Differential Equations

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Course: WI2180LR-I Differential Equations

Lecture 1

Separable variable method

$$\frac{dy}{dx} = f(y) \cdot g(x) \implies \int \frac{dy}{f(y)} = \int g(x)dx$$

Integrating factor method for first order linear DE

$$\frac{dy}{dt} + p(t)y = g(t)$$

Integrating factor:

$$I(t) = e^{\int p(t)dt}$$

$$I(t)\frac{dy}{dt} + I(t)p(t)y = I(t)g(t)$$

Integration by parts:

$$\frac{d}{dt}(I(t)y) = I(t)g(t)$$

$$y(t) = \frac{1}{I(t)}\left(\int I(t)g(t)dt + C\right)$$

Definition of linear DE

Looks like a polynomial with its coefficients being functions of x. Note that the only y or its derivatives can exist.

$$a_0(x)y + a_1(x)y' + a_2(x)y'' + \cdots + a_n(x)y^{(n)} = b(x)$$

Direction field for DE

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f[x_, y_] = (y + E^x)/(x + E^y)
StreamPlot[{1, f[x, y]}, {x, 1, 6}, {y, -20, 5}, Axes -> True, AspectRatio -> 1]
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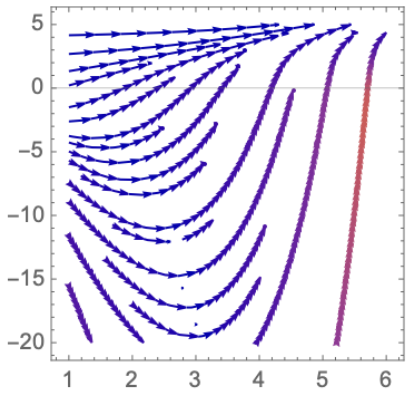


Lecture 2

NB: Define

$$\frac{dy(t)}{dt} = f(t, y)$$

Existence and uniqueness

- Whether a solution exists
- Whether it is unique

Existence and uniqueness for first order DE

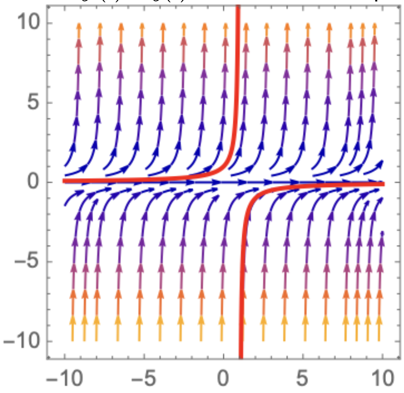
Let functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle containing the point (t0, y0); then in some interval contained in the width of the rectangle there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y)$$
$$y(t_0) = y_0$$

Implications:

- Is f and $\frac{\partial f}{\partial y}$ are continuous, then solutions to first order DE cannot intersect or cross. (Uniqueness)
  - May only be valid over a small region (some interval)
- A solution might only be defined on an interval.
- If f is continuous but partial f is not, it is still possible to show the solution exists. But the solution might not be unique.
- Should a discontinuity exists for a solution to IVP, the unique solution is only the part on the same side as the initial condition.

Field of $y'(t) = y(t)^2$. For the solution passing through (0,1), only its left half is valid:



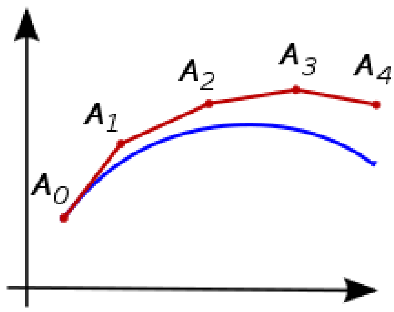
Forward Euler method for numerical approximation

- Take steps in the direction provided $(1, y'(t))$

$$y_i = y_{i-1} + f(t_{i-1}, y_{i-1}) \cdot \Delta t$$

or

$$y_i = y_{i-1} + \frac{dy(t_{i-1}, y_{i-1})}{dt} \cdot \Delta t$$



To determine a good delta t, continue decreasing the step size until there is almost no error between the solution with delta t and that with delta t/2.

Lecture 3

Second-order linear differential equations

Takes the form

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

NB: We only consider the intervals where P, Q, R, G are continuous.
Homogenous:

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Non-homogenous:

$$G(t) \neq 0$$

Here, we focus on homogeneous 2nd order DE with constant coefficients:

$$ay'' + by' + cy = 0$$

Uniqueness of solution to 2nd order DE

If $Q/P, R/P, G/P$ are continuous on an open interval I containing the initial point t_0, then the solution exists over I and is unique.

Solutions to $ay'' + by' + cy = 0$

Try

$$y = e^{rt}$$

$$(ar^2 + br + c)e^{rt} = 0$$

$$ar^2 + br + c = 0$$

This means the 2 roots of r will produce 2 solutions. There are 3 scenarios.
Define the differential operator

$$L[\phi] = \phi'' + p\phi' + q\phi$$

Principle of superposition

If $y_1(t)$ and $y_2(2)$ are 2 solutions to the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

Their linear combination

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

is also a solution for any constants.

Initial value problems

We can find

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$

$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$

If the coefficient matrix of this system has a non-0 determinant

$$W = \det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \neq 0$$

then its solution is unique.

The determinant above is called The Wronskians:

$$W = W[y_1, y_2]\Big| = y_1y_2' - y_1'y_2$$

Fundamental set of solutions

$$\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lecture 4

Implication of Wronskians

Basically,

- y1 and y2 are solutions to a 2nd order linear DE
- $W[y_1, y_2] \neq 0$ at any point
- Then any solution to the DE can be written as c1y1+c2y2

Abel's theorem

An interesting theorem that doesn't require you to find the explicit solutions.
If y1 and y2 are solutions to the 2nd order linear DE

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

and p, q are continuous on an open interval I, then the Wronskain is

$$W(t) = c \exp\left(-\int p(t)dt\right)$$

Interestingly, $W(t)$ is either 0 or never 0 over I.

