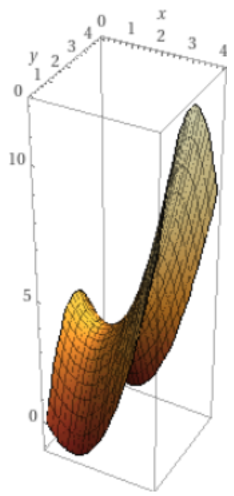


Figure 4.1: *The function in questionf*

We start with our critical points in D. From the previous chapters we can find them as:

$$f_x(x,y) = 0 = -2 + 2x$$
$$f_y(x,y) = 0 = 4 - 2y$$
$$\Rightarrow x = 1,\ y = 2$$
$$f(1,2) = 3$$

This is one of our candidate values. From here we check the boundaries of the domain, or, simply along the sides of the triangle. Starting with Boundary 1:

$$f(0,y) = 4y - y^2$$

$$f' = 4 - 2y = 0 \Rightarrow y = 2$$

$$f(0,2) = 4$$

Boundary 2:

$$f(x,0) = -2x + x^2$$

$$f' = -2 + 2x = 0 \Rightarrow x = 1$$

$$f(1,0) = -1$$

Boundary 3:

$$f(x,4-x) = -2x + 4(4-x) + x^2 - (4-x)^2$$

$$f(x,4-x) = 2x$$

This is linear and has no maximum

The last points we need to check are the vertices, and thankfully there is no differentiation needed here.

$$f(0,0) = 0$$

$$f(4,0) = 8$$

$$f(0,4) = 0$$

Now the final step is collect these candidate points and to compare them:

(x, y)	$f(x, y)$
1,2	3
0,2	4
1,0	-1 (Abs min)
0,0	0
4,0	8 (Abs max)
0,4	0

5 Double Integrals

5.1. The Basics

For a function of one variable, when we integrate we're essentially finding the area under the curve of the function, and we do this with the use of a Riemann sum. Go back and revise this from Calculus 1 A if you don't remember this. For functions of 2 variables though, we're basically finding the volume under the surface of the function using a Riemann sum again.

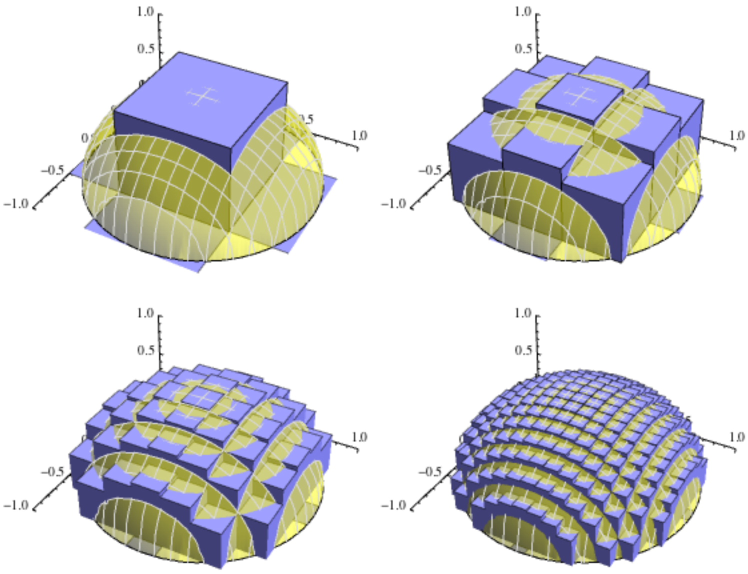


Figure 5.1: *A 3D Riemann Sum*

In this section of the course we're focusing on integrating over a rectangular domain R. So, the interval of $[a, b]$ on the x axis, and $[c, d]$ on the y axis. Thus; $R = [a, b][c, d]$ which can also be written as: $\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$.

Much like in a function of 1 variable, if we divide the domain into smaller and smaller sections we get a more accurate result. As you can see in (fig 5.1). We divide the interval over x into n sections, and the interval over y into m sections to get:

$$\Delta x = \frac{b-a}{n}, \ \Delta y = \frac{d-c}{m}$$

.

This divides up the domain R into little segments that we'll call $R_{ij} = [x_{i-1}, x_i][y_{j-1}, y_j]$. We take the midpoint of these segments and evaluate the function at these points:

$$f(x_i^*, y_j^*)$$

This is the height of the cuboid, and to get the volume of the cuboid we multiply by the area; $R_{ij} = \Delta x \Delta y$.

$$V_{ij} = f(x_i^*, y_j^*)\Delta x \Delta y \tag{5.1}$$

Thus we can say that the integral of the function over the area of the domain is:

$$\int\int f dA \approx \sum_{i=0}^{n}\sum_{j=0}^{n} f(x_i^*, y_j^*)\Delta x \Delta y \tag{5.2}$$

As is usual, we want it to be as accurate as possible so we take a limit:

$$\int\int f dA = \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} f(x_i^*, y_j^*)\Delta x \Delta y$$

5.2. Practicalities

Cool. That's all well and good but do we need to know this off for the course? No not really. But what you do need to know is that a double integral is of the format:

$$\int_a^b \int_c^d f(x,y)dydx \tag{5.3}$$

To do this calculation you first take the internal integral, in this case $\int_c^d f(x,y)dy$, and you integrate with a constant x. Then you integrate with the outer limit $\int_a^b$ with respect to x, and with a constant y. According to Fubini's Theorem, it doesn't matter which order you take the integrals, so long as the function is continuous over the domain. So:

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

5.3. Non-Rectangular Domains

Excellent, we've covered rectangular domains, now let's get a bit more complex and deal with a domain that's sandwiched between functions of x (which we call Type 1) or functions of y (which we call Type 2). For Type 1 Domains we have:

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, g_1(x) \le y \le g_2(x)\} \tag{5.4}$$

With g_1, g_2 continuous on $[a,b]$.Type 2 is:

$$D = \{(x,y) \in \mathbb{R}^2 \mid h_1(y) \le x \le h_2(y), c \le y \le d)\} \tag{5.5}$$

With h_1, h_2 continuous on $[c,d]$.

