Differential equations

and how to solve them

by Marco DS

October 28, 2015

Introduction:

This document provides information on several methods used to solve differential equation. Note that not all topics of the TU-Delft Bsc differential course are covered here, however the most basic ones that will provide you with the basic necessary understanding on the nature of differential equations.

The aim of this document is not to provide you simply a set of rules to follow for the exam, (for that you have already the other document in Aerostudents), but actually help you to understand the given material. If you put a bit of effort, you will be surprised how simple actually all the material is, and yet how enjoyable it all becomes. This will help you to recall functions even after the exam and broaden your horizon as an engineer.

Another useful source I would recommend for people studying differential equations is the online website: Paul‘s online notes. In fact, most of what I learned was thanks to the author of this website.

Contents

Chapter 1

Solving diff eq the standard way:

This chapter you should already be familiarized with from previous math lectures. I give in here a short but detailed review.

1.1 First order diff equations:

1.1.1 Separable differential equations:

These are probably the easiest differential equations to solve. The trick here is that one is able to seperate the variables to each side. Lets use an example to clarify.

Example 1: We have the following equation:

$$\frac{dy}{dx} - 5y = 0 \tag{1.1}$$

We can therefore easily rearrange the equation to:

$$\frac{dy}{dx} = 5y \tag{1.2}$$

Our aim now is to get all the y terms to one side, all the x terms to the other. So:

$$\frac{1}{5y}dy = 1dx \tag{1.3}$$

This is now an equation we can integrate in order to eliminate the dy and dx and finding out the unknown y:

$$\int \frac{1}{5y}dy = \int dx \tag{1.4}$$

This solves to:

$$\frac{1}{5}ln(y) = x \tag{1.5}$$

$$ln(y) = 5x \tag{1.6}$$

$$y = e^{5x} \tag{1.7}$$

1.1.2 Non separable differential equations:

The previous method only works for the simplest cases where we can actually seperate the variables. However there are several non seperable first order differential equations that can be solved with the following method:

We start with an equation of the following form:

$$\frac{dy}{dt} + a(t)y = g(t) \tag{1.8}$$

Our aim here is to get rid of that annoying dt. How do we do this? We first multiply it with an unknown function. So:

$$\mu\frac{dy}{dt} + \mu a(t)y = \mu g(t) \tag{1.9}$$

What is the use of this? Well, if we assume that $\frac{d}{dt}(\mu y) = \mu\frac{dy}{dt} + \frac{d\mu}{dt}y$ due to the product rule, we actually know that:

$$\mu a(t) = \frac{d\mu}{dt} \tag{1.10}$$

And this we can solve very easily, since it is a simple seperable equation! Yey!

$$\frac{1}{\mu}d\mu = a(t)dt \tag{1.11}$$

$$\int \frac{1}{\mu}d\mu = \int a(t)dt \tag{1.12}$$

$$ln(\mu) = \int a(t)dt \tag{1.13}$$

$$\mu = e^{\int a(t)dt} \tag{1.14}$$

Now we can go back to our original equation and rewrite it as:

$$\frac{d}{dt}(\mu y) = \mu g(t) \tag{1.15}$$

Integrating we get:

$$\mu y = \int \mu g(t)dt \tag{1.16}$$

Or else:

$$y = \frac{\int \mu g(t)dt}{\mu} \tag{1.17}$$

Tadaa! And there we go, we have the solution. Note that this method can be used with the cases of seperable equations.

1.2 Second order differential equations:

1.2.1 Homogenous equations:

We start with an equation of the following form:

$$ay“ + by‘ + cy = 0 \tag{1.18}$$

In order to solve this we are going to assume that y is going to be a solution such that $y = e^{rt}$. Why this solution? The genious in this idea is that by differentating this solution, the base never changes:

$$y = e^{rt} \tag{1.19}$$

$$y‘ = re^{rt} \tag{1.20}$$

$$y“ = r^2e^{rt} \tag{1.21}$$

We can use insert this phenomenon into our equation. [1] So:

$$a(r^2e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0 \tag{1.22}$$

We can cancel out e^{rt} and end up with the simple polynomial:

$$ar^2 + br + c = 0 \tag{1.23}$$

We can now solve for r by using simply the polynomial rule:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = r_{1,2} \tag{1.24}$$

There are now three possible cases for the results of r:

- **Case 1:** Both r are real and distinct. Therefore the solution is:

$$y = c_1e^{r_1t} + c_2e^{r_2t} \tag{1.25}$$

- **Case 2:** Both r are real, but same (so only one r). Therefore the solution is:

$$y = c_1e^{r_1t} + c_2te^{r_2t} \tag{1.26}$$

- **Case 3:** Both r are complex in the form of $\lambda \pm i\mu$. Therefore the solution is:

$$y = c_1e^{\lambda t}cos(\mu t) + c_2e^{\lambda t}sin(\mu t) \tag{1.27}$$

[1] This type of phenomenon is derived into several other techniques that will be shown later, but they all come with the same idea.

