Differential Equations Problems and Solutions

1 The Laplace Transform

1.1 Question 1 - January 19, 2007 (4 points)

Use the Laplace transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 4y = \delta\left(t - \pi\right), \qquad y(0) = 1, \qquad \frac{dy}{dt}(0) = 0. \tag{1.1}$$

1.1 Solution

Taking the Laplace transform will give

$$\begin{aligned} L\{y'' + 2y' + 4y\} &= L\{\delta\left(t - \pi\right)\}, & (1.2)\\ L\{y''\} + 2L\{y'\} + 4L\{y\} &= e^{-\pi s}, & (1.3)\\ s^2F(s) - s + 2\left(sF(s) - 1\right) + 4F(s) &= e^{-\pi s}, & (1.4)\\ F(s)\left(s^2 + 2s + 4\right) - (s + 2) &= e^{-\pi s}, & (1.5) \end{aligned}$$

Solving for $F(s)$ will give

$$\begin{aligned} F(s) &= \frac{e^{-\pi s}}{s^2 + 2s + 4} + \frac{s + 2}{s^2 + 2s + 4}, & (1.6)\\ F(s) &= \frac{s + 1}{(s + 1)^2 + \sqrt{3}^2} + \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s + 1)^2 + \sqrt{3}^2} + \frac{e^{-\pi s}}{s^2 + 2s + 4}. & (1.7) \end{aligned}$$

Now we split the equation up in three different parts, being

$$\begin{aligned} F(s)_1 &= \frac{s + 1}{(s + 1)^2 + \sqrt{3}^2}, & (1.8)\\ F(s)_2 &= \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s + 1)^2 + \sqrt{3}^2}, & (1.9)\\ F(s)_3 &= \frac{e^{-\pi s}}{s^2 + 2s + 4}. & (1.10) \end{aligned}$$

We solve for them individually. The first two are relatively easy. The last one requires an extra step.

$$\begin{aligned} f(t)_1 &= e^{-t}\cos\sqrt{3}t, & (1.11)\\ f(t)_2 &= e^{-t}\frac{1}{\sqrt{3}}\sin\sqrt{3}t, & (1.12)\\ f(t)_3 &= L\{e^{-\pi s}H(s)\} = u_\pi(t)h(t - \pi), & (1.13)\\ h(t) &= e^{-t}\frac{1}{\sqrt{3}}\sin\sqrt{3}t, & (1.14)\\ f(t)_3 &= u_\pi(t)e^{-(t-\pi)}\frac{1}{\sqrt{3}}\sin\sqrt{3}(t - \pi). & (1.15) \end{aligned}$$

Finally we add it all up to get

$$y = f(t) = f(t)_1 + f(t)_2 + f(t)_3 = e^{-t}\cos\sqrt{3}t + e^{-t}\frac{1}{\sqrt{3}}\sin\sqrt{3}t + u_\pi(t)e^{-(t-\pi)}\frac{1}{\sqrt{3}}\sin\sqrt{3}(t-\pi). \quad (1.16)$$

1.2 Question 1 - April 2, 2007 (6 points)

Use the Laplace transform to solve the initial value problem

$$y' + 4y = \sin 2t + \delta\left(t - \pi\right), \qquad y(0) = 1. \quad (1.17)$$

1.2 Solution

Taking the Laplace transform will give

$$\begin{aligned} L\{y' + 4y\} &= L\{\sin 2t + \delta\left(t-\pi\right)\}, && (1.18)\\ L\{y'\} + 4L\{y\} &= \frac{2}{s^2+2^2} + e^{-\pi s}, && (1.19)\\ sF(s) - 1 + 4F(s) &= \frac{2}{s^2+2^2} + e^{-\pi s}, && (1.20)\\ F(s)\left(s+4\right) - 1 &= \frac{2}{s^2+2^2} + e^{-\pi s}, && (1.21) \end{aligned}$$

Solving for $F(s)$ will give

$$F(s) = \frac{1}{s+4} + \frac{2}{\left(s^2+4\right)\left(s+4\right)} + \frac{e^{-\pi s}}{s+4}. \quad (1.22)$$

The second part can be split up into multiple factors. These factors are (s^2+4) and $(s+4)$. So we can make two fractions out of them. The fraction (s^2+4) has as highest power s^2, so we will place $as+b$ above it. The fraction $(s+4)$ has as highest power just s, so we will place c above it. We can now rewrite the second part to

$$\frac{2}{\left(s^2+4\right)\left(s+4\right)} = \frac{as+b}{s^2+4} + \frac{c}{s+4} = \frac{(as+b)(s+4)}{\left(s^2+4\right)\left(s+4\right)} + \frac{c(s^2+4)}{\left(s^2+4\right)\left(s+4\right)} = \frac{as^2+bs+4as+4b+cs^2+4c}{\left(s^2+4\right)\left(s+4\right)}. \quad (1.23)$$

So now we know that

$$\begin{aligned} a + c &= 0, && (1.24)\\ b + 4a &= 0, && (1.25)\\ 4b + 4c &= 2. && (1.26) \end{aligned}$$

If we use this, and also rewrite the fraction to a more useful form, we will find

$$\frac{2}{\left(s^2+4\right)\left(s+4\right)} = -\frac{1}{10}\frac{s}{s^2+2^2} + \frac{1}{5}\frac{2}{s^2+2^2} + \frac{1}{10}\frac{1}{s+4}. \quad (1.27)$$

Now we can take the inverse Laplace transform of all the terms in the equation. This is simple for most terms. Only the term $e^{-\pi s}/(s+4)$ is slightly difficult. The inverse Laplace transform of this one is found by

$$L\{F(s)\} = L\{e^{-\pi s}/(s+4)\} = L\{e^{-\pi s}H(s)\} = u_\pi(t)h(t-\pi), \qquad \text{where} \qquad h(t) = e^{-4t}. \quad (1.28)$$