Solutions to $ay'' + by' + cy = 0$ (Continued)

2 Distinct real roots

This is trivial
Obtain $r1, r2$

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then plug into

$$y(t) = c_1 e^{rt} + c_2 e^{rt}$$

Complex conjugate roots

If $b^2 - 4ac < 0$, then

$$r = \lambda \pm i\mu$$

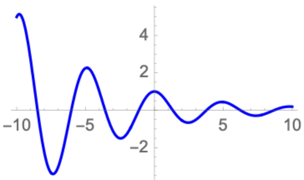
And applying superposition principle twice, we obtain

$$y = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$$

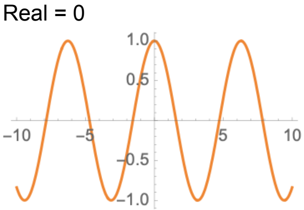
The Wronskain:

$$W(t) = \mu e^{2\lambda t}$$

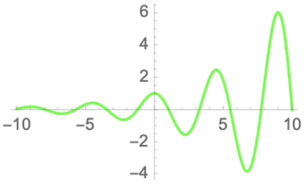
The real part of r determines (damping, excitation), and the complex part of r determines oscillation frequency.
Real < 0



Real = 0



Real > 0



Repeated real roots

Obviously, one direct solution is

$$y_1(t) = e^{-bt/(2a)}$$

Through trying, it is found that the general solution

$$y = c_1 e^{-bt/(2a)} + c_2 t e^{-bt/(2a)}$$

Wronskain:

$$W(t) = e^{-bt/a}$$

Lecture 5

Reduction of order

Suppose now that we have a linear 2nd homogeneous differential equations with non-constant coefficients:

$$y'' + p(t)y' + q(t)y = 0$$

If one solution y1 is known, then the second solution can be assumed to be

$$y_2 = v(t)y_1$$

Substituting into the ODE, DE will be reduced to a first order for v'.
It's like a change of variable for the ODE.

$$v''(t)y_1 + v'(t)(2y_1' + p(t)y_1) = 0$$

Let $u = v'(t)$

$$u'(t)y_1 + u(t)(2y_1' + p(t)y_1) = 0$$

Solve the DE, then y_2 can be found.

Solutions for non-homogeneous ODE

Variation of parameters is a general method for solving non-homogeneous differential equations
Non-homogeneous equation:

$$y'' + p(t)y' + q(t)y = g(t)$$

Method:
If Y1, Y2 are the 2 solutions of the non-homo DE, then their difference

$$Y1 - Y2$$

is a solution to the corresponding homo differential equation

$$y'' + p(t)y' + q(t)y = 0$$

The general solution to the non-homo can be written in the form

$$y(t) = c_1 y_1 + c_2 y_2 + Y(t)$$

where y1, y2 are from the set of solutions of the homo DE, and Y(t) are particular solutions of the non-homo DE.

Variation of parameters

This is a method for solving 2nd order DE if already have a fundamental set of solutions to the homo DE.

- Take the general solution to the homo DE

- $y_g(t) = c_1y_1(t) + c_2y_2(t)$
- Look for a solution to the non-homo DE in the form (u1 and u2 are not known)
  - $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$
- Differentiate once and making the following assumption: $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$
- Plug this particular solution and its derivatives into the DE
- Solve $u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$ and $u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$
- If you can solve for u_1 and u_2, then $y_p = u_1y_1 + u_1y_2$
- The general solution to the non-homo for DE can be written as
  - $y(t) = y_p(t) + y_g(t)$

Generally, the particular solution of

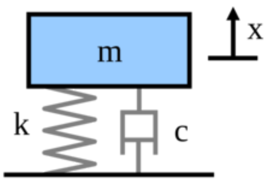
$$y'' + p(t)y' + q(t)y = g(t)$$

is

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}ds$$

Lecture 6

Model of a spring system



$$mx''(t) = mg - k(L + x(t)) - \gamma x'(t) + F(t)$$

$kL = mg$, $F(t)$ is the driving force.

$$mx''(t) + \gamma x'(t) + kx(t) = F(t)$$

For undamped, not driven case:

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \sigma)$$

$$R = \sqrt{A^2 + B^2},\ \tan\delta = \frac{B}{A}$$

For damped, not driven:

- $\gamma^2 > 4km$, overdamped, distinct roots
- $\gamma^2 = 4km$, critically damped, repeated roots
- $\gamma^2 < 4km$, underdamped, sinusoidal + exponential

For damped and driven:

$$mx''(t) + \gamma x'(t) + kx(t) = F_0cos(\omega t)$$

Solution:

$$u(t) = u_c(t) + U(t)$$

- $u_c(t) = c_1u_1 + c_2u_2$ is the general solution
  - It dies out as $t \to \infty$, and is called the transient solution
- $U(t) = R\cos(\omega t - \delta)$ is the particular solution
  - It is the steady state solution
  The R in the steady state solution can be expressed as follows:

  $$R = \frac{F_0}{\sqrt{m^2(\omega_0 - \omega^2)^2 + \gamma^2\omega^2}}$$

  where $\omega_0^2 = k/m$.
  Maximum amplitude of vibration occurs when

$$\omega_{max}^2 = \omega_0^2 \left(1 - \frac{\gamma^2}{2mk}\right)$$

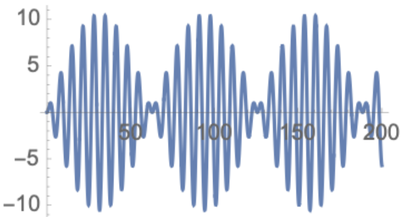
$$R_{max} \approx \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right)$$

Resonance: For a lightly damped situation, if the ω for driving force gets too close to ω_{max}.
Damping factor: $\Gamma = \gamma^2/(mk)$

- $\omega \sim 0 \implies \delta = 0$
- $\omega = \omega_0 \implies \delta = \frac{\pi}{2}$
- $\omega \sim \infty \implies \delta = \pi$
  δ is the phase between driving force and response.

For undamped, driven:

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(cos(\omega t) - \cos(\omega_0 t))$$



Beat: As shown in the graph, the oscillatory behavior of the amplitude is apparent.
The variation of amplitude w.r.t. time is called amplitude modulation.

Lecture 7

Laplace transforms

La PlaCE

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Main idea behind Laplace transform:

- Transform an initial value problem of an unknown function f in t-domain to an algebraic problem for F in s-domain.
- Recover f by its inverse transform.
- s may be complex, and $F(s)$ can be treated as a complex function.
- Essentially a change of variable.

Notation: Capitalize the function's name. For example $g(t) \to G(s)$.

Existence of Laplace

Suppose that $f(t)$ is piece-wise continuous for $0 \leq t$ and there exist real constants $K, M > 0$, a such that

$$|f(t)| \leq Ke^{at},\ t \geq M$$

For $s > a$, then the Laplace transform $L\{f(t)\} = F(s)$ is define as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Properties of Laplace

Linear operator:

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}$$

First derivative for IVP:

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Second derivative for IVP:

$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

For nth derivative for IVP:

$$L\{f^{(n)}(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Lecture 8

Properties of Inverse Laplace Transform

Inverse Laplace Transform:

$$L^{-1}\{F(s)\} = f(t)$$

It is also a linear operator:

$$L^{-1}\{F(s)\} = L^{-1}\{F_1(s)\} + \cdots + L^{-1}\{F_n(s)\} = f_1(t) + \cdots + f_n(t)$$

How to solve a DE with initial conditions with Laplace transforms:

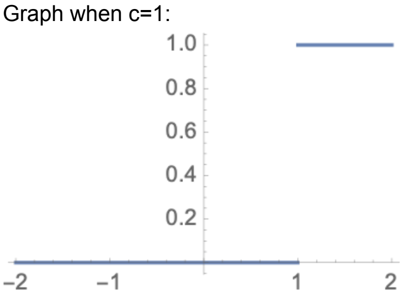
- Take Laplace of both sides
- Convert the derivatives to Laplace of the function and the initial conditions
- Rearrange algebraically to obtain $Y(s) = L\{y\}$
- Rearrange, apply partial fraction whenever needed, and use the inverse table to find the solutions

Step functions

Heaviside step function

$$u_c(t) = \begin{cases} 0, \ t < c \\ 1, \ t \geq c \end{cases}$$

Graph when c=1:



It essentially represents switch-on. Similarly, $1 - u_c(t)$ models switch-off.