1.2.2 Non-homogenous equations:

In the non-homogenous case, like the equations below:

$$ay`` + by` + cy = g(t) \tag{1.28}$$

We obtain a solution of the form $y = y_c + y_p$, where y_c is the complementary solution or the solution from the homogenous case, which we already know how to solve, with the added term y_p.

From previous years the method of *Undetermined coefficients* was used. This method however limits itself to simple cases such as a polynomial and is even then rather lengthy. Since it has already been covered before, and in my opinion its far more tidious than the alternative, I will jump ahead and explain the second method, the *Variation of parameters.*

Variation of parameters:

For an equation of the form:

$$ay`` + by` + cy = g(t) \tag{1.29}$$

We already know that the solution to y_c is of the form:

$$y_c = c_1y_1 + c_2y_2 \tag{1.30}$$

Therefore, lets assume that the particular solution is of the similar form:

$$y_p = u_1y_1 + u_2y_2 \tag{1.31}$$

Where we have the two unknowns, u_1 and u_2. Thus we will need two equations in order to solve them. One simple thing to do is differentiate this equation, which we can then plug into our differential equation:

$$y_p` = u_1`y_1 + u_1y_1` + u_2`y_2 + u_2y_2` \tag{1.32}$$

To simplify our lives we are going to assume that the derivatives of u_1 and u_2 are going to be such that:

$$u_1`y_1 + u_2`y_2 = 0 \tag{1.33}$$

Therefore, we write again:

$$y_p` = u_1y_1` + u_2y_2` \tag{1.34}$$

and differentiate again:

$$y_p`` = u_1y_1`` + u_1`y_1` + u_2y_2`` + u_2`y_2` \tag{1.35}$$

Inserting these into our differential equation we obtain:

$$a(u_1y_1``+u_1`y_1`+u_2y_2``+u_2`y_2`)+b(u_1y_1`+u_2y_2`)+c(u_1y_1+u_2y_2) = g(t) \tag{1.36}$$

Dont get too scared of this long equation. You will see in a moment how simple it all actually is. We can rearrange our equation the following (nicenice) way:

$$a(u_1`y_1` + u_2`y_2`) + u_1(ay`` + by` + cy) + u_2(ay`` + by` + cy) = g(t) \qquad (1.37)$$

Observe! We already know that $ay`` + by` + cy = 0$! Therefore our equation really is:

$$a(u_1`y_1` + u_2`y_2`) = g(t) \qquad (1.38)$$

or:

$$(u_1`y_1` + u_2`y_2`) = \frac{g(t)}{a} \qquad (1.39)$$

In the case where $a = 1$, we can write this equation to:

$$u_1`y_1` + u_2`y_2` = g(t) \qquad (1.40)$$

Therefore we end up with the two equations:

- $$u_1`y_1 + u_2`y_2 = 0 \qquad (1.41)$$
- $$u_1`y_1` + u_2`y_2` = g(t) \qquad (1.42)$$

Just solve them! I usually like to use Linear Algebra for this, so it would all look like:

$$\begin{bmatrix} y_1 & y_2 \\ y_1` & y_2` \end{bmatrix} \begin{bmatrix} u_1` \\ u_2` \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix} \qquad (1.43)$$

You may also notice how the first matrix is actually the identical to a *Wronskian* matrix. What you see in the books is usually the rearrangement of the same equations above, but with the Wronskian included... Really though, why make your life any harder?

Now that you know $u_1`$ and $u_2`$, we integrate them. Thus we can state that:

$$y_p = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \int \begin{bmatrix} u_1` \\ u_2` \end{bmatrix} dt \qquad (1.44)$$

With today's calculators (such as the TI-nspire) this method becomes even more of a breeze. All you need to know to solve these problems is equation 1.43 and 1.44.

Chapter 2

Laplace transform:

Laplace transform is a wonderful way to solve differential equations. Its magic really lies that it "transforms" the nasty equation with differentials to a simple algebraic equation in the laplace space.

This is extremely useful! (and you will be learning more about this in Systems and Control.) Imagine you have to superimpose and add several differential equations to create a "super-system". With laplace this all becomes an algebraic equation and you can go ahead and work in the laplace space up until the very end without ever the need to solve the equations except at the end.

For now though we shall only focus on how to solve differential equations by converting them into the Laplace space and then converting them back to "our" space. (Totally sounds like Star Trek.)Note that there are many more transform out there such as the Fourier transform (not Fourier series!) that are very commonly used. Lets start!