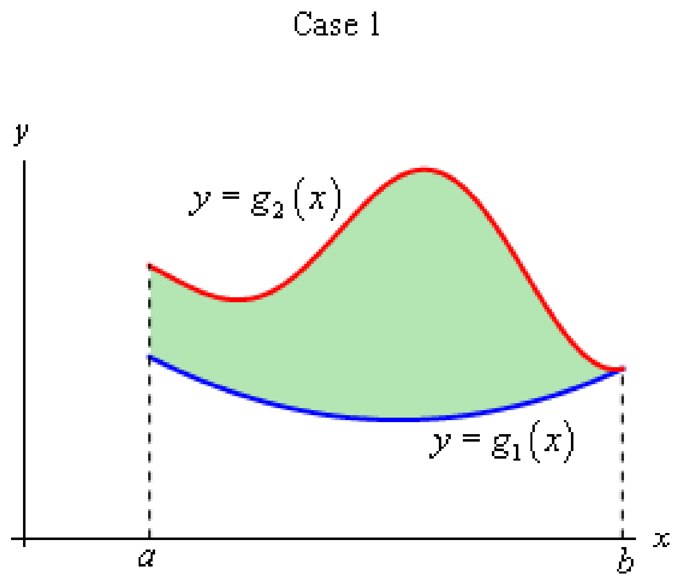


Figure 5.2: *A Type 1 Domain*

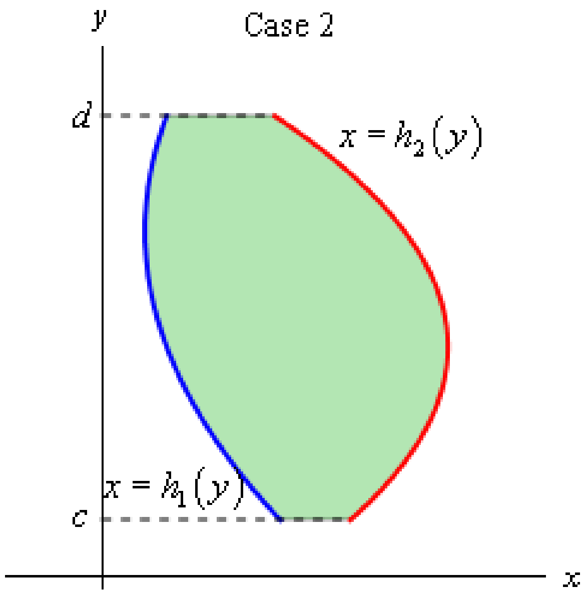


Figure 5.3: *A Type 2 Domain*

So how do you go about integrating a function of 2 variables $f(x, y)$ over this domain? Well if you recall from earlier, what we want to do is find the volume of this region, so we "cut it up" into slices. Taking a Type 1 what we'd do is make a load of cuts parallel to the y axis. Then the volume of this will be: Area of the slice * thickness

$$= A_{\text{slice}} * \Delta x$$

$$= \int f(x_i, y)dy * \Delta x$$

I want to pay a little bit of attention to that integral there. x_i is a constant, and it's the midpoint of each slice we take. What we do is add up all of the heights of the slice to find the area, much like if we take a 2D integral.

So that explains the integral for that. But what are the limits of this? Well, they're the functions that define the boundaries $g_1(x_i)$ and $g_2(x_i)$. Meaning that the volume of the slice is:

$$\int_{g_1(x_i)}^{g_2(x_i)} f(x_i, y)dy * \Delta x$$

As I'm sure you're used to by now, what we want to do is add up all of these slices, and we want them to be as thin as possible to get ans accurate result, thus $\Delta x \Rightarrow dx$. So the end result for the volume of the solid defined by the domain is:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx \tag{5.6}$$

The logic for a Type 2 integral is pretty much exactly the same, except we take slices parallel to the x axis instead. And we end up with something like:

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy \tag{5.7}$$

What about more complex domains? Good Question! in that case, we just divide it into more, smaller, domains. Because any domain can be divided into convenient Type 1 and Type 2 domains. Then integrate over these new domains and add up the resulting integrals to find the total.

5.4. Polar Regions

So now we've covered the basics of integrating over any sort of shape in Cartesian coordinates, but now let's kick things up a notch by introducing polar coordinates. I'm going to assume you remember how these work, because we did it at the end of Calculus 1 A when dealing with complex numbers, but if you don't understand them, then go back and revise them.

first let's introduce the concept of a *polar rectangle*. This is a shape with 4 right angles in it, defined by two angles, and two radii. See Figure (5.4). With previous integrations we divided our domain into small, normal, rectangles but for polar regions we will be dividing it into **polar** rectangles.

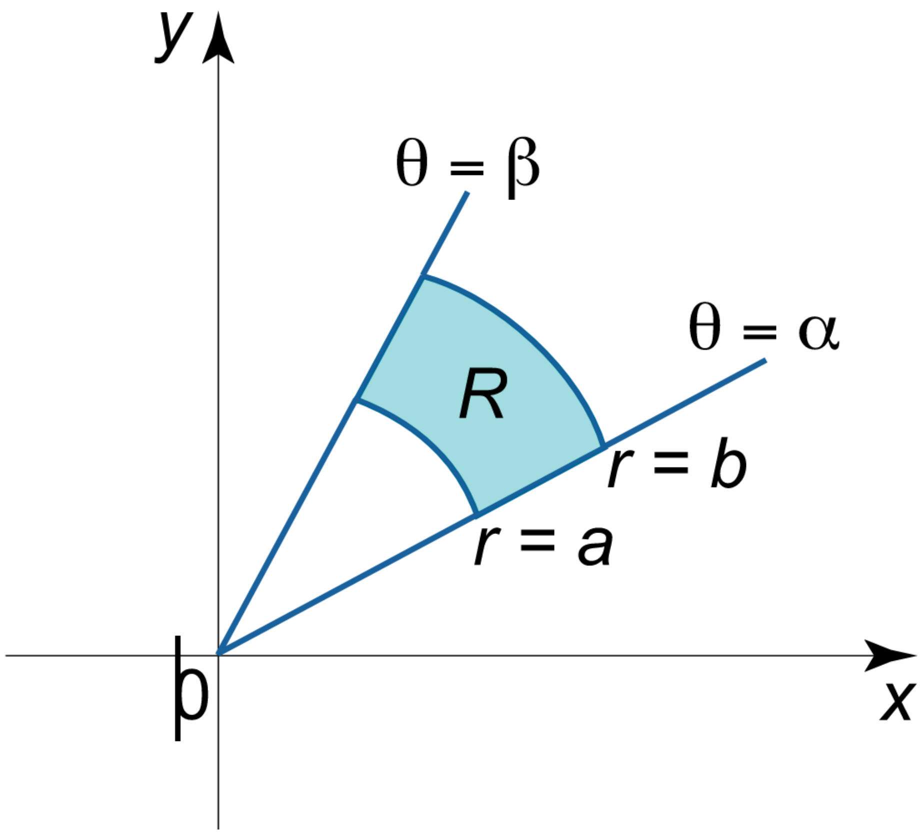


Figure 5.4: *A Polar Rectangle*

$$R = \{(r\cos\theta, r\sin\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\} \tag{5.8}$$

I'm really hoping you understand how Riemann sums work by now, because the process is really similar for each type of integration. Divide the intervals into subsections, evaluate each interval at the midpoint, multiply this evaluation by the interval, and then add up all of these results. That's the same here.

But what's the area of a polar rectangle? Well, the area of a ring around the origin, is $\pi r_i^2 - \pi r_{i-j}^2$, and we want just a fraction of this, so the total area is:

$$\pi(r_i^2 - r_{i-1}^2)\frac{\theta_j - \theta_{j-1}}{2\pi}$$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})(\theta_j - \theta_{j-1})$$

$$\Delta A_{ij} = r_i^* \Delta r \Delta\theta \tag{5.9}$$

As is the usual next step, we want to add up all of these regions and we take limits so the areas become infinitesimally small, giving us the final result that:

$$\int\int f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta \tag{5.10}$$

A few things to note from this:

- $dxdy = dA = rdrd\theta$
- $f(x,y) = f(r\cos\theta, r\sin\theta)$
- Integration boundaries are: $R = \{(r\cos\theta, r\sin\theta) \mid a \le r \le b, \alpha \le \theta \le \beta\} \Rightarrow \{a, b, \alpha, \beta\}$

if you want to have a more general region, for instance if you want to have a radius r defined by a function of θ you can do that too! That can be:

$$D = \{(r\cos\theta, r\sin\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

$$\int\int f(x,y)dA = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta \tag{5.11}$$

5.5. Applications of Double Integrals

So what can we do with double integrals? Assuming we have density in units of mass per unit area, we can calculate a load of things! For instance:

Mass:

$$m = \int\int \rho(x,y)dA \tag{5.12}$$

Moments:

$$M_y = \int\int x\rho(x,y)dA \tag{5.13}$$

$$M_x = \int\int y\rho(x,y)dA \tag{5.14}$$

Centre of Mass:

$$\bar{x} = \frac{M_y}{m} \tag{5.15}$$

$$\bar{y} = \frac{M_x}{m} \tag{5.16}$$

Moment of Inertia:

$$I_x = \int\int y^2\rho(x,y)dA \tag{5.17}$$

$$I_y = \int\int x^2\rho(x,y)dA \tag{5.18}$$

$$I_o = I_x + I_y \tag{5.19}$$

6 Triple Integrals

6.1. The Basics

Wonderful! We've integrated over 2D domains, but now we're going to integrate over a 3D domain. If we have a single integral, we integrate over a 1D domain to get an area. If we integrate over a 2D domain we get a volume, and if we integrate over a 3D domain we get... a hypervolume? Well, yes, but we don't exist in a 4D universe so we're not able to visualise what that looks like.

