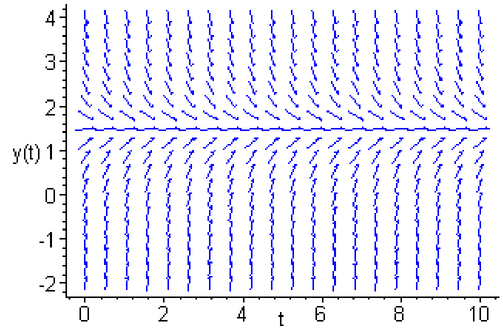
Chapter One

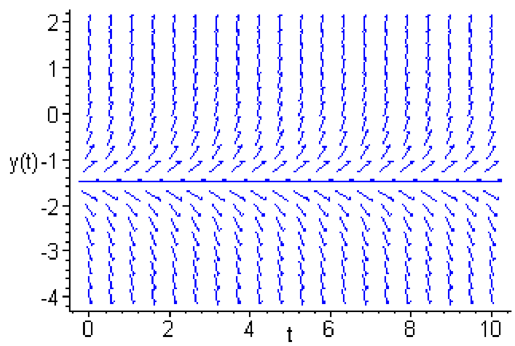
Section 1.1

1.



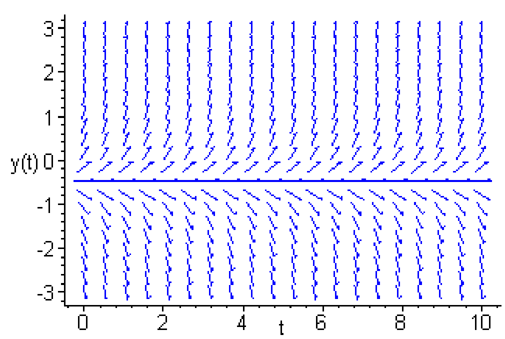
For $y > 1.5$, the slopes are *negative*, and hence the solutions decrease. For $y < 1.5$, the slopes are *positive*, and hence the solutions increase. The equilibrium solution appears to be $y(t) = 1.5$, to which all other solutions converge.

3.



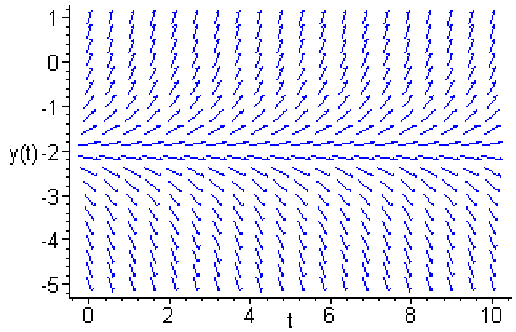
For $y > -1.5$, the slopes are *positive*, and hence the solutions increase. For $y < -1.5$, the slopes are *negative*, and hence the solutions decrease. All solutions appear to diverge away from the equilibrium solution $y(t) = -1.5$.

5.



For $y > -1/2$, the slopes are *positive*, and hence the solutions increase. For $y < -1/2$, the slopes are *negative*, and hence the solutions decrease. All solutions diverge away from the equilibrium solution $y(t) = -1/2$.

6.



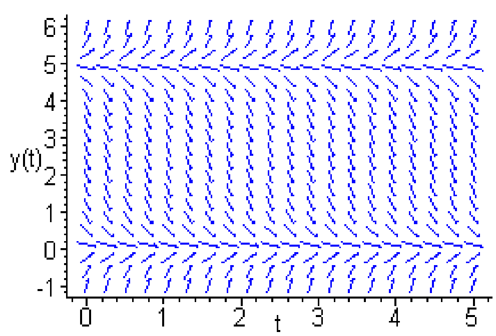
For $y > -2$, the slopes are *positive*, and hence the solutions increase. For $y < -2$, the slopes are *negative*, and hence the solutions decrease. All solutions diverge away from
the equilibrium solution $y(t) = -2$.

8. For *all* solutions to approach the equilibrium solution $y(t) = 2/3$, we must have $y' < 0$ for $y > 2/3$, and $y' > 0$ for $y < 2/3$. The required rates are satisfied by the differential equation $y' = 2 - 3y$.

9. For solutions *other* than $y(t) = 2$ to diverge from $y = 2$, $y(t)$ must be an *increasing* function for $y > 2$, and a *decreasing* function for $y < 2$. The simplest differential equation
whose solutions satisfy these criteria is $y' = y - 2$.

10. For solutions *other* than $y(t) = 1/3$ to diverge from $y = 1/3$, we must have $y' < 0$ for $y < 1/3$, and $y' > 0$ for $y > 1/3$. The required rates are satisfied by the differential equation $y' = 3y - 1$.

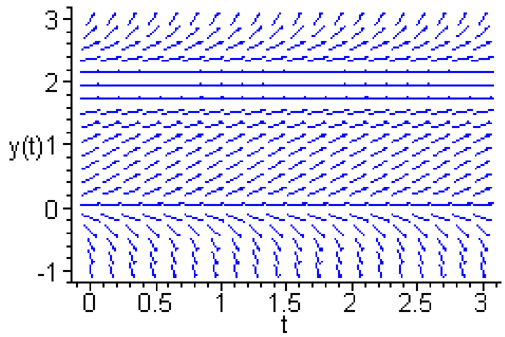
12.



Note that $y' = 0$ for $y = 0$ and $y = 5$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 5$. Based on the direction field, $y' > 0$ for $y > 5$; thus solutions with initial values *greater* than 5 diverge from the solution $y(t) = 5$. For $0 < y < 5$, the slopes are *negative*, and hence solutions with initial values *between* 0 and 5 all decrease toward the

solution $y(t) = 0$. For $y < 0$, the slopes are all *positive*; thus solutions with initial values
less than 0 approach the solution $y(t) = 0$.

14.

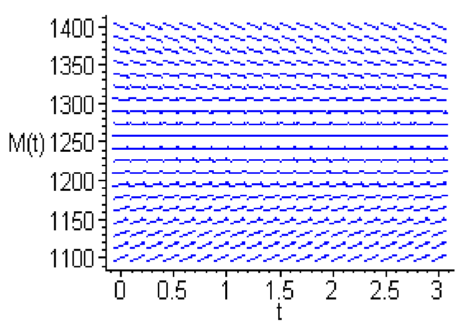


Observe that $y' = 0$ for $y = 0$ and $y = 2$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 2$. Based on the direction field, $y' > 0$ for $y > 2$; thus solutions with initial values *greater* than 2 diverge from $y(t) = 2$. For $0 < y < 2$, the slopes are also *positive*, and hence solutions with initial values *between* 0 and 2 all increase toward the solution
$y(t) = 2$. For $y < 0$, the slopes are all *negative*; thus solutions with initial values *less* than 0 diverge from the solution $y(t) = 0$.

16. (a) Let $M(t)$ be the total amount of the drug $($*in milligrams*$)$ in the patient's body at any
given time t (hrs). The drug is administered into the body at a *constant* rate of 500 mg/hr.
The rate at which the drug *leaves* the bloodstream is given by $0.4M(t)$. Hence the accumulation rate of the drug is described by the differential equation

$$\frac{dM}{dt} = 500 - 0.4\,M \quad (mg/hr)\,.$$

(b)



Based on the direction field, the amount of drug in the bloodstream approaches the equilibrium level of $1250\,mg$ $(within\,a\,few\,hours)$.

18. (a) Following the discussion in the text, the differential equation is

$$m\frac{dv}{dt} = mg - \gamma v^2$$

or equivalently,

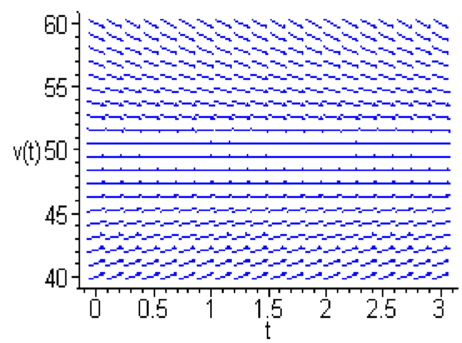
$$\frac{dv}{dt} = g - \frac{\gamma}{m}v^2 .$$

(b) After a long time, $\frac{dv}{dt} \approx 0$. Hence the object attains a *terminal velocity* given by

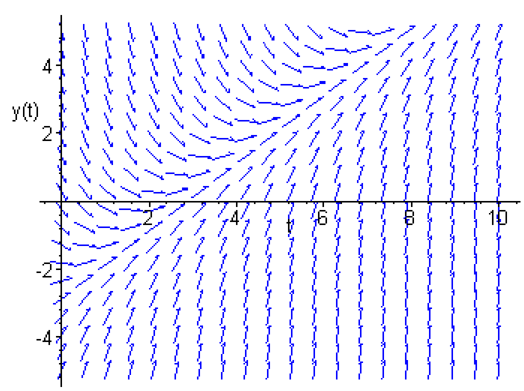
$$v_\infty = \sqrt{\frac{mg}{\gamma}} .$$

(c) Using the relation $\gamma\, v_\infty^2 = mg$, the required *drag coefficient* is $\gamma = 0.0408\, kg/sec$.

(d)

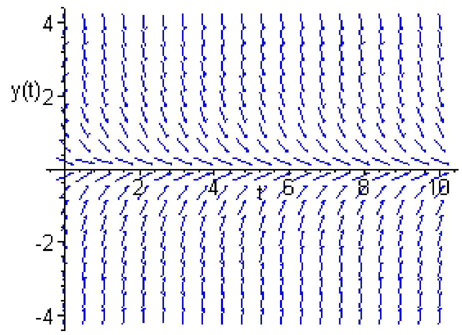


19.



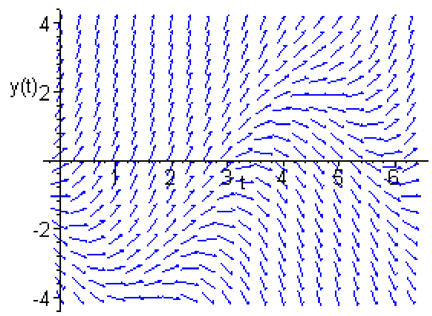
All solutions appear to approach a linear asymptote $(with\, slope\, equal\, to\, 1)$. It is easy to verify that $y(t) = t - 3$ is a solution.

20.



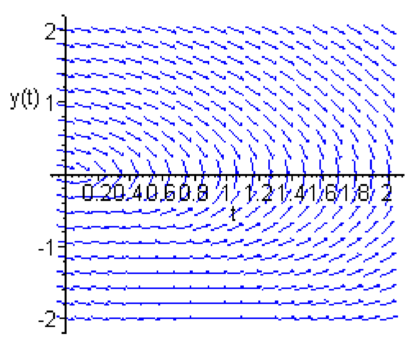
All solutions approach the equilibrium solution $y(t) = 0$.

23.



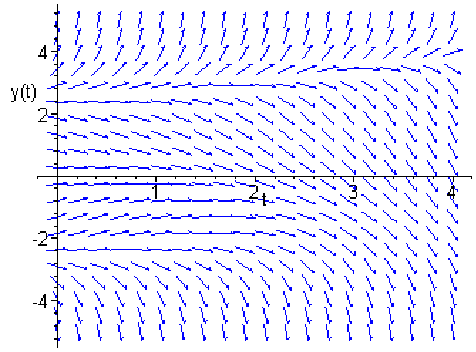
All solutions appear to *diverge* from the sinusoid $y(t) = -\frac{3}{\sqrt{2}} sin(t + \frac{\pi}{4}) - 1$, which is also a solution corresponding to the initial value $y(0) = -5/2$.

25.

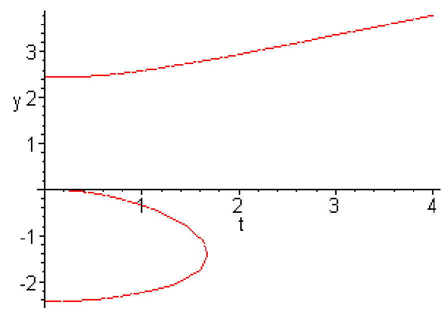


All solutions appear to converge to $y(t) = 0$. First, the rate of change is small. The slopes
eventually increase very rapidly in *magnitude*.

26.



The direction field is rather complicated. Nevertheless, the collection of points at which the slope field is *zero*, is given by the implicit equation $y^3 - 6y = 2t^2$. The graph of these points is shown below:



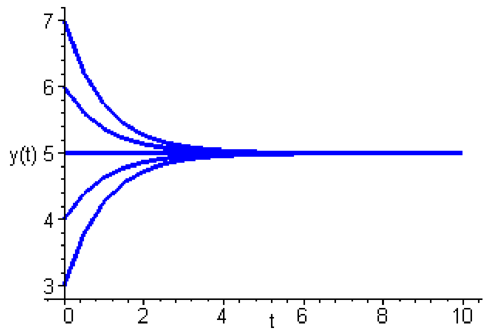
The *y-intercepts* of these curves are at $y = 0\,, \pm\sqrt{6}$. It follows that for solutions with initial values $y > \sqrt{6}$, all solutions increase without bound. For solutions with initial values in the range $y < -\sqrt{6}$ *and* $0 < y < \sqrt{6}$, the slopes remain *negative*, and hence
these solutions decrease without bound. Solutions with initial conditions in the range $-\sqrt{6} < y < 0$ initially increase. Once the solutions reach the critical value, given by the equation $y^3 - 6y = 2t^2$, the slopes become negative and *remain* negative. These solutions eventually decrease without bound.

Section 1.2

$1(a)$ The differential equation can be rewritten as

$$\frac{dy}{5-y} = dt\,.$$

Integrating both sides of this equation results in $-ln|5-y| = t + c_1$, or equivalently, $5 - y = c\,e^{-t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{-t}$.

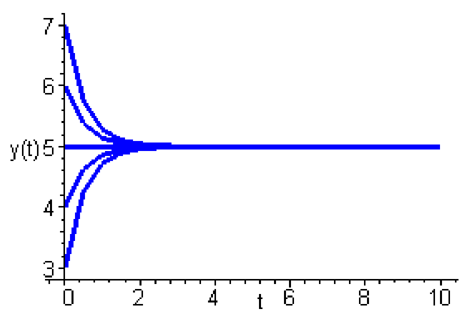


All solutions appear to converge to the equilibrium solution $y(t) = 5$.

$1(c)$. Rewrite the differential equation as

$$\frac{dy}{10-2y} = dt\,.$$

Integrating both sides of this equation results in $-\frac{1}{2}ln|10-2y| = t + c_1$, or equivalently,
$5 - y = c\,e^{-2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{-2t}$.

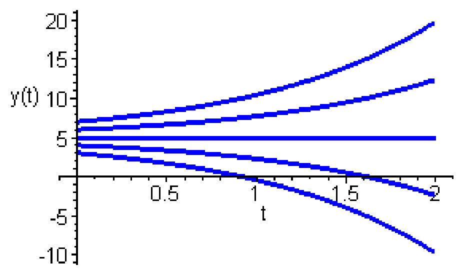


All solutions appear to converge to the equilibrium solution $y(t) = 5$, but at a *faster* rate than in Problem 1a .

$2(a)$. The differential equation can be rewritten as

$$\frac{dy}{y-5} = dt\,.$$

Integrating both sides of this equation results in $ln|y-5| = t + c_1$, or equivalently, $y - 5 = c\,e^t$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^t$.

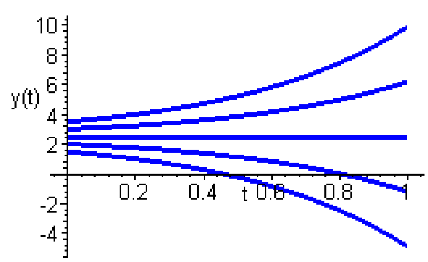


All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

$2(b)$. Rewrite the differential equation as

$$\frac{dy}{2y-5} = dt\,.$$

Integrating both sides of this equation results in $\frac{1}{2}ln|2y-5| = t + c_1$, or equivalently, $2y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 2y_0 - 5$. Hence the solution is $y(t) = 2.5 + (y_0 - 2.5)e^{2t}$.

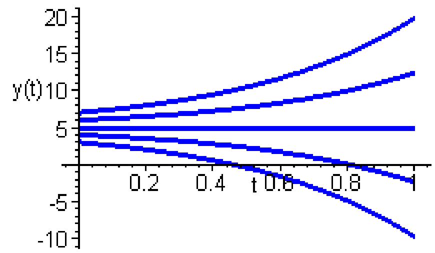


All solutions appear to diverge from the equilibrium solution $y(t) = 2.5$.

$2(c)$. The differential equation can be rewritten as

$$\frac{dy}{2y-10} = dt\,.$$

Integrating both sides of this equation results in $\frac{1}{2}ln|2y-10| = t + c_1$, or equivalently, $y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{2t}$.



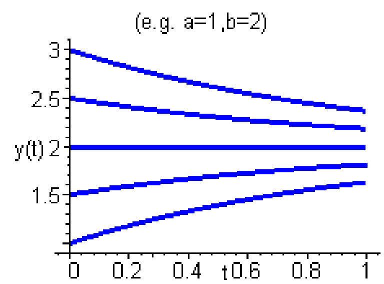
All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

$3(a)$. Rewrite the differential equation as

$$\frac{dy}{b - ay} = dt\,,$$

which is valid for $y \neq b\,/a$. Integrating both sides results in $\frac{-1}{a}ln|b - ay| = t + c_1$, or equivalently, $b - ay = c\,e^{-at}$. Hence the general solution is $y(t) = (b - c\,e^{-at})/a$. Note that if $y = b/a$, then $dy/dt = 0$, and $y(t) = b/a$ is an equilibrium solution.

(b)



(i) As a increases, the equilibrium solution gets closer to $y(t) = 0$, from above. Furthermore, the *convergence rate* of all solutions, that is, a, also increases.
(ii) As b increases, then the equilibrium solution $y(t) = b/a$ also becomes larger. In this case, the convergence rate remains the same.
(iii) If a and b both increase (*but* $b/a =$ *constant*), then the equilibrium solution $y(t) = b/a$ remains the same, but the *convergence rate* of all solutions increases.

$5(a)$. Consider the simpler equation $dy_1/dt = -ay_1$. As in the previous solutions, rewrite the equation as

$$\frac{dy_1}{y_1} = -a\,dt\,.$$

Integrating both sides results in $y_1(t) = c\,e^{-at}$.
(b). Now set $y(t) = y_1(t) + k$, and substitute into the original differential equation. We find that

$$-\,ay_1 + 0 = \,-\,a(y_1 + k) + b\,.$$

That is, $-\,ak + b = 0$, and hence $k = b/a$.
(c). The general solution of the differential equation is $y(t) = c\,e^{-at} + b/a$. This is exactly the form given by Eq. (17) in the text. Invoking an initial condition $y(0) = y_0$, the solution may also be expressed as $y(t) = b/a + (y_0 - b/a)e^{-at}$.

$6(a)$. The general solution is $p(t) = 900 + c\,e^{t/2}$, that is, $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the specific solution becomes $p(t) = 900 - 50e^{t/2}$. This solution is a *decreasing* exponential, and hence the time of extinction is equal to the number of months
it takes, say t_f, for the population to reach *zero*. Solving $900 - 50e^{t_f/2} = 0$, we find that $t_f = 2\,ln(900/50) = 5.78$ *months*.
(b) The solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, is a *decreasing* exponential as long as $p_0 < 900$. Hence $900 + (p_0 - 900)e^{t_f/2} = 0$ has only *one* root, given by

$$t_f = 2\,ln\left(\frac{900}{900 - p_0}\right).$$

(c). The answer in part (b) is a general equation relating time of extinction to the value of
the initial population. Setting $t_f = 12$ *months*, the equation may be written as

$$\frac{900}{900 - p_0} = e^6,$$

which has solution $p_0 = 897.7691$. Since p_0 is the initial population, the appropriate answer is $p_0 = 898$ *mice*.

$7(a)$. The general solution is $p(t) = p_0\,e^{rt}$. Based on the discussion in the text, time t is measured in *months*. Assuming 1 *month* $= 30$ *days*, the hypothesis can be expressed as $p_0\,e^{r\cdot 1} = 2p_0$. Solving for the rate constant, $r = ln(2)$, with units of *per month*.

(b). N *days* $= N/30$ *months*. The hypothesis is stated mathematically as $p_0 e^{rN/30} = 2p_0$.
It follows that $rN/30 = ln(2)$, and hence the rate constant is given by $r = 30\,ln(2)/N$. The units are understood to be *per month*.

$9(a)$. Assuming *no air resistance*, with the positive direction taken as *downward*, Newton's
Second Law can be expressed as

$$m\frac{dv}{dt} = mg$$

in which g is the *gravitational constant* measured in appropriate units. The equation can be

written as $dv/dt = g$, with solution $v(t) = gt + v_0$. The object is released with an initial
velocity v_0.

(b). Suppose that the object is released from a height of h *units* above the ground. Using the
fact that $v = dx/dt$, in which x is the *downward displacement* of the object, we obtain the
differential equation for the displacement as $dx/dt = gt + v_0$. With the origin placed at the point of release, direct integration results in $x(t) = gt^2/2 + v_0\,t$. Based on the chosen
coordinate system, the object reaches the ground when $x(t) = h$. Let $t = T$ be the time that it takes the object to reach the ground. Then $gT^2/2 + v_0T = h$. Using the quadratic
formula to solve for T,

$$T = \frac{-\,v_0 \pm \sqrt{v_0 + 2gh}}{g}\,.$$

The *positive* answer corresponds to the time it takes for the object to fall to the ground. The
negative answer represents a previous instant at which the object could have been launched
upward (*with the same impact speed*), only to ultimately fall downward with speed v_0, from a height of h *units* above the ground.

(c). The impact speed is calculated by substituting $t = T$ into $v(t)$ in part (a). That is, $v(T) = \sqrt{v_0 + 2gh}$.

10$(a$,$b)$. The general solution of the differential equation is $Q(t) = c\,e^{-rt}$. Given that $Q(0) = 100$ *mg*, the value of the constant is given by $c = 100$. Hence the amount of thorium-234 present at any time is given by $Q(t) = 100\,e^{-rt}$. Furthermore, based on the hypothesis, setting $t = 1$ results in $82.04 = 100\,e^{-r}$. Solving for the rate constant, we find that $r = -\,ln(82.04/100) = .19796$/*week* or $r = .02828$/*day*.

(c). Let T be the time that it takes the isotope to decay to *one-half* of its original amount.
From part (a), it follows that $50 = 100\,e^{-rT}$, in which $r = .19796$/*week*. Taking the natural logarithm of both sides, we find that $T = 3.5014$ *weeks* or $T = 24.51$ $days$.

11. The general solution of the differential equation $dQ/dt = -\,r\,Q$ is $Q(t) = Q_0e^{-rt}$, in which $Q_0 = Q(0)$ is the initial amount of the substance. Let τ be the time that it takes the substance to decay to *one-half* of its original amount, Q_0. Setting $t = \tau$ in the solution,
we have $0.5\,Q_0 = Q_0e^{-r\tau}$. Taking the natural logarithm of both sides, it follows that $-\,r\tau = ln(0.5)$ or $r\tau = ln\,2$.

12. The differential equation governing the amount of radium-226 is $dQ/dt = -r\,Q$, with solution $Q(t) = Q(0)e^{-rt}$. Using the result in Problem 11, and the fact that the half-life $\tau = 1620$ *years*, the decay rate is given by $r = ln(2)/1620$ *per year*. The amount of radium-226, after t years, is therefore $Q(t) = Q(0)e^{-0.00042786t}$. Let T be the time that it takes the isotope to decay to $3/4$ of its original amount. Then setting $t = T$,
and $Q(T) = \frac{3}{4}Q(0)$, we obtain $\frac{3}{4}Q(0) = Q(0)e^{-0.00042786T}$. Solving for the decay time, it follows that $-0.00042786\,T = ln(3/4)$ or $T = 672.36$ *years*.

13. The solution of the differential equation, with $Q(0) = 0$, is
$Q(t) = CV(1 - e^{-t/CR})$.
As $t \to \infty$, the exponential term vanishes, and hence the limiting value is $Q_L = CV$.

14(a). The *accumulation* rate of the chemical is $(0.01)(300)$ *grams per hour*. At any given time t, the *concentration* of the chemical in the pond is $Q(t)/10^6$ *grams per gallon*.
Consequently, the chemical *leaves* the pond at a rate of $(3 \times 10^{-4})Q(t)$ *grams per hour*. Hence, the rate of change of the chemical is given by

$$\frac{dQ}{dt} = 3 - 0.0003\,Q(t) \ \ \textit{gm/hr}\,.$$

Since the pond is initially free of the chemical, $Q(0) = 0$.

(b). The differential equation can be rewritten as

$$\frac{dQ}{10000 - Q} = 0.0003\,dt\,.$$

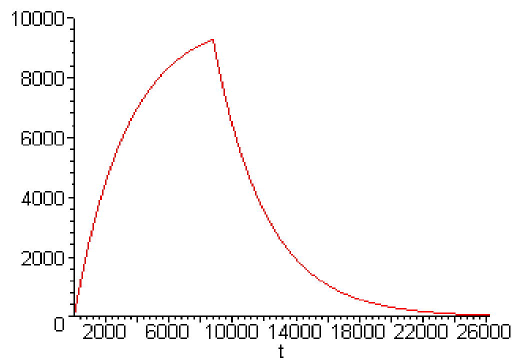
Integrating both sides of the equation results in $-ln|10000 - Q| = 0.0003t + C$. Taking
the natural logarithm of both sides gives $10000 - Q = c\,e^{-0.0003t}$. Since $Q(0) = 0$, the value of the constant is $c = 10000$. Hence the amount of chemical in the pond at any time
is $Q(t) = 10000(1 - e^{-0.0003t})$ *grams*. Note that 1 *year* $= 8760$ *hours*. Setting $t = 8760$, the amount of chemical present after *one year* is $Q(8760) = 9277.77$ *grams*, that is, 9.27777 *kilograms*.

(c). With the *accumulation* rate now equal to *zero*, the governing equation becomes $dQ/dt = -0.0003\,Q(t)$ *gm/hr*. Resetting the time variable, we now assign the new initial value as $Q(0) = 9277.77$ *grams*.

(d). The solution of the differential equation in Part (c) is $Q(t) = 9277.77\,e^{-0.0003t}$. Hence, one year *after* the source is removed, the amount of chemical in the pond is $Q(8760) = 670.1$ *grams*.

(e). Letting t be the amount of time after the source is removed, we obtain the equation $10 = 9277.77\, e^{-0.0003t}$. Taking the natural logarithm of both sides, $-\, 0.0003\, t =$ $= ln(10/9277.77)$ or $t = 22,776$ *hours* $= 2.6$ *years* .

(f)



$15(a)$. It is assumed that dye is no longer entering the pool. In fact, the rate at which the dye *leaves* the pool is $200 \cdot [q(t)/60000]$ *kg/min* $= 200(60/1000)[q(t)/60]$ *gm per hour* .

Hence the equation that governs the amount of dye in the pool is

$$\frac{dq}{dt} = -\, 0.2\, q \quad (gm/hr)\,.$$

The initial amount of dye in the pool is $q(0) = 5000$ *grams* .

(b). The solution of the governing differential equation, with the specified initial value, is $q(t) = 5000\, e^{-0.2\, t}$.

(c). The amount of dye in the pool after four hours is obtained by setting $t = 4$. That is, $q(4) = 5000\, e^{-0.8} = 2246.64$ *grams* . Since size of the pool is $60,000$ *gallons* , the *concentration* of the dye is 0.0374 *grams/gallon* .

(d). Let T be the time that it takes to reduce the concentration level of the dye to 0.02 *grams/gallon* . At that time, the amount of dye in the pool is $1,200$ *grams* . Using the answer in part (b), we have $5000\, e^{-0.2\, T} = 1200$. Taking the natural logarithm of both sides of the equation results in the required time $T = 7.14$ *hours* .

(e). Note that $0.2 = 200/1000$. Consider the differential equation

$$\frac{dq}{dt} = -\, \frac{r}{1000}\, q\,.$$

Here the parameter r corresponds to the *flow rate*, measured in *gallons per minute* . Using the same initial value, the solution is given by $q(t) = 5000\, e^{-r\, t/1000}$. In order to determine the appropriate flow rate, set $t = 4$ and $q = 1200$. (Recall that 1200 *gm* of dye has a concentration of 0.02 *gm/gal*). We obtain the equation $1200 = 5000\, e^{-r/250}$. Taking the natural logarithm of both sides of the equation results in the required flow rate $r = 357$ *gallons per minute*.

Section 1.3

1. The differential equation is *second order*, since the highest derivative in the equation is of order *two*. The equation is *linear*, since the left hand side is a linear function of y and
its derivatives.

3. The differential equation is *fourth order*, since the highest derivative of the function y is of order *four*. The equation is also *linear*, since the terms containing the dependent variable is linear in y and its derivatives.

4. The differential equation is *first order*, since the only derivative is of order *one*. The dependent variable is *squared*, hence the equation is *nonlinear*.

5. The differential equation is *second order*. Furthermore, the equation is *nonlinear*, since the dependent variable y is an argument of the *sine function*, which is *not* a linear function.

7. $y_1(t) = e^t \Rightarrow y_1'(t) = y_1''(t) = e^t$. Hence $y_1'' - y_1 = 0$.
Also, $y_2(t) = cosh\,t \Rightarrow y_1'(t) = sinh\,t$ and $y_2''(t) = cosh\,t$. Thus $y_2'' - y_2 = 0$.

9. $y(t) = 3t + t^2 \Rightarrow y'(t) = 3 + 2t$. Substituting into the differential equation, we have $t(3+2t) - (3t+t^2) = 3t + 2t^2 - 3t - t^2 = t^2$. Hence the given function is a solution.

10. $y_1(t) = t/3 \Rightarrow y_1'(t) = 1/3$ and $y_1''(t) = y_1'''(t) = y_1''''(t) = 0$. Clearly, $y_1(t)$ is a solution. Likewise, $y_2(t) = e^{-t} + t/3 \Rightarrow y_2'(t) = -e^{-t} + 1/3$, $y_2''(t) = e^{-t}$, $y_2'''(t) = -e^{-t}$, $y_2''''(t) = e^{-t}$. Substituting into the left hand side of the equation, we find that $e^{-t} + 4(-e^{-t}) + 3(e^{-t} + t/3) = e^{-t} - 4e^{-t} + 3e^{-t} + t = t$. Hence both functions are solutions of the differential equation.

11. $y_1(t) = t^{1/2} \Rightarrow y_1'(t) = t^{-1/2}/2$ and $y_1''(t) = -t^{-3/2}/4$. Substituting into the left hand side of the equation, we have

$$\begin{aligned} 2t^2\left(-t^{-3/2}/4\right) + 3t\left(t^{-1/2}/2\right) - t^{1/2} &= -t^{1/2}/2 + 3\,t^{1/2}/2 - t^{1/2} \\ &= 0 \end{aligned}$$

Likewise, $y_2(t) = t^{-1} \Rightarrow y_2'(t) = -t^{-2}$ and $y_2''(t) = 2\,t^{-3}$. Substituting into the left hand side of the differential equation, we have $2t^2(2\,t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4\,t^{-1} -$ $- 3\,t^{-1} - t^{-1} = 0$. Hence both functions are solutions of the differential equation.

12. $y_1(t) = t^{-2} \Rightarrow y_1'(t) = -2t^{-3}$ and $y_1''(t) = 6\,t^{-4}$. Substituting into the left hand side of the differential equation, we have $t^2(6\,t^{-4}) + 5t(-2t^{-3}) + 4\,t^{-2} = 6\,t^{-2} -$ $- 10\,t^{-2} + 4\,t^{-2} = 0$. Likewise, $y_2(t) = t^{-2}ln\,t \Rightarrow y_2'(t) = t^{-3} - 2t^{-3}ln\,t$ and $y_2''(t) = -5\,t^{-4} + 6\,t^{-4}ln\,t$. Substituting into the left hand side of the equation, we have $t^2(-5\,t^{-4} + 6\,t^{-4}ln\,t) + 5t(t^{-3} - 2t^{-3}ln\,t) + 4(t^{-2}ln\,t) = -5\,t^{-2} + 6\,t^{-2}ln\,t +$

$+\,5\,t^{-2}-10\,t^{-2}ln\,t+4\,t^{-2}ln\,t=0$. Hence both functions are solutions of the differential equation.

13. $y(t)=(cos\,t)ln\,cos\,t+t\,sin\,t\Rightarrow y'(t)=\,-\,(sin\,t)ln\,cos\,t+t\,cos\,t$ and $y''(t)=\,-\,(cos\,t)ln\,cos\,t-t\,sin\,t+sec\,t$. Substituting into the left hand side of the differential equation, we have $(\,-\,(cos\,t)ln\,cos\,t-t\,sin\,t+sec\,t)+(cos\,t)ln\,cos\,t+$ $+\,t\,sin\,t=\,-\,(cos\,t)ln\,cos\,t-t\,sin\,t+sec\,t+(cos\,t)ln\,cos\,t+t\,sin\,t=sec\,t$. Hence the function $y(t)$ is a solution of the differential equation.

15. Let $y(t)=e^{rt}$. Then $y''(t)=r^2e^{rt}$, and substitution into the differential equation results in $r^2e^{rt}+2\,e^{rt}=0$. Since $e^{rt}\neq 0$, we obtain the algebraic equation $r^2+2=0$. The roots of this equation are $r_{1,2}=\pm\,i\sqrt{2}$.

17. $y(t)=e^{rt}\Rightarrow y'(t)=r\,e^{rt}$ and $y''(t)=r^2e^{rt}$. Substituting into the differential equation, we have $r^2e^{rt}+re^{rt}-6\,e^{rt}=0$. Since $e^{rt}\neq 0$, we obtain the algebraic equation $r^2+r-6=0$, that is, $(r-2)(r+3)=0$. The roots are $r_{1,2}=\,-\,3\,,2$.

18. Let $y(t)=e^{rt}$. Then $y'(t)=re^{rt}$, $y''(t)=r^2e^{rt}$ and $y'''(t)=r^3e^{rt}$. Substituting the derivatives into the differential equation, we have $r^3e^{rt}-3r^2e^{rt}+2re^{rt}=0$. Since $e^{rt}\neq 0$, we obtain the algebraic equation $r^3-3r^2+2r=0$. By inspection, it follows that $r(r-1)(r-2)=0$. Clearly, the roots are $r_1=0$, $r_2=1$ and $r_3=2$.

20. $y(t)=t^r\Rightarrow y'(t)=r\,t^{r-1}$ and $y''(t)=r(r-1)t^{r-2}$. Substituting the derivatives into the differential equation, we have $t^2[r(r-1)t^{r-2}]-4t(r\,t^{r-1})+4\,t^r=0$. After some algebra, it follows that $r(r-1)t^r-4r\,t^r+4\,t^r=0$. For $t\neq 0$, we obtain the algebraic equation $r^2-5r+4=0$. The roots of this equation are $r_1=1$ and $r_2=4$.

21. The order of the partial differential equation is *two*, since the highest derivative, in fact each one of the derivatives, is of *second order*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

23. The partial differential equation is *fourth order*, since the highest derivative, and in fact each of the derivatives, is of order *four*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

24. The partial differential equation is *second order*, since the highest derivative of the function $u(x,y)$ is of order *two*. The equation is *nonlinear*, due to the product $u\cdot u_x$ on the left hand side of the equation.

25. $u_1(x,y)=cos\,x\,cosh\,y\Rightarrow\frac{\partial^2u_1}{\partial x^2}=\,-\,cos\,x\,cosh\,y$ and $\frac{\partial^2u_1}{\partial y^2}=cos\,x\,cosh\,y$. It is evident that $\frac{\partial^2u_1}{\partial x^2}+\frac{\partial^2u_1}{\partial y^2}=0$. Likewise, given $u_2(x,y)=ln(x^2+y^2)$, the second derivatives are

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$
$$\frac{\partial^2 u_2}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}$$

Adding the partial derivatives,

$$\begin{aligned}\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} &= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\ &= \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} \\ &= 0\,.\end{aligned}$$

Hence $u_2(x,y)$ is also a solution of the differential equation.

27. Let $u_1(x,t) = sin\,\lambda x\, sin\,\lambda a t$. Then the second derivatives are

$$\frac{\partial^2 u_1}{\partial x^2} = -\lambda^2 sin\,\lambda x\, sin\,\lambda a t$$
$$\frac{\partial^2 u_1}{\partial t^2} = -\lambda^2 a^2 sin\,\lambda x\, sin\,\lambda a t$$

It is easy to see that $a^2 \frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2}$. Likewise, given $u_2(x,t) = sin(x - at)$, we have

$$\frac{\partial^2 u_2}{\partial x^2} = -\,sin(x-at)$$
$$\frac{\partial^2 u_2}{\partial t^2} = -\,a^2 sin(x-at)$$

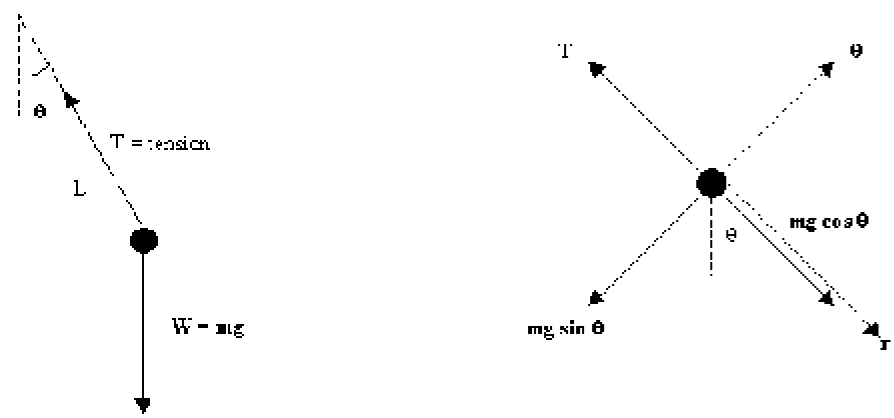
Clearly, $u_2(x,t)$ is also a solution of the partial differential equation.

28. Given the function $u(x,t) = \sqrt{\pi/t}\; e^{-x^2/4\alpha^2 t}$, the partial derivatives are

$$u_{xx} = -\frac{\sqrt{\pi/t}\; e^{-x^2/4\alpha^2 t}}{2\alpha^2 t} + \frac{\sqrt{\pi/t}\; x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^4 t^2}$$
$$u_t = -\frac{\sqrt{\pi t}\; e^{-x^2/4\alpha^2 t}}{2t^2} + \frac{\sqrt{\pi}\; x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2 \sqrt{t}}$$

It follows that $\alpha^2\, u_{xx} = u_t = -\frac{\sqrt{\pi}\,(2\alpha^2 t - x^2) e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2 \sqrt{t}}$.
Hence $u(x,t)$ is a solution of the partial differential equation.

29(a).



(b). The path of the particle is a circle, therefore *polar coordinates* are intrinsic to the problem. The variable r is radial distance and the angle θ is measured from the vertical. Newton's Second Law states that $\sum \mathbf{F} = m\mathbf{a}$. In the *tangential* direction, the equation of motion may be expressed as $\sum F_\theta = m\, a_\theta$, in which the *tangential acceleration*, that is, the linear acceleration *along* the path is $a_\theta = L\, d^2\theta/dt^2$. (a_θ is *positive* in the direction of increasing θ). Since the only force acting in the tangential direction is the component of weight, the equation of motion is

$$-\,mg\, sin\,\theta = mL\frac{d^2\theta}{dt^2}\,.$$

(Note that the equation of motion in the radial direction will include the tension in the rod).

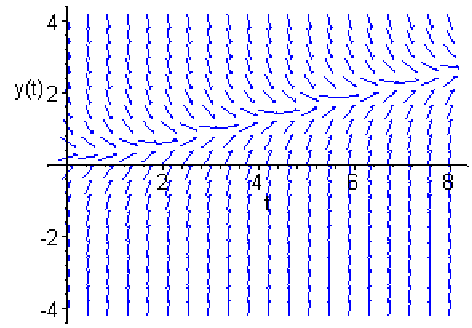
(c). Rearranging the terms results in the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\,\theta = 0\,.$$

Chapter Two

Section 2.1

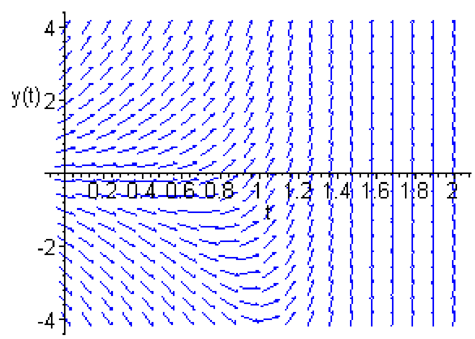
$1(a)$.



(b). Based on the direction field, all solutions seem to converge to a specific increasing function.

(c). The integrating factor is $\mu(t) = e^{3t}$, and hence $y(t) = t/3 - 1/9 + e^{-2t} + c\,e^{-3t}$. It follows that all solutions converge to the function $y_1(t) = t/3 - 1/9$.

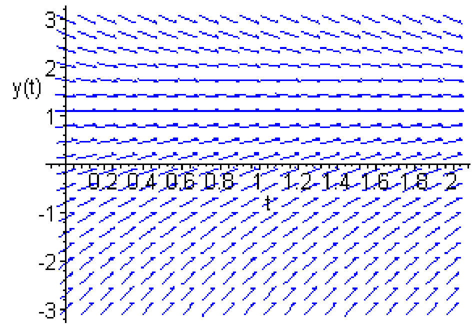
$2(a)$.



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = e^{-2t}$, and hence $y(t) = t^3e^{2t}/3 + c\,e^{2t}$. It is evident that all solutions increase at an exponential rate.

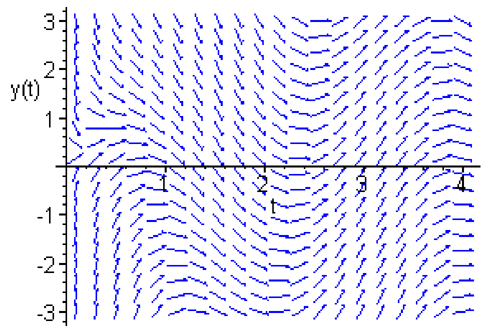
$3(a)$



(b). All solutions seem to converge to the function $y_0(t) = 1$.

(c). The integrating factor is $\mu(t) = e^{2t}$, and hence $y(t) = t^2e^{-t}/2 + 1 + c\,e^{-t}$. It is clear that all solutions converge to the specific solution $y_0(t) = 1$.

4(a).



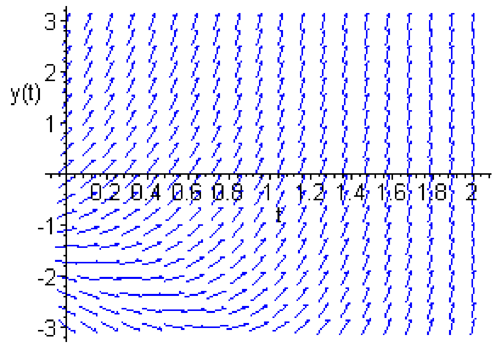
(b). Based on the direction field, the solutions eventually become oscillatory.

(c). The integrating factor is $\mu(t) = t$, and hence the general solution is

$$y(t) = \frac{3cos(2t)}{4t} + \frac{3}{2}sin(2t) + \frac{c}{t}$$

in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_1(t) = 3sin(2t)/2$.

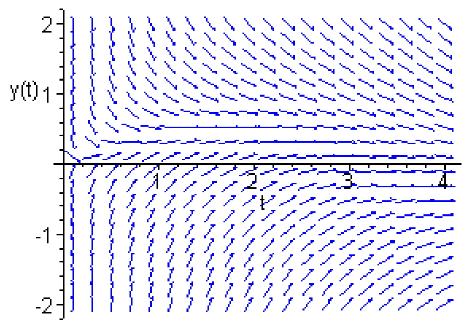
5(a).



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = exp(-\int 2dt) = e^{-2t}$. The differential equation can
be written as $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, that is, $(e^{-2t}y)' = 3e^{-t}$. Integration of both sides of the equation results in the general solution $y(t) = -3e^{t} + c\,e^{2t}$. It follows that all solutions will increase exponentially.

6(a)



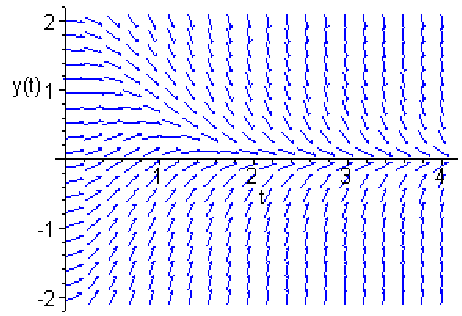
(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). The integrating factor is $\mu(t) = t^2$, and hence the general solution is

$$y(t) = -\frac{cos(t)}{t} + \frac{sin(2t)}{t^2} + \frac{c}{t^2}$$

in which c is an arbitrary constant. As t becomes large, all solutions converge to the function $y_0(t) = 0$.

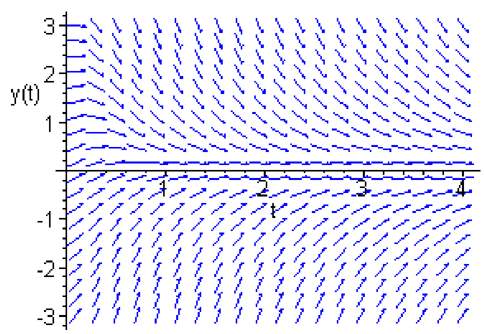
7(a).



(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). The integrating factor is $\mu(t) = exp(t^2)$, and hence $y(t) = t^2 e^{-t^2} + c\, e^{-t^2}$. It is clear that all solutions converge to the function $y_0(t) = 0$.

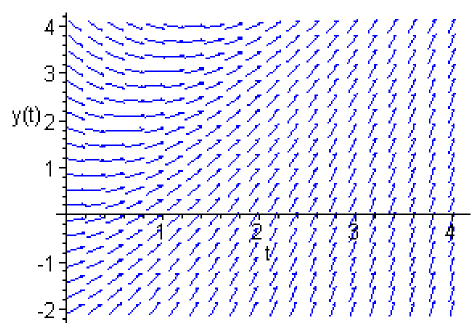
$8(a)$



(b). All solutions seem to converge to the function $y_0(t) = 0$.

(c). Since $\mu(t) = (1+t^2)^2$, the general solution is $y(t) = [tan^{-1}(t) + C]/(1+t^2)^2$. It follows that all solutions converge to the function $y_0(t) = 0$.

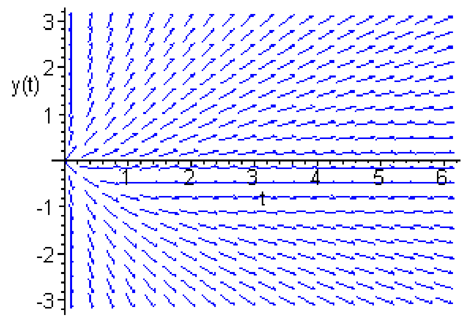
$9(a)$.



(b). All slopes *eventually* become positive, hence all solutions will increase without bound.

(c). The integrating factor is $\mu(t) = exp\left(\int \frac{1}{2}\,dt\right) = e^{t/2}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t\,e^{t/2}/2$, that is, $(e^{t/2}\,y/2)' = 3t\,e^{t/2}/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t - 6 + c\,e^{-t/2}$. All solutions approach the specific solution $y_0(t) = 3t - 6$.

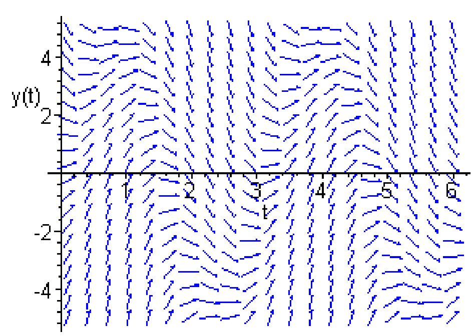
10(a).



(b). For $y > 0$, the slopes are *all* positive, and hence the corresponding solutions increase
without bound. For $y < 0$, almost all solutions have negative slopes, and hence solutions tend to decrease without bound.

(c). First divide both sides of the equation by t. From the resulting *standard form*, the integrating factor is $\mu(t) = exp\left(-\int \frac{1}{t}\,dt\right) = 1/t$. The differential equation can be written as $y'/t - y/t^2 = t\,e^{-t}$, that is, $(y/t)' = t\,e^{-t}$. Integration leads to the general solution $y(t) = -te^{-t} + c\,t$. For $c \neq 0$, solutions *diverge*, as implied by the direction field. For the case $c = 0$, the specific solution is $y(t) = -te^{-t}$, which evidently approaches *zero* as $t \to \infty$.

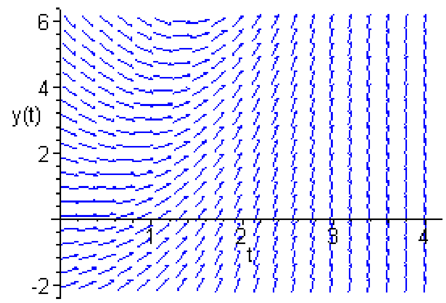
11(a).



(b). The solutions appear to be oscillatory.

(c). The integrating factor is $\mu(t) = e^t$, and hence $y(t) = sin(2t) - 2\,cos(2t) + c\,e^{-t}$. It is evident that all solutions converge to the specific solution $y_0(t) = sin(2t) - 2$ $cos(2t)$.

$12(a)$.



(b). All solutions *eventually* have positive slopes, and hence increase without bound.

(c). The integrating factor is $\mu(t) = e^{2t}$. The differential equation can be written as $e^{t/2}y' + e^{t/2}y/2 = 3t^2/2$, that is, $\left(e^{t/2}\,y/2\right)' = 3t^2/2$. Integration of both sides of the equation results in the general solution $y(t) = 3t^2 - 12t + 24 + c\,e^{-t/2}$. It follows that all solutions converge to the specific solution $y_0(t) = 3t^2 - 12t + 24$.

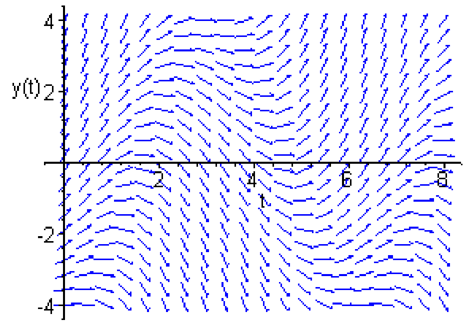
14. The integrating factor is $\mu(t) = e^{2t}$. After multiplying both sides by $\mu(t)$, the equation can be written as $\left(e^{2t}\,y\right)' = t$. Integrating both sides of the equation results in the general solution $y(t) = t^2e^{-2t}/2 + c\,e^{-2t}$. Invoking the specified condition, we require that $e^{-2}/2 + c\,e^{-2} = 0$. Hence $c = -1/2$, and the solution to the initial value problem is $y(t) = (t^2 - 1)e^{-2t}/2$.

16. The integrating factor is $\mu(t) = exp\left(\int \frac{2}{t}\,dt\right) = t^2$. Multiplying both sides by $\mu(t)$, the equation can be written as $(t^2\,y)' = cos(t)$. Integrating both sides of the equation results in the general solution $y(t) = sin(t)/t^2 + c\,t^{-2}$. Substituting $t = \pi$ and setting the value equal to *zero* gives $c = 0$. Hence the specific solution is $y(t) = sin(t)/t^2$.

17. The integrating factor is $\mu(t) = e^{-2t}$, and the differential equation can be written as $\left(e^{-2t}\,y\right)' = 1$. Integrating, we obtain $e^{-2t}\,y(t) = t + c$. Invoking the specified initial condition results in the solution $y(t) = (t+2)e^{2t}$.

19. After writing the equation in *standard form*, we find that the integrating factor is $\mu(t) = exp\left(\int \frac{4}{t}\,dt\right) = t^4$. Multiplying both sides by $\mu(t)$, the equation can be written as $\left(t^4\,y\right)' = t\,e^{-t}$. Integrating both sides results in $t^4y(t) = -(t+1)e^{-t} + c$. Letting $t = -1$ and setting the value equal to *zero* gives $c = 0$. Hence the specific solution of the initial value problem is $y(t) = -\left(t^{-3} + t^{-4}\right)e^{-t}$.

$21(a)$.

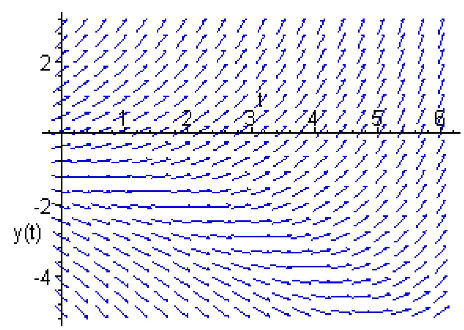


The solutions appear to diverge from an *apparent* oscillatory solution. From the direction field, the critical value of the initial condition seems to be $a_0 = -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, solutions decrease without bound.

(b). The integrating factor is $\mu(t) = e^{-t/2}$. The general solution of the differential equation is $y(t) = (8sin(t) - 4cos(t))/5 + c\,e^{t/2}$. The solution is sinusoidal as long as $c = 0$. The *initial value* of this sinusoidal solution is $a_0 = (8sin(0) - 4cos(0))/5 = -4/5$.

(c). See part (b).

$22(a)$.



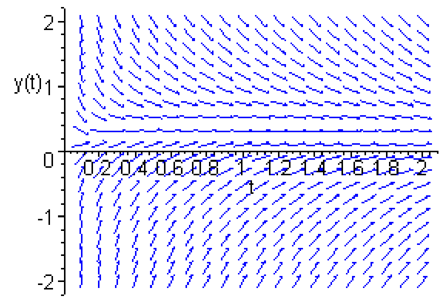
All solutions appear to *eventually* increase without bound. The solutions *initially* increase or decrease, depending on the initial value a. The critical value seems to be $a_0 = -1$.

(b). The integrating factor is $\mu(t) = e^{-t/2}$, and the general solution of the differential equation is $y(t) = -3e^{t/3} + c\,e^{t/2}$. Invoking the initial condition $y(0) = a$, the solution may also be expressed as $y(t) = -3e^{t/3} + (a+3)\,e^{t/2}$. Differentiating, follows that $y'(0) = -1 + (a+3)/2 = (a+1)/2$. The critical value is evidently $a_0 = -1$.

(c). For $a_0 = -1$, the solution is $y(t) = -3e^{t/3} + 2\,e^{t/2}$, which (for large t) is dominated by the term containing $e^{t/2}$.

is $y(t) = (8sin(t) - 4cos(t))/5 + c\,e^{t/2}$.

$23(a)$.

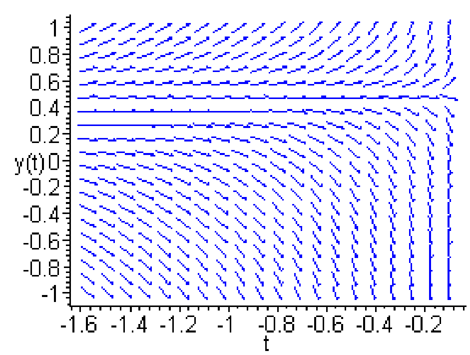


As $t \to 0$, solutions increase without bound if $y(1) = a > .4$, and solutions decrease without bound if $y(1) = a < .4$.

(b). The integrating factor is $\mu(t) = exp\left(\int \frac{t+1}{t} dt\right) = t\,e^t$. The general solution of the differential equation is $y(t) = t\,e^{-t} + c\,e^{-t}/t$. Invoking the specified value $y(1) = a$, we have $1 + c = a\,e$. That is, $c = a\,e - 1$. Hence the solution can also be expressed as $y(t) = t\,e^{-t} + (a\,e - 1)\,e^{-t}/t$. For *small* values of t, the second term is dominant. Setting $a\,e - 1 = 0$, critical value of the parameter is $a_0 = 1/e$.

(c). For $a > 1/e$, solutions increase without bound. For $a < 1/e$, solutions decrease without bound. When $a = 1/e$, the solution is $y(t) = t\,e^{-t}$, which approaches 0 as $t \to 0$
.

$24(a)$.



As $t \to 0$, solutions increase without bound if $y(1) = a > .4$, and solutions decrease without bound if $y(1) = a < .4$.

(b). Given the initial condition, $y(-\pi/2) = a$, the solution is $y(t) = (a\,\pi^2/4 - cos\,t)/t$.
Since $\lim_{t\to 0} cos\,t = 1$, solutions increase without bound if $a > 4/\pi^2$, and solutions decrease without bound if $a < 4/\pi^2$. Hence the critical value is
$a_0 = 4/\pi^2 = 0.452847....$

(c). For $a = 4/\pi^2$, the solution is $y(t) = (1 - cos\,t)/t$, and $\lim_{t\to 0} y(t) = 1/2$. Hence the solution is bounded.

25. The integrating factor is $\mu(t) = exp\left(\int \frac{1}{2}dt\right) = e^{t/2}$. Therefore general solution is $y(t) = [4cos(t) + 8sin(t)]/5 + c\,e^{-t/2}$. Invoking the initial condition, the specific solution is $y(t) = [4cos(t) + 8sin(t) - 9\,e^{t/2}]/5$. Differentiating, it follows that

$$y'(t) = \left[-4sin(t) + 8cos(t) + 4.5\,e^{-t/2}\right]/5$$
$$y''(t) = \left[-4cos(t) - 8sin(t) - 2.25\,e^{-t/2}\right]/5$$

Setting $y'(t) = 0$, the first solution is $t_1 = 1.3643$, which gives the location of the *first* stationary point. Since $y''(t_1) < 0$, the first stationary point in a local *maximum*. The coordinates of the point are $(1.3643\,,.82008)$.

26. The integrating factor is $\mu(t) = exp\left(\int \frac{2}{3}dt\right) = e^{2t/3}$, and the differential equation can
be written as $\left(e^{2t/3}\,y\right)' = e^{2t/3} - t\,e^{2t/3}/2$. The general solution is $y(t) = (21 - 6t)/8 +$ $+\,c\,e^{-2t/3}$. Imposing the initial condition, we have $y(t) = (21 - 6t)/8 + (y_0 - 21/8)e^{-2t/3}$. Since the solution is smooth, the desired intersection will be a point of tangency. Taking the derivative, $y'(t) = -3/4 - (2y_0 - 21/4)e^{-2t/3}/3$. Setting $y'(t) = 0$, the solution is $t_1 = \frac{3}{2}ln[(21 - 8y_0)/9]$. Substituting into the solution, the respective *value* at the stationary point is $y(t_1) = \frac{3}{2} + \frac{9}{4}ln\,3 - \frac{9}{8}ln(21 - 8y_0)$. Setting this result equal to *zero*, we obtain the required initial value $y_0 = (21 - 9\,e^{4/3})/8 = -1.643$.

27. The integrating factor is $\mu(t) = e^{t/4}$, and the differential equation can be written as $\left(e^{t/4}\,y\right)' = 3\,e^{t/4} + 2\,e^{t/4}cos(2t)$. The general solution is

$$y(t) = 12 + [8cos(2t) + 64sin(2t)]/65 + c\;e^{-t/4}.$$

Invoking the initial condition, $y(0) = 0$, the specific solution is

$$y(t) = 12 + \left[8cos(2t) + 64sin(2t) - 788\,e^{-t/4}\right]/65\,.$$

As $t \to \infty$, the exponential term will decay, and the solution will oscillate about an *average value* of 12, with an *amplitude* of $8/\sqrt{65}$.

29. The integrating factor is $\mu(t) = e^{-3t/2}$, and the differential equation can be written as $(e^{-3t/2}\,y)' = 3t\,e^{-3t/2} + 2\,e^{-t/2}$. The general solution is $y(t) = -2t - 4/3 - 4\,e^t + {}$ $+ c\,e^{3t/2}$. Imposing the initial condition, $y(t) = -2t - 4/3 - 4\,e^t + (y_0 + 16/3)\,e^{3t/2}$. As $t \to \infty$, the term containing $e^{3t/2}$ will *dominate* the solution. Its *sign* will determine the divergence properties. Hence the critical value of the initial condition is
$y_0 = -16/3$.
The corresponding solution, $y(t) = -2t - 4/3 - 4\,e^t$, will also decrease without bound.

Note on Problems 31-34 :

Let $g(t)$ be *given*, and consider the function $y(t) = y_1(t) + g(t)$, in which $y_1(t) \to \infty$ as $t \to \infty$. Differentiating, $y'(t) = y_1'(t) + g'(t)$. Letting a be a *constant*, it follows that $y'(t) + ay(t) = y_1'(t) + ay_1(t) + g'(t) + ag(t)$. Note that the hypothesis on the function $y_1(t)$ will be satisfied, if $y_1'(t) + ay_1(t) = 0$. That is, $y_1(t) = c\,e^{-at}$. Hence $y(t) = c\,e^{-at} + g(t)$, which is a solution of the equation $y' + ay = g'(t) + ag(t)$. For convenience, choose $a = 1$.

31. Here $g(t) = 3$, and we consider the linear equation $y' + y = 3$. The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\,y)' = 3e^t$. The general solution is $y(t) = 3 + c\,e^{-t}$.

33. $g(t) = 3 - t$. Consider the linear equation $y' + y = -1 + 3 - t$.The integrating factor is $\mu(t) = e^t$, and the differential equation can be written as $(e^t\,y)' = (2 - t)e^t$. The general solution is $y(t) = 3 - t + c\,e^{-t}$.

34. $g(t) = 4 - t^2$. Consider the linear equation $y' + y = 4 - 2t - t^2$.The integrating factor is $\mu(t) = e^t$, and the equation can be written as $(e^t\,y)' = (4 - 2t - t^2)e^t$. The general solution is $y(t) = 4 - t^2 + c\,e^{-t}$.

Section 2.2

2. For $x \neq -1$, the differential equation may be written as $y\,dy = [x^2/(1+x^3)]dx$. Integrating both sides, with respect to the appropriate variables, we obtain the relation $y^2/2 = \frac{1}{3}ln|1+x^3| + c$. That is, $y(x) = \pm\sqrt{\frac{2}{3}ln|1+x^3| + c}$.

3. The differential equation may be written as $y^{-2}dy = -\sin x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-1} = \cos x + c$. That is, $(C - \cos x)y = 1$, in which C is an arbitrary constant. Solving for the dependent variable, explicitly, $y(x) = 1/(C - \cos x)$.

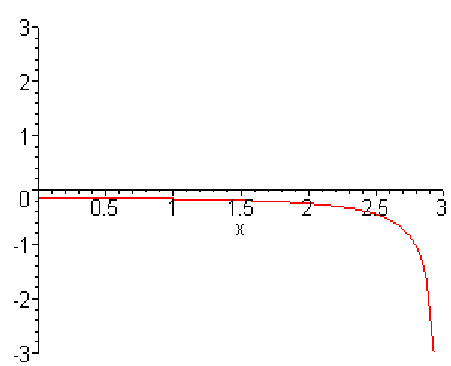
5. Write the differential equation as $\cos^{-2} 2y\,dy = \cos^2 x\,dx$, or $\sec^2 2y\,dy = \cos^2 x\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $\tan 2y = \sin x \cos x + x + c$.

7. The differential equation may be written as $(y + e^y)dy = (x - e^{-x})dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation
$y^2 + 2\,e^y = x^2 + 2\,e^{-x} + c$.

8. Write the differential equation as $(1+y^2)dy = x^2\,dx$. Integrating both sides of the equation, we obtain the relation $y + y^3/3 = x^3/3 + c$, that is, $3y + y^3 = x^3 + C$.

9(a). The differential equation is separable, with $y^{-2}dy = (1-2x)dx$. Integration yields $-y^{-1} = x - x^2 + c$. Substituting $x = 0$ and $y = -1/6$, we find that $c = 6$. Hence the specific solution is $y^{-1} = x^2 - x - 6$. The *explicit form* is
$y(x) = 1/(x^2 - x - 6)$.

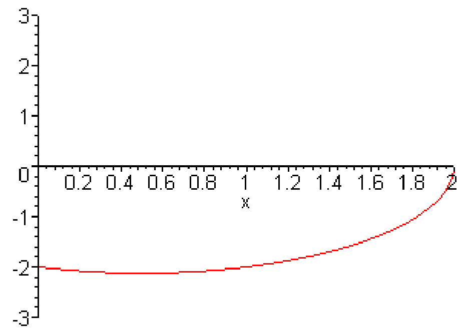
(b)



(c). Note that $x^2 - x - 6 = (x+2)(x-3)$. Hence the solution becomes *singular* at $x = -2$ and $x = 3$.

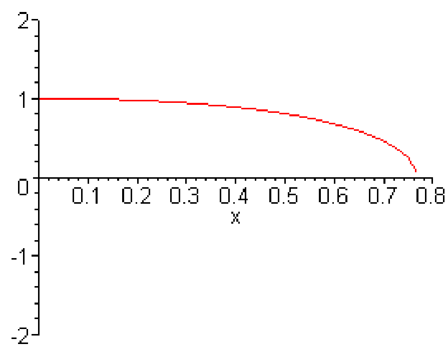
10(a). $y(x) = -\sqrt{2x - 2x^2 + 4}$.

$10(b)$.



$11(a)$. Rewrite the differential equation as $x\,e^x dx = -y\,dy$. Integrating both sides of the equation results in $x\,e^x - e^x = -y^2/2 + c$. Invoking the initial condition, we obtain $c = -1/2$. Hence $y^2 = 2e^x - 2x\,e^x - 1$. The *explicit form* of the solution is $y(x) = \sqrt{2e^x - 2x\,e^x - 1}$. The *positive* sign is chosen, since $y(0) = 1$.

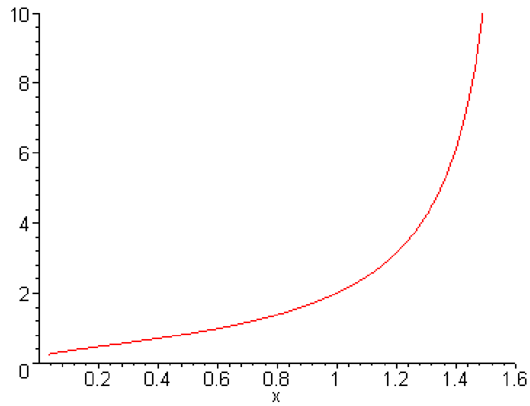
(b).



(c). The function under the radical becomes *negative* near $x = -1.7$ and $x = 0.76$.

$11(a)$. Write the differential equation as $r^{-2}dr = \theta^{-1}\,d\theta$. Integrating both sides of the equation results in the relation $-r^{-1} = \ln\theta + c$. Imposing the condition $r(1) = 2$, we obtain $c = -1/2$. The *explicit form* of the solution is $r(\theta) = 2/(1 - 2\ln\theta)$.