Turning on a function at c:

$$g(t) = \begin{cases} 0, \ t < c \\ f(t-c), \ t \geq c \end{cases}$$

$$g(t) = u_c(t) f(t-c)$$

Laplace transform of shifted functions

Translation in t-domain

If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$ and c is a positive constant, then

$$L\{u_c(t) f(t-c)\} = e^{-cs} L\{f(t)\} = e^{-cs} F(s), \ s > a$$

e^{-cs} shifts the function in the s-domain.
If $f(t) = L^{-1}\{F(s)\}$, then

$$u_c(t) f(t-c) = L^{-1}\{e^{-cs} F(s)\}$$

Translation in s-domain

Assume that $F(s) = L\{f(t)\}$, then

$$L\{e^{ct}f(t)\} = F(s-c),\ s > a + C$$
$$e^{ct}f(t) = L^{-1}\{F(s-c)\}$$

Differential equations with discontinuous forcing

The non-homogeneous is discontinuous in this case. Aka forcing function.
The solution itself and its derivatives* is continuous, however its highest derivative has discontinuities due to forcing.
The procedure is the same as its homogenous counterpart.
Use cases: e.g. Turning on the voltage of a LRC circuit.

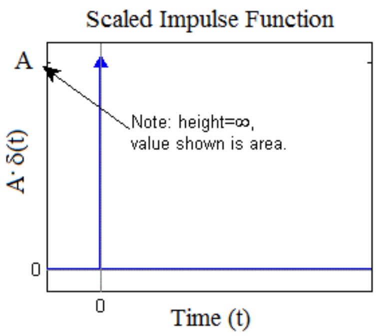
Lecture 9

Impulse functions

The forcing function $g(t)$ is large for $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and is 0 otherwise.
Define

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt = 1$$

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} \\ 0 \end{cases}$$



Dirac Delta function: Taking $d_\tau(t)$ as $\tau \to 0$

- $d_\tau(t)$ is 0 everywhere but for a small interval of length 2τ around $t = 0$
- Over the small interval, $d_\tau(t)$ equals $1/(2\tau)$
- $I(\tau) = 1$ for all $\tau \neq 0$
  Shifting the impulse to t_0 instead:

$$\delta(t) = 0,\ t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

Laplace transform of the impulse function

$$L[\delta(t-t_0)] = \lim_{\tau \to 0^+} L[d_\tau(t-t_0)]$$
$$L[d_\tau(t-t_0)] = \int_{t_0-\tau}^{t_0+\tau} e^{-st}\frac{1}{2\tau}\,dt$$

Simplying,

$$\frac{1}{2\tau s}e^{-st}\Big|_{t_0-\tau}^{t_0+\tau} = \frac{\sinh(s\tau)}{s\tau}e^{-st_0}$$

Taking $\tau \to 0$,

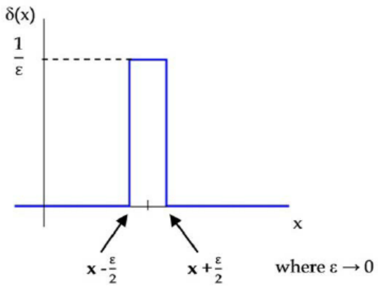
$$\lim_{\tau \to 0^+} \frac{\sinh(s\tau)}{s\tau}e^{-st_0} = e^{-st_0}$$

Therefore,

$$L[\delta(t-t_0)] = e^{-st_0},\ t_0 > 0$$

When $t_0 = 1$,

$$L[\delta(t)] = 1$$



(Here, $2\tau = \epsilon$)

Similarly, the integral of the product of Delta with another function is

$$\int_{-\infty}^{\infty} \delta(t - t_0) f(t)\, dt = f(t_0)$$

Or,

$$f(t)\delta(t - a) = f(a)\delta(t - a)$$

Convolution Integral

Convolution theorem
If $F(s) = L[f(t)]$ and $G(s) = L[g(t)]$ both exists, then

$$H(s) = F(s)G(s) = L\{h(t)\}$$

Where

$$h(t) = \int_0^t f(t - \tau) g(\tau)\, d\tau = \int_0^t f(\tau) g(t - \tau) d\tau$$

h is the convolution of f and g, and the integral in $h(t)$ is called convolution integral.

Properties of convolutions

Notation:

$$(f * g)(t) = h(t)$$

- $f * g = g * f$
- $f * (g_1 + g_2) = f * g_1 + f * g_2$
- $(f * g) * h = f * (g * h)$
- $f * 0 = 0 * f = 0$

Transfer function

Suppose a DE system

$$ay'' + by' + cy = g(t),\ y(0) = y_0,\ y'(0) = y'_0$$

This can be the model for an input-output problem: $g(t)$ is the input to the system, and solution $y(t)$ is the output.
Taking Laplace,

$$(as^2 + bs + c)Y(s) - (as + b)y_0 - ay'_0 = G(s)$$
$$Y(s) = \Phi(s) + \Psi(s)$$

In particular,

$$\Psi(s) = \frac{G(s)}{as^2 + bs + c}$$

$\phi(t)$ is the solution of the homog. IVP, and $\psi(t)$ is the solution of the non-homog with initial conditions set to 0s.

To find $\psi(t) = L^{-1}[\Psi(s)]$, let

$$\Psi(s) = H(s)G(s)$$

$H(s) = 1/(as^2 + bs + c)$ is know as the **transfer function** and is not affected by input $g(t)$.
By Convolution,

$$\psi(t) = L^{-1}[H(s)G(s)] = \int_0^t h(t-\tau)g(\tau)d\tau$$

Here $h(t)$ is the inverse Laplace of $H(s)$.

Replacing $g(t)$ by $\delta(t)$, then $G(s) = 1$ and solution $\Psi(s) = H(s)$, meaning that $y = h(t)$ is the solution to the IVP with the non-homog. term be $\delta(t)$ and the initial conditions being 0s. Thus, $h(t)$ is the response of the system to a unit impulse at 0, and is called the impulse response of the system.

Lecture 10

Systems of 1st-order DEs

Principle of superposition:
If the vector functions $x_1, x_2, \dots, x_n$ are linearly independent solutions of the system

$$\vec{x}'(t) = \vec{P}(t)\vec{x}$$

Then the general solution is

$$\vec{x}(t) = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n$$

The Wronskian:

$$W(t) = \det(\vec{x}_1, \dots, \vec{x}_n) \neq 0$$

The Abel's theorem also applies here.