Another useful function that we can get with a triple integral is a density function. That is, we have mass per unit volume, and it's given as a function of (x, y, z). We use this to find the mass of a volume.

$$\int\int\int \rho(x, y, z)dV$$

Once again we take a Riemann sum. We're integrating over a region $E = [a, b]\times[c, d]\times[e, f]$. Divide the volume into small sub boxes. Find the mass of each sub box, by evaluating at the midpoints. Add up all point masses, and take a limit. Thus giving us an integral.

Fubini's Theorem applies for triple integrals too! So, if $f(x, y, z)$ is continuous on a rectangular box $E = [a, b] \times [c, d] \times [e, f]$ then the order of integration doesn't matter.

Sometimes we end up with a separable integral, for example: $\int\int\int xyzdxdydz$. Notice how x, y, z are all independent of each other? That means that y, z are constants when we integrate over x and so on. And when we integrate, we take the constants out of the integral. Thus:

$$\int\int\int xyzdxdydz = \int xdx \int ydy \int zdz$$

6.2. Region Types

As with double integrals we have region types. A triple integral over a type 1 region is: a solid region in $\mathbb{R}^3$ that lies between the graphs of 2 continuous functions of (x, y):

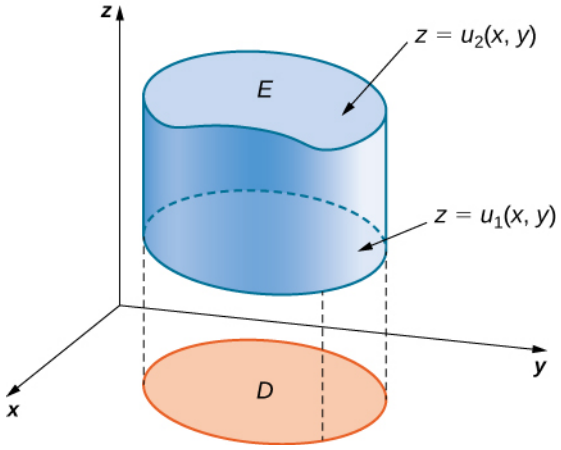


Figure 6.1: *A Type 1 Region in 3D*

$$\int\int\int f(x,y,z)dV = \int\int\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdA \tag{6.1}$$

To solve a triple integral of this type, first integrate over z, then solve the double integral over D. Similarly, we have Type 2, and Type 3 integrals. A Type 2 is one where D is on the $y-z$ plane, and a Type 3 is one where D is on the $x-z$ plane.

6.3. Cylindrical Coordinates

For double integrals we sometimes integrated over polar regions. For this we introduced the idea of a polar rectangle. For triple integrals we'll be doing something similar. *Cylindrical coordinates* are basically just polar coordinates, but extended up with a z component:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

Converting from Cartesian coordinates to cylindrical coordinates is a little bit trickier but it's just given by the formulae:

- $r = \sqrt{x^2+y^2}$
- $\theta = \arctan\frac{y}{x}$ for when $x > 0$
- $\theta = \arctan\frac{y}{x} + \pi$ for when $x < 0$

I wont bore you with the details of deriving an integral, but what we're doing is just taking a polar double integral, and adding a dimension. It's just another Riemann sum.

To integrate over the region E we'll define it with the domain D on the $x-y$ plane.

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$

$$D = \{(r\cos\theta, r\sin\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

$$\int\int\int f(x,y,z)dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r dz dr d\theta \tag{6.2}$$

6.4. Spherical Coordinates

When we measure the position of something on the surface of the earth, we use cylindrical coordinates. Similarly we can generalise this to any position in space as long as we have a distance, ρ, to the origin.

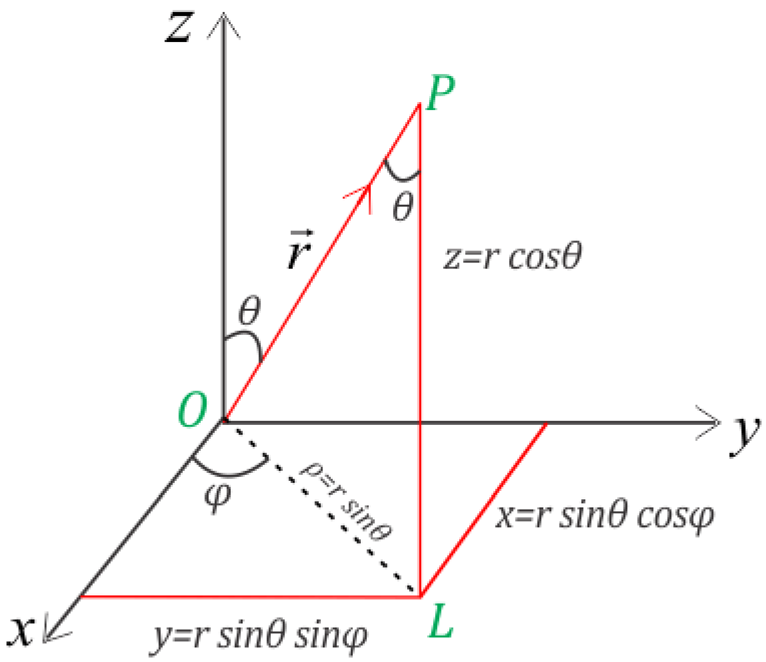


Figure 6.2: *Spherical Coordinates of a Point*

To convert from Cartesian to Spherical coordinates it is as follows:

- $x = \rho \sin\Phi \cos\theta$
- $y = \rho \sin\Phi \sin\theta$
- $z = \rho \cos\Phi$

Or the other way around;

- $\rho = \sqrt{x^2 + y^2 + z^2}$
- $\Phi = \arccos\frac{z}{\rho}$
- $\theta = \arctan\frac{y}{x}$m Or if $x < 0$; $\arctan\frac{y}{x} + \pi$

The way that we integrate with spherical coordinates is just the same as always. Divide the volume into small elements dV and then sum them all together. There's no new ideas to understand here, you just need to know what dV is.