So the solution is

$$y = f(t) = e^{-4t} - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t + \frac{1}{10}e^{-4t} + u_\pi(t)e^{-4(t-\pi)}. \quad (1.29)$$

1.3 Question 1 - January 21, 2005 (7 points)

Use the Laplace Transform to solve

$$\frac{d^2y}{dt^2} + y = g(t), \qquad g(t) = \begin{cases} \frac{1}{2}t & \text{if } 0 \le t < 6 \\ 3 & \text{if } 6 \le t \end{cases}, \qquad y(0) = 0, \qquad y'(0) = 1. \tag{1.30}$$

1.3 Solution

Note that we can write

$$g(t) = \frac{1}{2}t - u_6(t)\left(\frac{1}{2}t - 3\right). \tag{1.31}$$

Taking the Laplace transform now will give

$$L\{y'' + y\} = L\left\{\frac{1}{2}t - u_6(t)\left(\frac{1}{2}t - 3\right)\right\}, \tag{1.32}$$

$$L\{y''\} + L\{y\} = \frac{1}{2s^2} - \frac{1}{2s^2}e^{-6s}, \tag{1.33}$$

$$s^2F(s) - 1 + F(s) = \frac{1}{2s^2} - \frac{1}{2s^2}e^{-6s}, \tag{1.34}$$

$$F(s)\left(s^2 + 1\right) - 1 = \frac{1}{2s^2} - \frac{1}{2s^2}e^{-6s}, \tag{1.35}$$

Solving for $F(s)$ will give

$$F(s) = \frac{1}{s^2 + 1} + \frac{1}{s^2\left(s^2 + 1\right)}\frac{1 - e^{-6s}}{2}. \tag{1.36}$$

We can rewrite this to

$$F(s) = \frac{1}{s^2 + 1} + \left(\frac{1}{s^2} - \frac{1}{s^2 + 1}\right)\frac{1 - e^{-6s}}{2}, \tag{1.37}$$

$$F(s) = \frac{1}{2}\frac{1}{s^2 + 1} + \frac{1}{2}\frac{1}{s^2} - \frac{1}{2}\frac{1}{s^2}e^{-6s} + \frac{1}{2}\frac{1}{s^2 + 1}e^{-6s}. \tag{1.38}$$

Taking the inverse Laplace transform (we will leave the intermediate steps, as they have been shown in previous problems) will give

$$y = f(t) = \frac{1}{2}\sin t + \frac{1}{2}t - \frac{1}{2}u_6(t)\left(t - 6\right) + \frac{1}{2}u_6(t)\sin\left(t - 6\right). \tag{1.39}$$

2 Second Order Linear Differential Equations

2.1 Question 2 - January 19, 2007 (4 points)

Find (using the method of variation of parameters) the general solution of

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \frac{e^{-2t}}{t^2}. \tag{2.1}$$

2.1 Solution

First we solve the characteristic equation $r^2 + 4r + 4 = 0$. The only solution is $r = -2$. So we find that

$$y_1 = e^{-2t}, \qquad \text{and} \qquad y_2 = te^{-2t}. \tag{2.2}$$

Taking a derivative gives

$$y_1' = -2e^{-2t}, \qquad \text{and} \qquad y_2' = -2te^{-2t} + e^{-2t}. \tag{2.3}$$

The Wronskian now becomes

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = -2te^{-4t} + e^{-4t} + 2te^{-4t} = e^{-4t}. \tag{2.4}$$

Now we can find $u_1'(t)$ and $u_2'(t)$. These are

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(t)} = -\frac{1}{t}, \qquad \text{and} \qquad u_2'(t) = \frac{y_1(t)g(t)}{W(t)} = \frac{1}{t^2}. \tag{2.5}$$

The particular solution now is

$$Y(t) = y_1(t)\int_{t_0}^{t} u_1'(s)\,ds + y_2(t)\int_{t_0}^{t} u_2'(s)\,ds = e^{-2t}\ln\frac{t_0}{t} + te^{-2t}\left(\frac{1}{t_0} - \frac{1}{t}\right). \tag{2.6}$$

We may choose t_0. It seems to be convenient to choose $t_0 = 1$. So we get

$$Y(t) = e^{-2t}\left(-\ln t + t - 1\right). \tag{2.7}$$

Note that te^{-2t} and $-e^{-2t}$ are also part of the general solution, so we may ignore them. Therefore $Y(t) = -e^{-2t}\ln t$. This makes the general solution set

$$y(t) = y_1(t) + y_2(t) + Y(t) = e^{-2t} + te^{-2t} - e^{-2t}\ln t. \tag{2.8}$$

3 Systems of First Order Linear Differential Equations

3.1 Question 3 - January 19, 2007 (7 points)

Find the general solution of the system of differential equations

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{3.1}$$

3.1 Solution

Let's call the matrix A and the vector on the right $\mathbf{g}$. The equation then becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$. First we need to find the general solution set of $\mathbf{x}' = A\mathbf{x}$. The matrix has as eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$. Corresponding eigenvectors are

$$\xi_{\mathbf{1}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \xi_{\mathbf{2}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.2}$$

The set of solutions now is $\mathbf{x} = c_1 e^{\lambda_1 t}\xi_{\mathbf{1}} + c_2 e^{\lambda_2 t}\xi_{\mathbf{2}}$. So we can assemble the fundamental matrix Ψ. We will find

$$\Psi = \begin{bmatrix} 1 & e^{2t} \\ 3 & e^{2t} \end{bmatrix}. \tag{3.3}$$

Now let's find the specific solution to the nonhomogeneous problem. We will use the method of variation of parameters for that. First we need to find Ψ^{-1}. This is

$$\Psi^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2}e^{-2t} & \frac{1}{2}e^{-2t} \end{bmatrix}. \tag{3.4}$$