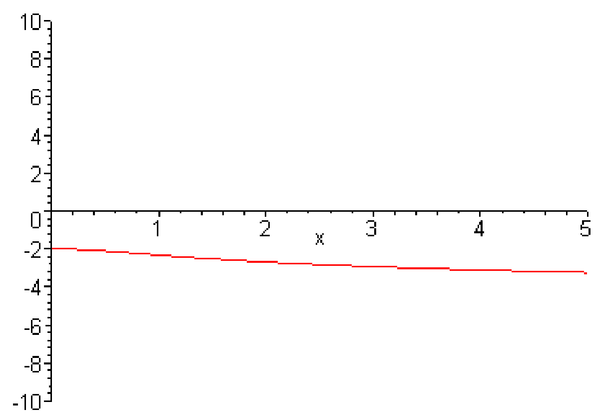
(b).



(c). Clearly, the solution makes sense only if $\theta > 0$. Furthermore, the solution becomes singular when $ln\,\theta = 1/2$, that is, $\theta = \sqrt{e}$.

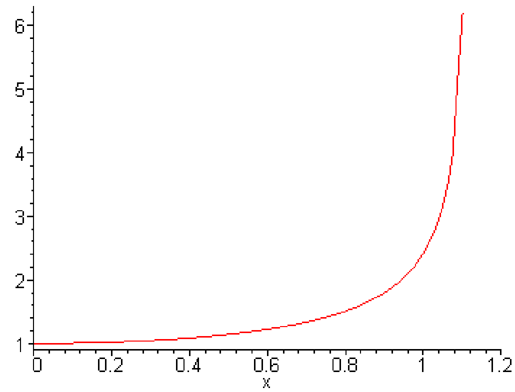
$13(a)$. $y(x) = -\sqrt{2\,ln(1+x^2)+4}$.

(b).



$14(a)$. Write the differential equation as $y^{-3}dy = x(1+x^2)^{-1/2}\,dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $-y^{-2}/2 = \sqrt{1+x^2} + c$. Imposing the initial condition, we obtain $c = -3/2$. Hence the specific solution can be expressed as $y^{-2} = 3 - 2\sqrt{1+x^2}$. The *explicit form* of the solution is $y(x) = 1/\sqrt{3-2\sqrt{1+x^2}}$. The *positive* sign is chosen to satisfy the initial condition.

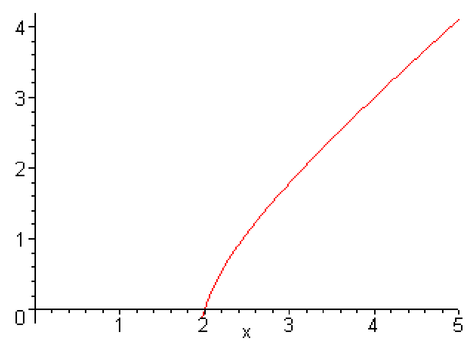
(b).



(c). The solution becomes singular when $2\sqrt{1+x^2}=3$. That is, at $x=\pm\sqrt{5}/2$.

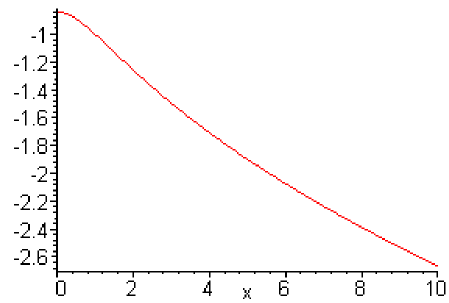
$15(a)$. $y(x)=-1/2+\sqrt{x^2-15/4}$.

(b).



$16(a)$. Rewrite the differential equation as $4y^3dy=x(x^2+1)dx$. Integrating both sides
of the equation results in $y^4=(x^2+1)^2/4+c$. Imposing the initial condition, we obtain $c=0$. Hence the solution may be expressed as $(x^2+1)^2-4y^4=0$. The *explicit* form of the solution is $y(x)=-\sqrt{(x^2+1)/2}$. The *sign* is chosen based on $y(0)=-1/\sqrt{2}$.

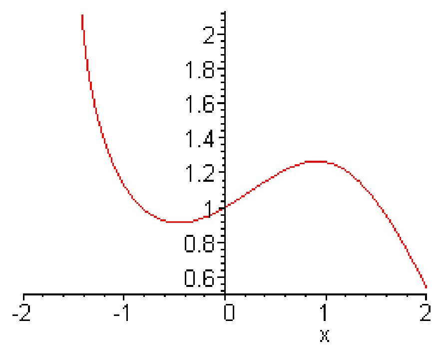
(b).



(c). The solution is valid for all $x \in \mathbb{R}$.

$17(a)$. $y(x) = -5/2 - \sqrt{x^3 - e^x + 13/4}$.

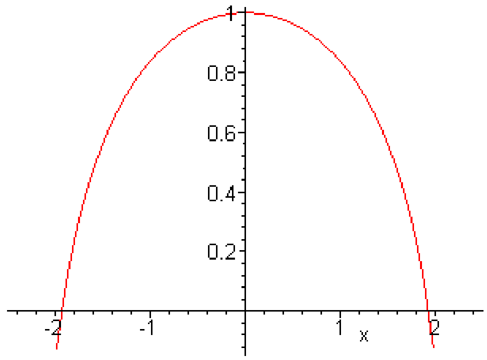
(b).



(c). The solution is valid for $x > -1.45$. This value is found by estimating the root of $4x^3 - 4e^x + 13 = 0$.

$18(a)$. Write the differential equation as $(3 + 4y)dy = (e^{-x} - e^x)dx$. Integrating both sides of the equation, with respect to the appropriate variables, we obtain the relation $3y + 2y^2 = -(e^x + e^{-x}) + c$. Imposing the initial condition, $y(0) = 1$, we obtain $c = 7$. Thus, the solution can be expressed as $3y + 2y^2 = -(e^x + e^{-x}) + 7$. Now by *completing the square* on the left hand side, $2(y + 3/4)^2 = -(e^x + e^{-x}) + 65/8$. Hence the *explicit* form of the solution is $y(x) = -3/4 + \sqrt{65/16 - cosh\, x}$.

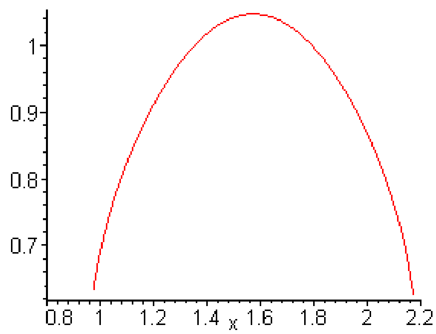
(b).



(c). Note the $65 - 16\,cosh\,x \geq 0$, as long as $|x| > 2.1$. Hence the solution is valid on the interval $-2.1 < x < 2.1$.

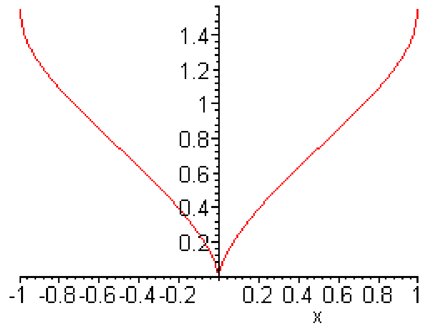
19(a). $y(x) = -\pi/3 + \frac{1}{3}sin^{-1}(3\,cos^2x)$.

(b).



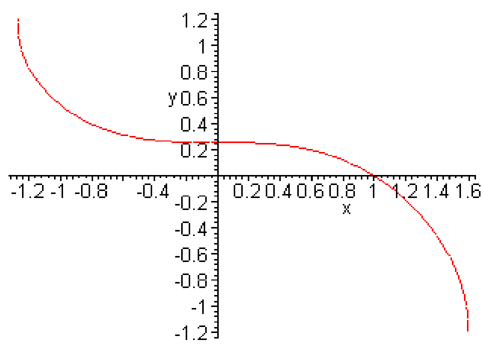
20(a). Rewrite the differential equation as $y^2dy = arcsin\,x/\sqrt{1-x^2}\,dx$. Integrating both sides of the equation results in $y^3/3 = (arcsin\,x)^2/2 + c$. Imposing the condition $y(0) = 0$, we obtain $c = 0$. The *explicit* form of the solution is $y(x) = \sqrt[3]{\frac{3}{2}}(arcsin\,x)^{2/3}$.

(b).



(c). Evidently, the solution is defined for $-1 \le x \le 1$.

22. The differential equation can be written as $(3y^2 - 4)dy = 3x^2 dx$. Integrating both sides, we obtain $y^3 - 4y = x^3 + c$. Imposing the initial condition, the specific solution is $y^3 - 4y = x^3 - 1$. Referring back to the differential equation, we find that $y' \to \infty$ as $y \to \pm 2/\sqrt{3}$. The respective values of the abscissas are $x = -1.276\,,\,1.598$.



Hence the solution is valid for $-1.276 < x < 1.598$.

24. Write the differential equation as $(3 + 2y)dy = (2 - e^x)dx$. Integrating both sides, we obtain $3y + y^2 = 2x - e^x + c$. Based on the specified initial condition, the solution can be written as $3y + y^2 = 2x - e^x + 1$. *Completing the square*, it follows that $y(x) = -3/2 + \sqrt{2x - e^x + 13/4}$. The solution is defined if $2x - e^x + 13/4 \ge 0$, that is, $-1.5 \le x \le 2$ (*approximately*). In that interval, $y' = 0$, for $x = ln\,2$. It can be verified that $y''(ln\,2) < 0$. In fact, $y''(x) < 0$ on the interval of definition. Hence the solution attains a global maximum at $x = ln\,2$.

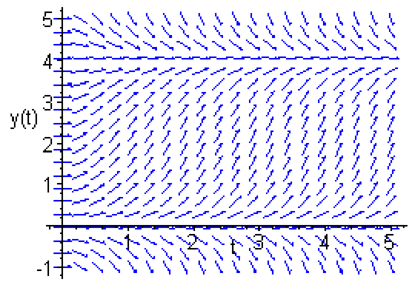
26. The differential equation can be written as $(1 + y^2)^{-1}dy = 2(1 + x)dx$. Integrating both sides of the equation, we obtain $arctan y = 2x + x^2 + c$. Imposing the given initial
condition, the specific solution is $arctan y = 2x + x^2$. Therefore, $y(x) = tan(2x + x^2)$. Observe that the solution is defined as long as $-\pi/2 < 2x + x^2 < \pi/2$. It is easy to see that $2x + x^2 \ge -1$. Furthermore, $2x + x^2 = \pi/2$ for $x = -2.6$ and 0.6. Hence the solution is valid on the interval $-2.6 < x < 0.6$. Referring back to the differential

equation, the solution is *stationary* at $x = -1$. Since $y''(x) > 0$ on the entire interval of
definition, the solution attains a global minimum at $x = -1$.

$28(a)$. Write the differential equation as $y^{-1}(4-y)^{-1}dy = t(1+t)^{-1}dt$. Integrating both sides of the equation, we obtain $ln\,|y| - ln|y-4| = 4t - 4ln|1+t| + c$. Taking the *exponential* of both sides, it follows that $|y/(y-4)| = C\,e^{4t}/(1+t)^4$. It follows that as $t \to \infty$, $|y/(y-4)| = |1 + 4/(y-4)| \to \infty$. That is, $y(t) \to 4$.

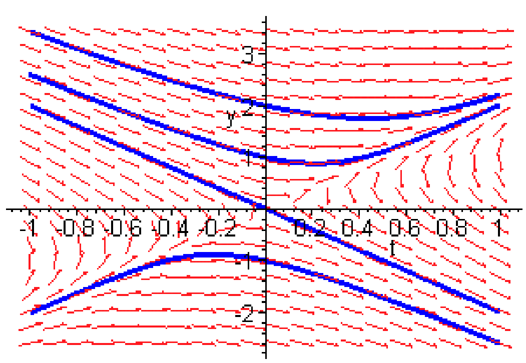
(b). Setting $y(0) = 2$, we obtain that $C = 1$. Based on the initial condition, the solution may be expressed as $y/(y-4) = -e^{4t}/(1+t)^4$. Note that $y/(y-4) < 0$, for all $t \geq 0$. Hence $y < 4$ for all $t \geq 0$. Referring back to the differential equation, it follows that y' is always *positive*. This means that the solution is *monotone increasing*. We find that the root of the equation $e^{4t}/(1+t)^4 = 399$ is near $t = 2.844$.

(c). Note the $y(t) = 4$ is an equilibrium solution. Examining the local direction field,

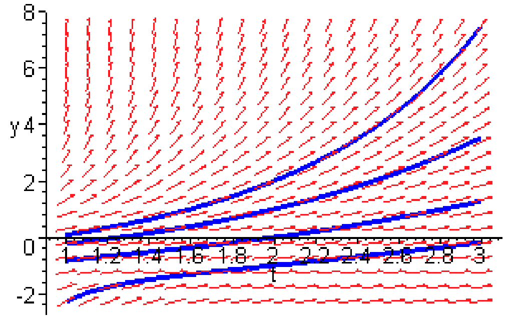


we see that if $y(0) > 0$, then the corresponding solutions converge to $y = 4$. Referring back to part (a), we have $y/(y-4) = [y_0/(y_0-4)]e^{4t}/(1+t)^4$, for $y_0 \neq 4$. Setting $t = 2$, we obtain $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Now since the function $f(y) = y/(y-4)$ is *monotone* for $y < 4$ and $y > 4$, we need only solve the equations $y_0/(y_0-4) = -399(3/e^2)^4$and $y_0/(y_0-4) = 401(3/e^2)^4$. The respective solutions are $y_0 = 3.6622$ and $y_0 = 4.4042$.

$30(f)$.



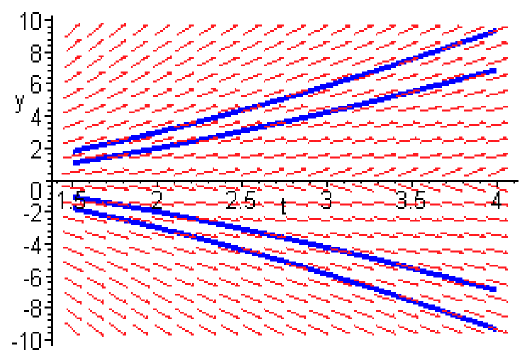
$31(c)$



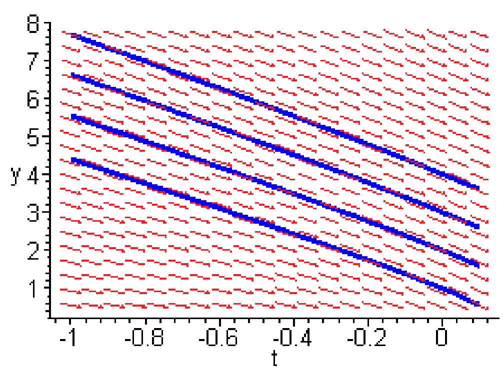
$32(a)$. Observe that $(x^2+3y^2)/2xy = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} + \frac{3}{2}\frac{y}{x}$. Hence the differential equation is *homogeneous*.

(b). The substitution $y = x\,v$ results in $v + x\,v' = (x^2+3x^2v^2)/2x^2v$. The transformed equation is $v' = (1+v^2)/2xv$. This equation is *separable*, with general solution $v^2+1 = c\,x$. In terms of the original dependent variable, the solution is $x^2+y^2 = c\,x^3$.

(c).



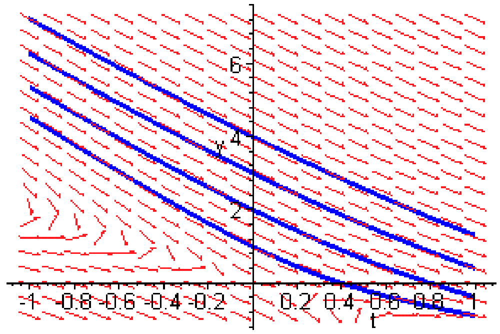
$33(c)$.



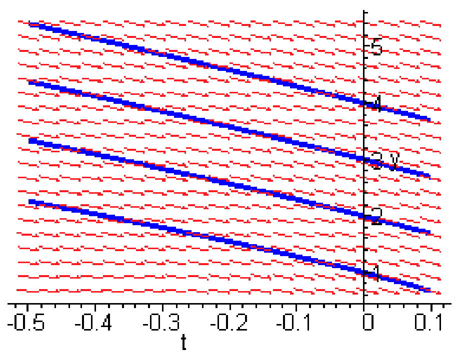
34(a). Observe that $-(4x+3y)/(2x+y) = -2 - \frac{y}{x}\left[2+\frac{y}{x}\right]^{-1}$. Hence the differential equation is *homogeneous*.

(b). The substitution $y = x\,v$ results in $v + x\,v' = -2 - v/(2+v)$. The transformed equation is $v' = -(v^2+5v+4)/(2+v)x$. This equation is *separable*, with general solution $(v{+}4)^2|v{+}1| = C/x^3$. In terms of the original dependent variable, the solution is $(4x+y)^2|x{+}y| = C$.

(c).



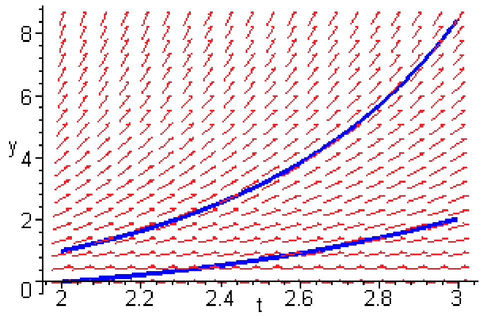
35(c).



36(a). Divide by x^2 to see that the equation is homogeneous. Substituting $y = x\,v$, we obtain $x\,v' = (1+v)^2$. The resulting differential equation is separable.

(b). Write the equation as $(1+v)^{-2}dv = x^{-1}dx$. Integrating both sides of the equation, we obtain the general solution $-1/(1+v) = ln|x| + c$. In terms of the original dependent variable, the solution is $y = x\,[C - ln|x|]^{-1} - x$.

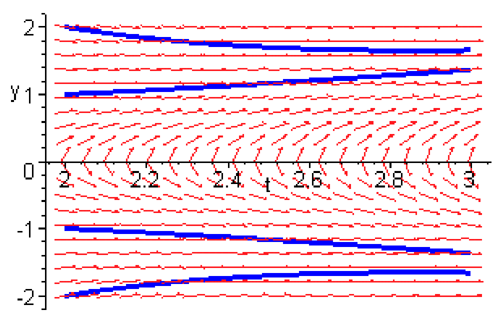
(c).



$37(a)$. The differential equation can be expressed as $y' = \frac{1}{2}\left(\frac{y}{x}\right)^{-1} - \frac{3}{2}\frac{y}{x}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (1-5v^2)/2v$. Separating variables, we have $\frac{2v}{1-5v^2}dv = \frac{1}{x}dx$.

(b). Integrating both sides of the transformed equation yields $-\frac{1}{5}$
$ln|1-5v^2| = ln|x| + c$,
that is, $1-5v^2 = C/|x|^5$. In terms of the original dependent variable, the general solution is $5y^2 = x^2 - C/|x|^3$.

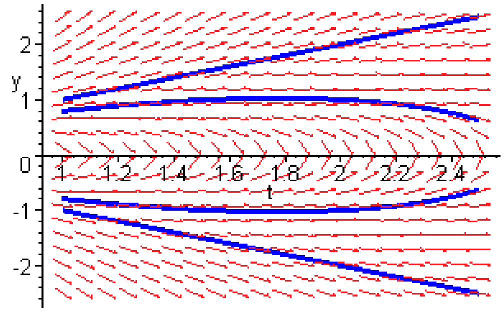
(c).



$38(a)$. The differential equation can be expressed as $y' = \frac{3}{2}\frac{y}{x} - \frac{1}{2}\left(\frac{y}{x}\right)^{-1}$. Hence the equation is homogeneous. The substitution $y = x\,v$ results in $x\,v' = (v^2-1)/2v$, that is, $\frac{2v}{v^2-1}dv = \frac{1}{x}dx$.

(b). Integrating both sides of the transformed equation yields $ln|v^2-1| = ln|x| + c$, that is, $v^2 - 1 = C|x|$. In terms of the original dependent variable, the general solution is $y^2 = C\,x^2|x| + x^2$.

(c).



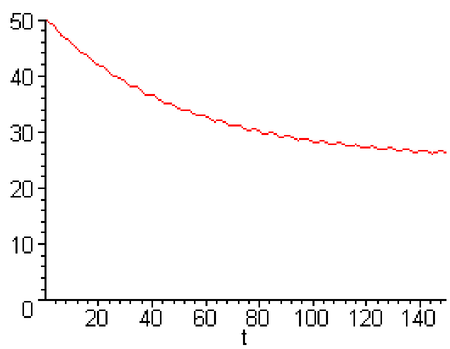
Section 2.3

$5(a)$. Let Q be the amount of salt in the tank. Salt enters the tank of water at a rate of $2\,\frac{1}{4}\left(1+\frac{1}{2}sin\,t\right)=\frac{1}{2}+\frac{1}{4}sin\,t\ \ oz/min$. It leaves the tank at a rate of $2\,Q/100\ \ oz/min$. Hence the differential equation governing the amount of salt at any time is

$$\frac{dQ}{dt}=\frac{1}{2}+\frac{1}{4}sin\,t-Q/50\,.$$

The initial amount of salt is $Q_0=50\;oz$. The governing ODE is *linear*, with integrating factor $\mu(t)=e^{t/50}$. Write the equation as $\left(e^{t/50}Q\right)'=e^{t/50}\left(\frac{1}{2}+\frac{1}{4}sin\,t\right)$. The specific solution is $Q(t)=25+\left[12.5sin\,t-625cos\,t+63150\,e^{-t/50}\right]/2501\ \ oz$.

(b).



(c). The amount of salt approaches a *steady state*, which is an oscillation of amplitude $1/4$ about a level of $25\;oz$.

$6(a)$. The equation governing the value of the investment is $dS/dt=r\,S$. The value of the investment, at any time, is given by $S(t)=S_0e^{rt}$. Setting $S(T)=2S_0$, the required time is $T=ln(2)/r$.

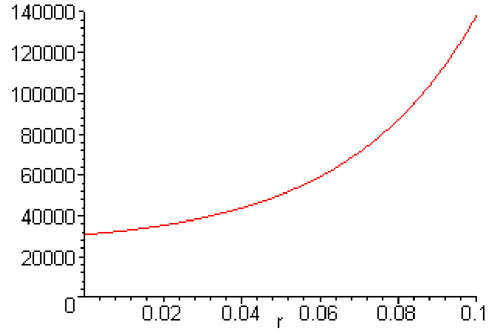
(b). For the case $r=7\%=.07$, $T\approx 9.9\;yrs$.

(c). Referring to Part(a), $r=ln(2)/T$. Setting $T=8$, the required interest rate is to be approximately $r=8.66\,\%$.

$8(a)$. Based on the solution in *Eq.*(16), with $S_0=0$, the value of the investments *with* contributions is given by $S(t)=25,000(e^{rt}-1)$. After *ten* years, person A has $S_A=\$25,000(1.226)=\$\,30,640$. Beginning at age 35, the investments can now be analyzed using the equations $S_A=30,640\,e^{.08t}$ and $S_B=25,000(e^{.08t}-1)$. After *thirty* years, the balances are $S_A=\$\,337,734$ and $S_B=\$\,250,579$.

(b). For an *unspecified* rate r, the balances after *thirty* years are $S_A=30,640\,e^{30r}$ and $S_B=25,000(e^{30r}-1)$.

(c).



(d). The two balances can *never* be equal.

$11(a)$. Let S be the value of the mortgage. The debt accumulates at a rate of rS, in which $r = .09$ is the *annual* interest rate. Monthly payments of $\$\,800$ are equivalent to $\$\,9,600$ *per year*. The differential equation governing the value of the mortgage is $dS/dt = .09\,S - 9,600$. Given that S_0 is the original amount borrowed, the debt is $S(t) = S_0 e^{.09t} - 106,667(e^{.09t} - 1)$. Setting $S(30) = 0$, it follows that $S_0 = \$\,99,500$.

(b). The *total* payment, over 30 years, becomes $\$\,288,000$. The interest paid on this purchase is $\$\,188,500$.

$13(a)$. The balance *increases* at a rate of $r\,S$ *\$/yr*, and *decreases* at a constant rate of k *\$ per year*. Hence the balance is modeled by the differential equation $dS/dt = rS - k$. The balance at any time is given by $S(t) = S_0 e^{rt} - \frac{k}{r}(e^{rt} - 1)$.

(b). The solution may also be expressed as $S(t) = (S_0 - \frac{k}{r})e^{rt} + \frac{k}{r}$. Note that if the withdrawal rate is $k_0 = r\,S_0$, the balance will remain at a constant level S_0.

(c). Assuming that $k > k_0$, $S(T_0) = 0$ for $T_0 = \frac{1}{r} ln\left[\frac{k}{k - k_0}\right]$.

(d). If $r = .08$ and $k = 2k_0$, then $T_0 = 8.66$ *years*.

(e). Setting $S(t) = 0$ and solving for e^{rt} in Part(b), $e^{rt} = \frac{k}{k - rS_0}$. Now setting $t = T$ results in $k = rS_0 e^{rT}/(e^{rT} - 1)$.

(f). In part(e), let $k = 12,000$, $r = .08$, and $T = 20$. The required investment becomes $S_0 = \$\,119,715$.

$14(a)$. Let $Q' = -\,r\,Q$. The general solution is $Q(t) = Q_0 e^{-rt}$. Based on the definition of *half-life*, consider the equation $Q_0/2 = Q_0 e^{-5730\,r}$. It follows that

$-5730\,r = ln(1/2)$, that is, $r = 1.2097 \times 10^{-4}$ *per year.*

(b). Hence the amount of carbon-14 is given by $Q(t) = Q_0\, e^{-1.2097\times 10^{-4} t}$.

(c). Given that $Q(T) = Q_0/5$, we have the equation $1/5 = e^{-1.2097\times 10^{-4} T}$. Solving for the *decay time*, the apparent age of the remains is approximately $T = 13,304.65$ *years*.

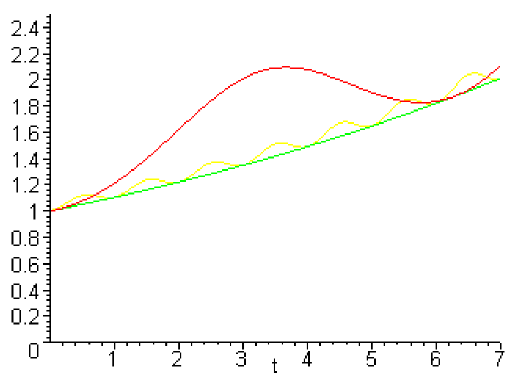
15. Let $P(t)$ be the population of mosquitoes at any time t. The rate of *increase* of the mosquito population is rP. The population *decreases* by $20,000$ *per day*. Hence the equation that models the population is given by $dP/dt = rP - 20,000$. Note that the variable t represents *days*. The solution is $P(t) = P_0 e^{rt} - \frac{20,000}{r}(e^{rt} - 1)$. In the absence of predators, the governing equation is $dP_1/dt = rP_1$, with solution $P_1(t) = P_0 e^{rt}$. Based on the data, set $P_1(7) = 2P_0$, that is, $2P_0 = P_0 e^{7r}$. The growth rate is determined as $r = ln(2)/7 = .09902$ *per day*. Therefore the population, including the *predation* by birds, is $P(t) = 2\times 10^5 e^{.099t} - 201,997(e^{.099t} - 1) =$
$= 201,997.3 - 1977.3\, e^{.099t}$.

16(a). $y(t) = exp[2/10 + t/10 - 2cos(t)/10]$. The *doubling-time* is $\tau \approx 2.9632$.

(b). The differential equation is $dy/dt = y/10$, with solution $y(t) = y(0)e^{t/10}$. The *doubling-time* is given by $\tau = 10ln(2) \approx 6.9315$.

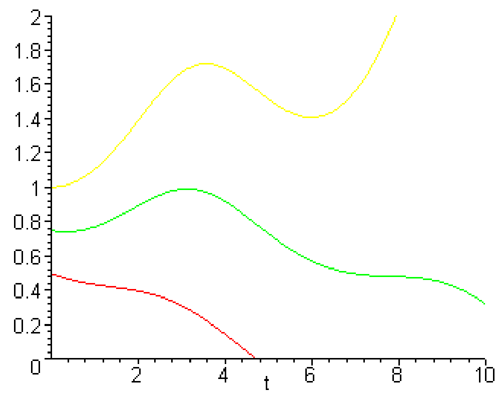
(c). Consider the differential equation $dy/dt = (0.5 + sin(2\pi t))\, y/5$. The equation is *separable*, with $\frac{1}{y}dy = \left(0.1 + \frac{1}{5}sin(2\pi t)\right)dt$. Integrating both sides, with respect to the appropriate variable, we obtain $ln\, y = (\pi t - cos(2\pi t))/10\pi + c$. Invoking the initial condition, the solution is $y(t) = exp[(1 + \pi t - cos(2\pi t))/10\pi]$. The *doubling-time* is $\tau \approx 6.3804$. The *doubling-time* approaches the value found in part(b).

(d).



17(a). The differential equation $dy/dt = r(t)\, y - k$ is *linear*, with integrating factor $\mu(t) = exp\left[-\int r(t)dt\right]$. Write the equation as $(\mu\, y)' = -k\,\mu(t)$. Integration of both

sides yields the general solution $y = \left[-k\int \mu(\tau)d\tau + y_0\,\mu(0)\right]/\mu(t)$. In this problem, the
integrating factor is $\mu(t) = exp[(\cos t - t)/5]$.



(b). The population becomes *extinct*, if $y(t^*) = 0$, for some $t = t^*$. Referring to part(a),
we find that $y(t^*) = 0 \Rightarrow$

$$\int_0^{t^*} exp[(\cos\tau - \tau)/5]d\tau = 5\,e^{1/5}y_c.$$

It can be shown that the integral on the left hand side increases *monotonically*, from *zero* to a limiting value of approximately 5.0893. Hence extinction can happen *only if* $5\,e^{1/5}y_c < 5.0893$, that is, $y_c < 0.8333$.

(c). Repeating the argument in part(b), it follows that $y(t^*) = 0 \Rightarrow$

$$\int_0^{t^*} exp[(\cos\tau - \tau)/5]d\tau = \frac{1}{k}\,e^{1/5}y_c.$$

Hence extinction can happen *only if* $e^{1/5}y_c/k < 5.0893$, that is, $y_c < 4.1667\,k$.

(d). Evidently, y_c is a *linear* function of the parameter k.

19(a). Let $Q(t)$ be the *volume* of carbon monoxide in the room. The rate of *increase* of *CO* is $(.04)(0.1) = 0.004\ ft^3/min$. The amount of *CO leaves the room* at a rate of $(0.1)Q(t)/1200 = Q(t)/12000\ \ ft^3/min$. Hence the total rate of change is given by the differential equation $dQ/dt = 0.004 - Q(t)/12000$. This equation is *linear* and separable, with solution $Q(t) = 48 - 48\,exp(-t/12000)\ \ ft^3$. Note that $Q_0 = 0\ \ ft^3$. Hence the *concentration* at any time is given by $x(t) = Q(t)/1200 = Q(t)/12\,\%$.

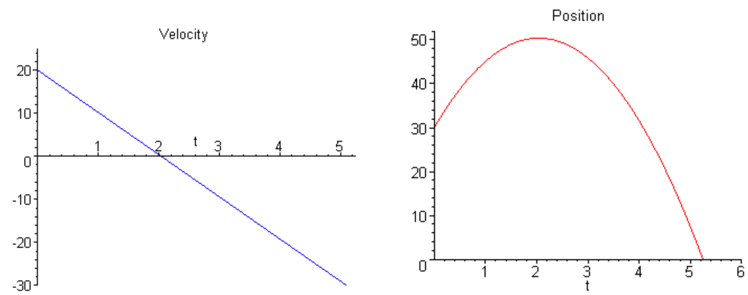
(b). The *concentration* of *CO* in the room is $x(t) = 4 - 4exp(-t/12000)\ \%$. A level of 0.00012 corresponds to 0.012 %. Setting $x(\tau) = 0.012$, the solution of the equation $4 - 4exp(-t/12000) = 0.012$ is $\tau \approx 36\ \ minutes$.

$20(a)$. The concentration is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. It is easy to see that $c(t \to \infty) = k + P/r$.

(b). $c(t) = c_0\, e^{-rt/V}$. The *reduction times* are $T_{50} = ln(2)V/r$ and $T_{10} = ln(10)V/r$.

(c). The *reduction times*, in *years*, are $T_S = ln(10)(65.2)/12,200 = 430.85$
$T_M = ln(10)(158)/4,900 = 71.4$; $T_E = ln(10)(175)/460 = 6.05$
$T_O = ln(10)(209)/16,000 = 17.63$.

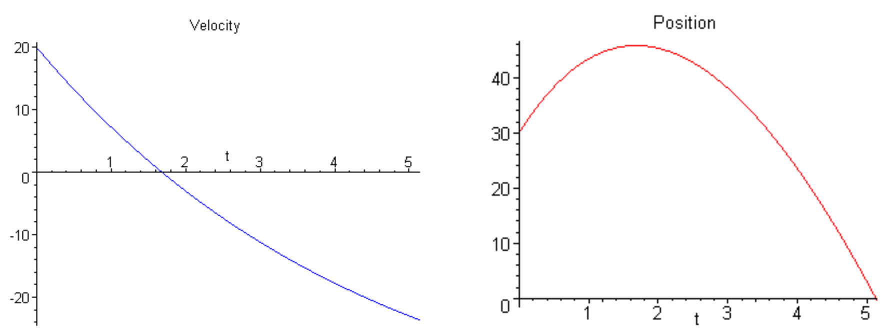
$21(c)$.



$22(a)$. The differential equation for the motion is $m\, dv/dt = -v/30 - mg$. Given the initial condition $v(0) = 20$ *m/s*, the solution is $v(t) = -44.1 + 64.1\, exp(-t/4.5)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.683$ *sec*. Integrating $v(t)$, the position is given by $x(t) = 318.45 - 44.1\, t - 288.45\, exp(-t/4.5)$. Hence the *maximum height* is $x(t_1) = 45.78$ *m*.

(b). Setting $x(t_2) = 0$, the ball hits the ground at $t_2 = 5.128$ *sec*.

(c).

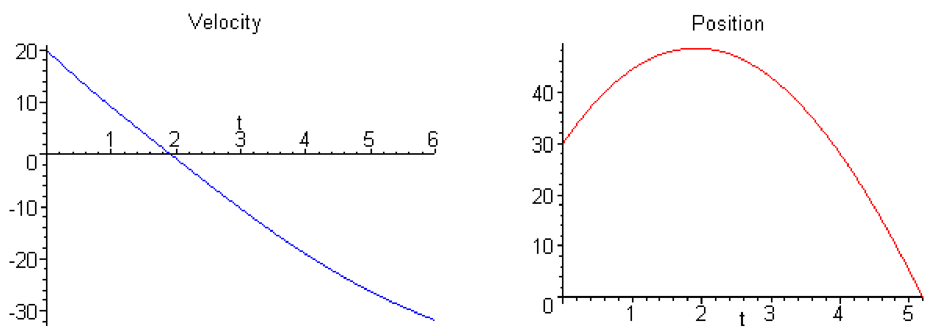


$23(a)$. The differential equation for the *upward* motion is $m\, dv/dt = -\mu v^2 - mg$, in which $\mu = 1/1325$. This equation is *separable*, with $\frac{m}{\mu v^2 + mg} dv = -dt$. Integrating both sides and invoking the initial condition, $v(t) = 44.133\, tan(.425 - .222\, t)$. Setting $v(t_1) = 0$, the ball reaches the maximum height at $t_1 = 1.916\, sec$. Integrating $v(t)$, the position is given by $x(t) = 198.75\, ln[cos(0.222\, t - 0.425)] + 48.57$. Therefore the *maximum height* is $x(t_1) = 48.56\ m$.

(b). The differential equation for the *downward* motion is $m\, dv/dt = +\mu v^2 - mg$. This equation is also separable, with $\frac{m}{mg - \mu v^2} dv = -dt$. For convenience, set $t = 0$ at the *top* of the trajectory. The new initial condition becomes $v(0) = 0$. Integrating both sides and invoking the initial condition, we obtain $ln[(44.13 - v)/(44.13 + v)] = t/2.25$.

Solving for the velocity, $v(t) = 44.13(1 - e^{t/2.25})/(1 + e^{t/2.25})$. Integrating $v(t)$, the position is given by $x(t) = 99.29\, ln\left[e^{t/2.25}/(1 + e^{t/2.25})^2\right] + 186.2$. To estimate the *duration* of the downward motion, set $x(t_2) = 0$, resulting in $t_2 = 3.276\, sec$. Hence the *total time* that the ball remains in the air is $t_1 + t_2 = 5.192\ sec$.

(c).



$24(a)$. Measure the positive direction of motion *downward*. Based on Newton's $2nd$ law,
the equation of motion is given by

$$m\,\frac{dv}{dt} = \begin{cases} -0.75\, v + mg & , \ 0 < t < 10 \\ -12\, v + mg & , \ t > 10 \end{cases} .$$

Note that gravity acts in the *positive* direction, and the drag force is *resistive*. During the first ten seconds of fall, the initial value problem is $dv/dt = -v/7.5 + 32$, with initial velocity $v(0) = 0\, fps$. This differential equation is separable and linear, with solution $v(t) = 240(1 - e^{-t/7.5})$. Hence $v(10) = 176.7\, fps$.

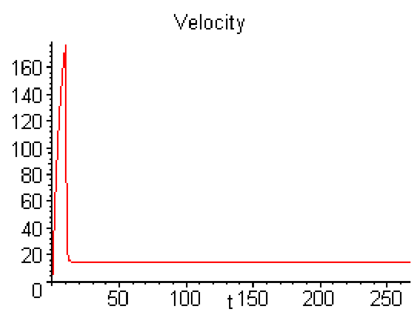
(b). Integrating the velocity, with $x(t) = 0$, the distance fallen is given by

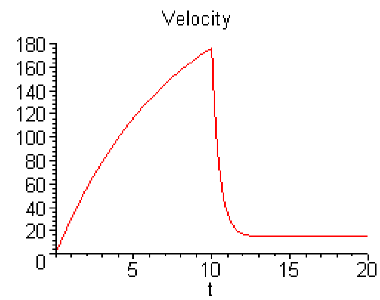
$$x(t) = 240\, t + 1800\, e^{-t/7.5} - 1800 .$$

Hence $x(10) = 1074.5\, ft$.

(c). For computational purposes, reset time to $t = 0$. For the remainder of the motion, the initial value problem is $dv/dt = -32v/15 + 32$, with specified initial velocity $v(0) = 176.7\, fps$. The solution is given by $v(t) = 15 + 161.7\, e^{-32t/15}$. As $t \to \infty$, $v(t) \to v_L = 15\, fps$. Integrating the velocity, with $x(0) = 1074.5$, the distance fallen after the parachute is open is given by $x(t) = 15\,t - 75.8\, e^{-32t/15} + 1150.3$. To find the duration of the second part of the motion, estimate the root of the transcendental equation $15\,T - 75.8\, e^{-32T/15} + 1150.3 = 5000$. The result is $T = 256.6\ sec$.

(d).





$25(a)$. Measure the positive direction of motion *upward*. The equation of motion is given by $m dv/dt = -k\,v - mg$. The initial value problem is $dv/dt = -kv/m - g$, with $v(0) = v_0$. The solution is $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Setting $v(t_m) = 0$, the maximum height is reached at time $t_m = (m/k)ln[(mg + k\,v_0)/mg]$. Integrating the velocity, the position of the body is

$$x(t) = -mg\,t/k + \left[\left(\frac{m}{k}\right)^2 g + \frac{m\,v_0}{k}\right](1 - e^{-kt/m}).$$

Hence the maximum height reached is

$$x_m = x(t_m) = \frac{m\,v_0}{k} - g\left(\frac{m}{k}\right)^2 ln\left[\frac{mg + k\,v_0}{mg}\right].$$

(b). Recall that for $\delta \ll 1$, $ln(1+\delta) = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4 + \ldots$

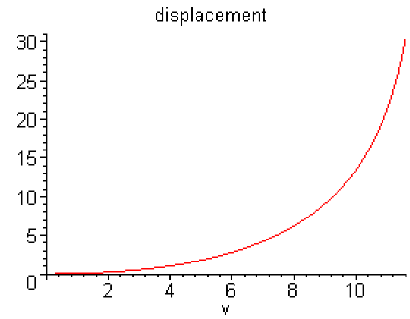
$26(b)$. $\lim\limits_{k \to 0} \frac{-mg + (k\,v_0 + mg)e^{-kt/m}}{k} = \lim\limits_{k \to 0} -\frac{t}{m}(k\,v_0 + mg)e^{-kt/m} = -gt$.

(c). $\lim\limits_{m \to 0}\left[-\frac{mg}{k} + \left(\frac{mg}{k} + v_0\right)e^{-kt/m}\right] = 0$, since $\lim\limits_{m \to 0} e^{-kt/m} = 0$.

$28(a)$. In terms of displacement, the differential equation is $mv\,dv/dx = -k\,v + mg$. This follows from the *chain rule*: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dt}$. The differential equation is separable, with

$$x(v) = -\frac{mv}{k} - \frac{m^2 g}{k^2} ln\left|\frac{mg - k\,v}{mg}\right|.$$

The inverse *exists*, since both x and v are monotone increasing. In terms of the given parameters, $x(v) = -1.25\,v - 15.31\,ln|0.0816\,v - 1|$.



(b). $x(10) = 13.45$ *meters*. The required value is $k = 0.24$.

(c). In part(a), set $v = 10$ *m/s* and $x = 10$ *meters*.

29(a). Let x represent the height above the earth's surface. The equation of motion is given by $m\frac{dv}{dt} = -G\frac{Mm}{(R+x)^2}$, in which G is the universal gravitational constant. The symbols M and R are the *mass* and *radius* of the earth, respectively. By the chain rule,

$$mv\frac{dv}{dx} = -G\frac{Mm}{(R+x)^2}.$$

This equation is separable, with $v\,dv = -GM(R+x)^{-2}dx$. Integrating both sides, and
invoking the initial condition $v(0) = \sqrt{2gR}$, the solution is $v^2 = 2GM(R+x)^{-1} +$ $+\,2gR - 2GM/R$. From elementary physics, it follows that $g = GM/R^2$. Therefore $v(x) = \sqrt{2g}\left[R/\sqrt{R+x}\right]$. (*Note that* $g = 78,545$ *mi/hr*^{2}.)

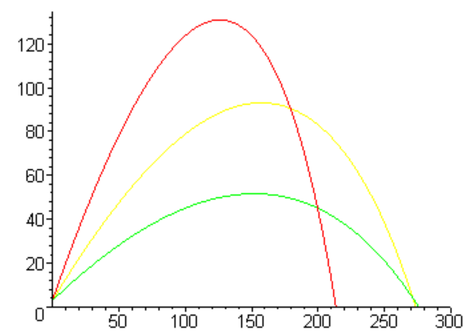
(b). We now consider $dx/dt = \sqrt{2g}\left[R/\sqrt{R+x}\right]$. This equation is also separable, with $\sqrt{R+x}\,dx = \sqrt{2g}\,R\,dt$. By definition of the variable x, the initial condition is $x(0) = 0$. Integrating both sides, we obtain $x(t) = \left[\frac{3}{2}\left(\sqrt{2g}\,R\,t + \frac{2}{3}R^{3/2}\right)\right]^{2/3} - R$. Setting the distance $x(T) + R = 240,000$, and solving for T, the duration of such a flight would be $T \approx 49$ *hours*.

32(a). Both equations are linear and separable. The initial conditions are $v(0) = u\,cos$ A
and $w(0) = u\,sin\,A$. The two solutions are $v(t) = u\,cos\,A\,e^{-rt}$ and $w(t) = -g/r +$ $+\,(u\,sin\,A + g/r)e^{-rt}$.

(b). Integrating the solutions in part(a), and invoking the initial conditions, the coordinates are $x(t) = \frac{u}{r} cos\, A(1 - e^{-rt})$ and

$$y(t) = -gt/r + \left(g + ur\, sin\, A + hr^2\right)/r^2 - \left(\frac{u}{r} sin\, A + g/r^2\right)e^{-rt}.$$

(c).



(d). Let T be the time that it takes the ball to go 350 *ft* horizontally. Then from above, $e^{-T/5} = (u\, cos\, A - 70)/u\, cos\, A$. At the same time, the height of the ball is given by $y(T) = -160\,T + 267 + 125u sin\, A - (800 + 5u\, sin\, A)[(u\, cos\, A - 70)/u\, cos\, A]$. Hence A and u must satisfy the inequality

$$800 ln\left[\frac{u\, cos\, A - 70}{u\, cos\, A}\right] + 267 + 125u sin\, A - (800 + 5u\, sin\, A)[(u\, cos\, A - 70)/u\, cos\, A] \geq 10\,.$$

33(a). Solving equation (i), $y'(x) = [(k^2 - y)/y]^{1/2}$. The *positive* answer is chosen, since y is an *increasing* function of x.

(b). Let $y = k^2 sin^2 t$. Then $dy = 2k^2 sin\, t\, cos\, t\, dt$. Substituting into the equation in part(a), we find that

$$\frac{2k^2 sin\, t\, cos\, t\, dt}{dx} = \frac{cos\, t}{sin\, t}\,.$$

Hence $2k^2 sin^2 t\, dt = dx$.

(c). Letting $\theta = 2t$, we further obtain $k^2 sin^2 \frac{\theta}{2}\, d\theta = dx$. Integrating both sides of the equation and noting that $t = \theta = 0$ corresponds to the *origin*, we obtain the solutions $x(\theta) = k^2(\theta - sin\, \theta)/2$ and [from part(b)] $y(\theta) = k^2(1 - cos\, \theta)/2$.

(d). Note that $y/x = (1 - cos\, \theta)/(\theta - sin\, \theta)$. Setting $x = 1$, $y = 2$, the solution of the equation $(1 - cos\, \theta)/(\theta - sin\, \theta) = 2$ is $\theta \approx 1.401$. Substitution into either of the expressions yields $k \approx 2.193$.

Section 2.4

2. Considering the roots of the coefficient of the leading term, the ODE has unique solutions on intervals *not* containing 0 or 4 . Since $2 \in (0\,,4)$, the initial value problem has a unique solution on the interval $(0\,,4)$.

3. The function $tan\, t$ is discontinuous at *odd multiples* of $\frac{\pi}{2}$. Since $\frac{\pi}{2} < \pi < \frac{3\pi}{2}$, the initial value problem has a unique solution on the interval $\left(\frac{\pi}{2}\,, \frac{3\pi}{2}\right)$.

5. $p(t) = 2t/(4 - t^2)$ and $g(t) = 3t^2/(4 - t^2)$. These functions are discontinuous at $x = \pm 2$. The initial value problem has a unique solution on the interval $(-2\,,2)$.

6. The function $ln\, t$ is defined and continuous on the interval $(0\,,\infty)$. Therefore the initial value problem has a unique solution on the interval $(0\,,\infty)$.

7. The function $f(t\,,y)$ is continuous everywhere on the plane, *except* along the straight line $y = -2t/5$. The partial derivative $\partial f/\partial y = -7t/(2t+5y)^2$ has the *same* region of continuity.

9. The function $f(t\,,y)$ is discontinuous along the coordinate axes, and on the hyperbola $t^2 - y^2 = 1$. Furthermore,

$$\frac{\partial f}{\partial y} = \frac{\pm 1}{y(1 - t^2 + y^2)} - 2\frac{y\, ln|ty|}{(1 - t^2 + y^2)^2}$$

has the *same* points of discontinuity.

10. $f(t\,,y)$ is continuous everywhere on the plane. The partial derivative $\partial f/\partial y$ is also continuous everywhere.

12. The function $f(t\,,y)$ is discontinuous along the lines $t = \pm k\,\pi$ and $y = -1$. The partial derivative $\partial f/\partial y = cot(t)/(1+y)^2$ has the *same* region of continuity.

14. The equation is separable, with $dy/y^2 = 2t\, dt$. Integrating both sides, the solution is given by $y(t) = y_0/(1 - y_0 t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \leq 0$, solutions are defined for *all* t .

15. The equation is separable, with $dy/y^3 = -\,dt$. Integrating both sides and invoking the initial condition, $y(t) = y_0/\sqrt{2y_0 t + 1}$. Solutions exist as long as $2y_0 t + 1 > 0$, that is, $2y_0 t > -1$. If $y_0 > 0$, solutions exist for $t > -1/2y_0$. If $y_0 = 0$, then the solution $y(t) = 0$ exists for all t . If $y_0 < 0$, solutions exist for $t < -1/2y_0$.

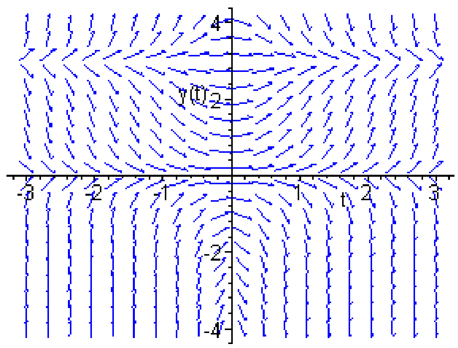
16. The function $f(t\,,y)$ is discontinuous along the straight lines $t = -1$ and $y = 0$. The partial derivative $\partial f/\partial y$ is discontinuous along the same lines. The equation is

separable, with $y\,dy = t^2\,dt/(1+t^3)$. Integrating and invoking the initial condition, the solution is $y(t) = \left[\frac{2}{3}ln|1+t^3| + y_0^2\right]^{1/2}$. Solutions exist as long as

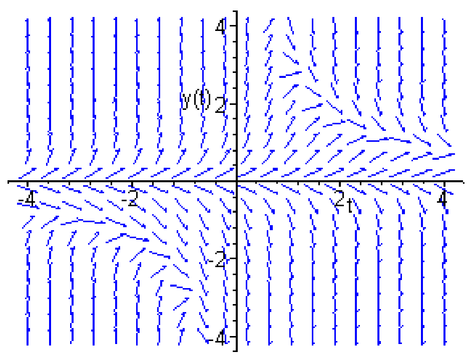
$$\frac{2}{3}ln|1+t^3| + y_0^2 \geq 0\,,$$

that is, $y_0^2 \geq -\frac{2}{3}ln|1+t^3|$. For all y_0 (it can be verified that $y_0 = 0$ yields a valid solution, even though Theorem 2.4.2 does not guarantee one) , solutions exists as long as $|1+t^3| \geq exp(-3y_0^2/2)$. From above, we must have $t > -1$. Hence the inequality may be written as $t^3 \geq exp(-3y_0^2/2) - 1$. It follows that the solutions are valid for $[exp(-3y_0^2/2)-1]^{1/3} < t < \infty$.

17.

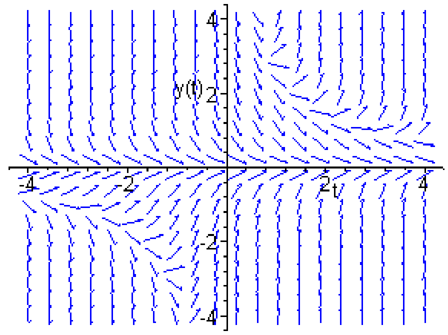


18.



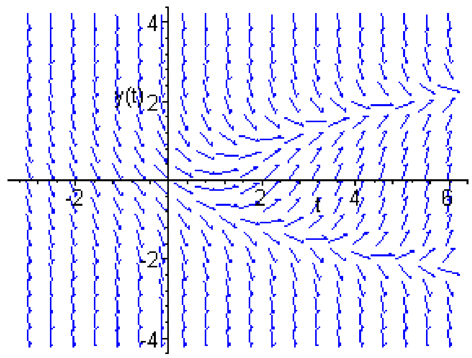
Based on the direction field, and the differential equation, for $y_0 < 0$, the slopes *eventually* become negative, and hence solutions tend to $-\infty$. For $y_0 < 0$, solutions increase without bound if $t_0 < 0$. Otherwise, the slopes *eventually* become negative, and solutions tend to *zero*. Furthermore, $y_0 = 0$ is an *equilibrium solution*. Note that slopes are *zero* along the curves $y = 0$ and $ty = 3$.

19.



For initial conditions (t_0, y_0) satisfying $ty < 3$, the respective solutions all tend to *zero*. Solutions with initial conditions *above or below* the hyperbola $ty = 3$ eventually tend to $\pm\infty$. Also, $y_0 = 0$ is an *equilibrium solution*.

20.



Solutions with $t_0 < 0$ all tend to $-\infty$. Solutions with initial conditions (t_0, y_0) to the *right* of the parabola $t = 1 + y^2$ asymptotically approach the parabola as $t \to \infty$. Integral curves with initial conditions *above* the parabola (and $y_0 > 0$) also approach the curve. The slopes for solutions with initial conditions *below* the parabola (and $y_0 < 0$) are all negative. These solutions tend to $-\infty$.

21. Define $y_c(t) = \frac{2}{3}(t-c)^{3/2}u(t-c)$, in which $u(t)$ is the Heaviside step function. Note that $y_c(c) = y_c(0) = 0$ and $y_c\big(c + (3/2)^{2/3}\big) = 1$.

(a). Let $c = 1 - (3/2)^{2/3}$.

(b). Let $c = 2 - (3/2)^{2/3}$.

(c). Observe that $y_0(2) = \frac{2}{3}(2)^{3/2}$, $y_c(t) < \frac{2}{3}(2)^{3/2}$ for $0 < c < 2$, and that $y_c(2) = 0$ for $c \geq 2$. So for any $c \geq 0$, $\pm y_c(2) \in [-2, 2]$.

26(a). Recalling Eq. (35) in Section 2.1,

$$y = \frac{1}{\mu(t)} \int \mu(s) g(s)\, ds + \frac{c}{\mu(t)}.$$

It is evident that $y_1(t) = \frac{1}{\mu(t)}$ and $y_2(t) = \frac{1}{\mu(t)} \int \mu(s) g(s)\, ds$.

(b). By definition, $\frac{1}{\mu(t)} = exp\big(- \int p(t)dt \big)$. Hence $y_1' = - p(t)\, \frac{1}{\mu(t)} = - p(t) y_1$. That is, $y_1' + p(t) y_1 = 0$.

(c). $y_2' = \Big(- p(t)\, \frac{1}{\mu(t)} \Big) \int_0^t \mu(s) g(s)\, ds + \Big(\frac{1}{\mu(t)} \Big) \mu(t) g(t) = - p(t) y_2 + g(t)$. That is, $y_2' + p(t) y_2 = g(t)$.

30. Since $n = 3$, set $v = y^{-2}$. It follows that $\frac{dv}{dt} = - 2y^{-3} \frac{dy}{dt}$ and $\frac{dy}{dt} = - \frac{y^3}{2} \frac{dv}{dt}$. Substitution into the differential equation yields $- \frac{y^3}{2} \frac{dv}{dt} - \varepsilon y = - \sigma y^3$, which further results in $v' + 2\varepsilon v = 2\sigma$. The latter differential equation is linear, and can be written as $(e^{2\varepsilon t})' = 2\sigma$. The solution is given by $v(t) = 2\sigma t\, e^{-2\varepsilon t} + c e^{-2\varepsilon t}$. Converting back to the original dependent variable, $y = \pm v^{-1/2}$.

31. Since $n = 3$, set $v = y^{-2}$. It follows that $\frac{dv}{dt} = - 2y^{-3} \frac{dy}{dt}$ and $\frac{dy}{dt} = - \frac{y^3}{2} \frac{dv}{dt}$. The differential equation is written as $- \frac{y^3}{2} \frac{dv}{dt} - (\Gamma cos\, t + T) y = \sigma y^3$, which upon further substitution is $v' + 2(\Gamma cos\, t + T) v = 2$. This ODE is linear, with integrating factor $\mu(t) = exp\big(2 \int (\Gamma cos\, t + T) dt \big) = exp(- 2\Gamma sin\, t + 2Tt)$. The solution is

$$v(t) = 2exp(2\Gamma sin\, t - 2Tt) \int_0^t exp(- 2\Gamma sin\, \tau + 2T\tau) d\tau + c\, exp(- 2\Gamma sin\, t + 2Tt).$$

Converting back to the original dependent variable, $y = \pm v^{-1/2}$.

33. The solution of the initial value problem $y_1' + 2y_1 = 0$, $y_1(0) = 1$ is $y_1(t) = e^{-2t}$. Therefore $y(1^-) = y_1(1) = e^{-2}$. On the interval $(1, \infty)$, the differential equation is $y_2' + y_2 = 0$, with $y_2(t) = ce^{-t}$. Therefore $y(1^+) = y_2(1) = ce^{-1}$. Equating the limits $y(1^-) = y(1^+)$, we require that $c = e^{-1}$. Hence the global solution of the initial value problem is

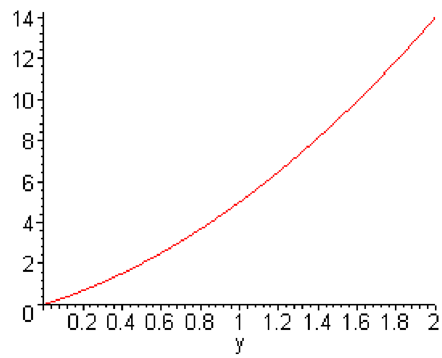
$$y(t) = \begin{cases} e^{-2t}, & 0 \le t \le 1 \\ e^{-1-t}, & t > 1 \end{cases}.$$

Note the discontinuity of the derivative

$$y(t) = \begin{cases} - 2e^{-2t}, & 0 < t < 1 \\ - e^{-1-t}, & t > 1 \end{cases}.$$

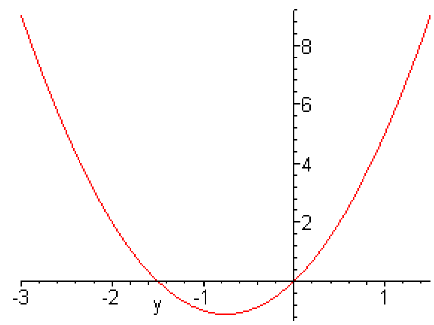
Section 2.5

1.



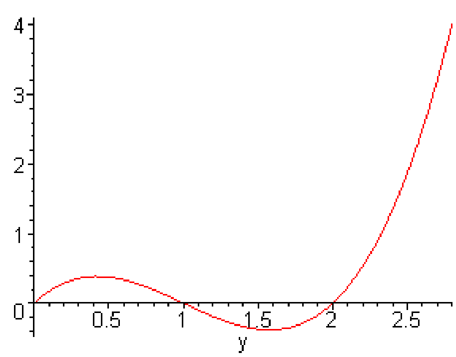
For $y_0 \geq 0$, the only equilibrium point is $y^* = 0$. $f'(0) = a > 0$, hence the equilibrium solution $\phi(t) = 0$ is *unstable*.

2.

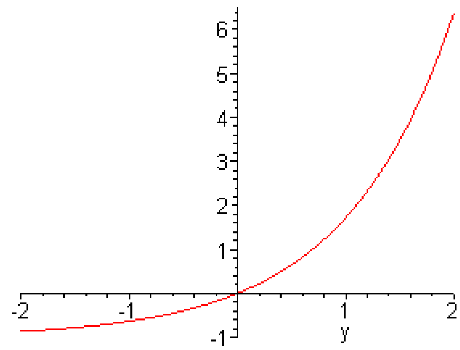


The equilibrium points are $y^* = -a/b$ and $y^* = 0$. $f'(-a/b) < 0$, therefore the equilibrium solution $\phi(t) = -a/b$ is *asymptotically stable*.

3.

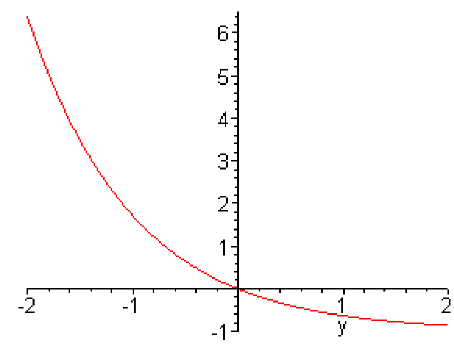


4.



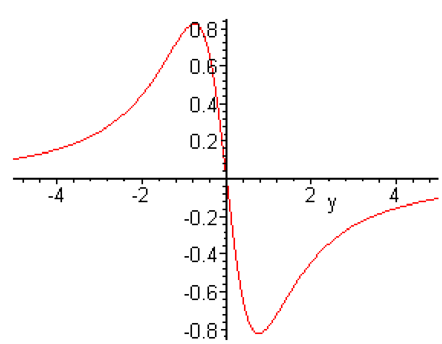
The only equilibrium point is $y^* = 0$. $f'(0) > 0$, hence the equilibrium solution $\phi(t) = 0$
is *unstable*.

5.

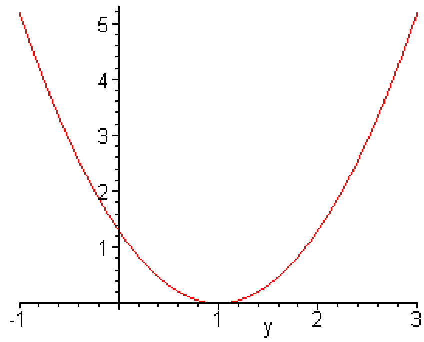


The only equilibrium point is $y^* = 0$. $f'(0) < 0$, hence the equilibrium solution $\phi(t) = 0$
is $asymptotically\, stable$.

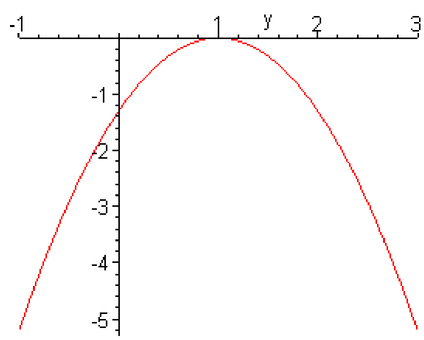
6.



$7(b)$.

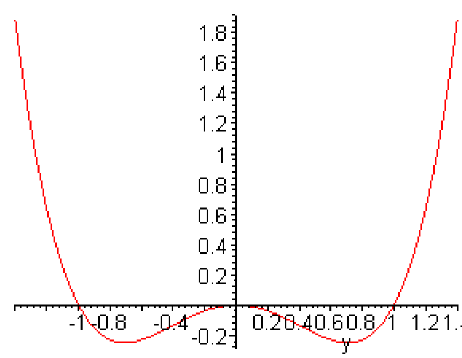


8.

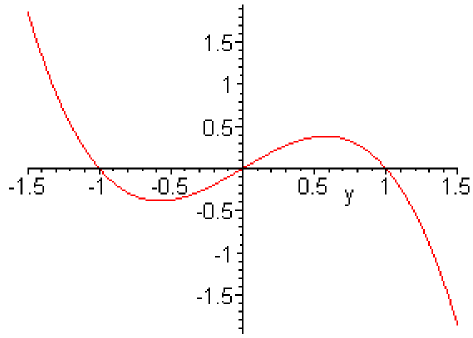


The only equilibrium point is $y^* = 1$. Note that $f'(1) = 0$, and that $y' < 0$ for $y \neq 1$. As long as $y_0 \neq 1$, the corresponding solution is *monotone decreasing*. Hence the equilibrium solution $\phi(t) = 1$ is *semistable*.

9.

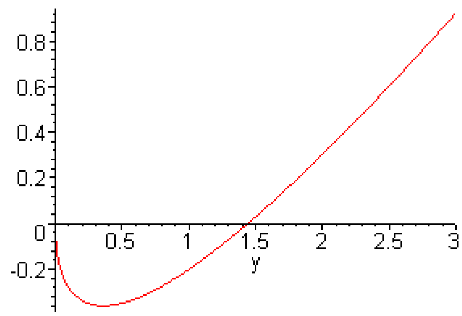


10.

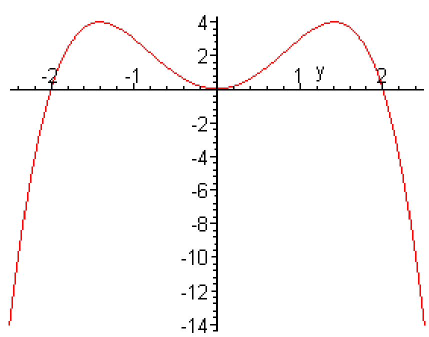


The equilibrium points are $y^* = 0, \pm 1$. $f'(y) = 1 - 3y^2$. The equilibrium solution $\phi(t) = 0$ is *unstable*, and the remaining two are *asymptotically stable*.

11.

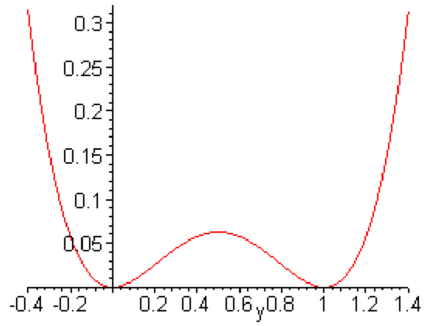


12.



The equilibrium points are $y^* = 0, \pm 2$. $f'(y) = 8y - 4y^3$. The equilibrium solutions $\phi(t) = -2$ and $\phi(t) = +2$ are *unstable* and *asymptotically stable*, respectively. The equilibrium solution $\phi(t) = 0$ is *semistable*.

13.



The equilibrium points are $y^* = 0$ and 1 . $f'(y) = 2y - 6y^2 + 4y^3$. Both equilibrium solutions are *semistable*.

$15(a)$. Inverting the Solution (11), Eq. (13) shows t as a function of the population y and
the carrying capacity K. With $y_0 = K/3$,

$$t = -\frac{1}{r} ln \left| \frac{(1/3)[1-(y/K)]}{(y/K)[1-(1/3)]} \right|.$$

Setting $y = 2y_0$,

$$\tau = -\frac{1}{r} ln \left| \frac{(1/3)[1-(2/3)]}{(2/3)[1-(1/3)]} \right|.$$

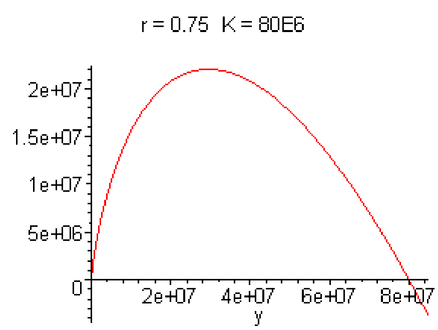
That is, $\tau = \frac{1}{r} ln\, 4$. If $r = 0.025$ *per year*, $\tau = 55.45$ *years*.