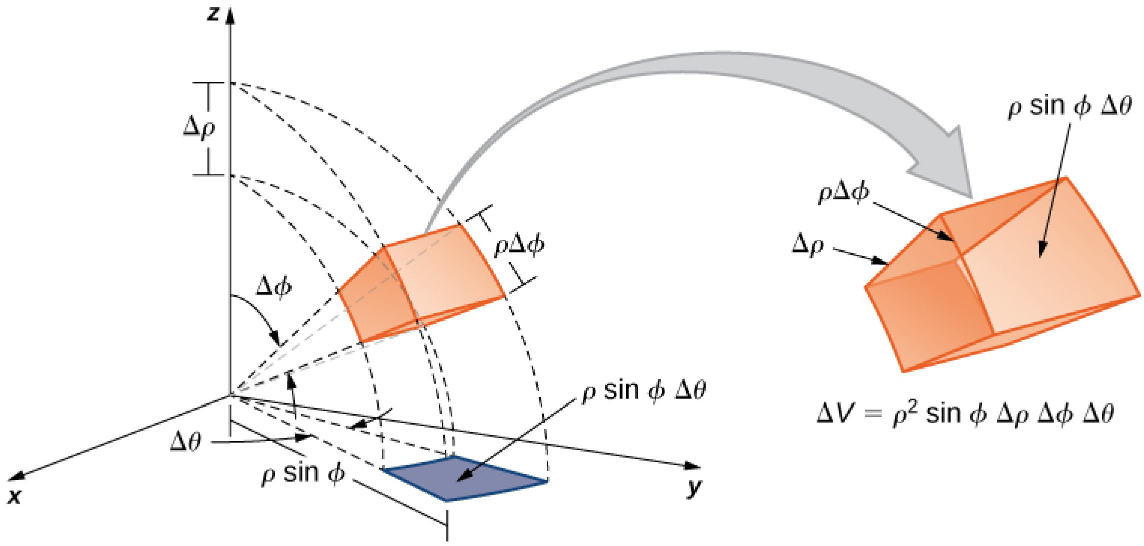


Figure 6.3: *Volume Element in Spherical Coordinates (it says $\vec{r}$ but it should say rho)*

From (fig. 6.3) you can see that the element is defined by the difference between two angles, the difference between two *other*, angles, and the difference between two radii. As such To find the volume of this "Polar Cuboid" we get:

$$\Delta V = \Delta\rho \cdot \rho\Delta\Phi \cdot \rho \sin\Phi\Delta\theta$$

Which simplifies down to:

$$dV = \rho^2 \sin\Phi d\rho d\Phi d\theta \tag{6.3}$$

If we want to integrate over a region E which is defined by:

$$E = \{(\rho \sin\Phi \cos\theta, \rho \sin\Phi \sin\theta, \rho \cos\Phi) \mid a \leq \rho \leq b, \alpha \leq \theta\beta, \gamma \leq \Phi \leq \delta\}$$

Then the integral is:

$$\int\int\int f(x,y,z)dV = \int_\alpha^\beta \int_\gamma^\delta \int_a^b f(\rho \sin\Phi \cos\theta, \rho \sin\Phi \sin\theta, \rho \cos\Phi) \cdot \rho^2 \sin\Phi d\rho d\Phi d\theta \tag{6.4}$$

7

Coordinate Transformation

7.1. The Basics

Okay in my opinion this stuff is the hardest in this module, We cover more of this sort of stuff later on in 3D, but for now we're dealing with the very basics. This will be our introduction to *Parametisation*. If we have a region defined by functions of x and y. We might want to change these to be functions of u and v to make integrating easier. Essentially U-Substitution, but on steroids.

A Transformation function: $T : S \to R$ is called one-to-one if each point (x, y) in R is the image of exactly one point (u, v) in S.If this is the case then the transformation T is invertable.

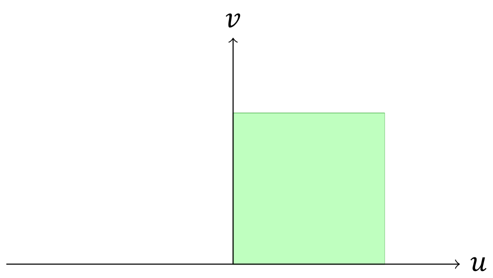
$$T(u,v) = (x,y) : \begin{Bmatrix} x = g(u,v) \\ y = h(u,v) \end{Bmatrix} \tag{7.1}$$

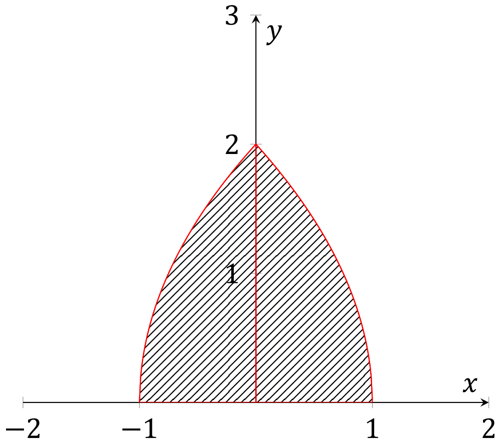
As an example let's take a look at a rectangle defined in $u - v$. this region would be really easy to integrate over. We've covered it earlier in this summary. But anyway, what would this look like in the $x - y$ plane?

$$T = \begin{Bmatrix} x = u^2 - v^2 \\ y = 2uv \end{Bmatrix}$$

With:

$$S : \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$$





To find the area of the region S it's simply $\Delta u \cdot \Delta v$. It's a rectangle. But what about finding the area of R? That's a weird shape. Well if we have divide the area into small regions of R, we can draw position vectors $\vec{r}$ as functions of u and v. With these little parallelograms, we can draw vectors to each vertex.

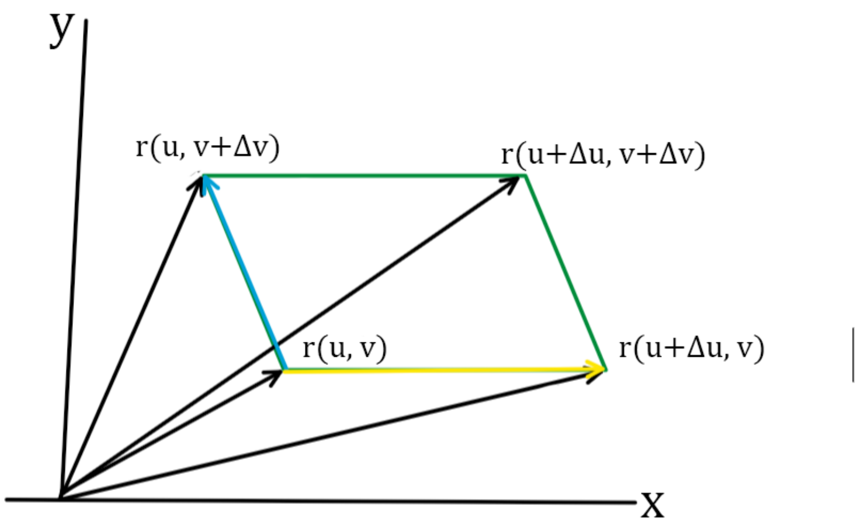


Figure 7.1: *A Crude Drawing of a Parallelogram Defined by Position Vectors*

- $\vec{r}(u_0, v_0)$
- $\vec{r}(u_0, v_1)$
- $\vec{r}(u_1, v_0)$
- $\vec{r}(u_1, v_1)$

But, as is usual, we want to define u_1 as $u_0 + \Delta u$ so that we can do some integration. This gives us the new position vectors:

- $\vec{r}(u_0, v_0)$
- $\vec{r}(u_0, v_0 + \Delta v)$

- $\vec{r}(u_0 + \Delta u, v_0)$
- $\vec{r}(u_0 + \Delta u, v_0 + \Delta v)$

If you remember back from the very beginning of first year, you'll remember that the area of a parallelogram is the magnitude of the cross product of two vectors spanning the sides of the shape. these vectors are:

$$\vec{r}(u_0 + \Delta u, v_0) - \vec{r}\vec{r}(u_0, v_0)$$
$$\vec{r}(u_0, v_0 + \Delta v) - \vec{r}\vec{r}(u_0, v_0)$$

These look familiar, don't they. Well, the partial derivatives with respect to u and v are:

$$\vec{r}_u = \frac{\vec{r}(u_0 + \Delta u, v_0) - \vec{r}\vec{r}(u_0, v_0)}{\Delta u}$$
$$\vec{r}_v = \frac{\vec{r}(u_0, v_0 + \Delta v) - \vec{r}\vec{r}(u_0, v_0)}{\Delta u}$$

So we can describe the area of R as $|\vec{r}_u \Delta u \times \vec{r}_v \Delta v|$.