To find $\mathbf{u}(t)$ we will use

$$\mathbf{u}(t) = \int \Psi^{-1}\mathbf{g} dt = \int \begin{bmatrix} -1 \\ 3e^{-2t} \end{bmatrix} dt = \begin{bmatrix} -t \\ -\frac{3}{2}e^{-2t} \end{bmatrix}. \tag{3.5}$$

The specific solution can then be found using

$$\mathbf{x} = \Psi(t)\mathbf{u}(t) = \begin{bmatrix} 1 & e^{2t} \\ 3 & e^{2t} \end{bmatrix} \begin{bmatrix} -t \\ -\frac{3}{2}e^{-2t} \end{bmatrix} = \begin{bmatrix} -t - \frac{3}{2} \\ -3t - \frac{3}{2} \end{bmatrix}. \tag{3.6}$$

We can now assemble the general solution set to be

$$\mathbf{x} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} + \begin{bmatrix} -t - 1 \\ -3t \end{bmatrix}. \tag{3.7}$$

Note that we have added 1/2 times the first solution to the specific solution, to make it a bit easier to write.

3.2 Question 3 - April 2, 2007 (8 points)

Find the general solution of the system of differential equations (thereby expressing the General solution of the corresponding homogeneous system in terms of real-valued functions)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -3 & 5 \\ -1 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}. \tag{3.8}$$

3.2 Solution

Let's call the matrix A and the vector on the right $\mathbf{g}$. The equation then becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$. First we need to find the general solution set of $\mathbf{x}' = A\mathbf{x}$. The matrix has as eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Corresponding eigenvectors are

$$\xi_{\mathbf{1}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i = \mathbf{a} + \mathbf{b}i, \qquad \text{and} \qquad \xi_{\mathbf{2}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} i = \mathbf{a} - \mathbf{b}i. \tag{3.9}$$

The two solutions needed for the general solution set now become

$$\mathbf{x_1} = e^{-t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t\right), \qquad \text{and} \qquad \mathbf{x_2} = e^{-t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t\right). \tag{3.10}$$

Now we need to find a specific solution to the nonhomogeneous equation. Since there is an exponential in $\mathbf{g}$, the specific solution will also probably have an exponential in it. So we assume that the specific solution will look like

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} e^{-t}. \tag{3.11}$$

It's easy to see that $\mathbf{x}' = -\mathbf{x}$. Inserting $\mathbf{x}$ in the system of differential equations gives

$$-\begin{bmatrix} a \\ b \end{bmatrix} e^{-t} = \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \left(\begin{bmatrix} -3a + 5b \\ -a + b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^{-t}. \tag{3.12}$$

Solving will give $a = -2$ and $b = -1$, so we know that the specific solution is

$$\mathbf{x} = -\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}. \tag{3.13}$$

The general solution set therefore becomes

$$\mathbf{x} = c_1 e^{-t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t\right) + c_2 e^{-t}\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t\right) - \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \tag{3.14}$$

3.3 Question 2 - January 21, 2005 (7 points)

Find the general solution of the system of equations (thereby expressing the general solution of the corresponding homogeneous system in terms of real-valued functions)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}. \tag{3.15}$$

3.3 Solution

Let's call the matrix A and the vector on the right $\mathbf{g}$. The equation then becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$. First we need to find the general solution set of $\mathbf{x}' = A\mathbf{x}$. The matrix has as eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. Corresponding eigenvectors are

$$\xi_{\mathbf{1}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} i = \mathbf{a} + \mathbf{b}i, \qquad \text{and} \qquad \xi_{\mathbf{2}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} i = \mathbf{a} - \mathbf{b}i. \tag{3.16}$$

The two solutions needed for the general solution set now become

$$\mathbf{x_1} = \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t\right), \qquad \text{and} \qquad \mathbf{x_2} = \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t\right). \tag{3.17}$$

Now we need to find a specific solution to the nonhomogeneous equation. Since there is an exponential in $\mathbf{g}$, the specific solution will also probably have an exponential in it. So we assume that the specific solution will look like

$$\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} e^{-2t}. \tag{3.18}$$

It's easy to see that $\mathbf{x}' = -2\mathbf{x}$. Inserting $\mathbf{x}$ in the system of differential equations gives

$$-2 \begin{bmatrix} a \\ b \end{bmatrix} e^{-2t} = \mathbf{x}' = A\mathbf{x} + \mathbf{g} = \left(\begin{bmatrix} 2a - 5b \\ a - 2b \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{-2t}. \tag{3.19}$$

Solving will give $a = -1$ and $b = -4/5$, so we know that the specific solution is

$$\mathbf{x} = - \begin{bmatrix} 1 \\ 4/5 \end{bmatrix} e^{-2t}. \tag{3.20}$$

The general solution set therefore becomes

$$\mathbf{x} = c_1 \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t \right) + c_2 \left(\begin{bmatrix} 5 \\ 2 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t \right) - \begin{bmatrix} 1 \\ 4/5 \end{bmatrix} e^{-2t} \tag{3.21}$$

4 Stability of Systems of Differential Equations

4.1 Question 4 - January 19, 2007 (5 points)

Consider the system of nonlinear equations

$$\frac{dx}{dt} = x - xy, \tag{4.1}$$

$$\frac{dy}{dt} = x^2 + y^2 + y. \tag{4.2}$$

Determine type and (in-)stability of each critical point of this almost linear system (for the linearised case as well as for the nonlinear case).