(b). In Eq. (13), set $y_0/K = \alpha$ and $y/K = \beta$. As a result, we obtain

$$T = -\frac{1}{r} ln \left| \frac{\alpha[1-\beta]}{\beta[1-\alpha]} \right|.$$

Given $\alpha = 0.1$, $\beta = 0.9$ and $r = 0.025$ *per year*, $\tau = 175.78$ *years*.

$16(a)$.



17. Consider the change of variable $u = ln(y/K)$. Differentiating both sides with respect
to t, $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$, with solution $u = u_0 e^{-rt}$. It follows that $ln(y/K) = ln(y_0/K)e^{-rt}$. That is,

$$\frac{y}{K} = exp\big[ln(y_0/K)e^{-rt}\big].$$

(a). Given $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$ *per year*, $y(2) = 57.58 \times 10^6$.

(b). Solving for t,

$$t = -\frac{1}{r} ln\left[\frac{ln(y/K)}{ln(y_0/K)}\right].$$

Setting $y(\tau) = 0.75K$, the corresponding time is $\tau = 2.21$ *years*.

$19(a)$. The rate of *increase* of the volume is given by rate of *flow in* $-$ rate of *flow out*. That is, $dV/dt = k - \alpha a\sqrt{2gh}$. Since the cross section is *constant*, $dV/dt = Adh/dt$. Hence the governing equation is $dh/dt = \left(k - \alpha a\sqrt{2gh}\right)/A$.

(b). Setting $dh/dt = 0$, the equilibrium height is $h_e = \frac{1}{2g}\left(\frac{k}{\alpha a}\right)^2$. Furthermore, since $f'(h_e) < 0$, it follows that the equilibrium height is *asymptotically stable*.

(c). Based on the answer in part(b), the water level will intrinsically tend to approach h_e. Therefore the height of the tank must be *greater* than h_e; that is, $h_e < V/A$.

$22(a)$. The equilibrium points are at $y^* = 0$ and $y^* = 1$. Since $f'(y) = \alpha - 2\alpha y$, the equilibrium solution $\phi = 0$ is *unstable* and the equilibrium solution $\phi = 1$ is *asymptotically stable*.

(b). The ODE is separable, with $[y(1-y)]^{-1}dy = \alpha\, dt$. Integrating both sides and invoking the initial condition, the solution is

$$y(t) = \frac{y_0\, e^{\alpha t}}{1 - y_0 + y_0\, e^{\alpha t}}.$$

It is evident that (independent of y_0) $\lim_{t\to -\infty} y(t) = 0$ and $\lim_{t\to\infty} y(t) = 1$.

$23(a)$. $y(t) = y_0\, e^{-\beta t}$.

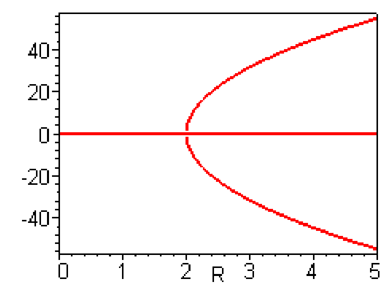
(b). From part(a), $dx/dt = \alpha\, x\, y_0\, e^{-\beta t}$. Separating variables, $dx/x = \alpha\, y_0\, e^{-\beta t}dt$. Integrating both sides, the solution is $x(t) = x_0\, exp\big[\alpha\, y_0/\beta\big(1 - e^{-\beta t}\big)\big]$.

(c). As $t \to \infty$, $y(t) \to 0$ and $x(t) \to x_0\, exp(\alpha\, y_0/\beta)$. Over a *long* period of time, the proportion of carriers *vanishes*. Therefore the proportion of the population that escapes the epidemic is the proportion of *susceptibles* left at that time, $x_0\, exp(\alpha\, y_0/\beta)$.

25(a). Note that $f(x) = x[(R - R_c) - a\,x^2]$, and $f'(x) = (R - R_c) - 3a\,x^2$. So if $(R - R_c) < 0$, the only equilibrium point is $x^* = 0$. $f'(0) < 0$, and hence the solution $\phi(t) = 0$ is *asymptotically stable*.

(b). If $(R - R_c) > 0$, there are *three* equilibrium points $x^* = 0\,, \pm\sqrt{(R - R_c)/a}$. Now $f'(0) > 0$, and $f'\left(\pm\sqrt{(R - R_c)/a}\,\right) < 0$. Hence the solution $\phi = 0$ is *unstable*, and the solutions $\phi = \pm\sqrt{(R - R_c)/a}$ are *asymptotically stable*.

(c).



Section 2.6

1. $M(x,y) = 2x + 3$ and $N(x,y) = 2y - 2$. Since $M_y = N_x = 0$, the equation is *exact*. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) =$ $= x^2 + 3x + h(y)$. Now $\psi_y = h'(y)$, and equating with N results in the possible function $h(y) = y^2 - 2y$. Hence $\psi(x,y) = x^2 + 3x + y^2 - 2y$, and the solution is defined *implicitly* as $x^2 + 3x + y^2 - 2y = c$.

2. $M(x,y) = 2x + 4y$ and $N(x,y) = 2x - 2y$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

4. First divide both sides by $(2xy + 2)$. We now have $M(x,y) = y$ and $N(x,y) = x$. Since $M_y = N_x = 0$, the resulting equation is *exact*. Integrating M with respect to x, while holding y constant, results in $\psi(x,y) = xy + h(y)$. Differentiating with respect to y, $\psi_y = x + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 0$, and hence $h(y) = 0$ is acceptable. Therefore the solution is defined *implicitly* as $xy = c$. Note that if $xy + 1 = 0$, the equation is trivially satisfied.

6. Write the given equation as $(ax - by)dx + (bx - cy)dy$. Now $M(x,y) = ax - by$ and $N(x,y) = bx - cy$. Since $M_y \neq N_x$, the differential equation is *not* exact.

8. $M(x,y) = e^x sin\, y + 3y$ and $N(x,y) = -3x + e^x sin\, y$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

10. $M(x,y) = y/x + 6x$ and $N(x,y) = ln\, x - 2$. Since $M_y = N_x = 1/x$, the given equation is *exact*. Integrating N with respect to y, while holding x constant, results in $\psi(x,y) = y\, ln\, x - 2y + h(x)$. Differentiating with respect to x, $\psi_x = y/x + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = 6x$, and hence $h(x) = 3x^2$. Therefore the solution
is defined *implicitly* as $3x^2 + y\, ln\, x - 2y = c$.

11. $M(x,y) = x\, ln\, y + xy$ and $N(x,y) = y\, ln\, x + xy$. Note that $M_y \neq N_x$, and hence the differential equation is *not* exact.

13. $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Since $M_y = N_x = -1$, the equation is *exact*. Integrating M with respect to x, while holding y constant, yields $\psi(x,y) =$ $= x^2 - xy + h(y)$. Now $\psi_y = -x + h'(y)$. Equating ψ_y with N results in $h'(y) = 2y$, and hence $h(y) = y^2$. Thus $\psi(x,y) = x^2 - xy + y^2$, and the solution is given *implicitly* as $x^2 - xy + y^2 = c$. Invoking the initial condition $y(1) = 3$, the specific solution is $x^2 - xy + y^2 = 7$. The *explicit* form of the solution is $y(x) = \frac{1}{2}\left[x + \sqrt{28 - 3x^2}\,\right]$. Hence the solution is valid as long as $3x^2 \leq 28$.

16. $M(x,y) = y\, e^{2xy} + x$ and $N(x,y) = bx\, e^{2xy}$. Note that $M_y = e^{2xy} + 2xy\, e^{2xy}$, and $N_x = b\, e^{2xy} + 2bxy\, e^{2xy}$. The given equation is *exact*, as long as $b = 1$. Integrating

N with respect to y, while holding x constant, results in $\psi(x,y) = e^{2xy}/2 + h(x)$. Now differentiating with respect to x, $\psi_x = y\,e^{2xy} + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = x$, and hence $h(x) = x^2/2$. Conclude that $\psi(x,y) = e^{2xy}/2 + x^2/2$. Hence the solution is given *implicitly* as $e^{2xy} + x^2 = c$.

17. Integrating $\psi_y = N$, while holding x constant, yields
$\psi(x,y) = \int N(x,y)dy + h(x)$.
Taking the partial derivative with respect to x, $\psi_x = \int \frac{\partial}{\partial x} N(x,y)dy + h'(x)$. Now set $\psi_x = M(x,y)$ and therefore $h'(x) = M(x,y) - \int \frac{\partial}{\partial x} N(x,y)dy$. Based on the fact that $M_y = N_x$, it follows that $\frac{\partial}{\partial y}[h'(x)] = 0$. Hence the expression for $h'(x)$ can be integrated to obtain

$$h(x) = \int M(x,y)dx - \int \left[\int \frac{\partial}{\partial x} N(x,y)dy\right] dx\,.$$

18. Observe that $\frac{\partial}{\partial y}[M(x)] = \frac{\partial}{\partial x}[N(y)] = 0$.

20. $M_y = y^{-1}\cos y - y^{-2}\sin y$ and $N_x = -2\,e^{-x}(\cos x + \sin x)/y$. Multiplying both sides by the integrating factor $\mu(x,y) = y\,e^x$, the given equation can be written as $(e^x \sin y - 2y\sin x)dx + (e^x\cos y + 2\cos x)dy = 0$. Let $\overline{M} = \mu M$ and $\overline{N} = \mu N$. Observe that $\overline{M}_y = \overline{N}_x$, and hence the latter ODE is *exact*. Integrating $\overline{N}$ with respect to y, while holding x constant, results in $\psi(x,y) = e^x \sin y + 2y\cos x + h(x)$. Now differentiating with respect to x, $\psi_x = e^x\sin y - 2y\sin x + h'(x)$. Setting $\psi_x = \overline{M}$, we find that $h'(x) = 0$, and hence $h(x) = 0$ is feasible. Hence the solution of the given equation is defined *implicitly* by $e^x \sin y + 2y\cos x = \beta$.

21. $M_y = 1$ and $N_x = 2$. Multiply both sides by the integrating factor $\mu(x,y) = y$ to obtain $y^2dx + (2xy - y^2e^y)dy = 0$. Let $\overline{M} = yM$ and $\overline{N} = yN$. It is easy to see that $\overline{M}_y = \overline{N}_x$, and hence the latter ODE is *exact*. Integrating $\overline{M}$ with respect to x yields $\psi(x,y) = xy^2 + h(y)$. Equating ψ_y with $\overline{N}$ results in $h'(y) = -y^2e^y$, and hence $h(y) = -e^y(y^2 - 2y + 2)$. Thus $\psi(x,y) = xy^2 - e^y(y^2 - 2y + 2)$, and the solution is defined *implicitly* by $xy^2 - e^y(y^2 - 2y + 2) = c$.

24. The equation $\mu M + \mu N y' = 0$ has an integrating factor if $(\mu M)_y = (\mu N)_x$, that is, $\mu_y M - \mu_x N = \mu N_x - \mu M_y$. Suppose that $N_x - M_y = R\,(xM - yN)$, in which R is some function depending *only* on the quantity $z = xy$. It follows that the modified form of the equation is *exact*, if $\mu_y M - \mu_x N = \mu R\,(xM - yN) = R\,(\mu\, xM - \mu\, yN)$. This relation is satisfied if $\mu_y = (\mu\, x)R$ and $\mu_x = (\mu\, y)R$. Now consider $\mu = \mu(xy)$. Then the partial derivatives are $\mu_x = \mu' y$ and $\mu_y = \mu' x$. Note that $\mu' = d\mu/dz$. Thus μ must satisfy $\mu'(z) = R(z)$. The latter equation is *separable*, with $d\mu = R(z)dz$, and $\mu(z) = \int R(z)dz$. Therefore, given $R = R(xy)$, it is possible to determine $\mu = \mu(xy)$ which becomes an integrating factor of the differential equation.

28. The equation is not exact, since $N_x - M_y = 2y - 1$. However, $(N_x - M_y)/M =$ $= (2y-1)/y$ is a function of y alone. Hence there exists $\mu = \mu(y)$, which is a solution of the differential equation $\mu' = (2 - 1/y)\mu$. The latter equation is *separable*, with $d\mu/\mu = 2 - 1/y$. One solution is $\mu(y) = exp(2y - ln\, y) = e^{2y}/y$. Now rewrite the given ODE as $e^{2y}dx + (2x\, e^{2y} - 1/y)dy = 0$. This equation is *exact*, and it is easy to see that $\psi(x,y) = x\, e^{2y} - ln\, y$. Therefore the solution of the given equation is defined implicitly by $x\, e^{2y} - ln\, y = c$.

30. The given equation is not exact, since $N_x - M_y = 8x^3/y^3 + 6/y^2$. But note that $(N_x - M_y)/M = 2/y$ is a function of y alone, and hence there is an integrating factor $\mu = \mu(y)$. Solving the equation $\mu' = (2/y)\mu$, an integrating factor is $\mu(y) = y^2$. Now rewrite the differential equation as $(4x^3 + 3y)dx + (3x + 4y^3)dy = 0$. By inspection, $\psi(x,y) = x^4 + 3xy + y^4$, and the solution of the given equation is defined implicitly by $x^4 + 3xy + y^4 = c$.

32. Multiplying both sides of the ODE by $\mu = [xy(2x+y)]^{-1}$, the given equation is equivalent to $[(3x+y)/(2x^2+xy)]dx + [(x+y)/(2xy+y^2)]dy = 0$. Rewrite the differential equation as

$$\left[\frac{2}{x} + \frac{2}{2x+y}\right]dx + \left[\frac{1}{y} + \frac{1}{2x+y}\right]dy = 0\,.$$

It is easy to see that $M_y = N_x$. Integrating M with respect to x, while keeping y constant, results in $\psi(x,y) = 2ln|x| + ln|2x+y| + h(y)$. Now taking the partial derivative with respect to y , $\psi_y = (2x+y)^{-1} + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 1/y$, and hence $h(y) = ln\,|y|$. Therefore

$$\psi(x,y) = 2ln|x| + ln|2x+y| + ln\,|y|\,,$$

and the solution of the given equation is defined implicitly by $2x^3y + x^2y^2 = c$.

Section 2.7

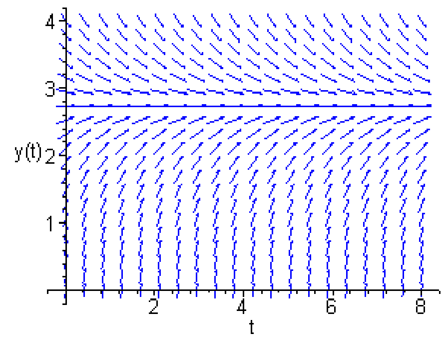
$2(a)$. The Euler formula is $y_{n+1} = y_n + h(2y_n - 1) = (1 + 2h)y_n - h$.

(d). The differential equation is *linear*, with solution $y(t) = (1 + e^{2t})/2$.

$4(a)$. The Euler formula is $y_{n+1} = (1 - 2h)y_n + 3h\, cos\, t_n$.

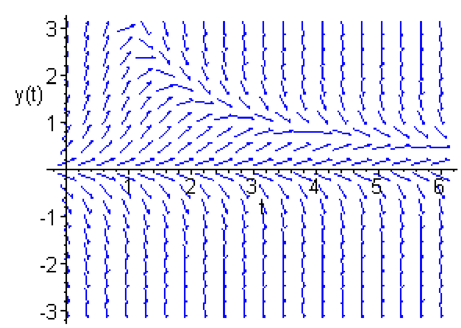
(d). The exact solution is $y(t) = (6cos\, t + 3sin\, t - 6\, e^{-2t})/5$.

5.



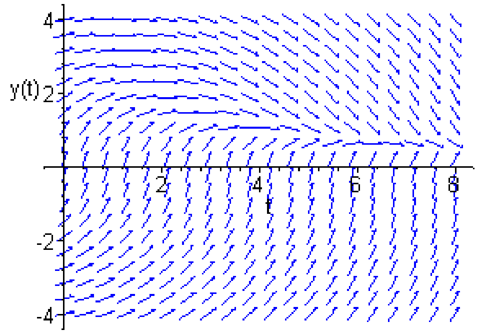
All solutions seem to converge to $\phi(t) = 25/9$.

6.



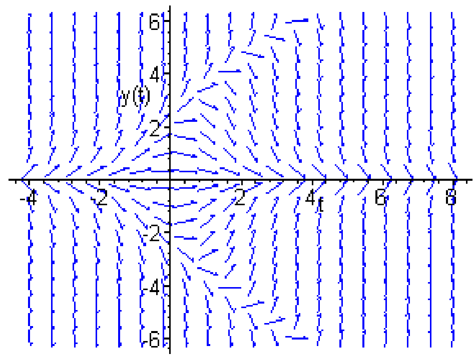
Solutions with *positive* initial conditions seem to converge to a specific function. On the other hand, solutions with *negative* coefficients decrease without bound. $\phi(t) = 0$ is an equilibrium solution.

7.



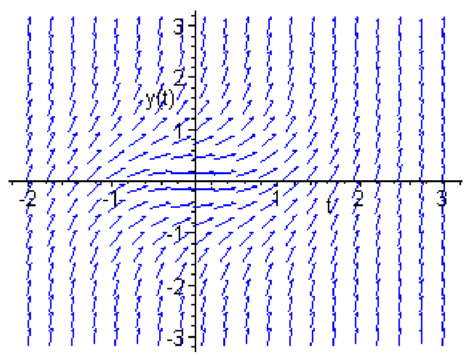
All solutions seem to converge to a specific function.

8.



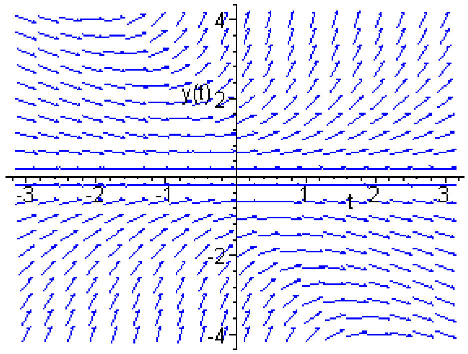
Solutions with initial conditions to the *'left'* of the curve $t = 0.1y^2$ seem to diverge. On the other hand, solutions to the *'right'* of the curve seem to converge to *zero*. Also, $\phi(t)$ is an equilibrium solution.

9.



All solutions seem to diverge.

10.



Solutions with *positive* initial conditions increase without bound. Solutions with
negative
initial conditions decrease without bound. Note that $\phi(t) = 0$ is an equilibrium solution.

11. The Euler formula is $y_{n+1} = y_n - 3h\sqrt{y_n} + 5h$. The initial value is $y_0 = 2$.

12. The iteration formula is $y_{n+1} = (1 + 3h)y_n - h\, t_n\, y_n^2$. $(t_0, y_0) = (0\,, 0.5)$.

14. The iteration formula is $y_{n+1} = (1 - h\, t_n\,)y_n + h\, y_n^3\,/10$. $(t_0, y_0) = (0\,, 1)$.

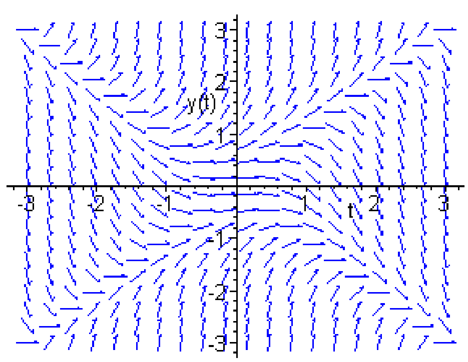
17. The Euler formula is

$$y_{n+1} = y_n + \frac{h(y_n^2 + 2t_n\, y_n)}{3 + t_n^2}\,.$$

The initial point is $(t_0, y_0) = (1\,, 2)$.

$18(a)$. See Problem 8.

$19(a)$.



(b). The iteration formula is $y_{n+1} = y_n + h\, y_n^2 - h\, t_n^2$. The critical value of α appears to be near $\alpha_0 \approx 0.6815$. For $y_0 > \alpha_0$, the iterations diverge.

$20(a)$. The ODE is *linear*, with general solution $y(t) = t + c\,e^t$. Invoking the specified initial condition, $y(t_0) = y_0$, we have $y_0 = t_0 + c\,e^{t_0}$. Hence $c = (y_0 - t_0)e^{-t_0}$. Thus the solution is given by $\phi(t) = (y_0 - t_0)e^{t-t_0} + t$.

(b). The Euler formula is $y_{n+1} = (1+h)y_n + h - h\,t_n$. Now set $k = n+1$.

(c). We have $y_1 = (1+h)y_0 + h - h\,t_0 = (1+h)y_0 + (t_1 - t_0) - h\,t_0$. Rearranging the terms, $y_1 = (1+h)(y_0 - t_0) + t_1$. Now suppose that $y_k = (1+h)^k(y_0 - t_0) + t_k$, for some $k \geq 1$. Then $y_{k+1} = (1+h)y_k + h - h\,t_k$. Substituting for y_k, we find that $y_{k+1} = (1+h)^{k+1}(y_0 - t_0) + (1+h)\,t_k + h - h\,t_k = (1+h)^{k+1}(y_0 - t_0) + t_k + h$. Noting that $t_{k+1} = t_k + k$, the result is verified.

(d). Substituting $h = (t - t_0)/n$, with $t_n = t$,

$$y_n = \left(1 + \frac{t - t_0}{n}\right)^n (y_0 - t_0) + t\,.$$

Taking the limit of both sides, as $n \to \infty$, and using the fact that $\lim_{n\to\infty}(1 + a/n)^n = e^a$, pointwise convergence is proved.

21. The exact solution is $\phi(t) = e^t$. The Euler formula is $y_{n+1} = (1+h)y_n$. It is easy to see that $y_n = (1+h)^n y_0 = (1+h)^n$. Given $t > 0$, set $h = t/n$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}(1 + t/n)^n = e^t$.

23. The exact solution is $\phi(t) = t/2 + e^{2t}$. The Euler formula is $y_{n+1} = (1+2h)y_n +$ $+ h/2 - h\,t_n$. Since $y_0 = 1$, $y_1 = (1+2h) + h/2 = (1+2h) + t_1/2$. It is easy to show by mathematical induction, that $y_n = (1+2h)^n + t_n/2$. For $t > 0$, set $h = t/n$ and thus $t_n = t$. Taking the limit, we find that $\lim_{n\to\infty} y_n = \lim_{n\to\infty}[(1 + 2t/n)^n + t/2] =$ $= e^{2t} + t/2$. Hence pointwise convergence is proved.

Section 2.8

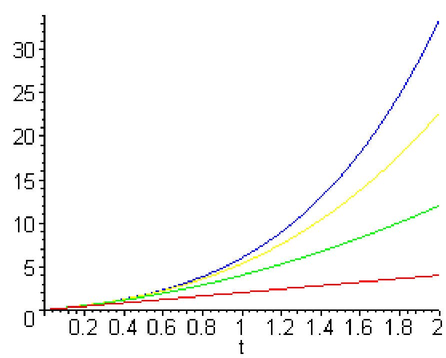
2. Let $z = y - 3$ and $\tau = t + 1$. It follows that $dz/d\tau = (dz/dt)(dt/d\tau) = dz/dt$. Furthermore, $dz/dt = dy/dt = 1 - y^3$. Hence $dz/d\tau = 1 - (z+3)^3$. The new initial condition is $z(\tau = 0) = 0$.

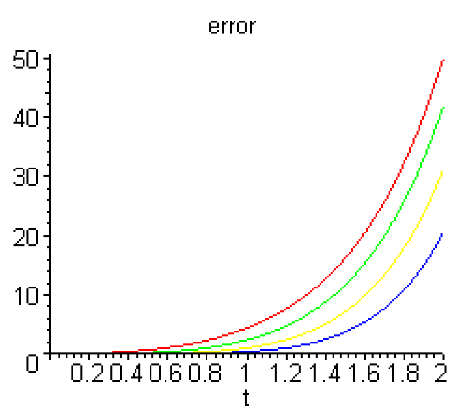
3. The approximating functions are defined recursively by $\phi_{n+1}(t) = \int_0^t 2[\phi_n(s) + 1]ds$. Setting $\phi_0(t) = 0$, $\phi_1(t) = 2t$. Continuing, $\phi_2(t) = 2t^2 + 2t$, $\phi_3(t) = \frac{4}{3}t^3 + 2t^2 + 2t$, $\phi_4(t) = \frac{2}{3}t^4 + \frac{4}{3}t^3 + 2t^2 + 2t$, $\cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= 2t + \sum_{k=2}^{\infty}\frac{a_k}{k\,!}t^k .\end{aligned}$$

Comparing coefficients, $a_3/3! = 4/3$, $a_4/4! = 2/3$, $\cdots$. It follows that $a_3 = 8$, $a_4 = 16$,
and so on. We find that in general, that $a_k = 2^k$. Hence

$$\begin{aligned}\phi(t) &= \sum_{k=1}^{\infty}\frac{2^k}{k\,!}t^k \\ &= e^{2t} - 1 .\end{aligned}$$





5. The approximating functions are defined recursively by

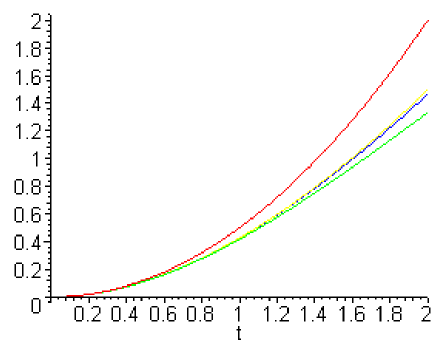
$$\phi_{n+1}(t) = \int_0^t [-\phi_n(s)/2 + s]ds .$$

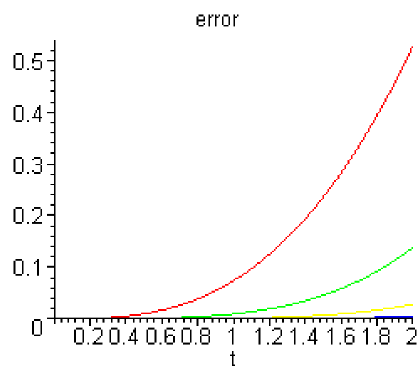
Setting $\phi_0(t) = 0$, $\phi_1(t) = t^2/2$. Continuing, $\phi_2(t) = t^2/2 - t^3/12$, $\phi_3(t) = t^2/2 - - t^3/12 + t^4/96$, $\phi_4(t) = t^2/2 - t^3/12 + t^4/96 - t^5/960$, $\cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= t^2/2 + \sum_{k=3}^{\infty}\frac{a_k}{k\,!}t^k\,.\end{aligned}$$

Comparing coefficients, $a_3/3! = -1/12\,, a_4/4! = 1/96\,, a_5/5! = -1/960\,, \cdots$. We find that $a_3 = -1/2\,, a_4 = 1/4, a_5 = -1/8\,, \cdots$. In general, $a_k = 2^{-k+1}$. Hence

$$\begin{aligned}\phi(t) &= \sum_{k=2}^{\infty}\frac{2^{-k+2}}{k\,!}(-t)^k \\ &= 4\,e^{-t/2} + 2t - 4\,.\end{aligned}$$





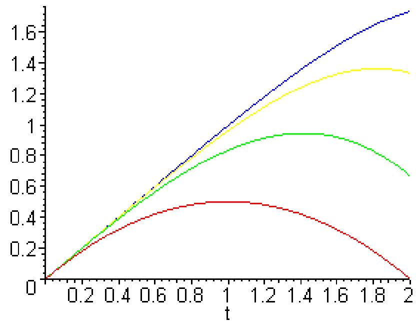
6. The approximating functions are defined recursively by

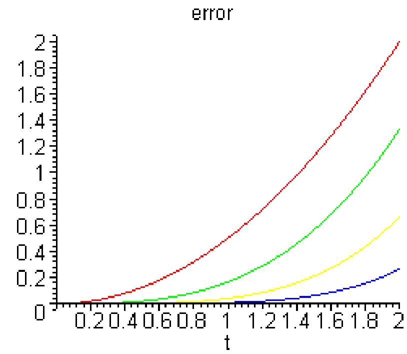
$$\phi_{n+1}(t) = \int_0^t [\phi_n(s) + 1 - s]ds\,.$$

Setting $\phi_0(t) = 0\,,\ \phi_1(t) = t - t^2/2\,, \phi_2(t) = t - t^3/6\,, \phi_3(t) = t - t^4/24\,, \phi_4(t) = $ $= t - t^5/120\,, \cdots$. Given convergence, set

$$\begin{aligned}\phi(t) &= \phi_1(t) + \sum_{k=1}^{\infty}[\phi_{k+1}(t) - \phi_k(t)] \\ &= t - t^2/2 + \left[t^2/2 - t^3/6\right] + \left[t^3/6 - t^4/24\right] + \cdots \\ &= t + 0 + 0 + \cdots\,.\end{aligned}$$

Note that the terms can be rearranged, as long as the series converges *uniformly*.





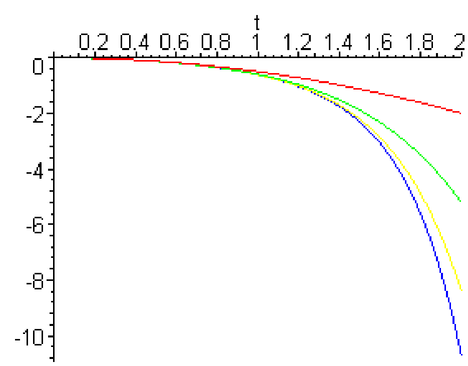
$8(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2\phi_n(s) - s\right] ds\,.$$

Set $\phi_0(t) = 0$. The iterates are given by $\phi_1(t) = -t^2/2$, $\phi_2(t) = -t^2/2 - t^5/10$, $\phi_3(t) = -t^2/2 - t^5/10 - t^8/80$, $\phi_4(t) = -t^2/2 - t^5/10 - t^8/80 - t^{11}/880, \cdots$. Upon inspection, it becomes apparent that

$$\begin{aligned}\phi_n(t) &= -t^2\left[\frac{1}{2} + \frac{t^3}{2\cdot 5} + \frac{t^6}{2\cdot 5\cdot 8} + \cdots + \frac{(t^3)^{n-1}}{2\cdot 5\cdot 8\cdots[2+3(n-1)]}\right] \\ &= -t^2\sum_{k=1}^{n}\frac{(t^3)^{k-1}}{2\cdot 5\cdot 8\cdots[2+3(k-1)]}\,.\end{aligned}$$

(b).



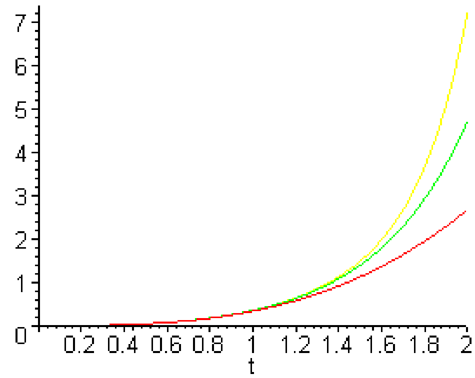
The iterates appear to be converging.

$9(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[s^2 + \phi_n^2(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t^3/3$, $\phi_2(t) = t^3/3 + t^7/63$, $\phi_3(t) = t^3/3 + t^7/63 + 2t^{11}/2079 + t^{15}/59535$.

(b).



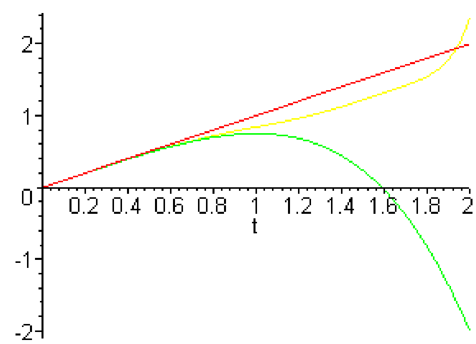
The iterates appear to be converging.

$10(a)$. The approximating functions are defined recursively by

$$\phi_{n+1}(t) = \int_0^t \left[1 - \phi_n^3(s)\right] ds\,.$$

Set $\phi_0(t) = 0$. The first three iterates are given by $\phi_1(t) = t$, $\phi_2(t) = t - t^4/4$, $\phi_3(t) = t - t^4/4 + 3t^7/28 - 3t^{10}/160 + t^{13}/833$.

(b).



The approximations appear to be diverging.

$12(a)$. The approximating functions are defined recursively by

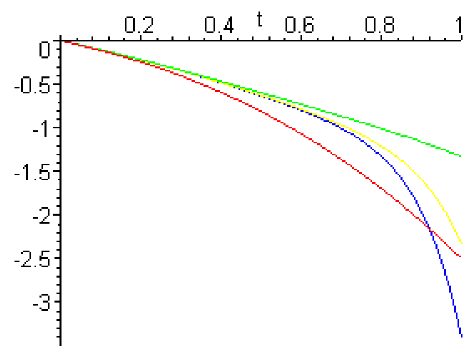
$$\phi_{n+1}(t) = \int_0^t \left[\frac{3s^2 + 4s + 2}{2(\phi_n(s) - 1)}\right] ds\,.$$

Note that $1/(2y-2) = -\frac{1}{2}\sum_{k=0}^{6} y^k + O(y^7)$. For computational purposes, replace the above iteration formula by

$$\phi_{n+1}(t) = -\frac{1}{2}\int_0^t \left[\left(3s^2+4s+2\right)\sum_{k=0}^{6} \phi_n^k(s) \right] ds \,.$$

Set $\phi_0(t) = 0$. The first four approximations are given by $\phi_1(t) = -t - t^2 - t^3/2$,
$\phi_2(t) = -t - t^2/2 + t^3/6 + t^4/4 - t^5/5 - t^6/24 + \cdots$,
$\phi_3(t) = -t - t^2/2 + t^4/12 - 3t^5/20 + 4t^6/45 + \cdots$,
$\phi_4(t) = -t - t^2/2 + t^4/8 - 7t^5/60 + t^6/15 + \cdots$

(b).



The approximations appear to be converging to the exact solution,

$$\phi(t) = 1 - \sqrt{1 + 2t + 2t^2 + t^3} \,.$$

13. Note that $\phi_n(0) = 0$ and $\phi_n(1) = 1\,, \forall\, n \geq 1$. Let $a \in (0\,,1)$. Then $\phi_n(a) = a^n$. Clearly, $\lim_{n\to\infty} a^n = 0$. Hence the assertion is true.

14(a). $\phi_n(0) = 0\,, \forall\, n \geq 1$. Let $a \in (0\,,1]$. Then $\phi_n(a) = 2na\, e^{-na^2} = 2na/e^{na^2}$. Using l'Hospital's rule, $\lim_{z\to\infty} 2az/e^{az^2} = \lim_{z\to\infty} 1/ze^{az^2} = 0$. Hence $\lim_{n\to\infty} \phi_n(a) = 0$.

(b). $\int_0^1 2nx\, e^{-nx^2} dx = -e^{-nx^2}\big|_0^1 = 1 - e^{-n}$. Therefore,

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)dx \neq \int_0^1 \lim_{n\to\infty} \phi_n(x)dx \,.$$

15. Let t be fixed, such that $(t\,, y_1), (t\,, y_2) \in D$. Without loss of generality, assume that $y_1 < y_2$. Since f is differentiable with respect to y, the mean value theorem asserts that $\exists\, \xi \in (y_1\,, y_2)$ such that $f(t\,, y_1) - f(t\,, y_2) = f_y(t\,, \xi)(y_1 - y_2)$. Taking the absolute value of both sides, $|f(t\,, y_1) - f(t\,, y_2)| = |f_y(t\,, \xi)|\,|y_1 - y_2|$. Since, by assumption, $\partial f/\partial y$ is continuous in D, f_y attains a *maximum* on any closed and bounded subset of D
.

Hence $|f(t\,,y_1) - f(t\,,y_2)| \leq K\,|y_1 - y_2|$.

16. For a *sufficiently small* interval of t, $\phi_{n-1}(t)\,,\phi_n(t) \in D$. Since f satisfies a Lipschitz condition, $|f(t\,,\phi_n(t)) - f(t\,,\phi_{n-1}(t))| \leq K\,|\phi_n(t) - \phi_{n-1}(t)|$. Here $K = max\,|f_y|$.

17(a). $\phi_1(t) = \int_0^t f(s\,,0)ds$. Hence $|\phi_1(t)| \leq \int_0^{|t|}|f(s\,,0)|ds \leq \int_0^{|t|}M ds = M|t|$, in which M is the maximum value of $|f(t\,,y)|$ on D.

(b). By definition, $\phi_2(t) - \phi_1(t) = \int_0^t [f(s\,,\phi_1(s)) - f(s\,,0)]ds$. Taking the absolute value of both sides, $|\phi_2(t) - \phi_1(t)| \leq \int_0^{|t|}|[f(s\,,\phi_1(s)) - f(s\,,0)]|ds$. Based on the results in Problems 16 and 17, $|\phi_2(t) - \phi_1(t)| \leq \int_0^{|t|}K|\phi_1(s) - 0|ds \leq KM\int_0^{|t|}|s|ds$. Evaluating the last integral, we obtain $|\phi_2(t) - \phi_1(t)| \leq MK|t|^2/2$.

(c). Suppose that

$$|\phi_i(t) - \phi_{i-1}(t)| \leq \frac{MK^{\,i-1}|t|^i}{i\,!}$$

for some $i \geq 1$. By definition, $\phi_{i+1}(t) - \phi_i(t) = \int_0^t [f(t\,,\phi_i(s)) - f(s\,,\phi_{i-1}(s))]ds$. It follows that

$$\begin{aligned}
|\phi_{i+1}(t) - \phi_i(t)| &\leq \int_0^{|t|}|f(s\,,\phi_i(s)) - f(s\,,\phi_{i-1}(s))|ds \\
&\leq \int_0^{|t|}K|\phi_i(s) - \phi_{i-1}(s)|ds \\
&\leq \int_0^{|t|}K\frac{MK^{\,i-1}|s|^i}{i\,!}ds \\
&= \frac{MK^{\,i}|t|^{i+1}}{(i+1)\,!} \leq \frac{MK^{\,i}h^{i+1}}{(i+1)\,!}\,.
\end{aligned}$$

Hence, by mathematical induction, the assertion is true.

18(a). Use the triangle inequality, $|a+b| \leq |a| + |b|$.

(b). For $|t| \leq h$, $|\phi_1(t)| \leq Mh$, and $|\phi_n(t) - \phi_{n-1}(t)| \leq MK^{\,n-1}h^n/(n\,!)$. Hence

$$\begin{aligned}
|\phi_n(t)| &\leq M\sum_{i=1}^{n}\frac{K^{\,i-1}h^i}{i\,!} \\
&= \frac{M}{K}\sum_{i=1}^{n}\frac{(Kh)^i}{i\,!}\,.
\end{aligned}$$

(c). The sequence of partial sums in (b) converges to $\frac{M}{K}\left(e^{Kh}-1\right)$. By the *comparison test*, the sums in (a) also converge. Furthermore, the sequence $|\phi_n(t)|$ is *bounded*, and hence has a convergent subsequence. Finally, since individual terms of the series must tend to zero, $|\phi_n(t)-\phi_{n-1}(t)|\to 0$, and it follows that the sequence $|\phi_n(t)|$ is convergent.

19(a). Let $\phi(t)=\int_0^t f(s\,,\phi(s))ds$ and $\psi(t)=\int_0^t f(s\,,\psi(s))ds$. Then by *linearity* of the integral, $\phi(t)-\psi(t)=\int_0^t[f(s\,,\phi(s))-f(s\,,\psi(s))]ds$.

(b). It follows that $|\phi(t)-\psi(t)|\le\int_0^t|f(s\,,\phi(s))-f(s\,,\psi(s))|ds$.

(c). We know that f satisfies a Lipschitz condition,

$$|f(t\,,y_1)-f(t\,,y_2)|\le K\,|y_1-y_2|\,,$$

based on $|\partial f/\partial y|\le K$ in D. Therefore,

$$\begin{aligned}|\phi(t)-\psi(t)| &\le \int_0^t|f(s\,,\phi(s))-f(s\,,\psi(s))|ds\\ &\le \int_0^t K|\phi(s)-\psi(s)|ds\,.\end{aligned}$$

Section 2.9

1. Writing the equation for each $n \geq 0$, $y_1 = -0.9\, y_0$, $y_2 = -0.9\, y_1$, $y_3 = -0.9\, y_2$ and so on, it is apparent that $y_n = (-0.9)^n\, y_0$. The terms constitute an *alternating series*, which converge to *zero*, regardless of y_0.

3. Write the equation for each $n \geq 0$, $y_1 = \sqrt{3}\, y_0$, $y_2 = \sqrt{4/2}\, y_1$, $y_3 = \sqrt{5/3}\, y_2, \cdots$. Upon substitution, we find that $y_2 = \sqrt{(4 \cdot 3)/2}\, y_1$, $y_3 = \sqrt{(5 \cdot 4 \cdot 3)/(3 \cdot 2)}\, y_0, \cdots$. It can be proved by mathematical induction, that

$$\begin{aligned} y_n &= \frac{1}{\sqrt{2}} \sqrt{\frac{(n+2)!}{n!}}\, y_0 \\ &= \frac{1}{\sqrt{2}} \sqrt{(n+1)(n+2)}\, y_0 . \end{aligned}$$

This sequence is *divergent*, except for $y_0 = 0$.

4. Writing the equation for each $n \geq 0$, $y_1 = -y_0$, $y_2 = y_1$, $y_3 = -y_2$, $y_4 = y_3$, and so on, it can be shown that

$$y_n = \begin{cases} y_0 & , \text{ for } n = 4k \text{ or } n = 4k-1 \\ -y_0 & , \text{ for } n = 4k-2 \text{ or } n = 4k-3 \end{cases}$$

The sequence is convergent *only* for $y_0 = 0$.

6. Writing the equation for each $n \geq 0$,

$$\begin{aligned} y_1 &= 0.5\, y_0 + 6 \\ y_2 &= 0.5\, y_1 + 6 = 0.5(0.5\, y_0 + 6) + 6 = (0.5)^2 y_0 + 6 + (0.5)6 \\ y_3 &= 0.5\, y_2 + 6 = 0.5(0.5\, y_1 + 6) + 6 = (0.5)^3 y_0 + 6[1 + (0.5) + (0.5)^2] \\ &\vdots \\ y_n &= (0.5)^n y_0 + 12[1 - (0.5)^n\,] \end{aligned}$$

which can be verified by mathematical induction. The sequence is convergent for all y_0, and in fact $y_n \to 12$.

7. Let y_n be the balance at the end of the *n-th* day. Then $y_{n+1} = (1 + r/356)\, y_n$. The solution of this difference equation is $y_n = (1 + r/365)^n\, y_0$, in which y_0 is the initial balance. At the end of *one year*, the balance is $y_{365} = (1 + r/365)^{365}\, y_0$. Given that $r = .07$, $y_{365} = (1 + r/365)^{365}\, y_0 = 1.0725\, y_0$. Hence the effective annual yield is $(1.0725\, y_0 - y_0)/y_0 = 7.25\,\%$.

8. Let y_n be the balance at the end of the *n-th* month. Then $y_{n+1} = (1 + r/12)\, y_n + 25$. As in the previous solutions, we have

$$y_n = \rho^n \left[y_0 - \frac{25}{1-\rho} \right] + \frac{25}{1-\rho} \,,$$

in which $\rho = (1 + r/12)$. Here r is the annual interest rate, given as $8\,\%$. Therefore $y_{36} = (1.0066)^{36} \left[1000 + \frac{(12)25}{r}\right] - \frac{(12)25}{r} = 2,283.63$ *dollars*.

9. Let y_n be the balance due at the end of the n-*th* month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$. Here r is the annual interest rate and P is the monthly payment. The solution, in terms of the amount borrowed, is given by

$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,,$$

in which $\rho = (1 + r/12)$ and $y_0 = 8,000$. To figure out the monthly payment, P, we require that $y_{36} = 0$. That is,

$$\rho^{36} \left[y_0 + \frac{P}{1-\rho} \right] = \frac{P}{1-\rho} \,.$$

After the specified amounts are substituted, we find the $P = \$\,258.14$.

11. Let y_n be the balance due at the end of the n-*th* month. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = .09$ and P is the monthly payment. The initial value of the mortgage is $y_0 = 100,000$ *dollars*. Then the balance due at the end of the n-*th* month is

$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,.$$

where $\rho = (1 + r/12)$. In terms of the specified values,

$$y_n = (0.0075)^n \left[10^5 - \frac{12P}{r} \right] + \frac{12P}{r} \,.$$

Setting $n = 30(12) = 360$, and $y_{360} = 0$, we find that $P = 804.62$ *dollars*. For the monthly payment corresponding to a 20 year mortgage, set $n = 240$ and $y_{240} = 0$.

12. Let y_n be the balance due at the end of the n-*th* month, with y_0 the initial value of the mortgage. The appropriate difference equation is $y_{n+1} = (1 + r/12)\, y_n - P$, in which $r = 0.1$ and $P = 900$ *dollars* is the *maximum* monthly payment. Given that the life of the mortgage is 20 *years*, we require that $y_{240} = 0$. The balance due at the end of the n-*th* month is

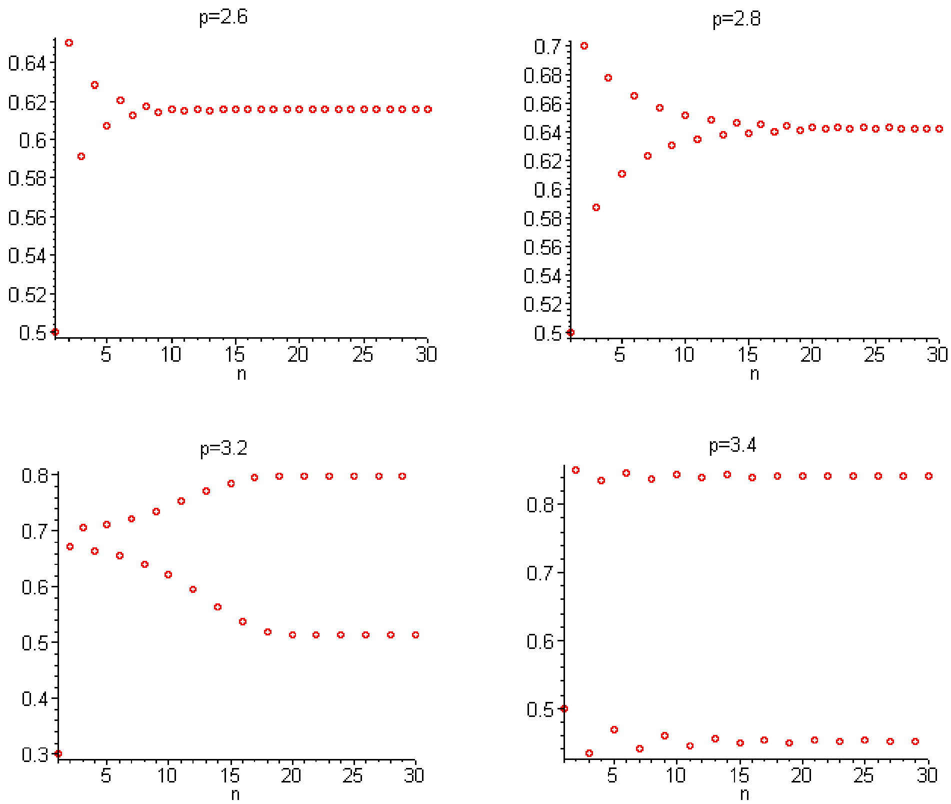
$$y_n = \rho^n \left[y_0 + \frac{P}{1-\rho} \right] - \frac{P}{1-\rho} \,.$$

In terms of the specified values for the parameters, the solution of

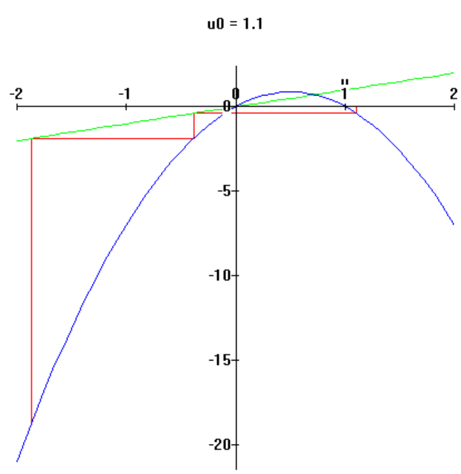
$$(.00833)^{240}\left[y_0 - \frac{12(1000)}{0.1}\right] = -\frac{12(1000)}{0.1}$$

is $y_0 = 103,624.62$ *dollars*.

15.



16. For example, take $\rho = 3.5$ and $u_0 = 1.1$:

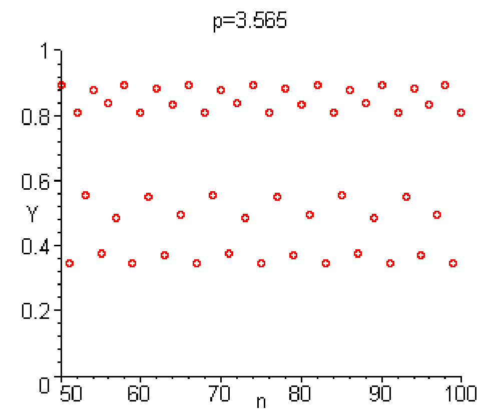


$19(a)$. $\delta_2 = (\rho_2 - \rho_1)/(\rho_3 - \rho_2) = (3.449 - 3)/(3.544 - 3.449) = 4.7263$.

(b). $\%\ diff = \frac{|\delta - \delta_2|}{\delta} \times 100 = \frac{|4.6692 - 4.7363|}{4.6692} \times 100 \approx 1.22\ \%$.

(c). Assuming $(\rho_3 - \rho_2)/(\rho_4 - \rho_3) = \delta$, $\rho_4 \approx 3.5643$

(d). A period 16 solutions appears near $\rho \approx 3.565$.



(e). Note that $(\rho_{n+1} - \rho_n) = \delta_n^{-1}(\rho_n - \rho_{n-1})$. With the assumption that $\delta_n = \delta$, we have $(\rho_{n+1} - \rho_n) = \delta^{-1}(\rho_n - \rho_{n-1})$, which is of the form $y_{n+1} = \alpha\, y_n$, $n \geq 3$. It follows that $(\rho_k - \rho_{k-1}) = \delta^{3-k}(\rho_3 - \rho_2)$ for $k \geq 4$. Then

$$\begin{aligned} \rho_n &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2) + (\rho_4 - \rho_3) + \cdots + (\rho_n - \rho_{n-1}) \\ &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)[1 + \delta^{-1} + \delta^{-2} + \cdots + \delta^{3-n}] \\ &= \rho_1 + (\rho_2 - \rho_1) + (\rho_3 - \rho_2)\left[\frac{1 - \delta^{4-n}}{1 - \delta^{-1}}\right]. \end{aligned}$$

Hence $\lim_{n\to\infty} \rho_n = \rho_2 + (\rho_3 - \rho_2)\left[\frac{\delta}{\delta - 1}\right]$. Substitution of the appropriate values yields

$$\lim_{n\to\infty} \rho_n = 3.5699$$

Miscellaneous Problems

1. Linear $[\, y = c/x^2 + x^3/5 \,]$.
2. Homogeneous $[arctan(y/x) - ln\sqrt{x^2+y^2} = c\,]$.
3. Exact $[\, x^2 + xy - 3y - y^3 = 0\,]$.
4. Linear in $x(y)$ $[\, x = c\, e^y + y\, e^y\,]$.
5. Exact $[\, x^2y + xy^2 + x = c\,]$.
6. Linear $[\, y = x^{-1}(1 - e^{1-x})\,]$.
7. Let $u = x^2$ $[\, x^2 + y^2 + 1 = c\, e^{y^2}\,]$.
8. Linear $[\, y = (4 + cos\, 2 - cos\, x)/x^2\,]$.
9. Exact $[\, x^2y + x + y^2 = c\,]$.
10. $\mu = \mu(x)$ $[\, y^2/x^3 + y/x^2 = c\,]$.
11. Exact $[\, x^3/3 + xy + e^y = c\,]$.
12. Linear $[\, y = c\, e^{-x} + e^{-x} ln(1 + e^x)\,]$.
13. Homogeneous $[\, 2\sqrt{y/x} - ln\,|x| = c\,]$.
14. Exact/Homogeneous $[\, x^2 + 2xy + 2y^2 = 34\,]$.
15. Separable $[\, y = c/cosh^2(x/2)\,]$.
16. Homogeneous $[\, \left(2/\sqrt{3}\right) arctan\left[(2y - x)/\sqrt{3}\, x\right] - ln\,|x| = c\,]$.
17. Linear $[\, y = c\, e^{3x} - e^{2x}\,]$.
18. Linear/Homogeneous $[\, y = c\, x^{-2} - x\,]$.
19. $\mu = \mu(x)$ $[\, 3y - 2xy^3 - 10x = 0\,]$.
20. Separable $[\, e^x + e^{-y} = c\,]$.
21. Homogeneous $[\, e^{-y/x} + ln\,|x| = c\,]$.
22. Separable $[\, y^3 + 3y - x^3 + 3x = 2\,]$.
23. Bernoulli $[\, 1/y = -x\int x^{-2}e^{2x}\, dx + cx\,]$.
24. Separable $[\, sin^2x\, sin\, y = c\,]$.
25. Exact $[\, x^2/y + arctan(y/x) = c\,]$.
26. $\mu = \mu(x)$ $[\, x^2 + 2x^2y - y^2 = c\,]$.
27. $\mu = \mu(x)$ $[\, sin\, x\, cos\, 2y - \frac{1}{2}\, sin^2x = c\,]$.
28. Exact $[\, 2xy + xy^3 - x^3 = c\,]$.
29. Homogeneous $[\, arcsin(y/x) - ln\,|x| = c\,]$.
30. Linear in $x(y)$ $[\, xy^2 - ln\,|y| = 0\,]$.
31. Separable $[\, x + ln\,|x| + x^{-1} + y - 2\, ln\,|y| = c\,]$.
32. $\mu = \mu(y)$ $[\, x^3y^2 + xy^3 = -4\,]$.

Chapter Three

Section 3.1

1. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3\,, 1$. Hence the general solution is $y = c_1 e^t + c_2 e^{-3t}$.

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 3r + 2 = 0$. The roots of the equation are $r = -2\,, -1$. Hence the general solution is $y = c_1 e^{-t} + c_2 e^{-2t}$.

4. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $2r^2 - 3r + 1 = 0$. The roots of the equation are $r = 1/2\,, 1$. Hence the general solution is $y = c_1 e^{t/2} + c_2 e^t$.

6. The characteristic equation is $4r^2 - 9 = 0$, with roots $r = \pm 3/2$. Therefore the general solution is $y = c_1 e^{-3t/2} + c_2 e^{3t/2}$.

8. The characteristic equation is $r^2 - 2r - 2 = 0$, with roots $r = 1 \pm \sqrt{3}$. Hence the general solution is $y = c_1 exp\left(1 - \sqrt{3}\right)t + c_2 exp\left(1 + \sqrt{3}\right)t$.

9. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + r - 2 = 0$. The roots of the equation are $r = -2\,, 1$. Hence the general solution is $y = c_1 e^{-2t} + c_2 e^t$. Its derivative is $y' = -2c_1 e^{-2t} + c_2 e^t$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-2c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^t$.

11. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $6r^2 - 5r + 1 = 0$. The roots of the equation are $r = 1/3\,, 1/2$. Hence the general solution is $y = c_1 e^{t/3} + c_2 e^{t/2}$. Its derivative is $y' = c_1 e^{t/3}/3 + c_2 e^{t/2}/2$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 4$. In order to satisfy the condition $y'(0) = 1$, we find that $c_1/3 + c_2/2 = 0$. Solving for the constants, $c_1 = 12$ and $c_2 = -8$. Hence the specific solution is $y(t) = 12\,e^{t/3} - 8\,e^{t/2}$.

12. The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3\,, 0$. Therefore the general solution is $y = c_1 + c_2 e^{-3t}$, with derivative $y' = -3\,c_2 e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = -2$, and $-3\,c_2 = 3$. Hence the specific solution is $y(t) = -1 - e^{-3t}$.

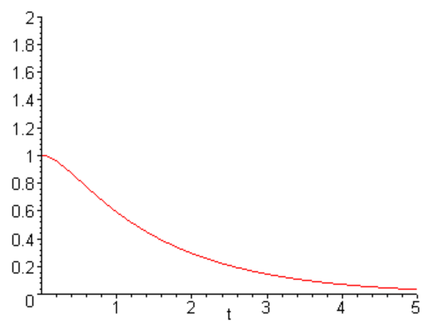
13. The characteristic equation is $r^2 + 5r + 3 = 0$, with roots

$$r_{1,2} = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}\,.$$

The general solution is $y = c_1 exp\left(-5-\sqrt{13}\right)t/2 + c_2 exp\left(-5+\sqrt{13}\right)t/2$, with derivative

$$y' = \frac{-5-\sqrt{13}}{2}\,c_1 exp\left(-5-\sqrt{13}\right)t/2 + \frac{-5+\sqrt{13}}{2}\,c_2 exp\left(-5+\sqrt{13}\right)t/2\,.$$

In order to satisfy the initial conditions, we require that $c_1 + c_2 = 1$, and $\frac{-5-\sqrt{13}}{2}\,c_1 + \frac{-5+\sqrt{13}}{2}\,c_2 = 0$. Solving for the coefficients, $c_1 = \left(1 - 5/\sqrt{13}\right)/2$ and $c_2 = \left(1 + 5/\sqrt{13}\right)/2$.



14. The characteristic equation is $2r^2 + r - 4 = 0$, with roots

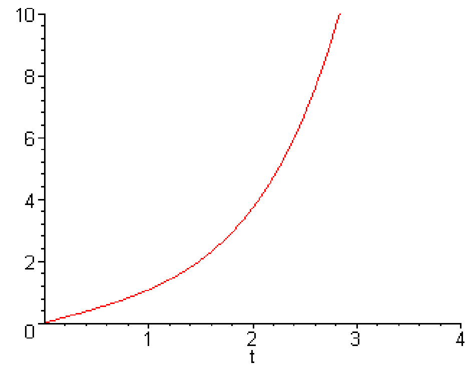
$$r_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}\,.$$

The general solution is $y = c_1 exp\left(-1-\sqrt{33}\right)t/4 + c_2 exp\left(-1+\sqrt{33}\right)t/4$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4}\,c_1 exp\left(-1-\sqrt{33}\right)t/4 + \frac{-1+\sqrt{33}}{4}\,c_2 exp\left(-1+\sqrt{33}\right)t/4\,.$$

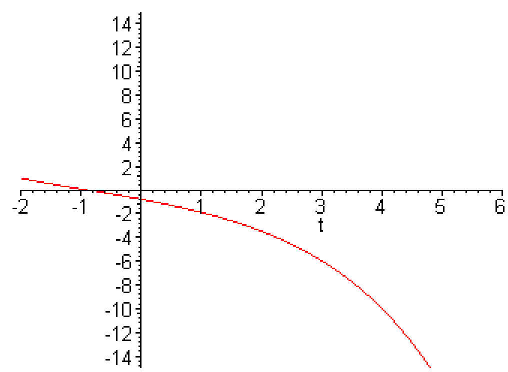
In order to satisfy the initial conditions, we require that $c_1 + c_2 = 0$, and $\frac{-1-\sqrt{33}}{4}\,c_1 + \frac{-1+\sqrt{33}}{4}\,c_2 = 1$. Solving for the coefficients, $c_1 = -2/\sqrt{33}$ and $c_2 = 2/\sqrt{33}$. The specific solution is

$$y(t) = -2\left[exp\left(-1-\sqrt{33}\right)t/4 - exp\left(-1+\sqrt{33}\right)t/4\right]/\sqrt{33}\,.$$



16. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Therefore the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1 e^{-(t+2)/2} + d_2 e^{(t+2)/2}$. The derivative is given by $y' = -\left[d_1 e^{-(t+2)/2}\right]/2 + \left[d_2 e^{(t+2)/2}\right]/2$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/2 + d_2/2 = -1$. Solving for the coefficients, $d_1 = 3/2$, and $d_2 = -1/2$. The specific solution is

$$\begin{aligned} y(t) &= \frac{3}{2} e^{-(t+2)/2} - \frac{1}{2} e^{(t+2)/2} \\ &= \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2}. \end{aligned}$$



18. An algebraic equation with roots -2 and $-1/2$ is $2r^2 + 5r + 2 = 0$. This is the characteristic equation for the ODE $2y'' + 5y' + 2y = 0$.

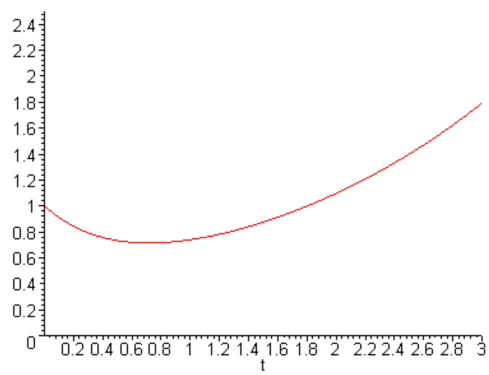
20. The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2, 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^t$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^t$. In order to satisfy the initial conditions, we require $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$, and $c_2 = -1$. The specific solution is $y(t) = 3e^{t/2} - e^t$. To find the *stationary point*, set $y' = 3e^{t/2}/2 - e^t = 0$. There is a unique solution, with $t_1 = ln(9/4)$. The maximum value is then $y(t_1) = 9/4$. To find

the *x-intercept*, solve the equation $3e^{t/2} - e^t = 0$. The solution is readily found to be $t_2 = ln9 \approx 2.1972$.

22. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Hence the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$, with derivative $y' = -c_1 e^{-t/2}/2 + c_2 e^{t/2}/2$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$ and $-c_1 + c_2 = \beta$. The specific solution is $y(t) = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t \to \infty$, $y(t) \to 0$ as long as $\beta = -1$.

23. The characteristic equation is $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$. Examining the coefficients, the roots are $r = \alpha$, $\alpha - 1$. Hence the general solution of the differential equation is $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha - 1)t}$. Assuming $\alpha \in \mathbb{R}$, all solutions will tend to *zero* as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is, $\alpha > 1$.

25. $y(t) = 2\, e^{t/2}/5 + 3\, e^{-2t}/5$.



The minimum occurs at $(t_0 \, , y_0) = (0.7167 \, , 0.7155)$.

26(a). The characteristic roots are $r = -3, -2$. The solution of the initial value problem is $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$.

(b). The maximum point has coordinates $t_0 = ln\left[\frac{3(4+\beta)}{2(6+\beta)}\right]$, $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2}$.

(c). $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2} \geq 4$, as long as $\beta \geq 6 + 6\sqrt{3}$.

(d). $\lim_{\beta \to \infty} t_0 = ln\,\frac{3}{2}$. $\lim_{\beta \to \infty} y_0 = \infty$.

29. Set $v = y'$ and $v' = y''$. Substitution into the ODE results in the first order equation $t\,v' + v = 1$. The equation is *linear*, and can be written as $(t\,v)' = 1$. Hence the general solution is $v = 1 + c_1/t$. Hence $y' = 1 + c_1/t$, and $y = t + c_1 ln\, t + c_2$.

31. Setting $v = y'$ and $v' = y''$, the transformed equation is $2t^2 v' + v^3 = 2t\,v$. This

is a *Bernoulli* equation, with $n = 3$. Let $w = v^{-2}$. Substitution of the new dependent variable yields $-t^2 w' + 1 = 2t\,w$, or $t^2 w' + 2t\,w = \;= 1$. Integrating, we find that $w = (t + c_1)/t^2$. Hence $v = \pm t/\sqrt{t + c_1}$, that is, $y' = \pm t/\sqrt{t + c_1}$. Integrating one more time results in $y(t) = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2$. (Note that $v = 0$ is also a solution of the transformed equation).

32. Setting $v = y'$ and $v' = y''$, the transformed equation is $v' + v = e^{-t}$. This ODE is *linear*, with integrating factor $\mu(t) = e^t$. Hence $v = y' = (t + c_1)e^{-t}$. Integrating, we obtain $y(t) = -(t + c_1)e^{-t} + c_2$.

33. Set $v = y'$ and $v' = y''$. The resulting equation is $t^2 v' = v^2$. This equation is *separable*, with solution $v = y' = t/(1 + c_1 t)$. Integrating, the general solution is

$$y(t) = t/c_1 - c_1^{-2} ln|1 + c_1 t| + c_2 ,$$

as long as $c_1 \neq 0$. For $c_1 = 0$, the solution is $y(t) = t^2/2 + c_2$. Note that $v = 0$ is also a solution of the transformed equation.

35. Let $y' = v$ and $y'' = v\,dv/dy$. Then $v\,dv/dy + y = 0$ is the transformed equation for $v = v(y)$. This equation is *separable*, with $v\,dv = -y\,dy$. The solution is given by $v^2 = -y^2 + c_1$. Substituting for v, we find that $y' = \pm\sqrt{c_1 - y^2}$. This equation is *also*
separable, with solution $arcsin\big(y/\sqrt{c_1}\big) = \pm\,t + c_2$, or $y(t) = d_1 sin(t + d_2)$.

36. Let $y' = v$ and $y'' = v\,dv/dy$. It follows that $vdv/dy + yv^3 = 0$ is the differential equation for $v = v(y)$. This equation is *separable*, with $v^{-2}\,dv = -y\,dy$. The solution is given by $v = \left[y^2/2 + c_1\right]^{-1}$. Substituting for v, we find that $y' = \left[y^2/2 + c_1\right]^{-1}$. This equation is *also* separable, with $(y^2/2 + c_1)dy = dt$. The solution is defined *implicitly* by $y^3/6 + c_1 y + c_2 = t$.

38. Setting $y' = v$ and $y'' = vdv/dy$, the transformed equation is $y\,vdv/dy - v^3 = 0$. This equation is *separable*, with $v^{-2}\,dv = dy/y$. The solution is $v(y) = [c_1 - ln|y|]^{-1}$. Substituting for v, we obtain a *separable* equation, $(c_1 - ln|y|)dy = dx$. The solution is given *implicitly* by $c_2 y - y\,ln|y| + c_3 = t$.

39. Let $y' = v$ and $y'' = vdv/dy$. It follows that $vdv/dy + v^2 = 2e^{-y}$ is the equation for $v = v(y)$. Inspection of the left hand side suggests a substitution $w = v^2$. The resulting
equation is $dw/dy + 2w = 4e^{-y}$. This equation is *linear*, with integrating factor $\mu = e^{2y}$.
We obtain $d(e^{2y}\,w)/dy = 4\,e^y$, which upon integration yields $w(y) = 4\,e^{-y} + c_1 e^{-2y}$. Converting back to the original dependent variable, $y' = \pm e^{-y}\sqrt{4\,e^y + c_1}$. Separating variables, $e^y(4\,e^y + c_1)^{-1/2}dy = \pm\,dt$. Integration yields $\sqrt{4\,e^y + c_1} = \pm\,2t + c_2$.