Note that the solution to systems of DE can be complex:

$$\vec{x} = \underbrace{\vec{u}(t)}_{real} + i\underbrace{\vec{v}(t)}_{imag}$$

2nd-order linear DE to two 1st-order DE

Suppose non-homog DE

$$ax''(t) + bx'(t) + c(t) = g(t)$$

Let

$$u_1'(t) = u_2(t)$$

The other equation

$$u_2'(t) = -\frac{c}{a}u_1(t) - \frac{b}{a}u_2(t) + \frac{1}{a}g(t)$$

For a n-th order linear DE with solution $y(t)$

$$x_1 = y,\, x_2 = y,\, \dots,\, x_n = y_n$$

Homogeneous linear system with constant coefficients

Consider a system

$$\vec{x}'(t) = A\vec{x}(t)$$

The matrix A can be diagonalized as PDP^{-1}, where D is a diagonal matrix formed by the eigenvalues, P matrix is formed by the corresponding eigenvectors. Then we can decouple the system in eigenspace; let $\vec{x}(t) = P\vec{y}(t)$ and $\vec{y}(t) = P^{-1}\vec{x}(t)$.

$$\frac{d}{dt}(P\vec{y}) = A(P\vec{y}) = (PDP^{-1})P\vec{y} = PD\vec{y}$$

$$\therefore \vec{y}' = D\vec{y} \implies \vec{y}(t) = [c_1e^{\lambda_1}, \dots, c_ne^{\lambda_n}]^T$$

and use $P = \{\vec{v}_1, \dots, \vec{v}_n\}$ to tranform the solutions to the standard basis. The solution (standard) should take the form

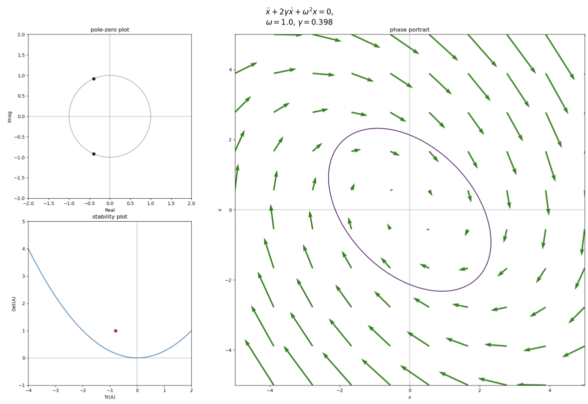
$$\vec{x} = \vec{v}_1e^{\lambda_1 t} + \cdots + \vec{v}_ne^{\lambda_n t}$$

For reference,

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

Phase space

Also called state space. It's a space in which all possible states of a dynamical/control system are represented, with each possible state corresponding to one unique point in the phase plane. It could be constructed with x_1 and x_2 or $\dot{x}_1$ and x_1.



(https://en.wikipedia.org/wiki/Phase_space#/media/File:Phase_portrait_of_damped_oscillator,_with_increasing_damping_strength.gif)