4.1 Solution

First we need to find the critical points. So we set $x' = 0$ and $y' = 0$ and solve for x and y. We find the points $(0,0)$ and $(0,-1)$. Now we define $F(x,y) = x - xy$ and $G(x,y) = x^2 + y^2 + y$.

First examine point 1, being $(0,0)$. The Jacobian matrix now is

$$J = \begin{bmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.3}$$

The eigenvalues of this matrix are $\lambda_1 = \lambda_2 = 1$. We have two the same eigenvalues, so we better determine the eigenvectors as well. It turns out that every vector is an eigenvector. So it's possible to choose two eigenvectors that are linearly independent. The type of critical point in the linear system is therefore a proper node. Since $\lambda_1 = \lambda_2 > 0$, the stability of the linear system is unstable. In the almost linear system, the critical point is a node or a spiral point. The stability is still unstable.

Now let's examine the other critical point. The Jacobian matrix now becomes

$$J = \begin{bmatrix} F_x(0,-1) & F_y(0,-1) \\ G_x(0,-1) & G_y(0,-1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.4}$$

The eigenvalues are now $\lambda_1 = 2$ and $\lambda_2 = -1$. The critical point is therefore an unstable saddle point, both in the linear as in the almost linear system.

4.2 Question 4 - April 2, 2007 (5 points)

Consider the system of nonlinear equations

$$\frac{dx}{dt} = (1+x)\sin y, \tag{4.5}$$

$$\frac{dy}{dt} = 1 - x - \cos y. \tag{4.6}$$

Points $(0,0)$ en $(2,\pi)$ are critical points. Determine type and (in-)stability of these two points of the given almost linear system (for the linearized case as well as for the nonlinear case).

4.2 Solution

The critical points are given. So we only need to examine their types. First examine the point $(0,0)$. We define $F(x,y) = (1+x)\sin y$ and $G(x,y) = 1 - x - \cos y$. The Jacobian matrix now is

$$J = \begin{bmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4.7}$$

The corresponding eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. So the critical point in the linear system is a center. Centers are always stable. In the almost linear system, the critical point is a center or a spiral point. The stability is indeterminate.

Now let's examine the critical point $(2, \pi)$.The Jacobian matrix now is

$$J = \begin{bmatrix} F_x(2,\pi) & F_y(2,\pi) \\ G_x(2,\pi) & G_y(2,\pi) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -1 & 0 \end{bmatrix}. \tag{4.8}$$

The corresponding eigenvalues are $\lambda_1 = \sqrt{3}$ and $\lambda_2 = -\sqrt{3}$. So the critical point is an unstable saddle point, both in the linear system as in the almost linear system.

4.3 Question 3 - April 4, 2005 (6 points)

Consider the system of nonlinear equations

$$\frac{dx}{dt} = x + x^2 + y^2, \tag{4.9}$$

$$\frac{dy}{dt} = y - xy. \tag{4.10}$$

Determine type and (in-)stability of each critical point of this almost linear system (linearized case and nonlinear case).

4.3 Solution

The critical points are $(0, 0)$ and $(-1, 0)$. Let's look at $(0, 0)$ first. We can find the Jacobian matrix to be

$$J = \begin{bmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.11}$$

Eigenvalues are $\lambda_1 = \lambda_2 = 1$. Eigenvectors are linearly independent, so we are dealing with an unstable proper node in the linear system. In the almost linear system we have an unstable node or spiral point.

For the point $(-1, 0)$ we can find the Jacobian matrix to be

$$J = \begin{bmatrix} F_x(-1,0) & F_y(-1,0) \\ G_x(-1,0) & G_y(-1,0) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}. \tag{4.12}$$

Corresponding eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$. So we are dealing with an unstable saddle point, both in the linear system as in the almost linear system.

5 Eigenfunctions

5.1 Question 6 - January 19, 2007 (7 points)

Determine the normalised eigenfunctions (assume that all eigenvalues are real) of the following problem:

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y'(1) = 0. \tag{5.1}$$

5.1 Solution

We consider the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. First assume $\lambda < 0$. If we define $\mu = \sqrt{-\lambda}$, we find the general solution set to be

$$y = c_1 e^{\mu t} + c_2 e^{-\mu t}. \tag{5.2}$$

Differentiating gives

$$y' = \mu c_1 e^{\mu t} - \mu c_2 e^{-\mu t}. \tag{5.3}$$

The first boundary condition implies that $c_1 = c_2$. If we then use the second boundary conditions, we find that

$$y'(1) = \mu c_1 \left(e^{\mu} - e^{-\mu}\right) = 0. \tag{5.4}$$

Since $\mu \neq 0$ we know that $c_1 = 0$. The solution therefore is $y = 0$, which is the trivial solution. There are no non-trivial solutions for $\lambda < 0$.

Now let's consider $\lambda = 0$. In this case the differential equations becomes $y'' = 0$ with as solution $y = c_1 t + c_2$. From both boundary conditions we find that $c_1 = 0$ and c_2 is undetermined. So we have a non-trivial solution, being $y = c$, with $c \neq 0$ a constant.

Now let's consider $\lambda > 0$. If we define $\mu = \sqrt{\lambda}$ we will find as general solution set

$$y = c_1 \sin \mu t + c_2 \cos \mu t. \tag{5.5}$$

Differentiating gives

$$y' = \mu c_1 \cos \mu t - \mu c_2 \sin \mu t. \tag{5.6}$$

The first boundary condition implies $c_1 = 0$. The second boundary conditions implies that

$$y'(1) = -\mu c_2 \sin \mu = 0. \tag{5.7}$$

This is only true if $c_2 = 0$ or $\mu = n\pi$, with n an integer. Since $c_2 = 0$ gives the trivial solution, we know that $\mu = n\pi$ and thus $\lambda = \sqrt{n\pi}$. The eigenfunctions are therefore $y_n(t) = c \cos n\pi t$, corresponding to $\lambda_n = n^2\pi^2$.