7.2. Application to Integrals

The area of R is the same as ΔA which means that if we're to put this into an integral we would get the following:

$$\int\int_R f(x,y)dA = \int\int_S f(x(u,v), y(u,v))\ |\vec{r}_u \times \vec{r}v|\ dudv \tag{7.2}$$

This is how we translate from one form to another. We need to have the cross product of the partial derivatives. This is so important that we get to have a new name for something here!

$$R = \Delta A = |\vec{r}_u \times \vec{r}v|\ \Delta u \Delta v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u \Delta v \tag{7.3}$$

The Result of the cross product will give us the *Jacobian*: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ Which can be written in many ways. key among them: $\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$

Suppose that T is a one-to-one transformation ans its Jacobian is non-zero, and that T maps a region S (on $u - v$) onto a region R (on $x - y$). Suppose also that f is a continuous function on R and that RS are plane regions of Type I or Type II, Then:

$$\int\int_R f(x,y)dA = \int\int_S f(x(u,v), y(u,v))\ \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ dudv \tag{7.4}$$

This is how we use it in double integrals, but if we want to generalise into triple integrals, then we simply add in a third variable.

$$\int\int_R f(x,y,z)dV = \int\int_S f(x(u,v,w), y(u,v,w), z(u,v,w))\ \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\ dudv \tag{7.5}$$

8 Vector Fields

8.1. The Basics

8.1.1. 2D Vector Fields

A vector field is a function of a space whose value at each point is a vector quantity. It is a vector function of x and y.

$$\vec{F}(x,y) \tag{8.1}$$

Let D be a set in $\mathbb{R}^2$, A vector field on $\mathbb{R}^2$ is a "function $\vec{F}$ that assigns to each point (x,y) in D, a 2D vector; $\vec{F}(x,y)$.

$$\vec{F} = \langle P(x,y),\ Q(x,y)\rangle = P\vec{\imath} + Q\vec{\jmath} \tag{8.2}$$

The functions P and Q are called the component functions of the vector fields.

Examples of such vector fields are:

- Velocity fields in fluids or gasses, like weather maps.
- Velocity fields of rotating rigid objects.
- Gradient Fields

8.1.2. 3D Vector Fields

Let E be a set in $\mathbb{R}^3$, A vector field on $\mathbb{R}^3$ is a function, $\vec{F}$, that assigns to each point (x,y,z) in E, a 3D vector; $\vec{F}(x,y,z)$

$$\vec{F} = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle = P\vec{\imath} + Q\vec{\jmath} + R\vec{k} \tag{8.3}$$

Examples of such vector fields are:

- Velocity fields in 3D fluids or gasses, like explosions, turbulence, and aerodynamic simulations
- Force fields of Gravitation and electromagnetism

8.1.3. Gradient Vector Fields

If f is a **scalar** function of 2 variables then;

$$\vec{\nabla} f(x,y) = f_x(x,y)\vec{\imath} + f_y(x,y)\vec{\jmath} \tag{8.4}$$

is a gradient vector field on $\mathbb{R}^2$. Similarly, this can be expanded out to 3D to give the form:

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\vec{\imath} + f_y(x,y,z)\vec{\jmath} + f_z(x,y,z)\vec{k} \tag{8.5}$$

8.1.4. Conservative Vector Fields

A Vector field $\vec{F}$ is conservative if it is the gradient of some scalar function that, there is a function of f such that $\vec{F} = \vec{\nabla} f$. f is, under these conditions, called a potential function for $\vec{F}$

The properties of a conservative vector field are:

Vectors in a conservative vector field do not make loops. (This is because $\vec{\nabla} f$ points to the highest point, and it ends at the peak)

line integrals over conservative vector fields are *independent* of the path of integration.

Force fields are usually described by conservative vector fields

8.2. Parametric Equations

8.2.1. 2D Plane Curves

So far we've become quite used to drawing curves in 2D, it's a simple $y = mx + c$, secondary school things. but oftentimes it's nice for us to be able to describe a curve by the positions of all points with $\vec{r}$. You'll remember that $\vec{r}$ is a position vector.

We introduce a new variable t which could be time, or whatever, it doesn't matter. We now define x and y as functions of t. A Plane Curve C in $\mathbb{R}^2$ is the set of points (x, y) where:

$$\begin{Bmatrix} x = f(t) \\ y = G(t) \end{Bmatrix}$$

With $a \leq t \leq b$. This is called a *Parametrisation of* C. For C we have:

$$\vec{r}(t) = \langle x(t), y(t) \rangle = f(t)\vec{\imath} + g(t)\vec{\jmath} \tag{8.6}$$

With the limitation of $a \leq t \leq b$.

8.2.2. 3D Curves

Similarly we are able to extrapolate this out into $\mathbb{R}^3$ as:

$$C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \tag{8.7}$$

8.3. Calculating Arc Length

Something you might want to do from time to time is to calculate the length of an arc. This becomes much easier if you first parametrize the function for C.

Let C be a curve defined by the vector function $\vec{r}(t)$ with $a \leq t \leq b$. If the curve C is traversed (that means to be travelled along) exactly once as t increases from a to b, then the length L can be calculated as:

$$L = \int_a^b \left|\vec{r}'(t)\right| dt = \int_a^b ds \tag{8.8}$$

In this equation, the variable s is a parameter along C that measures length, as you will be familiar with from Mechanics. This should make sense if you think about it, If you recall the fundamental theorem of calculus, the integral of a derivative is the same as the original function. But because we're finding length (a scalar variable) from a vector function, we must take the magnitude.

From here, we can make a function of the arc length. Let C be a curve defined by the vector $\vec{r}(t)$ with $a \leq t \leq b$. If $\vec{r}'(t)$ is continuous , and C is traversed exactly once as t increases from a to b, then:

$$s(t) = \int_a^t \left|\vec{r}'(u)\right| du = \int_C^{\prime} ds \tag{8.9}$$

Note, that u is just representing a position in the same way that t is, but because t is the upper limit, we use u to avoid confusion. Also note that C' is the curve $a \leq u \leq t$; it's a part of the full curve C.

The length s is always positive, Negative length is not possible, that would be weird. If you want to integrate in the opposite direction, then you need to change the parametrization such that you integrate from a smaller t to a larger t.

8.4. Line Integrals

We're used to line integrals already. When we do a 2D line integral we integrate along a line on the x axis. But what if the line is in 3D, and what if it's wiggly? In that case, the function will look a bit like a curtain, and we want to find the area of this weird shape.