For $dx/dt = 4/3x + 2/3y$ and $dy/dt = 1/3x + 5/3y$:

```
StreamPlot[{4/3 x + 2/3 y, 1/3 x + 5/3 y}, {x, -8, 8}, {y, -8, 8},
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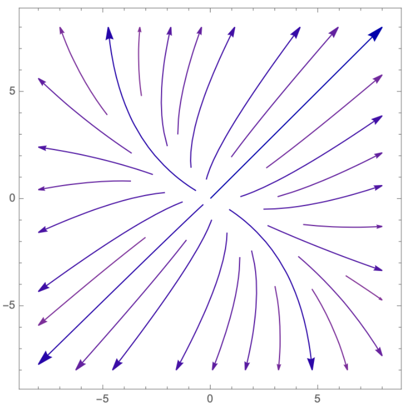


Lecture 11

Repeated lecture 10

Lecture 12

Modes in phase plane

The eigenvectors r1 and r2 of matrix A in $\vec{x}' = A\vec{x}$ determine the outline of the system of DEs in the phase plane.
Attractors as stable, while repellers are unstable.

DEs with complex eigenvalues

The real part scales the solution vectors, and the imaginary rotates them.
Suppose we have solutions

$$\vec{x}_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{at+bit}$$

$$\vec{x}_2(t) = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{at-bit}$$

Equivalently,

$$\vec{x}_1(t) = e^{at}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix} + ie^{at}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}$$
$$\vec{x}_2(t) = e^{at}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix} - ie^{at}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}$$

Let

$$\vec{u} = e^{at}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix}$$
$$\vec{v} = e^{at}\begin{pmatrix}\sin(t)\\cos(t)\end{pmatrix}$$

And yes, the solutions do form a fundamental set:

$$W[u,v] = e^{-t}$$

Therefore, the general solution is

$$\vec{x} = c_1e^{at}\begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix} + c_2e^{at}\begin{pmatrix}\sin(t)\\\cos(t)\end{pmatrix}$$

The solution in the direction field has a clockwise rotation. To see this a few test points can be drawn in the phase plane. The direction of the vectors is $A\vec{x}$ and the position is $\vec{x}$.

If multiple eigenvalues have negative real parts, the eigenvalue representing the least damping, also known as the one with the real part closest to 0, dominates the behavior of the system. For the complex parts of eigenvalues, the one with higher frequency (less damping) dominates in the short term; however, the eigenvalue associated with lower frequency will be more prominent in the long run.

Lecture 13

Repeated eigenvalues

It is possible that an eigenvalue λ with algebraic multiplicity (power in the characteristic equation) n has less than n eigenvectors.

Hermetian matrix (complex):

$$A = \bar{A}^T$$

A real and symmetric matrix is also Hermetian.

If it is not possible to find n linearly independent eigenvectors, we need to find generalized eigenvectors. The First solution is obviously $\vec{x}(t) = \vec{v}e^{\lambda t}$, and the other solution looks like

$$\vec{x} = \vec{\xi}te^{\lambda t} + \vec{\eta}e^{\lambda t}$$

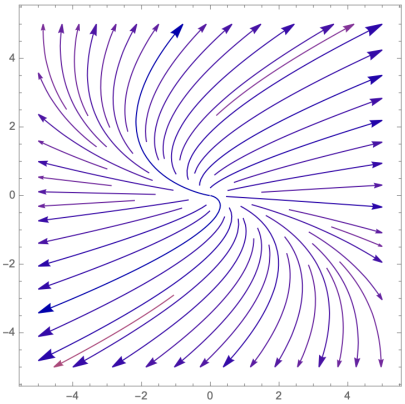
Plugging into $x' = Ax$, we have $2\xi = A\xi$ and $\xi + 2\eta = A\eta$. Finding them is the same as solving the following matrix equations:

$$(A - \lambda I)\xi = 0$$
$$(A - \lambda I)\eta = \xi$$

Actually, ξ is the eigenvector; now η can be found.

The origin in the phase plane of a repeated eigenvector system is called an improper node. Conversely, proper node means the eigenvectors are independent.

The phase portrait of a system w. repeated eigenvalues looks like a spiral:



Lecture 14

Fundamental matrix

Assuming we have found a fundamental set of solutions $\{\vec{x}_1(t), \vec{x}_2(t), \vec{x}_n(t)\}$ we can construct the fundamental matrix

$$\Phi(t) = (\vec{x}_1(t) \dots \vec{x}_n(t))$$

This matrix solves the DE

$$\Psi'(t) = \vec{P}(t)\vec{\Psi}(t)$$

This matrix will be used to explain variation of parameters.

Nonhomogeneous linear systems

They take the form

$$\vec{x}'(t) = \vec{P}(t)\vec{x} + \vec{g}(t)$$

The solution can be expressed as

$$\vec{x} = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t) + \vec{v}(t)$$

The focus here is $\vec{v}(t)$.

Techniques for nonhomo linear systems

Variation of parameters

If we assume fundamental matrix exists and P(t) and g(t) are continuous, then the general solution to homogeneous problem is

$$\vec{x}_h(t) = \vec{\Psi}(t)\vec{c}$$

To solve its homogeneous counterpart, let c be u(t)

$$\vec{x}_p(t) = \vec{\Psi}(t)\vec{u}(t)$$

As before (Lecture 5), we need to look for u(t) by plugging into the system:

$$P(t)\Psi(t)u(t) + g(t) = x_p(t) = \Psi'(t)t(t) + \Psi\vec{u}'(t)$$

Since Ψ is a fundamental matrix, we have

$$\Psi' = P\Psi \implies \Psi u' = g \implies \vec{u}' = \Psi^{-1}g$$

Therefore,

$$\vec{u}(t) = \int_{t_1}^{t} \Psi^{-1}(s)\vec{g}(s)\, ds + \vec{c}$$

$$\vec{x}_{full}(t) = \Psi(t)\vec{u}(t) = \Psi(t)\vec{c} + \Psi(t)\int_{t_1}^{t} \Psi^{-1}(s)g(s)\, ds$$

Procedure for solving nonhomo linear systems using variation of parameters

- Find fundamental matrix

    - $x = \Psi c$
- Substitute $u(t)$ for c
    - $x = \Psi u$
- Substitute $x = \Psi u$ into the nonhomo system
- Solve for $u(t)$
- Substitute to find particular solution $x = \Psi u$

Laplace transforms

$$s\vec{X}(s) - \vec{x}(0) = A\vec{X}(s) + \vec{G}(s)$$

$$(sI - A)\vec{X}(s) = \vec{G}(s)$$

$$\vec{X}(s) = \underbrace{(sI - A)^{-1}}_{\text{Transfer Matrix}}\vec{G}(s)$$

We are now done with systems of linear DEs.

Nonlinear differential equations

Any nonlinear DE systems that cannot be expressed as $\dot{\mathbf{x}} = A\mathbf{x}$.

Phase plane for linear systems

- If the eigenvectors are distinct, then there is no time dependency for the solutions in the phase plane.
- However, if the system has repeated eigenvectors, then the solutions vary with time.

Autonomous systems and stability

An autonomous system does not depend on parameter t.

$$\frac{dx}{dt} = F(x, y)$$

$$\frac{dy}{dt} = G(x, y)$$

$$\frac{d\vec{x}}{dt} = f(\vec{x})$$

$$\vec{x}(t_0) = \vec{x}_0$$

Critical points of autonomous sytems are points where the state does not change:

$$\frac{d\vec{x}}{dt} = f(\vec{x}) = \vec{0}$$

Critical points can be either stable, asymptotically stable, or unstable.

TABLE 9.1.1 Stability Properties of Linear Systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - r\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically stable
$r_2 < 0 < r_1$	Saddle point	Unstable
$r_1 = r_2 > 0$	Proper or improper node	Unstable
$r_1 = r_2 < 0$	Proper or improper node	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$		
$\lambda > 0$	Spiral point	Unstable
$\lambda < 0$	Spiral point	Asymptotically stable
$\lambda = 0$	Center	Stable