Now we need to normalize the eigenfunctions. We do this by using

$$\int_0^1 y_n^2(t)dt = 1. \tag{5.8}$$

For $y_0(t)$ we simply find that $c = 1$ and so $y_0(t) = 1$. For $y_n(t)$ with $n \geq 1$ we find that

$$\int_0^1 y_n^2(t)dt = c^2 \int_0^1 \cos^2 n\pi t = \frac{1}{2}c^2 = 1. \tag{5.9}$$

So evidently $c = \sqrt{2}$. The normalized eigenfunctions therefore are

$$y_0(t) = 1, \qquad \text{and} \qquad y_n(t) = \sqrt{2}\cos n\pi t, \qquad \text{for} \qquad n \geq 1. \tag{5.10}$$

6 Power Series

6.1 Question 6 - January 21, 2005 (5 points)

Solve the following initial value problem by means of a power series expansion near $x_0 = 0$

$$\frac{dy}{dx} + xy = 1, \qquad y(0) = 0. \tag{6.1}$$

(You may stop after three non-zero terms.)

6.1 Solution

We will use a power series. So we first write down

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x - x_0)^n, && (6.2)\\ y' &= \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n. && (6.3) \end{aligned}$$

This makes the differential equation

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + x\sum_{n=0}^{\infty} a_n x^n &= 1, && (6.4)\\ \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_n x^{n+1} &= 1, && (6.5)\\ \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} a_{n-1} x^n &= 1, && (6.6)\\ a_1 + \sum_{n=1}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} a_{n-1} x^n &= 1, && (6.7)\\ a_1 + \sum_{n=1}^{\infty} \left(a_{n+1}(n+1) + a_{n-1}\right) x^n &= 1. && (6.8) \end{aligned}$$

Equating coefficients on both sides gives $a_1 = 1$. We also find the recurrence relation

$$a_{n+1} = -\frac{a_{n-1}}{n+1}. \tag{6.9}$$

From the boundary condition we also find that $a_0 = 0$. So we have $a_{2k} = 0$ for every integer k. We can also find $a_3 = -1/3$, $a_5 = 1/15$, $a_7 = -1/105$ and so on. This makes the power series

$$y = x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 + \ldots. \tag{6.10}$$

6.2 Question 6 - January 10, 2003 (6 points) (adjusted)

Find the general solution of the following differential equation by means of a power series expansion about the point $x_0 = 0$

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + xy = 0, \qquad x > 0. \tag{6.11}$$

[You may stop after some non-zero terms].

6.2 Solution

We will use a power series. So we first write down

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{6.12}$$

$$y' = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n, \tag{6.13}$$

$$y'' = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x - x_0)^n. \tag{6.14}$$

Notice that since $x > 0$ we can remove one factor x from the differential equation. If you don't do this, the equations below will differ slightly, but the final answer will be the same. Now we can find that the differential equation is

$$x \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + 4\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0, \tag{6.15}$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^{n+1} + \sum_{n=0}^{\infty} 4a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0, \tag{6.16}$$

$$\sum_{n=1}^{\infty} a_{n+1}(n+1)nx^n + \sum_{n=0}^{\infty} 4a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0, \tag{6.17}$$

$$4a_1 + a_0 + \sum_{n=1}^{\infty} a_{n+1}(n+1)nx^n + \sum_{n=1}^{\infty} 4a_{n+1}(n+1)x^n + \sum_{n=1}^{\infty} a_n x^n = 0, \tag{6.18}$$

$$4a_1 + a_0 + \sum_{n=1}^{\infty} \left(a_{n+1}(n+1)n + 4a_{n+1}(n+1) + a_n\right) x^n = 0. \tag{6.19}$$

So we find that $a_1 = -\frac{1}{4}a_0$. We can also determine the recurrence relation

$$a_{n+1} = -\frac{a_n}{(n+1)(n+4)}. \tag{6.20}$$

So we can determine the first couple of coefficients. These are $a_1 = -\frac{1}{4}a_0$, $a_2 = \frac{1}{40}a_0$, $a_3 = -\frac{1}{720}a_0$ and so on. The power series therefore is

$$y = a_0 \left(1 - \frac{1}{4}x + \frac{1}{40}x^2 - \frac{1}{720}x^3 + \dots\right). \tag{6.21}$$

6.3 Question 6 - April 2, 2007 (6 points)

Find the general solution of the following differential equation by means of a power series expansion about the point $x_0 = 1$

$$\frac{d^2y}{dx^2} - (x-1)^2 y = 0. \tag{6.22}$$

[You may stop after some non-zero terms].

6.3 Solution

We will use a power series. So we first write down

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \tag{6.23}$$

$$y' = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n, \tag{6.24}$$

$$y'' = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x - x_0)^n. \tag{6.25}$$

This makes the differential equation

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - (x-1)^2 \sum_{n=0}^{\infty} a_n (x-1)^n = 0, \tag{6.26}$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^{n+2} = 0, \tag{6.27}$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=2}^{\infty} a_{n-2} (x-1)^n = 0, \tag{6.28}$$

$$2a_2 + 6a_3(x-1) + \sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=2}^{\infty} a_{n-2} (x-1)^n = 0, \tag{6.29}$$

$$2a_2 + 6a_3(x-1) + \sum_{n=2}^{\infty} \left(a_{n+2}(n+2)(n+1) - a_{n-2}\right)(x-1)^n = 0. \tag{6.30}$$

Equating coefficients on both sides, we find that $a_2 = 0$ and $a_3 = 0$. We also find the recurrence relation