41. Setting $y' = v$ and $y'' = vdv/dy$, the transformed equation is $vdv/dy - 3y^2 = 0$.

This equation is *separable*, with $vdv = 3y^2dy$. The solution is $y' = v = \sqrt{2y^3 + c_1}$. The *positive* root is chosen based on the initial conditions. Furthermore, when $t = 0$, $y = 2$, and $y' = v = 4$. The initial conditions require that $c_1 = 0$. It follows that $y' = \sqrt{2y^3}$. Separating variables and integrating, $1/\sqrt{y} = -t/\sqrt{2} + c_2$. Hence the solution is $y(t) = 2/(1 - t)^2$.

42. Setting $v = y'$ and $v' = y''$, the transformed equation is $(1 + t^2)v' + 2t\,v =$ $= -3t^{-2}$. Rewrite the equation as $v' + 2t\,v/(1 + t^2) = -3t^{-2}/(1 + t^2)$. This equation is *linear*, with integrating factor $\mu = 1 + t^2$. Hence we have

$$\left[(1 + t^2)v\right]' = -3t^{-2}.$$

Integrating both sides, $v = 3t^{-1}/(1 + t^2) + c_1/(1 + t^2)$. Invoking the initial condition $v(1) = -1$, we require that $c_1 = -5$. Hence $y' = (3 - 5t)/(t + t^3)$. Integrating, we obtain $y(t) = \frac{3}{2}ln[t^2/(1 + t^2)] - 5\,arctan(t) + c_2$. Based on the initial condition $y(1) = 2$, we find that $c_2 = \frac{3}{2}ln\,2 + \frac{5}{4}\pi + 2$.

Section 3.2

1.

$$W\left(e^{2t}, e^{-3t/2}\right) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

3.

$$W\left(e^{-2t}, t\,e^{-2t}\right) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}.$$

5.

$$W\left(e^t sin\,t\,, e^t cos\,t\right) = \begin{vmatrix} e^t sin\,t & e^t cos\,t \\ e^t(sin\,t + cos\,t) & e^t(cos\,t - sin\,t) \end{vmatrix} = -e^{2t}.$$

6.

$$W\left(cos^2\theta\,, 1 + cos\,2\theta\right) = \begin{vmatrix} cos^2\theta & 1 + cos\,2\theta \\ -2sin\,\theta\,cos\,\theta & -2\,sin\,2\theta \end{vmatrix} = 0.$$

7. Write the equation as $y'' + (3/t)y' = 1$. $p(t) = 3/t$ is continuous for all $t > 0$. Since $t_0 > 0$, the IVP has a unique solution for all $t > 0$.

9. Write the equation as $y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$. The coefficients are not continuous at $t = 0$ and $t = 4$. Since $t_0 \in (0\,,4)$, the largest interval is $0 < t < 4$.

10. The coefficient $3ln|t|$ is discontinuous at $t = 0$. Since $t_0 > 0$, the largest interval of existence is $0 < t < \infty$.

11. Write the equation as $y'' + \frac{x}{x-3}y' + \frac{ln|x|}{x-3}y = 0$. The coefficients are discontinuous at $x = 0$ and $x = 3$. Since $x_0 \in (0\,,3)$, the largest interval is $0 < x < 3$.

13. $y_1'' = 2$. We see that $t^2(2) - 2(t^2) = 0$. $y_2'' = 2\,t^{-3}$, with $t^2(y_2'') - 2(y_2) = 0$. Let $y_3 = c_1t^2 + c_2t^{-1}$, then $y_3'' = 2c_1 + 2c_2t^{-3}$. It is evident that y_3 is also a solution.

16. No. Substituting $y = sin(t^2)$ into the differential equation,

$$-4t^2 sin\left(t^2\right) + 2cos\left(t^2\right) + 2t\,cos\left(t^2\right)p(t) + sin\left(t^2\right)q(t) = 0.$$

For the equation to be valid, we must have $p(t) = -1/t$, which is *not* continuous, or even defined, at $t = 0$.

17. $W(e^{2t}, g(t)) = e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$. Dividing both sides by e^{2t}, we find that g must satisfy the ODE $g' - 2g = 3e^{2t}$. Hence $g(t) = 3t\, e^{2t} + c\, e^{2t}$.

19. $W(f\,, g) = fg' - f'g$. Also, $W(u\,, v) = W(2f - g\,, f + 2g)$. Upon evaluation, $W(u\,, v) = 5fg' - 5f'g = 5W(f\,, g)$.

20. $W(f\,, g) = fg' - f'g = t\,cos\,t - sin\,t$, and $W(u\,, v) = -4fg' + 4f'g$. Hence $W(u\,, v) = -4t\,cos\,t + 4sin\,t$.

22. The general solution is $y = c_1e^{-3t} + c_2e^{-t}$. $W(e^{-3t}, e^{-t}) = 2e^{-4t}$, and hence the exponentials form a *fundamental set* of solutions. On the other hand, the *fundamental solutions* must also satisfy the conditions $y_1(1) = 1\,, y_1'(1) = 0\,; y_2(1) = 0\,, y_2'(1) = 1$. For y_1, the initial conditions require $c_1 + c_2 = e\,,\ -3c_1 - c_2 = 0$. The coefficients are $c_1 = -e^3/2\,, c_2 = 3e/2$. For the solution, y_2, the initial conditions require $c_1 + c_2 = 0$, $-3c_1 - c_2 = e$. The coefficients are $c_1 = -e^3/2\,, c_2 = e/2$. Hence the fundamental solutions are $\left\{y_1 = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)},\ y_2 = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}\right\}$.

23. Yes. $y_1'' = -4\,cos\,2t\,;\ y_2'' = -4\,sin\,2t$. $W(cos\,2t\,, sin\,2t) = 2$.

24. Clearly, $y_1 = e^t$ is a solution. $y_2' = (1+t)e^t, y_2'' = (2+t)e^t$. Substitution into the ODE results in $(2+t)e^t - 2(1+t)e^t + t\,e^t = 0$. Furthermore, $W(e^t, te^t) = e^{2t}$. Hence the solutions form a fundamental set of solutions.

26. Clearly, $y_1 = x$ is a solution. $y_2' = cos\,x\,, y_2'' = -sin\,x$. Substitution into the ODE results in $(1 - x\,cot\,x)(-sin\,x) - x(cos\,x) + sin\,x = 0$. $W(y_1, y_2) = x\,cos\,x - sin\,x$,
which is *nonzero* for $0 < x < \pi$. Hence $\{x\,, sin\,x\}$ is a fundamental set of solutions.

28. $P = 1\,, Q = x\,, R = 1$. We have $P'' - Q' + R = 0$. The equation is *exact*. Note that $(y')' + (xy)' = 0$. Hence $y' + xy = c_1$. This equation is *linear*, with integrating factor $\mu = e^{x^2/2}$. Therefore the general solution is

$$y(x) = c_1exp\big(-x^2/2\big)\int_{x_0}^{x} exp\big(u^2/2\big)du + c_2exp\big(-x^2/2\big).$$

29. $P = 1\,, Q = 3x^2\,, R = x$. Note that $P'' - Q' + R = -5x$, and therefore the differential equation is *not* exact.

31. $P = x^2\,, Q = x\,, R = -1$. We have $P'' - Q' + R = 0$. The equation is *exact*. Write the equation as $(x^2y')' - (xy)' = 0$. Integrating, we find that $x^2y' - x\,y = c$. Divide both sides of the ODE by x^2. The resulting equation is *linear*, with integrating factor $\mu = 1/x$. Hence $(y/x)' = c\,x^{-3}$. The solution is $y(t) = c_1x^{-1} + c_2x$.

33. $P = x^2\,, Q = x\,, R = x^2 - \nu^2$. Hence the coefficients are $2P' - Q = 3x$ and $P'' - Q' + R = x^2{+}1 - \nu^2$. The *adjoint* of the original differential equation is given by $x^2\mu'' + 3x\,\mu'{+}(x^2{+}1 - \nu^2)\mu = 0\,.$

35. $P = 1\,, Q = 0\,, R = \,-\,x\,.$ Hence the coefficients are given by $2P' - Q = 0$ and $P'' - Q' + R = \,-\,x\,.$ Therefore the *adjoint* of the original equation is $\mu'' - x\,\mu = 0\,.$

Section 3.3

1. Suppose that $\alpha\, f(t) + \beta\, g(t) = 0$, that is, $\alpha(t^2 + 5t) + \beta(t^2 - 5t) = 0$ on some interval I. Then $(\alpha + \beta)t^2 + 5(\alpha - \beta)t = 0\,, \forall\, t \in I$. Since a quadratic .has at most *two*
roots, we must have $\alpha + \beta = 0$ and $\alpha - \beta = 0$. The *only* solution is $\alpha = \beta = 0$. Hence the two functions are linearly *independent*.

3. Suppose that $e^{\lambda t} cos\, \mu t = A\, e^{\lambda t} sin\, \mu t$, for some $A \neq 0$, on an interval I. Since the function $sin\, \mu t \neq 0$ on some *subinterval* $I_0 \subset I$, we conclude that $tan\, \mu t = A$ on I_0. This is clearly a contradiction, hence the functions are linearly *independent*.

4. Obviously, $f(x) = e\, g(x)$ for all real numbers x. Hence the functions are linearly *dependent*.

5. Here $f(x) = 3g(x)$ for all real numbers. Hence the functions are linearly *dependent*.

8. Note that $f(x) = g(x)$ for $x \in [\,0\,, \infty)$, and $f(x) = -g(x)$ for $x \in (-\infty\,, 0\,]$. It follows that the functions are linearly *dependent* on $\mathbb{R}^+$ and $\mathbb{R}^-$. Nevertheless, they are linearly *independent* on *any* open interval containing *zero*.

9. Since $W(t) = t\, sin^2 t$ has only *isolated* zeros, $W(t)$ cannot identically vanish on any open interval. Hence the functions are linearly *independent*.

10. Same argument as in Prob. 9.

11. By linearity of the differential operator, $c_1 y_1$ and $c_2 y_2$ are also solutions. Calculating
the Wronskian, $W(c_1 y_1\,, c_2 y_2) = (c_1 y_1)(c_2 y_2)' - (c_1 y_1)'(c_2 y_2) = c_1 c_2\, W(y_1\,, y_2)$. Since $W(y_1\,, y_2)$ is not *identically zero*, neither is $W(c_1 y_1\,, c_2 y_2)$.

13. Direct calculation results in

$$\begin{aligned} W(a_1 y_1 + a_2 y_2, b_1 y_1 + b_2 y_2) &= a_1 b_2 W(y_1\,, y_2) - b_1 a_2 W(y_1\,, y_2) \\ &= (a_1 b_2 - a_2 b_1) W(y_1\,, y_2)\,. \end{aligned}$$

Hence the combinations are also linearly independent as long as $a_1 b_2 - a_2 b_1 \neq 0$.

14. Let $\alpha(\mathbf{i} + \mathbf{j}) + \beta(\mathbf{i} - \mathbf{j}) = 0\,\mathbf{i} + 0\,\mathbf{j}$. Then $\alpha + \beta = 0$ and $\alpha - \beta = 0$. The only solution is $\alpha = \beta = 0$. Hence the given vectors are linearly independent. Furthermore, any vector $a_1 \mathbf{i} + a_2 \mathbf{j} = \left(\frac{a_1}{2} + \frac{a_2}{2}\right)(\mathbf{i} + \mathbf{j}) + \left(\frac{a_1}{2} - \frac{a_2}{2}\right)(\mathbf{i} - \mathbf{j})$.

16. Writing the equation in standard form, we find that $P(t) = sin\, t / cos\, t$. Hence the Wronskian is $W(t) = b\, exp\left(-\int \frac{sin\, t}{cos\, t} dt\right) = b\, exp(ln|cos\, t|) = b\, cos\, t$, in which b is some constant.

17. After writing the equation in standard form, we have $P(x) = 1/x$. The Wronskian is $W(t) = c\,exp\big(-\int \frac{1}{x}dx\big) = c\,exp(-ln|x|) = c/|x|$, in which c is some constant.

18. Writing the equation in standard form, we find that $P(x) = -2x/(1-x^2)$. The Wronskian is $W(t) = c\,exp\big(-\int \frac{-2x}{1-x^2}dx\big) = c\,exp(-ln|1-x^2|) = c|1-x^2|^{-1}$, in which c is some constant.

19. Rewrite the equation as $p(t)y'' + p'(t)y' + q(t)y = 0$. After writing the equation in standard form, we have $P(t) = p'(t)/p(t)$. Hence the Wronskian is

$$W(t) = c\,exp\bigg(-\int \frac{p'(t)}{p(t)}dt\bigg) = c\,exp(-ln\,p(t)) = c/p(t)\,.$$

21. The Wronskian associated with the solutions of the differential equation is given by $W(t) = c\,exp\big(-\int \frac{-2}{t^2}dt\big) = c\,exp(-2/t)$. Since $W(2) = 3$, it follows that for the hypothesized set of solutions, $c = 3\,e$. Hence $W(4) = 3\sqrt{e}$.

22. For the given differential equation, the Wronskian satisfies the first order differential equation $W' + p(t)W = 0$. Given that W is *constant*, it is necessary that $p(t) \equiv 0$.

23. Direct calculation shows that

$$\begin{aligned} W(f\,g\,,f\,h) &= (fg)(fh)' - (fg)'(fh) \\ &= (fg)(f'h + fh') - (f'g + fg')(fh) \\ &= f^2\,W(g\,,h)\,. \end{aligned}$$

25. Since y_1 and y_2 are solutions, they are differentiable. The hypothesis can thus be restated as $y_1'(t_0) = y_2'(t_0) = 0$ at some point t_0 in the interval of definition. This implies that $W(y_1\,,y_2)(t_0) = 0$. But $W(y_1\,,y_2)(t_0) = c\,exp\big(-\int p(t)dt\big)$, which *cannot* be equal to zero, unless $c = 0$. Hence $W(y_1\,,y_2) \equiv 0$, which is ruled out for a fundamental set of solutions.

Section 3.4

2. $exp(2-3i) = e^2 e^{-3i} = e^2(cos\,3 - i\,sin\,3)$.

3. $e^{i\pi} = cos\,\pi + i\,sin\,\pi = -1$.

4. $exp\big(2 - \frac{\pi}{2} i\big) = e^2\big(cos\,\frac{\pi}{2} - i\,sin\,\frac{\pi}{2}\big) = -e^2\,i$.

6. $\pi^{-1+2i} = exp[(-1+2i)ln\,\pi] = exp(-ln\,\pi)exp(2\,ln\,\pi\,i) = \frac{1}{\pi}\,exp(2\,ln\,\pi\,i) =$
$= \frac{1}{\pi}[cos\,(2\,ln\,\pi) + i\,sin\,(2\,ln\,\pi)]$.

8. The characteristic equation is $r^2 - 2r + 6 = 0$, with roots $r = 1 \pm i\sqrt{5}$. Hence the general solution is $y = c_1 e^t cos\,\sqrt{5}\,t + c_2\,e^t sin\,\sqrt{5}\,t$.

9. The characteristic equation is $r^2 + 2r - 8 = 0$, with roots $r = -4\,, 2$. The roots are *real* and different, hence the general solution is $y = c_1 e^{-4t} + c_2\,e^{2t}$.

10. The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Hence the general solution is $y = c_1 e^{-t} cos\,t + c_2\,e^{-t} sin\,t$.

12. The characteristic equation is $4r^2 + 9 = 0$, with roots $r = \pm\frac{3}{2}\,i$. Hence the general solution is $y = c_1 cos\,\frac{3}{2}t + c_2\,sin\,\frac{3}{2}t$.

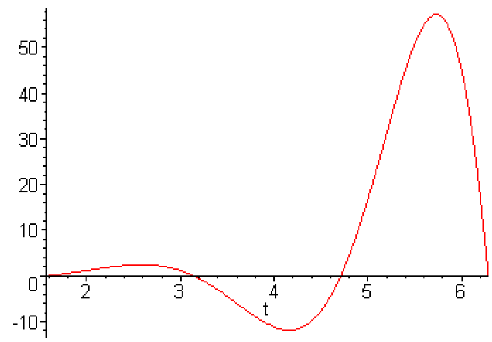
13. The characteristic equation is $r^2 + 2r + 1.25 = 0$, with roots $r = -1 \pm \frac{1}{2}i$. Hence the general solution is $y = c_1 e^{-t} cos\,\frac{1}{2}t + c_2\,e^{-t} sin\,\frac{1}{2}t$.

15. The characteristic equation is $r^2 + r + 1.25 = 0$, with roots $r = -\frac{1}{2} \pm i$. Hence the general solution is $y = c_1 e^{-t/2} cos\,t + c_2\,e^{-t/2} sin\,t$.

16. The characteristic equation is $r^2 + 4r + 6.25 = 0$, with roots $r = -2 \pm \frac{3}{2}\,i$. Hence the general solution is $y = c_1 e^{-2t} cos\,\frac{3}{2}t + c_2\,e^{-2t} sin\,\frac{3}{2}t$.

17. The characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm\,2i$. Hence the general solution is $y = c_1 cos\,2t + c_2\,sin\,2t$. Its derivative is $y' = -2c_1 sin\,2t + 2c_2\,cos\,2t$. Based on the first condition, $y(0) = 0$, we require that $c_1 = 0$. In order to satisfy the condition $y'(0) = 1$, we find that $2c_2 = 1$. The constants are $c_1 = 0$ and $c_2 = 1/2$. Hence the specific solution is $y(t) = \frac{1}{2} sin\,2t$.

19. The characteristic equation is $r^2 - 2r + 5 = 0$, with roots $r = 1 \pm 2i$. Hence the general solution is $y = c_1 e^t cos\,2t + c_2\,e^t sin\,2t$. Based on the condition, $y(\pi/2) = 0$, we require that $c_1 = 0$. It follows that $y = c_2\,e^t sin\,2t$, and so the first derivative is $y' = c_2\,e^t sin\,2t + 2c_2\,e^t cos\,2t$. In order to satisfy the condition $y'(\pi/2) = 2$, we find that $-2e^{\pi/2} c_2 = 2$. Hence we have $c_2 = -e^{-\pi/2}$. Therefore the specific solution is $y(t) = -e^{t-\pi/2}\,sin\,2t$.

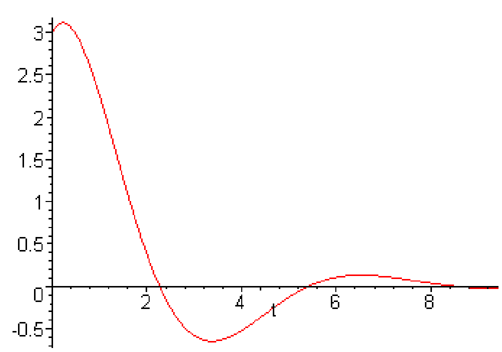


20. The characteristic equation is $r^2+1=0$, with roots $r=\pm i$. Hence the general solution is $y=c_1 cos\,t+c_2\,sin\,t$. Its derivative is $y'=-c_1 sin\,t+c_2\,cos\,t$. Based on the first condition, $y(\pi/3)=2$, we require that $c_1+\sqrt{3}\,c_2=4$. In order to satisfy the condition $y'(\pi/3)=-4$, we find that $-\sqrt{3}\,c_1+c_2=-8$. Solving these for the constants, $c_1=1+2\sqrt{3}$ and $c_2=\sqrt{3}-2$. Hence the specific solution is a steady oscillation, given by $y(t)=\left(1+2\sqrt{3}\right)cos\,t+\left(\sqrt{3}-2\right)sin\,t$.

21. From Prob. 15, the general solution is $y=c_1 e^{-t/2}cos\,t+c_2\,e^{-t/2}sin\,t$. Invoking the first initial condition, $y(0)=3$, which implies that $c_1=3$. Substituting, it follows that $y=3e^{-t/2}cos\,t+c_2\,e^{-t/2}sin\,t$, and so the first derivative is

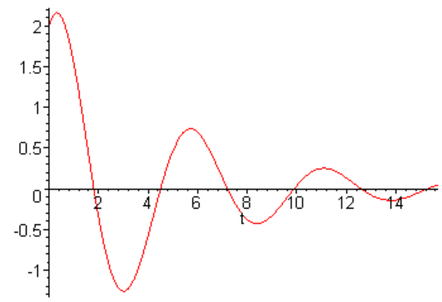
$$y'=-\frac{3}{2}e^{-t/2}cos\,t-3e^{-t/2}sin\,t+c_2\,e^{-t/2}cos\,t-\frac{c_2}{2}e^{-t/2}sin\,t\,.$$

Invoking the initial condition, $y'(0)=1$, we find that $-\frac{3}{2}+c_2=1$, and so $c_2=\frac{5}{2}$. Hence the specific solution is $y(t)=3e^{-t/2}cos\,t+\frac{5}{3}\,e^{-t/2}sin\,t$.



24(a). The characteristic equation is $5r^2+2r+7=0$, with roots $r=-\frac{1}{5}\pm i\frac{\sqrt{34}}{5}$. The solution is $u=c_1 e^{-t/5}cos\,\frac{\sqrt{34}}{5}t+c_2 e^{-t/5}sin\,\frac{\sqrt{34}}{5}t$. Invoking the given initial conditions, we obtain the equations for the coefficients: $c_1=2\,,\ -2+\sqrt{34}\,c_2=5$. That is, $c_1=2\,,\ c_2=7/\sqrt{34}$. Hence the specific solution is

$$u(t) = 2e^{-t/5}cos\,\frac{\sqrt{34}}{5}t + \frac{7}{\sqrt{34}}e^{-t/5}sin\,\frac{\sqrt{34}}{5}t\,.$$



(b). Based on the graph of $u(t)$, T is in the interval $14 < t < 16$. A numerical solution on that interval yields $T \approx 14.5115$.

26(a). The characteristic equation is $r^2 + 2a\,r + (a^2+1) = 0$, with roots $r = -a \pm i$. Hence the general solution is $y(t) = c_1 e^{-at}cos\,t + c_2 e^{-at}sin\,t$. Based on the initial conditions, we find that $c_1 = 1$ and $c_2 = a$. Therefore the specific solution is given by

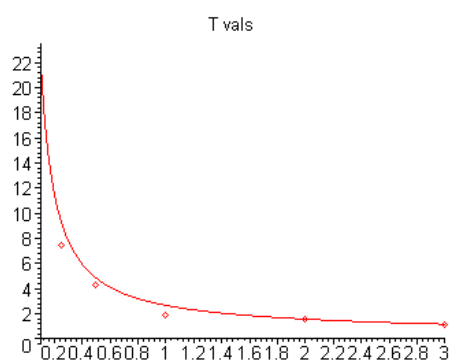
$$\begin{aligned} y(t) &= e^{-at}cos\,t + a\,e^{-at}sin\,t \\ &= \sqrt{1+a^2}\;e^{-at}cos\,(t-\phi), \end{aligned}$$

in which $\phi = tan^{-1}(a)$.

(b). For estimation, note that $|y(t)| \leq \sqrt{1+a^2}\;e^{-at}$. Now consider the inequality $\sqrt{1+a^2}\;e^{-at} \leq 1/10$. The inequality holds for $t \geq \frac{1}{a}ln\left[10\sqrt{1+a^2}\right]$. Therefore $T \leq \frac{1}{a}ln\left[10\sqrt{1+a^2}\right]$. Setting $a = 1$, numerical analysis gives $T \approx 1.8763$.

(c). Similarly, $T_{1/4} \approx 7.4284$, $T_{1/2} \approx 4.3003$, $T_2 \approx 1.5116$, $T_3 \approx 1.1496$.

(d).



Note that the estimates T_a approach the graph of $\frac{1}{a} ln\left[10\sqrt{1+a^2}\right]$ as a gets large.

27. Direct calculation gives the result. On the other hand, it was shown in Prob. 3.3.23 that $W(f\,g\,,f\,h) = f^2 W(g\,,h)$. Hence

$$\begin{aligned} W\left(e^{\lambda t} cos\,\mu t\,, e^{\lambda t} sin\,\mu t\right) &= e^{2\lambda t}\, W(cos\,\mu t\,, sin\,\mu t) \\ &= e^{2\lambda t}\left[cos\,\mu t(sin\,\mu t)' - (cos\,\mu t)' sin\,\mu t\right] \\ &= \mu\, e^{2\lambda t}. \end{aligned}$$

28(a). Clearly, y_1 and y_2 are solutions. Also, $W(cos\,t, sin\,t) = cos^2 t + sin^2 t = 1$.

(b). $y' = i\,e^{it}\,, y'' = i^2\,e^{it} = \,-\,e^{it}$. Evidently, y is a solution and so $y = c_1 y_1 + c_2 y_2$.

(c). Setting $t = 0\,,\; 1 = c_1 cos\,0 + c_2\, sin\,0$, and $c_1 = 0$. Differentiating, $i\,e^{it} = c_2\,cos\,t$. Setting $t = 0\,,\; i = c_2\,cos\,0$ and hence $c_2 = i$. Therefore $e^{it} = cos\,t + i\,sin\,t$.

29. *Euler's formula* is $e^{it} = cos\,t + i\,sin\,t$. It follows that $e^{-it} = cos\,t - i\,sin\,t$. Adding these equation, $e^{it} + e^{-it} = 2\,cos\,t$. Subtracting the two equations results in $e^{it} - e^{-it} = 2i\,sin\,t$.

30. Let $r_1 = \lambda_1 + i\mu_1$, and $r_2 = \lambda_2 + i\mu_2$. Then

$$\begin{aligned} exp(r_1 + r_2)t &= exp[(\lambda_1 + \lambda_2)t + i(\mu_1 + \mu_2)t] \\ &= e^{(\lambda_1+\lambda_2)t}[cos(\mu_1 + \mu_2)t + i\,sin(\mu_1 + \mu_2)t] \\ &= e^{(\lambda_1+\lambda_2)t}[(cos\,\mu_1 t +\; isin\,\mu_1 t)(cos\,\mu_2 t +\; isin\,\mu_2 t)] \\ &= e^{\lambda_1 t}(cos\,\mu_1 t +\; isin\,\mu_1 t)\cdot e^{\lambda_2 t}(cos\,\mu_1 t +\; isin\,\mu_1 t) \end{aligned}$$

Hence $e^{(r_1+r_2)t} = e^{r_1 t}\,e^{r_2 t}$.

32. If $\phi(t) = u(t) + i\,v(t)$ is a solution, then

$$(u+iv)'' + p(t)(u+iv)' + q(t)(u+iv) = 0\,,$$

and $(u'' + iv'') + p(t)(u' + iv') + q(t)(u+iv) = 0$. After expanding the equation and separating the *real* and *imaginary* parts,

$$\begin{aligned} u'' + p(t)u' + q(t)u = 0 \\ v'' + p(t)v' + q(t)v = 0 \end{aligned}$$

Hence both $u(t)$ and $v(t)$ are solutions.

34(a). By the *chain rule*, $y(x)' = \frac{dy}{dx}\,x'$. In general, $\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt}$. Setting $z = \frac{dy}{dt}$, we have $\frac{d^2y}{dt^2} = \frac{dz}{dx}\frac{dx}{dt} = \frac{d}{dx}\left[\frac{dy}{dx}\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2y}{dx^2}\frac{dx}{dt}\right]\frac{dx}{dt} + \frac{dy}{dx}\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt}$. However, $\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt} = \left[\frac{d^2x}{dt^2}\right]\frac{dt}{dx}\cdot\frac{dx}{dt} = \frac{d^2x}{dt^2}$. Hence $\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}$.

(b). Substituting the results in Part(a) into the general ODE, $y'' + p(t)y' + q(t)y = 0$, `we find that

$$\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + p(t)\frac{dy}{dx}\frac{dx}{dt} + q(t)y = 0\,.$$

Collecting the terms,

$$\left[\frac{dx}{dt}\right]^2\frac{d^2y}{dx^2} + \left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\frac{dy}{dx} + q(t)y = 0\,.$$

(c). Assuming $\left[\frac{dx}{dt}\right]^2 = k\,q(t)$, and $q(t) > 0$, we find that $\frac{dx}{dt} = \sqrt{k\,q(t)}$, which can be integrated. That is, $x = \xi(t) = \int \sqrt{k\,q(t)}\;dt$.

(d). Let $k = 1$. It follows that $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} = \frac{d\xi}{dt} + p(t)\xi(t) = \frac{q'}{2\sqrt{q}} + p\sqrt{q}$. Hence

$$\left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\Big/\left[\frac{dx}{dt}\right]^2 = \frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\,.$$

As long as $dx/dt \neq 0$, the differential equation can be expressed as

$$\frac{d^2y}{dx^2} + \left[\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\right]\frac{dy}{dx} + y = 0\,.$$

* For the case $q(t) < 0$, write $q(t) = -[-q(t)]$, and set $\left[\frac{dx}{dt}\right]^2 = -q(t)$.

36. $p(t) = 3t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \left(1 + 3t^2\right)/t^2\,.$$

The ratio is *not* constant, and therefore the equation cannot be transformed.

37. $p(t) = t - 1/t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = 1\,.$$

The ratio is constant, and therefore the equation can be transformed. From Prob. 35, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $r^2 + r + 1 = 0$, with roots $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution is

$$y(x) = c_1 e^{-x/2} cos\,\sqrt{3}\,x/2 + c_2\, e^{-x/2} sin\,\sqrt{3}\,x/2\,.$$

Since $x = t^2/2$, the solution in the original variable t is

$$y(t) = e^{-t^2/4}\left[c_1 cos\left(\sqrt{3}\,t^2/4\right) + c_2\, sin\left(\sqrt{3}\,t^2/4\right)\right].$$

40. $p(t) = 4/t$ and $q(t) = 2/t^2$. We have $x = \sqrt{2}\int t^{-1}dt = \sqrt{2}\,ln\,t$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \frac{3}{\sqrt{2}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{3}{\sqrt{2}}\frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{2}\,r^2 + 3r + \sqrt{2} = 0$, with roots $r = -\sqrt{2}\,,\ -1/\sqrt{2}$. The general solution is

$$y(x) = c_1 e^{-\sqrt{2}x} + c_2\, e^{-x/\sqrt{2}}.$$

Since $x = \sqrt{2}\,ln\,t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{-2\,ln\,t} + c_2\, e^{-ln\,t} \\ &= c_1 t^{-2} + c_2\, t^{-1}. \end{aligned}$$

41. $p(t) = 3/t$ and $q(t) = 1.25/t^2$. We have $x = \sqrt{1.25}\int t^{-1}dt = \sqrt{1.25}\,ln\,t$.
Checking the feasibility of the transformation,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \frac{4}{\sqrt{5}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{4}{\sqrt{5}}\frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is
$\sqrt{5}\,r^2 + 4r + \sqrt{5} = 0$, with roots $r = -\frac{2}{\sqrt{5}} \pm i\frac{1}{\sqrt{5}}$. The general solution is

$$y(x) = c_1 e^{-2x/\sqrt{5}} cos\, x/\sqrt{5} + c_2\, e^{-2x/\sqrt{5}} sin\, x/\sqrt{5}\,.$$

Since $2x/\sqrt{5} = ln\,t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{-ln\,t} cos\left(ln\sqrt{t}\right) + c_2\, e^{-ln\,t} sin\left(ln\sqrt{t}\right) \\ &= t^{-1}\left[c_1 cos\left(ln\sqrt{t}\right) + c_2\, sin\left(ln\sqrt{t}\right)\right]. \end{aligned}$$

42. $p(t) = -4/t$ and $q(t) = -6/t^2$. Set $x = \sqrt{6}\int t^{-1}dt = \sqrt{6}\, ln\, t$.
Checking the feasibility of the transformation (**see Prob. 34 d, with* $q < 0$),

$$\frac{-q'(t) - 2p(t)q(t)}{2[-q(t)]^{3/2}} = \frac{-5}{\sqrt{6}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{-5}{\sqrt{6}}\frac{dy}{dx} - y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{6}\, r^2 - 5$
$r - \sqrt{6} = 0$,
with roots $r = \sqrt{6}\,,\ -1/\sqrt{6}$. The general solution is

$$y(x) = c_1 e^{\sqrt{6}x} + c_2\, e^{-x/\sqrt{6}}.$$

Since $x = \sqrt{6}\, ln\, t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{6 ln\, t} + c_2\, e^{-ln\, t} \\ &= c_1 t^6 + c_2\, t^{-1}. \end{aligned}$$

Section 3.5

2. The characteristic equation is $9r^2 + 6r + 1 = 0$, with the *double* root $r = -1/3$. Based on the discussion in this section, the general solution is $y(t) = c_1 e^{-t/3} + c_2 t\, e^{-t/3}$.

3. The characteristic equation is $4r^2 - 4r - 3 = 0$, with roots $r = -1/2\,, 3/2$. The general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$.

4. The characteristic equation is $4r^2 + 12r + 9 = 0$, with the *double* root $r = -3/2$. Based on the discussion in this section, the general solution is $y(t) = (c_1 + c_2\, t)e^{-3t/2}$.

5. The characteristic equation is $r^2 - 2r + 10 = 0$, with complex roots $r = 1 \pm 3i$. The general solution is $y(t) = c_1 e^t cos\, 3t + c_2 e^t sin\, 3t$.

6. The characteristic equation is $r^2 - 6r + 9 = 0$, with the *double* root $r = 3$. The general solution is $y(t) = c_1 e^{3t} + c_2 t\, e^{3t}$.

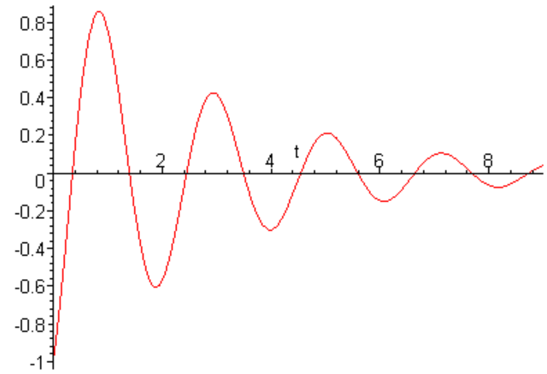
7. The characteristic equation is $4r^2 + 17r + 4 = 0$, with roots $r = -1/4\,, -4$. The general solution is $y(t) = c_1 e^{-t/4} + c_2 e^{-4t}$.

8. The characteristic equation is $16r^2 + 24r + 9 = 0$, with the *double* root $r = -3/4$. The general solution is $y(t) = c_1 e^{-3t/4} + c_2 t\, e^{-3t/4}$.

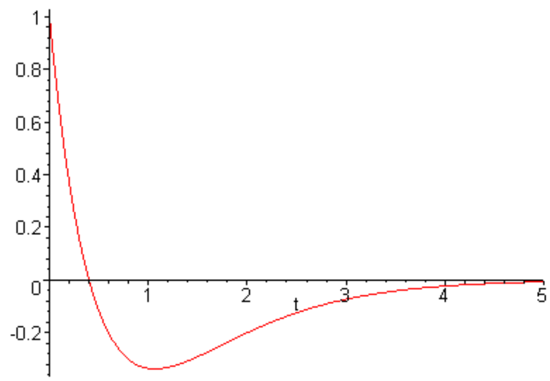
10. The characteristic equation is $2r^2 + 2r + 1 = 0$, with complex roots $r = -\frac{1}{2} \pm \frac{1}{2}\, i$. The general solution is $y(t) = c_1 e^{-t/2} cos\, t/2 + c_2 e^{-t/2} sin\, t/2$.

11. The characteristic equation is $9r^2 - 12r + 4 = 0$, with the *double* root $r = 2/3$. The general solution is $y(t) = c_1 e^{2t/3} + c_2 t\, e^{2t/3}$. Invoking the first initial condition, it follows that $c_1 = 2$. Now $y'(t) = (4/3 + c_2)e^{2t/3} + 2c_2 t\, e^{2t/3}/3$. Invoking the second initial condition, $4/3 + c_2 = -1$, or $c_2 = -7/3$. Hence $y(t) = 2e^{2t/3} - \frac{7}{3} t\, e^{2t/3}$. Since the *second* term dominates for large t, $y(t) \to -\infty$.

13. The characteristic equation is $9r^2 + 6r + 82 = 0$, with complex roots $r = -\frac{1}{3} \pm 3\, i$. The general solution is $y(t) = c_1 e^{-t/3} cos\, 3t + c_2 e^{-t/3} sin\, 3t$. Based on the first initial condition, $c_1 = -1$. Invoking the second initial condition, $1/3 + 3c_2 = 2$, or $c_2 = \frac{5}{9}$. Hence $y(t) = -e^{-t/3} cos\, 3t + \frac{5}{9} e^{-t/3} sin\, 3t$.



$15(a)$. The characteristic equation is $4r^2 + 12r + 9 = 0$, with the *double* root $r = -\frac{3}{2}$. The general solution is $y(t) = c_1 e^{-3t/2} + c_2 t\, e^{-3t/2}$. Invoking the first initial condition, it follows that $c_1 = 1$. Now $y'(t) = (-3/2 + c_2)e^{2t/3} - \frac{3}{2}c_2 t\, e^{2t/3}$. The second initial condition requires that $-3/2 + c_2 = -4$, or $c_2 = -5/2$. Hence the specific solution is $y(t) = e^{-3t/2} - \frac{5}{2}t\, e^{-3t/2}$.



(b). The solution crosses the *x-axis* at $t = 0.4$.

(c). The solution has a minimum at the point $(16/15\,,\,-5e^{-8/5}/3)$.

(d). Given that $y'(0) = b$, we have $-3/2 + c_2 = b$, or $c_2 = b + 3/2$. Hence the solution is $y(t) = e^{-3t/2} + \left(b + \frac{3}{2}\right)t\, e^{-3t/2}$. Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $b + \frac{3}{2}$. The critical value is $b = -\frac{3}{2}$.

16. The characteristic roots are $r_1 = r_2 = 1/2$. Hence the general solution is given by $y(t) = c_1 e^{t/2} + c_2 t\, e^{t/2}$. Invoking the initial conditions, we require that $c_1 = 2$, and that $1 + c_2 = b$. The specific solution is $y(t) = 2e^{t/2} + (b-1)t\, e^{t/2}$. Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $b - 1$. The critical value is $b = 1$.

$18(a)$. The characteristic roots are $r_1 = r_2 = -2/3$. Therefore the general solution is given by $y(t) = c_1 e^{-2t/3} + c_2 t\, e^{-2t/3}$. Invoking the initial conditions, we require that $c_1 = a$, and that $-2a/3 + c_2 = -1$. After solving for the coefficients, the specific solution is $y(t) = ae^{-2t/3} + \left(\frac{2a}{3} - 1\right) t\, e^{-2t/3}$.

(b). Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $\frac{2a}{3} - 1$. The critical value is $a = 3/2$.

$20(a)$. The characteristic equation is $r^2 + 2ar + a^2 = 0$, with *double* root $r = -a$. Hence one solution is $y_1(t) = c_1 e^{-at}$.

(b). Recall that the Wronskian satisfies the differential equation $W' + 2aW = 0$. The solution of this equation is $W(t) = c\, e^{-2at}$.

(c). By definition, $W = y_1 y_2' - y_1' y_2$. Hence $c_1 e^{-at} y_2' + ac_1 e^{-at} y_2 = c\, e^{-2at}$. That is, $y_2' + a\, y_2 = c_2 e^{-at}$. This equation is first order *linear*, with general solution $y_2(t) = c_2 t e^{-at} + c_3 e^{-at}$. Setting $c_2 = 1$ and $c_3 = 0$, we obtain $y_2(t) = te^{-at}$.

$22(a)$. Write $ar^2 + br + c = a\left(r^2 + \frac{b}{a} r + \frac{c}{a}\right)$. It follows that $\frac{b}{a} = -2r_1$ and $\frac{c}{a} = r_1^2$. Hence $ar^2 + br + c = ar^2 - 2ar_1 r + ar_1^2 = a(r^2 - 2r_1 r + r_1^2) = a(r - r_1)^2$. We find that $L[e^{rt}] = (ar^2 + br + c)e^{rt} = a(r - r_1)^2 e^{rt}$. Setting $r = r_1$, $L[e^{r_1 t}] = 0$.

(b). Differentiating Eq.(i) with respect to r,

$$\frac{\partial}{\partial r} L\left[e^{rt}\right] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1).$$

Now observe that

$$\begin{aligned}
\frac{\partial}{\partial r} L\left[e^{rt}\right] &= \frac{\partial}{\partial r}\left[a\frac{\partial^2}{\partial t^2}\left(e^{rt}\right) + b\frac{\partial}{\partial t}\left(e^{rt}\right) + c\left(e^{rt}\right)\right] \\
&= \left[a\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial r}e^{rt}\right) + b\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}e^{rt}\right) + c\left(\frac{\partial}{\partial r}e^{rt}\right)\right] \\
&= a\frac{\partial^2}{\partial t^2}\left(te^{rt}\right) + b\frac{\partial}{\partial t}\left(te^{rt}\right) + c\left(te^{rt}\right).
\end{aligned}$$

Hence $L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1)$. Setting $r = r_1$, $L[te^{r_1 t}] = 0$.

23. Set $y_2(t) = t^2\, v(t)$. Substitution into the ODE results in

$$t^2\left(t^2 v'' + 4tv' + 2v\right) - 4t\left(t^2 v' + 2tv\right) + 6t^2 v = 0\,.$$

After collecting terms, we end up with $t^4 v'' = 0$. Hence $v(t) = c_1 + c_2 t$, and thus $y_2(t) = c_1 t^2 + c_2 t^3$. Setting $c_1 = 0$ and $c_2 = 1$, we obtain $y_2(t) = t^3$.

24. Set $y_2(t) = t\, v(t)$. Substitution into the ODE results in

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0\,.$$

After collecting terms, we end up with $t^3v'' + 4t^2v' = 0$. This equation is *linear* in the variable $w = v'$. It follows that $v'(t) = c\,t^{-4}$, and $v(t) = c_1t^{-3} + c_2$. Thus $y_2(t) = c_1t^{-2} + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = t^{-2}$.

26. Set $y_2(t) = t\,v(t)$. Substitution into the ODE results in $v'' - v' = 0$. This ODE is *linear* in the variable $w = v'$. It follows that $v'(t) = c_1e^t$, and $v(t) = c_1e^t + c_2$. Thus $y_2(t) = c_1te^t + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = te^t$.

28. Set $y_2(x) = e^xv(x)$. Substitution into the ODE results in

$$v'' + \frac{x-2}{x-1}v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is

$$\begin{aligned}\mu &= exp\left(\int \frac{x-2}{x-1}dx\right)\\ &= \frac{e^x}{x-1}\,.\end{aligned}$$

Rewrite the equation as $\left[\frac{e^xv'}{x-1}\right]' = 0$, from which it follows that $v'(x) = c(x-1)e^{-x}$. Hence $v(x) = c_1xe^{-x} + c_2$ and $y_2(x) = c_1x + c_2e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

29. Set $y_2(x) = y_1(x)\,v(x)$, in which $y_1(x) = x^{1/4}exp\big(2\sqrt{x}\big)$. It can be verified that y_1 is a solution of the ODE, that is, $x^2y_1'' - (x - 0.1875)y_1 = 0$. Substitution of the given form of y_2 results in the differential equation

$$2x^{9/4}v'' + \big(4x^{7/4} + x^{5/4}\big)v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is

$$\begin{aligned}\mu &= exp\left(\int \left[2x^{-1/2} + \frac{1}{2x}\right]dx\right)\\ &= \sqrt{x}\;exp\big(4\sqrt{x}\big).\end{aligned}$$

Rewrite the equation as $\left[\sqrt{x}\;exp\big(4\sqrt{x}\big)\;v'\right]' = 0$, from which it follows that

$$v'(x) = c\,exp\big(-4\sqrt{x}\big)/\sqrt{x}\,.$$

Integrating, $v(x) = c_1exp\big(-4\sqrt{x}\big) + c_2$ and as a result,

$$y_2(x) = c_1x^{1/4}exp\big(-2\sqrt{x}\big) + c_2x^{1/4}exp\big(2\sqrt{x}\big).$$

Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{1/4}exp\big(-2\sqrt{x}\big)$.

32. Direct substitution verifies that $y_1(t) = exp(-\delta x^2/2)$ is a solution of the ODE. Now set $y_2(x) = y_1(x)\, v(x)$. Substitution of y_2 into the ODE results in

$$v'' - \delta x v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is $\mu = exp(-\delta x^2/2)$. Rewrite the equation as $[\, exp(-\delta x^2/2)v']' = 0$, from which it follows that

$$v'(x) = c_1\, exp\big(\delta x^2/2\big)\,.$$

Integrating, we obtain

$$v(x) = c_1 \int_{x_0}^{x} exp\big(\delta u^2/2\big)du + v(x_0).$$

Hence

$$y_2(x) = c_1 exp\big(-\delta x^2/2\big)\int_{x_0}^{x} exp\big(\delta u^2/2\big)du + c_2 exp\big(-\delta x^2/2\big).$$

Setting $c_2 = 0$, we obtain a second independent solution.

34. After writing the ODE in standard form, we have $p(t) = 3/t$. Based on *Abel's identity*, $W(y_1, y_2) = c_1 exp\big(-\int \frac{3}{t}dt\big) = c_1 t^{-3}$. As shown in Prob. 33, two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2\,.$$

In the given problem, $y_1(t) = t^{-1}$. Hence $(t\, y_2)' = c_1 t^{-1}$. Integrating both sides of the equation, $y_2(t) = c_1 t^{-1} ln\, t + c_2 t^{-1}$.

36. After writing the ODE in standard form, we have $p(x) = -x/(x-1)$. Based on *Abel's identity*, $W(y_1, y_2) = c\ exp\big(\int \frac{x}{x-1}dx\big) = c\ e^x(x-1)$. Two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2\,.$$

In the given problem, $y_1(x) = e^x$. Hence $(e^{-x}\, y_2)' = c\, e^{-x}(x-1)$. Integrating both sides of the equation, $y_2(x) = c_1 x + c_2 e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

37. Write the ODE in standard form to find $p(x) = 1/x$. Based on *Abel's identity*, $W(y_1, y_2) = c\ exp\big(-\int \frac{1}{x}dx\big) = c\, x^{-1}$. Two solutions of a second order linear ODE satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(x) = x^{-1/2} sin\, x$. Hence

$$\left(\frac{\sqrt{x}}{sin\, x}\, y_2\right)' = c\, \frac{1}{sin^2 x}\,.$$

Integrating both sides of the equation, $y_2(x) = c_1 x^{-1/2} cos\, x + c_2 x^{-1/2} sin\, x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{-1/2} cos\, x$.

39(a). The characteristic equation is $ar^2 + c = 0$. If $a\,, c > 0$, then the roots are $r_{1,2} = \pm\, i\sqrt{c/a}$. The general solution is

$$y(t) = c_1 cos\sqrt{\frac{c}{a}}\, t + c_2 sin\sqrt{\frac{c}{a}}\, t\,,$$

which is bounded.

(b). The characteristic equation is $ar^2 + br = 0$. The roots are $r_{1,2} = 0\,,\, -b/a\,,$ and hence the general solution is $y(t) = c_1 + c_2 exp(-bt/a)$. Clearly, $y(t) \to c_1$.

40. Note that $cos\, t\, sin\, t = \frac{1}{2} sin\, 2t$. So that $1 - k\, cos\, t\, sin\, t = 1 - \frac{k}{2} sin\, 2t$. If $0 < k < 2$, then $\frac{k}{2} sin\, 2t < |sin\, 2t|$ and $-\frac{k}{2} sin\, 2t > -|sin\, 2t|$. Hence

$$\begin{aligned} 1 - k\, cos\, t\, sin\, t &= 1 - \frac{k}{2} sin\, 2t \\ &> 1 - |sin\, 2t| \\ &\geq 0\,. \end{aligned}$$

41. $p(t) = -3/t$ and $q(t) = 4/t^2$. We have $x = 2\int t^{-1} dt = 2\, ln\, t$, and $t = e^{x/2}$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = -2\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0\,.$$

The general solution of this ODE is $y(x) = c_1 e^x + c_2 x e^x$. In terms of the original independent variable, $y(t) = c_1 t^2 + c_2 t^2\, ln\, t$.

Section 3.6

2. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. Since the function $g(t) = 3\, sin\, 2t$ is not proportional to the solutions of the homogeneous equation, set $Y = A\, cos\, 2t + B\, sin\, 2t$. Substitution into the given ODE, and comparing the coefficients, results in the system of equations $B - 4A = 3$ and $A + 4B = 0$. Hence $Y = -\frac{12}{17} cos\, 2t + \frac{3}{17} sin\, 2t$. The general solution is $y(t) = y_c(t) + Y$.

3. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1, 3$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is *not* sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2 e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -3$. Hence $Y = \frac{3}{16} te^{-t} + \frac{3}{8} t^2 e^{-t}$. The general
solution is $y(t) = y_c(t) + Y$.

5. The characteristic equation for the homogeneous problem is $r^2 + 9 = 0$, with complex roots $r = \pm 3i$. Hence $y_c(t) = c_1 cos\, 3t + c_2 sin\, 3t$. To simplify the analysis, set $g_1(t) = 6$ and $g_2(t) = t^2 e^{3t}$. By inspection, we have $Y_1 = 2/3$. Based on the form of g_2, set $Y_2 = Ae^{3t} + Bte^{3t} + Ct^2 e^{3t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $18A + 6B + 2C = 0$, $18B + 12C = 0$, and $18C = 1$. Hence

$$Y_2 = \frac{1}{162} e^{3t} - \frac{1}{27} te^{3t} + \frac{1}{18} t^2 e^{3t}.$$

The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

7. The characteristic equation for the homogeneous problem is $2r^2 + 3r + 1 = 0$, with roots $r = -1, -1/2$. Hence $y_c(t) = c_1 e^{-t} + c_2\, e^{-t/2}$. To simplify the analysis, set $g_1(t) = t^2$ and $g_2(t) = 3\, sin\, t$. Based on the form of g_1, set $Y_1 = A + Bt + Ct^2$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $A + 3B + 4C = 0$, $B + 6C = 0$, and $C = 1$. Hence we obtain $Y_1 = 14 - 6t + t^2$. On the other hand, set $Y_2 = D\, cos\, t + E\, sin\, t$. After substitution into the ODE, we find that $D = -9/10$ and $E = -3/10$. The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

9. The characteristic equation for the homogeneous problem is $r^2 + \omega_0^2 = 0$, with complex roots $r = \pm\, \omega_0 i$. Hence $y_c(t) = c_1 cos\, \omega_0 t + c_2 sin\, \omega_0 t$. Since $\omega \neq \omega_0$, set $Y = A\, cos\, \omega t + B\, sin\, \omega t$. Substitution into the ODE and comparing the coefficients results in the system of equations $(\omega_0^2 - \omega^2) A = 1$ and $(\omega_0^2 - \omega^2) B = 0$. Hence

$$Y = \frac{1}{\omega_0^2 - \omega^2}\, cos\, \omega t\,.$$

The general solution is $y(t) = y_c(t) + Y$.

10. From Prob. 9, $y_c(t) = c$. Since $cos\,\omega_0 t$ is a solution of the homogeneous problem, set $Y = At\,cos\,\omega_0 t + Bt\,sin\,\omega_0 t$. Substitution into the given ODE and comparing the coefficients results in $A = 0$ and $B = \frac{1}{2\omega_0}$. Hence the general solution is $y(t) = c_1 cos\,\omega_0 t + c_2 sin\,\omega_0 t + \frac{t}{2\omega_0} sin\,\omega_0 t$.

12. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1, 2$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{2t}$. Based on the form of the right hand side, that is, $cosh(2t) = (e^{2t} + e^{-2t})/2$, set $Y = At\,e^{2t} + Be^{-2t}$. Substitution into the given ODE and comparing the coefficients results in $A = 1/6$ and $B = 1/8$. Hence the general solution is $y(t) = c_1 e^{-t} + c_2 e^{2t} + t\,e^{2t}/6 + e^{-2t}/8$.

14. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 cos\,2t + c_2 sin\,2t$. Set $Y_1 = A + Bt + Ct^2$. Comparing the coefficients of the respective terms, we find that $A = -1/8$, $B = 0$, $C = 1/4$. Now set $Y_2 = D\,e^t$, and obtain $D = 3/5$. Hence the general solution is

$$y(t) = c_1 cos\,2t + c_2 sin\,2t - 1/8 + t^2/4 + 3\,e^t/5\,.$$

Invoking the initial conditions, we require that $19/40 + c_1 = 0$ and $3/5 + 2c_2 = 2$. Hence $c_1 = -19/40$ and $c_2 = 7/10$.

15. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\,e^t$. Consider $g_1(t) = t\,e^t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1 = At^2e^t + Bt^3e^t$ (the *first* term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\,e^t + t^3 e^t/6 + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 1$ and $c_1 + c_2 = 1$. Hence $c_1 = -3$ and $c_2 = 4$.

17. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 cos\,2t + c_2 sin\,2t$. Since the function $sin\,2t$ is a solution of the homogeneous problem, set $Y = At\,cos\,2t + Bt\,sin\,2t$. Upon substitution, we obtain $Y = -\frac{3}{4}t\,cos\,2t$. Hence the general solution is $y(t) = c_1 cos\,2t + c_2 sin\,2t - \frac{1}{4}t\,cos\,2t$. Invoking the initial conditions, we require that $c_1 = 2$ and $2c_2 - \frac{3}{4} = -1$. Hence $c_1 = 2$ and $c_2 = -1/8$.

18. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} cos\,2t + c_2 e^{-t} sin\,2t$. Based on the form of $g(t)$, set $Y = At\,e^{-t} cos\,2t + Bt\,e^{-t} sin\,2t$. After comparing coefficients, we obtain $Y = t\,e^{-t} sin\,2t$. Hence the general solution is

$$y(t) = c_1 e^{-t} cos\,2t + c_2 e^{-t} sin\,2t + t\,e^{-t} sin\,2t\,.$$

Invoking the initial conditions, we require that $c_1 = 1$ and $-c_1 + 2c_2 = 0$. Hence $c_1 = 1$ and $c_2 = 1/2$.

20. The characteristic equation for the homogeneous problem is $r^2 + 1 = 0$, with complex roots $r = \pm\, i$. Hence $y_c(t) = c_1 cos\, t + c_2 sin\, t$. Let $g_1(t) = t\, sin\, t$ and $g_2(t) = t$. By inspection, it is easy to see that $Y_2(t) = 1$. Based on the form of $g_1(t)$, set $Y_1(t) = At\, cos\, t + Bt\, sin\, t + Ct^2 cos\, t + Dt^2 sin\, t$. Substitution into the equation and comparing the coefficients results in $A = 0$, $B = 1/4$, $C = -1/4$, and $D = 0$. Hence $Y(t) = 1 + \frac{1}{4}t\, sin\, t - \frac{1}{4}t^2 cos\, t$.

21. The characteristic equation for the homogeneous problem is $r^2 - 5r + 6 = 0$, with roots $r = 2\,, 3$. Hence $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$. Consider $g_1(t) = e^{2t}(3t+4)sin\, t$, and $g_2(t) = e^t cos\, 2t$. Based on the form of these functions on the right hand side of the ODE,
set $Y_2(t) = e^t(A_1 cos\, 2t + A_2 sin\, 2t)$, $Y_1(t) = (B_1 + B_2 t)e^{2t} sin\, t + (C_1 + C_2 t)e^{2t} cos\, t$. Substitution into the equation and comparing the coefficients results in

$$Y(t) = -\frac{1}{20}\left(e^t cos\, 2t + 3e^t sin\, 2t\right) + \frac{3}{2}te^{2t}(cos\, t - sin\, t) + e^{2t}\left(\frac{1}{2}cos\, t - 5 sin\, t\right).$$

23. The characteristic roots are $r = 2\,, 2$. Hence $y_c(t) = c_1 e^{2t} + c_2 te^{2t}$. Consider the functions $g_1(t) = 2t^2$, $g_2(t) = 4te^{2t}$, and $g_3(t) = t\, sin\, 2t$. The corresponding forms of the respective parts of the particular solution are $Y_1(t) = A_0 + A_1 t + A_2 t^2$, $Y_2(t) = = e^{2t}(B_2 t^2 + B_3 t^3)$, and $Y_3(t) = t(C_1 cos\, 2t + C_2 sin\, 2t) + (D_1 cos2t + D_2 sin2t)$. Substitution into the equation and comparing the coefficients results in

$$Y(t) = \frac{1}{4}\left(3 + 4t + 2t^2\right) + \frac{2}{3}t^3 e^{2t} + \frac{1}{8}t\, cos\, 2t + \frac{1}{16}(cos\, 2t - sin\, 2t).$$

24. The homogeneous solution is $y_c(t) = c_1 cos\, 2t + c_2 sin\, 2t$. Since $cos\, 2t$ and $sin\, 2t$ are both solutions of the homogeneous equation, set

$$Y(t) = t\left(A_0 + A_1 t + A_2 t^2\right)cos\, 2t + t\left(B_0 + B_1 t + B_2 t^2\right)sin\, 2t.$$

Substitution into the equation and comparing the coefficients results in

$$Y(t) = \left(\frac{13}{32}t - \frac{1}{12}t^3\right)cos\, 2t + \frac{1}{16}\left(28t + 13t^2\right)sin\, 2t.$$

25. The homogeneous solution is $y_c(t) = c_1 e^{-t} + c_2 te^{-2t}$. None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider $Y(t) = e^t(A_0 + A_1 t + A_2 t^2)cos\, 2t + + e^t(B_0 + B_1 t + B_2 t^2)sin\, 2t + e^{-t}(C_1 cos\, t + C_2 sin\, t) + De^t$. Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned} Y(t) &= e^t\left(A_0 + A_1 t + A_2 t^2\right)cos\, 2t + \ + e^t\left(B_0 + B_1 t + B_2 t^2\right)sin\, 2t + \\ &+ e^{-t}\left(-\frac{2}{3}cos\, t + \frac{2}{3}sin\, t\right) + 2e^t/3\,, \end{aligned}$$

in which $A_0 = -4105/35152$, $A_1 = 73/676$, $A_2 = -5/52$, $B_0 = -1233/35152$, $B_1 = 10/169$, $B_2 = 1/52$.

26. The homogeneous solution is $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. None of the terms on the right hand side are solutions of the homogenous equation. In order to include the appropriate combinations of derivatives, consider $Y(t) = e^{-t}(A_1 t + A_2 t^2) cos\, 2t +$ $+ e^{-t}(B_1 t + B_2 t^2) sin\, 2t + e^{-2t}(C_0 + C_1 t) cos\, 2t + e^{-2t}(D_0 + D_1 t) sin\, 2t$. Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned} Y(t) &= \frac{3}{16} t e^{-t} cos\, 2t + \frac{3}{8} t^2 e^{-t} sin\, 2t - \frac{1}{25} e^{-2t}(7 + 10t) cos\, 2t + \\ &+ \frac{1}{25} e^{-2t}(1 + 5t) sin\, 2t \,. \end{aligned}$$

27. The homogeneous solution is $y_c(t) = c_1 cos\, \lambda t + c_2 sin\, \lambda t$. Since the differential operator does not contain a *first derivative* (and $\lambda \neq m\pi$), we can set

$$Y(t) = \sum_{m=1}^{N} C_m sin\, m\pi t \,.$$

Substitution into the ODE yields

$$-\sum_{m=1}^{N} m^2 \pi^2 C_m sin\, m\pi t + \lambda^2 \sum_{m=1}^{N} C_m sin\, m\pi t = \sum_{m=1}^{N} a_m sin\, m\pi t \,.$$

Equating coefficients of the individual terms, we obtain

$$C_m = \frac{a_m}{\lambda^2 - m^2 \pi^2} \,, \; m = 1, 2 \cdots N.$$

29. The homogeneous solution is $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. The input function is *independent* of the homogeneous solutions, on any interval. Since the right hand side is
piecewise constant, it follows by inspection that

$$Y(t) = \begin{cases} 1/5 \,, & 0 \le t \le \pi/2 \\ 0 \quad , & t > \pi/2 \end{cases} .$$

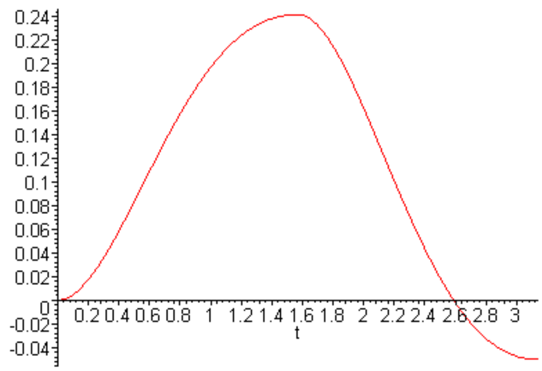
For $0 \le t \le \pi/2$, the general solution is $y(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t + 1/5$. Invoking the initial conditions $y(0) = y'(0) = 0$, we require that $c_1 = -1/5$, and that $c_2 = -1/10$. Hence

$$y(t) = \frac{1}{5} - \frac{1}{10}\left(2e^{-t} cos\, 2t + e^{-t} sin\, 2t\right)$$

on the interval $0 \le t \le \pi/2$. We now have the values $y(\pi/2) = \left(1 + e^{-\pi/2}\right)/5$, and $y'(\pi/2) = 0$. For $t > \pi/2$, the general solution is $y(t) = d_1 e^{-t} cos\, 2t + d_2 e^{-t} sin\, 2t$. It follows that $y(\pi/2) = -e^{-\pi/2} d_1$ and $y'(\pi/2) = e^{-\pi/2} d_1 - 2e^{-\pi/2} d_2$. Since the solution is continuously differentiable, we require that

$$-e^{-\pi/2}d_1 = \left(1+e^{-\pi/2}\right)/5$$
$$e^{-\pi/2}d_1 - 2e^{-\pi/2}d_2 = 0\,.$$

Solving for the coefficients, $d_1 = 2d_2 = -\left(e^{\pi/2}+1\right)/5$.



31. Since $a\,,b,c>0$, the roots of the characteristic equation has *negative* real parts. That is, $r=\alpha\pm\beta\,i$, where $\alpha<0$. Hence the homogeneous solution is

$$y_c(t) = c_1e^{\alpha t}cos\,\beta t + c_2e^{\alpha t}sin\,\beta t\,.$$

If $g(t)=d$, then the general solution is

$$y(t) = d/c + c_1e^{\alpha t}cos\,\beta t + c_2e^{\alpha t}sin\,\beta t\,.$$

Since $\alpha<0$, $y(t)\to d/c$ as $t\to\infty$. If $c=0$, then that characteristic roots are $r=0$ and $r=-b/a$. The ODE becomes $ay''+by'=d$. Integrating both sides, we find that $ay'+by=d\,t+c_1$. The general solution can be expressed as

$$y(t) = d\,t/b + c_1 + c_2e^{-bt/a}.$$

In this case, the solution grows without bound. If $b=0$, *also*, then the differential equation
can be written as $y''=d/a$, which has general solution $y(t)=d\,t^2/2a+c_1+c_2$. Hence the assertion is true only if the coefficients are *positive*.

32(a). Since D is a linear operator,

$$\begin{aligned} D^2y+bDy+cy &= D^2y-(r_1+r_2)Dy+r_1r_2y \\ &= D^2y-r_2Dy-r_1Dy+r_1r_2y \\ &= D(Dy-r_2y)-r_1(Dy-r_2y) \\ &= (D-r_1)(D-r_2)y\,. \end{aligned}$$

(b). Let $u=(D-r_2)y$. Then the ODE (i) can be written as $(D-r_1)u=g(t)$, that is,

$u' - r_1 u = g(t)$. The latter is a linear *first order* equation in u. Its general solution is

$$u(t) = e^{r_1 t}\int_{t_0}^{t} e^{-r_1\tau} g(\tau)d\tau + c_1 e^{r_1 t}\,.$$

From above, we have $y' - r_2 y = u(t)$. This equation is also a first order ODE. Hence the general solution of the original second order equation is

$$y(t) = e^{r_2 t}\int_{t_0}^{t} e^{-r_2\tau} u(\tau)d\tau + c_2 e^{r_2 t}\,.$$

Note that the solution $y(t)$ contains *two* arbitrary constants.

34. Note that $(2D^2 + 3D + 1)y = (2D+1)(D+1)y$. Let $u = (D+1)y$, and solve the ODE $2u' + u = t^2 + 3sin\,t$. This equation is a linear first order ODE, with solution

$$\begin{aligned} u(t) &= e^{-t/2}\int_{t_0}^{t} e^{\tau/2}\Big[\tau^2/2 + \frac{3}{2}sin\,\tau\Big]d\tau + c\,e^{-t/2} \\ &= t^2 - 4t + 8 - \frac{6}{5}cos\,t + \frac{3}{5}sin\,t + c\,e^{-t/2}. \end{aligned}$$

Now consider the ODE $y' + y = u(t)$. The general solution of this first order ODE is

$$y(t) = e^{-t}\int_{t_0}^{t} e^{\tau} u(\tau)d\tau + c_2 e^{-t}\,,$$

in which $u(t)$ is given above. Substituting for $u(t)$ and performing the integration,

$$y(t) = t^2 - 6t + 14 - \frac{9}{10}cos\,t - \frac{3}{10}sin\,t + c_1 e^{-t/2} + c_2 e^{-t}.$$

35. We have $(D^2 + 2D + 1)y = (D+1)(D+1)y$. Let $u = (D+1)y$, and consider the ODE $u' + u = 2e^{-t}$. The general solution is $u(t) = 2t\,e^{-t} + c\,e^{-t}$. We therefore have the first order equation $u' + u = 2t\,e^{-t} + c_1 e^{-t}$. The general solution of the latter differential equation is

$$\begin{aligned} y(t) &= e^{-t}\int_{t_0}^{t}[2\tau + c_1]d\tau + c_2 e^{-t} \\ &= e^{-t}\big(t^2 + c_1 t + c_2\big). \end{aligned}$$

36. We have $(D^2 + 2D)y = D(D+2)y$. Let $u = (D+2)y$, and consider the equation $u' = 3 + 4sin\,2t$. Direct integration results in $u(t) = 3t - 2cos\,2t + c$. The problem is reduced to solving the ODE $y' + 2y = 3t - 2cos\,2t + c$. The general solution of this first order differential equation is

$$
\begin{aligned}
y(t) &= e^{-2t}\int_{t_0}^{t} e^{2\tau}[3\tau - 2cos\,2\tau + c\,]d\tau + c_2e^{-2t} \\
&= \frac{3}{2}t - \frac{1}{2}(cos\,2t + sin\,2t) + c_1 + c_2e^{-2t}.
\end{aligned}
$$

Section 3.7

1. The solution of the homogeneous equation is $y_c(t) = c_1 e^{2t} + c_2 e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^{3t}(2e^t)}{W(t)}dt \\ &= 2e^{-t} \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{2t}(2e^t)}{W(t)}dt \\ &= -\, e^{-2t} \end{aligned}$$

Hence the particular solution is $Y(t) = 2e^t - e^t = e^t$.