Lecture 15

Autonomous systems

Slope in phase space is

$$\frac{dy}{dx} = \frac{G(x, y)}{F(x, y)}$$

Solving this ODE sometimes results in

$$H(x, y) = C$$

In this case, the trajectory of the system in phase space is a level curve.

The critical point $\vec{x} = \vec{0}$ of the linear system

$$\vec{x}' = A\vec{x}$$

Modes of critical points

- Asmp. stable spiral point
- Stable center
- Unstable spiral point
- Asmp. stable node
- Unstable node
- Unstable saddle point

Equilibrium points

- Node
- Saddle point
- Improper node
    - Repeated λ, dependent $\vec{v}$
- Proper node
    - Repeated λ, independent $\vec{v}$
- Spiral point
- Center

Locally linear systems

Consider a nonlinear system

$$\vec{x}' = f(\vec{x})$$

Critical points:

$$\vec{x}' = \vec{0} = f(\vec{x})$$

To translate the origin to a critical point, let $\vec{u} = \vec{x} - \vec{x}_0$, then

$$\vec{u}' = f(\vec{u})$$

Locally linear system takes the form

$$\vec{x}' = A\vec{x} + g(\vec{x})$$

where $g(x)$ has continuous partial derivatives and

$$\lim_{x \to 0} \frac{|g(\vec{x})|}{|\vec{x}|} = 0$$

In essence, if $g(x)$ is small, then it can be considered a perturbation of the system. We can thus assume the behavior of this nonlinear system is similar to a linear system around the equilibrium points.

Linearization of nonlinear systems

Check for local linearity

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}$$

If F and G have **continuous 2nd order partial derivatives** in the neighborhood of a critical point, then the system is locally linear about that point.

Jacobian

$$J = J[F, G](x, y) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

If $\det(J) \neq 0$ when at critical point (x_0, y_0), then it's an isolated critical point.

Linearized system
The linearized system takes the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_x(x_0) & F_y(x_0) \\ G_x(x_0) & G_y(x_0) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \vec{\eta}(x, y)$$

where $u = x - x_0$ and $v = y - y_0$, it's a change of origin's coordinates.

The main idea behind is that if we take the 2nd order Taylor polynomial of $f(x, y)$, then

$$\dot{x} = f(x, y) = f(x_0, u_0) + \frac{\partial f(x, y)}{\partial x}\bigg|_{x=x_0} \delta x + \frac{\partial f(x, y)}{\partial y}\bigg|_{u=u_0} \delta y + \eta$$

and $\eta \to 0$.

Stability of a locally linear system

Let r_1 and r_2 be the eigenvalues of the linear system $x' = Ax$ corresponding to the locally linear system $x' = Ax + g(x)$, then the types of stability of the critical point $(0, 0)$:

TABLE 9.3.1 Stability and Instability Properties of Linear and Locally Linear Systems

	Linear System		Locally Linear System	
Eigenvalues	Type	Stability	Type	Stability
$r_1 > r_2 > 0$	N	Unstable	N	Unstable
$r_1 < r_2 < 0$	N	Asymptotically stable	N	Asymptotically stable
$r_2 < 0 < r_1$	SP	Unstable	SP	Unstable
$r_1 = r_2 > 0$	*PN or IN*	Unstable	*N or SpP*	Unstable
$r_1 = r_2 < 0$	*PN or IN*	Asymptotically stable	*N or SpP*	Asymptotically stable
$r_1, r_2 = \lambda \pm i\mu$				
$\lambda > 0$	SpP	Unstable	SpP	Unstable
$\lambda < 0$	SpP	Asymptotically stable	SpP	Asymptotically stable
$\lambda = 0$	*C*	Stable	*C or SpP*	Indeterminate

Key: N, node; IN, improper node; PN, proper node; SP, saddle point; SpP, spiral point; C, center.

Lecture 16

Competing species

Exponential growth model

$$\frac{dx}{dt} = \alpha x$$

Logistic equation

$$\frac{dx}{dt} = x(\epsilon - \sigma x)$$

Model for competing species (finite resources)

$$\frac{dx}{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$\frac{dy}{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x)$$

Predatory-prey model: Lotka-Volterra model

$$\frac{dx}{dt} = ax - (\alpha y)x$$
$$\frac{dy}{dt} = (\beta x)y - cy$$

x is the prey here and y is the predator. You can see that when there is more predator (y), the growth rate of x (x dot) is less.

Lecture 17

Partial differential equations

To solve PDEs, we will use separation of variables by assuming

$$u(x,t) = X(x)T(t)$$

- Make assumption that $u(x,t) = X(x)T(t)$.
- Substitute into the PDE.
- Separate so that the terms involving x are on on side and those involving t are on the other side.
- Equate the fractions to a constant, and convert the PDE to 2 linear ODEs.

Two-point boundary value problems

Well... just add boundaries conditions instead of an initial condition.
Like $y(\alpha) = y_\alpha$ and $y(\beta) = y_\beta$
When $g(x)$ and $y(\alpha) = y(\beta) = 0$

Recall that for $A\vec{x} = \vec{b}$, if A is nonsingular, then only the trivial solution exists, otherwise, there are infinitely many solutions for a singular A.

Eigenvalues problem

$$y'' + \lambda y = 0$$

with BV $y(0) = 0$, $y(\pi) = 0$. The λs where solutions exist are called eigenvalues, and the solutions are called the eigenfuctions.

Main idea behind Fourier series (Khan academy)

Ok, the lecture and the textbook made no sense to me, so I watched Khan academy instead.
The main idea of Fourier series is that you can approximate a periodic function across an infinite domain, or literarily any non-periodic function within a finite domain, by approximating is as a bunch of sines and cosines, plus a constant term to shift the solution up or down:

$$f(t) \approx a_0 + \sum_{i=1}^{n} a_i \cos(nt) + \sum_{i=1}^{n} b_i \sin(nt)$$

Note that here we assume the function has a period of 2π. If it's different then we can change n to $\frac{n\pi}{L}$. Intuitively, if we take $n \to \infty$, then $f(t)$ is perfectly equal to the sines and cosines.
To find the constant term a_0, we can integrate the above equation from 0 to the end of its first period 2π:

$$\int_0^{2\pi} f(t)dt = \int_0^{2\pi} a_0\,dt + \sum_{i=1}^{\infty}\int_0^{2\pi} a_i\cos(nt)\,dt + \sum_{i=1}^{\infty}\int_0^{2\pi} b_i\sin(nt)\,dt = a_0 2\pi + 0 + 0$$

$$\therefore a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt$$

which is also the average. Now it's time to find a_i and b_i. To obtain a_i, we can do a trick by multiplying by $cos(nt)$ and then integrate:

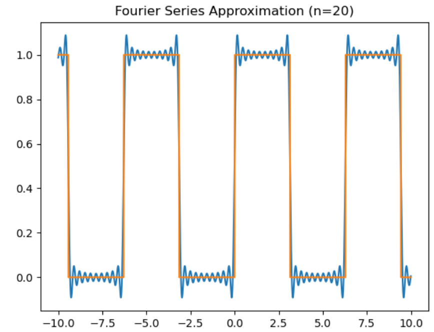
$$\int_0^{2\pi} f(t)\cos(nt)dt = \int_0^{2\pi} a_0\cos(nt)\,dt + \sum_{i=1}^{\infty}\int_0^{2\pi} a_i\cos(nt)\cos(nt)\,dt + \sum_{i=1}^{\infty}\int_0^{2\pi} a_i\sin(nt)\cos(nt)\,dt = a_0 2\pi = 0 + a_n\pi + 0$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)dt$$