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)}. \tag{6.31}$$

Using this, we can determine that $a_4 = a_0/12$, $a_8 = a_4/56 = a_0/672$, $a_5 = a_1/20$, $a_9 = a_5/72 = a_1/1440$, $a_{10} = a_6 = a_2 = 0$, $a_{11} = a_7 = a_3 = 0$ and so on. The power series therefore is

$$y = a_0 + a_1(x-1)^1 + \frac{1}{12}a_0(x-1)^4 + \frac{1}{20}a_1(x-1)^5 + \frac{1}{672}a_0(x-1)^8 + \frac{1}{1440}a_1(x-1)^9 + \ldots. \tag{6.32}$$

7 Fourier Series

7.1 Question 5b - January 19, 2007 (3 points)

Find the Fourier series of the function (first, sketch its graph!) given by

$$g(x+2) = g(x), \qquad g(x) = x \text{ for } 0 \leq x < 1, \qquad g(x) = 0 \text{ for } 1 \leq x < 2. \tag{7.1}$$

7.1 Solution

We know that the solution will have the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{7.2}$$

The period $T = 2$ so we know that $L = T/2 = 1$. First we find a_0. This will be

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \cos 0 \, dx = \int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} 0 \, dx + \int_{0}^{1} x \, dx = \frac{1}{2}. \tag{7.3}$$

Then we will look at the a-coefficients. These are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \tag{7.4}$$

$$a_n = \int_{-1}^{0} 0 \cos n\pi x \, dx + \int_{0}^{1} x \cos n\pi x \, dx, \tag{7.5}$$

$$a_n = \left[x \frac{\sin n\pi x}{n\pi} \right]_0^1 - \int_0^1 \frac{\sin n\pi x}{n\pi} dx, \tag{7.6}$$

$$a_n = (0-0) - \left[-\frac{\cos n\pi x}{n^2\pi^2} \right]_0^1, \tag{7.7}$$

$$a_n = \frac{-1 + \cos n\pi}{n^2\pi^2}. \tag{7.8}$$

Note that we have used integration by parts. Now let's look at the b-coefficients. We find

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \tag{7.9}$$

$$b_n = \int_{-1}^{0} 0 \sin n\pi x \, dx + \int_{0}^{1} x \sin n\pi x \, dx, \tag{7.10}$$

$$b_n = \left[-x \frac{\cos n\pi x}{n\pi} \right]_0^1 - \int_0^1 -\frac{\cos n\pi x}{n\pi} dx, \tag{7.11}$$

$$b_n = \left(-\frac{\cos n\pi}{n\pi} + 0 \right) + (0-0), \tag{7.12}$$

$$b_n = -\frac{\cos n\pi}{n\pi}. \tag{7.13}$$

So the Fourier series becomes

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left(\frac{-1 + \cos n\pi}{n^2\pi^2} \cos n\pi x - \frac{\cos n\pi}{n\pi} \sin n\pi x \right). \tag{7.14}$$

7.2 Question 5 - April 4, 2005 (2 points)

Find the Fourier series of the function (at first, sketch its graph!) given by

$$f(x) = |x|, \qquad -1 \leq x < 1, \qquad f(x+2) = f(x). \tag{7.15}$$

7.2 Solution

We know that the solution will have the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{7.16}$$

The period $T = 2$ so we know that $L = T/2 = 1$. First we find a_0. This will be

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \cos 0 \, dx = \int_{-1}^{1} f(x) \, dx = \int_{-1}^{0} -x \, dx + \int_{0}^{1} x \, dx = \frac{1}{2} + \frac{1}{2} = 1. \tag{7.17}$$

Then we will look at the a-coefficients. These are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \tag{7.18}$$

$$a_n = \int_{-1}^{0} -x \cos n\pi x \, dx + \int_{0}^{1} x \cos n\pi x \, dx, \tag{7.19}$$

$$a_n = \left(\left[-x \frac{\sin n\pi x}{n\pi} \right]_{-1}^{0} - \int_{-1}^{0} - \frac{\sin n\pi x}{n\pi} dx \right) + \left(\left[x \frac{\sin n\pi x}{n\pi} \right]_{0}^{1} - \int_{0}^{1} \frac{\sin n\pi x}{n\pi} dx \right), \tag{7.20}$$

$$a_n = (0-0) - \left[\frac{\cos n\pi x}{n^2\pi^2} \right]_{-1}^{0} + (0-0) - \left[- \frac{\cos n\pi x}{n^2\pi^2} \right]_{0}^{1}, \tag{7.21}$$

$$a_n = 2 \frac{-1 + \cos n\pi}{n^2\pi^2}. \tag{7.22}$$

Note that we have used integration by parts. Now let's look at the b-coefficients. We find

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \tag{7.23}$$

$$b_n = \int_{-1}^{0} -x \sin n\pi x \, dx + \int_{0}^{1} x \sin n\pi x \, dx, \tag{7.24}$$

$$b_n = \left(\left[x \frac{\cos n\pi x}{n\pi} \right]_{-1}^{0} - \int_{-1}^{0} \frac{\cos n\pi x}{n\pi} dx \right) + \left(\left[-x \frac{\cos n\pi x}{n\pi} \right]_{0}^{1} - \int_{0}^{1} - \frac{\cos n\pi x}{n\pi} dx \right), \tag{7.25}$$

$$b_n = \left(\frac{\cos n\pi}{n\pi} + 0 \right) + (0-0) + \left(- \frac{\cos n\pi}{n\pi} + 0 \right) + (0-0), \tag{7.26}$$

$$b_n = 0. \tag{7.27}$$

So the Fourier series becomes

$$f(x) = \frac{1}{2} + 2 \sum_{n=1}^{\infty} \left(\frac{-1 + \cos n\pi}{n^2\pi^2} \cos n\pi x \right). \tag{7.28}$$