3. The solution of the homogeneous equation is $y_c(t) = c_1 e^{-t} + c_2 t e^{-t}$. The functions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-2t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{-t}(3e^{-t})}{W(t)}dt \\ &= -\,3t^2/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{-t}(3e^{-t})}{W(t)}dt \\ &= 3t \end{aligned}$$

Hence the particular solution is $Y(t) = -\,3t^2e^{-t}/2 + 3t^2e^{-t} = 3t^2e^{-t}/2\,$.

4. The functions $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^t$. First write the equation in standard form, so that $g(t) = 4e^{t/2}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{t/2}\left(4e^{t/2}\right)}{W(t)}dt \\ &= -\,2t^2 \end{aligned}$$

$$u_2(t) = \int \frac{e^{t/2}\left(4e^{t/2}\right)}{W(t)}dt$$
$$= 4t$$

Hence the particular solution is $Y(t) = -2t^2e^{t/2} + 4t^2e^{t/2} = 2t^2e^{t/2}$.

6. The solution of the homogeneous equation is $y_c(t) = c_1 cos\,3t + c_2 sin\,3t$. The two functions $y_1(t) = cos\,3t$ and $y_2(t) = sin\,3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{sin\,3t(9\,sec^2 3t)}{W(t)}dt$$
$$= -csc\,3t$$

$$u_2(t) = \int \frac{cos\,3t(9\,sec^2 3t)}{W(t)}dt$$
$$= ln|sec\,3t + tan\,3t|$$

Hence the particular solution is $Y(t) = -1 + (sin\,3t)ln|sec\,3t + tan\,3t|$. The general solution is given by $y(t) = c_1 cos\,3t + c_2 sin\,3t + (sin\,3t)ln|sec\,3t + tan\,3t| - 1$.

7. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}dt$$
$$= -ln\,t$$

$$u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}dt$$
$$= -1/t$$

Hence the particular solution is $Y(t) = -e^{-2t}ln\,t - e^{-2t}$. Since the *second term* is a solution of the homogeneous equation, the general solution is given by $y(t) = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}ln\,t$.

8. The solution of the homogeneous equation is $y_c(t) = c_1 cos\,2t + c_2 sin\,2t$. The two functions $y_1(t) = cos\,2t$ and $y_2(t) = sin\,2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{sin\,2t(3\,csc\,2t)}{W(t)}dt$$
$$= -3t/2$$

$$u_2(t) = \int \frac{cos\,2t(3\,csc\,2t)}{W(t)}dt$$
$$= \frac{3}{4}ln|sin\,2t|$$

Hence the particular solution is $Y(t) = -\frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$. The general solution is given by $y(t) = c_1cos\,2t + c_2sin\,2t - \frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$.

9. The functions $y_1(t) = cos\,(t/2)$ and $y_2(t) = sin(t/2)$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = 1/2$. First write the ODE in standard form, so that $g(t) = sec(t/2)/2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{cos\,(t/2)[sec(t/2)]}{2W(t)}dt$$
$$= 2\,ln[cos\,(t/2)]$$

$$u_2(t) = \int \frac{sin(t/2)[sec(t/2)]}{2W(t)}dt$$
$$= t$$

The particular solution is $Y(t) = 2cos(t/2)ln[cos\,(t/2)] + t\,sin(t/2)$. The general solution is given by

$$y(t) = c_1cos\,(t/2) + c_2sin(t/2) + 2\,cos(t/2)\,ln[cos\,(t/2)] + t\,sin(t/2).$$

10. The solution of the homogeneous equation is $y_c(t) = c_1e^t + c_2te^t$. The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions, with $W(y_1, y_2) = e^{2t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^t(e^t)}{W(t)(1+t^2)}dt$$
$$= -\frac{1}{2}ln\left(1+t^2\right)$$

$$u_2(t) = \int \frac{e^t(e^t)}{W(t)(1+t^2)}dt$$
$$= arctan\,t$$

The particular solution is $Y(t) = -\frac{1}{2}e^t\,ln(1+t^2) + te^t\,arctan(t)$. Hence the general solution is given by $y(t) = c_1 e^t + c_2 t e^t - \frac{1}{2} e^t \, ln(1+t^2) + t e^t \, arctan(t)$.

12. The functions $y_1(t) = cos\, 2t$ and $y_2(t) = sin\, 2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)$ $y_1(t) + u_2(t)\, y_2(t)$,
in which

$$u_1(t) = -\frac{1}{2}\int^t g(s)\, sin\, 2s\, ds$$

$$u_2(t) = \frac{1}{2}\int^t g(s)\, cos\, 2s\, ds$$

Hence the particular solution is

$$Y(t) = -\frac{1}{2} cos\, 2t \int^t g(s)\, sin\, 2s\, ds + \frac{1}{2} sin\, 2t \int^t g(s)\, cos\, 2s\, ds\,.$$

Note that $sin\, 2t\, cos\, 2s - cos\, 2t\, sin\, 2s = sin(2t - 2s)$. It follows that

$$Y(t) = \frac{1}{2}\int^t g(s) sin(2t - 2s) ds\,.$$

The general solution of the differential equation is given by

$$y(t) = c_1 cos\, 2t + c_2 sin\, 2t + \frac{1}{2}\int^t g(s) sin(2t - 2s) ds\,.$$

13. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (3t^2 - 1)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1, y_2) = -3$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{t^{-1}(3t^2 - 1)}{t^2\, W(t)} dt \\ &= t^{-2}/6 + ln\, t \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{t^2(3t^2 - 1)}{t^2\, W(t)} dt \\ &= -t^3/3 + t/3 \end{aligned}$$

Therefore $Y(t) = 1/6 + t^2 ln\, t - t^2/3 + 1/3$. Hence the general solution is

$$y(t) = c_1 t^2 + c_2 t^{-1} + t^2 ln\, t + 1/2\,.$$

15. Observe that $g(t) = t\,e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1,y_2) = t\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^t(t\,e^{2t})}{W(t)}dt \\ &= -\,e^{2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{(1+t)(t\,e^{2t})}{W(t)}dt \\ &= t\,e^t \end{aligned}$$

Therefore $Y(t) = -\,(1+t)e^{2t}/2 + t\,e^{2t} = -\,e^{2t}/2 + t\,e^{2t}/2$.

16. Observe that $g(t) = 2(1-t)\,e^{-t}$. Direct substitution of $y_1(t) = e^t$ and $y_2(t) = t$ verifies that they are solutions of the homogeneous equation. The Wronskian of the two solutions is $W(y_1,y_2) = (1-t)\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{2t(1-t)e^{-t}}{W(t)}dt \\ &= te^{-2t} + e^{-2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{2(1-t)}{W(t)}dt \\ &= -\,2\,e^{-t} \end{aligned}$$

Therefore $Y(t) = te^{-t} + e^{-t}/2 - 2t\,e^{-t} = -\,te^{-t} + \,e^{-t}/2$.

17. Note that $g(x) = ln\,x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 ln\,x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = x^3$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$\begin{aligned} u_1(x) &= -\int \frac{x^2 ln\,x(ln\,x)}{W(x)}dx \\ &= -\,(ln\,x)^3/3 \end{aligned}$$

$$\begin{aligned} u_2(x) &= \int \frac{x^2(ln\, x)}{W(x)} dx \\ &= (ln\, x)^2/2 \end{aligned}$$

Therefore $Y(x) = -x^2(ln\, x)^3/3 + x^2(ln\, x)^3/2 = x^2(ln\, x)^3/6$.

19. First write the equation in *standard form*. Note that the forcing function becomes $g(x)/(1-x)$. The functions $y_1(x) = e^x$ and $y_2(x) = x$ are a fundamental set of solutions,
as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = (1-x)e^x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

in which

$$u_1(x) = -\int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau$$

$$u_2(x) = \int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau$$

Therefore

$$\begin{aligned} Y(x) &= -e^x \int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau + x\int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)} d\tau \\ &= \int^x \frac{(xe^\tau - e^x\tau)g(\tau)}{(1-\tau)^2 e^\tau} d\tau\,. \end{aligned}$$

20. First write the equation in *standard form*. The forcing function becomes $g(x)/x^2$. The functions $y_1(x) = x^{-1/2} sin\, x$ and $y_2(x) = x^{-1/2} cos\, x$ are a fundamental set of solutions. The Wronskian of the solutions is $W(y_1,y_2) = -1/x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x),$$

in which

$$u_1(x) = \int^x \frac{cos\,\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau$$

$$u_2(x) = -\int^x \frac{sin\,\tau\,(g(\tau))}{\tau\sqrt{\tau}} d\tau$$

Therefore

$$\begin{aligned} Y(x) &= \frac{sin\,x}{\sqrt{x}}\int^{x}\frac{cos\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}dt - \frac{cos\,x}{\sqrt{x}}\int^{x}\frac{sin\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}d\tau \\ &= \frac{1}{\sqrt{x}}\int^{x}\frac{sin(x-\tau)\,g(\tau)}{\tau\sqrt{\tau}}d\tau\,. \end{aligned}$$

21. Let $y_1(t)$ and $y_2(t)$ be a fundamental set of solutions, and $W(t) = W(y_1, y_2)$ be the corresponding Wronskian. Any solution, $u(t)$, of the homogeneous equation is a linear combination $u(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$. Invoking the initial conditions, we require that

$$\begin{aligned} y_0 &= \alpha_1\, y_1(t_0) + \alpha_2\, y_2(t_0) \\ y_0' &= \alpha_1\, y_1'(t_0) + \alpha_2\, y_2'(t_0) \end{aligned}$$

Note that this system of equations has a unique solution, since $W(t_0) \neq 0$. Now consider the *nonhomogeneous* problem, $L[v] = g(t)$,with *homogeneous* initial conditions. Using the method of variation of parameters, the particular solution is given by

$$Y(t) = \; - y_1(t)\int_{t_0}^{t}\frac{y_2(s)\,g(s)}{W(s)}ds + y_2(t)\int_{t_0}^{t}\frac{y_1(s)\,g(s)}{W(s)}ds\,.$$

The general solution of the IVP (iii) is

$$\begin{aligned} v(t) &= \beta_1 y_1(t) + \beta_2 y_2(t) + Y(t) \\ &= \beta_1 y_1(t) + \beta_2 y_2(t) + y_1(t)u_1(t) + y_2(t)u_2(t) \end{aligned}$$

in which u_1 and u_2 are defined above. Invoking the initial conditions, we require that

$$\begin{aligned} 0 &= \beta_1 y_1(t_0) + \beta_2 y_2(t_0) + Y(t_0) \\ 0 &= \beta_1 y_1'(t_0) + \beta_2 y_2'(t_0) + Y'(t_0) \end{aligned}$$

Based on the definition of u_1 and u_2 , $Y(t_0) = 0$. Furthermore, since $y_1 u_1' + y_2 u_2' = 0$, it follows that $Y'(t_0) = 0$. Hence the only solution of the above system of equations is the *trivial solution*. Therefore $v(t) = Y(t)$. Now consider the function $y = u{+}v$. Then $L[y] = L[u+v] = L[u] + L[v] = g(t)$. That is, $y(t)$ is a solution of the nonhomogeneous
problem. Further, $y(t_0) = u(t_0) + v(t_0) = y_0$, and similarly, $y'(t_0) = y_0'$. By the uniqueness theorems, $y(t)$ is the unique solution of the initial value problem.

23. A fundamental set of solutions is $y_1(t) = cos\,t$ and $y_2(t) = sin\,t$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = 1$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t}\frac{cos(s)\,sin(t) - cos(t)\,sin(s)}{W(s)}g(s)ds \\ &= \int_{t_0}^{t}[cos(s)\,sin(t) - cos(t)\,sin(s)]g(s)ds\,. \end{aligned}$$

Finally, we have $cos(s)\,sin(t) - cos(t)\,sin(s) = sin(t-s)$.

24. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = e^{bt}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = (b-a)exp[(a+b)t]$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{W(s)} g(s)ds \\ &= \frac{1}{b-a}\int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{exp[(a+b)s]} g(s)ds\,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^{t} \left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds\,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{W(s)} g(s)ds \\ &= \frac{1}{b-a}\int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{exp[(a+b)s]} g(s)ds\,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^{t} \left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds\,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{te^{as+at} - s\,e^{at+as}}{W(s)} g(s)ds \\ &= \int_{t_0}^{t} \frac{(t-s)e^{as+at}}{e^{2as}} g(s)ds\,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \int_{t_0}^{t} (t-s)e^{a(t-s)} g(s)ds\,.$$

27. Depending on the values of a, b and c, the operator $aD^2 + bD + c$ can have *three* types of fundamental solutions.

(i) The characteristic roots $r_{1,2} = \alpha\,,\beta\,;\ \alpha \neq \beta\,.$ $y_1(t) = e^{\alpha t}$ and $y_2(t) = e^{\beta t}$.

$$K(t) = \frac{1}{\beta - \alpha}\left[e^{\beta t} - e^{\alpha t}\right].$$

(ii) The characteristic roots $r_{1,2} = \alpha\,,\beta\,;\ \alpha = \beta\,.\ \ y_1(t) = e^{\alpha t}$ and $y_2(t) = te^{\alpha t}$.

$$K(t) = t\,e^{\alpha t}.$$

(iii) The characteristic roots $r_{1,2} = \lambda \pm i\,\mu\,.\ y_1(t) = e^{\lambda t} cos\,\mu t$ and $y_2(t) = e^{\lambda t} sin\,\mu t$.

$$K(t) = \frac{1}{\mu} e^{\lambda t} sin\,\mu t\,.$$

28. Let $y(t) = v(t)y_1(t)$, in which $y_1(t)$ is a solution of the *homogeneous equation*. Substitution into the given ODE results in

$$v'' y_1 + 2v' y_1' + v y_1'' + p(t)[v' y_1 + v y_1'] + q(t) v y_1 = g(t)\,.$$

By assumption, $y_1'' + p(t)y_1 + q(t)y_1 = 0$, hence $v(t)$ must be a solution of the ODE

$$v'' y_1 + [2y_1' + p(t)y_1]v' = g(t)\,.$$

Setting $w = v'$, we also have $w' y_1 + [2y_1' + p(t)y_1]w = g(t)$.

30. First write the equation as $y'' + 7t^{-1}y + 5t^{-2}\,y = t^{-1}$. As shown in Prob. 28, the function $y(t) = t^{-1}v(t)$ is a solution of the given ODE as long as v is a solution of

$$t^{-1}\,v'' + [\,-2t^{-2} + 7t^{-2}]v' = t^{-1}\,,$$

that is, $v'' + 5t^{-1}\,v' = 1$. This ODE is *linear and first order* in v'. The integrating factor is $\mu = t^5$. The solution is $v' = t/6 + c\,t^{-5}$. Direct integration now results in $v(t) = t^2/12 + c_1 t^{-4} + c_2$. Hence $y(t) = t/12 + c_1 t^{-5} + c_2 t^{-1}$.

31. Write the equation as $y'' - t^{-1}(1+t)y + t^{-1}\,y = t\,e^{2t}$. As shown in Prob. 28, the function $y(t) = (1+t)v(t)$ is a solution of the given ODE as long as v is a solution of

$$(1+t)\,v'' + \left[2 - t^{-1}(1+t)^2\right]v' = t\,e^{2t},$$

that is, $v'' - \frac{1+t^2}{t(t+1)}\,v' = \frac{t}{t+1}e^{2t}$. This equation is first order linear in v', with integrating factor $\mu = t^{-1}(1+t)^2 e^{-t}$. The solution is $v' = (t^2 e^{2t} + c_1 t e^t)/(1+t)^2$. Integrating, we obtain $v(t) = e^{2t}/2 - e^{2t}/(t+1) + c_1 e^t/(t+1) + c_2$. Hence the solution of the original ODE is $y(t) = (t-1)e^{2t}/2 + c_1 e^t + c_2(t+1)$.

32. Write the equation as $y'' + t(1-t)^{-1}y - (1-t)^{-1}\,y = 2(1-t)\,e^{-t}$. The function $y(t) = e^t v(t)$ is a solution to the given ODE as long as v is a solution of

$$e^t v'' + \left[2e^t + t(1-t)^{-1}e^t\right]v' = 2(1-t)\,e^{-t},$$

that is, $v'' + [(2-t)/(1-t)]v' = 2(1-t)\,e^{-2t}$. This equation is first order linear in v', with integrating factor $\mu = e^t/(t-1)$. The solution is

$$v' = (t-1)\left(2e^{-2t} + c_1 e^{-t}\right).$$

Integrating, we obtain $v(t) = (1/2-t)e^{-2t} - c_1 te^{-t} + c_2$. Hence the solution of the original ODE is $y(t) = (1/2-t)e^{-t} - c_1 t + c_2 e^t$.

Section 3.8

1. $R cos\,\delta = 3$ and $R sin\,\delta = 4 \Rightarrow R = \sqrt{25} = 5$ and $\delta = arctan(4/3)$. Hence

$$u = 5\,cos(2t - 0.9273).$$

3. $R cos\,\delta = 4$ and $R sin\,\delta = -2 \Rightarrow R = \sqrt{20} = 2\sqrt{5}$ and $\delta = -arctan(1/2)$. Hence

$$u = 2\sqrt{5}\;cos(3t + 0.4636).$$

4. $R cos\,\delta = -2$ and $R sin\,\delta = -3 \Rightarrow R = \sqrt{13}$ and $\delta = \pi + arctan(3/2)$. Hence

$$u = \sqrt{13}\,cos(\pi t - 4.1244).$$

5. The spring constant is $k = 2/(1/2) = 4$ *lb/ft*. Mass $m = 2/32 = 1/16$ *lb-s²/ft*. Since there is no damping, the equation of motion is

$$\frac{1}{16}u'' + 4u = 0\,,$$

that is, $u'' + 64u = 0$. The initial conditions are $u(0) = 1/4\,ft$, $u'(0) = 0\,fps$. The general solution is $u(t) = A\,cos\,8t + B\,sin\,8t$. Invoking the initial conditions, we have $u(t) = \frac{1}{4}\,cos\,8t$. $R = 3$ *inches*, $\delta = 0$ *rad*, $\omega_0 = 8$ *rad/s*, and $T = \pi/4$ *sec*.

7. The spring constant is $k = 3/(1/4) = 12$ *lb/ft*. Mass $m = 3/32$ *lb-s²/ft*. Since there is no damping, the equation of motion is

$$\frac{3}{32}u'' + 12u = 0\,,$$

that is, $u'' + 128u = 0$. The initial conditions are $u(0) = -1/12\,ft$, $u'(0) = 2\,fps$. The general solution is $u(t) = A\,cos\,8\sqrt{2}\,t + B\,sin\,8\sqrt{2}\,t$. Invoking the initial conditions, we have

$$u(t) = -\frac{1}{12}\,cos\,8\sqrt{2}\,t + \frac{1}{4\sqrt{2}}\,sin\,8\sqrt{2}\,t\,.$$

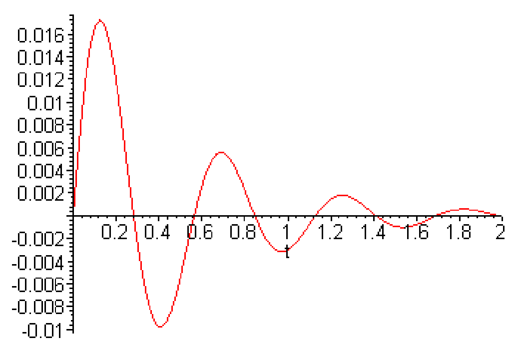
$R = \sqrt{11}/12\,ft$, $\delta = \pi - atan\left(3/\sqrt{2}\right)\,rad$, $\omega_0 = 8\sqrt{2}\,rad/s$, and $T = \pi/\left(4\sqrt{2}\right)\,sec$.

10. The spring constant is $k = 16/(1/4) = 64\,lb/ft$. Mass $m = 1/2\,lb\text{-}s^2/ft$. The damping coefficient is $\gamma = 2\,lb\text{-}sec/ft$. Hence the equation of motion is

$$\frac{1}{2}u'' + 2u' + 64u = 0\,,$$

that is, $u'' + 4u' + 128u = 0$. The initial conditions are $u(0) = 0\,ft$, $u'(0) = 1/4\,fps$. The general solution is $u(t) = A\,cos\,2\sqrt{31}\,t + B\,sin\,2\sqrt{31}\,t$. Invoking the initial conditions, we have

$$u(t) = \frac{1}{8\sqrt{31}}e^{-2t}\,sin\,2\sqrt{31}\,t\,.$$



Solving $u(t) = 0$, on the interval $[0.2\,,\,0.4]$, we obtain $t = \pi/2\sqrt{31} = 0.2821\,sec$. Based on the graph, and the solution of $u(t) = 0.01$, we have $|u(t)| \leq 0.01$ for $t \geq \tau = 0.2145$.

11. The spring constant is $k = 3/(.1) = 30\,N/m$. The damping coefficient is given as $\gamma = 3/5\,N\text{-}sec/m$. Hence the equation of motion is

$$2u'' + \frac{3}{5}u' + 30u = 0\,,$$

that is, $u'' + 0.3u' + 15u = 0$. The initial conditions are $u(0) = 0.05\,m$ and $u'(0) = 0.01\,m/s$. The general solution is $u(t) = A\,cos\,\mu t + B\,sin\,\mu t$, in which $\mu = 3.87008\,rad/s$. Invoking the initial conditions, we have

$$u(t) = e^{-0.15t}(0.05cos\,\mu t + 0.00452sin\,\mu t)\,.$$

Also, $\mu/\omega_0 = 3.87008/\sqrt{15} \approx 0.99925$.

13. The frequency of the *undamped* motion is $\omega_0 = 1$. The quasi frequency of the damped
motion is $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$. Setting $\mu = \frac{2}{3}\omega_0$, we obtain $\gamma = \frac{2}{3}\sqrt{5}$.

14. The spring constant is $k = mg/L$. The equation of motion for an undamped system is

$$mu'' + \frac{mg}{L}u = 0\,.$$

Hence the natural frequency of the system is $\omega_0 = \sqrt{\frac{g}{L}}$. The period is $T = 2\pi/\omega_0$.

15. The general solution of the system is $u(t) = A\,cos\,\gamma(t-t_0) + B\,sin\,\gamma(t-t_0)$. Invoking the initial conditions, we have $u(t) = u_0 cos\gamma(t-t_0) + (u_0'/\gamma)sin\gamma(t-t_0)$. Clearly, the functions $v = u_0 cos\gamma(t-t_0)$ and $w = (u_0'/\gamma)sin\gamma(t-t_0)$ satisfy the given criteria.

16. Note that $r\,sin(\,\omega_0 t - \theta) = r\,sin\omega_0 t\,cos\,\theta - r\,cos\omega_0 t\,sin\theta$. Comparing the given expressions, we have $A = -r\,sin\,\theta$ and $B = r\,cos\,\theta$. That is, $r = R = \sqrt{A^2+B^2}$, and $tan\,\theta = -A/B = -1/tan\,\delta$. The latter relation is also $tan\,\theta + cot\,\delta = 1$.

18. The system is *critically damped*, when $R = 2\sqrt{L/C}$. Here $R = 1000$ *ohms*.

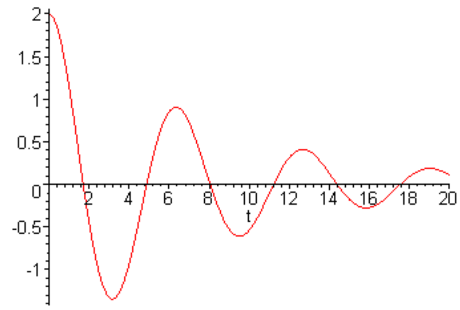
$21(a)$. Let $u = Re^{-\gamma t/2m}cos(\mu t - \delta)$. Then attains a *maximum* when $\mu t_k - \delta = 2k\pi$. Hence $T_d = t_{k+1} - t_k = 2\pi/\mu$.

(b). $u(t_k)/u(t_{k+1}) = exp(-\gamma t_k/2m)/exp(-\gamma t_{k+1}/2m) = exp[(\gamma t_{k+1} - \gamma t_k)/2m]$. Hence $u(t_k)/u(t_{k+1}) = exp[\gamma(2\pi/\mu)/2m] = exp(\gamma T_d/2m)$.

(c). $\Delta = ln[u(t_k)/u(t_{k+1})] = \gamma(2\pi/\mu)/2m = \pi\gamma/\mu m$.

22. The spring constant is $k = 16/(1/4) = 64$ *lb/ft*. Mass $m = 1/2$ *lb-s²/ft*. The damping coefficient is $\gamma = 2$ *lb-sec/ft*. The quasi frequency is $\mu = 2\sqrt{31}$ *rad/s*. Hence $\Delta = \frac{2\pi}{\sqrt{31}} \approx 1.1285$.

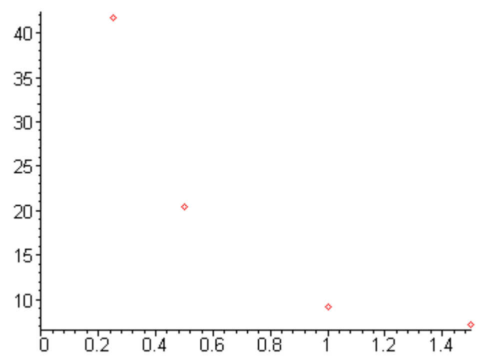
$25(a)$. The solution of the IVP is $u(t) = e^{-t/8}\left(2\,cos\,\frac{3}{8}\sqrt{7}\,t + 0.252sin\,\frac{3}{8}\sqrt{7}\,t\right)$.



Using the plot, and numerical analysis, $\tau \approx 41.715$.

(b). For $\gamma = 0.5$, $\tau \approx 20.402$; for $\gamma = 1.0$, $\tau \approx 9.168$; for $\gamma = 1.5$, $\tau \approx 7.184$.

(c).



(d). For $\gamma = 1.6$, $\tau \approx 7.218$; for $\gamma = 1.7$, $\tau \approx 6.767$; for $\gamma = 1.8$, $\tau \approx 5.473$; for $\gamma = 1.9$, $\tau \approx 6.460$. τ steadily decreases to about $\tau_{min} \approx 4.873$, corresponding to the critical value $\gamma_0 \approx 1.73$.

(e). We have $u(t) = \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}} cos(\mu t - \delta)$, in which $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$, and $\delta = tan^{-1}\frac{\gamma}{\sqrt{4-\gamma^2}}$. Hence $|u(t)| \leq \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}}$.

$26(a)$. The characteristic equation is $mr^2 + \gamma r + k = 0$. Since $\gamma^2 < 4km$, the roots are $r_{1,2} = -\frac{\gamma}{2m} \pm i\frac{\sqrt{4mk-\gamma^2}}{2m}$. The general solution is

$$u(t) = e^{-\gamma t/2m}\left[A\,cos\frac{\sqrt{4mk-\gamma^2}}{2m}t + B\,sin\frac{\sqrt{4mk-\gamma^2}}{2m}t\right].$$

Invoking the initial conditions, $A = u_0$ and

$$B = \frac{(2mv_0 - \gamma u_0)}{\sqrt{4mk-\gamma^2}}.$$

(b). We can write $u(t) = R\,e^{-\gamma t/2m} cos(\mu t - \delta)$, in which

$$R = \sqrt{u_0^2 + \frac{(2mv_0 - \gamma u_0)^2}{4mk - \gamma^2}}\,,$$

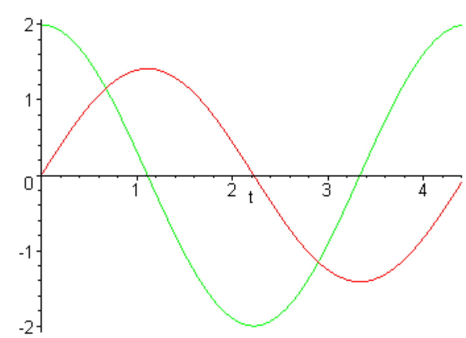
and

$$\delta = arctan\left[\frac{(2mv_0 - \gamma u_0)}{u_0\sqrt{4mk - \gamma^2}}\right].$$

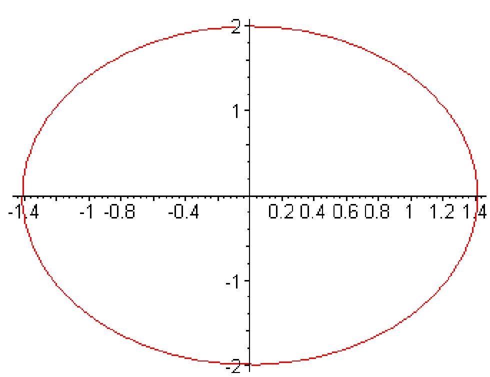
(c). $R = \sqrt{u_0^2 + \frac{(2mv_0-\gamma u_0)^2}{4mk-\gamma^2}} = 2\sqrt{\frac{m(ku_0^2+\gamma u_0 v_0+mv_0^2)}{4mk-\gamma^2}} = \sqrt{\frac{a+b\gamma}{4mk-\gamma^2}}$.
It is evident that R increases (*monotonically*) without bound as $\gamma \to \left(2\sqrt{mk}\right)^-$.

28(a). The general solution is $u(t) = Acos\,\sqrt{2}\,t + Bsin\,\sqrt{2}\,t$. Invoking the initial conditions, we have $u(t) = \sqrt{2}\ sin\,\sqrt{2}\ t$.

(b).

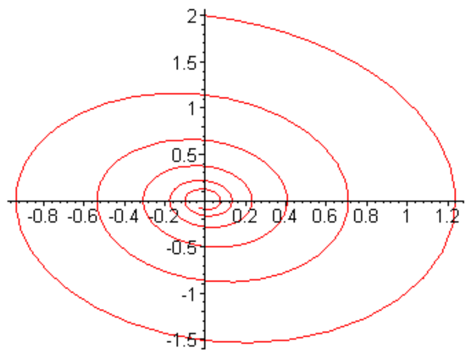


(c).



The condition $u'(0) = 2$ implies that $u(t)$ *initially* increases. Hence the phase point travels *clockwise*.

29. $u(t) = \frac{16}{\sqrt{127}} e^{-t/8} sin\, \frac{\sqrt{127}}{8} t$.



31. Based on *Newton's second law*, with the positive direction to the right,

$$\sum F = mu''$$

where

$$\sum F = -ku - \gamma u'.$$

Hence the equation of motion is $mu'' + \gamma u' + ku = 0$. The only difference in this problem is that the equilibrium position is located at the *unstretched* configuration of the spring.

$32(a)$. The *restoring* force exerted by the spring is $F_s = -(ku + \varepsilon u^3)$. The *opposing* viscous force is $F_d = -\gamma u'$. Based on *Newton's second law*, with the positive direction to the right,

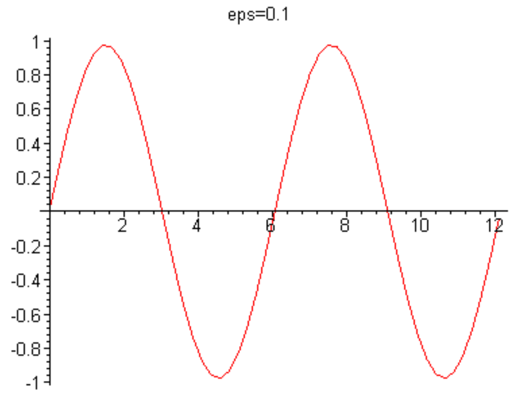
$$F_s + F_d = mu''.$$

Hence the equation of motion is $mu'' + \gamma u' + ku + \varepsilon u^3 = 0$.

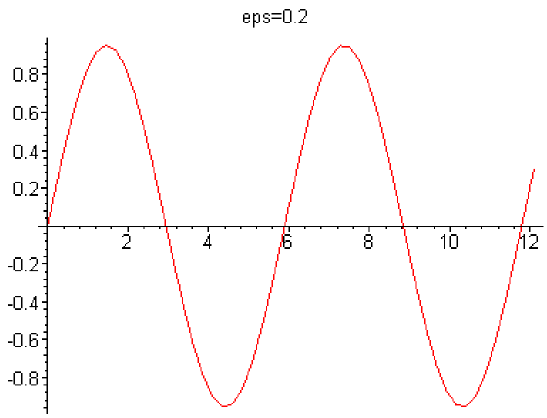
(b). With the specified parameter values, the equation of motion is $u'' + u = 0$. The general solution of this ODE is $u(t) = A\, cos\, t + B\, sin\, t$. Invoking the initial conditions,
the specific solution is $u(t) = sin\, t$. Clearly, the amplitude is $R = 1$, and the period of the motion is $T = 2\pi$.

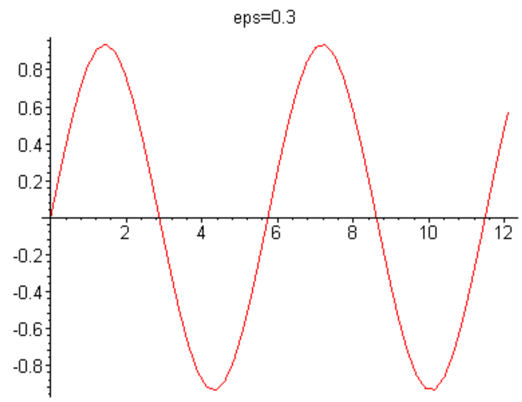
(c). Given $\varepsilon = 0.1$, the equation of motion is $u'' + u + 0.1\, u^3 = 0$. A solution of the

IVP can be generated numerically:



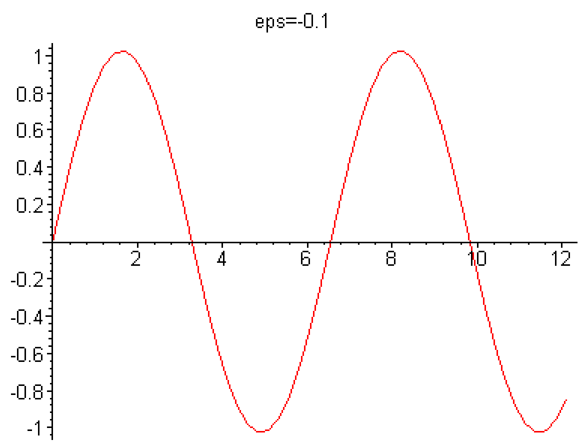
(d).

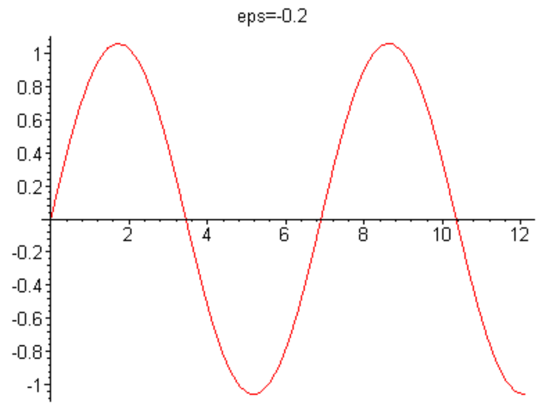


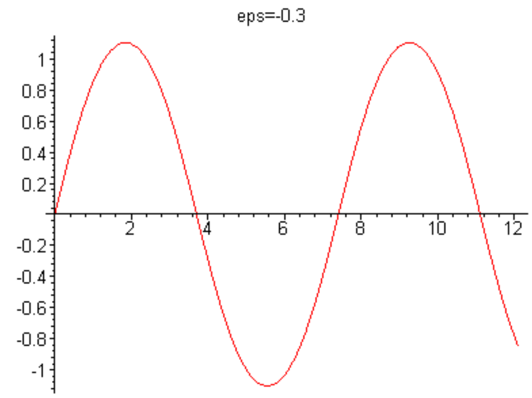


(e). The amplitude and period both seem to *decrease*.

(f).




eps=-0.2


eps=-0.3

Section 3.9

2. We have $sin(\alpha \pm \beta) = sin\,\alpha\, cos\,\beta \pm cos\,\alpha\, sin\,\beta$. Subtracting the two identities, we obtain $sin(\alpha + \beta) - sin(\alpha - \beta) = 2\,cos\,\alpha\, sin\,\beta$. Setting $\alpha + \beta = 7t$ and $\alpha - \beta = 6t$, $\alpha = 6.5t$ and $\beta = 0.5t$. Hence $sin\,7t - sin\,6t = 2\,sin\,\frac{t}{2}\,cos\,\frac{13t}{2}$.

3. Consider the trigonometric identity $cos(\alpha \pm \beta) = cos\,\alpha\, cos\,\beta \mp sin\,\alpha\, sin\,\beta$. Adding the two identities, we obtain $cos(\alpha - \beta) + cos(\alpha + \beta) = 2\,cos\,\alpha\, cos\,\beta$. Comparing the expressions, set $\alpha + \beta = 2\pi t$ and $\alpha - \beta = \pi t$. Hence $\alpha = 3\pi t/2$ and $\beta = \pi t/2$. Upon substitution, we have $cos(\pi t) + cos(2\pi t) = 2\,cos(3\pi t/2)\,cos(\pi t/2)$.

4. Adding the two identities $sin(\alpha \pm \beta) = sin\,\alpha\, cos\,\beta \pm cos\,\alpha\, sin\,\beta$, it follows that $sin(\alpha - \beta) + sin(\alpha + \beta) = 2sin\,\alpha\, cos\,\beta$. Setting $\alpha + \beta = 4t$ and $\alpha - \beta = 3t$, we have $\alpha = 7t/2$ and $\beta = t/2$. Hence $sin\,3t + sin\,4t = 2\,sin(7t/2)\,cos(t/2)$.

6. Using *mks* units, the spring constant is $k = 5(9.8)/0.1 = 490$ *N/m*, and the damping coefficient is $\gamma = 2/0.04 = 50$ *N-sec/m*. The equation of motion is

$$5u'' + 50u' + 490u = 10\,sin(t/2)\,.$$

The initial conditions are $u(0) = 0$ *m* and $u'(0) = 0.03$ *m/s*.

8(a). The homogeneous solution is $u_c(t) = Ae^{-5t}cos\sqrt{73}\,t + Be^{-5t}sin\sqrt{73}\,t$. Based on the method of *undetermined coefficients*, the particular solution is

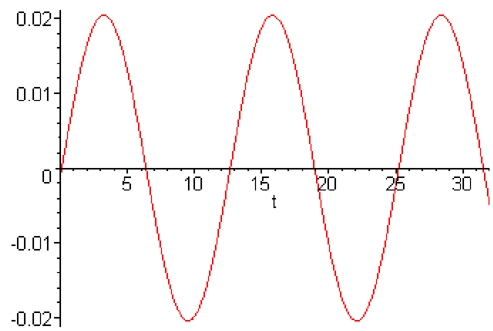
$$U(t) = \frac{1}{153281}[-160\,cos(t/2) + 3128\,sin(t/2)].$$

Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 160/153281$ and $B = 383443\sqrt{73}/1118951300$. Hence the response is

$$u(t) = \frac{1}{153281}\left[160\,e^{-5t}cos\,\sqrt{73}\,t + \frac{383443\sqrt{73}}{7300}e^{-5t}sin\,\sqrt{73}\,t\right] + U(t).$$

(b). $u_c(t)$ is the transient part and $U(t)$ is the steady state part of the response.

(c).



(d). Based on Eqs. (9) and (10), the amplitude of the forced response is given by $R = 2/\Delta$, in which

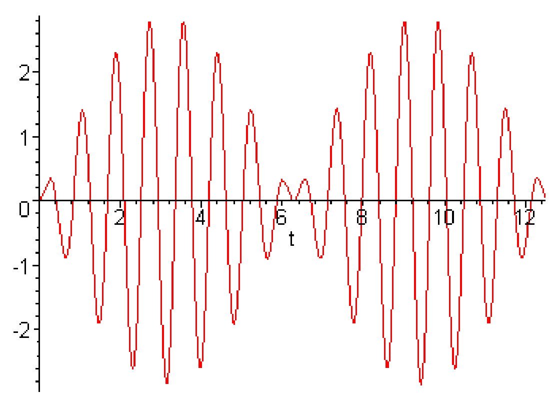
$$\Delta = \sqrt{25(98 - \omega^2)^2 + 2500\,\omega^2}\,.$$

The maximum amplitude is attained when Δ is a *minimum*. Hence the amplitude is maximum at $\omega = 4\sqrt{3}$ *rad/s* .

9. The spring constant is $k = 12$ *lb/ft* and hence the equation of motion is

$$\frac{6}{32}u'' + 12u = 4\,cos\,7t\,,$$

that is, $u'' + 64u = \frac{64}{3}cos\,7t$. The initial conditions are $u(0) = 0$ *ft*, $u'(0) = 0$ *fps*. The general solution is $u(t) = Acos\,8t + Bsin\,8t + \frac{64}{45}cos\,7t$. Invoking the initial conditions, we have $u(t) = -\frac{64}{45}cos\,8t + \frac{64}{45}cos\,7t = \frac{128}{45}sin(t/2)sin(15t/2)$.



12. The equation of motion is

$$2u'' + u' + 3u = 3cos\,3t - 2sin\,3t\,.$$

Since the system is *damped*, the steady state response is equal to the particular solution. Using the method of *undetermined coefficients*, we obtain

$$u_{ss}(t) = \frac{1}{6}(sin\,3t - cos\,3t).$$

Further, we find that $R = \sqrt{2}/6$ and $\delta = arctan(-1) = 3\pi/4$. Hence we can write $u_{ss}(t) = \frac{\sqrt{2}}{6}cos(3t - 3\pi/4)$.

13. The amplitude of the steady-state response is given by

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2}}\,.$$

Since F_0 is constant, the amplitude is *maximum* when the denominator of R is *minimum*. Let $z = \omega^2$, and consider the function $f(z) = m^2(\omega_0^2 - z)^2 + \gamma^2 z$. Note that $f(z)$ is a quadratic, with *minimum* at $z = \omega_0^2 - \gamma^2/2m^2$. Hence the amplitude R attains a maximum at $\omega_{max}^2 = \omega_0^2 - \gamma^2/2m^2$. Furthermore, since $\omega_0^2 = k/m$, and therefore

$$\omega_{max}^2 = \omega_0^2\left[1 - \frac{\gamma^2}{2km}\right].$$

Substituting $\omega^2 = \omega_{max}^2$ into the expression for the amplitude,

$$\begin{aligned} R &= \frac{F_0}{\sqrt{\gamma^4/4m^2 + \gamma^2\,(\omega_0^2 - \gamma^2/2m^2)}} \\ &= \frac{F_0}{\sqrt{\omega_0^2\gamma^2 - \gamma^4/4m^2}} \\ &= \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/4mk}}\,. \end{aligned}$$

14(a). The forced response is $u_{ss}(t) = Acos\,\omega t + Bsin\,\omega t$. The constants are obtain by the method of *undetermined coefficients*. That is, comparing the coefficients of $cos\,\omega t$ and $sin\,\omega t$, we find that

$$-m\omega^2 A + \gamma\omega B + kA = F_0\,,\text{ and } -m\omega^2 B - \gamma\omega A + kB = 0\,.$$

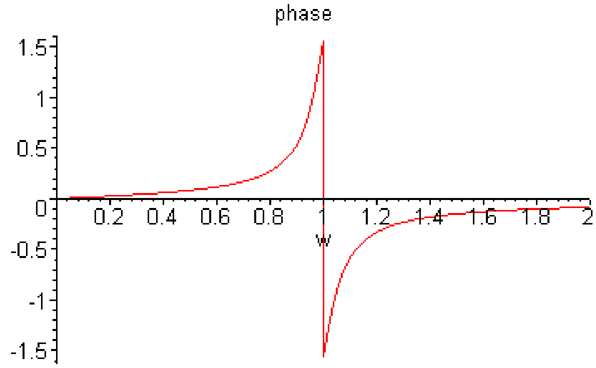
Solving this system results in

$$A = m\big(\omega_0^2 - \omega^2\big)/\Delta \quad\text{and}\quad B = \gamma\omega/\Delta\,,$$

in which $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2}$. It follows that

$$tan\,\delta = B/A = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}\,.$$

(b). Here $m = 1$, $\gamma = 0.125$, $\omega_0 = 1$. Hence $tan\,\delta = 0.125\omega/(1 - \omega^2)$.

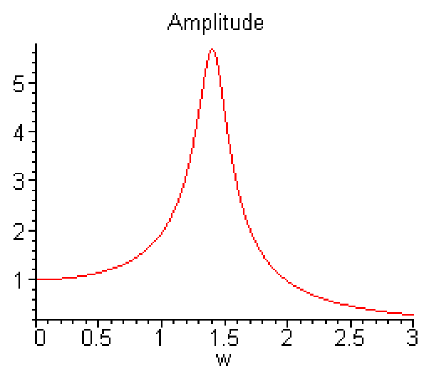


$17(a)$. Here $m = 1\,, \gamma = 0.25\,, \omega_0^2 = 2\,, F_0 = 2$. Hence $u_{ss}(t) = \frac{2}{\Delta} cos(\omega t - \delta)\,$, where $\Delta = \sqrt{(2-\omega^2)^2 + \omega^2/16} \;= \frac{1}{4}\sqrt{64 - 63\omega^2 + 16\,\omega^4}$, and $tan\,\delta = \frac{\omega}{4(2-\omega^2)}$.

(b). The amplitude is

$$R = \frac{8}{\sqrt{64 - 63\omega^2 + 16\,\omega^4}}\,.$$

(c).



(d). *See Prob. 13*. The amplitude is maximum when the denominator of R is minimum. That is, when $\omega = \omega_{max} = 3\sqrt{14}/8 \approx 1.4031$. Hence $R(\omega = \omega_{max}) = 64/\sqrt{127}$.

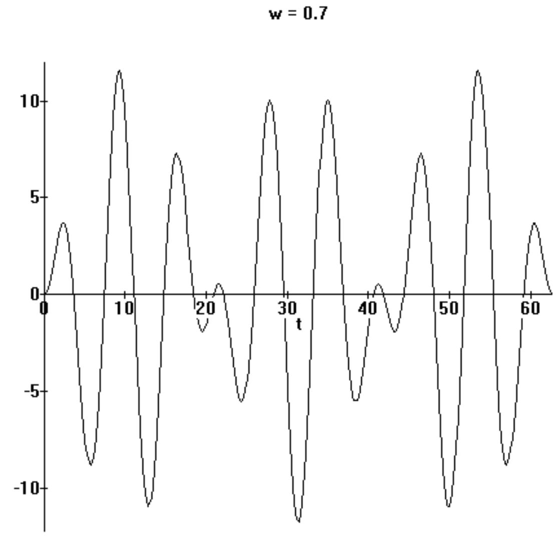
$18(a)$. The homogeneous solution is $u_c(t) = Acos\,t + Bsin\,t$. Based on the method of *undetermined coefficients*, the particular solution is

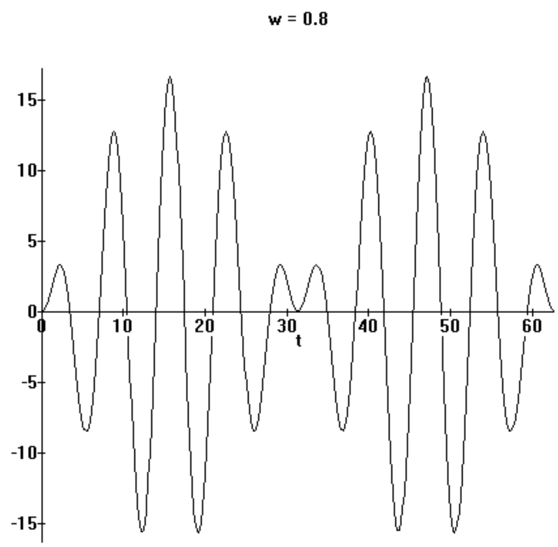
$$U(t) = \frac{3}{1-\omega^2}\,cos\,\omega t\,.$$

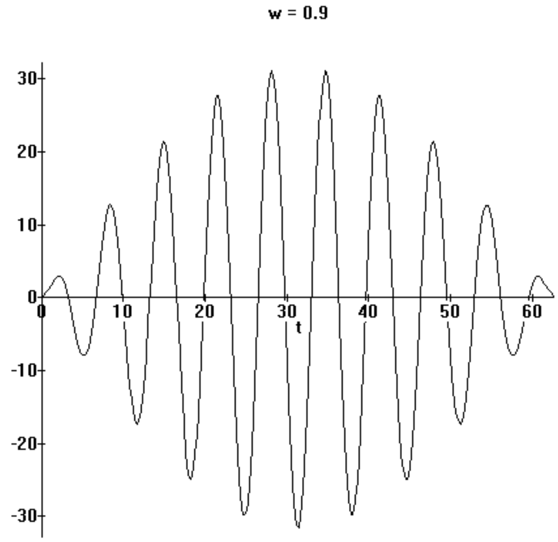
Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 3/(\omega^2 - 1)$ and $B = 0$. Hence the response is

$$u(t) = \frac{3}{1-\omega^2}[\,cos\,\omega t - cos\,t\,].$$

(b).







Note that

$$u(t) = \frac{6}{1-\omega^2}\, sin\left[\frac{(1-\omega)t}{2}\right] sin\left[\frac{(\omega+1)t}{2}\right].$$

$19(a)$. The homogeneous solution is $u_c(t) = A cos\, t + B sin\, t$. Based on the method of *undetermined coefficients*, the particular solution is

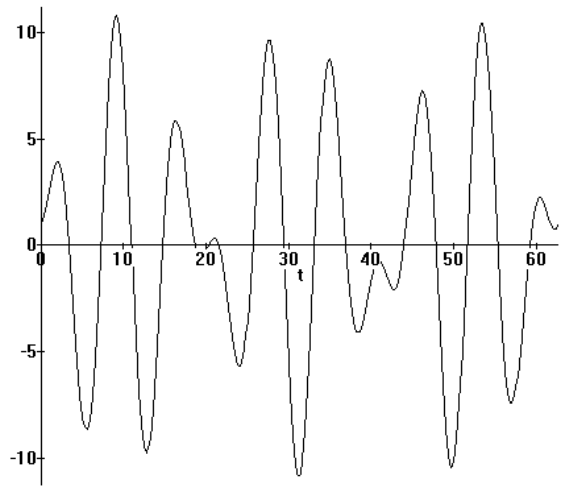
$$U(t) = \frac{3}{1-\omega^2}\, cos\, \omega t\, .$$

Hence the general solution is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = (\omega^2 + 2)/(\omega^2 - 1)$ and $B = 1$. Hence the response is

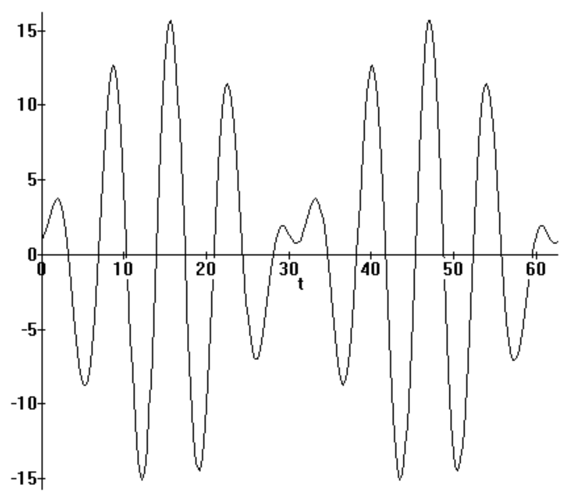
$$u(t) = \frac{1}{1-\omega^2}\left[\, 3\, cos\, \omega t - \left(\omega^2 + 2\right) cos\, t\,\right] + sin\, t\, .$$

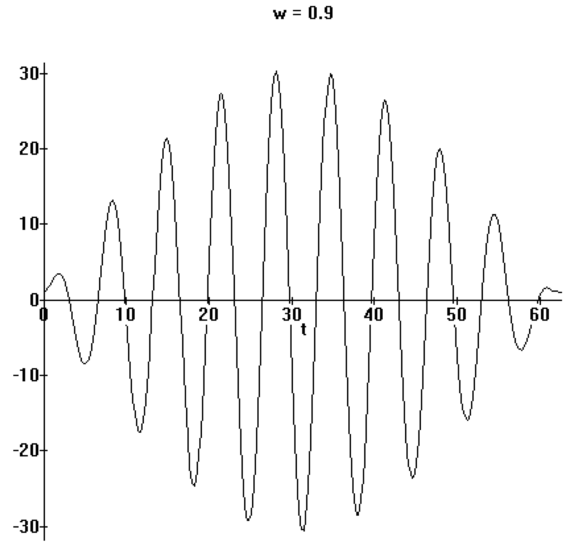
$(b.)$

w = 0.7



w = 0.8

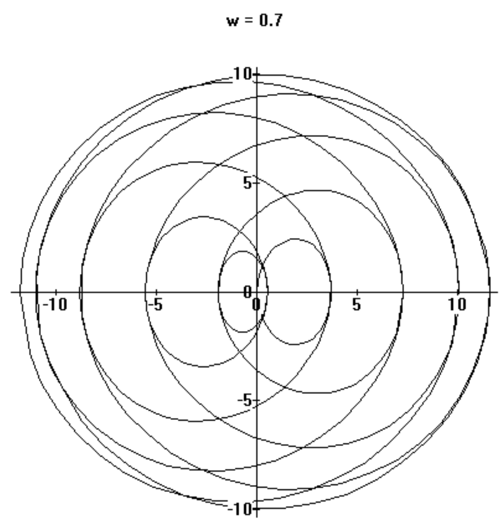


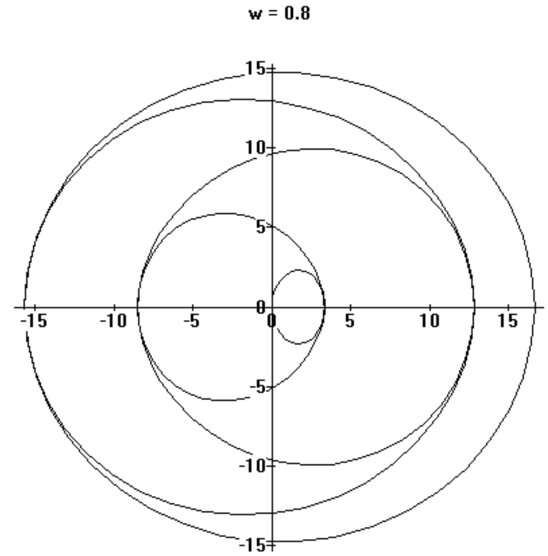


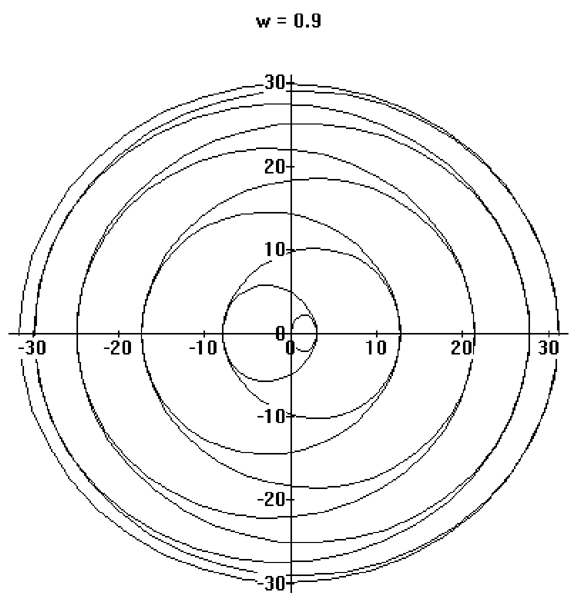
Note that

$$u(t) = \frac{6}{1-\omega^2}\, sin\left[\frac{(1-\omega)t}{2}\right] sin\left[\frac{(\omega+1)t}{2}\right] + cos\, t + sin\, t\,.$$

20.







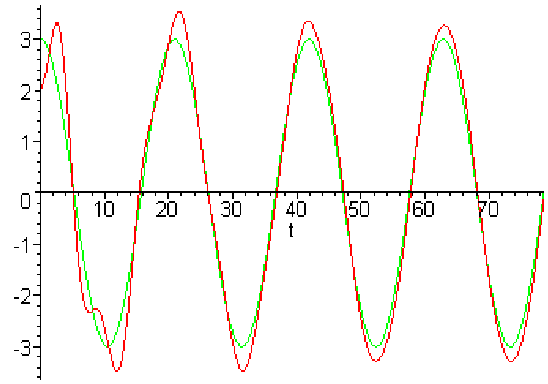
21. The general solution is $u(t) = u_c(t) + U(t)$, in which

$$u_c(t) = e^{-t/16}\left[-\frac{171358}{132721}\,cos\,\frac{\sqrt{255}}{16}t - \frac{257758}{132721\sqrt{255}}\,sin\,\frac{\sqrt{255}}{16}t \right]$$

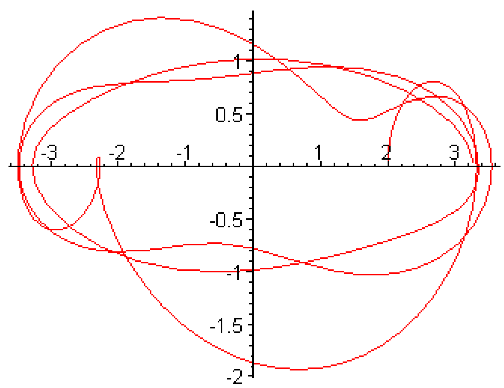
and

$$U(t) = \frac{1}{132721}\left[436800\,cos(.3t) + 18000\,sin(.3t)\right].$$

(a).



(b).



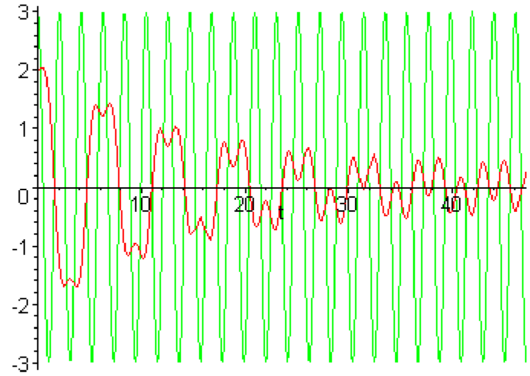
23. The general solution is $u(t) = u_c(t) + U(t)$, in which

$$u_c(t) = e^{-t/16}\left[\frac{9746}{4105}\,cos\,\frac{\sqrt{255}}{16}t + \frac{1258}{821\sqrt{255}}\,sin\,\frac{\sqrt{255}}{16}t\right]$$

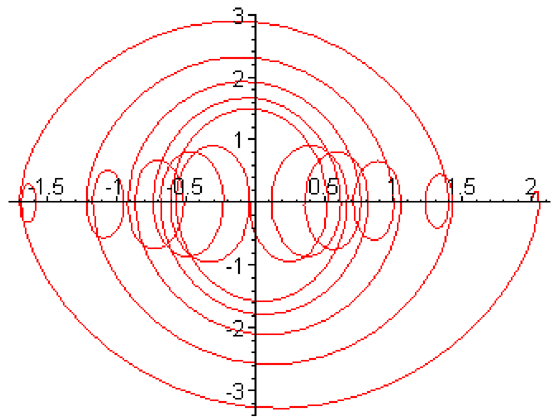
and

$$U(t) = \frac{1}{4105}[\,-1536\,cos(3t) + 72\,sin(3t)]\,.$$

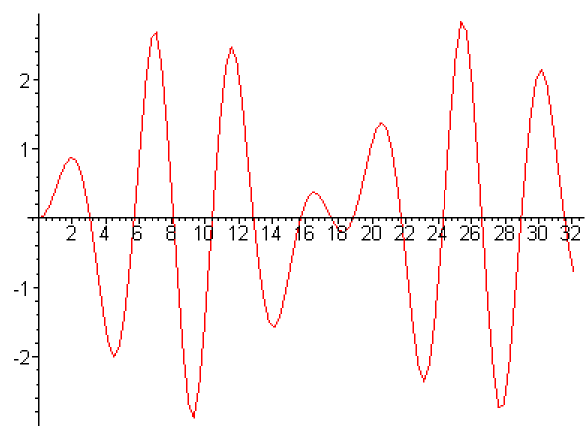
(a).


3
2
1
0
-1
-2
-3
10
20
t
30
40

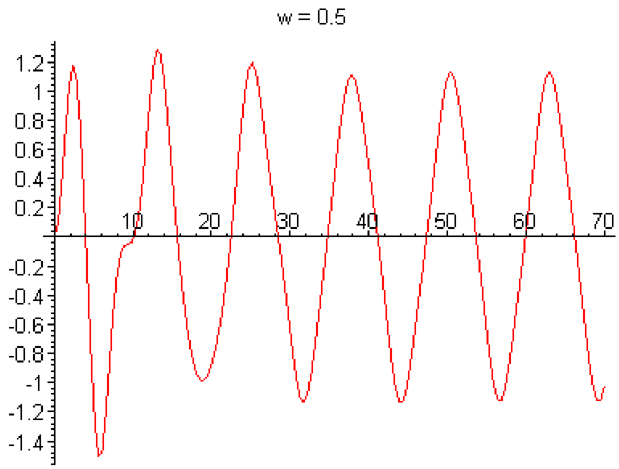
(b).

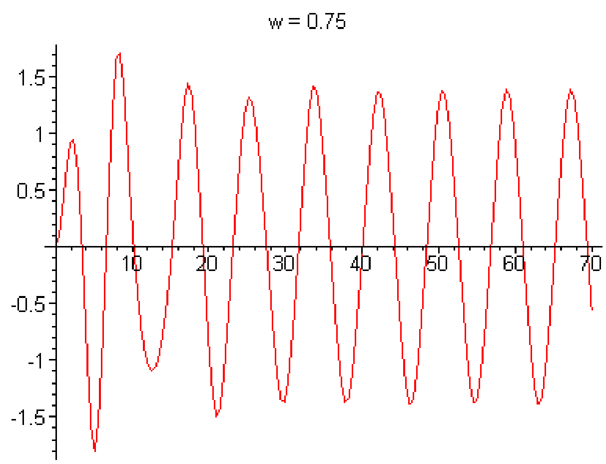

3
2
1
-1
-2
-3
-1.5
-1
-0.5
0.5
1
1.5
2

24.


2
1
-1
-2
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32

25(a).


w = 0.5
1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
10
20
30
40
50
60
70


w = 0.75
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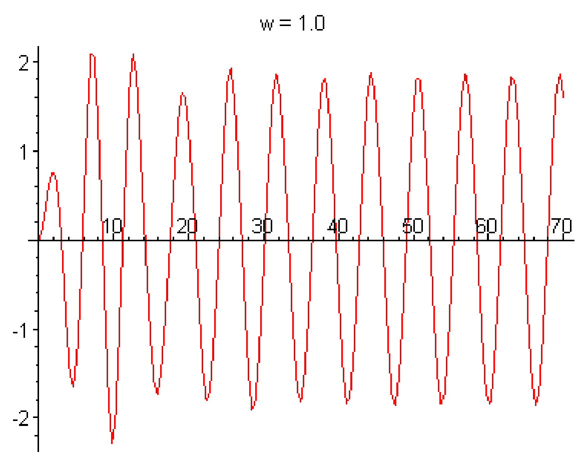
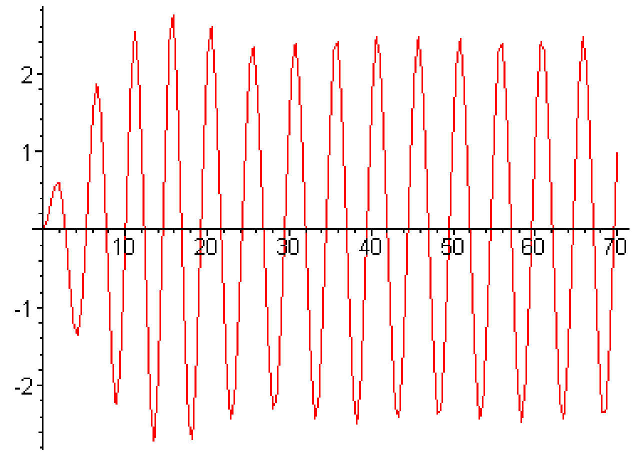
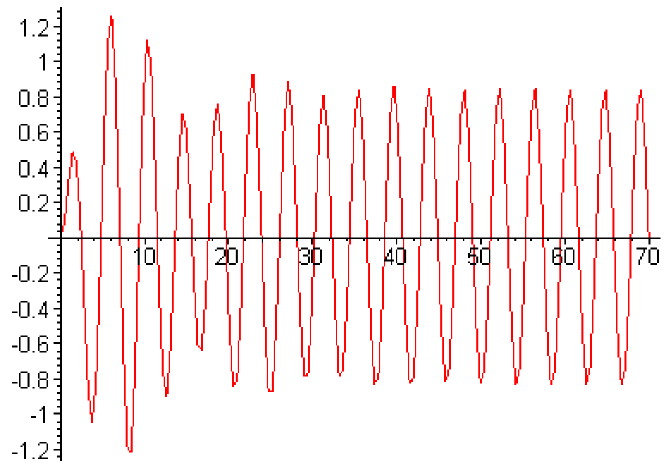
1
0.5
-0.5
-1
-1.5
10
20
30
40
50
60
70


w = 1.0
2
1
-1
-2
10
20
30
40
50
60
70

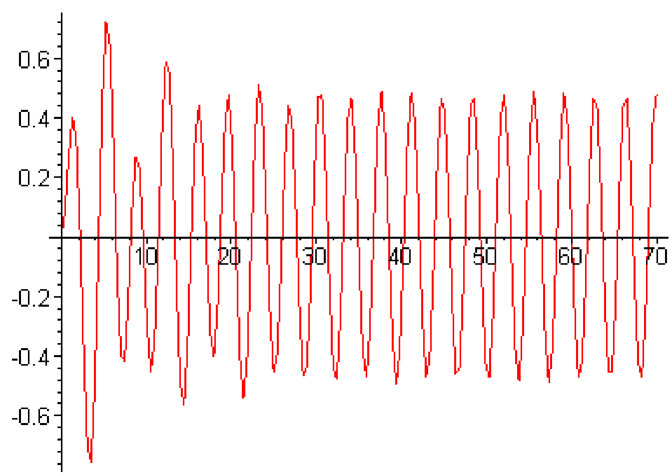
w = 1.25

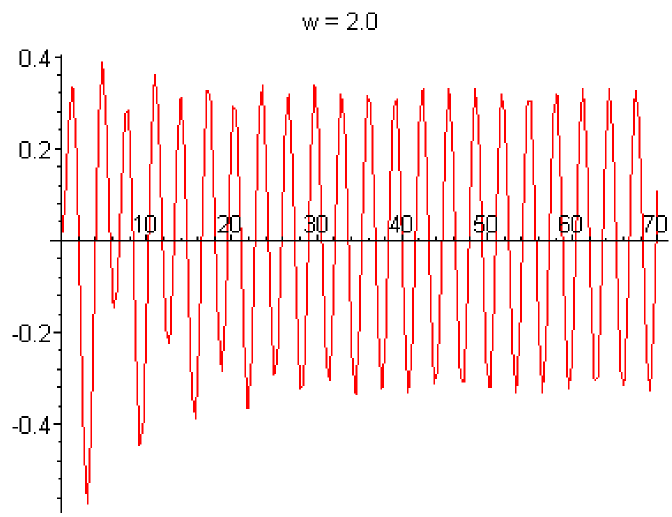


w = 1.5

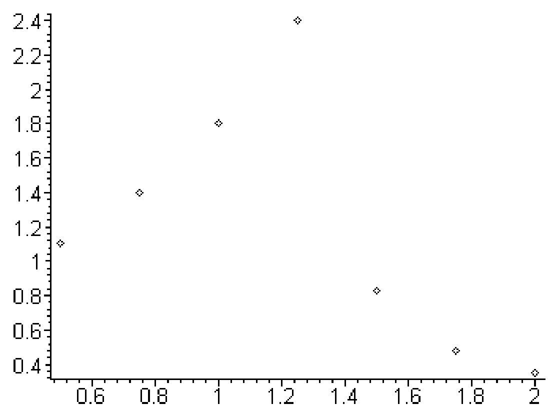


w = 1.75



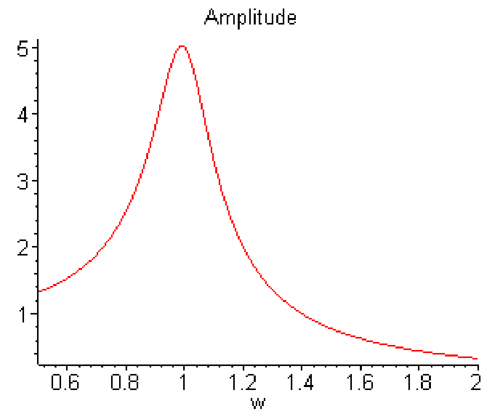


(b).