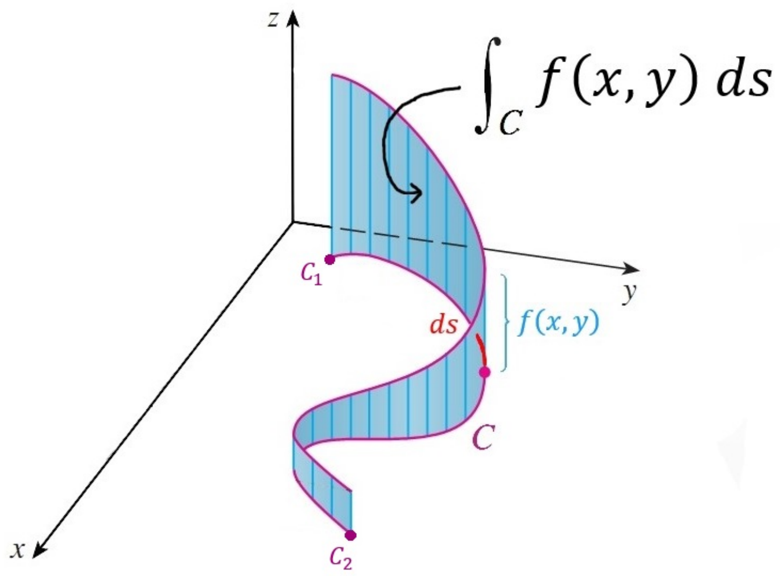


Figure 8.1: *Integrate to find the area of the "curtain"*

If f is a scalar function defined on a smooth curve C, which is in turn defined by the parametrization $\vec{r}(t)$ with $a \leq t \leq b$. Then the integral along the line C is defined as:

$$\int_C f(x,y,z) ds = \int_a^b f(\vec{r}(t)) \left|\vec{r}'(t)\right| dt \tag{8.10}$$

From (Eq. 8.8) we know that $ds = \left|\vec{r}'(t)\right| dt$. Note that because this is a scalar function, the integral is independent of the orientation.

If we want the line integral of a Vector Field, then this *will* be dependant of the orientation. It will be defined as:

$$\int_C \vec{F}(x, y, z) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \tag{8.11}$$

8.4.1. Fundamental Theorem for Line Integrals

Let C be a smooth curve given by the vector field $\vec{F}(t)$, with $a \leq t \leq b$. Let f be a differentiable function of two or three variables, which has a gradient $\vec{\nabla} f$ is continuous on C. This in turn implies:

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \tag{8.12}$$

Compare this to the similarities of the normal fundamental theorem of calculus:

$$\int_a^b f'(x) = f(b) - f(a)$$

8.4.2. Conservative Vector Fields and Line Integrals

In Section (8.1.4) we had a look at conservative vector fields. When it comes to integrating over such vector fields there are a few interesting characteristics to pick up on. Take the Vector field

- $\vec{F} = \vec{\nabla} f$
- $\int_C \vec{F} \cdot d\vec{r}$ Is independent of the Path C
- $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
- $\oint \vec{F} \cdot d\vec{r} = 0$ (integral over a closed path)

The fun thing about these qualities is that they are all exclusive to conservative vector fields. So, if you know that one of these applies, then you can infer that the other 3 qualities must also be true.

8.5. Green's Theorem

8.5.1. Orientation

Let's start with a definition of orientation. A simple closed curve C has a positive orientation if the enclosed region D is always on the left, as you traverse along C. The orientation of the curve is thus described by the parametric equation that you use.

This is important to keep in mind, because for the duration of this section, we'll be integrating over a closed loop.

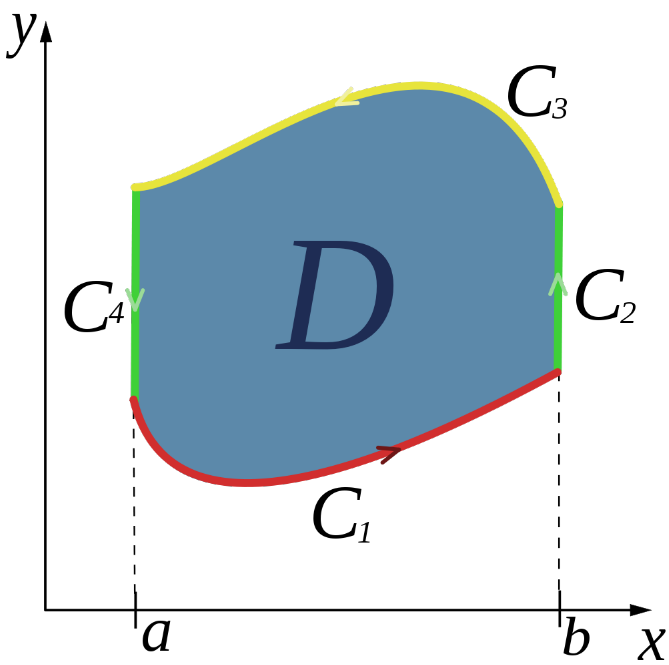


Figure 8.2: *An Example of a Positively Oriented Curve*

8.5.2. The Theorem

Green's Theorem is as such: Let C be a positively oriented, piece-wise (this means that can be constructed neatly out of smaller surfaces. A lot like 3D computer models, for example figure 8.3),smooth, simple, closed curve in the plane. And let D be the region enclosed by the curve C.

Figure 8.3: *A Piece-wise Surface*

If P and Q have continuous partial derivatives on an open region that contains D then:

$$\oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{8.13}$$

In this case the curve C is in fact, the partial derivative of the region D.

If we have a vector field $\vec{F} = P\vec{\imath} + Q\vec{\jmath}$Then we can say that

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

8.5.3. Calculating Area

If we have a fun case where $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then we can write a conclusion about the area of the region D:

$$\iint_D dA = \oint_C Pdx + Qdy$$

8.5.4. General Areas

As you should know from, geometry, a region can be cut up into smaller regions. What you can do is cut up a region and then integrate over these regions, and the result will still be the same.

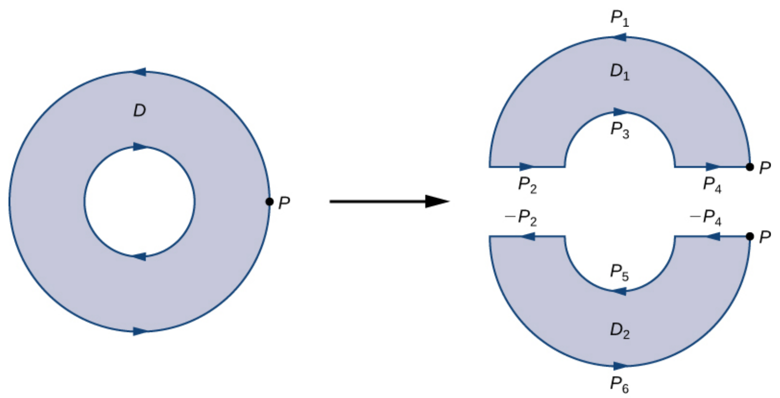


Figure 8.4: *A Domain cut up Into Smaller Domains*

You can see an example of this sort of thing in figure (8.4). If you'll notice, because we retain the direction of integration when we cut the domain, the boundaries along where we cut (In this case P_2 and P_4) cancel each other out, because we integrate in opposite directions there. As such we can say:

$$\iint_{D_1 \cup D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \tag{8.14}$$

8.6. Curl and Divergence

8.6.1. What Is Curl?

Curl describes the localised rotation at a point in a vector field. A good analogy is that of an object floating on water. You can think of a the surface of the water as being a vector field, with the vectors describing the direction that the water flows.