8 Fourier Series Applications

8.1 Question 5a - January 19, 2007 (5 points)

Find a formal solution (using the method of separation of variables) $u(x,t)$ of the initial-boundary value heat conduction problem

$$\frac{1}{\alpha^2}u_t = u_{xx}, \qquad u(0,t)=0, \qquad u_x(l,t)=0, \qquad u(x,0)=f(x). \tag{8.1}$$

8.1 Solution

We assume that $u(x,t) = X(x)T(t)$. This separates the problem into the two differential equations

$$X'' + \lambda X = 0, \qquad T' + \alpha^2 \lambda T = 0. \tag{8.2}$$

Let's look for the eigenvalues of the first differential equation. First consider $\lambda < 0$. Let's define $\mu = \sqrt{-\lambda}$. Now the solution becomes $X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$. We know that $0 = u(0,t) = X(0)T(t)$. Since $T(t) = 0$ isn't true in general, we know that $X(0) = 0$. This implies that $c_2 = -c_1$. From $0 = u_x(l,t) = X'(l)T(t)$ we find that $X'(l) = 0$. From this follows that $c_1 \mu \left(e^{\mu l} + e^{-\mu l}\right) = 0$. We Since $\mu > 0$ and $e^{\mu l} + e^{-\mu l} > 0$, we have $c_1 = 0$ and thus also $c_2 = 0$. So there are no non-trivial solutions for $\lambda < 0$.

Let's consider $\lambda = 0$ now. We find the function $X(x) = c_1 x + c_2$. From $X(0) = 0$ follows that $c_2 = 0$. From $X'(l) = 0$ follows that $c_1 = 0$. So we only find the trivial solution.

Now let's consider $\lambda > 0$. Let's redefine $\mu = \sqrt{\lambda}$. The solution now is $X(x) = c_1 \cos \mu x + c_2 \sin \mu x$. From $X(0) = 0$ follows that $c_1 = 0$. From $X'(l) = 0$ follows that

$$\mu_n = \frac{n - 1/2}{l}\pi \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n-1/2}{l}\pi\right)^2. \tag{8.3}$$

The corresponding eigenfunction is

$$X_n(x) = \sin \mu_n x. \tag{8.4}$$

Using λ_n and the second differential equation, we will find

$$T_n(t) = e^{-\alpha^2 \mu_n^2 t}. \tag{8.5}$$

Thus the solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n \sin\left(\mu_n x\right) e^{-\alpha^2 \mu_n^2 t}. \tag{8.6}$$

The coefficients then need to be found. For that, we use the initial condition, stating that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\mu_n x\right). \tag{8.7}$$

We recognize a sine series in the equation above. So the coefficients c_n correspond to the coefficients b_n in the Fourier series, for which we have an equation. This is

$$c_n = \frac{2}{l}\int_0^l f(x) \sin\left(\mu_n x\right) dx. \tag{8.8}$$

8.2 Question 4 - January 9, 2004 (7 points)

Find a formal solution (using the method of separation of variables) $u(x,t)$ to the initial-boundary value heat conduction problem

$$u_t = u_{xx}, \qquad u_x(0,t) = 0, \qquad u_x(l,t) = 0, \qquad u(x,0) = f(x). \tag{8.9}$$

8.2 Solution

We assume that $u(x,t) = X(x)T(t)$. This separates the problem into the two differential equations

$$X'' + \lambda X = 0, \qquad T' + \lambda T = 0. \tag{8.10}$$

Let's look for the eigenvalues of the first differential equation. First consider $\lambda < 0$. Let's define $\mu = \sqrt{-\lambda}$. Now the solution becomes $X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$. We know that $0 = u_x(0,t) = X'(0)T(t)$. Since $T(t) = 0$ isn't true in general, we know that $X'(0) = 0$. This implies that $c_2 = c_1$. From $0 = u_x(l,t) = X'(l)T(t)$ we find that $X'(l) = 0$. From this follows that $c_1\mu\left(e^{\mu l} - e^{-\mu l}\right) = 0$. We know that $\mu > 0$. Also $e^{\mu l} - e^{-\mu l} = 0$ if $\mu l = 0$, which isn't possible since $l > 0$. So we have $c_1 = 0$ and thus also $c_2 = 0$. So there are no non-trivial solutions for $\lambda < 0$.

Let's consider $\lambda = 0$ now. We find the function $X(x) = c_1 x + c_2$. From $X'(0) = 0$ follows that $c_1 = 0$. From $X'(l) = 0$ follows that $c_1 = 0$ as well. So we find one solution, being $X_0(x) = c_0$, where c_0 is a constant. The corresponding T_n function (for $\lambda = 0$) is $T_n(t) = 1$. So $u_0(x,t) = X_0(x)T_0(t) = c_0$.

Now let's consider $\lambda > 0$. Let's redefine $\mu = \sqrt{\lambda}$. The solution now is $X(x) = c_1\cos\mu x + c_2\sin\mu x$. From $X'(0) = 0$ follows that $c_2 = 0$. From $X'(l) = 0$ follows that

$$\mu_n = \frac{n}{l}\pi \qquad \Rightarrow \qquad \lambda_n = \left(\frac{n}{l}\pi\right)^2. \tag{8.11}$$

The corresponding eigenfunction is

$$X_n(x) = c_n\cos\mu_n x. \tag{8.12}$$

Using λ_n and the second differential equation, we will find

$$T_n(t) = e^{-\mu_n^2 t}. \tag{8.13}$$

We can now also find that for $n \geq 1$

$$u_n(x,t) = X(x)T(t) = c_n\cos(\mu_n x)\, e^{-\mu_n^2 t}. \tag{8.14}$$

Thus the solution becomes

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) = c_0 + \sum_{n=1}^{\infty} c_n\cos(\mu_n x)\, e^{-\mu_n^2 t}. \tag{8.15}$$