2.0.1 Laplace transform definition:

The definition of the laplace transform is as following:

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt \tag{2.1}$$

where the function in terms of t is converted to a function in terms of s.

Example 1: Lets convert the function $f(t) = 1$ to the laplace space:

$$\int_0^\infty e^{-st} 1 dt \tag{2.2}$$

$$\int_0^\infty e^{-st} dt \tag{2.3}$$

$$[\frac{-1}{s}e^{-st}]_0^\infty \tag{2.4}$$

$$\frac{-1}{s}*0-\frac{-1}{s}*1=\frac{1}{s} \tag{2.5}$$

In a similar manner the laplace table (which you are going to have in the exam) is created. Additionally the derivatives of a function y can be expressed by using integration by parts. I wont show the integration here but just the result of them:

$$y => \mathcal{L}[y] \tag{2.6}$$

$$y` => s\mathcal{L}[y]-y(0) \tag{2.7}$$

$$y`` => s^2\mathcal{L}[y]-sy(0)-y`(0) \tag{2.8}$$

As you can see a nice trend appears, which we can make use of for even bigger differential equations. Notice also how the laplace transform directly uses the initial conditions! There is no need to plug them in afterwards unlike the previous method learned so far.

2.0.2 Laplace transform applied:

One very important note: You will need to know how to solve partial fractions in order to complete most exercises given to you that involve laplace.
Lets have an example to see how one would solve a differential equation with laplace.

Example 2: Solve the following equation with laplace:

$$y``+3y`+2y=0 \tag{2.9}$$

with I.C. $y(0)=1$ and $y`(0)=2$.

Lets apply the laplace transform:

$$(s^2\mathcal{L}[y]-sy(0)-y`(0))+3(s\mathcal{L}[y]-y(0))+2\mathcal{L}[y]=0 \tag{2.10}$$

Inserting the I.C. we have:

$$(s^2\mathcal{L}[y]-s-2)+3(s\mathcal{L}[y]-1)+2\mathcal{L}[y]=0 \tag{2.11}$$

$$s^2\mathcal{L}[y]-s-2+3s\mathcal{L}[y]-3+2\mathcal{L}[y]=0 \tag{2.12}$$

$$\mathcal{L}[y](s^2+3s+2)=s+5 \tag{2.13}$$

$$\mathcal{L}[y]=\frac{s+5}{s^2+3s+2} \tag{2.14}$$

Tadaa! Wasnt that easy?! For this you already get half of the points! And in practical terms, you could now already go ahead and design a whole system for an airplane with this transform! Lets not get out of hand though, we need now to convert this equation back.

Unfortunately this part may be a bit trickier, but we have a good friend with us, the laplace table. What we essentially try to do is to fit our laplace equation into an equation that resembles one of the equations in the laplace table. If we mananage that, we can convert back. So lets do that:

$$\frac{s+5}{s^2+3s+2} = \frac{s+5}{(s+2)(s+1)} = \frac{A}{(s+2)} + \frac{B}{s+1} \tag{2.15}$$

Therefore:

$$s+5 = \frac{A(s+2)(s+1)}{(s+2)} + \frac{B(s+2)(s+1)}{s+1} \tag{2.16}$$

$$s+5 = A(s+1) + B(s+2) \tag{2.17}$$

One way of solving this equation is to use the fact that at s=-1, we cancel out all A-terms, thus being able to find B (and same thing vice versa). There are some other ways to solve this last bit, but I let that for you to review. Thus we can write our laplace expression as:

$$\mathcal{L}[y] = \frac{-3}{s+2} + \frac{4}{s+1} \tag{2.18}$$

Using the laplace table, we can convert easily to:

$$y[t] = -3e^{-2t} + 4e^{-3t} \tag{2.19}$$

And voilá! We are done!

Remember to look also into special functions such as step functions or impulse functions. The real power of laplace comes in here, where one can actually go ahead and solve all these functions! Just remember, the laplace table is your friend.

Chapter 3

Systems of diff eq:

Similar to laplace, you will be seeing systems of differential equations many many many more times. Reason for this is that you can describe a whole system by a set of matrices. So in Systems and Control, you will be seeing this again.

I therefore highly recommend you that you learn how to convert a differential equation into a system of differential equations. However, during this exam you wont need to know how to do that as you will be given the system with the request to solve it.