The item on the water, will move, like if it's a river, then the item will flow downstream. But, as well as that, the item will rotate about itself. This is curl.

8.6.2. Defenitions

We're going to start by defining the ∇ operator. $\vec{\nabla}$ is not a vector, however, we can think of it as such so that we can do our calculations.

$$\vec{\nabla} = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{8.15}$$

To use this you need to take the vector field $\vec{F}$ Which is defined as: $\vec{F} = P\vec{\imath} + Q\vec{\jmath} + R\vec{k}$. The curl of $\vec{F}$ is:

$$\vec{\nabla} \times \vec{F} \tag{8.16}$$

Remember how the operator isn't a vector but we can treat is as such? Yeah, well you can just use the amsterdam method to find the "cross product", giving you

$$\vec{\nabla} \times \vec{F} = \begin{matrix} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} \\ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} \\ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \end{matrix}$$

8.6.3. Useful Relations

If f is a function of 3 variables that has continuous second order partial derivatives then the curl of $(\vec{\nabla}f)$ will be zero. Similar to how $\vec{a} \times \vec{a} = 0$. The rotation of the gradient of a function is always zero. because at this point you're already at the max slope.

If we imagine the river flowing again, the gradient vector would be the vector in the centre of the river, where the water is flowing fastest. to either side of this point, the vectors are slower than the gradient vector.

If the Curl of a Vector field is zero, then we can say it's a conservative vector field, and vice versa.

$$\vec{\nabla} \times \vec{F} = 0 \Leftrightarrow \vec{F} = \vec{\nabla}f \tag{8.17}$$

8.6.4. What is Divergence?

So if Curl is a measure of rotation, then Divergence is a measure of *expansion*. It describes how much something is radiation out. We define it as:

$$div\ \vec{F} = \vec{\nabla} \cdot \vec{F} = \frac{\partial P}{\partial x}\vec{\imath} + \frac{\partial Q}{\partial y}\vec{\jmath} + \frac{\partial R}{\partial z}\vec{k} \tag{8.18}$$

The Divergence of Curl, is zero, for obvious reasons. In such cases, where the divergence of a vector field is zero, we call the field *incompressible*.

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{F}\right) = 0 \tag{8.19}$$

8.7. Useful Relations

The Curl of a gradient vector field is always zero, but the divergence of a gradient vector field is not. In such instance we get a new operator, the Laplace operator, $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$

$$\nabla^2 f = \vec{\nabla} \cdot \left(\vec{\nabla} f\right) \tag{8.20}$$

If we want to get the vector form of Green's Theorem, it is as follows:

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{A} \tag{8.21}$$

9

Parametric Surfaces

9.1. The Basics

9.1.1. Equations

Much like with parametric curves from the previous chapter, what we're trying to do here is, we want a parametric way of describing a surface similar to Parametric Curves. In a curve of 2 dimensions we have a the function $\vec{r}$, of one variable, which then gives us two coordinates (Eq. 9.1). A Parametric Surface would instead be a function of two variables, and would turn this into 3 coordinates (Eq. 9.2).

$$\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta) \rangle \tag{9.1}$$

$$\vec{r}(x, y) = \langle x, y, f(x, y) \rangle \tag{9.2}$$

You can also limit the domains to alter the surface, consider, a portion of a cylinder. With the given parametrisation $\vec{r}(u, v) = 2\cos u\vec{\imath} + v\vec{\jmath} + 2\sin u\vec{k}$. This will give us a cylinder along the y axis., but if we but the limits of $0 \leq u \leq \frac{\pi}{2}$ and $0 \leq v \leq 3$, we can see how that would limit the surface.

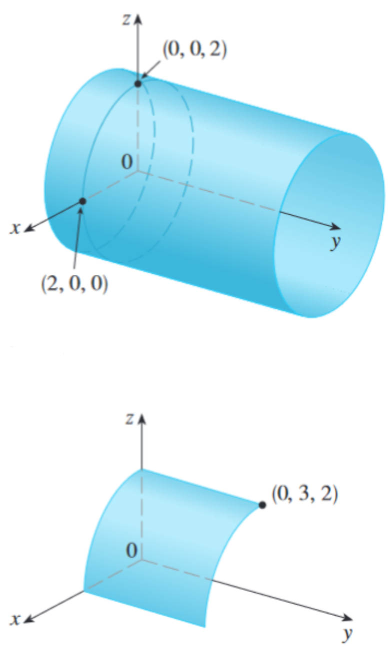


Figure 9.1: *A Cylindrical parametric Surface*

9.1.2. Grid Curves

Grid curves are lines with a constant u or a constant v. In the domain on the $u-v$ plane, they make up an orthogonal grid. but then, these lines get projected onto the parametric surface with the functions $\vec{r}(u_0, v)$ and $\vec{r}(u, v_0)$

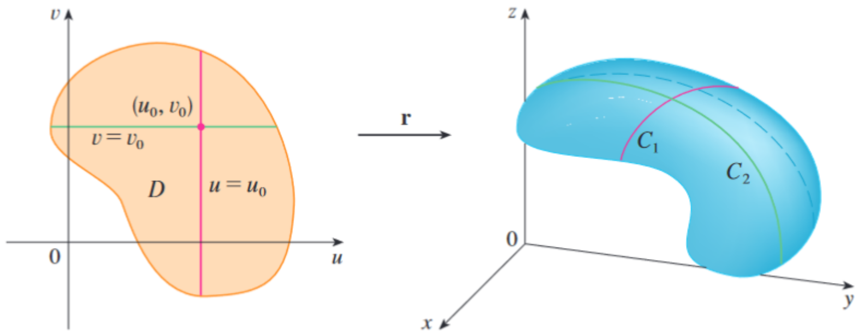


Figure 9.2: *Grid Curves on a Parametric Surface*

9.1.3. Area of a Parametric Surface

If we want to calculate the area of a parametric surface we'll need the following: Let S be the surface

$$\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$$

With a given domain D such that $(u, v) \in D$. then the area of this surface S will be the integral, of the magnitude of the cross product, of the partial derivatives with respect to u and v (Eq. 9.3). Think about it. The partial derivatives will be vectors perpendicular to each other, and the magnitude of a cross product is the area of a parallelogram spanned by the vectors.

$$\iint_D |\vec{r}_u \times \vec{r}_v| \, dudv \tag{9.3}$$

9.2. Surface Integrals

9.2.1. Scalar Functions

the integral of a scalar function on a parametric surface in $\mathbb{R}^3$ is:

$$\iint_S f(x,y,z) = \iint_D f(\vec{r}(u,v)) \, |\vec{r}_u \times \vec{r}_v| \, dA \tag{9.4}$$

The way that we arrive at this conclusion is basically the same way we always found an integral, we divided up the surface into small sections (with the grid curves) and we evaluated the function for each section. The Riemann sum, as usual.

The term $|\vec{r}_u \times \vec{r}_v| \, dA$ in on account of the fact that the small sub areas spanned by the grid curves is the same as the small section of the surface dS.

9.2.2. Orientation

We don't pay attention to non-orientable surfaces like mobius strips or Klein bottles. Instead we prefer much nicer surfaces like a sphere, which has 2 distinct sides to the surface, and as such, can have a positive direction, and a negative direction.

To define the positive direction, we use the unit normal vector $\vec{n}$. This vector, will point in the positive outward direction from the surface.