The coefficients then need to be found. For that, we use the initial condition, stating that

$$f(x) = u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n\cos(\mu_n x). \tag{8.16}$$

We recognize a cosine series in the equation above. So the coefficients c_n correspond to the coefficients a_n in the Fourier series, for which we have an equation. Also note that $a_0 \neq c_0$. In fact,

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2}\frac{2}{l}\int_0^l f(x)dx. \tag{8.17}$$

We can do the same for c_n and find

$$c_n = \frac{2}{l}\int_0^l f(x)\cos(\mu_n x)\, dx. \tag{8.18}$$

8.3 Question 4 - April 4, 2005 (7 points)

Find a formal solution $u(x, y)$ of the potential equation

$$u_{xx} + u_{yy} = 0 \qquad \text{in the rectangle } 0 < x < a, 0 < y < b, \tag{8.19}$$

that satisfies the boundary conditions

$$u(0, y) = 0, \qquad u(a, y) = 0, \qquad 0 < y < b, \tag{8.20}$$

$$u_y(x, 0) = 0, \qquad u(x, b) = g(x), \qquad 0 < x < a. \tag{8.21}$$

8.3 Solution

We first assume that $u(x, y)$ can be written as

$$u(x, y) = X(x)Y(y). \tag{8.22}$$

Inserting this in the differential equation gives

$$X''Y + XY'' = 0. \tag{8.23}$$

The boundary condition with the unknown function $g(x)$ is given for constant y and varying x. So, according to the problem solving guide, we should rewrite this such that we get the familiar eigenfunction equation for X. So we will have

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda, \tag{8.24}$$

from which follows that

$$X'' + \lambda X = 0. \tag{8.25}$$

And then of course we also have an equation for Y, being

$$Y'' - \lambda Y = 0. \tag{8.26}$$

Now let's transform the boundary conditions. From $u(0, y) = 0$ follows $X(0) = 0$. From $u(a, y) = 0$ follows $X(a) = 0$. Finally, from $u_y(x, 0) = 0$ follows $Y'(0) = 0$.

Now it's time to find the eigenfunctions X_n. First consider $\lambda < 0$. We define $\mu = \sqrt{-\lambda}$. The general solution becomes

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}. \tag{8.27}$$

From $X(0) = 0$ follows $c_1 = -c_2$. Combining this fact with $X(a) = 0$ results in $c_1 \left(e^{\mu a} - e^{-\mu a}\right)$. The term inside the brackets is only zero if $\mu a = 0$. This isn't possible, since $\mu > 0$ and $a > 0$. So we have $c_1 = 0$ and thus also $c_2 = 0$. There are therefore no non-trivial solutions.

Now let's consider $\lambda = 0$. We get as general solution

$$X(x) = c_1 x + c_2. \tag{8.28}$$

From $X(0) = 0$ follows that $c_2 = 0$. Also, from $X(a) = 0$ follows that $c_1 = 0$. So we once more have no non-trivial solutions.

Now let's consider $\lambda > 0$. We define $\mu = \sqrt{\lambda}$ (thus having $\lambda = \mu^2$). The general solution now is

$$X(x) = c_1 \sin \mu x + c_2 \cos \mu x. \tag{8.29}$$

From $X(0) = 0$ follows that $c_2 = 0$. From $X(a) = 0$ follows that

$$c_1 \sin \mu a = 0 \qquad \Rightarrow \qquad \mu_n = \frac{n\pi}{a}. \tag{8.30}$$

The corresponding eigenfunction becomes

$$X_n(x) = \sin \mu_n x. \tag{8.31}$$

Note that we have ignored the constant here, since every constant multiple of an eigenfunction is automatically also an eigenfunction.

Now that we have determined $X_n(x)$, let's look for $Y_n(y)$. We know that the eigenvalue $\lambda_n = \mu_n^2$, so we need to solve the differential equation

$$Y'' - \mu_n^2 Y = 0. \tag{8.32}$$

The general solution is

$$Y(y) = c_1 e^{\mu_n y} + c_2 e^{-\mu_n y}, \tag{8.33}$$

with as derivative

$$Y'(y) = c_1 \mu_n e^{\mu_n y} - c_2 \mu_n e^{\mu_n y}. \tag{8.34}$$

From $Y'(0) = 0$ follows that $c_1 = c_2$ (since $\mu \neq 0$). So we have as function Y_n

$$Y_n(y) = e^{\mu_n y} + e^{-\mu_n y} = \cosh \mu_n y. \tag{8.35}$$

Note that we have once more ignored the constant. We have also used the definition of the hyperbolic cosine here to write it a bit shorter. (This is not obligatory to do though. Using the exponentials is fine as well. It just costs you some extra ink.)

So the general solution to our problem now becomes

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} c_n \cosh(\mu_n y) \sin(\mu_n x) = \sum_{n=1}^{\infty} c_n \cosh\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n\pi}{a} x\right). \tag{8.36}$$

We still need to find an expression for the constants c_n. We have one remaining condition for that. We know that

$$u(x,b) = \sum_{n=1}^{\infty} c_n \cosh(\mu_n b) \sin(\mu_n x) = g(x). \tag{8.37}$$

This looks just like a sine series on the interval $[0, a]$. So we finally write

$$c_n \cosh(\mu_n b) = \frac{2}{a} \int_0^a g(x) \sin(\mu_n x)\, dx. \tag{8.38}$$

We now rewrite this equation slightly, and insert the value for μ_n, to find

$$c_n = \frac{2}{a \cosh(\mu_n b)} \int_0^a g(x) \sin\left(\frac{n\pi}{a} x\right) dx. \tag{8.39}$$

And we're done!