3.1 Solving sys:

3.1.1 homogenous sys:

The system of diff.eg. is going to consist of matrices and look as follows:

$$x` = Ax \tag{3.1}$$

Similarly to chapter one, we are going to assume a solution $y = \vec{v}e^{\lambda t}$. (Where $\vec{v}$ is going to be the eigenvector and λ the eigenvalue of the matrix A.)
Plugging that into our differential equation, we get:

$$\vec{v}\lambda e^{\lambda t} = A\vec{v}e^{\lambda t} \tag{3.2}$$

Cancelling terms we get to the well known expression:

$$(A - \lambda)\vec{v} = 0 \tag{3.3}$$

Since we are dealing with matrices, we cant just differentiate $A - \lambda$. However if we rewrite our expression as the following:

$$(A - \lambda I)\vec{v} = 0 \tag{3.4}$$

Since we assume that the vector $\vec{v}$ is non-zero, the matrix expressed inside the brackets will become singular. This can be seen here:

$$A = \lambda I \tag{3.5}$$

There is no way one can take the inverse of one matrix, thus defining singularity. From Linear Algebra we know that this phenomenon occurs only when:

$$det(A - \lambda I) = 0 \tag{3.6}$$

Thus we can use this property to find the eigenvalues λ and solve our system. The solution for an n^{th} order differential equation would be of the form:

$$x = c_1\vec{v_1}e^{\lambda_1 t} + c_2\vec{v_1}e^{\lambda_2 t}...c_n\vec{v_n}e^{\lambda_n t} \tag{3.7}$$

Just like in chapter one, there are three cases to consider and depending on the values of lambda the solution will be a bit different.

- **Case 1:** Both λ are real and distinct. In this case the solution is just of the same form as above:

$$x = c_1\vec{v_1}e^{\lambda_1 t} + c_2\vec{v_1}e^{\lambda_2 t} \tag{3.8}$$

- **Case 2:** There is only one real λ. This of course is a bit of a problem for us, because we require for a 2nd order differential equation at least a 2 term solution. So what we do is to use the basic equation from before to create a second term:

$$(A - \lambda I)\vec{v} = 0 \tag{3.9}$$

$$(A - \lambda I)\vec{\eta} = \vec{v} \tag{3.10}$$

- **Case 3:** The λ's are complex. Most of the people will usually try to remember that the equation is formed in terms of cosine and sines, however all that really happens is the same as in the first case, but we continue by using *de Moivres* law:

$$e^{i\alpha t} = \cos(\alpha t) + i\sin(\alpha t) \tag{3.11}$$

If you have done an exercise involving complex eigenvalues, you will have noticed that the imaginary part is turned real. The reason why the imaginary part disappears in the solution is because of the following simple reason: Lets assume that the solution for the equation $x' = Ax$ is:

$$x = u + iv \tag{3.12}$$

since we know that $x' - Ax = 0$, we can insert our solution into our equation:

$$u' + iv' - A(u + iv) = 0 \tag{3.13}$$

Rewrite this:

$$u` - Au + i(v` - Av) = 0 \tag{3.14}$$

In order for this solution to be indeed zero, both the u terms and the v terms must be zero and thus are a solution:

$$u` - Au = 0 \tag{3.15}$$

$$v` - Av = 0 \tag{3.16}$$

Therefore in the solution we can convert the imaginary part to the real part, since this equivalent with the solution.

3.1.2 non-homogenous sys:

Lets move on to the non homogenous systems. These would be of the form:

$$x` = Ax + g(t) \tag{3.17}$$

Just like in chapter one, we already know how to solve the homogenous case. This would solve for us here as the complementary solution. To this one, we still require the particular solution.

$$x = x_c + x_p \tag{3.18}$$

In the following I will be showing the method of *Variation of parameters* as I believe that it is far easier and shorter than the other methods shown in this course, such as the method *Undetermined coefficients*.

We make the assumption that the particular solution is formed from the complementary solution with an according transformation. That would like this:

$$x_p = Xv \tag{3.19}$$

(where X is a matrix formed from the complementary solutions and v is a transformation vector.) If we plug that into our differential equation, we get:

$$X`v + Xv` = AXv + g(t) \tag{3.20}$$

where we can assume that $AX = X`$. Thus cancelling terms we end up with:

$$X`v + Xv` = X`v + g(t) \tag{3.21}$$

$$Xv` = g(t) \tag{3.22}$$

Our job now is to find v. So lets get to it; its quite straight forward:

$$v` = X^{-1}g(t) \tag{3.23}$$

$$v = \int X^{-1}g(t)dt \tag{3.24}$$

Therefore we can rewrite equation 3.19 as:

$$X \int X^{-1} g(t) dt = x_p \tag{3.25}$$

And that is it! We have found an expression for x_p and thus are able to complete our solution! Easy, wasnt it?

3.2 Analyzing sys:

Another great thing about systems of differential equations is that with their solution, we can make a phase plot, which will describe how our system behaves. How cool is that? You can actually observe whether your system is stable or not, and how it interacts.