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \tag{9.5}$$

For closed surfaces however, the positive orientation will always be outward from the enclosed region.

9.2.3. Vector Fields

If $\vec{F}$ is a continuous vector field defined on an orientated surface S with unit normal vector $\vec{n}$, then the integral is:

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F}\left(\vec{r}(u,v)\right) \cdot \left(\vec{r}_u \times \vec{r}_v\right) dA \tag{9.6}$$

If you'll remember from earlier:

$$\vec{n} dS = \left(\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\right) |\vec{r}_u \times \vec{r}_v| \, dudv = \vec{r}_u \times \vec{r}_v dudv$$

9.3. Stokes' Theorem

9.3.1. The Basics

In general, Stokes' Theorem is a pretty central concept in calculus. It states that the total change on the outside, is the same as the sum of all small changes on the inside. This is a very general statement, but again, it can be applied in an awful lot of circumstances.

Think of a standard integral, one of the basic 2D integrals you're used to from Q1 or even secondary school. How much does the function change from one limit a to the other limit b?

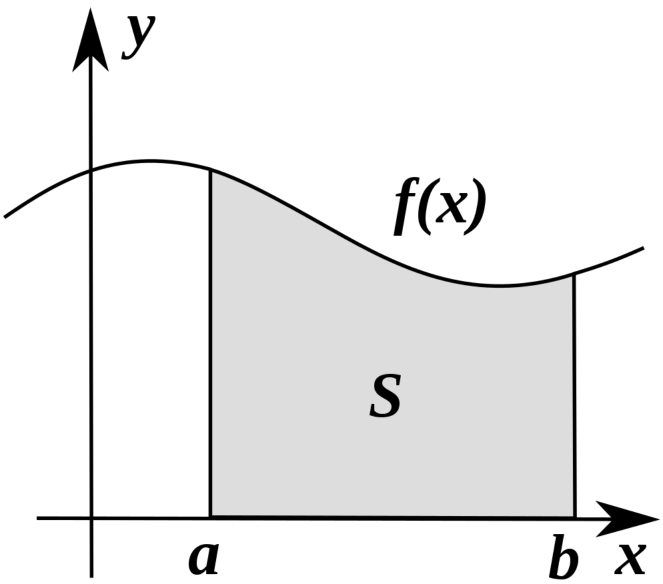


Figure 9.3: *A Basic Integral*

$$\int df = \int_a^b \frac{df}{dx} dx = f(b) - f(a) \tag{9.7}$$

We can see even here, that the sum of small changes $\int df$, is the same as the total change from a to b, $f(b) - f(a)$. Remember Green's Theorem? That's basically the 2D version of what Stokes' theorem. What we're aiming to do here then, is to find a way of applying Green's theorem in $\mathbb{R}^3$.

9.3.2. Green's Theorem

Recall Green's Theorem and see how similar it is to stokes theorem:

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{A}$$

In this case $d\vec{A}$ is on the $x - y$ plane. Stokes theorem is very very similar, but instead of being on the $x - y$ plane, it is on any 2D surface in $\mathbb{R}^3$.

$$\oint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{S} \tag{9.8}$$

In fact, if the Vector field $\vec{F}$ is such that $\vec{F} = \langle P, Q, 0\rangle$ then Stokes' theorem is Green's theorem. So, we can see that Green's Theorem is just a special case of Stokes's Theorem.

9.3.3. Orientation

So, Finally, let's consider the effects of orientation. If you remember from earlier, the positive side of a surface will be to the left, as you traverse along the boundary. In 3D, this is more fun, because as you traverse along the boundary, there will be some surface on both the left and right. so the positive normal vector $\vec{n}$ will depend on the parametisation.

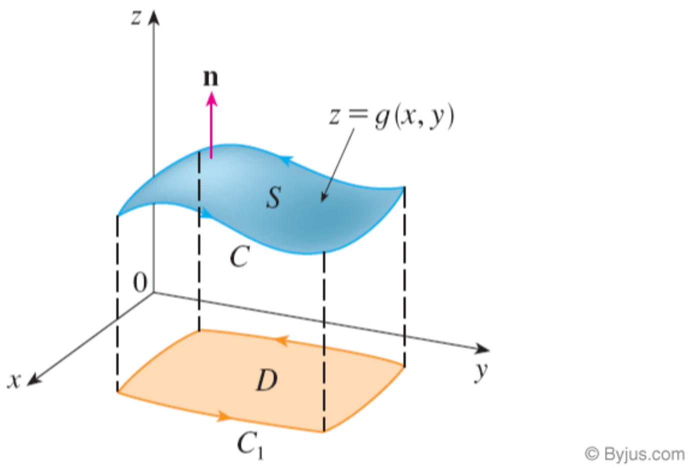


Figure 9.4: *The normal vector shows the positive orientation*

At last, we can describe Stokes' theorem. Let S be an oriented, piece-wise, smooth surface, bounded by a piece-wise smooth boundary curve,
S=C.$With positive orientation. Let \vec{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then:

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{Curl}\vec{F} \cdot d\vec{S} = \iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot d\vec{S} \tag{9.9}$$

9.4. Divergence Theorem

9.4.1. The Basics

So while in Stokes' Theorem the surface S, did not enclose a region, for the Divergence Theorem (Also called Gauss' Theorem) the Surface S **does** encloses a solid region, like the outer surface of a solid object, Or maybe a balloon if you will. Consider the latex to the surface and the air inside it to be the enclosed region E.

9.4.2. Specifics

Let E be a simple solid region and let S (or ∂E if you like), given with positive (the direction pointing outwards) orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives on an open region containing E. Then it is true that:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \vec{\nabla} \cdot \vec{F} dV \tag{9.10}$$

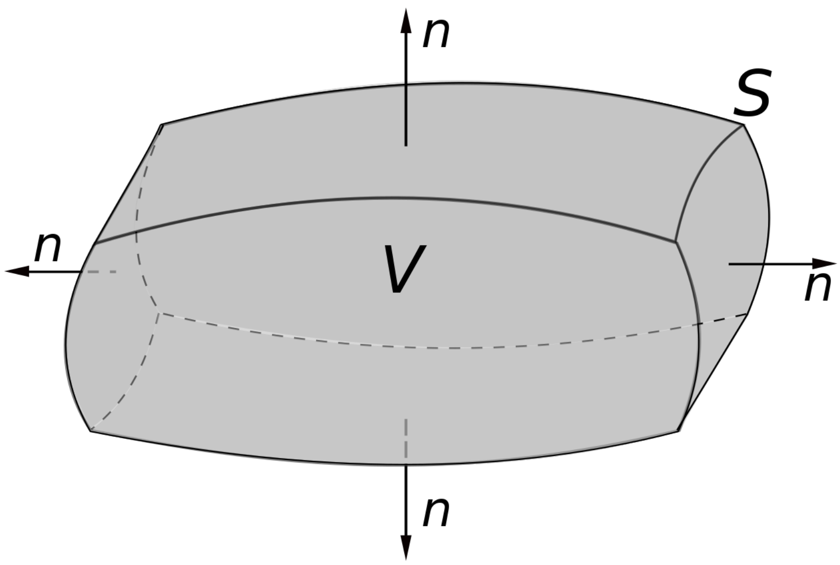


Figure 9.5: *The Expansion of a Solid Object*

For this course you really don't need to know how to derive or prove this, but once again, this is another example of "sum of small changes on the inside is the same as the total change on the outside".