# Approximations and their errors

Some phenomena can't be described algebraically. Then numerical techniques offer an outcome. However, these techniques are approximations. And where there are approximations, there are errors. We want to know in what way these errors behave when we refine our model. That's what we'll be looking at in this chapter.

## 1 Introducing the Landau symbols

### 1.1 Definitions of the Landau symbols

When examining approximation errors, we often use the two **Landau symbols** $O$ (**big-o**) and $o$ (**little-o**). We will examine their definitions now. Let's suppose we have two functions $u(x)$ and $v(x)$. We say that $u(x) = O(v(x))$ as $x \to a$ if

$$\lim_{x \to a} \frac{|u(x)|}{|v(x)|} = C, \tag{1.1}$$

with $C$ a constant. In words this means that the function $u(x)$ starts behaving more or less like $v(x)$ as $x$ gets close to $a$. Similarly, we say that $u(x) = o(v(x))$ as $x \to a$ if

$$\lim_{x \to a} \frac{|u(x)|}{|v(x)|} = 0. \tag{1.2}$$

Now how does it work? Suppose $u(x)$ is for example $2x^3 + 7x^4 + 5x^5$. Then the order of $u(x)$ is simply the term with the lowest power of $x$. (So in this case $u(x) = O(2x^3)$.) And we're even allowed to drop the constant! So in fact $u(x) = O(x^3)$. We also say that $u(x)$ is of order 3. By the way, the little-o of $u(x)$ is always one power smaller. So $u(x) = o(x^2)$.

There are a few handy rules which you can use with Landau symbols. As we just saw, we can ignore constants. So we can just say that $c_1 O(c_2 h^n) = O(h^n)$. The same, of course, works for additions. So $O(h^n) + O(h^n) = O(h^n)$, but also $O(h^n) - O(h^n) = O(h^n) \neq 0$. We can also multiply $O(h^n)$ with powers of $h$. We then get $h^m O(h^n) = O(h^{m+n})$. (This also works if $m < 0$.) Finally we have $O(h^n) + O(h^m) = O(h^n)$ if $n \leq m$.

There is one final addition which you should know. Suppose a function $u(x) = O(x^n)$ as $x \to a$. Now, as $x$ gets 2 times as close to $a$, then $u(x)$ will be $2^n$ times as close to $u(a)$. In other words, the deviation of $u(x)$ from $u(a)$ has changed by a factor $(1/2)^n$.

### 1.2 The connection to Taylor polynomials

The **Taylor polynomial to the** $n^{th}$ **degree** near $x = a$ of a function $u(x)$ is given by

$$p_n(x) = u(a) + \frac{Du(a)}{1!}(x-a) + \frac{D^2u(a)}{2!}(x-a)^2 + \ldots + \frac{D^n u(a)}{n!}(x-a)^n. \tag{1.3}$$

To find this polynomial, the function $u(x)$ must of course be differentiable, for as many times as necessary.

The above polynomial doesn't include all terms of the entire Taylor expansion $P_\infty(x)$. It is therefore only an approximation. We can now ask ourselves, how does the error $u(x) - p_n(x)$ behave as $x \to a$? To answer that question, we look at the entire Taylor expansion $p_\infty(x)$ of $u(x)$. By subtracting $p_n(x)$, we have removed all terms with a power $x^i$, where $i \leq n$. The term with the smallest power of $x$ will then be $x^{n+1}$. The function $u(x) - p_n(x)$ is therefore $O(x^{n+1})$ as $x \to a$.

# 2 The finite-difference method

## 2.1 Basic finite-difference equations

It is possible to approximate an $n^{th}$ derivative $D^n u(x)$ of a function $u(x)$, using only the function $u(x)$. This process is known as the **finite-difference method**. We can, for example, approximate $Du(x)$ as

$$\delta_h(x) = \frac{u(x+h) - u(x-h)}{2h}. \tag{2.1}$$

This approximation increases in accuracy as $h \to 0$. But there of course still is an error in this approximation. And once more we would like to know the order of this error. To find it, we have to use Taylor expansions of $u(x+h)$. The general equation for the Taylor expansion of $u(x+ih)$ is

$$u(x+ih) = u(x) + Du(x)(ih) + \frac{1}{2}D^2u(x)(ih)^2 + \frac{1}{6}D^3u(x)(ih)^3 + \frac{1}{24}D^4u(x)(ih)^4 + O(h^5). \tag{2.2}$$

In the above equation, we have cut off the expansion after the fourth term. You can of course also cut it off at any other term. Using this equation, we can find the order of $\delta_h(x)$. We do this according to

$$\delta_h(x) = \frac{u(x) + Du(x)h + \frac{1}{2}Du(x)h^2 + O(h^3) - u(x) + Du(x)h - \frac{1}{2}Du(x)h^2 + O(h^3)}{h} = Du(x) + O(h^2). \tag{2.3}$$

So the error $\delta_h(x) - Du(x)$ is $O(h^2)$. If we thus decrease $h$ by a factor 2, then the error decreases approximately by a factor $2^2 = 4$.