Unfortunately, for this course most people just memorize the table of stabilities presented by the teacher. Not only do these people forget that table once the exam is over, but they have little idea of what is actually going on!

I hope in the following to show you how easy it actually is to determine how the differential equation behaves and that you do not need to learn a whole freaking table with weird numbers and symbols for your exam. Lets take a look at how the solution of a system is represented onto the phase plot.

As you are aware, a solution consists of eigenvectors and eigenvalues. Kinda like this:

$$x = c_1 \vec{v_1} e^{\lambda_1 t} + c_2 \vec{v_1} e^{\lambda_2 t} \tag{3.26}$$

Now, lets focus only on the first term of the solution. There are several cases that could appear:

- **Case 1:** The λ is real and positive. Just look at the equation! What happens as t grows bigger? Exactly! They scale the eigenvector to bigger values, so away from the origin!
- **Case 2:** The λ is negative. Again, looking at the equation, you can now recognize how the eigenvector is scaled smaller as t grows, so it would move toward the origin.
- **Case 3:** The λ is complex. In this case we have a sin and cos term in our equation. That means that as t grows bigger, each eigenvector is scaled *periodically!* This is what causes complex solutions to spiral around the origin, because as one periodic shift grows smaller for one eigenvalue, the other one increases.

Now a solution can be a combination of any of these three cases. The table essentially shows you all the names for each possible combination. What is more important though, now you are able to draw a phase plot without needing a table. Sure, you might need the table to remember how the case was called, but that is all.

Chapter 4

Power series:

This might be one of the most intimidating ways to solve a differential equation, and yet it is incredibly simple. It might be useful for you if you take a short review on how to manipulate series, but the example below should clarify those issues.

Example 1: Lets solve the following equation for $x_0 = 0$:

$$y'' + xy = 0 \tag{4.1}$$

In order to solve this equation we assume y is a solution to a power series. Why? For same reason as in chapter one, and that is that the derivates never change base. Below the series and its derivatives:

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{4.2}$$

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \tag{4.3}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} \tag{4.4}$$

Notice where the series starts! Also, since we are taking $x_0 = 0$, we can take out the x_0 term. So what are we waiting for? Lets just plug in these equations into our original differential equation. That would give the following result:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0 \tag{4.5}$$

First thing we wanna do is to insert the x on the second term into the sum series.

$$x \sum_{n=0}^{\infty} a_n x^n \tag{4.6}$$

$$\sum_{n=0}^{\infty} a_n x^{n+1} \tag{4.7}$$

So all together:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \tag{4.8}$$

Now though, what we have to get to is to have for both sums the same power term, so that we can join them together later on. Lets make the common term x^n. Then for the first term we have to convert:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \tag{4.9}$$

we insert the dummy variable: $i = n - 2$, or $n = i + 2$:

$$\sum_{i=0}^{\infty} (i+2)(i+1)a_{i+2} x^{i} \tag{4.10}$$

and then rewrite i to n:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} \tag{4.11}$$

Now we have managed to get the power term into x^n for the first term. Lets do the same thing for the second term:

$$\sum_{n=0}^{\infty} a_n x^{n+1} \tag{4.12}$$

we insert the dummy variable: $i = n + 1$, or $n = i - 1$:

$$\sum_{i=1}^{\infty} a_{i-1} x^{i} \tag{4.13}$$

and convert again:

$$\sum_{n=1}^{\infty} a_{n-1} x^{n} \tag{4.14}$$

Therefore our equation becomes:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^{n} + \sum_{n=1}^{\infty} a_{n-1} x^{n} = 0 \tag{4.15}$$

Good job! Now we have both sums with the same power term! The last thing we have to do before being able to join both sums is to make sure both sums start at the same integer. We can do this easily by taking out the first number from the first term series:

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n = 0 \tag{4.16}$$

$$2a_2 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + a_{n-1}]x^n = 0 \tag{4.17}$$

Since this equation is true for all n-integers, we know that for $n = 0$:

$$2a_2 = 0 \tag{4.18}$$

so we know that $a_2 = 0$, and for the rest of n, we need:

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0 \tag{4.19}$$

Which we can rewrite as:

$$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \tag{4.20}$$

Plugging in then a few more integers, we may be able to observe a trend:

- **n = 1:**
$$a_3 = \frac{-a_0}{(3)(2)} \tag{4.21}$$

- **n = 2:**
$$a_4 = \frac{-a_1}{(4)(3)} \tag{4.22}$$

- **n = 3:**
$$a_5 = \frac{-a_2}{(5)(4)} \tag{4.23}$$

  but since $a_2 = 0$, $a_5 = 0$.