Likewise, by multiplying by $sin(nt)$, we can obtain b_i:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

The Fourier series for a square wave looks like



Fourier series (formal, textbook)

Definition

Now, a more general method is studied. Fourier series of function f

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(\frac{m\pi x}{L}) + b_m \sin(\frac{m\pi x}{L}) \right)$$

Q: Which functions can be written as Fourier series?
A: Periodic functions (well, not necessary)! Like

$$f(T + x) = f(x)$$

Here, T is a fundamental period of f.
The linear combination of functions both with the period T also has the same period T.

$\cos(\frac{m\pi x}{L})$ and $\sin(\frac{\mathbf{m}\pi x}{L})$ have period $\frac{2L}{m}$.
$\cos(\frac{m\pi x}{L})$ and $\sin(\frac{\mathbf{n}\pi x}{L})$ have common period $2L$.

Orthogonality

Inner product of functions equal to 0

$$(u, v) = \int_{\alpha}^{\beta} u(x)v(x)dx = 0$$

Sets of functions are mutually orthogonal if each pari of functions in the set is orthogonal.
For example,

$$\{\cos(\frac{m\pi x}{L}), \cos(\frac{m\pi x}{L})\}_{m\in\mathbb{N}}$$

is mutually orthogonal.
Some useful properties

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{matrix} 0, \; m \neq n \\ L, \; m = n \end{matrix}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{matrix} 0, \; m \neq n \\ 0, \; m = n \end{matrix}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{matrix} 0, \; m \neq n \\ L, \; m = n \end{matrix}$$

Lecture 18

Derivation of Euler-Fourier

Suppose we have the Fourier series of f(x)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}) \right)$$

How to find a_n, b_n? We can use orthogonality:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, , n = 0, 1\ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,\ n = 1, 2\ldots$$

Fourier convergence theorem

Suppose f and f' are piecewise continuous $\in [-L, L]$ and is periodic with $T = 2L$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m\cos(\frac{m\pi x}{L}) + b_m\sin(\frac{m\pi x}{L})\right)$$

whose coefficients are given by the Euler-Fourier formula above.

The Fourier series converges to the midpoint $\frac{1}{2}(f(x^+) + f(x^-))$ at all points where continuous.

Gibbs phenomenon: Errors of Fourier series grow around discontinuities.

Extensions of Fourier series

The goal of this section is to demonstrate that one can apply Fourier to any functions like Taylor series so long as the domain $[-L, L]$ for fitting is finite.

Odd and even functions

The sum/difference and product/quotient of two even functions are even.
The sum/difference of two odd functions is odd; the product/quotient of two odd functions is even.
The sum/difference of an odd function and an even function is nether even nor odd; the product/quotient of an odd function and an even function is odd.

If f even

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

If f is odd

$$\int_{-L}^{L} f(x)dx = 0$$

Fourier cos series

If f is even with period $2L$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Fourier sin series

If f is odd with period $2L$,

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Periodic extensions

So basically, we can create
Suppose f is defined $\in [0, L]$, we can create still an even/odd periodic extension of f to use Fourier
Define function g with period $2L$ such that
Even extension:

$$g(x) = \begin{matrix} f(x),\, 0 \le x \le L \\ f(-x),\, -L \le x \le 0 \end{matrix}$$

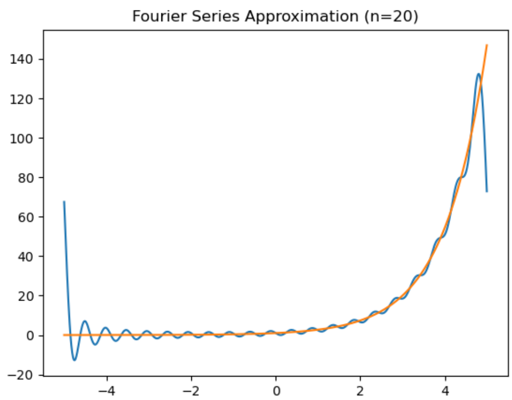
Odd extension:

$$h(x) = \begin{matrix} f(x),\, 0 \le x \le L \\ -f(-x),\, -L \le x \le 0 \end{matrix}$$

Third type
Define any function k with period 2L such that

$$h(x) = f(x),\, 0 \le x \le L$$

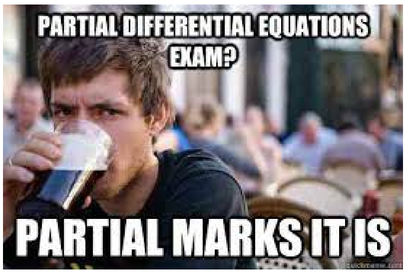
This allows us to approximate any integratable function within a finite domain. The Fourier series third type extension with 20 terms for $\exp(x)$ is shown below, which is a reasonable approximation.



Convergence of periodic extensions
Although all Fourier series for periodic extensions of f will converge, their rates will be different. In general, the smoother the function (more continuous derivative), the faster the series will converge.

Lecture 19

Partial differential equations (continued)



Some of the canonical examples:

Heat equations

$$\alpha^2 u_{xx} = u_t$$

The equations describes the space distribution and time evolution of temperature of a rod.
The boundary values $u(0,t) = u(L,t) = 0$ of the heat equation represent constant source temperatures.

With the above BV conditions, the solution takes the form

$$u(x,t) = c_2 e^{-\left(\frac{n^2\pi^2\alpha^2}{L^2}\right)t} \sin\left(\frac{n\pi\alpha}{L}\right)$$

However, this does not necessarily satisfy the BV condition $u(x,t) = f(x)$, as the solution does not necessarily resemble a sine function. To make it more generic, we can use Fourier series. Isn't it neat?
By superposition of solutions,

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\left(\frac{n^2\pi^2\alpha^2}{L^2}\right)t} \sin\left(\frac{n\pi\alpha}{L}\right)$$

where

$$c_n = \frac{2}{L}\int_0^L (f(x) - v(x))\sin\left(\frac{n\pi x}{L}\right)dx$$

To solve this system, we can write the solution as $u(x,t) = v(x) + w(x,t)$
$v(x)$ steady-state solution satisfying boundary conditions $u_t = 0$. It concerns the evolution in space.
$w(x,t)$ transient solution satisfying heat equation and homo $u(0,t) = u(L,t) = 0$. It concerns the evolution in time and space.

Wave equation

$$\alpha^2 u_{xx} = u_{tt}$$

Laplace equation

$$\begin{aligned} u_{xx} + u_{yy} = 0 \\ \Delta u = 0 \end{aligned}$$

Lecture 20

Heat Equations (Continued)

Boundary Conditions

- Dirichlet: $u(0,t) = u(L,t) = c$, value known.
- Neumann: $u_x(0,t) = u_x(L,t) = c$, its special case: no flux -- insulated endpoints.
- Robin: linear combinations of u and u_x.

Heat equation with Nonhomogensous Dirichlet BV

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Boundary conditions:

$$u(0,t) = u(L,t) = 0$$

Initial conditions:

$$u(x,0) = f(x)$$

To solve, we assume the solution takes the form (ansatz)

$$u(x,t) = X(x)T(t)$$

$$\alpha^2 X''(t)T(t) = X(x)T'(t)$$

$$\frac{X''(t)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\lambda$$