## 2.2 Finite-difference equations in matrix form

In our previous example, we only had the terms $u(x+h)$ and $u(x-h)$. It turns out that if you want to approximate higher derivatives of $u(x)$, or if you want more accurate approximations of $Du(x)$, then you also need more terms. Although we could continue to write these terms in equations, it would be a lot better to put it in vector form. (At least in this way our equations will still fit on one sheet of paper.) So we say that $\mathbf{u} = M\mathbf{d} + O(h^{\alpha+1})$, where

$$\mathbf{u} = \begin{bmatrix} u(x-jh) \\ \vdots \\ u(x-h) \\ u(x) \\ u(x+h) \\ \vdots \\ u(x+kh) \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} u(x) \\ Du(x)h \\ D^2u(x)h^2 \\ D^3u(x)h^3 \\ \vdots \\ D^{\alpha-1}u(x)h^{\alpha-1} \\ D^{\alpha}u(x)h^{\alpha} \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & -j/1! & (-j)^2/2! & \dots & (-j)^\alpha/\alpha! \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & -1 & 1/2 & \dots & (-1)^\alpha/\alpha! \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1/2 & \dots & a/\alpha! \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & k/1! & k^2/2! & \dots & k^\alpha/\alpha! \end{bmatrix} \tag{2.4}$$

Note that $M$ is a $(j+k+1) \times (\alpha+1)$ matrix. Let's suppose that $M$ is a square matrix, and thus that $j+k = \alpha$. In this case it can be shown that $M$ is invertible, and thus that $\mathbf{d} = M^{-1}\mathbf{u} + O(h^{\alpha+1}$. From this equation we can derive expressions for $D^i u(x)$ (with $0 \leq i \leq \alpha$). This expression will then usually have an order of accuracy $O(h^{\alpha+1-i})$. So what can we note from this? If we increase $\alpha$, we can either get very accurate approximations, or we can approximate higher-order derivatives.

# 3 Numerically solving differential equations

## 3.1 An outline of the solving method

Let's suppose we have some differential equation we want to solve. For example, we want to solve

$$Du(x) = f(x) \quad \text{on the interval } [0,1], \text{ with the initial value } u(0) = \phi. \tag{3.1}$$

To solve this, we divide the interval $[0,1]$ in a **grid** of $N+1$ equally spaced points. For our example, these points are $x_i = ih$, where $i = 0, 1, 2, \ldots, N$ and $h = 1/N$. We will now try to find the corresponding values of $u(x_i)$. We will denote our approximations by $u_i$.

But the question remains, how can we find these approximations $u_i$? For that, we use the finite-difference approximation. For our example, we therefore approximate that

$$\frac{u_i - u_{i-1}}{h} = f(x_i). \tag{3.2}$$

The value of $u_0$ follows from the initial condition. (In fact, $u_0 = u(0) = \phi$.) Then, using the above equation, we can find every other value of $u_i$. And even when we have a more difficult equation, with other initial conditions, it is usually possible to find a solution in this way.

The above method is based on the idea of **convergence**. This means that, as $h$ decreases, our approximations converge to the actual solutions. In other words, the more data points we use, the more accurate we are.

## 3.2 Using residuals to determine the approximation accuracy

Of course we are interested in the error of our approximation. To examine this error, we define two residuals. First we define the **residual of the differential equation** $R_i(u)$. This is the error in our finite-difference approximation. For our example, we thus have

$$R_i(u) = \frac{u(x_i) - u(x_{i-1})}{h} - f(x) = \frac{u(x_i) - \big(u(x_i) - Du(x_i)h + O(h^2)\big)}{h} - f(x_i) = O(h). \tag{3.3}$$

(The latter part follows from the relation $Du(x_i) = f(x_i)$, which was our differential equation.) We of course want that $R_i(u) \to 0$ as $h \to 0$. If this is indeed the case, we say that our finite-difference approximation is **consistent** with the differential equation.

The second residual we will define, is the **residual of the initial conditions** $R_{IC}(u)$. This residual is the difference between the given initial conditions $u(x_i)$, and our approximations $u_i$ to them. For our example we thus have

$$R_{IC}(u) = u_0 - u(0) = u_0 - \phi. \tag{3.4}$$

Although for our example we have $R_{IC}(u) = 0$, this is of course not always the case. It may very well occur that $R_{IC}(u) = O(h^2)$, for example.

Of course we want our actual solution to be accurate. But then both our initial conditions and our finite-difference approximation should be accurate. If only one of these is accurate, while the other is inaccurate, then our solution isn't accurate either. In fact, our solution is as accurate as the worst of these two residuals. If, for example, $R_i(u) = O(h^3)$ and $R_{IC}(u) = O(h^2)$, then our solution has an error which is of order 2 (meaning it is $O(h^2)$). This order is referred to as the **order of consistency**.

## 3.3 Using plots to find the order of consistency

Another way to find the order of consistency, is by making a plot. Let's suppose $r$ is the error in our approximation. We will now plot ${}^2\log r$ with respect to ${}^2\log h$. As $h \to 0$, this graph converges to a straight line. Let's call the slope of this line $n$. It turns out that this $n$ is actually the order of consistency of our approximation!

Don't believe me? Then let's show it. We have thus defined $n$ as

$$n = \frac{d\left({}^2\log r\right)}{d\left({}^2\log h\right)}, \qquad \text{which, after working out, gives} \qquad \frac{1}{r}dr = \frac{n}{h}dh. \tag{3.5}$$

Let's suppose our error $r$ is $O(h^m)$. As $h \to 0$, we can thus approximate $r$ as $ch^m$. It follows that $dr = mch^{m-1}dh$. By inserting this in the above equation, we find

$$\frac{mch^{m-1}}{ch^m}dh = \frac{n}{h}dh, \qquad \text{which implies that} \qquad m = n. \tag{3.6}$$

In other words, the slope of this graph $n$ is equal to the order of our approximation error $m$. And this is what we wanted to show.