- **n = 4:**
$$a_6 = \frac{-a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)} \tag{4.24}$$

- **n = 5:**
$$a_7 = \frac{-a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)} \tag{4.25}$$

As you can see, a pattern has arised. Therefore our solution is a set of the sum of all these terms. In the exam and in practice its fine if you write down the first terms, however if you can, try to describe the solution as a pattern. In this case the patterns would be:

$$a_{3n} = 0 \tag{4.26}$$

$$a_{3n+1} = \frac{a_1}{(3n)(3n+1)} \tag{4.27}$$

$$a_{3n+2} = \frac{a_0}{(3n-1)(3n)} \tag{4.28}$$

Now that all this has been done, we remember how the solution of this differential equation was:

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{4.29}$$

which expanded for n integers is:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 ... a_n x^n \tag{4.30}$$

Inserting the patterns that we have just found:

$$y = a_{3(n=0)} + a_{3(n=0)+1} x + a_{3(n=0)+2} x^2 + a_{3(n=1)} x^3 ... \tag{4.31}$$

$$y = a_1 [1 + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3n)(3n+1)}] + a_0 [1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(3n-1)(3n)}] \tag{4.32}$$

And there you go! It might take some time and require good attention, but if done carefully, it should not be "out of this world!" neither.

Chapter 5

Fourier Series:

5.0.1 Exploring Fourier series:

This is quite a nice chapter. Although this method does not provide you with a way to solve a differential equation, the results of a differential equation may appear as an expression in terms of cos and/or sin, so a Fourier series.

In other words, Fourier series occur many times in the world of engineering, and the beauty of them is that it can describe what otherwise would be a set of step functions to simply a (periodic) function.

What is a Fourier series? Really, its only the superposition of periodic functions. Most often we make use of cos and sin functions, (which are really the same function shifted by 90 degrees), but you could apply other periodic functions if you like to make your life harder. Point is, by taking several of the cos and/or sin functions we are able to create periodic other more complicated functions that usually can only be described for example by step functions.

Keeping that in mind, the actual Fourier series takes an obvious form of:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L}) \tag{5.1}$$

Now, before we jump right into finding the constants a_0, a_n and b_n with the integral forms presented to you in class, lets just take a look at the equation.

The first term, $a_0/2$, clearly is the **average of the function**. Just like in a linear function $y = mx + c$, the c is responsible for shifting the given equation up or down. Same thing with the $a_0/2$: It shifts the equation up or down. Since we are dealing with periodic functions, it states where the function average lies. For a normal cosine function $f(x) = cos(x)$, the average would be zero.
In most cases therefore, you dont even need to compute the integral to find a_0. Its simply the double of the average of the given curve.

Next, we have the constants a_n and b_n, which affect the amplitude of each cosine or sine term. However, in most cases we can determine whether a function is **even** or **odd**. A simple example for each case: cosine functions are even, sine functions are odd. The definition for even or odd functions can also be expressed as:

$$even : f(x) = f(-x) \tag{5.2}$$

$$odd : f(x) = -f(-x) \tag{5.3}$$

What is the use of this? In some cases you will be provided with a function that is either odd or even. Thus you can describe it either fully only with cosine terms or sine terms. You therefore have even less work to do and only need to determine the a_n or b_n.

5.0.2 Finding the series constants:

So now that we have understood what the fourier series are all about, we can go ahead and try to find the needed constants.

Once again, lets start with the general expression and try to find a_0, a_n and b_n:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n sin(\frac{n\pi x}{L}) \tag{5.4}$$

- **Finding a_0:** So we need to find a_0. How do we do this? Well, notice that if we integrate the equation from the start of a period to the end of a period, so from -L to L, the cosine and sine terms all become zero. We end up thus with the following scenario:

  $$\int_{-L}^{L} f(x)dx = \frac{a_0}{2} \int_{-L}^{L} dx + 0 + 0 \tag{5.5}$$

  $$\int_{-L}^{L} f(x)dx = \frac{a_0}{2}(2L) \tag{5.6}$$

  Rearranging to a_0:

  $$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)dx \tag{5.7}$$

- **Finding a_n:** Again we will try to "filter" the term we are interested in. If we multiply the whole equation with a cosine and then integrate it, we observe how the first term and the last term dissapears. In case it is unclear why last term dissapears after integrating it, just graph the function and you will see that the integral across a whole period is zero. Thus we end up with:

$$\int_{-L}^{L} f(x)cos(x)dx = \int_{-L}^{L} a_n cos(\frac{n\pi x}{L})cos(\frac{m\pi x}{L})dx \tag{5.8}$$

The integral to a double cosine (or sine) function has the following possible solutions:

$$(m \neq n) = 0 \tag{5.9}$$

$$(m = n = 0) = 2L \tag{5.10}$$

$$(m = n \neq 0) = L \tag{5.11}$$

Since our series (and that is the actual reason) starts at n = 1, the integral solution for the double cosine is L. Therefore we can write:

$$\int_{-L}^{L} f(x)cos(\frac{m\pi x}{L})dx = a_n L \tag{5.12}$$

Rearrange it to a_n:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{m\pi x}{L})dx \tag{5.13}$$

- **Finding b_n:** This is the same thing we did with a_n, just using sines instead. The solution would finally be:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{m\pi x}{L})dx \tag{5.14}$$

And there you go, that is all the magic to Fourier series. Just like with the power series, mind your work: be careful and organized, and then everything should come out smoothly.