(c). The amplitude for a similar system with a *linear* spring is given by

$$R = \frac{5}{\sqrt{25 - 49\omega^2 + 25\omega^4}}\,.$$


Amplitude
5
4
3
2
1
0.6
0.8
1
1.2
1.4
1.6
1.8
2
w

Chapter Four

Section 4.1

1. The differential equation is in standard form. Its coefficients, as well as the function $g(t) = t$, are continuous *everywhere*. Hence solutions are valid on the entire real line.

3. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty,0)$, $(0\,,1)$, and $(1\,,\infty)$.

4. The coefficients are continuous everywhere, but the function $g(t) = ln\,t$ is defined and
continuous only on the interval $(0\,,\infty)$. Hence solutions are defined for positive reals.

5. Writing the equation in standard form, the coefficients are *rational* functions with a singularity at $x_0 = 1$. Furthermore, $p_4(x) = tan\,x/(x-1)$ is *undefined*, and hence not continuous, at $x_k = \pm(2k+1)\pi/2$, $k = 0,1,2,\cdots$. Hence solutions are defined on any *interval* that *does not* contain x_0 or x_k.

6. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $x = \pm\,2$. Hence the solutions are valid on the intervals $(-\infty,-2)$, $(-2\,,2)$, and $(2\,,\infty)$.

7. Evaluating the Wronskian of the three functions, $W(f_1,f_2,f_3) = -14$. Hence the functions are linearly *independent*.

9. Evaluating the Wronskian of the four functions, $W(f_1,f_2,f_3,f_4) = 0$. Hence the functions are linearly *dependent*. To find a linear relation among the functions, we need to find constants c_1,c_2,c_3,c_4, not all zero, such that

$$c_1f_1(t) + c_2f_2(t) + c_3f_3(t) + c_4f_4(t) = 0\,.$$

Collecting the common terms, we obtain

$$(c_2+2c_3+c_4)t^2 + (2c_1-c_3+c_4)t + (-3c_1+c_2+c_4) = 0\,,$$

which results in *three* equations in *four* unknowns. Arbitrarily setting $c_4 = -1$, we can solve the equations $c_2+2c_3 = 1$, $2c_1-c_3 = 1$, $-3c_1+c_2 = 1$, to find that $c_1 = 2/7$, $c_2 = 13/7$, $c_3 = -3/7$. Hence

$$2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0\,.$$

10. Evaluating the Wronskian of the three functions, $W(f_1,f_2,f_3) = 156$. Hence the functions are linearly *independent*.

11. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have
$W(1, cos\, t, sin\, t) = 1$.

12. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, cos\, t, sin\, t) = 1$.

14. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, e^{-t}, t\, e^{-t}) = e^{-2t}$.

15. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, x, x^3) = 6x$.

16. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

18. The operation of taking a derivative is linear, and hence

$$(c_1 y_1 + c_2 y_2)^{(k)} = c_1 y_1^{(k)} + c_2 y_2^{(k)}.$$

It follows that

$$L[c_1 y_1 + c_2 y_2] = c_1 y_1^{(n)} + c_2 y_2^{(n)} + p_1 \left[c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)}\right] + \cdots + p_n [c_1 y_1 + c_2 y_2].$$

Rearranging the terms, we obtain $L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$. Since y_1 and y_2 are solutions, $L[c_1 y_1 + c_2 y_2] = 0$. The rest follows by induction.

19(a). Note that $d^k(t^n)/dt^k = n(n-1)\cdots(n-k+1)t^{n-k}$, for $k = 1, 2, \cdots, n$.
Hence

$$L[t^n] = a_0\, n! + a_1 [n(n-1)\cdots 2]t + \cdots a_{n-1}\, n\, t^{n-1} + a_n\, t^n.$$

(b). We have $d^k(e^{rt})/dt^k = r^k e^{rt}$, for $k = 0, 1, 2, \cdots$. Hence

$$\begin{aligned} L\left[e^{rt}\right] &= a_0\, r^n e^{rt} + a_1 r^{n-1} e^{rt} + \cdots a_{n-1} r\, e^{rt} + a_n\, e^{rt} \\ &= \left[a_0\, r^n + a_1 r^{n-1} + \cdots a_{n-1}\, r\; + a_n\right] e^{rt}. \end{aligned}$$

(c). Set $y = e^{rt}$, and substitute into the ODE. It follows that $r^4 - 5r^2 + 4 = 0$, with $r = \pm 1, \pm 2$. Furthermore, $W(e^t, e^{-t}, e^{2t},\, e^{-2t}) = 72$.

20(a). Let $f(t)$ and $g(t)$ be arbitrary functions. Then $W(f, g) = fg' - f'g$. Hence $W'(f, g) = f'g' + fg'' - f''g - f'g' = fg'' - f''g$. That is,

$$W'(f, g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}.$$

Now expand the 3-*by*-3 determinant as

$$W(y_1,y_2,y_3)=y_1\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

Differentiating, we obtain

$$\begin{aligned} W'(y_1,y_2,y_3) &= y_1'\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2'\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3'\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} + \\ &+ y_1\begin{vmatrix} y_2' & y_3' \\ y_2''' & y_3''' \end{vmatrix} - y_2\begin{vmatrix} y_1' & y_3' \\ y_1''' & y_3''' \end{vmatrix} + y_3\begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}. \end{aligned}$$

The *second* line follows from the observation above. Now we find that

$$W'(y_1,y_2,y_3)=\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hence the assertion is true, since the first determinant is equal to *zero*.

(b). Based on the properties of determinants,

$$p_2(t)p_3(t)W'=\begin{vmatrix} p_3\,y_1 & p_3\,y_2 & p_3\,y_3 \\ p_2\,y_1' & p_2\,y_2' & p_2\,y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Adding the *first two* rows to the *third* row does not change the value of the determinant. Since the functions are assumed to be solutions of the given ODE, addition of the rows results in

$$p_2(t)p_3(t)W'=\begin{vmatrix} p_3\,y_1 & p_3\,y_2 & p_3\,y_3 \\ p_2\,y_1' & p_2\,y_2' & p_2\,y_3' \\ -\,p_1\,y_1'' & -\,p_1\,y_2'' & -\,p_1\,y_3'' \end{vmatrix}.$$

It follows that $p_2(t)p_3(t)W' = -\,p_1(t)p_2(t)p_3(t)W$. As long as the coefficients are not zero, we obtain $W' = -\,p_1(t)W$.

(c). The first order equation $W' = -\,p_1(t)W$ is linear, with integrating factor $\mu(t) =$ $= exp\big(\int p_1(t)dt\big)$. Hence $W(t) = c\,exp\big(-\int p_1(t)dt\big)$. Furthermore, $W(t)$ is *zero* only if $c = 0$.

(d). It can be shown, by mathematical induction, that

$$W'(y_1,y_2,\cdots,y_n)=\begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & \cdots & y_{n-1}' & y_n' \\ \vdots & & & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_{n-1}^{(n)} & y_n^{(n)} \end{vmatrix}.$$

Based on the reasoning in Part(b), it follows that

$$p_2(t)p_3(t)\cdots p_n(t)W' = -p_1(t)p_2(t)p_3(t)\cdots p_n(t)W,$$

and hence $W' = -p_1(t)W$.

22. Inspection of the coefficients reveals that $p_1(t) = 0$. Based on Prob. 20, we find that $W' = 0$, and hence $W = c$.

23. After writing the equation in standard form, observe that $p_1(t) = 2/t$. Based on the results in Prob. 20, we find that $W' = (-2/t)W$, and hence $W = c/t^2$.

24. Writing the equation in standard form, we find that $p_1(t) = 1/t$. Using *Abel's formula*, the Wronskian has the form $W(t) = c\,exp\big(-\int \frac{1}{t}dt\big) = c/t$.

25(a). Assuming that $c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0$, then taking the first $n-1$ derivatives of this equation results in

$$c_1y_1^{(k)}(t) + c_2y_2^{(k)}(t) + \cdots + c_ny_n^{(k)}(t) = 0$$

for $k = 0, 1, \cdots, n-1$. Setting $t = t_0$, we obtain a system of n algebraic equations with unknowns $c_1, c_2, \cdots, c_n$. The Wronskian, $W(y_1, y_2, \cdots, y_n)(t_0)$, is the determinant of the coefficient matrix. Since system of equations is homogeneous, $W(y_1, y_2, \cdots, y_n)(t_0) \neq 0$ implies that the only solution is the *trivial* solution, $c_1 = c_2 = \cdots = c_n = 0$.

(b). Suppose that $W(y_1, y_2, \cdots, y_n)(t_0) = 0$ for some t_0. Consider the system of algebraic
equations

$$c_1y_1^{(k)}(t_0) + c_2y_2^{(k)}(t_0) + \cdots + c_ny_n^{(k)}(t_0) = 0\,,$$

$k = 0, 1, \cdots, n-1$, with unknowns $c_1, c_2, \cdots, c_n$. Vanishing of the Wronskian, which is the determinant of the coefficient matrix, implies that there is a *nontrivial* solution of the system of homogeneous equations. That is, there exist constants $c_1, c_2, \cdots, c_n$, not all zero, which satisfy the above equations. Now let

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t).$$

Since the ODE is linear, $y(t)$ is also a *nonzero* solution. Based on the system of algebraic equations above, $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. This contradicts the uniqueness of the *identically zero* solution.

26. Let $y(t) = y_1(t)v(t)$. Then $y' = y_1'v + y_1v'$, $y'' = y_1''v + 2y_1'v' + y_1v''$, and $y''' = y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$. Substitution into the ODE results in

$$y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v''' + p_1[y_1''v + 2y_1'v' + y_1v''] + p_2[y_1'v + y_1v'] + p_3y_1v = 0.$$

Since y_1 is assumed to be a solution, all terms containing the factor $v(t)$ vanish. Hence

$$y_1v''' + [p_1y_1 + 3y_1']v'' + [3y_1'' + 2p_1y_1' + p_2y_1]v' = 0\,,$$

which is a *second order* ODE in the variable $u = v'$.

28. First write the equation in standard form:

$$y''' - 3\frac{t+2}{t(t+3)}y'' + 6\frac{t+1}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = 0\,.$$

Let $y(t) = t^2v(t)$. Substitution into the given ODE results in

$$t^2v''' + 3\frac{t(t+4)}{t+3}v'' = 0\,.$$

Set $w = v''$. Then w is a solution of the first order differential equation

$$w' + 3\frac{t+4}{t(t+3)}w = 0\,.$$

This equation is *linear*, with integrating factor $\mu(t) = t^4/(t+3)$. The general solution is $w = c(t+3)/t^4$. Integrating twice, it follows that $v(t) = c_1t^{-1} + c_1t^{-2} + c_2t + c_3$. Hence $y(t) = c_1t + c_1 + c_2t^3 + c_3t^2$. Finally, since $y_1(t) = t^2$ and $y_2(t) = t^3$are given solutions, the *third* independent solution is $y_3(t) = c_1t + c_1$.

Section 4.2

1. The *magnitude* of $1+i$ is $R=\sqrt{2}$ and the *polar angle* is $\pi/4$. Hence the polar form is given by $1+i=\sqrt{2}\,e^{i\pi/4}$.

3. The *magnitude* of -3 is $R=3$ and the *polar angle* is π. Hence $-3=3\,e^{i\pi}$.

4. The *magnitude* of $-i$ is $R=1$ and the *polar angle* is $3\pi/2$. Hence $-i=e^{3\pi i/2}$.

5. The *magnitude* of $\sqrt{3}-i$ is $R=2$ and the *polar angle* is $-\pi/6=11\pi/6$. Hence the polar form is given by $\sqrt{3}-i=2\,e^{11\pi i/6}$.

6. The *magnitude* of $-1-i$ is $R=\sqrt{2}$ and the *polar angle* is $5\pi/4$. Hence the polar form is given by $-1-i=\sqrt{2}\,e^{5\pi i/4}$.

7. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/3}=e^{2m\pi i/3}$. Setting $m=0,1,2$ successively, we obtain the three roots as $1^{1/3}=1$, $1^{1/3}=e^{2\pi i/3}$, $1^{1/3}=e^{4\pi i/3}$. Equivalently, the roots can also be written as 1, $cos(2\pi/3)+i\,sin(2\pi/3)=\frac{1}{2}\left(-1+\sqrt{3}\right)$, $cos(4\pi/3)+isin(4\pi/3)=\frac{1}{2}\left(-1+\sqrt{3}\right)$.

9. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/4}=e^{2m\pi i/4}$. Setting $m=0,1,2,3$ successively, we obtain the three roots as $1^{1/4}=1$, $1^{1/4}=e^{\pi i/2}$, $1^{1/4}=e^{\pi i}$, $1^{1/4}=e^{3\pi i/2}$. Equivalently, the roots can also be written as 1, $cos(\pi/2)+i\,sin(\pi/2)=i$, $cos(\pi)+isin(\pi)=-1$, $cos(3\pi/2)+$ $+isin(3\pi/2)=-i$.

10. In polar form, $2(cos\,\pi/3+i\,sin\,\pi/3)=2\,e^{i\pi/3+2m\pi}$, in which m is any integer. Thus $[2(cos\,\pi/3+i\,sin\,\pi/3)]^{1/2}=2^{1/2}\,e^{i\pi/6+m\pi}$. With $m=0$, one square root is given by $2^{1/2}\,e^{i\pi/6}=\left(\sqrt{3}+i\right)/\sqrt{2}$. With $m=1$, the other root is given by $2^{1/2}\,e^{i7\pi/6}=\left(-\sqrt{3}-i\right)/\sqrt{2}$.

11. The characteristic equation is $r^3-r^2-r+1=0$. The roots are $r=-1,1,1$. One root is *repeated*, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3te^t$.

13. The characteristic equation is $r^3-2r^2-r+2=0$, with roots $r=-1,1,2$. The roots are real and *distinct*, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3e^{2t}$.

14. The characteristic equation can be written as $r^2(r^2-4r+4)=0$. The roots are $r=0,0,2,2$. There are two repeated roots, and hence the general solution is given by $y=c_1+c_2t+c_3e^{2t}+c_4te^{2t}$.

15. The characteristic equation is $r^6+1=0$. The roots are given by $r=(-1)^{1/6}$, that is, the six *sixth roots* of -1. They are $e^{-\pi i/6+m\pi i/3}$, $m=0,1,\cdots,5$. Explicitly,

$r = \left(\sqrt{3} - i\right)/2\,,\, \left(\sqrt{3} + i\right)/2\,,\, i\,,\, -i\,,\, \left(-\sqrt{3} + i\right)/2\,,\, \left(-\sqrt{3} - i\right)/2$. Hence the general solution is given by $y = e^{\sqrt{3}\,t/2}[c_1 cos\,(t/2) + c_2 sin\,(t/2)] + c_3 cos\,t + c_4 sin\,t + e^{-\sqrt{3}\,t/2}[c_5 cos\,(t/2) + c_6 sin\,(t/2)]$.

16. The characteristic equation can be written as $(r^2 - 1)(r^2 - 4) = 0$. The roots are given by $r = \pm 1, \pm 2$. The roots are real and *distinct*, hence the general solution is $y = c_1 e^{-t} + c_2 e^{t} + c_3 e^{-2t} + c_4 e^{2t}$.

17. The characteristic equation can be written as $(r^2 - 1)^3 = 0$. The roots are given by $r = \pm 1$, each with *multiplicity three*. Hence the general solution is

$$y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} + c_4 e^{t} + c_5 t e^{t} + c_6 t^2 e^{t}.$$

18. The characteristic equation can be written as $r^2\left(r^4 - 1\right) = 0$. The roots are given by $r = 0, 0, \pm 1, \pm i$. The general solution is $y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^{t} + c_5 cos\,t +$ $+ c_6 sin\,t$.

19. The characteristic equation can be written as $r\left(r^4 - 3r^3 + 3r^2 - 3r + 2\right) = 0$. Examining the coefficients, it follows that $r^4 - 3r^3 + 3r^2 - 3r + 2 = (r-1)(r-2) \times (r^2+1)$. Hence the roots are $r = 0, 1, 2, \pm i$. The general solution of the ODE is given by $y = c_1 + c_2 e^{t} + c_3 e^{2t} + c_4 cos\,t + c_5 sin\,t$.

20. The characteristic equation can be written as $r(r^3 - 8) = 0$, with roots $r = 0$, $2\,e^{2m\pi i/3}, m = 0, 1, 2$. That is, $r = 0, 2, -1 \pm i\sqrt{3}$. Hence the general solution is $y = c_1 + c_2 e^{2t} + e^{-t}\left[c_3 cos\sqrt{3}\,t + c_4 sin\sqrt{3}\,t\right]$.

21. The characteristic equation can be written as $\left(r^4 + 4\right)^2 = 0$. The roots of the equation $r^4 + 4 = 0$ are $r = 1 \pm i\,,\, -1 \pm i$. Each of these roots has $multiplicity\,two$. The general solution is $y = e^{t}[c_1 cos\,t + c_2 sin\,t\,] + te^{t}[c_3 cos\,t + c_4 sin\,t\,] +$ $+ e^{-t}[c_5 cos\,t + c_6 sin\,t\,] + te^{-t}[c_7 cos\,t + c_8 sin\,t\,]$.

22. The characteristic equation can be written as $(r^2 + 1)^2 = 0$. The roots are given by $r = \pm i$, each with $multiplicity\,two$. The general solution is $y = c_1 cos\,t + c_2 sin$ $t +$
$+ t[c_3 cos\,t + c_4 sin\,t\,]$.

24. The characteristic equation is $r^3 + 5r^2 + 6r + 2 = 0$. Examining the coefficients, we find that $r^3 + 5r^2 + 6r + 2 = (r+1)(r^2 + 4r + 2)$. Hence the roots are deduced as $r = -1\,,\, -2 \pm \sqrt{2}$. The general solution is $y = c_1 e^{-t} + c_2 e^{\left(-2+\sqrt{2}\right)t} + c_3 e^{\left(-2-\sqrt{2}\right)t}$.

25. The characteristic equation is $18r^3 + 21r^2 + 14r + 4 = 0$. By examining the first and last coefficients, we find that $18r^3 + 21r^2 + 14r + 4 = (2r+1)(9r^2 + 6r + 4)$.

Hence the roots are $r = -1/2\,,\left(-1\pm\sqrt{3}\right)/3$. The general solution of the ODE is given by $y = c_1e^{-t/2} + e^{-t/3}\left[c_2cos\left(t/\sqrt{3}\right) + c_3sin\left(t/\sqrt{3}\right)\right]$.

26. The characteristic equation is $r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$. By examining the first and last coefficients, we find that

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = (r-3)(r+2)\left(r^2 - 6r + 6\right).$$

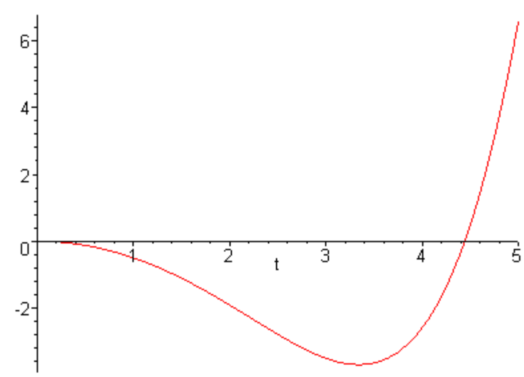
The roots are $r = -2, 3, 3\pm\sqrt{3}$. The general solution is

$$y = c_1e^{-2t} + c_2e^{3t} + c_3e^{\left(3-\sqrt{3}\right)t} + c_4e^{\left(3+\sqrt{3}\right)t}.$$

28. The characteristic equation is $r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$. It can be shown that $r^4 + 6r^3 + 17r^2 + 22r + 14 = (r^2 + 2r + 2)(r^2 + 4r + 7)$. Hence the roots are $r = -1\pm i\,,\ -2\pm i\sqrt{3}$. The general solution is

$$y = e^{-t}[c_1cos\,t + c_2sin\,t\,] + e^{-2t}\left[c_3cos\sqrt{3}\,t + c_4sin\sqrt{3}\,t\,\right].$$

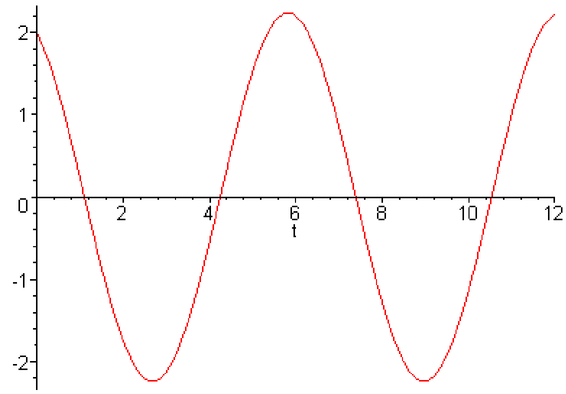
30. $y(t) = \frac{1}{2}e^{-t/\sqrt{2}}sin\left(t/\sqrt{2}\right) - \frac{1}{2}e^{t/\sqrt{2}}sin\left(t/\sqrt{2}\right)$.



32. The characteristic equation is $r^3 - r^2 + r - 1 = 0$, with roots $r = 1\,,\pm i$. Hence the general solution is $y(t) = c_1e^t + c_2cos\,t + c_3sin\,t$. Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= -1 \\ c_1 - c_2 &= -2 \end{aligned}$$

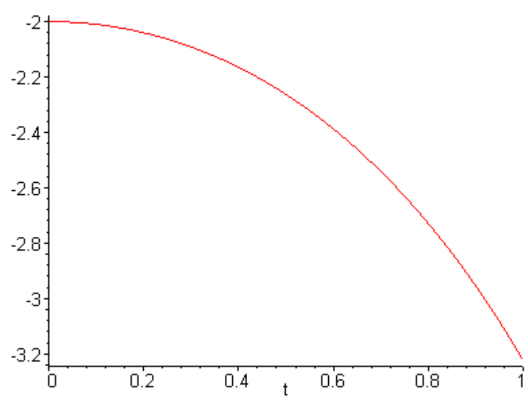
with solution $c_1 = 0\,, c_2 = 2\,, c_3 = -1$. Therefore the solution of the initial value problem is $y(t) = 2cos\,t - sin\,t$.



33. The characteristic equation is $2r^4 - r^3 - 9r^2 + 4r + 4 = 0$, with roots $r = -1/2$, 1, ± 2. Hence the general solution is $y(t) = c_1 e^{-t/2} + c_2 e^t + c_3 e^{-2t} + c_4 e^{2t}$. Applying the initial conditions, we obtain the system of equations

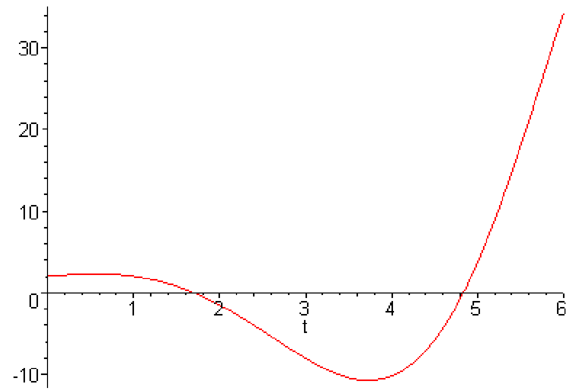
$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= -2 \\ -\frac{1}{2}c_1 + c_2 - 2c_3 + 2c_4 &= 0 \\ \frac{1}{4}c_1 + c_2 + 4c_3 + 4c_4 &= -2 \\ -\frac{1}{8}c_1 + c_2 - 8c_3 + 8c_4 &= 0 \end{aligned}$$

with solution $c_1 = -16/15$, $c_2 = -2/3$, $c_3 = -1/6$, $c_4 = -1/10$. Therefore the solution of the initial value problem is $y(t) = -\frac{16}{15}e^{-t/2} - \frac{2}{3}e^t - \frac{1}{6}e^{-2t} - \frac{1}{10}e^{2t}$.



The solution decreases without bound.

34. $y(t) = \frac{2}{13}e^{-t} + e^{t/2}\left[\frac{24}{13}cos\,t + \frac{3}{13}sin\,t\right]$.

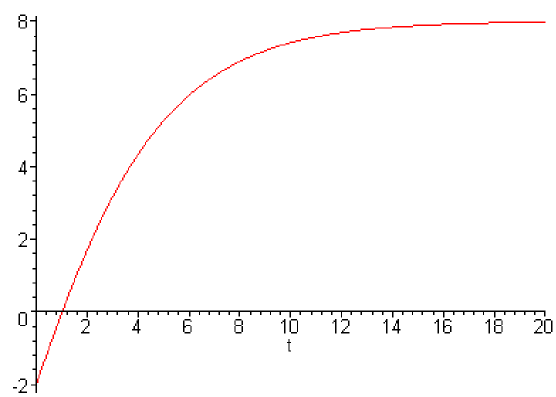


The solution is an oscillation with *increasing* amplitude.

35. The characteristic equation is $6\,r^3 + 5r^2 + r = 0$, with roots $r = 0\,, -1/3\,, -1/2$. The general solution is $y(t) = c_1 + c_2 e^{-t/3} + c_3 e^{-t/2}$. Invoking the initial conditions, we require that

$$c_1 + c_2 + c_3 = -2$$
$$-\frac{1}{3}c_2 - \frac{1}{2}c_3 = 2$$
$$\frac{1}{9}c_2 + \frac{1}{4}c_3 = 0$$

with solution $c_1 = 8\,, c_2 = -18\,, c_3 = 8$. Therefore the solution of the initial value problem is $y(t) = 8 - 18e^{-t/3} + 8e^{-t/2}$.



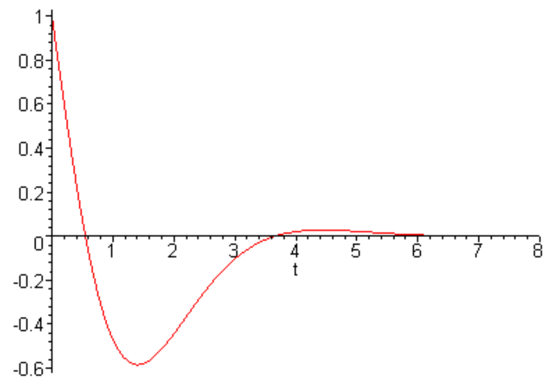
36. The general solution is derived in Prob.(28) as

$$y(t) = e^{-t}[c_1 cos\,t + c_2 sin\,t] + e^{-2t}\left[c_3 cos\sqrt{3}\,t + c_4 sin\sqrt{3}\,t\right].$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_3 = 1$$
$$-c_1 + c_2 - 2c_3 + \sqrt{3}\, c_4 = -2$$
$$-2c_2 + c_3 - 4\sqrt{3}\, c_4 = 0$$
$$2c_1 + 2c_2 + 10c_3 + 9\sqrt{3}\, c_4 = 3$$

with solution $c_1 = 21/13$, $c_2 = -38/13$, $c_3 = -8/13$, $c_4 = 17\sqrt{3}/39$.



The solution is a rapidly-decaying oscillation.

38.

$$W\left(e^t, e^{-t}, cos\, t, sin\, t\right) = -8$$
$$W(cosh\, t, sinh\, t, cos\, t, sin\, t) = 4$$

40. Suppose that $c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t} = 0$, and each of the r_k are real and different. Multiplying this equation by $e^{-r_1 t}$, $c_1 + c_2 e^{(r_2 - r_1)t} + \cdots + c_n e^{(r_n - r_1)t} = 0$. Differentiation results in

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0\,.$$

Now multiplying the latter equation by $e^{-(r_2 - r_1)t}$, and differentiating, we obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3 - r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n - r_2)t} = 0\,.$$

Following the above steps in a similar manner, it follows that

$$c_n(r_n - r_{n-1})\cdots(r_n - r_1)e^{(r_n - r_{n-1})t} = 0\,.$$

Since these equations hold for all t, and all the r_k are different, we have $c_n = 0$. Hence

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_{n-1} e^{r_{n-1} t} = 0\,, \quad -\infty < t < \infty\,.$$

The same procedure can now be repeated, successively, to show that

$$c_1 = c_2 = \cdots = c_n = 0\,.$$

Section 4.3

2. The general solution of the homogeneous equation is $y_c = c_1e^t + c_2e^{-t} + c_3cos\,t +$ $+ c_4sin\,t$. Let $g_1(t) = 3t$ and $g_2(t) = cos\,t$. By inspection, we find that $Y_1(t) = -3t$. Since $g_2(t)$ is a solution of the homogeneous equation, set $Y_2(t) = t(Acos\,t + Bsin\,t)$. Substitution into the given ODE and comparing the coefficients of similar term results in $A = 0$ and $B = -1/4$. Hence the general solution of the nonhomogeneous problem is

$$y(t) = y_c(t) - 3t - \frac{t}{4}sin\,t\,.$$

3. The characteristic equation corresponding to the homogeneous problem can be written as $(r+1)(r^2+1) = 0$. The solution of the homogeneous equation is $y_c = c_1e^{-t} +$ $+ c_2cos\,t + c_3sin\,t$. Let $g_1(t) = e^{-t}$ and $g_2(t) = 4t$. Since $g_1(t)$ is a solution of the homogeneous equation, set $Y_1(t) = Ate^{-t}$. Substitution into the ODE results in $A = 1/2$. Now let $Y_2(t) = Bt + C$. We find that $B = -C = 4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) + te^{-t}/2 + 4(t-1)$.

4. The characteristic equation corresponding to the homogeneous problem can be written as $r(r+1)(r-1) = 0$. The solution of the homogeneous equation is $y_c = c_1 + c_2e^t +$ $+ c_3e^{-t}$. Since $g(t) = 2\,sin\,t$ is not a solution of the homogeneous problem, we can set $Y(t) = A\,cos\,t + B\,sin\,t$. Substitution into the ODE results in $A = 1$ and $B = 0$. Thus
the general solution is $y(t) = c_1 + c_2e^t + c_3e^{-t} + cos\,t$.

6. The characteristic equation corresponding to the homogeneous problem can be written as $(r^2+1)^2 = 0$. It follows that $y_c = c_1cos\,t + c_2sin\,t + t(c_3cos\,t + c_4sin\,t)$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A + Bcos\,2t + Csin\,2t$. Substitution into the ODE results in $A = 3$, $B = 1/9$, $C = 0$. Thus the general solution is $y(t) = y_c(t) + 3 + \frac{1}{9}cos\,2t$.

7. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1) = 0$. Thus the homogeneous solution is

$$y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t} + e^{t/2}\left[c_5cos\left(\sqrt{3}\,t/2\right) + c_5sin\left(\sqrt{3}\,t/2\right)\right].$$

Note the $g(t) = t$ is a solution of the homogenous problem. Consider a particular solution
of the form $Y(t) = t^3(At + B)$. Substitution into the ODE results in $A = 1/24$ and $B = 0$. Thus the general solution is $y(t) = y_c(t) + t^4/24$.

8. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r+1) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t}$. Since $g(t)$ is *not* a solution of the homogeneous problem, set $Y(t) = Acos\,2t + Bsin\,2t$. Substitution into the ODE results in $A = 1/40$ and $B = 1/20$. Thus the general solution

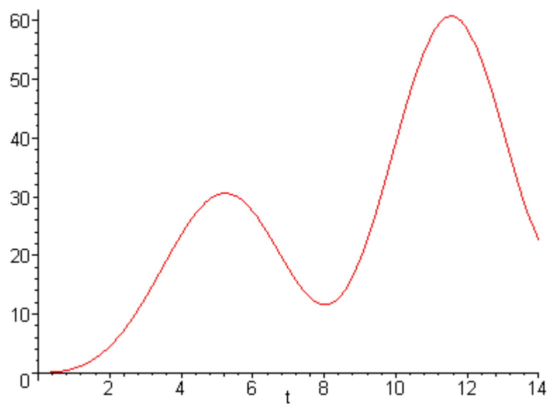
is $y(t) = y_c(t) + (cos\,2t + 2sin\,2t)/40$.

10. From Prob. 22 in Section 4.2 , the homogeneous solution is

$$y_c = c_1 cos\,t + c_2 sin\,t \; + \; + t[c_3 cos\,t + c_4 sin\,t].$$

Since $g(t)$ is *not* a solution of the homogeneous problem, substitute $Y(t) = At + B$ into the ODE to obtain $A = 3$ and $B = 4$. Thus the general solution is $y(t) = y_c(t) + 3t + 4$. Invoking the initial conditions, we find that $c_1 = -4$, $c_2 = -4$, $c_3 = 1$, $c_4 = -3/2$. Therefore the solution of the initial value problem is

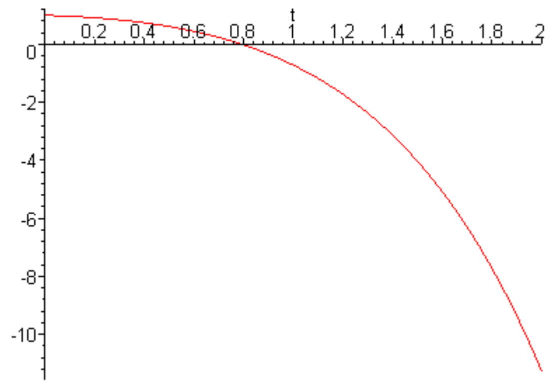
$$y(t) = (t-4)cos\,t - (3t/2+4)sin\,t + 3t + 4\,.$$



11. The characteristic equation can be written as $r(r^2 - 3r + 2) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{2t}$. Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1(t) = Ate^t$. Substitution into the ODE results in $A = -1$. Now let $Y_2(t) = Bt^2 + Ct$. Substitution into the ODE results in $B = 1/4$ and $C = 3/4$. Therefore the general solution is

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + \left(t^2 + 3t\right)/4\,.$$

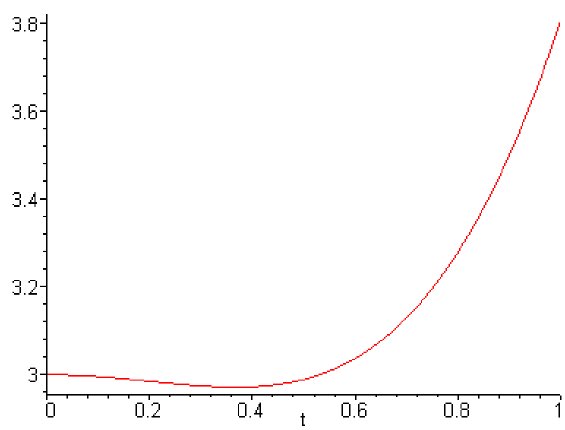
Invoking the initial conditions, we find that $c_1 = 1$, $c_2 = c_3 = 0$. The solution of the initial value problem is $y(t) = 1 - te^t + (t^2 + 3t)/4$.



12. The characteristic equation can be written as $(r-1)(r+3)(r^2+4)=0$. Hence the homogeneous solution is $y_c = c_1e^t + c_2e^{-3t} + c_3cos\,2t + c_4sin\,2t$. None of the terms in $g(t)$ is a solution of the homogeneous problem. Therefore we can assume a form $Y(t) = Ae^{-t} + Bcos\,t + Csin\,t$. Substitution into the ODE results in $A = 1/20$, $B = -2/5$, $C = -4/5$. Hence the general solution is

$$y(t) = c_1e^t + c_2e^{-3t} + c_3cos\,2t + c_4sin\,2t + e^{-t}/20 - (2cos\,t + 4sin\,t)/5\,.$$

Invoking the initial conditions, we find that $c_1 = 81/40$, $c_2 = 73/520$, $c_3 = 77/65$, $c_4 = -49/130$.



14. From Prob. 4, the homogeneous solution is $y_c = c_1 + c_2e^t + c_3e^{-t}$. Consider the terms $g_1(t) = te^{-t}$ and $g_2(t) = 2cos\,t$. Note that since $r = -1$ is a *simple* root of the characteristic equation, Table 4.3.1 suggests that we set $Y_1(t) = t(At+B)e^{-t}$. The function $2cos\,t$ is *not* a solution of the homogeneous equation. We can simply choose $Y_2(t) = Ccos\,t + Dsin\,t$. Hence the particular solution has the form

$$Y(t) = t(At+B)e^{-t} + Ccos\,t + Dsin\,t\,.$$

15. The characteristic equation can be written as $(r^2-1)^2 = 0$. The roots are given

as $r = \pm 1$, each with *multiplicity two*. Hence the solution of the homogeneous problem is $y_c = c_1e^t + c_2te^t + c_3e^{-t} + c_4te^{-t}$. Let $g_1(t) = e^t$ and $g_2(t) = sin\,t$. The function e^t is a solution of the homogeneous problem. Since $r = 1$ has multiplicity two, we set $Y_1(t) = At^2e^t$. The function $sin\,t$ is *not* a solution of the homogeneous equation. We can set $Y_2(t) = Bcos\,t + Csin\,t$. Hence the particular solution has the form

$$Y(t) = At^2e^t + Bcos\,t + Csin\,t\,.$$

16. The characteristic equation can be written as $r^2(r^2+4) = 0$, with roots $r = 0, \pm 2i$. The root $r = 0$ has multiplicity two, hence the homogeneous solution is $y_c = c_1 + c_2t +$ $+ c_3cos\,2t + c_4sin\,2t$. The functions $g_1(t) = sin\,2t$ and $g_2(t) = 4$ are solutions of the homogenous equation. The complex roots have multiplicity *one*, therefore we need to set $Y_1(t) = At\,cos\,2t + Bt\,sin\,2t$. Now $g_2(t) = 4$ is associated with the *double* root $r = 0$. Based on Table 4.3.1, set $Y_2(t) = Ct^2$. Finally, $g_3(t) = te^t$ (and its derivatives) is independent of the homogeneous solution. Therefore set $Y_3(t) = (Dt+E)e^t$. Conclude that the particular solution has the form

$$Y(t) = At\,cos\,2t + Bt\,sin\,2t + Ct^2 + (Dt+E)e^t.$$

18. The characteristic equation can be written as $r^2(r^2+2r+2) = 0$, with roots $r = 0$, with multiplicity *two*, and $r = -1 \pm i$. The homogeneous solution is $y_c = c_1 + c_2t +$ $+ c_3e^{-t}cos\,t + c_4e^{-t}sin\,t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt+C)e^{-t}$. Now $g_2(t) = e^{-t}sin\,t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D\,e^{-t}cos\,t + E\,e^{-t}sin\,t)$. It follows that the particular solution has the form

$$Y(t) = Ae^t + (Bt+C)e^{-t} + t\big(D\,e^{-t}cos\,t + E\,e^{-t}sin\,t\big).$$

19. Differentiating $y = u(t)v(t)$, successively, we have

$$\begin{aligned} y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \\ &\vdots \\ y^{(n)} &= \sum_{j=0}^{n} \binom{n}{j} u^{(n-j)}v^{(j)} \end{aligned}$$

Setting $v(t) = e^{\alpha t}$, $v^{(j)} = \alpha^j e^{\alpha t}$. So for any $p = 1, 2, \cdots, n$,

$$y^{(p)} = e^{\alpha t}\sum_{j=0}^{p} \binom{p}{j} \alpha^j\, u^{(p-j)}.$$

It follows that

$$L\big[e^{\alpha t}u\big] = e^{\alpha t}\sum_{p=0}^{n}\left[a_{n-p}\sum_{j=0}^{p}\binom{p}{j}\alpha^j\, u^{(p-j)}\right] \qquad (*).$$

It is evident that the right hand side of Eq. $(*)$ is of the form

$$e^{\alpha t}\big[k_0\, u^{(n)} + k_1\, u^{(n-1)} + \cdots + k_{n-1}u' + k_n\, u\big]\,.$$

Hence operator equation $L[e^{\alpha t}u] = e^{\alpha t}(b_0\, t^m + b_1\, t^{m-1} + \cdots + b_{m-1}t + b_m\,)$ can be written as

$$k_0\, u^{(n)} + k_1\, u^{(n-1)} + \cdots + k_{n-1}u' + k_n\, u =$$

$$= b_0\, t^m + b_1\, t^{m-1} + \cdots + b_{m-1}t + b_m\,.$$

The coefficients $k_i\,, i = 0, 1, \cdots, n$ can be determined by collecting the like terms in the double summation in Eq. $(*)$. For example, k_0 is the coefficient of $u^{(n)}$. The *only* term that contains $u^{(n)}$ is when $p = n$ and $j = 0$. Hence $k_0 = a_0$. On the other hand, k_n is the coefficient of $u(t)$. The inner summation in $(*)$ contains terms with u, given by $\alpha^p u$ (when $j = p$), for each $p = 0, 1, \cdots, n$. Hence

$$k_n = \sum_{p=0}^{n} a_{n-p}\,\alpha^p\,.$$

$21(a)$. Clearly, e^{2t} is a solution of $y' - 2y = 0$, and te^{-t} is a solution of the differential equation $y'' + 2y' + y = 0$. The latter ODE has characteristic equation $(r+1)^2 = 0$. Hence $(D-2)[3e^{2t}] = 3(D-2)[e^{2t}] = 0$ and $(D+1)^2[te^{-t}] = 0$. Furthermore, we have $(D-2)(D+1)^2[te^{-t}] = (D-2)[0] = 0$, and $(D-2)(D+1)^2[3e^{2t}] =$ $= (D+1)^2(D-2)[3e^{2t}] = (D+1)^2[0] = 0$.

(b). Based on Part (a),

$$\begin{aligned}(D-2)(D+1)^2\big[(D-2)^3(D+1)Y\big] &= (D-2)(D+1)^2\big[3e^{2t} - te^{-t}\big]\\ &= 0\,,\end{aligned}$$

since the operators are linear. The implied operations are associative and commutative. Hence

$$(D-2)^4(D+1)^3 Y = 0\,.$$

The operator equation corresponds to the solution of a linear homogeneous ODE with characteristic equation $(r-2)^4(r+1)^3 = 0$. The roots are $r = 2$, with multiplicity 4 and $r = -1$, with multiplicity 3. It follows that the given homogeneous solution is

$$Y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + c_4t^3e^{2t} + c_5e^{-t} + c_6te^{-t} + c_7t^2e^{-t},$$

which is a linear combination of seven independent solutions.

22(15). Observe that $(D-1)[e^t]=0$ and $(D^2+1)[sin\,t]=0$. Hence the operator $H(D)=(D-1)(D^2+1)$ is an annihilator of $e^t+sin\,t$. The operator corresponding to the left hand side of the given ODE is $(D^2-1)^2$. It follows that

$$(D+1)^2(D-1)^3\big(D^2+1\big)Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$Y(t)=c_1e^{-t}+c_2te^{-t}+c_3e^t+c_4te^t+c_5t^3e^t+c_6cos\,t+c_7sin\,t\,.$$

After examining the homogeneous solution of Prob. 15, and eliminating duplicate terms, we have

$$Y(t)=c_5t^3e^t+c_6cos\,t+c_7sin\,t\,.$$

22(16). We find that $D[4]=0$, $(D-1)^2[te^t]=0$, and $(D^2+4)[sin\,2t]=0$.
The operator $H(D)=D(D-1)^2(D^2+4)$is an annihilator of $t^2+te^t+sin\,2t$. The operator corresponding to the left hand side of the ODE is $D^2(D^2+4)$. It follows that

$$D^3(D-1)^2\big(D^2+4\big)^2Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$Y(t)=c_1+c_2t+c_3t^2+c_4e^t+c_5te^t+c_6cos\,2t+c_7sin\,2t+c_8tcos\,2t+c_9tsin\,2t\,.$$

After examining the homogeneous solution of Prob. 16, and eliminating duplicate terms, we have

$$Y(t)=c_3t^2+c_4e^t+c_5te^t+c_8tcos\,2t+c_9tsin\,2t\,.$$

22(18). Observe that $(D-1)[e^t]=0$, $(D+1)^2[te^{-t}]=0$. The function $e^{-t}sin\,t$ is a solution of a second order ODE with characteristic roots $r=-1\pm i$. It follows that $(D^2+2D+2)[e^{-t}sin\,t]=0$. Therefore the operator

$$H(D)=(D-1)(D+1)^2\big(D^2+2D+2\big)$$

is an annihilator of $3e^t+2te^{-t}+e^{-t}sin\,t$. The operator corresponding to the left hand side of the given ODE is $D^2(D^2+2D+2)$. It follows that

$$D^2(D-1)(D+1)^2\big(D^2+2D+2\big)^2Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$\begin{aligned}Y(t)=\;&c_1+c_2t+c_3e^t+c_4e^{-t}+c_5te^{-t}+\\&+e^{-t}(c_6cos\,t+c_7sin\,t)+te^{-t}(c_8cos\,t+c_9sin\,t)\,.\end{aligned}$$

After examining the homogeneous solution of Prob. 18, and eliminating duplicate terms, we have

$$Y(t) = c_3e^t + c_4e^{-t} + c_5te^{-t} + te^{-t}(c_8 cos\, t + c_9 sin\, t\,).$$

Section 4.4

2. The characteristic equation is $r(r^2-1)=0$. Hence the homogeneous solution is $y_c(t)=c_1+c_2e^t+c_3e^{-t}$. The Wronskian is evaluated as $W(1,e^t,e^{-t})=2$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{-t}\\ 0 & e^t & -e^{-t}\\ 1 & e^t & e^{-t}\end{vmatrix}=-2$$

$$W_2(t)=\begin{vmatrix}1 & 0 & e^{-t}\\ 0 & 0 & -e^{-t}\\ 0 & 1 & e^{-t}\end{vmatrix}=e^{-t}$$

$$W_3(t)=\begin{vmatrix}1 & e^t & 0\\ 0 & e^t & 0\\ 0 & e^t & 1\end{vmatrix}=e^t$$

The solution of the system of equations (10) is

$$u_1'(t)=\frac{t\,W_1(t)}{W(t)}=-t$$

$$u_2'(t)=\frac{t\,W_2(t)}{W(t)}=te^{-t}/2$$

$$u_3'(t)=\frac{t\,W_3(t)}{W(t)}=te^t/2$$

Hence $u_1(t)=-t^2/2$, $u_2(t)=-e^{-t}(t+1)/2$, $u_3(t)=e^t(t-1)/2$. The particular solution becomes $Y(t)=-t^2/2-(t+1)/2+(t-1)/2=-t^2/2-1$. The constant is a solution of the homogeneous equation, therefore the general solution is

$$y(t)=c_1+c_2e^t+c_3e^{-t}-t^2/2\,.$$

3. From Prob. 13 in Section 4.2, $y_c(t)=c_1e^{-t}+c_2e^t+c_3e^{2t}$. The Wronskian is evaluated as $W(e^{-t},e^t,e^{2t})=6\,e^{2t}$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{2t}\\ 0 & e^t & 2e^{2t}\\ 1 & e^t & 4e^{2t}\end{vmatrix}=e^{3t}$$

$$W_2(t) = \begin{vmatrix} e^{-t} & 0 & e^{2t} \\ -e^{-t} & 0 & 2e^{2t} \\ e^{-t} & 1 & 4e^{2t} \end{vmatrix} = -3e^t$$

$$W_3(t) = \begin{vmatrix} e^{-t} & e^t & 0 \\ -e^{-t} & e^t & 0 \\ e^{-t} & e^t & 1 \end{vmatrix} = 2$$

Hence $u_1'(t) = e^{5t}/6$, $u_2'(t) = -e^{3t}/2$, $u_3'(t) = e^{2t}/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= e^{-t}\left[e^{5t}/30\right] - e^t\left[e^{3t}/6\right] + e^{2t}\left[e^{2t}/6\right] \\ &= e^{4t}/30\,. \end{aligned}$$

6. From Prob. 22 in Section 4.2, $y_c(t) = c_1 cos\, t + c_2 sin\, t + t[c_3 cos\, t + c_4 sin\, t]$. The Wronskian is evaluated as $W(cos\, t, sin\, t, t\, cos\, t, t\, sin\, t) = 4$. Now compute the four auxiliary determinants

$$W_1(t) = \begin{vmatrix} 0 & sin\, t & t\, cos\, t & t\, sin\, t \\ 0 & cos\, t & cos\, t - t\, sin\, t & sin\, t + t\, cos\, t \\ 0 & -sin\, t & -2sin\, t - t\, cos\, t & 2cos\, t - t\, sin\, t \\ 1 & -cos\, t & -3cos\, t + t\, sin\, t & -3sin\, t - t\, cos\, t \end{vmatrix} = -2sin\, t + 2t\, cos\, t$$

$$W_2(t) = \begin{vmatrix} cos\, t & 0 & t\, cos\, t & t\, sin\, t \\ -sin\, t & 0 & cos\, t - t\, sin\, t & sin\, t + t\, cos\, t \\ -cos\, t & 0 & -2sin\, t - t\, cos\, t & 2cos\, t - t\, sin\, t \\ sin\, t & 1 & -3cos\, t + t\, sin\, t & -3sin\, t - t\, cos\, t \end{vmatrix} = 2t\, sin\, t + 2cos\, t$$

$$W_3(t) = \begin{vmatrix} cos\, t & sin\, t & 0 & t\, sin\, t \\ -sin\, t & cos\, t & 0 & sin\, t + t\, cos\, t \\ -cos\, t & -sin\, t & 0 & 2cos\, t - t\, sin\, t \\ sin\, t & -cos\, t & 1 & -3sin\, t - t\, cos\, t \end{vmatrix} = -2cos\, t$$

$$W_4(t) = \begin{vmatrix} cos\, t & sin\, t & t\, cos\, t & 0 \\ -sin\, t & cos\, t & cos\, t - t\, sin\, t & 0 \\ -cos\, t & -sin\, t & -2sin\, t - t\, cos\, t & 0 \\ sin\, t & -cos\, t & -3cos\, t + t\, sin\, t & 1 \end{vmatrix} = -2sin\, t$$

It follows that $u_1'(t) = [-sin^2 t + t\, sin\, t\, cos\, t]/2$, $u_2'(t) = [t sin^2 t + sin\, t\, cos\, t]/2$, $u_3'(t) = -sin\, t\, cos\, t/2$, $u_4'(t) = -sin^2 t/2$. Hence

$$u_1(t) = \left[3sin\, t\, cos\, t - 2t\, cos^2 t - t\right]/8$$

$$u_2(t) = \left[sin^2 t - 2cos^2 t - 2t\, sin\, t\, cos\, t + t^2\,\right]/8$$

$$u_3(t) = \,-\, sin^2 t/4$$

$$u_4(t) = [cos\, t\, sin\, t - t]/4$$

Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= cos\, t\,[u_1(t)] + sin\, t\,[u_2(t)] + t\, cos\, t\,[u_3(t)] + t\, sin\, t\,[u_4(t)] \\ &= \left[sin\, t - 3t\, cos\, t - t^2 sin\, t\right]/8\,. \end{aligned}$$

Note that only the *last term* is not a solution of the homogeneous equation. Hence the general solution is

$$y(t) = c_1 cos\, t + c_2 sin\, t\; + t[c_3 cos\, t + c_4 sin\, t\,] - t^2 sin\, t\,/8\,.$$

8. Based on the results in Prob. 2, $y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$. It was also shown that $W(1, e^t, e^{-t}) = 2$, with $W_1(t) = \,-\,2\,, W_2(t) = e^{-t}, W_3(t) = e^t$. Therefore we have $u_1'(t) = \,-\,csc\, t$, $u_2'(t) = e^{-t} csc\, t\,/2$, $u_3'(t) = e^t csc\, t\,/2$. The particular solution can be expressed as $Y(t) = [u_1(t)] + e^{-t}[u_2(t)] + e^t\,[u_3(t)]$. More specifically,

$$\begin{aligned} Y(t) &= ln|csc(t) + cot(t)| + \frac{e^t}{2}\int_{t_0}^{t} e^{-s} csc(s)ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s} csc(s)ds \\ &= ln|csc(t) + cot(t)| + \int_{t_0}^{t} cosh(t-s)csc(s)ds\,. \end{aligned}$$

9. Based on Prob. 4, $u_1'(t) = sec\, t$, $u_2'(t) = \,-\,1$, $u_3'(t) = \,-\,tan\, t$. The particular solution can be expressed as $Y(t) = [u_1(t)] + cos\, t\,[u_2(t)] + sin\, t\,[u_3(t)]$. That is,

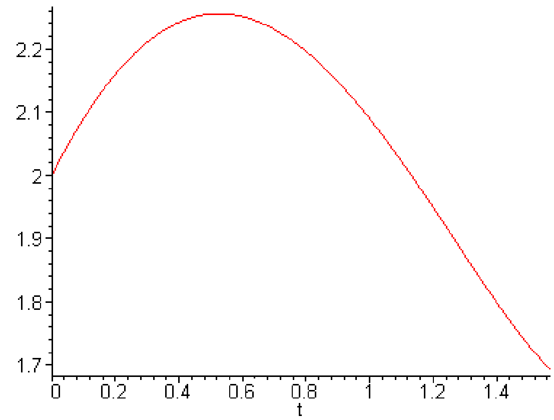
$$Y(t) = ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|.$$

Hence the general solution of the initial value problem is

$$y(t) = c_1 + c_2 cos\, t + c_3 sin\, t\; + ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|.$$

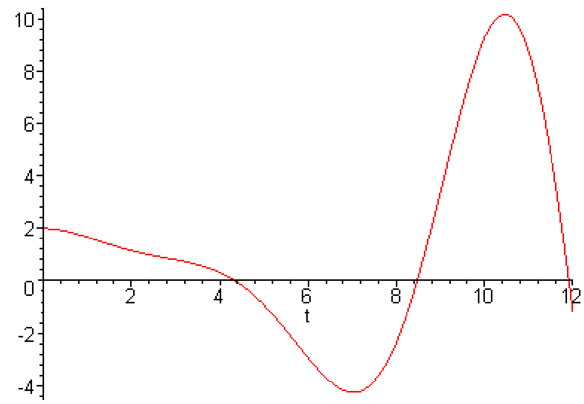
Invoking the initial conditions, we require that $c_1 + c_2 = 2\,, c_3 = 1\,,\, -\,c_2 = \,-\,2\,.$ Therefore

$$y(t) = 2cos\, t + sin\, t + ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|$$



10. From Prob. 6, $y(t) = c_1 cos\, t + c_2 sin\, t + c_3 t\, cos\, t + c_4 t\, sin\, t - t^2 sin\, t/8$. In order to satisfy the initial conditions, we require that $c_1 = 2$, $c_2 + c_3 = 0$, $-c_1 + 2c_4 = -1$, $-3/4 - c_2 - 3c_3 = 1$. Therefore

$$y(t) = 2cos\, t + \left[7sin\, t - 7t\, cos\, t + 4t\, sin\, t - t^2 sin\, t\right]/8\,.$$



12. From Prob. 8, the general solution of the initial value problem is

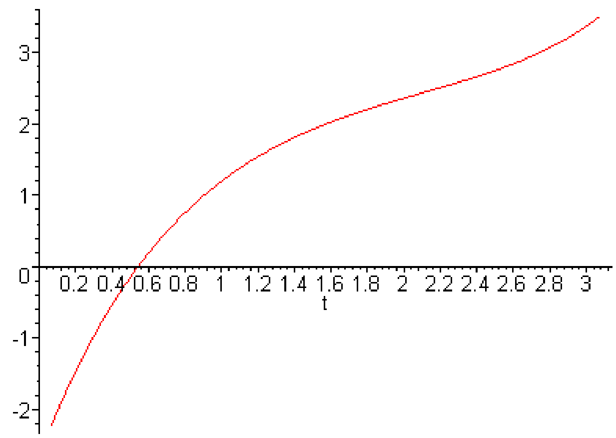
$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} + ln|csc(t) + cot(t)| + \frac{e^t}{2}\int_{t_0}^{t} e^{-s} csc(s)ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s} csc(s)ds.$$

In this case, $t_0 = \pi/2$. Observe that $y(\pi/2) = y_c(\pi/2)$, $y'(\pi/2) = y_c'(\pi/2)$, and $y''(\pi/2) = y_c''(\pi/2)$. Therefore we obtain the system of equations

$$\begin{aligned} c_1 + c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= 2 \\ c_2 e^{\pi/2} - c_3 e^{-\pi/2} &= 1 \\ c_2 e^{\pi/2} + c_3 e^{-\pi/2} &= -1 \end{aligned}$$

Hence the solution of the initial value problem is

$$y(t) = 3 - e^{-t+\pi/2} + ln|csc(t) + cot(t)| + \int_{t_0}^{t} cosh(t-s)csc(s)ds\,.$$



13. First write the equation as $y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$. The Wronskian is evaluated as $W(x, x^2, 1/x) = 6/x$. Now compute the three determinants

$$W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3$$

$$W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Hence $u_1'(x) = -x^2$, $u_2'(x) = 2x/3$, $u_3'(x) = x^4/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(x) &= x\left[-x^3/3\right] + x^2\left[x^2/3\right] + \frac{1}{x}\left[x^5/15\right] \\ &= x^4/15\,. \end{aligned}$$

15. The homogeneous solution is $y_c(t) = c_1 cos\,t + c_2 sin\,t + c_3 cosh\,t + c_4 sinh\,t$. The Wronskian is evaluated as $W(cos\,t, sin\,t, cosh\,t, sinh\,t) = 4$. Now the four additional determinants are given by $W_1(t) = 2\,sin\,t$, $W_2(t) = -2\,cos\,t$, $W_3(t) = -2\,sinh\,t$, $W_4(t) = 2\,cosh\,t$. If follows that $u_1'(t) = g(t)\,sin(t)/2$, $u_2'(t) = -g(t)\,cos(t)/2$, $u_3'(t) = -g(t)\,sinh(t)/2$, $u_4'(t) = g(t)\,cosh(t)/2$. Therefore the particular solution

can be expressed as

$$\begin{aligned} Y(t) &= \frac{cos(t)}{2}\int_{t_0}^{t} g(s)\,sin(s)\,ds - \frac{sin(t)}{2}\int_{t_0}^{t} g(s)\,cos(s)\,ds - \\ &- \frac{cosh(t)}{2}\int_{t_0}^{t} g(s)\,sinh(s)\,ds + \frac{sinh(t)}{2}\int_{t_0}^{t} g(s)\,cosh(s)\,ds\,. \end{aligned}$$

Using the appropriate identities, the integrals can be combined to obtain

$$Y(t) = \frac{1}{2}\int_{t_0}^{t} g(s)\,sinh(t-s)\,ds - \frac{1}{2}\int_{t_0}^{t} g(s)\,sin(t-s)\,ds\,.$$

17. First write the equation as $y''' - 3\,x^{-1}y'' + 6\,x^{-2}y' - 6\,x^{-3}y = g(x)\,/\,x^3$. It can be shown that $y_c(x) = c_1 x + c_2\,x^2 + c_3\,x^3$ is a solution of the homogeneous equation. The Wronskian of this fundamental set of solutions is $W(x, x^2, x^3) = 2x^3$. The three additional determinants are given by $W_1(x) = x^4$, $W_2(x) = -2x^3$, $W_3(x) = x^2$. Hence $u_1'(x) = g(x)/2x^2$, $u_2'(x) = -g(x)\,/x^3$, $u_3'(x) = g(x)/2x^4$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(x) &= x\int_{x_0}^{x} \frac{g(t)}{2t^2}dt - x^2\int_{x_0}^{x} \frac{g(t)}{t^3}dt + x^3\int_{x_0}^{x} \frac{g(t)}{2t^4}dt \\ &= \frac{1}{2}\int_{x_0}^{x} \left[\frac{x}{t^2} - \frac{2\,x^2}{t^3} + \frac{x^3}{t^4}\right] g(t)dt\,. \end{aligned}$$

Chapter Five

Section 5.1

1. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|(x-3)^{n+1}\right|}{\left|(x-3)^{n}\right|}=\lim_{n\to\infty}|x-3|=|x-3|.$$

Hence the series converges absolutely for $|x-3|<1$. The radius of convergence is $\rho=1$. The series diverges for $x=2$ and $x=4$, since the *n-th* term does not approach zero.

3. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!\,x^{2n+2}|}{|(n+1)!\,x^{2n}|}=\lim_{n\to\infty}\frac{x^2}{n+1}=0\,.$$

The series converges absolutely for *all* values of x . Thus the radius of convergence is $\rho=\infty$.

4. Apply the ratio test :

$$\lim_{n\to\infty}\frac{|2^{n+1}x^{n+1}|}{|2^n x^n|}=\lim_{n\to\infty}2|x|=2|x|.$$

Hence the series converges absolutely for $2|x|$, or $|x|<1/2$. The radius of convergence is $\rho=1/2$. The series diverges for $x=\pm 1/2$, since the *n-th* term does not approach zero.

6. Applying the ratio test,

$$\lim_{n\to\infty}\frac{\left|n(x-x_0)^{n+1}\right|}{\left|(n+1)(x-x_0)^{n}\right|}=\lim_{n\to\infty}\frac{n}{n+1}|(x-x_0)|=|(x-x_0)|.$$

Hence the series converges absolutely for $|(x-x_0)|<1$. The radius of convergence is $\rho=1$. At $x=x_0+1$, we obtain the *harmonic series*, which is *divergent*. At the other endpoint, $x=x_0-1$, we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\,,$$

which is *conditionally* convergent.

7. Apply the ratio test :

$$\lim_{n\to\infty}\frac{\left|3^{n}(n+1)^2(x+2)^{n+1}\right|}{\left|3^{n+1}n^2(x+2)^n\right|} = \lim_{n\to\infty}\frac{(n+1)^2}{3\,n^2}|(x+2)| = \frac{1}{3}|(x+2)|.$$

Hence the series converges absolutely for $\frac{1}{3}|x+2| < 1$, or $|x+2| < 3$. The radius of convergence is $\rho = 3$. At $x = -5$ and $x = +1$, the series diverges, since the *n-th* term does not approach zero.

8. Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n^n(n+1)!\,x^{n+1}|}{\left|(n+1)^{n+1}n!\,x^n\right|} = \lim_{n\to\infty}\frac{n^n}{(n+1)^n}|x| = \frac{1}{e}|x|,$$

since

$$\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = e^{-1}.$$

Hence the series converges absolutely for $|x| < e$. The radius of convergence is $\rho = e$. At $x = \pm e$, the series *diverges*, since the *n-th* term does not approach zero. This follows from the fact that

$$\lim_{n\to\infty}\frac{n!\,e^n}{n^n\sqrt{2\pi n}} = 1\,.$$

10. We have $f(x) = e^x$, with $f^{(n)}(x) = e^x$, for $n = 1, 2, \cdots$. Therefore $f^{(n)}(0) = 1$. Hence the Taylor expansion about $x_0 = 0$ is

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}\,.$$

Applying the ratio test,

$$\lim_{n\to\infty}\frac{|n!x^{n+1}|}{|(n+1)!\,x^n|} = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0.$$

The radius of convergence is $\rho = \infty$.

11. We have $f(x) = x$, with $f'(x) = 1$ and $f^{(n)}(x) = 0$, for $n = 2, \cdots$. Clearly, $f(1) = 1$ and $f'(1) = 1$, with all other derivatives equal to *zero*. Hence the Taylor expansion about $x_0 = 1$ is

$$x = 1 + (x-1).$$

Since the series has only a finite number of terms, the converges absolutely for all x.

14. We have $f(x) = 1/(1+x)$, $f'(x) = -1/(1+x)^2$, $f''(x) = 2/(1+x)^3, \cdots$ with $f^{(n)}(x) = (-1)^n n!/(1+x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = (-1)^n n!$

for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n\to\infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

15. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(0) = n!$, for $n \geq 0$. Hence the Taylor expansion about $x_0 = 0$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{|x^{n+1}|}{|x^n|} = \lim_{n\to\infty} |x| = |x|.$$

The series converges absolutely for $|x| < 1$, but diverges at $x = \pm 1$.

16. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \cdots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1} n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = -\sum_{n=0}^{\infty} (-1)^n (x-2)^n.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{|(x-2)^{n+1}|}{|(x-2)^n|} = \lim_{n\to\infty} |x-2| = |x-2|.$$

The series converges absolutely for $|x-2| < 1$, but diverges at $x = 1$ and $x = 3$.