$$\begin{aligned} X'' + \lambda X = 0,\, 0 < x < L \\ T' + \alpha^2 \lambda T = 0,\, t > 0 \end{aligned}$$

From boundary conditions,

$$X'(0) = X'(L) = 0$$

The nontrivial solution exists only if $\lambda \in \mathbb{R}$.
To find the sign of lambda, we can use the boundary conditions and find the one that gives a non-trivial solution.
If $\lambda < 0$

$$X = K_1 \sinh(\sqrt{\lambda}x) + K_2 \cosh(\sqrt{\lambda}x)$$

Negative λ results in a trivial solution.

If $\lambda = 0$

$$X = K_1 x + K_2$$

If $\lambda > 0$

$$X = K_1 \sin(\sqrt{\lambda}x) + K_2 \cos(\sqrt{\lambda}x)$$

$$c_1 \sin(\sqrt{\lambda}L) = 0 \implies \lambda_n = -\frac{n^2\pi^2}{L^2},\, \sqrt{\lambda_n} = \frac{n\pi}{L}$$

Now T is solved:

$$T_n(t) = D_n e^{\lambda_n} kt$$

Combining the solution of X and T,

$$u_n(x,t) = B_n e^{\lambda_n kt} \sin(\sqrt{\lambda_n} x), \quad \lambda_n = -\frac{n^2\pi^2}{L^2}, \sqrt{\lambda_n}$$

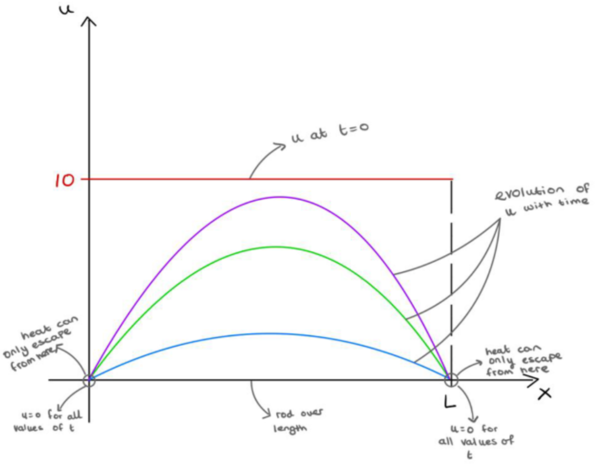
By linearity,

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{\lambda_n kt} \sin(\sqrt{\lambda_n} x)$$

The initial condition can be used to simplify the expression, then the Fourier sine can be used:

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x$$

If u(x,0)=10, then the evolution of solution wrt time is



Heat equation with Nonhomogensous Dirichlet BV

We now solve the time-varying heat equation

$$\alpha^2 u_{xx} = u_t$$

Assume the solution takes the form

$$u(x,t) = v(x) + w(x,t)$$

Again, the heat equation:

$$\alpha^2 u_{xx} = u_t$$

Yet the ends of the rod are held at different temperatures

$$u(0,t) = T_1,\, u(L,t) = T_2,\, t > 0$$

Again, the solution can be expressed in the form

$$u(x,t) = v(x) + w(x,t)$$

We observe for steady-state solution $v(x)$ for $t \to \infty$,

$$v''(x) = 0$$

The steady-state temperature distribution across the rod should be linear:

$$v(x) = (T_2 - T_1)\frac{x}{L} + T_1$$

From the PDE we have

$$\alpha^2 (v+w)_{xx} = (v+w)_t \implies \alpha^2 w_{xx} = w_t$$

where $w(x,t)$ is the transient solution. It follows that

$$\alpha^2 w_{xx} = w_t$$

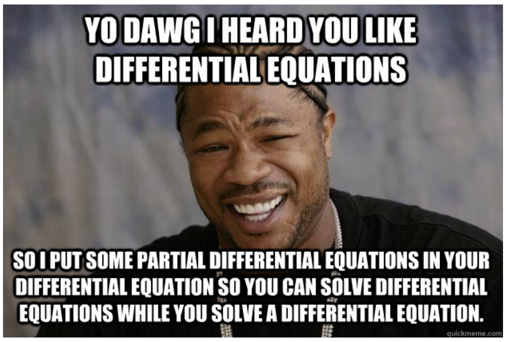
BV & IV:

$$
\begin{aligned}
w(0,t) &= u(0,t) - v(0) = T_1 - T_1 = 0, \\
w(L,t) &= u(L,t) - v(L) = T_2 - T_2 = 0 \\
w(x,0) &= u(x,0) - v(x) = f(x) - v(x)
\end{aligned}
$$

The above PDE and the boundary values and its boundary/initial conditions gives the transient solution. Adding transient sol. and steady-state sol. together:

$$u(x,t) = \left(\frac{T_2 - T_1}{L}\right)x + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right),$$

$$c_n = \frac{2}{L}\int_0^L \left(f(x) - \left(\frac{T_2 - T_1}{L}\right)x - T_1\right)\sin\left(\frac{n\pi x}{L}\right)dx$$



Heat equation under Neumann BV (Bar with insulated ends)

Boundary conditions

$$u_x(0,t) = 0,\ u_x(L,t) = 0,\ t > 0$$

Initial condition

$$u(x,0) = f(x)$$

The procedure for solving is almost identical to the one demonstrated in the homo. Dirichlet.
We can obtain the fundamental solutions:

$$
\begin{aligned}
&u_0(x,t) = 1, \\
&u_n(x,t) = e^{-n^2\alpha^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots
\end{aligned}
$$

The solution takes the form

$$u(x,t) = \frac{c_0}{2}u_0(x,t) + \sum_{n=1}^{\infty} c_n u_n(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\alpha^2 t/L^2}\cos\left(\frac{n\pi x}{L}\right)$$

c_n can be determined form the following expression, which is a Fourier cosine series with $T = 2L$.

$$u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$$

$c_0/2$ can be understood as the steady-state temperature distribution which is constant and the average of the original temperature since there is no heat exchange.

Lecture 21

Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

- Steady-state for heat conduction
- Potential fields/equations
- Elasticity -- warping function

Dirichlet problem on a rectangle

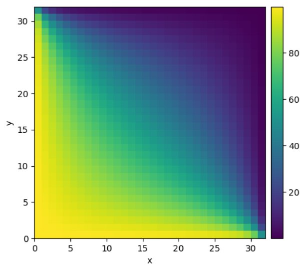
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$u(x,0) = 0,\ u(x,b) = 0$$
$$u(0,y) = 0,\ u(a,y) = f(y)$$

where $0 < x < a$ and $0 < y < b$.
Solutions:

$$Y_n = \sin\left(\frac{n\pi y}{b}\right), \quad X_n = K_1 \cosh\left(\frac{n\pi x}{b}\right) + K_2 \sinh\left(\frac{n\pi x}{b}\right),\ K_1 = 0$$

By linearity,

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right), \quad c_n \sinh\left(\frac{n\pi x}{b}\right) = \frac{2}{b}\int_0^b f(y) \sin\left(\frac{n\pi y}{b}\right) dy$$



2D heat diffusion result with BV of 100 at lower x and lower y borders.

Wave equation (elastic string)

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad a^2 = \frac{T}{\rho}$$

$u(x,t)$ can be viewed as the vertical displacement of an elastic string, T is the tension, ρ is the mass per unit length.
The wave equation with fixed end positions of the string gives the following boundary conditions:

$$u(0,t) = u(L,t) = 0$$

we also need the initial values in t:

$$u(x,0) = f(x),\ u_t(x,0) = g(x)$$

Solution steps:
Assume solution form

$$u(x,t) = X(x)T(t)$$

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

$$T''(t) + \frac{n^2\pi^2 a^2}{L^2} T(t) = 0$$

Solving

$$T_n(t) = K_1 \cos\left(\frac{n\pi at}{L}\right) + K_2 \cos\left(\frac{n\pi at}{L}\right),\ K_2 = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi at}{L}\right)$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right), \quad c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$\frac{n\pi a}{L}$ is the natural frequency of the string, at which it vibrates freely.
$\frac{2L}{n}$ is the wavelength (from $\omega = 2\pi f = \frac{n\pi}{L}$)

An ODE to DE

In the land of calculus, quite divine,
Lies a subject that makes equations intertwine.
Differential Equations, we call the game,
For systems complex, it stakes its claim.

ODEs first, with variables one,
Simple yet complex, never outdone.
Separate, integrate, solutions we find,
In time and space, to reality they're aligned.

PDEs next, dimensions galore,
Heat, waves, and more to explore.
Boundary conditions set the stage,
For Fourier and Laplace to engage.

Euler's method, a numerical feat,
Making approximations that are quite neat.
Though not exact, it gives insight,
Into systems complex, it sheds some light.

So here's to Diff Eq, may we all find,
The elegance and power it has designed.
In engineering and beyond, it plays a role,
In understanding systems, it's the ultimate goal.