Also, if you are up to some fun, I would recommend to you to actually graph a Fourier series. You can do that easily with a graphical calculator by inserting the first few terms of the Fourier series. It really helps to visualize what all this equations are actually doing.

Chapter 6

Partial differential equations:

Partial differential equations are a further step into making differential equations even more complicated. Instead of having only one dimension to worry about (such as f(t) is a function of *time*), we include a second one (such as g(x,t), a function of both *position and time*).

Another important difference of these equations is that to describe them to a certain situation, we make use of boundary conditions. (Before the differential equations only required initial conditions, here we require some more constraints.) Depending on how the boundary conditions are set, we may call a differential equation homogenous (if they are equal to zero) or non-homogenous (if they are non-zero).

6.0.1 Separation of variables:

In order to explain this method, lets just directly go over an example:

Example 1: Solve the heat equation:

$$u_t = ku_{xx} \tag{6.1}$$

with initial and boundary conditions: $u(x,0) = f(x)$; $u(0,t) = 0$ and $u(L,t) = 0$.

As you can see this equation is homogenous. Thus we may proceed with using the seperation of variables. We will assume that:

$$u = X(x)T(t) \tag{6.2}$$

Why we choose this is mainly because it works. We can check that by computing whether the determinant of the Wronskian for the solution is non-zero. If it is, we are cool. Anyways, lets just keep working and insert our expression into our differential equation:

$$XT` = kX``T \tag{6.3}$$

We can rearrange now all X terms to one side, T terms to the other:

$$\frac{X}{kX``} = \frac{T}{T`} \tag{6.4}$$

Now this is a condition that is always true, thus we can express these ratios as a constant:

$$\frac{X}{kX``} = \frac{T}{T`} = -\lambda \tag{6.5}$$

The minus is more there for "comfort" for the continuation of the calculations we require.
Thus we end up with the two equations to solve:

$$X + \lambda kX`` = 0 \tag{6.6}$$

$$T + \lambda T` = 0 \tag{6.7}$$

Now in order to solve this and find a general solution, we require to check how λ affects the solution. It may be that if λ is negative, there is no solution, whereas if it is positive there is. We dont know, so we need to check that. We shall pick the X-equation, mainly because we can then make use of the boundary conditions.

- **Case 1:** $\lambda > 0$: In that case, we can just use the rules from the first chapter to realize that the solution will be of the form:

$$\lambda kr^2 + 1 = 0 \tag{6.8}$$

$$r = \pm\sqrt{\frac{-1}{\lambda k}} \tag{6.9}$$

Since λ is positive:

$$r = \pm i\sqrt{\frac{1}{\lambda k}} \tag{6.10}$$

So the solution would be:

$$X(x) = c_1 cos(\sqrt{\frac{1}{\lambda k}}x) + sin(\sqrt{\frac{1}{\lambda k}}x) \tag{6.11}$$

Applying the second boundary condition:

$$X(0) = 0 = c_1 cos(0) + sin(0) \tag{6.12}$$

$$c_1 = 0 \tag{6.13}$$

Applying now the third condition:

$$X(L) = 0 = c_1 cos(\sqrt{\frac{1}{\lambda k}}L) + sin(\sqrt{\frac{1}{\lambda k}}L) \tag{6.14}$$

but we know that c_1 is zero, so:

$$X(L) = 0 = sin(\sqrt{\frac{1}{\lambda k}}L) \tag{6.15}$$

$$\arcsin(0) = \sqrt{\frac{1}{\lambda k}}L \tag{6.16}$$

Knowing that the sine function is zero at $0, \pi, 2\pi, 3\pi ... n\pi$ we can rewrite our last expression as:

$$n\pi = \sqrt{\frac{1}{\lambda k}}L \tag{6.17}$$

Rearranging for λ, we have:

$$k(\frac{n\pi}{L})^2 = \frac{1}{\lambda} \tag{6.18}$$

and the solution would be:

$$X(x) = sin(\sqrt{(\frac{n\pi}{L})^2}x) \tag{6.19}$$

$$X(x) = sin(\frac{n\pi x}{L}) \tag{6.20}$$

Notice how this is a Fourier series expression. (Told you they would come up.)

- **Case 2:** $\lambda = 0$: We already know that:

$$r = \pm i\sqrt{\frac{1}{\lambda k}} \tag{6.21}$$

Since λ is zero, we have $r = 0$. This would give us only the trivial solution[1], thus for this case we have no (non-trivial) solution.

- **Case 3:** $\lambda < 0$: In that case we would have:

$$r = \pm\sqrt{\frac{1}{\lambda k}} \tag{6.22}$$

This would give us the following solution:

$$X(x) = c_1 e^{\sqrt{\frac{1}{\lambda k}}x} + c_2 e^{-\sqrt{\frac{1}{\lambda k}}x} \tag{6.23}$$

[1] trivial solution means that a solution is equal to zero.

Applying the second condition, we can see that $0 = c_1 + c_2$, and therefore $c_1 = -c_2$.

Applying the third condition, we have:

$$X(x) = 0 = -c_2 e^{\sqrt{\frac{1}{\lambda k}}L} + c_2 e^{-\sqrt{\frac{1}{\lambda k}}L} \tag{6.24}$$

So:

$$c_2 e^{\sqrt{\frac{1}{\lambda k}}L} = c_2 e^{-\sqrt{\frac{1}{\lambda k}}L} \tag{6.25}$$

Since this is NOT true unless L = 0 as well, there is no solution. Thus the only solution is when λ was positive!

So now that we know the solution to the first equation, written in x-terms, we move on to the second equation, now with the known λ:

$$T + \lambda T' = 0 \tag{6.26}$$

We shall just not yet plug in λ to prevent a huge mess. This equation is a simple first order (even seperable!) differential equation. Thus we know that the solution is:

$$T(t) = c_1 e^{-\frac{1}{\lambda}t} \tag{6.27}$$

Now inserting λ:

$$T(t) = c_1 e^{-k(\frac{n\pi}{L})^2 t} \tag{6.28}$$

Remembering that we said that:

$$u(x,t) = X(x)T(t) \tag{6.29}$$

We can include now both of our found solutions:

$$u(x,t) = c_1 sin(\frac{n\pi x}{L})e^{-k(\frac{n\pi}{L})^2 t} \tag{6.30}$$

Notice that for different n-values, the constant c_1 would also change. So a better way of writing it is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L})e^{-k(\frac{n\pi}{L})^2 t} \tag{6.31}$$

And there you go. A long road, but we are done! How to find A_n is just like what we had done in the chapter of fourier series.

6.0.2 Limiting the equation:

So what happens when we have a non homogenous equation? If you think about it, in the least of the cases, we actually end up with a homogenous case. Take for example the heat equation for a rod: The temperature at the end of the rods is most of the times *not* zero.

Instead the boundary and initial conditions would look more like this in reality:
$u(x,0) = f(x)$; $u(0,t) = T_1$ and $u(L,t) = T_2$.

So lets try to solve the same heat equation as last time, but with these new boundary conditions:

Example 2: Solve the heat equation:

$$u_t = ku_{xx} \tag{6.32}$$

Unlike last time, we are not able to use separation of variables, as this method only works for homogeneous cases. Similarly to previous cases, we are able to find a solution for the homogeneous case, but still have to find the particular solution.

However we notice that as time progresses, and there is no source term, the system described by the heat equation will arrive to an equilibrium point. In mathematical words:

$$\lim_{t\to\infty} u(x,t) = u_E(x) \tag{6.33}$$

Using this assumption, our differential equation would convert to:

$$0 = k(u_E)_{xx} \tag{6.34}$$

and the boundary conditions would change to: $u_E(0) = T_1$ and $u_E(L) = T_2$.

What we are dealing with now is of course a very simple second order homogeneous differential equation. (Not even partial!) By integrating it twice, we solve it to:

$$k[c_1x + c_2] = 0 \tag{6.35}$$

or simply:

$$c_1x + c_2 = 0 \tag{6.36}$$

and including the boundary conditions, we end up with the following equation:

$$\frac{T_2 - T_1}{L}x + T_1 = 0 \tag{6.37}$$

and this is the particular solution! You might have seen the definition:

$$v(x,t) = u(x,t) - u_E \tag{6.38}$$

which is then rearranged to:

$$u(x,t) = v(x,t) + u_E \tag{6.39}$$

What does all that mean? Well, $v(x,t)$ is the solution for the homogenous case, so the complementary solution, and u_E is the particular solution. Thus the solution to our problem here is:

$$u(x,t) = (\sum A_n sin(\frac{n\pi x}{L})e^{-k(\frac{n\pi}{L})^2 t}) + (\frac{T_2 - T_1}{L}x + T_1) \tag{6.40}$$