17. Applying the ratio test,

$$\lim_{n\to\infty} \frac{|(n+1)x^{n+1}|}{|n\, x^n|} = \lim_{n\to\infty} \frac{n+1}{n}|x| = |x|.$$

The series converges absolutely for $|x| < 1$. Term-by-term differentiation results in

$$y' = \sum_{n=1}^{\infty} n^2 x^{n-1} = 1 + 4x + 9x^2 + 16\,x^3 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n^2(n-1)\,x^{n-2} = 4 + 18x + 48x^2 + 100x^3 + \cdots$$

Shifting the indices, we can also write

$$y' = \sum_{n=0}^{\infty} (n+1)^2 x^n \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} (n+2)^2(n+1)\,x^n.$$

20. Shifting the index in the *second* series, that is, setting $n = k + 1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1} x^n\,.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k \\ &= a_1 + \sum_{k=1}^{\infty} (a_{k+1} + a_{k-1}) x^{k+1}.\end{aligned}$$

21. Shifting the index by 2, that is, setting $m = n - 2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} &= \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2}\, x^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n\,.\end{aligned}$$

22. Shift the index *down* by 2, that is, set $m = n + 2$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^{n+2} &= \sum_{m=2}^{\infty} a_{m-2}\, x^m \\ &= \sum_{n=2}^{\infty} a_{n-2}\, x^n\,.\end{aligned}$$

24. Clearly,

$$\left(1-x^2\right)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=2}^{\infty}n(n-1)a_n x^n.$$

Shifting the index in the *first* series, that is, setting $k=n-2$,

$$\begin{aligned}\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}&=\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\,x^k\\&=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.\end{aligned}$$

Hence

$$(1-x^2)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n-\sum_{n=2}^{\infty}n(n-1)a_n\,x^n.$$

Note that when $n=0$ and $n=1$, the coefficients in the *second* series are *zero*. So that

$$(1-x^2)\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}=\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}-n(n-1)a_n]x^n.$$

26. Clearly,

$$\sum_{n=1}^{\infty}na_n\,x^{n-1}+x\sum_{n=0}^{\infty}a_n\,x^n=\sum_{n=1}^{\infty}na_n\,x^{n-1}+\sum_{n=0}^{\infty}a_n\,x^{n+1}.$$

Shifting the index in the *first* series, that is, setting $k=n-1$,

$$\sum_{n=1}^{\infty}na_n\,x^{n-1}=\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k.$$

Shifting the index in the *second* series, that is, setting $k=n+1$,

$$\sum_{n=0}^{\infty}a_n\,x^{n+1}=\sum_{k=1}^{\infty}a_{k-1}x^k.$$

Combining the series, and starting the summation at $n=1$,

$$\sum_{n=1}^{\infty}na_n\,x^{n-1}+x\sum_{n=0}^{\infty}a_n\,x^n=a_1+\sum_{n=1}^{\infty}[(n+1)a_{n+1}+a_{n-1}]x^n.$$

27. We note that

$$x\sum_{n=2}^{\infty} n(n-1)a_n\,x^{n-2}+\sum_{n=0}^{\infty} a_n\,x^n=\sum_{n=2}^{\infty} n(n-1)a_n\,x^{n-1}+\sum_{n=0}^{\infty} a_n\,x^n.$$

Shifting the index in the *first* series, that is, setting $k=n-1$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n\,x^{n-1} &= \sum_{k=1}^{\infty} k(k+1)a_{k+1}x^k\\ &= \sum_{k=0}^{\infty} k(k+1)a_{k+1}x^k,\end{aligned}$$

since the coefficient of the term associated with $k=0$ is *zero*. Combining the series,

$$x\sum_{n=2}^{\infty} n(n-1)a_n\,x^{n-2}+\sum_{n=0}^{\infty} a_n\,x^n=\sum_{n=0}^{\infty}[n(n+1)a_{n+1}+a_n]x^n.$$

Section 5.2

1. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - a_n]x^n = 0\,.$$

Equating all the coefficients to *zero*,

$$(n+2)(n+1)a_{n+2} - a_n = 0\,, \qquad n = 0, 1, 2, \cdots.$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}\,, \qquad n = 0, 1, 2, \cdots.$$

The subscripts differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots = \frac{a_1}{(2k+1)!}\,.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\,.$$

The linearly independent solutions are

$$y_1 = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right) = a_0 cosh\, x$$

$$y_2 = a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) = a_1 sinh\, x\,.$$

4. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + k^2 x^2 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Rewriting the *second* summation,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + \sum_{n=2}^{\infty} k^2 a_{n-2}\, x^n = 0\,,$$

that is,

$$2a_2 + 3\cdot 2\, a_3 x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + k^2 a_{n-2}\right] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we have $a_2 = 0$, $a_3 = 0$, and

$$(n+2)(n+1)a_{n+2} + k^2 a_{n-2} = 0\,, \quad \text{for } n = 2, 3, 4, \cdots .$$

The recurrence relation can be written as

$$a_{n+2} = -\frac{k^2 a_{n-2}}{(n+2)(n+1)}\,, \quad n = 2, 3, 4, \cdots .$$

The indices differ by *four*, so a_4, a_8, a_{12}, $\cdots$ are defined by

$$a_4 = -\frac{k^2 a_0}{4\cdot 3}\,,\; a_8 = -\frac{k^2 a_4}{8\cdot 7}\,, a_{12} = -\frac{k^2 a_8}{12\cdot 11}\,, \cdots .$$

Similarly, a_5, a_9, a_{13}, $\cdots$ are defined by

$$a_5 = -\frac{k^2 a_1}{5\cdot 4}\,,\; a_9 = -\frac{k^2 a_5}{9\cdot 8}\,, a_{13} = -\frac{k^2 a_9}{13\cdot 12}\,, \cdots .$$

The remaining coefficients are *zero*. Therefore the general solution is

$$\begin{aligned} y &= a_0\left[1 - \frac{k^2}{4\cdot 3}x^4 + \frac{k^4}{8\cdot 7\cdot 4\cdot 3}x^8 - \frac{k^6}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3}x^{12} + \cdots\right] + \\ &+ a_1\left[x - \frac{k^2}{5\cdot 4}x^5 + \frac{k^4}{9\cdot 8\cdot 5\cdot 4}x^9 - \frac{k^6}{13\cdot 12\cdot 9\cdot 8\cdot 4\cdot 4}x^{13} + \cdots\right]. \end{aligned}$$

Note that for the *even* coefficients,

$$a_{4m} = -\frac{k^2 a_{4m-4}}{(4m-1)4m}, \quad m = 1, 2, 3, \cdots$$

and for the *odd* coefficients,

$$a_{4m+1} = -\frac{k^2 a_{4m-3}}{4m(4m+1)}, \quad m = 1, 2, 3, \cdots.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{3 \cdot 4 \cdot 7 \cdot 8 \cdots (4m+3)(4m+4)}$$

$$y_2(x) = x\left[1 + \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\left(k^2 x^4\right)^{m+1}}{4 \cdot 5 \cdot 8 \cdot 9 \cdots (4m+4)(4m+5)}\right].$$

6. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(2 + x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x + \sum_{n=2}^{\infty} \left[2(n+2)(n+1) a_{n+2} + n(n-1) a_n - n\, a_n + 4a_n\right] x^n = 0\,.$$

Equating the coefficients to *zero*, we find that $a_2 = -a_0$, $a_3 = -a_1/4$, and

$$a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)}\,a_n\,, \quad n = 0,1,2,\cdots.$$

The indices differ by *two*, so for $k = 0,1,2,\cdots$

$$a_{2k+2} = -\frac{(2k)^2-4k+4}{2(2k+2)(2k+1)}\,a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2-4k+2}{2(2k+3)(2k+2)}\,a_{2k+1}.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \cdots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \cdots.$$

7. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty}a_nx^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_nx^n.$$

We then obtain

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + n\,a_n + 2a_n]x^n = 0\,.$$

It follows that $a_2 = -a_0$ and $a_{n+2} = -a_n/(n+1)$, $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = -\frac{a_{2k-2}}{2k-1} = \frac{a_{2k-4}}{(2k-3)(2k-1)} = \cdots = \frac{(-1)^k a_0}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$$

and

$$a_{2k+1} = -\frac{a_{2k-1}}{2k} = \frac{a_{2k-3}}{(2k-2)2k} = \cdots = \frac{(-1)^k a_1}{2 \cdot 4 \cdot 6 \cdots (2k)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{x^2}{1} + \frac{x^4}{1 \cdot 3} - \frac{x^6}{1 \cdot 3 \cdot 5} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)}\,.$$

9. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\left(1 + x^2\right) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 4x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4n\,a_n + 6a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}\,a_n\,, \quad n = 0, 1, 2, \cdots.$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by *two*, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1 x - 3a_0 x^2 - a_1 x^3/3\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3\,.$$

10. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$\left(4 - x^2\right)\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x^2\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\,x^n = \sum_{n=2}^{\infty} n(n-1)a_n x^n.$$

It follows that

$$2a_0 + 8a_2 + (2a_1 + 24a_3)x + \sum_{n=2}^{\infty} \left[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n\right]x^n = 0\,.$$

We obtain $a_2 = -a_0/4$, $a_3 = -a_1/12$ and

$$4(n+2)a_{n+2} = (n-2)a_n\,, \quad n = 0, 1, 2, \cdots.$$

Note that for $n = 2$, $a_4 = 0$. Since the indices differ by *two*, we also have $a_{2k} = 0$ for $k = 2, 3, \cdots$. On the other hand, for $k = 1, 2, \cdots$,

$$a_{2k+1} = \frac{(2k-3)a_{2k-1}}{4(2k+1)} = \frac{(2k-5)(2k-3)a_{2k-3}}{4^2(2k-1)(2k+1)} = \cdots = \frac{-a_1}{4^k(2k-1)(2k+1)}\,.$$

Therefore the general solution is

$$y = a_0 + a_1 x - a_0\frac{x^2}{4} - a_1\sum_{n=1}^{\infty}\frac{x^{2n+1}}{4^n(2n-1)(2n+1)}\,.$$

Hence the linearly independent solutions are $y_1(x) = 1 - x^2/4$ and

$$y_2(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \cdots = x - \sum_{n=1}^{\infty}\frac{x^{2n+1}}{4^n(2n-1)(2n+1)}\,.$$

11. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$\left(3 - x^2\right)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n - 3x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - \sum_{n=0}^{\infty}a_nx^n = 0\,.$$

Before proceeding, write

$$x^2\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n = \sum_{n=2}^{\infty}n(n-1)a_nx^n$$

and

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty}n\,a_nx^n.$$

It follows that

$$6a_2 - a_0 + (-4a_1 + 18a_3)x + \sum_{n=2}^{\infty}\left[3(n+2)(n+1)a_{n+2} - n(n-1)a_n - 3n\,a_n - a_n\right]x^n = 0.$$

We obtain $a_2 = a_0/6$, $2a_3 = a_1/9$, and

$$3(n+2)a_{n+2} = (n+1)a_n\,,\quad n = 0, 1, 2, \cdots.$$

The indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{(2k-1)a_{2k-2}}{3(2k)} = \frac{(2k-3)(2k-1)a_{2k-4}}{3^2(2k-2)(2k)} = \cdots = \frac{3\cdot 5\cdots(2k-1)\,a_0}{3^k\cdot 2\cdot 4\cdots(2k)}$$

and

$$a_{2k+1} = \frac{(2k)a_{2k-1}}{3(2k+1)} = \frac{(2k-2)(2k)a_{2k-3}}{3^2(2k-1)(2k+1)} = \cdots = \frac{2\cdot 4\cdot 6\cdots(2k)\,a_1}{3^k\cdot 3\cdot 5\cdots(2k+1)}\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{24} + \frac{5x^6}{432} + \cdots = 1 + \sum_{n=1}^{\infty} \frac{3\cdot 5\cdots(2n-1)\,x^{2n}}{3^n\cdot 2\cdot 4\cdots(2n)}$$

$$y_2(x) = x + \frac{2x^3}{9} + \frac{8x^5}{135} + \frac{16x^7}{945} + \cdots = x + \sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)\,x^{2n+1}}{3^n\cdot 3\cdot 5\cdots(2n+1)}\,.$$

12. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n = \sum_{n=1}^{\infty} (n+1)n\, a_{n+1} x^n$$

and

$$x\sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$2a_2 - a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1} + n\, a_n - a_n\right] x^n = 0.$$

We obtain $a_2 = a_0/2$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n\, a_{n+1} + (n-1)a_n = 0$$

for $n = 0, 1, 2, \cdots$. Writing out the individual equations,

$$\begin{aligned}
3\cdot 2\,a_3 - 2\cdot 1\,a_2 &= 0\\
4\cdot 3\,a_4 - 3\cdot 2\,a_3 + a_2 &= 0\\
5\cdot 4\,a_5 - 4\cdot 3\,a_4 + 2\,a_3 &= 0\\
6\cdot 5\,a_6 - 5\cdot 4\,a_5 + 3\,a_4 &= 0\\
&\vdots
\end{aligned}$$

The coefficients can be calculated successively as $a_3 = a_0/(2\cdot 3)$, $a_4 = a_3/2 - a_2/12$ $= a_0/24$, $a_5 = 3a_4/5 - a_3/10 = a_0/120$, $\cdots$. We can now see that for $n \geq 2$, a_n is proportional to a_0. In fact, for $n \geq 2$, $a_n = a_0/(n!)$. Therefore the general solution is

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!} + \frac{a_0 x^4}{4!} + \cdots .$$

Hence the linearly independent solutions are $y_2(x) = x$ and

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} .$$

13. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n + x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 3\sum_{n=0}^{\infty} a_n x^n = 0 .$$

First write

$$x\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$4a_2 + 3a_0 + \sum_{n=1}^{\infty} [2(n+2)(n+1) a_{n+2} + n\, a_n + 3a_n] x^n = 0 .$$

It follows that $a_2 = -3a_0/4$ and

$$2(n+2)(n+1) a_{n+2} + (n+3) a_n = 0$$

for $n = 0, 1, 2, \cdots$. The indices differ by *two*, so for $k = 1, 2, \cdots$

$$\begin{aligned} a_{2k} &= -\frac{(2k+1)a_{2k-2}}{2(2k-1)(2k)} = \frac{(2k-1)(2k+1)a_{2k-4}}{2^2(2k-3)(2k-2)(2k-1)(2k)} = \cdots \\ &= \frac{(-1)^k 3 \cdot 5 \cdots (2k+1)}{2^k\,(2k)!}\, a_0\,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k+2)a_{2k-1}}{2(2k)(2k+1)} = \frac{(2k)(2k+2)a_{2k-3}}{2^2(2k-2)(2k-1)(2k)(2k+1)} = \cdots \\ &= \frac{(-1)^k\, 4 \cdot 6 \cdots (2k)(2k+2)}{2^k\,(2k+1)!}\, a_1\,. \end{aligned}$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{4}x^2 + \frac{5}{32}x^4 - \frac{7}{384}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3 \cdot 5 \cdots (2n+1)}{2^n\,(2n)!} x^{2n}$$

$$y_2(x) = x - \frac{1}{3}x^3 + \frac{1}{20}x^5 - \frac{1}{210}x^7 + \cdots = x + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cdot 6 \cdots (2n+2)}{2^n\,(2n+1)!} x^{2n+1}.$$

$15(a)$. From Prob. 2, we have

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\, n!} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n\, n!\, x^{2n+1}}{(2n+1)!}\,.$$

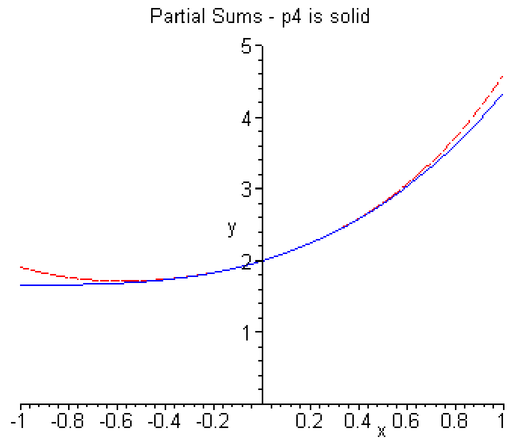
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 2\,y_1(x) + y_2(x)$. That is,

$$y(x) = 2 + x + x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 + \frac{1}{24}x^6 + \cdots.$$

The *four*- and *five*-term polynomial approximations are

$$\begin{aligned} p_4 &= 2 + x + x^2 + x^3/3 \\ p_5 &= 2 + x + x^2 + x^3/3 + x^4/4\,. \end{aligned}$$

(b).



(c). The *four-term* approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.7$.

$17(a)$. From Prob. 7, the linearly independent solutions are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2 \cdot 4 \cdot 6 \cdots (2n)} .$$

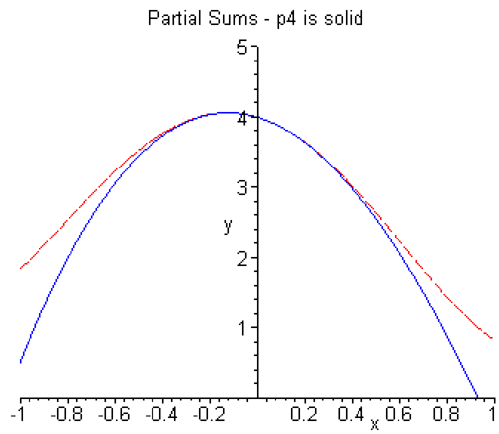
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = 4\,y_1(x) - y_2(x)$. That is,

$$y(x) = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4 - \frac{1}{8}x^5 - \frac{4}{15}x^6 + \cdots .$$

The *four-* and *five*-term polynomial approximations are

$$p_4 = 4 - x - 4x^2 + \frac{1}{2}x^3$$
$$p_5 = 4 - x - 4x^2 + \frac{1}{2}x^3 + \frac{4}{3}x^4.$$

(b).



(c). The *four-term* approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.5$.

$18(a)$. From Prob. 12 , we have

$$y_1(x) = 1 + \sum_{n=2}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad y_2(x) = x\,.$$

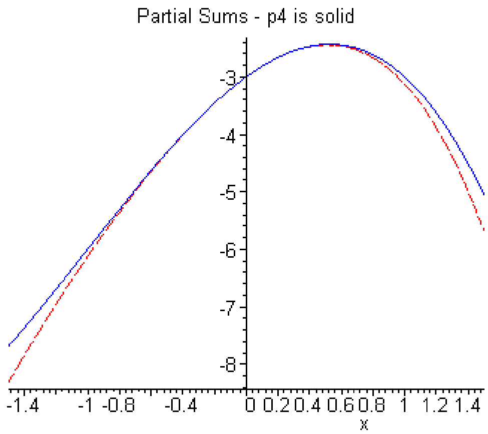
Since $a_0 = y(0)$ and $a_1 = y'(0)$, we have $y(x) = \,-3\,y_1(x) + 2\,y_2(x)$. That is,

$$y(x) = \,-3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 - \frac{1}{40}x^5 - \frac{1}{240}x^6 + \cdots.$$

The *four-* and *five-*term polynomial approximations are

$$\begin{aligned} p_4 &= \,-3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 \\ p_5 &= \,-3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4\,. \end{aligned}$$

(b).



(c). The *four-term* approximation p_4 appears to be reasonably accurate (within 10%) on the interval $|x| < 0.9$.

20. Two linearly independent solutions of *Airy's equation* (about $x_0 = 0$) are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)}.$$

Applying the *ratio test* to the terms of $y_1(x)$,

$$\lim_{n\to\infty} \frac{|2 \cdot 3 \cdots (3n-1)(3n)\, x^{3n+3}|}{|2 \cdot 3 \cdots (3n+2)(3n+3)\, x^{3n}|} = \lim_{n\to\infty} \frac{1}{(3n+1)(3n+2)(3n+3)} |x|^3 = 0.$$

Similarly, applying the *ratio test* to the terms of $y_2(x)$,

$$\lim_{n\to\infty} \frac{|3 \cdot 4 \cdots (3n)(3n+1)\, x^{3n+4}|}{|3 \cdot 4 \cdots (3n+3)(3n+4)\, x^{3n+1}|} = \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)(3n+4)} |x|^3 = 0.$$

Hence both series converge *absolutely* for all x.

21. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n - 2x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \lambda\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

First write

$$x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

We then obtain

$$2a_2 + \lambda\, a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - 2n\, a_n + \lambda\, a_n]x^n = 0\,.$$

Setting the coefficients equal to *zero*, it follows that

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)}\, a_n$$

for $n = 0, 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$\begin{aligned} a_{2k} &= \frac{(4k-4-\lambda)a_{2k-2}}{(2k-1)2k} = \frac{(4k-8-\lambda)(4k-4-\lambda)a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \cdots \\ &= (-1)^k \frac{\lambda\cdots(\lambda-4k+8)(\lambda-4k+4)}{(2k)!}\, a_0\,. \end{aligned}$$

and

$$\begin{aligned} a_{2k+1} &= \frac{(4k-2-\lambda)a_{2k-1}}{2k(2k+1)} = \frac{(4k-6-\lambda)(4k-2-\lambda)a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \cdots \\ &= (-1)^k \frac{(\lambda-2)\cdots(\lambda-4k+6)(\lambda-4k+2)}{(2k+1)!}\, a_1. \end{aligned}$$

Hence the linearly independent solutions of the *Hermite equation* (about $x_0 = 0$) are

$$y_1(x) = 1 - \frac{\lambda}{2!}x^2 + \frac{\lambda(\lambda-4)}{4!}x^4 - \frac{\lambda(\lambda-4)(\lambda-8)}{6\,!}x^6 + \cdots$$

$$y_2(x) = x - \frac{\lambda-2}{3\,!}x^3 + \frac{(\lambda-2)(\lambda-6)}{5\,!}x^5 - \frac{(\lambda-2)(\lambda-6)(\lambda-10)}{7\,!}x^7 + \cdots.$$

(b). Based on the recurrence relation

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)}\, a_n \,,$$

the series solution will *terminate* as long as λ is a *nonnegative* even integer. If $\lambda = 2m$, then *one or the other* of the solutions in Part (b) will contain at most $m/2+1$ terms. In particular, we obtain the polynomial solutions corresponding to $\lambda = 0, 2, 4, 6, 8, 10$:

$\lambda = 0$	$y_1(x) = 1$
$\lambda = 2$	$y_2(x) = x$
$\lambda = 4$	$y_1(x) = 1 - 2x^2$
$\lambda = 6$	$y_2(x) = x - 2x^3/3$
$\lambda = 8$	$y_1(x) = 1 - 4x^2 + 4x^4/3$
$\lambda = 10$	$y_2(x) = x - 4x^3/3 + 4x^5/15$

(c). Observe that if $\lambda = 2n$, and $a_0 = a_1 = 1$, then

$$a_{2k} = (\,-1)^k\, \frac{2n\cdots(2n-4k+8)(2n-4k+4)}{(2k)!}$$

and

$$a_{2k+1} = (\,-1)^k\, \frac{(2n-2)\cdots(2n-4k+6)(2n-4k+2)}{(2k+1)!}\,.$$

for $k = 1, 2, \cdots [n/2]$. It follows that the *coefficient* of x^n, in y_1 and y_2, is

$$a_n = \begin{cases} (\,-1)^k\, \frac{4^k\, k!}{(2k)!} & \text{for } n = 2k \\ (\,-1)^k\, \frac{4^k\, k!}{(2k+1)!} & \text{for } n = 2k+1 \end{cases}$$

Then by definition,

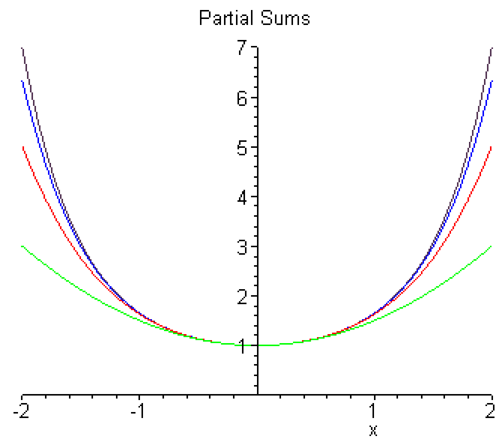
$$H_n(x) = \begin{cases} (\,-1)^k\, 2^n \frac{(2k)!}{4^k\, k!} y_1(x) = (\,-1)^k\, \frac{(2k)!}{k!} y_1(x) & \text{for } n = 2k \\ (\,-1)^k\, 2^n \frac{(2k+1)!}{4^k\, k!} y_2(x) = (\,-1)^k\, \frac{2\,(2k+1)!}{k!} y_2(x) & \text{for } n = 2k+1 \end{cases}$$

Therefore the first six *Hermite polynomials* are

$$\begin{aligned} &H_0(x) = 1 \\ &H_1(x) = 2x \\ &H_2(x) = 4x^2 - 2 \\ &H_3(x) = 8x^8 - 12x \\ &H_4(x) = 16x^4 - 48x^2 + 12 \\ &H_5(x) = 32x^5 - 160x^3 + 120x \end{aligned}$$

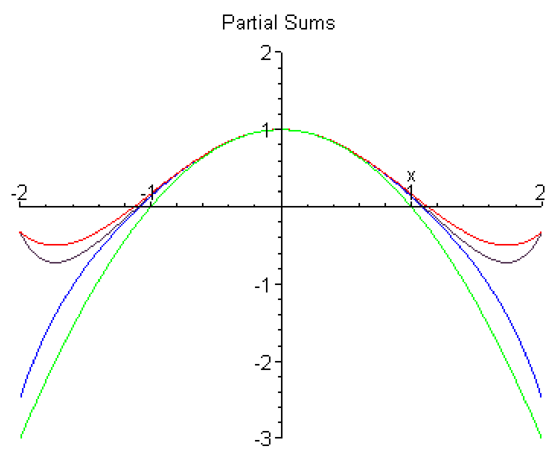
23. The series solution is given by

$$y(x) = 1 + \frac{1}{2}x^2 + \frac{1}{2^2\,2!}x^4 + \frac{1}{2^3 3!}x^6 + \frac{1}{2^4 4!}x^8 + \cdots.$$



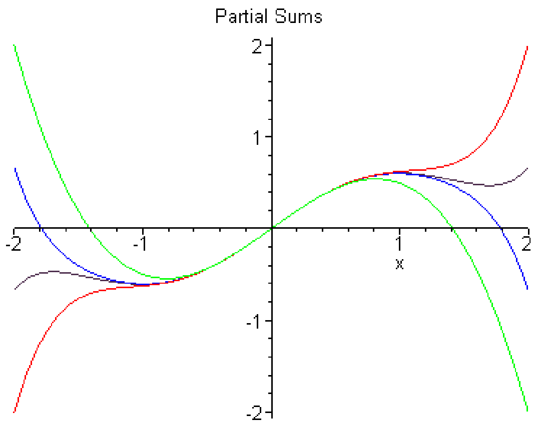
24. The series solution is given by

$$y(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \frac{x^8}{120} + \cdots.$$



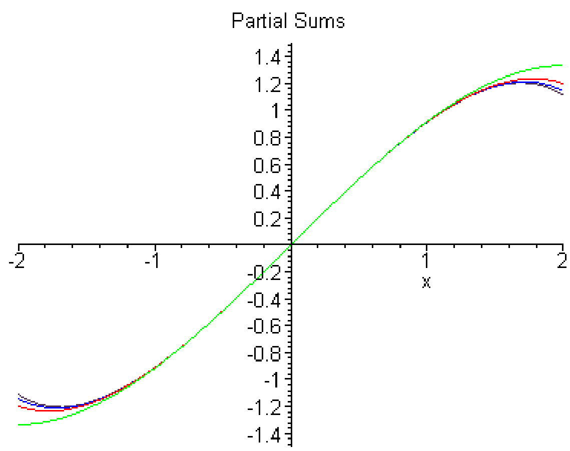
25. The series solution is given by

$$y(x) = x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \frac{x^9}{2\cdot 4\cdot 6\cdot 8} - \cdots.$$



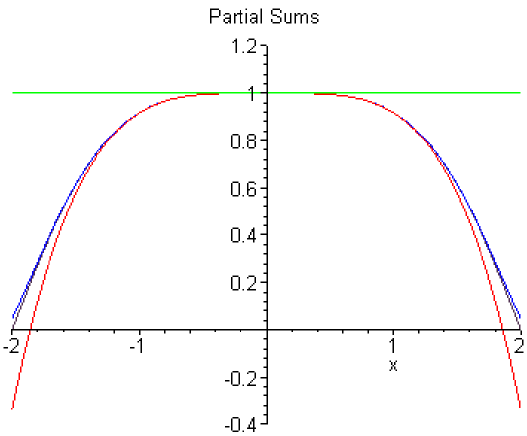
26. The series solution is given by

$$y(x) = x - \frac{x^3}{12} - \frac{x^5}{240} - \frac{x^7}{2240} - \frac{x^9}{16128} - \cdots.$$



27. The series solution is given by

$$y(x) = 1 - \frac{x^4}{12} + \frac{x^8}{672} - \frac{x^{12}}{88704} + \cdots.$$



28. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then

$$y' = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n.$$

Substitution into the ODE results in

$$(1-x)\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2\sum_{n=0}^{\infty}a_nx^n = 0\,.$$

After appropriately shifting the indices, it follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + n\,a_n - 2\,a_n\right]x^n = 0.$$

We find that $a_2 = a_0$ and

$$(n+2)(n+1)a_{n+2} - (n+1)n\,a_{n+1} + (n-2)a_n = 0$$

for $n = 1, 2, \cdots$. Writing out the individual equations,

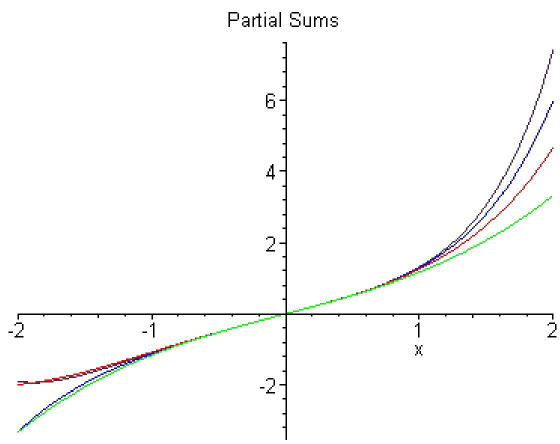
$$\begin{aligned}
3\cdot 2\,a_3 - 2\cdot 1\,a_2 - a_1 &= 0\\
4\cdot 3\,a_4 - 3\cdot 2\,a_3 &= 0\\
5\cdot 4\,a_5 - 4\cdot 3\,a_4 + a_3 &= 0\\
6\cdot 5\,a_6 - 5\cdot 4\,a_5 + 2\,a_4 &= 0\\
&\vdots
\end{aligned}$$

Since $a_0 = 0$ and $a_1 = 1$, the remaining coefficients satisfy the equations

$$
\begin{aligned}
3 \cdot 2\, a_3 - 1 &= 0 \\
4 \cdot 3\, a_4 - 3 \cdot 2\, a_3 &= 0 \\
5 \cdot 4\, a_5 - 4 \cdot 3\, a_4 + a_3 &= 0 \\
6 \cdot 5\, a_6 - 5 \cdot 4\, a_5 + 2\, a_4 &= 0 \\
&\vdots
\end{aligned}
$$

That is, $a_3 = 1/6\,, a_4 = 1/12\,, a_5 = 1/24\,, a_6 = 1/45\,, \cdots$. Hence the series solution of the initial value problem is

$$y(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{24}x^5 + \frac{1}{45}x^6 + \frac{13}{1008}x^7 + \cdots.$$



Section 5.3

2. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -(sin\,x)y' - (cos\,x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -(sin\,x)y'' - 2(cos\,x)y' + (sin\,x)y \\ y^{iv} &= -(sin\,x)y''' - 3(cos\,x)y'' + 3(sin\,x)y' + (cos\,x)y\,. \end{aligned}$$

Given that $\phi(0) = 0$ and $\phi'(0) = 1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -2$ and $\phi^{iv}(0) = 0$.

3. Let $y = \phi(x)$ be a solution of the initial value problem. First write

$$y'' = -\frac{1+x}{x^2}\,y' - \frac{3\,ln\,x}{x^2}\,y\,.$$

Differentiating twice,

$$y''' = \frac{-1}{x^3}\left[\left(x + x^2\right)y'' + (3x\,ln\,x - x - 2)y' + (3 - 6\,ln\,x)y\right].$$

$$\begin{aligned} y^{iv} = \frac{-1}{x^4}\Big[&\left(x^2 + x^3\right)y''' + \left(3x^2 ln\,x - 2x^2 - 4x\right)y'' + \\ &+ (6 + 8x - 12x\,ln\,x)y' + (18\,ln\,x - 15)y\Big]. \end{aligned}$$

Given that $\phi(1) = 2$ and $\phi'(1) = 0$, the *first* equation gives $\phi''(1) = 0$ and the last two equations give $\phi'''(0) = -6$ and $\phi^{iv}(0) = 42$.

4. Let $y = \phi(x)$ be a solution of the initial value problem. First note that

$$y'' = -x^2\,y' - (sin\,x)y\,.$$

Differentiating twice,

$$\begin{aligned} y''' &= -x^2\,y'' - (2x + sin\,x)y' - (cos\,x)y \\ y^{iv} &= -x^2\,y''' - (4x + sin\,x)y'' - (2 + 2cos\,x)y' + (sin\,x)y\,. \end{aligned}$$

Given that $\phi(0) = a_0$ and $\phi'(0) = a_1$, the *first* equation gives $\phi''(0) = 0$ and the last two equations give $\phi'''(0) = -a_0$ and $\phi^{iv}(0) = -4a_1$.

5. Clearly, $p(x) = 4$ and $q(x) = 6x$ are analytic for all x. Hence the series solutions converge *everywhere*.

7. The *zeroes* of $P(x) = 1 + x^3$ are the *three* cube roots of -1. They all lie on the unit circle in the complex plane. So for $x_0 = 0$, $\rho_{min} = 1$. For $x_0 = 2$, the *nearest*

root is $e^{i\pi/3} = \left(1 + i\sqrt{3}\right)/2$, hence $\rho_{min} = \sqrt{3}$.

8. The only root of $P(x) = x$ is *zero*. Hence $\rho_{min} = 1$.

9(b). $p(x) = -x$ and $q(x) = -1$ are analytic for all x.
(c). $p(x) = -x$ and $q(x) = -1$ are analytic for all x.
(d). $p(x) = 0$ and $q(x) = kx^2$ are analytic for all x.
(e). The only root of $P(x) = 1 - x$ is 1. Hence $\rho_{min} = 1$.
(g). $p(x) = x$ and $q(x) = 2$ are analytic for all x.
(i). The *zeroes* of $P(x) = 1 + x^2$ are $\pm i$. Hence $\rho_{min} = 1$.
(j). The *zeroes* of $P(x) = 4 - x^2$ are ± 2. Hence $\rho_{min} = 2$.
(k). The *zeroes* of $P(x) = 3 - x^2$ are $\pm\sqrt{3}$. Hence $\rho_{min} = \sqrt{3}$.
(l). The only root of $P(x) = 1 - x$ is 1. Hence $\rho_{min} = 1$.
(m). $p(x) = x/2$ and $q(x) = 3/2$ are analytic for all x.
(n). $p(x) = (1 + x)/2$ and $q(x) = 3/2$ are analytic for all x.

12. The Taylor series expansion of e^x, about $x_0 = 0$, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n\right] + x\sum_{n=0}^{\infty} a_n x^n = 0.$$

First note that

$$x\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \cdots + a_{n-1} x^n + \cdots.$$

The coefficient of x^n in the *product* of the two series is

$$c_n = 2a_2 \frac{1}{n!} + 6a_3 \frac{1}{(n-1)!} + 12a_4 \frac{1}{(n-2)!} + \cdots + (n+1)n\, a_{n+1} + (n+2)(n+1)a_{n+2}.$$

Expanding the individual series, it follows that

$$2a_2 + (2a_2 + 6a_3)x + (a_2 + 6a_3 + 12a_4)x^2 + (a_2 + 6a_3 + 12a_4 + 20a_5)x^3 + \cdots +$$
$$+ a_0 x + a_1 x^2 + a_2 x^3 + \cdots = 0.$$

Setting the coefficients equal to *zero*, we obtain the system $2a_2 = 0$, $2a_2 + 6a_3 + a_0 = 0$, $a_2 + 6a_3 + 12a_4 + a_1 = 0$, $a_2 + 6a_3 + 12a_4 + 20a_5 + a_2 = 0, \cdots$. Hence the general solution is

$$y(x) = a_0 + a_1 x - a_0 \frac{x^3}{6} + (a_0 - a_1)\frac{x^4}{12} + (2a_1 - a_0)\frac{x^5}{40} + \left(\tfrac{4}{3}a_0 - 2a_1\right)\frac{x^6}{120} + \cdots .$$

We find that two linearly independent solutions are

$$y_1(x) = 1 - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{40} + \cdots$$

$$y_2(x) = x - \frac{x^4}{12} + \frac{x^5}{20} - \frac{x^6}{60} + \cdots$$

Since $p(x) = 0$ and $q(x) = xe^{-x}$ converge everywhere, $\rho = \infty$.

13. The Taylor series expansion of $cos\, x$, about $x_0 = 0$, is

$$cos\, x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)\,!} .$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)\,!}\right]\left[\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\, x^n\right] + \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0 .$$

The coefficient of x^n in the *product* of the two series is

$$c_n = 2a_2 b_n + 6a_3 b_{n-1} + 12a_4 b_{n-2} + \cdots + (n+1)n\, a_{n+1} b_1 + (n+2)(n+1)a_{n+2} b_0 ,$$

in which $cos\, x = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$. It follows that

$$2a_2 - 2a_0 + \sum_{n=1}^{\infty} c_n x^n + \sum_{n=1}^{\infty}(n-2)a_n x^n = 0 .$$

Expanding the product of the series, it follows that

$$2a_2 - 2a_0 + 6a_3 x + (-a_2 + 12a_4)x^2 + (-3a_3 + 20a_5)x^3 + \cdots -$$
$$- a_1 x + a_3 x^3 + 2a_4 x^4 + \cdots = 0 .$$

Setting the coefficients equal to *zero*, $a_2 - a_0 = 0$, $6a_3 - a_1 = 0$, $-a_2 + 12a_4 = 0$, $-3a_3 + 20a_5 + a_3 = 0$, $\cdots$. Hence the general solution is

$$y(x) = a_0 + a_1 x + a_0 x^2 + a_1 \frac{x^3}{6} + a_0 \frac{x^4}{12} + a_1 \frac{x^5}{60} + a_0 \frac{x^6}{120} + a_1 \frac{x^7}{560} + \cdots .$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + x^2 + \frac{x^4}{12} + \frac{x^6}{120} + \cdots$$

$$y_2(x) = x + \frac{x^3}{6} + \frac{x^5}{60} + \frac{x^7}{560} + \cdots$$

The *nearest* zero of $P(x) = cos\, x$ is at $x = \pm\pi/2$. Hence $\rho_{min} = \pi/2$.

14. The Taylor series expansion of $ln(1+x)$, about $x_0 = 0$, is

$$ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n\,!}\right] \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n +$$
$$+ \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}\right] \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - x \sum_{n=0}^{\infty} a_n x^n = 0.$$

The *first* product is the series

$$2a_2 + (-2a_2 + 6a_3)x + (a_2 - 6a_3 + 12a_4)x^2 + (-a_2 + 6a_3 - 12a_4 + 20a_5)x^3 + \cdots.$$

The *second* product is the series

$$a_1 x + (2a_2 - a_1/2)x^2 + (3a_3 - a_2 + a_1/3)x^3 + (4a_4 - 3a_3/2 + 2a_2/3 - a_1/4)x^3 + \cdots.$$

Combining the series and equating the coefficients to *zero*, we obtain

$$\begin{aligned} 2a_2 &= 0 \\ -2a_2 + 6a_3 + a_1 - a_0 &= 0 \\ 12a_4 - 6a_3 + 3a_2 - 3a_1/2 &= 0 \\ 20a_5 - 12a_4 + 9a_3 - 3a_2 + a_1/3 &= 0 \\ &\vdots \end{aligned}$$

Hence the general solution is

$$y(x) = a_0 + a_1 x + (a_0 - a_1)\frac{x^3}{6} + (2a_0 + a_1)\frac{x^4}{24} + a_1 \frac{7x^5}{120} + \left(\frac{5}{3}a_1 - a_0\right)\frac{x^6}{120} + \cdots.$$

We find that two linearly independent solutions are

$$y_1(x) = 1 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{120} + \cdots$$

$$y_2(x) = x - \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + \cdots$$

The coefficient $p(x) = e^x ln(1+x)$ is analytic at $x_0 = 0$, but its power series has a radius of convergence $\rho = 1$.

15. If $y_1 = x$ and $y_2 = x^2$ are solutions, then substituting y_2 into the ODE results in

$$2\,P(x) + 2x\,Q(x) + x^2 R(x) = 0\,.$$

Setting $x = 0$, we find that $P(0) = 0$. Similarly, substituting y_1 into the ODE results in $Q(0) = 0$. Therefore $P(x)/Q(x)$ and $R(x)/P(x)$ may not be analytic. If they were, Theorem 3.2.1 would guarantee that y_1 and y_2 were the *only* two solutions. But note that an *arbitrary* value of $y(0)$ cannot be a linear combination of $y_1(0)$ and $y_2(0)$. Hence $x_0 = 0$ must be a singular point.

16. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}\left[(n+1)a_{n+1} - a_n\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain

$$a_{n+1} = \frac{a_n}{n+1}$$

for $n = 0, 1, 2, \cdots$. It is easy to see that $a_n = a_0/(n\,!)$. Therefore the general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots\right] \\ &= a_0 e^x. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

17. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - x\sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

Combining the series, we have

$$a_1 + \sum_{n=1}^{\infty}[(n+1)a_{n+1} - a_{n-1}]\, x^n = 0\,.$$

Setting the coefficient equal to *zero*, $a_1 = 0$ and $a_{n+1} = a_{n-1}/(n+1)$ for $n = 1, 2, \cdots$. Note that the indices differ by *two*, so for $k = 1, 2, \cdots$

$$a_{2k} = \frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{a_0}{2 \cdot 4 \cdots (2k)}$$

and

$$a_{2k+1} = 0\,.$$

Hence the general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} + \frac{x^6}{2^3\,3\,!} + \cdots + \frac{x^{2n}}{2^n\,n\,!} + \cdots\right] \\ &= a_0 exp\big(x^2/2\big). \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

19. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Substituting into the ODE,

$$(1-x)\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=0}^{\infty} a_nx^n = 0\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\, x^n - \sum_{n=1}^{\infty} n\, a_n\, x^n - \sum_{n=0}^{\infty} a_nx^n = 0\,.$$

Combining the series, we have

$$a_1 - a_0 + \sum_{n=1}^{\infty}[(n+1)a_{n+1} - n\, a_n - a_n]\, x^n = 0\,.$$

Setting the coefficients equal to *zero*, $a_1 = a_0$ and $a_{n+1} = a_n$ for $n = 0, 1, 2, \cdots$. Hence the general solution is

$$\begin{aligned} y(x) &= a_0\left[1 + x + x^2 + x^3 + \cdots + x^n + \cdots\right] \\ &= a_0 \frac{1}{1-x}\,. \end{aligned}$$

The coefficient $a_0 = y(0)$, which can be arbitrary.

21. Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Substituting into the ODE,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + x\sum_{n=0}^{\infty}a_n x^n = 1 + x\,.$$

That is,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}\,x^n + \sum_{n=1}^{\infty}a_{n-1}x^n = 1 + x\,.$$

Combining the series, and the nonhomogeneous terms, we have

$$(a_1 - 1) + (2a_2 + a_0 - 1)x + \sum_{n=2}^{\infty}\left[(n+1)a_{n+1} + a_{n-1}\right]x^n = 0\,.$$

Setting the coefficients equal to *zero*, we obtain $a_1 = 1$, $2a_2 + a_0 - 1 = 0$, and

$$a_n = -\,\frac{a_{n-2}}{n}\,, \quad n = 3, 4, \cdots.$$

The indices differ by *two*, so for $k = 2, 3, \cdots$

$$a_{2k} = -\,\frac{a_{2k-2}}{(2k)} = \frac{a_{2k-4}}{(2k-2)(2k)} = \cdots = \frac{(-1)^{k-1}\,a_2}{4\cdot 6\cdots(2k)} = \frac{(-1)^k(a_0 - 1)}{2\cdot 4\cdot 6\cdots(2k)}\,,$$

and for $k = 1, 2, \cdots$

$$a_{2k+1} = \, = -\,\frac{a_{2k-1}}{(2k+1)} = \frac{a_{2k-3}}{(2k-1)(2k+1)} = \cdots = \frac{(-1)^k}{3\cdot 5\cdots(2k+1)}\,.$$

Hence the general solution is

$$y(x) = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{x^3}{3} + a_0\frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} - a_0\frac{x^6}{2^3\,3\,!} - \cdots$$

Collecting the terms containing a_0,

$$\begin{aligned} y(x) \,&= a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{2^2\,2\,!} - \frac{x^6}{2^3\,3\,!} + \cdots\right] + \\ &+ \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right]. \end{aligned}$$

Upon inspection, we find that

$$y(x) = a_0 exp(-x^2/2) + \left[x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{2^2\,2\,!} + \frac{x^5}{3\cdot 5} + \frac{x^6}{2^3\,3\,!} - \frac{x^7}{3\cdot 5\cdot 7} + \cdots\right].$$

Note that the given ODE is *first order linear*, with integrating factor $\mu(t) = e^{x^2/2}$. The general solution is given by

$$y(x) = e^{-x^2/2}\int_0^x e^{u^2/2}du + (y(0)-1)e^{-x^2/2} + 1\,.$$

23. If $\alpha = 0$, then $y_1(x) = 1$. If $\alpha = 2n$, then $a_{2m} = 0$ for $m \geq n+1$. As a result,

$$y_1(x) = 1 + \sum_{m=1}^{n}(-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+1)(2n+3)\cdots(2n+2m-1)}{(2m)!}\,x^{2m}.$$

$\alpha = 0$	1
$\alpha = 2$	$1 - 3x^2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$

If $\alpha = 2n+1$, then $a_{2m+1} = 0$ for $m \geq n+1$. As a result,

$$y_2(x) = x + \sum_{m=1}^{n}(-1)^m \frac{2^m n(n-1)\cdots(n-m+1)(2n+3)(2n+5)\cdots(2n+2m+1)}{(2m+1)!}\,x^{2m+1}.$$

$\alpha = 1$	x
$\alpha = 3$	$x - \frac{5}{3}x^3$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$

24(a). Based on Prob. 23,

$\alpha = 0$	1	$y_1(1) = 1$
$\alpha = 2$	$1 - 3x^2$	$y_1(1) = -2$
$\alpha = 4$	$1 - 10x^2 + \frac{35}{3}x^4$	$y_1(1) = \frac{8}{3}$

Normalizing the polynomials, we obtain

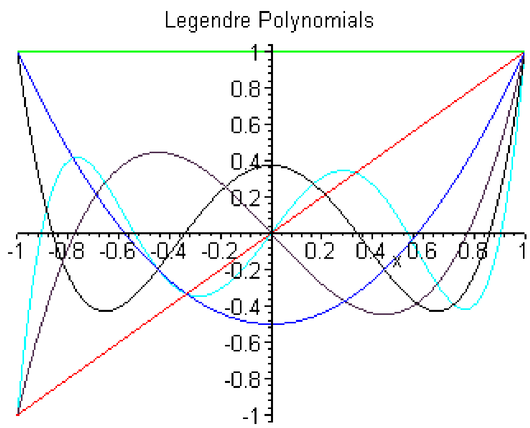
$$P_0(x) = 1$$
$$P_2(x) = -\frac{1}{2} + \frac{3}{2}x^2$$
$$P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

$\alpha = 1$	x	$y_2(1) = 1$
$\alpha = 3$	$x - \frac{5}{3}x^3$	$y_2(1) = -\frac{2}{3}$
$\alpha = 5$	$x - \frac{14}{3}x^3 + \frac{21}{5}x^5$	$y_2(1) = \frac{8}{15}$

Similarly,

$$P_1(x) = x$$
$$P_3(x) = -\frac{3}{2}x + \frac{5}{2}x^3$$
$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

(b).



(c). $P_0(x)$ has no roots. $P_1(x)$ has one root at $x = 0$. The *zeros* of $P_2(x)$ are at $x = \pm 1/\sqrt{3}$. The *zeros* of $P_3(x)$ are $x = 0, \pm\sqrt{3/5}$. The roots of $P_4(x)$ are given by $x^2 = \left(15 + 2\sqrt{30}\right)/35$, $\left(15 - 2\sqrt{30}\right)/35$. The roots of $P_5(x)$ are given by $x = 0$ and $x^2 = \left(35 + 2\sqrt{70}\right)/63$, $\left(35 - 2\sqrt{70}\right)/63$.

25. Observe that

$$\begin{aligned} P_n(-1) &= \frac{(-1)^n}{2^n} \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} \\ &= (-1)^n P_n(1). \end{aligned}$$

But $P_n(1) = 1$ for *all* nonnegative integers n.

27. We have

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}\, x^{2k},$$

which is a polynomial of *degree* $2n$. Differentiating *n times*,

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{k=\mu}^{n} \frac{(-1)^{n-k}\, n!}{k!(n-k)!}(2k)(2k-1)\cdots(2k-n+1)x^{2k-n},$$

in which the *lower index* is $\mu = [n/2] + 1$. Note that if $n = 2m+1$, then $\mu = m+1$.

Now shift the index, by setting

$$k = n - j\,.$$

Hence

$$\begin{aligned}\frac{d^n}{dx^n}\left(x^2-1\right)^n &= \sum_{j=0}^{[n/2]}\frac{(-1)^j\,n!}{(n-j)!j!}(2n-2j)(2n-2j-1)\cdots(n-2j+1)x^{n-2j}\\ &= n!\sum_{j=0}^{[n/2]}\frac{(-1)^j(2n-2j)!}{(n-j)!j!(n-2j)!}\,x^{n-2j}.\end{aligned}$$

Based on Prob. 25,

$$\frac{d^n}{dx^n}\left(x^2-1\right)^n = n!\,2^n P_n(x).$$

29. Since the $n+1$ polynomials $P_0, P_1, \cdots, P_n$ are *linearly independent*, and the *degree* of P_k is k, any polynomial, f, of degree n can be expressed as a linear combination

$$f(x) = \sum_{k=0}^{n} a_k P_k(x)\,.$$

Multiplying both sides by P_m and integrating,

$$\int_{-1}^{1} f(x)P_m(x)dx = \sum_{k=0}^{n} a_k \int_{-1}^{1} P_k(x)P_m(x)dx\,.$$

Based on Prob. 28,

$$\int_{-1}^{1} P_k(x)P_m(x)dx = \frac{2}{2m+1}\delta_{km}.$$

Hence

$$\int_{-1}^{1} f(x)P_m(x)dx = \frac{2}{2m+1}\,a_m\,.$$

Section 5.4

2. We see that $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no factors
in common, both of these points are singular points. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{2x}{x^2 (1-x)^2} = 2\,.$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{4}{x^2 (1-x)^2} = 4\,.$$

The singular point $x = 0$ is *regular*. Considering $x = 1$,

$$\lim_{x \to 1} (x-1) p(x) = \lim_{x \to 1} (x-1) \frac{2x}{x^2 (1-x)^2}\,.$$

The latter limit *does not exist*. Hence $x = 1$ is an *irregular* singular point.

3. $P(x) = 0$ when $x = 0$ and 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{x-2}{x^2 (1-x)}\,.$$

The limit *does not exist*, and so $x = 0$ is an *irregular* singular point. Considering $x = 1$,

$$\lim_{x \to 1} (x-1) p(x) = \lim_{x \to 1} (x-1) \frac{x-2}{x^2 (1-x)} = 1\,.$$

$$\lim_{x \to 1} (x-1)^2 q(x) = \lim_{x \to 1} (x-1)^2 \frac{-3x}{x^2 (1-x)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

4. $P(x) = 0$ when $x = 0$ and ± 1. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{2}{x^3 (1-x^2)}\,.$$

The limit *does not exist*, and so $x = 0$ is an *irregular* singular point. Near $x = -1$,

$$\lim_{x \to -1} (x+1) p(x) = \lim_{x \to -1} (x+1) \frac{2}{x^3 (1-x^2)} = -1\,.$$

$$\lim_{x\to-1}(x+1)^2q(x)=\lim_{x\to-1}(x+1)^2\frac{2}{x^3(1-x^2)}=0\,.$$

Hence $x=\,-1$ is a *regular* singular point. At $x=1$,

$$\lim_{x\to1}(x-1)p(x)=\lim_{x\to1}(x-1)\frac{2}{x^3(1-x^2)}=\,-1\,.$$

$$\lim_{x\to1}(x-1)^2q(x)=\lim_{x\to1}(x-1)^2\frac{2}{x^3(1-x^2)}=0\,.$$

Hence $x=1$ is a *regular* singular point.

6. The only singular point is at $x=0$. We find that

$$\lim_{x\to0}x\,p(x)=\lim_{x\to0}x\frac{x}{x^2}=1\,.$$

$$\lim_{x\to0}x^2q(x)=\lim_{x\to0}x^2\,\frac{x^2-\nu^2}{x^2}=\,-\nu^2.$$

Hence $x=0$ is a *regular* singular point.

7. The only singular point is at $x=\,-3$. We find that

$$\lim_{x\to-3}(x+3)p(x)=\lim_{x\to-3}(x+3)\frac{-2x}{x+3}=6\,.$$

$$\lim_{x\to-3}(x+3)^2q(x)=\lim_{x\to-3}(x+3)^2\,\frac{1-x^2}{x+3}=0.$$

Hence $x=\,-3$ is a *regular* singular point.

8. Dividing the ODE by $x(1-x^2)^3$, we find that

$$p(x)=\frac{1}{x(1-x^2)}\quad\text{and}\quad q(x)=\frac{2}{x(1+x)^2(1-x)^3}\,.$$

The singular points are at $x=0$ and ± 1. For $x=0$,

$$\lim_{x\to0}x\,p(x)=\lim_{x\to0}x\frac{1}{x(1-x^2)}=1\,.$$

$$\lim_{x\to0}x^2q(x)=\lim_{x\to0}x^2\,\frac{2}{x(1+x)^2(1-x)^3}=0\,.$$

Hence $x=0$ is a *regular* singular point. For $x=\,-1$,

$$\lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}(x+1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}(x+1)^2\,\frac{2}{x(1+x)^2(1-x)^3} = -\frac{1}{4}\,.$$

Hence $x = -1$ is a *regular* singular point. For $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{1}{x(1-x^2)} = -\frac{1}{2}\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\,\frac{2}{x(1+x)^2(1-x)^3}\,.$$

The latter limit *does not exist*. Hence $x = 1$ is an *irregular* singular point.

9. Dividing the ODE by $(x+2)^2(x-1)$, we find that

$$p(x) = \frac{3}{(x+2)^2} \quad \text{and} \quad q(x) = \frac{-2}{(x+2)(x-1)}\,.$$

The singular points are at $x = -2$ and 1. For $x = -2$,

$$\lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}(x+2)\frac{3}{(x+2)^2}\,.$$

The limit *does not exist*. Hence $x = -2$ is an *irregular* singular point. For $x = 1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{3}{(x+2)^2} = 0\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\,\frac{-2}{(x+2)(x-1)} = 0\,.$$

Hence $x = 1$ is a *regular* singular point.

10. $P(x) = 0$ when $x = 0$ and 3. Since the three coefficients have no common factors, both of these points are singular points. Near $x = 0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{x+1}{x(3-x)} = \frac{1}{3}\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\frac{-2}{x(3-x)} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. For $x = 3$,

$$\lim_{x\to 3}(x-3)p(x) = \lim_{x\to 3}(x-3)\frac{x+1}{x(3-x)} = -\frac{4}{3}\,.$$

$$\lim_{x\to 3}(x-3)^2 q(x) = \lim_{x\to 3}(x-3)^2\frac{-2}{x(3-x)} = 0\,.$$

Hence $x=3$ is a *regular* singular point.

11. Dividing the ODE by (x^2+x-2), we find that

$$p(x) = \frac{x+1}{(x+2)(x-1)} \quad \text{and} \quad q(x) = \frac{2}{(x+2)(x-1)}\,.$$

The singular points are at $x=-2$ and 1. For $x=-2$,

$$\lim_{x\to -2}(x+2)p(x) = \lim_{x\to -2}\frac{x+1}{x-1} = \frac{1}{3}\,.$$

$$\lim_{x\to -2}(x+2)^2 q(x) = \lim_{x\to -2}\frac{2(x+2)}{x-1} = 0\,.$$

Hence $x=-2$ is a *regular* singular point. For $x=1$,

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{x+1}{x+2} = \frac{2}{3}\,.$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\frac{2(x-1)}{(x+2)} = 0\,.$$

Hence $x=1$ is a *regular* singular point.

13. Note that $p(x) = ln|x|$ and $q(x)=3x$. Evidently, $p(x)$ is *not* analytic at $x_0=0$. Furthermore, the function $x\,p(x) = x\,ln|x|$ does *not* have a Taylor series about $x_0=0$. Hence $x=0$ is an *irregular* singular point.

14. $P(x)=0$ when $x=0$. Since the three coefficients have no common factors, $x=0$ is a singular point. The Taylor series of e^x-1, about $x=0$, is

$$e^x - 1 = x + x^2/2 + x^3/6 + \cdots.$$

Hence the function $x\,p(x) = 2(e^x-1)/x$ is analytic at $x=0$. Similarly, the Taylor series of $e^{-x}cos\,x$, about $x=0$, is

$$e^{-x}cos\,x = 1 - x + x^3/3 - x^4/6 + \cdots.$$

The function $x^2q(x) = e^{-x}cos\,x$ is also analytic at $x=0$. Hence $x=0$ is a *regular* singular point.

15. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. The Taylor series of $sin\,x$, about $x = 0$, is

$$sin\,x = x - x^3/3! + x^5/5! - \cdots .$$

Hence the function $x\,p(x) = -3sin\,x/x$ is analytic at $x = 0$. On the other hand, $q(x)$ is a rational function, with

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2 \frac{1+x^2}{x^2} = 1\,.$$

Hence $x = 0$ is a *regular* singular point.

16. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. We find that

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{1}{x} = 1\,.$$

Although the function $R(x) = cot\,x$ does not have a Taylor series about $x = 0$, note that $x^2 q(x) = x\,cot\,x = 1 - x^2/3 - x^4/45 - 2x^6/945 - \cdots$. Hence $x = 0$ is a *regular* singular point. Furthermore, $q(x) = cot\,x/x^2$ is undefined at $x = \pm\,n\pi$. Therefore the points $x = \pm\,n\pi$ are *also* singular points. First note that

$$\lim_{x\to \pm n\pi} (x\mp n\pi)p(x) = \lim_{x\to \pm n\pi} (x\mp n\pi)\frac{1}{x} = 0\,.$$

Furthermore, since $cot\,x$ has period π,

$$\begin{aligned} q(x) &= cot\,x/x = cot(x\mp n\pi)/x \\ &= cot(x\mp n\pi)\frac{1}{(x\mp n\pi)\pm n\pi}\,. \end{aligned}$$

Therefore

$$(x\mp n\pi)^2 q(x) = (x\mp n\pi)cot(x\mp n\pi)\left[\frac{(x\mp n\pi)}{(x\mp n\pi)\pm n\pi}\right].$$

From above,

$$(x\mp n\pi)cot(x\mp n\pi) = 1 - (x\mp n\pi)^2/3 - (x\mp n\pi)^4/45 - \cdots .$$

Note that the function in *brackets* is analytic *near* $x = \pm\,n\pi$. It follows that the function $(x\mp n\pi)^2 q(x)$ is also analytic near $x = \pm\,n\pi$. Hence all the singular points are *regular*.

18. The singular points are located at $x = \pm\,n\pi$, $n = 0, 1, \cdots$. Dividing the ODE by $x\,sin\,x$, we find that $x\,p(x) = 3\,csc\,x$ and $x^2 q(x) = x^2 csc\,x$. Evidently, $x\,p(x)$ is not even defined at $x = 0$. Hence $x = 0$ is an *irregular* singular point. On the other hand, the Taylor series of $x\,csc\,x$, about $x = 0$, is

$$x\,csc\,x = 1 + x^2/6 + 7x^4 360 + \cdots.$$

Noting that $csc(x \mp n\pi) = (-1)^n csc\,x$,

$$\begin{aligned}(x \mp n\pi)p(x) &= 3(-1)^n (x \mp n\pi) csc(x \mp n\pi)/x \\ &= 3(-1)^n (x \mp n\pi) csc(x \mp n\pi)\left[\frac{1}{(x \mp n\pi) \pm n\pi}\right].\end{aligned}$$

It is apparent that $(x \mp n\pi)p(x)$ is analytic at $x = \pm n\pi$. Similarly,

$$\begin{aligned}(x \mp n\pi)^2 q(x) &= (x \mp n\pi)^2 csc\,x \\ &= (-1)^n (x \mp n\pi)^2 csc(x \mp n\pi),\end{aligned}$$

which is also analytic at $x = \pm n\pi$. Hence all other singular points are *regular*.

20. $x = 0$ is the only singular point. Dividing the ODE by $2x^2$, we have $p(x) = 3/(2x)$ and $q(x) = -x^{-2}(1+x)/2$. It follows that

$$\lim_{x \to 0} x\,p(x) = \lim_{x \to 0} x \frac{3}{2x} = \frac{3}{2},$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{-(1+x)}{2x^2} = -\frac{1}{2}.$$

Hence $x = 0$ is a *regular* singular point. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$. Substitution into the ODE results in

$$2x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n + 3x \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n - (1+x) \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$2\sum_{n=2}^{\infty} n(n-1)a_n\, x^n + 3\sum_{n=1}^{\infty} n\, a_n x^n - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

It follows that

$$-a_0 + (2a_1 - a_0)x + \sum_{n=2}^{\infty} [2n(n-1)a_n + 3n\, a_n - a_n - a_{n-1}]x^n = 0.$$

Equating the coefficients to *zero*, we find that $a_0 = 0$, $2a_1 - a_0 = 0$, and

$$(2n-1)(n+1)a_n = a_{n-1}, \quad n = 2, 3, \cdots.$$

We conclude that *all* the a_n are *equal to zero*. Hence $y(x) = 0$ is the only solution that can be obtained.

22. Based on Prob. 21, the change of variable, $x = 1/\xi$, transforms the ODE into the

form

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + y = 0 \,.$$

Evidently, $\xi = 0$ is a singular point. Now $p(\xi) = 2/\xi$ and $q(\xi) = 1/\xi^4$. Since the value of $\lim\limits_{\xi \to 0} \xi^2 q(\xi)$ does not exist, $\xi = 0$, that is, $x = \infty$, is an *irregular* singular point.

24. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \left(1 - \frac{1}{\xi^2}\right) \frac{d^2y}{d\xi^2} + \left[2\xi^3 \left(1 - \frac{1}{\xi^2}\right) + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0 \,,$$

that is,

$$\left(\xi^4 - \xi^2\right) \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} + \alpha(\alpha+1)y = 0 \,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2\xi}{\xi^2 - 1} \text{ and } q(\xi) = \frac{\alpha(\alpha+1)}{\xi^2(\xi^2 - 1)} \,.$$

It follows that

$$\lim_{\xi \to 0} \xi \, p(\xi) = \lim_{\xi \to 0} \xi \frac{2\xi}{\xi^2 - 1} = 0 \,,$$

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \frac{\alpha(\alpha+1)}{\xi^2(\xi^2 - 1)} = -\alpha(\alpha+1) \,.$$

Hence $\xi = 0$ $(x = \infty)$ is a *regular* singular point.

26. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + \left[2\xi^3 + 2\xi^2 \frac{1}{\xi}\right] \frac{dy}{d\xi} + \lambda \, y = 0 \,,$$

that is,

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\left(\xi^3 + \xi\right) \frac{dy}{d\xi} + \lambda \, y = 0 \,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2(\xi^2 + 1)}{\xi^3} \text{ and } q(\xi) = \frac{\lambda}{\xi^4} \,.$$

It immediately follows that the limit $\lim\limits_{\xi \to 0} \xi \, p(\xi)$ *does not exist*. Hence $\xi = 0$ $(x = \infty)$

is an *irregular* singular point.

27. Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4 \frac{d^2y}{d\xi^2} + 2\xi^3 \frac{dy}{d\xi} - \frac{1}{\xi} y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{2}{\xi} \text{ and } q(\xi) = \frac{-1}{\xi^5}.$$

We find that

$$\lim_{\xi \to 0} \xi\, p(\xi) = \lim_{\xi \to 0} \xi \frac{2}{\xi} = 2\,,$$

but

$$\lim_{\xi \to 0} \xi^2 q(\xi) = \lim_{\xi \to 0} \xi^2 \frac{(-1)}{\xi^5}\,.$$

The latter limit *does not exist*. Hence $\xi = 0$ $(x = \infty)$ is an *irregular* singular point.

Section 5.5

1. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) + 4r + 2 \\ &= r^2 + 3r + 2\,. \end{aligned}$$

The roots are $r = -2\,,\ -1$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-2} + c_2\, x^{-1}.$$

3. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - 3r + 4 \\ &= r^2 - 4r + 4\,. \end{aligned}$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x^2.$$

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$\begin{aligned} F(r) &= r(r-1) - r + 1 \\ &= r^2 - 2r + 1\,. \end{aligned}$$

The root is $r = 1$, with multiplicity *two*. Hence the general solution, for $x \neq 0$, is

$$y = (c_1 + c_2\, ln|x|)\, x\,.$$

6. Substitution of $y = (x-1)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 7r + 12\,.$$

The roots are $r = -3\,,\ -4$. Hence the general solution, for $x \neq 1$, is

$$y = c_1 (x-1)^{-3} + c_2 (x-1)^{-4}.$$

7. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 5r - 1\,.$$

The roots are $r = -\left(5 \pm \sqrt{29}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{-\left(5+\sqrt{29}\right)/2} + c_2 |x|^{-\left(5-\sqrt{29}\right)/2}.$$

8. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 3r + 3\,.$$

The roots are complex, with $r = \left(3 \pm i\sqrt{3}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{3/2} cos\left(\frac{\sqrt{3}}{2}\,ln|x|\right) + c_2|x|^{3/2} sin\left(\frac{\sqrt{3}}{2}\,ln|x|\right).$$

10. Substitution of $y = (x-2)^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + 4r + 8\,.$$

The roots are complex, with $r = -2 \pm 2i$. Hence the general solution, for $x \neq 2$, is

$$y = c_1\,(x-2)^{-2} cos(2\,ln|x-2|) + c_2(x-2)^{-2} sin(2\,ln|x-2|).$$

11. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 + r + 4\,.$$

The roots are complex, with $r = -\left(1 \pm i\sqrt{15}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,|x|^{-1/2} cos\left(\frac{\sqrt{15}}{2}\,ln|x|\right) + c_2|x|^{-1/2} sin\left(\frac{\sqrt{15}}{2}\,ln|x|\right).$$

12. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 5r + 4\,.$$

The roots are $r = 1\,, 4$. Hence the general solution, for $x \neq 0$, is

$$y = c_1\,x + c_2\,x^4.$$

14. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = 4r^2 + 4r + 17\,.$$

The roots are complex, with $r = -1/2 \pm 2i$. Hence the general solution, for $x > 0$, is

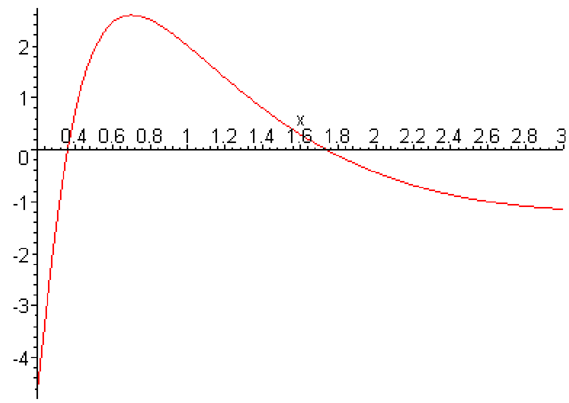
$$y = c_1\,x^{-1/2} cos(2\,ln\,x) + c_2\,x^{-1/2} sin(2\,ln\,x).$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-\frac{1}{2}c_1 + 2c_2 = -3$$

Hence the solution of the initial value problem is

$$y(x) = 2\,x^{-1/2}cos(2\,ln\,x) - x^{-1/2}sin(2\,ln\,x).$$



As $x \to 0^+$, the solution decreases without bound.

15. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where

$$F(r) = r^2 - 4r + 4\,.$$

The root is $r = 2$, with multiplicity *two*. Hence the general solution, for $x < 0$, is

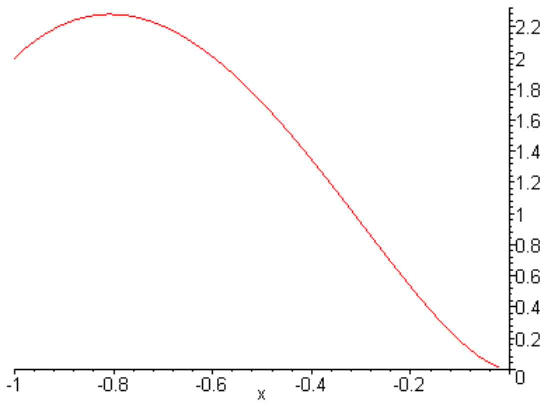
$$y = (c_1 + c_2\,ln\,|x|)\,x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 = 2$$
$$-2c_1 - c_2 = 3$$

Hence the solution of the initial value problem is

$$y(x) = (2 - 7\,ln\,|x|)\,x^2.$$



We find that $y(x) \to 0$ as $x \to 0^-$.

18. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r + \beta = 0$. The roots are

$$r = \frac{1 \pm \sqrt{1 - 4\beta}}{2}.$$

If $\beta > 1/4$, the roots are complex, with $r_{1,2} = \left(1 \pm i\sqrt{4\beta - 1}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2} cos\left(\frac{1}{2}\sqrt{4\beta - 1}\; ln|x|\right) + c_2 |x|^{1/2} sin\left(\frac{1}{2}\sqrt{4\beta - 1}\; ln|x|\right).$$

Since the trigonometric factors are *bounded*, $y(x) \to 0$ as $x \to 0$. If $\beta = 1/4$, the roots are *equal*, and

$$y = c_1 |x|^{1/2} + c_2 |x|^{1/2}\, ln\,|x|\,.$$

Since $\lim_{x \to 0} \sqrt{|x|}\, ln\,|x| = 0$, $y(x) \to 0$ as $x \to 0$. If $\beta < 1/4$, the roots are real, with $r_{1,2} = \left(1 \pm \sqrt{1 - 4\beta}\right)/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{1/2 + \sqrt{1-4\beta}/2} + c_2 |x|^{1/2 - \sqrt{1-4\beta}/2}.$$

Evidently, solutions approach *zero* as long as $1/2 - \sqrt{1 - 4\beta}/2 > 0$. That is,

$$0 < \beta < 1/4\,.$$

Hence *all* solutions approach *zero*, for $\beta > 0$.

19. Substitution of $y = x^r$ results in the quadratic equation $r^2 - r - 2 = 0$. The roots are $r = -1, 2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 x^{-1} + c_2 x^2.$$

Invoking the initial conditions, we obtain the system of equations

$$c_1 + c_2 = 1$$
$$-\,c_1 + 2c_2 = \gamma$$

Hence the solution of the initial value problem is

$$y(x) = \frac{2-\gamma}{3}x^{-1} + \frac{1+\gamma}{3}\,x^2.$$

The solution is *bounded*, as $x \to 0$, if $\gamma = 2$.

20. Substitution of $y = x^r$ results in the quadratic equation $r^2 + (\alpha - 1)r + 5/2 = 0$. Formally, the roots are given by

$$\begin{aligned} r &= \frac{1-\alpha \pm \sqrt{\alpha^2 - 2\alpha - 9}}{2} \\ &= \frac{1-\alpha \pm \sqrt{\left(\alpha - 1 - \sqrt{10}\right)\left(\alpha - 1 + \sqrt{10}\right)}}{2}\,. \end{aligned}$$

(i) The roots $r_{1,2}$ will be *complex*, if $|1-\alpha| < \sqrt{10}$. For solutions to approach *zero*, as $x \to \infty$, we need $-\sqrt{10} < 1 - \alpha < 0$.
(ii) The roots will be *equal*, if $|1-\alpha| = \sqrt{10}$. In this case, all solutions approach *zero* as long as $1 - \alpha = -\sqrt{10}$.
(iii) The roots will be real and *distinct*, if $|1-\alpha| > \sqrt{10}$. It follows that

$$r_{max} = \frac{1-\alpha + \sqrt{\alpha^2 - 2\alpha - 9}}{2}\,.$$

For solutions to approach *zero*, we need $1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9} < 0$. That is, $1 - \alpha < -\sqrt{10}$.

Hence all solutions approach *zero*, as $x \to \infty$, as long as $\alpha > 1$.

23(a). Given that $x = e^z$, $y(x) = y(e^z) = w(z)$. By the chain rule,

$$\frac{dy}{dx} = \frac{d}{dx}w(z) = \frac{dw}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dw}{dz}\,.$$

Similarly,

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dw}{dz}\right] &= -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x}\frac{d^2w}{dz^2}\frac{dz}{dx} \\ &= -\frac{1}{x^2}\frac{dw}{dz} + \frac{1}{x^2}\frac{d^2w}{dz^2}\,. \end{aligned}$$

(b). Direct substitution results in

$$x^2\left[\frac{1}{x^2}\frac{d^2w}{dz^2}-\frac{1}{x^2}\frac{dw}{dz}\right]+\alpha\,x\left[\frac{1}{x}\frac{dw}{dz}\right]+\beta\,w=0\,,$$

that is,

$$\frac{d^2w}{dz^2}+(\alpha-1)\frac{dw}{dz}+\beta\,w=0\,.$$

The associated *characteristic equation* is $r^2+(\alpha-1)r+\beta=0$. Since $z=ln\,x$, it follows that $y(x)=w(ln\,x)$.

(c). If the roots $r_{1,2}$ are real and *distinct*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2e^{r_2z}\\ &= c_1x^{r_1}+c_2x^{r_2}. \end{aligned}$$

(d). If the roots $r_{1,2}$ are real and *equal*, then

$$\begin{aligned} y &= c_1e^{r_1z}+c_2z\,e^{r_1z}\\ &= c_1x^{r_1}+c_2x^{r_1}ln\,x\,. \end{aligned}$$

(e). If the roots are *complex conjugates*, then $r=\lambda\pm i\mu$, and

$$\begin{aligned} y &= e^{\lambda z}(c_1cos\,\mu z+c_2\,sin\,\mu z)\\ &= x^{\lambda}[c_1cos(\mu\,ln\,x)+c_2\,sin(\mu\,ln\,x)]. \end{aligned}$$

24. Based on Prob. 23, the change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}-\frac{dw}{dz}-2w=0\,.$$

The associated *characteristic equation* is $r^2-r-2=0$, with roots $r=-1\,,2$. Hence $w(z)=c_1e^{-z}+c_2e^{2z}$, and $y(x)=c_1x^{-1}+c_2\,x^2$.

26. The change of variable $x=e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2}+6\frac{dw}{dz}+5w=e^z.$$

The associated *characteristic equation* is $r^2+6\,r+5=0$, with roots $r=-5\,,\,-1$. Hence $w_c(z)=c_1e^{-z}+c_2e^{-5z}$. Since the right hand side is *not* a solution of the homogeneous equation, we can use the *method of undetermined coefficients* to show that a particular solution is $W=e^z/12$. Therefore the general solution is given by $w(z)=c_1e^{-z}+c_2e^{-5z}+e^z/12$, that is, $y(x)=c_1x^{-1}+c_2\,x^{-5}+x/12$.

27. The change of variable $x=e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} - 3\frac{dw}{dz} + 2w = 3e^{2z} + 2z\,.$$

The associated *characteristic equation* is $r^2 - 3\,r + 2 = 0$, with roots $r = 1\,,2$. Hence $w_c(z) = c_1e^z + c_2e^{2z}$. Using the *method of undetermined coefficients*, let $W = Ae^{2z} + Bze^{2z} + Cz + D$. It follows that the general solution is given by $w(z) = c_1e^z + c_2e^{2z} + 3ze^{2z} + z + 3/2$, that is,

$$y(x) = c_1x + c_2\,x^2 + 3\,x^2 ln\,x + ln\,x + 3/2\,.$$

28. The change of variable $x = e^z$ transforms the given ODE into

$$\frac{d^2w}{dz^2} + 4w = sin\,z\,.$$

The solution of the homogeneous equation is $w_c(z) = c_1cos\,2z + c_2sin\,2z$. The right hand side is *not* a solution of the homogeneous equation. We can use the *method of undetermined coefficients* to show that a particular solution is $W = \frac{1}{3}sin\,z$. Hence the general solution is given by $w(z) = c_1cos\,2z + c_2sin\,2z + \frac{1}{3}sin\,z$, that is, $y(x) = c_1cos(2\,ln\,x) + c_2sin(2\,ln\,x) + \frac{1}{3}sin(ln\,x)$.

29. After dividing the equation by 3, the change of variable $x = e^z$ transforms the ODE into

$$\frac{d^2w}{dz^2} + 3\frac{dw}{dz} + 3w = 0\,.$$

The associated *characteristic equation* is $r^2 + 3\,r + 3 = 0$, with complex roots $r = -\left(3 \pm i\sqrt{3}\right)/2$. Hence the general solution is

$$w(z) = e^{-3z/2}\left[c_1cos\left(\sqrt{3}\;z/2\right) + c_2sin\left(\sqrt{3}\;z/2\right)\right],$$

and therefore

$$y(x) = x^{-3/2}\left[c_1cos\left(\frac{\sqrt{3}}{2}\,ln\,x\right) + c_2sin\left(\frac{\sqrt{3}}{2}\,ln\,x\right)\right].$$

30. Let $x < 0$. Setting $y = (-x)^r$, successive differentiation gives $y' = -r(-x)^{r-1}$ and $y'' = r(r-1)(-x)^{r-2}$. It follows that

$$L[(-x)^r] = r(r-1)x^2(-x)^{r-2} - \alpha\,r\,x(-x)^{r-1} + \beta(-x)^r.$$

Since $x^2 = (-x)^2$, we find that

$$\begin{aligned} L[(-x)^r] &= r(r-1)(-x)^r + \alpha\, r(-x)^r + \beta(-x)^r \\ &= (-x)^r[r(r-1) + \alpha\, r + \beta]. \end{aligned}$$

Given that r_1 and r_2 are roots of $F(r) = r(r-1) + \alpha\, r + \beta$, we have $L[(-x)^{r_i}] = 0$. Therefore $y_1 = (-x)^{r_1}$ and $y_2 = (-x)^{r_2}$ are *linearly independent* solutions of the differential equation, $L[y] = 0$, for $x < 0$, as long as $r_1 \neq r_2$.

Section 5.6

1. $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. Near $x = 0$,

$$\lim_{x \to 0} x\, p(x) = \lim_{x \to 0} x \frac{1}{2x} = \frac{1}{2}.$$

$$\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{2} = 0.$$

Hence $x = 0$ is a *regular* singular point. Let

$$y = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \right) = \sum_{n=0}^{\infty} a_n\, x^{r+n}.$$

Then

$$y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n-1} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+1} = 0.$$

That is,

$$2\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n} + \sum_{n=0}^{\infty} (r+n) a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2}\, x^{r+n} = 0.$$

It follows that

$$a_0[2r(r-1) + r]x^r + a_1[2(r+1)r + r + 1]x^{r+1} + \\ + \sum_{n=2}^{\infty} [2(r+n)(r+n-1)a_n + (r+n)a_n + a_{n-2}]x^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, we obtain the *indicial equation* $2r^2 - r = 0$, with roots $r_1 = 1/2$

and $r_2 = 0$. It immediately follows that $a_1 = 0$. Setting the remaining coefficients equal
to *zero*, we have

$$a_n = \frac{-\,a_{n-2}}{(r+n)[2(r+n)-1]}\,, \quad n = 2, 3, \cdots.$$

For $r = 1/2$, the recurrence relation becomes

$$a_n = \frac{-\,a_{n-2}}{n(1+2n)}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*. Furthermore, for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(1+4k)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-3)(4k+1)} = \frac{(-1)^k a_0}{2^k\, k!\, 5\cdot 9\cdot 13\cdots(4k+1)}\,.$$

For $r = 0$, the recurrence relation becomes

$$a_n = \frac{-\,a_{n-2}}{n(2n-1)}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*, and for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(4k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(4k-5)(4k-1)} = \frac{(-1)^k a_0}{2^k\, k!\, 3\cdot 7\cdot 11\cdots(4k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x) = \sqrt{x}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k\, x^{2k}}{2^k\, k!\, 5\cdot 9\cdot 13\cdots(4k+1)}\right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k\, x^{2k}}{2^k\, k!\, 3\cdot 7\cdot 11\cdots(4k-1)}\,.$$

3. Note that $x\,p(x) = 0$ and $x^2 q(x) = x$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,,$$

and after multiplying both sides of the equation by x,

$$\sum_{n=0}^{\infty} (r+n)(r+n-1) a_n\, x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0\,.$$

It follows that

$$a_0[r(r-1)]x^r + \sum_{n=1}^{\infty} [(r+n)(r+n-1)a_n + a_{n-1}]x^{r+n} = 0\,.$$

Setting the coefficients equal to *zero*, the *indicial equation* is $r(r-1) = 0$. The roots are $r_1 = 1$ and $r_2 = 0$. Here $r_1 - r_2 = 1$. The recurrence relation is

$$a_n = \frac{-\,a_{n-1}}{(r+n)(r+n-1)}\,, \quad n = 1, 2, \cdots.$$

For $r = 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)}\,, \quad n = 1, 2, \cdots.$$

Hence for $n \geq 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \cdots = \frac{(-1)^n a_0}{n!(n+1)!}\,.$$

Therefore one solution is

$$y_1(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n\,x^n}{n!(n+1)!}\,.$$

5. Here $x\,p(x) = 2/3$ and $x^2q(x) = x^2/3$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$3\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty} (r+n)a_n\,x^{r+n} + \sum_{n=0}^{\infty} a_nx^{r+n+2} = 0\,.$$

It follows that

$$a_0[3r(r-1) + 2r]x^r + a_1[3(r+1)r + 2(r+1)]x^{r+1} + \\ + \sum_{n=2}^{\infty}[3(r+n)(r+n-1)a_n + 2(r+n)a_n + a_{n-2}]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $3r^2 - r = 0$, with roots $r_1 = 1/3$, $r_2 = 0$. Setting the remaining coefficients equal to *zero*, we have $a_1 = 0$, and

$$a_n = \frac{-\,a_{n-2}}{(r+n)[3(r+n)-1]}\,, \quad n = 2, 3, \cdots.$$

It immediately follows that the *odd* coefficients are equal to *zero*. For $r = 1/3$,

$$a_n = \frac{-\,a_{n-2}}{n(1+3n)}\,, \quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(6k+1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-5)(6k{+}1)} = \frac{(-1)^k a_0}{2^k\,k!\,7\cdot 13\cdots(6k+1)}\,.$$

For $r = 0$,

$$a_n = \frac{-\,a_{n-2}}{n(3n-1)}\,,\quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{2k(6k-1)} = \frac{a_{2k-4}}{(2k-2)(2k)(6k-7)(6k-1)} = \frac{(-1)^k a_0}{2^k\,k!\,5\cdot 11\cdots(6k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x) = x^{1/3}\left[1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,7\cdot 13\cdots(6k+1)}\left(\frac{x^2}{2}\right)^k\right]$$

$$y_2(x) = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k}{k!\,5\cdot 11\cdots(6k-1)}\left(\frac{x^2}{2}\right)^k.$$

6. Note that $x\,p(x) = 1$ and $x^2 q(x) = x - 2$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \\ + \sum_{n=0}^{\infty}a_n x^{r+n+1} - 2\sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1) + r - 2]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n - 2\,a_n + a_{n-1}]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - 2 = 0$, with roots $r = \pm\sqrt{2}$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{-\,a_{n-1}}{(r+n)^2 - 2}\,,\quad n = 1, 2, \cdots.$$

First note that $(r+n)^2 - 2 = \left(r + n + \sqrt{2}\right)\left(r + n - \sqrt{2}\right)$. So for $r = \sqrt{2}$,

$$a_n = \frac{-\,a_{n-1}}{n\left(n + 2\sqrt{2}\right)}\,,\quad n = 1, 2, \cdots.$$

It follows that

$$a_n = \frac{(-1)^n a_0}{n!\left(1+2\sqrt{2}\right)\left(2+2\sqrt{2}\right)\cdots\left(n+2\sqrt{2}\right)}\,,\quad n = 1, 2, \cdots.$$

For $r = -\sqrt{2}\,$,

$$a_n = \frac{-\,a_{n-1}}{n\left(n-2\sqrt{2}\right)}\,,\quad n = 1, 2, \cdots,$$

and therefore

$$a_n = \frac{(-1)^n a_0}{n!\left(1-2\sqrt{2}\right)\left(2-2\sqrt{2}\right)\cdots\left(n-2\sqrt{2}\right)}\,,\quad n = 1, 2, \cdots.$$

The two linearly independent solutions are

$$y_1(x) = x^{\sqrt{2}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\,x^n}{n!\left(1+2\sqrt{2}\right)\left(2+2\sqrt{2}\right)\cdots\left(n+2\sqrt{2}\right)}\right]$$

$$y_2(x) = x^{-\sqrt{2}}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n\,x^n}{n!\left(1-2\sqrt{2}\right)\left(2-2\sqrt{2}\right)\cdots\left(n-2\sqrt{2}\right)}\right].$$

7. Here $x\,p(x) = 1 - x$ and $x^2 q(x) = -x$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n-1} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n-1} - \\ -\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} - \sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After multiplying both sides by x,

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} - \\ -\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n+1} - \sum_{n=0}^{\infty}a_n x^{r+n+1} = 0.$$

After adjusting the indices in the *last two* series, we obtain

$$a_0[r(r-1)+r]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n - (r+n)a_{n-1}]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 = 0$, with roots $r_1 = r_2 = 0$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{a_{n-1}}{r+n}\,, \quad n = 1, 2, \cdots.$$

With $r = 0$,

$$a_n = \frac{a_{n-1}}{n}\,, \quad n = 1, 2, \cdots.$$

Hence one solution is

$$y_1(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x.$$

8. Note that $x\,p(x) = 3/2$ and $x^2 q(x) = x^2 - 1/2$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + 3\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \\ + 2\sum_{n=0}^{\infty}a_n x^{r+n+2} - \sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[2r(r-1)+3r-1]x^r + a_1[2(r+1)r + 3(r+1) - 1] + \\ + \sum_{n=2}^{\infty}[2(r+n)(r+n-1)a_n + 3(r+n)a_n - a_n + 2\,a_{n-2}]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $2r^2 + r - 1 = 0$, with roots $r_1 = 1/2$ and $r_2 = -1$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{-2a_{n-2}}{(r+n+1)[2(r+n)-1]}\,, \quad n = 2, 3, \cdots.$$

Setting the remaining coefficients equal to *zero*, we have $a_1 = 0$, which implies that all of the *odd* coefficients are *zero*. With $r = 1/2$,

$$a_n = \frac{-2a_{n-2}}{n(2n+3)}\,, \quad n = 2, 3, \cdots.$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{k(4k+3)} = \frac{a_{2k-4}}{(k-1)k(4k-5)(4k+3)} = \frac{(-1)^k a_0}{k!\,7\cdot 11\cdots(4k+3)}\ .$$

With $r = -1$,

$$a_n = \frac{-\,2a_{n-2}}{n(2n-3)}\,,\quad n = 2, 3, \cdots .$$

So for $k = 1, 2, \cdots$,

$$a_{2k} = \frac{-\,a_{2k-2}}{k(4k-3)} = \frac{a_{2k-4}}{(k-1)k(4k-11)(4k-3)} = \frac{(-1)^k a_0}{k!\,5\cdot 9\cdots(4k-3)}\ .$$

The two linearly independent solutions are

$$y_1(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n\,x^{2n}}{n!\,7\cdot 11\cdots(4n+3)}\right]$$

$$y_2(x) = x^{-1}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n\,x^{2n}}{n!\,5\cdot 9\cdots(4n-3)}\right].$$

9. Note that $x\,p(x) = -x-3$ and $x^2q(x) = x+3$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n+1} - 3\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + \sum_{n=0}^{\infty}a_nx^{r+n+1} + 3\sum_{n=0}^{\infty}a_nx^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1) - 3r + 3]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n - (r+n-2)a_{n-1} - 3(r+n-1)a_n]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - 4r + 3 = 0$, with roots $r_1 = 3$ and $r_2 = 1$. Setting the remaining coefficients equal to *zero*, the recurrence relation is

$$a_n = \frac{(r+n-2)a_{n-1}}{(r+n-1)(r+n-3)}\,,\quad n = 1, 2, \cdots .$$

With $r = 3$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)}\,, \quad n = 1, 2, \cdots.$$

It follows that for $n \geq 1$,

$$a_n = \frac{(n+1)a_{n-1}}{n(n+2)} = \frac{a_{n-2}}{(n-1)(n+2)} = \cdots = \frac{2\,a_0}{n!\,(n+2)}\,.$$

Therefore one solution is

$$y_1(x) = x^3\left[1 + \sum_{n=1}^{\infty} \frac{2\,x^n}{n!\,(n+2)}\right].$$

10. Here $x\,p(x) = 0$ and $x^2 q(x) = x^2 + 1/4$, which are *both* analytic at $x = 0$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second* series, we obtain

$$a_0\left[r(r-1) + \frac{1}{4}\right]x^r \; + a_1\left[(r+1)r + \frac{1}{4}\right]x^{r+1} + \\ + \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + \frac{1}{4}a_n + a_{n-2}\right]x^{r+n} = 0\,.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 - r + \frac{1}{4} = 0$, with roots $r_1 = r_2 = 1/2$. Setting the remaining coefficients equal to *zero*, we find that $a_1 = 0$. The recurrence relation is

$$a_n = \frac{-\,4a_{n-2}}{(2r+2n-1)^2}\,, \quad n = 2, 3, \cdots.$$

With $r = 1/2$,

$$a_n = \frac{-\,a_{n-2}}{n^2}\,, \quad n = 2, 3, \cdots.$$

Since $a_1 = 0$, the *odd* coefficients are *zero*. So for $k \geq 1$,

$$a_{2k} = \frac{-\,a_{2k-2}}{4k^2} = \frac{a_{2k-4}}{4^2(k-1)^2 k^2} = \cdots = \frac{(-1)^k a_0}{4^k (k!)^2}\,.$$

Therefore one solution is

$$y_1(x) = \sqrt{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right].$$

$12(a)$. Dividing through by the leading coefficient, the ODE can be written as

$$y'' - \frac{x}{1-x^2}\,y' + \frac{\alpha^2}{1-x^2}\,y = 0\,.$$

For $x = 1$,

$$p_0 = \lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{x}{x+1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\frac{\alpha^2(1-x)}{x+1} = 0\,.$$

For $x = -1$,

$$p_0 = \lim_{x\to -1}(x+1)p(x) = \lim_{x\to -1}\frac{x}{x-1} = \frac{1}{2}\,.$$

$$q_0 = \lim_{x\to -1}(x+1)^2 q(x) = \lim_{x\to -1}\frac{\alpha^2(x+1)}{(1-x)} = 0\,.$$

Hence both $x = -1$ and $x = 1$ are *regular* singular points. As shown in Example **1**, the indicial equation is given by

$$r(r-1) + p_0 r + q_0 = 0\,.$$

In this case, *both* sets of roots are $r_1 = 1/2$ and $r_2 = 0$.

(b). Let $t = x - 1$, and $u(t) = y(t+1)$. Under this change of variable, the differential equation becomes

$$\left(t^2 + 2t\right)u'' + (t+1)u' - \alpha^2 u = 0\,.$$

Based on Part (a), $t = 0$ is a *regular* singular point. Set $u = \sum_{n=0}^{\infty} a_n\, t^{r+n}$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, t^{r+n-1} +$$
$$+ \sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty}a_n t^{r+n} = 0\,.$$

Upon inspection, we can also write

$$\sum_{n=0}^{\infty}(r+n)^2 a_n\, t^{r+n} + 2\sum_{n=0}^{\infty}(r+n)\left(r+n-\frac{1}{2}\right)a_n\, t^{r+n-1} - \alpha^2\sum_{n=0}^{\infty}a_n t^{r+n} = 0.$$

After adjusting the indices in the *second* series, it follows that

$$a_0\left[2r\left(r-\frac{1}{2}\right)\right]t^{r-1} + \sum_{n=0}^{\infty}\left[(r+n)^2 a_n + 2(r+n+1)\left(r+n+\frac{1}{2}\right)a_{n+1} - \alpha^2 a_n\right]t^{r+n} = 0.$$

Assuming that $a_0 \neq 0$, the *indicial equation* is $2r^2 - r = 0$, with roots $r = 0\,,\,1/2$. The recurrence relation is

$$(r+n)^2 a_n + 2(r+n+1)\left(r+n+\frac{1}{2}\right)a_{n+1} - \alpha^2 a_n = 0\,,\quad n = 0,1,2,\cdots.$$

With $r_1 = 1/2$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{4\alpha^2 - (2n-1)^2}{4n(2n+1)}\, a_{n-1} \\ &= (-1)^n \frac{[1-4\alpha^2][9-4\alpha^2]\cdots\left[(2n-1)^2 - 4\alpha^2\right]}{2^n(2n+1)!}\, a_0\,. \end{aligned}$$

With $r_2 = 0$, we find that for $n \geq 1$,

$$\begin{aligned} a_n &= \frac{\alpha^2 - (n-1)^2}{n(2n-1)}\, a_{n-1} \\ &= (-1)^n \frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2]\cdots\left[(n-1)^2 - \alpha^2\right]}{n!\cdot 3\cdot 5\cdots(2n-1)}\, a_0\,. \end{aligned}$$

The two linearly independent solutions of the *Chebyshev equation* are

$$y_1(x) = |x-1|^{1/2}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{[1-4\alpha^2][9-4\alpha^2]\cdots\left[(2n-1)^2-4\alpha^2\right]}{2^n(2n+1)!}(x-1)^n\right]$$

$$y_2(x) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{\alpha(-\alpha)[1-\alpha^2][4-\alpha^2]\cdots\left[(n-1)^2-\alpha^2\right]}{n!\cdot 3\cdot 5\cdots(2n-1)}(x-1)^n\,.$$

13. Here $x\,p(x) = 1 - x$ and $x^2 q(x) = \lambda\, x$, which are *both* analytic at $x = 0$. In fact,

$$p_0 = \lim_{x\to 0} x\, p(x) = 1 \text{ and } q_0 = \lim_{x\to 0} x^2 q(x) = 0\,.$$

Hence the *indicial equation* is $r(r-1) + r = 0$, with roots $r_{1,2} = 0$. Set

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots.$$

Substitution into the ODE results in

$$\sum_{n=2}^{\infty} n(n-1)a_n\, x^{n-1} + \sum_{n=1}^{\infty} na_n\, x^{n-1} - \\ - \sum_{n=0}^{\infty} na_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}\, x^{n} + \sum_{n=0}^{\infty} (n+1)a_{n+1}\, x^{n} - \\ - \sum_{n=1}^{\infty} na_n\, x^n + \lambda \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

It follows that

$$a_1 + \lambda\, a_0 + \sum_{n=1}^{\infty} \big[(n+1)^2 a_{n+1} - (n-\lambda)a_n\big] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we find that $a_1 = -\lambda a_0$, and

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1}\,, \quad n = 2, 3, \cdots.$$

That is, for $n \geq 2$,

$$a_n = \frac{(n-1-\lambda)}{n^2}\, a_{n-1} = \cdots = \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, a_0\,.$$

Therefore one solution of the *Laguerre equation* is

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, x^n.$$

Note that if $\lambda = m$, a *positive integer*, then $a_n = 0$ for $n \geq m+1$. In that case, the solution is a *polynomial*

$$y_1(x) = 1 + \sum_{n=1}^{m} \frac{(-\lambda)(1-\lambda)\cdots(n-1-\lambda)}{(n!)^2}\, x^n\,.$$

Section 5.7

2. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = -2 - x$ and $x^2 q(x) = 2 + x^2$. It follows that

$$p_0 = \lim_{x \to 0}(-2 - x) = -2$$
$$q_0 = \lim_{x \to 0}\left(2 + x^2\right) = 2$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - 2r + 2 = 0\,,$$

that is, $r^2 - 3r + 2 = 0$, with roots $r_1 = 2$ and $r_2 = 1$.

4. The coefficients $P(x)$, $Q(x)$, and $R(x)$ are analytic for all $x \in \mathbb{R}$. Hence there are *no* singular points.

5. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = 3\frac{sin\,x}{x}$ and $x^2 q(x) = -2$. It follows that

$$p_0 = \lim_{x \to 0} 3\,\frac{sin\,x}{x} = 3$$
$$q_0 = \lim_{x \to 0} -2 = -2$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + 3r - 2 = 0\,,$$

that is, $r^2 + 2r - 2 = 0$, with roots $r_1 = -1 + \sqrt{3}$ and $r_2 = -1 - \sqrt{3}$.

6. $P(x) = 0$ for $x = 0$ and $x = -2$. We note that $p(x) = x^{-1}(x+2)^{-1}/2$, and $q(x) = -(x+2)^{-1}/2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x \to 0} \frac{1}{2(x+2)} = \frac{1}{4}$$
$$q_0 = \lim_{x \to 0} \frac{-x^2}{2(x+2)} = 0$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{3}{4}r = 0$, with roots $r_1 = \frac{3}{4}$ and $r_2 = 0$. For the singularity at $x = -2$,

$$p_0 = \lim_{x \to -2} (x+2)p(x) = \lim_{x \to -2} \frac{1}{2x} = -\frac{1}{4}$$
$$q_0 = \lim_{x \to -2} (x+2)^2 q(x) = \lim_{x \to -2} \frac{-(x+2)}{2} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

7. $P(x) = 0$ only for $x = 0$. Furthermore, $x\,p(x) = \frac{1}{2} + \frac{\sin x}{2x}$ and $x^2 q(x) = 1$. It follows that

$$p_0 = \lim_{x \to 0} xp(x) = 1$$
$$q_0 = \lim_{x \to 0} x^2 q(x) = 1$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) + r + 1 = 0\,,$$

that is, $r^2 + 1 = 0$, with *complex conjugate* roots $r = \pm\, i$.

8. Note that $P(x) = 0$ only for $x = -1$. We find that $p(x) = 3(x-1)/(x+1)$, and $q(x) = 3/(x+1)^2$. It follows that

$$p_0 = \lim_{x \to -1} (x+1)p(x) = \lim_{x \to -1} 3(x-1) = -6$$
$$q_0 = \lim_{x \to -1} (x+1)^2 q(x) = \lim_{x \to -1} 3 = 3$$

and therefore $x = -1$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - 6r + 3 = 0\,,$$

that is, $r^2 - 7r + 3 = 0$, with roots $r_1 = \left(7 + \sqrt{37}\right)/2$ and $r_2 = \left(7 - \sqrt{37}\right)/2$.

10. $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x(x-2)^{-2}(x+2)^{-1}$, and $q(x) = 3(x-2)^{-1}(x+2)^{-1}$. For the singularity at $x = 2$,

$$\lim_{x \to 2} (x-2)p(x) = \lim_{x \to 2} \frac{2x}{x^2 - 4}\,,$$

which is *undefined*. Therefore $x = 0$ is an *irregular* singular point. For the singularity at $x = -2$,

$$p_0 = \lim_{x\to -2} (x+2)p(x) = \lim_{x\to -2} \frac{2x}{(x-2)^2} = -\frac{1}{4}$$
$$q_0 = \lim_{x\to -2} (x+2)^2 q(x) = \lim_{x\to -2} \frac{3(x+2)}{x-2} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - \frac{1}{4}r = 0\,,$$

that is, $r^2 - \frac{5}{4}r = 0$, with roots $r_1 = \frac{5}{4}$ and $r_2 = 0$.

11. $P(x) = 0$ for $x = 2$ and $x = -2$. We note that $p(x) = 2x/(4-x^2)$, and $q(x) = 3/(4-x^2)$. For the singularity at $x = 2$,

$$p_0 = \lim_{x\to 2} (x-2)p(x) = \lim_{x\to 2} \frac{-2x}{x+2} = -1$$
$$q_0 = \lim_{x\to 2} (x-2)^2 q(x) = \lim_{x\to 2} \frac{3(2-x)}{x+2} = 0$$

and therefore $x = 2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - r = 0\,,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$. For the singularity at $x = -2$,

$$p_0 = \lim_{x\to -2} (x+2)p(x) = \lim_{x\to -2} \frac{2x}{2-x} = -1$$
$$q_0 = \lim_{x\to -2} (x+2)^2 q(x) = \lim_{x\to -2} \frac{3(x+2)}{2-x} = 0$$

and therefore $x = -2$ is a *regular* singular point. The indicial equation is given by

$$r(r-1) - r = 0\,,$$

that is, $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$.

12. $P(x) = 0$ for $x = 0$ and $x = -3$. We note that $p(x) = -2x^{-1}(x+3)^{-1}$, and $q(x) = -1/(x+3)^2$. For the singularity at $x = 0$,

$$p_0 = \lim_{x\to 0} x\,p(x) = \lim_{x\to 0} \frac{-2}{x+3} = -\frac{2}{3}$$
$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{-x^2}{(x+3)^2} = 0$$

and therefore $x = 0$ is a *regular* singular point. The indicial equation is given by

$$r(r-1)-\frac{2}{3}r=0\,,$$

that is, $r^2-\frac{5}{3}r=0$, with roots $r_1=\frac{5}{3}$ and $r_2=0$. For the singularity at $x=-3$,

$$p_0=\lim_{x\to-3}(x+3)p(x)=\lim_{x\to-3}\frac{-2}{x}=\frac{2}{3}$$
$$q_0=\lim_{x\to-3}(x+3)^2q(x)=\lim_{x\to-3}(-1)=-1$$

and therefore $x=-3$ is a *regular* singular point. The indicial equation is given by

$$r(r-1)+\frac{2}{3}r-1=0\,,$$

that is, $r^2-\frac{1}{3}r-1=0$, with roots $r_1=\left(1+\sqrt{37}\right)/6$ and $r_2=\left(1-\sqrt{37}\right)/6$.

$13(a)$. Note the $p(x)=1/x$ and $q(x)=-1/x$. Furthermore, $x\,p(x)=1$ and $x^2q(x)=-x$. It follows that

$$p_0=\lim_{x\to0}(1)=1$$
$$q_0=\lim_{x\to0}(-x)=0$$

and therefore $x=0$ is a *regular* singular point.

(b). The indicial equation is given by

$$r(r-1)+r=0\,,$$

that is, $r^2=0$, with roots $r_1=r_2=0$.

(c). Let $y=a_0+a_1x+a_2x^2+\cdots+a_nx^n+\cdots$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^{n+1}+\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n-\sum_{n=0}^{\infty}a_nx^n=0\,.$$

After adjusting the indices in the *first* series, we obtain

$$a_1-a_0+\sum_{n=1}^{\infty}[n(n+1)a_{n+1}+(n+1)a_{n+1}-a_n]x^n=0\,.$$

Setting the coefficients equal to *zero*, it follows that for $n\ge0$,

$$a_{n+1}=\frac{a_n}{(n+1)^2}\,.$$

So for $n\ge1$,

$$a_n = \frac{a_{n-1}}{n^2} = \frac{a_{n-2}}{n^2(n-1)^2} = \cdots = \frac{1}{(n!)^2}\, a_0 \,.$$

With $a_0 = 1$, one solution is

$$y_1(x) = 1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \cdots + \frac{1}{(n!)^2}\, x^n + \cdots .$$

For a second solution, set $y_2(x) = y_1(x)\, ln\, x \,+\, b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$.
Substituting into the ODE, we obtain

$$L[y_1(x)] \cdot ln\, x + 2\, y_1'(x) + L\left[\sum_{n=1}^{\infty} b_n\, x^n\right] = 0\,.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[\sum_{n=1}^{\infty} b_n\, x^n\right] = \,-\,2\, y_1'(x)\,.$$

More specifically,

$$\begin{aligned} b_1 + \sum_{n=1}^{\infty}[n(n+1)b_{n+1} + (n+1)b_{n+1} - b_n]x^n &= \\ &= \,-\,2 - x - \frac{1}{6}x^2 - \frac{1}{72}x^3 - \frac{1}{1440}x^4 - \cdots . \end{aligned}$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} b_1 &= \,-\,2 \\ 4b_2 - b_1 &= \,-\,1 \\ 9b_3 - b_2 &= \,-\,1/6 \\ 16b_4 - b_3 &= \,-\,1/72 \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $b_1 = \,-\,2,\ b_2 = \,-\,3/4,\ b_3 = \,-\,11/108,$ $b_4 = \,-\,25/3456\,, \cdots$. Therefore a *second* solution is

$$y_2(x) = y_1(x)\, ln\, x + \left[\,-\,2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \cdots\right].$$

$14(a)$. Here $x\,p(x) = 2x$ and $x^2 q(x) = 6\, x e^x$. Both of these functions are *analytic* at $x = 0$, therefore $x = 0$ is a *regular* singular point. Note that $p_0 = q_0 = 0$.

(b). The indicial equation is given by

$$r(r-1)=0\,,$$

that is, $r^2-r=0$, with roots $r_1=1$ and $r_2=0$.

(c). In order to find the solution corresponding to $r_1=1$, set $y=x\sum_{n=0}^{\infty}a_nx^n$. Upon substitution into the ODE, we have

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+1}\,x^{n+1}+2\sum_{n=0}^{\infty}(n+1)a_nx^{n+1}+6\,e^x\sum_{n=0}^{\infty}a_nx^{n+1}=0\,.$$

After adjusting the indices in the *first* two series, and expanding the *exponential* function,

$$\sum_{n=1}^{\infty}n(n+1)a_n\,x^n+2\sum_{n=1}^{\infty}n\,a_{n-1}x^n+6\,a_0x+(6a_0+6a_1)x^2+$$
$$+(6a_2+6a_1+3a_0)x^3+(6a_3+6a_2+3a_1+a_0)x^4+\cdots=0\,.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned}2a_1+2a_0+6a_0&=0\\6a_2+4a_1+6a_0+6a_1&=0\\12a_3+6a_2+6a_2+6a_1+3a_0&=0\\20a_4+8a_3+6a_3+6a_2+3a_1+a_0&=0\\&\vdots\end{aligned}$$

Setting $a_0=1$, solution of the system results in $a_1=-4$, $a_2=17/3$, $a_3=-47/12$, $a_4=191/120$, $\cdots$. Therefore one solution is

$$y_1(x)=x-4x^2+\frac{17}{3}x^3-\frac{47}{12}x^4+\cdots.$$

The exponents differ by an integer. So for a second solution, set

$$y_2(x)=a\,y_1(x)\,ln\,x\,+1+c_1x+c_2x^2+\cdots+c_nx^n+\cdots.$$

Substituting into the ODE, we obtain

$$a\,L[y_1(x)]\cdot ln\,x+2a\,y_1'(x)+2a\,y_1(x)-a\frac{y_1(x)}{x}+L\left[1+\sum_{n=1}^{\infty}c_n\,x^n\right]=0\,.$$

Since $L[y_1(x)]=0$, it follows that

$$L\left[1+\sum_{n=1}^{\infty}c_n\,x^n\right]=-2a\,y_1'(x)-2a\,y_1(x)+a\frac{y_1(x)}{x}\,.$$

More specifically,

$$\sum_{n=1}^{\infty} n(n+1)c_{n+1}x^n + 2\sum_{n=1}^{\infty} n\,c_n x^n + 6 \,+ (6+6c_1)x +$$
$$+ (6c_2+6c_1+3)x^2 + \cdots = -a + 10ax - \frac{61}{3}ax^2 + \frac{193}{12}ax^3 + \cdots .$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 6 &= -a \\ 2c_2 + 8c_1 + 6 &= 10a \\ 6c_3 + 10c_2 + 6c_1 + 3 &= -\frac{61}{3}a \\ 12c_4 + 12c_3 + 6c_2 + 3c_1 + 1 &= \frac{193}{12}a \\ &\vdots \end{aligned}$$

Solving these equations for the coefficients, $a = -6$. In order to solve the remaining equations, set $c_1 = 0$. Then $c_2 = -33,\ c_3 = 449/6$, $c_4 = -1595/24, \cdots$. Therefore a *second* solution is

$$y_2(x) = -6\,y_1(x)\,ln\,x + \left[1 - 33x^2 + \frac{449}{6}x^3 - \frac{1595}{24}x^4 + \cdots\right].$$

$15(a)$. Note the $p(x) = 6x/(x-1)$ and $q(x) = 3x^{-1}(x-1)^{-1}$. Furthermore, $x\,p(x) = 6x^2/(x-1)$ and $x^2q(x) = 3x/(x-1)$. It follows that

$$\begin{aligned} p_0 &= \lim_{x\to 0} \frac{6x^2}{x-1} = 0 \\ q_0 &= \lim_{x\to 0} \frac{3x}{x-1} = 0 \end{aligned}$$

and therefore $x = 0$ is a *regular* singular point.

(b). The indicial equation is given by

$$r(r-1) = 0\,,$$

that is, $r^2 - r = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c). In order to find the solution corresponding to $r_1 = 1$, set $y = x\sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n\, x^{n+1} \; - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + \\ + 6\sum_{n=0}^{\infty}(n+1)a_n x^{n+2} + 3\sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

After adjusting the indices, it follows that

$$\sum_{n=2}^{\infty} n(n-1)a_{n-1}\, x^n \; - \sum_{n=1}^{\infty} n(n+1)a_n\, x^n + \\ + 6\sum_{n=2}^{\infty}(n-1)a_{n-2} x^n + 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0\,.$$

That is,

$$-2a_1 + 3a_0 + \sum_{n=2}^{\infty}\left[-n(n+1)a_n + (n^2 - n + 3)a_{n-1} + 6(n-1)a_{n-2}\right]x^n = 0.$$

Setting the coefficients equal to *zero*, we have $a_1 = 3a_0/2$, and for $n \geq 2$,

$$n(n+1)a_n = \left(n^2 - n + 3\right)a_{n-1} + 6(n-1)a_{n-2}\,.$$

If we assign $a_0 = 1$, then we obtain $a_1 = 3/2$, $a_2 = 9/4$, $a_3 = 51/16$, $\cdots$.
Hence one solution is

$$y_1(x) = x + \frac{3}{2}x^2 + \frac{9}{4}x^3 + \frac{51}{16}x^4 + \frac{111}{40}x^5 + \cdots.$$

The exponents differ by an *integer*. So for a second solution, set

$$y_2(x) = a\, y_1(x)\, ln\, x \; + 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$$

Substituting into the ODE, we obtain

$$2ax\, y_1'(x) - 2a\, y_1'(x) + 6ax\, y_1(x) - a\, y_1(x) + a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 0,$$

since $L[y_1(x)] = 0$. It follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 2a\, y_1'(x) - 2ax\, y_1'(x) + a\, y_1(x) - 6ax\, y_1(x) - a\frac{y_1(x)}{x}\,.$$

Now

$$L\left[1 + \sum_{n=1}^{\infty} c_n\, x^n\right] = 3 + (-2c_2 + 3c_1)x + (-6c_3 + 5c_2 + 6c_1)x^2 + \\ + (-12c_4 + 9c_3 + 12c_2)x^3 + (-20c_5 + 15c_4 + 18c_3)x^4 + \cdots.$$

Substituting for $y_1(x)$, the *right hand side* of the ODE is

$$a + \frac{7}{2}ax + \frac{3}{4}ax^2 + \frac{33}{16}ax^3 - \frac{867}{80}ax^4 - \frac{441}{10}ax^5 + \cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 3 &= a \\ -2c_2 + 3c_1 &= \frac{7}{2}a \\ -6c_3 + 5c_2 + 6c_1 &= \frac{3}{4}a \\ -12c_4 + 9c_3 + 12c_2 &= \frac{33}{16}a \\ &\vdots \end{aligned}$$

We find that $a = 3$. In order to solve the second equation, set $c_1 = 0$. Solution of the remaining equations results in $c_2 = -21/4$, $c_3 = -19/4$, $c_4 = -597/64, \cdots$. Hence a second solution is

$$y_2(x) = 3\,y_1(x)\,ln\,x + \left[1 - \frac{21}{4}x^2 - \frac{19}{4}x^3 - \frac{597}{64}x^4 + \cdots\right].$$

16(a). After multiplying both sides of the ODE by x, we find that $x\,p(x) = 0$ and $x^2q(x) = x$. Both of these functions are *analytic* at $x = 0$, hence $x = 0$ is a *regular* singular point.

(b). Furthermore, $p_0 = q_0 = 0$. So the indicial equation is $r(r-1) = 0$, with roots $r_1 = 1$ and $r_2 = 0$.

(c). In order to find the solution corresponding to $r_1 = 1$, set $y = x\sum_{n=0}^{\infty} a_n x^n$. Upon substitution into the ODE, we have

$$\sum_{n=1}^{\infty} n(n+1)a_n\, x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0\,.$$

That is,

$$\sum_{n=1}^{\infty} \left[n(n+1)a_n + a_{n-1}\right] x^n = 0\,.$$

Setting the coefficients equal to *zero*, we find that for $n \geq 1$,

$$a_n = \frac{-\,a_{n-1}}{n(n+1)}\,.$$

It follows that

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \cdots = \frac{(-1)^n a_0}{(n!)^2(n+1)}\,.$$

Hence one solution is

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \frac{1}{2880}x^5 + \cdots.$$

The exponents differ by an *integer*. So for a second solution, set

$$y_2(x) = a\,y_1(x)\,ln\,x\ + 1 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots.$$

Substituting into the ODE, we obtain

$$a\,L[y_1(x)]\cdot ln\,x + 2a\,y_1'(x) - a\frac{y_1(x)}{x} + L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = 0\,.$$

Since $L[y_1(x)] = 0$, it follows that

$$L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] = -2a\,y_1'(x) + a\frac{y_1(x)}{x}\,.$$

Now

$$\begin{aligned} L\left[1 + \sum_{n=1}^{\infty} c_n\,x^n\right] &= 1 + (2c_2 + c_1)x + (6c_3 + c_2)x^2 + (12c_4 + c_3)x^3 + \\ &\quad + (20c_5 + c_4)x^4 + (30c_6 + c_5)x^5 + \cdots. \end{aligned}$$

Substituting for $y_1(x)$, the *right hand side* of the ODE is

$$-a + \frac{3}{2}ax - \frac{5}{12}ax^2 + \frac{7}{144}ax^3 - \frac{1}{320}ax^4 + \cdots.$$

Equating the coefficients, we obtain the system of equations

$$\begin{aligned} 1 &= -a \\ 2c_2 + c_1 &= \frac{3}{2}a \\ 6c_3 + c_2 &= -\frac{5}{12}a \\ 12c_4 + c_3 &= \frac{7}{144}a \\ &\vdots \end{aligned}$$

Evidently, $a = -1$. In order to solve the *second* equation, set $c_1 = 0$. We then find that $c_2 = -3/4$, $c_3 = 7/36$, $c_4 = -35/1728$, $\cdots$. Therefore a second solution is

$$y_2(x) = -y_1(x)\,ln\,x + \left[1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \cdots\right].$$

$19(a)$. After dividing by the leading coefficient, we find that

$$p_0 = \lim_{x\to 0} x\, p(x) = \lim_{x\to 0} \frac{\gamma - (1+\alpha+\beta)x}{1-x} = \gamma\,.$$

$$q_0 = \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} \frac{-\,\alpha\beta\, x}{1-x} = 0\,.$$

Hence $x = 0$ is a *regular* singular point. The indicial equation is $r(r-1) + \gamma\, r = 0$, with roots $r_1 = 1-\gamma$ and $r_2 = 0$.

(b). For $x = 1$,

$$p_0 = \lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1} \frac{-\,\gamma + (1+\alpha+\beta)x}{x} = 1-\gamma+\alpha+\beta\,.$$

$$q_0 = \lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1} \frac{\alpha\beta(x-1)}{x} = 0\,.$$

Hence $x = 1$ is a *regular* singular point. The indicial equation is

$$r^2 - (\gamma-\alpha-\beta)\, r = 0\,,$$

with roots $r_1 = \gamma-\alpha-\beta$ and $r_2 = 0$.

(c). Given that $r_1 - r_2$ is not a positive integer, we can set $y = \sum_{n=0}^{\infty} a_n x^n$. Substitution into the ODE results in

$$x(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + [\gamma - (1+\alpha+\beta)x]\sum_{n=1}^{\infty} n\, a_n x^{n-1} - \alpha\beta\sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n + \gamma\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n -$$
$$-\,(1+\alpha+\beta)\sum_{n=1}^{\infty} n\, a_n x^n - \alpha\beta\sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the series, we obtain

$$\gamma\, a_1 - \alpha\beta\, a_0 + [(2+2\gamma)a_2 - (1+\alpha+\beta+\alpha\beta)a_1]x + \sum_{n=2}^{\infty} A_n\, x^n = 0\,,$$

in which

$$A_n = (n+1)(n+\gamma)a_{n+1} - [n(n-1) + (1+\alpha+\beta)n + \alpha\beta]a_n \,.$$

Note that $n(n-1) + (1+\alpha+\beta)n + \alpha\beta = (n+\alpha)(n+\beta)$. Setting the coefficients equal to *zero*, we have $\gamma\, a_1 - \alpha\beta\, a_0 = 0$, and

$$a_{n+1} = \frac{(n+\alpha)(n+\beta)}{(n+1)(n+\gamma)}\, a_n$$

for $n \geq 1$. Hence one solution is

$$\begin{aligned} y_1(x) = \ & 1 + \frac{\alpha\beta}{\gamma\cdot 1!}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 2!}x^2 + \\ & + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\cdot 3!}x^3 + \cdots . \end{aligned}$$

Since the nearest other singularity is at $x = 1$, the radius of convergence of $y_1(x)$ will be *at least* $\rho = 1$.

(d). Given that $r_1 - r_2$ is not a positive integer, we can set $y = x^{1-\gamma}\sum_{n=0}^{\infty} b_n x^n$. Then Substitution into the ODE results in

$$\begin{aligned} & x(1-x)\sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma-1} + \\ & + [\gamma-(1+\alpha+\beta)x]\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0. \end{aligned}$$

That is,

$$\begin{aligned} & \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n+1-\gamma} + \\ & + \gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n+1-\gamma} - \alpha\beta\sum_{n=0}^{\infty}a_n x^{n+1-\gamma} = 0. \end{aligned}$$

After adjusting the indices,

$$\begin{aligned} & \sum_{n=0}^{\infty}(n+1-\gamma)(n-\gamma)a_n x^{n-\gamma} - \sum_{n=1}^{\infty}(n-\gamma)(n-1-\gamma)a_{n-1} x^{n-\gamma} + \\ & + \gamma\sum_{n=0}^{\infty}(n+1-\gamma)a_n x^{n-\gamma} - (1+\alpha+\beta)\sum_{n=1}^{\infty}(n-\gamma)a_{n-1} x^{n-\gamma} - \alpha\beta\sum_{n=1}^{\infty}a_{n-1} x^{n-\gamma} = 0. \end{aligned}$$

Combining the series, we obtain

$$\sum_{n=1}^{\infty} B_n\, x^{n-\gamma} = 0\,,$$

in which

$$B_n = n(n+1-\gamma)b_n - [(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta]b_{n-1}.$$

Note that $(n-\gamma)(n-\gamma+\alpha+\beta)+\alpha\beta = (n+\alpha-\gamma)(n+\beta-\gamma)$. Setting $B_n = 0$, it follows that for $n \geq 1$,

$$b_n = \frac{(n+\alpha-\gamma)(n+\beta-\gamma)}{n(n+1-\gamma)}\, b_{n-1}.$$

Therefore a second solution is

$$\begin{aligned} y_2(x) = \; & x^{1-\gamma}\Bigg[1 + \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{(2-\gamma)1!}x + \\ & + \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{(2-\gamma)(3-\gamma)2\,!}x^2 + \cdots\Bigg]. \end{aligned}$$

(e). Under the transformation $x = 1/\xi$, the ODE becomes

$$\xi^4\frac{1}{\xi}\left(1-\frac{1}{\xi}\right)\frac{d^2y}{d\xi^2} + \left\{2\xi^3\frac{1}{\xi}\left(1-\frac{1}{\xi}\right) - \xi^2\left[\gamma - (1+\alpha+\beta)\frac{1}{\xi}\right]\right\}\frac{dy}{d\xi} - \alpha\beta\, y = 0\,.$$

That is,

$$\left(\xi^3-\xi^2\right)\frac{d^2y}{d\xi^2} + \left[2\xi^2 - \gamma\,\xi^2 + (\,-1+\alpha+\beta)\xi\right]\frac{dy}{d\xi} - \alpha\beta\; y = 0\,.$$

Therefore $\xi = 0$ is a singular point. Note that

$$p(\xi) = \frac{(2-\gamma)\,\xi + (\,-1+\alpha+\beta)}{\xi^2-\xi} \text{ and } q(\xi) = \frac{-\,\alpha\beta}{\xi^3-\xi^2}\,.$$

It follows that

$$p_0 = \lim_{\xi\to 0}\xi\, p(\xi) = \lim_{\xi\to 0}\frac{(2-\gamma)\,\xi + (\,-1+\alpha+\beta)}{\xi-1} = 1-\alpha-\beta\,,$$

$$q_0 = \lim_{\xi\to 0}\xi^2 q(\xi) = \lim_{\xi\to 0}\frac{-\,\alpha\beta}{\xi-1} = \alpha\beta\,.$$

Hence $\xi = 0$ $(x = \infty)$ is a *regular* singular point. The indicial equation is

$$r(r-1) + (1-\alpha-\beta)r + \alpha\beta = 0\,,$$

or $r^2 - (\alpha+\beta)r + \alpha\beta = 0$. Evidently, the roots are $r = \alpha$ and $r = \beta$.

$21(a)$. Note that

$$p(x) = \frac{\alpha}{x^s} \text{ and } q(\xi) = \frac{\beta}{x^t}\,.$$

It follows that

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} \alpha\, x^{1-s},$$

$$\lim_{\xi\to 0} \xi^2 q(\xi) = \lim_{\xi\to 0} \beta\, x^{2-s}.$$

Hence if $s > 1$ or $t > 2$, one or both of the limits does not exist. Therefore $x = 0$ is an *irregular* singular point.

(c). Let $y = a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots$. Write the ODE as

$$x^3 y'' + \alpha\, x^2 y' + \beta\, y = 0\,.$$

Substitution of the assumed solution results in

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r+1} + \alpha\sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} + \beta\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Adjusting the indices, we obtain

$$\sum_{n=1}^{\infty}(n-1+r)(n+r-2)a_{n-1}x^{n+r} + \alpha\sum_{n=1}^{\infty}(n-1+r)a_{n-1}x^{n+r} + \beta\sum_{n=0}^{\infty}a_n x^{n+r} = 0.$$

Combining the series,

$$\beta\, a_0 + \sum_{n=1}^{\infty} A_n\, x^{n+r} = 0\,,$$

in which $A_n = \beta\, a_n + (n-1+r)(n+r+\alpha-2)a_{n-1}$. Setting the coefficients equal to *zero*, we have $a_0 = 0$. But for $n \geq 1$,

$$a_n = \frac{(n-1+r)(n+r+\alpha-2)}{\beta}\, a_{n-1}.$$

Therefore, regardless of the value of r, it follows that $a_n = 0$, for $n = 1, 2, \cdots$.

Section 5.8

3. Here $x\,p(x) = 1$ and $x^2q(x) = 2x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n} + 2\sum_{n=0}^{\infty}a_nx^{r+n+1} = 0\,.$$

After adjusting the indices in the *last* series, we obtain

$$a_0[r(r-1)+r]x^r + \sum_{n=1}^{\infty}[(r+n)(r+n-1)a_n + (r+n)a_n + 2\,a_{n-1}]x^{r+n} = 0.$$

Assuming $a_0 \neq 0$, the *indicial equation* is $r^2 = 0$, with *double* root $r = 0$. Setting the remaining coefficients equal to *zero*, we have for $n \geq 1$,

$$a_n(r) = -\frac{2}{(n+r)^2}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r) = \frac{(-1)^n\,2^n}{[(n+r)(n+r-1)\cdots(1+r)]^2}\,a_0\,,\quad n \geq 1\,.$$

Since $r = 0$, one solution is given by

$$y_1(x) = \sum_{n=0}^{\infty}\frac{(-1)^n\,2^n}{(n!)^2}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.7 . First
note that

$$\frac{a_n'(r)}{a_n(r)} = -2\left[\frac{1}{n+r} + \frac{1}{n+r-1} + \cdots + \frac{1}{1+r}\right].$$

Setting $r = 0$,

$$a_n'(0) = -2\,H_n\,a_n(0) = -2\,H_n\frac{(-1)^n\,2^n}{(n!)^2}\,.$$

Therefore,

$$y_2(x) = y_1(x)\,ln\,x - 2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^nH_n}{(n!)^2}\,x^n\,.$$

4. Here $x\,p(x) = 4$ and $x^2q(x) = 2 + x$, which are both analytic everywhere. We set $y = x^r(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n}+\,4\sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}+$$
$$+\sum_{n=0}^{\infty}a_n x^{r+n+1}+2\sum_{n=0}^{\infty}a_n x^{r+n}=0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1)+4r+2]x^r+\sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n+4(r+n)a_n+2\,a_n+a_{n-1}\right]x^{r+n}=0.$$

Assuming $a_0\neq 0$, the *indicial equation* is $r^2+3r+2=0$, with roots $r_1=\,-1$ and $r_2=\,-2$. Setting the remaining coefficients equal to *zero*, we have for $n\geq 1$,

$$a_n(r)=\,-\frac{1}{(n+r+1)(n+r+2)}\,a_{n-1}(r)\,.$$

It follows that

$$a_n(r)=\frac{(\,-1)^n}{[(n+r+1)(n+r)\cdots(2+r)][(n+r+2)(n+r)\cdots(3+r)]}\,a_0\,,\;n\geq 1\,.$$

Since $r_1=\,-1$, one solution is given by

$$y_1(x)=x^{-1}\sum_{n=0}^{\infty}\frac{(\,-1)^n}{(n)!(n+1)!}\,x^n\,.$$

For a second linearly independent solution, we follow the discussion in Section 5.7 . Since $r_1-r_2=N=1$, we find that

$$a_1(r)=\,-\frac{1}{(r+2)(r+3)}\,,$$

with $a_0=1$. Hence the leading coefficient in the solution is

$$a=\lim_{r\to-2}(r+2)\,a_1(r)=\,-1\,.$$

Further,

$$(r+2)\,a_n(r)=\frac{(\,-1)^n}{(n+r+2)\,[(n+r+1)(n+r)\cdots(3+r)]^2}\,.$$

Let $A_n(r)=(r+2)\,a_n(r)$. It follows that

$$\frac{A_n'(r)}{A_n(r)}=\,-\frac{1}{n+r+2}-2\left[\frac{1}{n+r+1}+\frac{1}{n+r}+\cdots+\frac{1}{3+r}\right].$$

Setting $r=r_2=\,-2$,

$$\begin{aligned}\frac{A_n'(-2)}{A_n(-2)} &= -\frac{1}{n} - 2\left[\frac{1}{n-1} + \frac{1}{n-2} + \cdots + 1\right] \\ &= -H_n - H_{n-1}\,.\end{aligned}$$

Hence

$$\begin{aligned}c_n(-2) &= -(H_n + H_{n-1})\,A_n(-2) \\ &= -(H_n + H_{n-1})\frac{(-1)^n}{n!(n-1)!}\,.\end{aligned}$$

Therefore,

$$y_2(x) = -y_1(x)\,ln\,x + x^{-2}\left[1 - \sum_{n=1}^{\infty}\frac{(-1)^n(H_n + H_{n-1})}{n!(n-1)!}\,x^n\right].$$

6. Let $y(x) = v(x)/\sqrt{x}$. Then $y' = x^{-1/2}\,v' - x^{-3/2}\,v/2$ and $y'' = x^{-1/2}\,v'' -$ $-\,x^{-3/2}\,v' + 3\,x^{-5/2}\,v/4$. Substitution into the ODE results in

$$\left[x^{3/2}\,v'' - x^{1/2}\,v' + 3\,x^{-1/2}\,v/4\right] + \left[x^{1/2}\,v' - x^{-1/2}\,v/2\right] + \left(x^2 - \frac{1}{4}\right)x^{-1/2}\,v = 0\,.$$

Simplifying, we find that

$$v'' + v = 0\,,$$

with *general solution* $v(x) = c_1 cos\,x + c_2\,sin\,x$. Hence

$$y(x) = c_1 x^{-1/2} cos\,x + c_2\,x^{-1/2} sin\,x\,.$$

8. The absolute value of the ratio of consecutive terms is

$$\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \frac{|x|^{2m+2}\,2^{2m}(m+1)!\,m!}{|x|^{2m}\,2^{2m+2}(m+2)!(m+1)!} = \frac{|x|^2}{4(m+2)(m+1)}\,.$$

Applying the *ratio test*,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \lim_{m\to\infty}\,\frac{|x|^2}{4(m+2)(m+1)} = 0\,.$$

Hence the series for $J_1(x)$ converges absolutely *for all* values of x . Furthermore, since the series for $J_0(x)$ also converges absolutely for all x, term-by-term differentiation results in

$$\begin{aligned} J_0'(x) &= \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1}\, m!(m-1)!} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\, x^{2m+1}}{2^{2m+1}(m+1)!\, m!} \\ &= -\frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2^{2m}(m+1)!\, m!} \,. \end{aligned}$$

Therefore, $J_0'(x) = -J_1(x)$.

$9(a)$. Note that $x\,p(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are *both* analytic at $x = 0$. Thus $x = 0$ is a *regular* singular point. Furthermore, $p_0 = 1$ and $q_0 = -\nu^2$. Hence the *indicial equation* is $r^2 - \nu^2 = 0$, with roots $r_1 = \nu$ and $r_2 = -\nu$.

(b). Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} + \\ + \sum_{n=0}^{\infty} a_n x^{r+n+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0\left[r(r-1) + r - \nu^2\right]x^r + a_1\left[(r+1)r + (r+1) - \nu^2\right] + \\ + \sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n - \nu^2 a_n + a_{n-2}\right]x^{r+n} = 0.$$

Setting the coefficients equal to *zero*, we find that $a_1 = 0$, and

$$a_n = \frac{-1}{(r+n)^2 - \nu^2}\, a_{n-2}\,,$$

for $n \geq 2$. It follows that $a_3 = a_5 = \cdots = a_{2m+1} = \cdots = 0$. Furthermore, with $r = \nu$,

$$a_n = \frac{-1}{n(n+2\nu)}\, a_{n-2}\,.$$

So for $m = 1, 2, \cdots$,

$$\begin{aligned} a_{2m} &= \frac{-1}{2m(2m+2\nu)}\, a_{2m-2} \\ &= \frac{(-1)^m}{2^{2m}\, m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)}\, a_0\,. \end{aligned}$$

Hence one solution is

$$y_1(x) = x^{\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\cdots(m-1+\nu)(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

(c). Assuming that $r_1 - r_2 = 2\nu$ is *not* an integer, simply setting $r = -\nu$ in the above results in a second *linearly independent* solution

$$y_2(x) = x^{-\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\cdots(m-1-\nu)(m-\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

(d). The absolute value of the ratio of consecutive terms in $y_1(x)$ is

$$\begin{aligned}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| &= \frac{|x|^{2m+2}\,2^{2m}\,m!(1+v)\cdots(m+\nu)}{|x|^{2m}\,2^{2m+2}(m+1)!(1+v)\cdots(m+1+\nu)} \\ &= \frac{|x|^2}{4(m+1)(m+1+\nu)}\,.\end{aligned}$$

Applying the *ratio test*,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \lim_{m\to\infty}\frac{|x|^2}{4(m+1)(m+1+\nu)} = 0\,.$$

Hence the series for $y_1(x)$ converges absolutely *for all* values of x. The same can be shown for $y_2(x)$. Note also, that if ν is a *positive* integer, then the coefficients in the series for $y_2(x)$ are *undefined*.

$10(a)$. It suffices to calculate $L[J_0(x)\,ln\,x]$. Indeed,

$$[J_0(x)\,ln\,x]' = J_0'(x)\,ln\,x + \frac{J_0(x)}{x}$$

and

$$[J_0(x)\,ln\,x]'' = J_0''(x)\,ln\,x + 2\frac{J_0'(x)}{x} - \frac{J_0(x)}{x^2}\,.$$

Hence

$$\begin{aligned}L[J_0(x)\,ln\,x] &= x^2 J_0''(x)\,ln\,x + 2x\,J_0'(x) - J_0(x) + \\ &\quad + x\,J_0'(x)\,ln\,x + J_0(x) + x^2 J_0(x)\,ln\,x\,.\end{aligned}$$

Since $x^2 J_0''(x) + x\,J_0'(x) + x^2 J_0(x) = 0$,

$$L[J_0(x)\,ln\,x] = 2x\,J_0'(x)\,.$$

(b). Given that $L[y_2(x)] = 0$, after adjusting the indices in Part (a), we have

$$b_1 x + 2^2 b_2\, x^2 + \sum_{n=3}^{\infty}\big(n^2 b_n + b_{n-2}\big)x^n = -2x\, J_0'(x)\,.$$

Using the series representation of $J_0'(x)$ in Problem 8,

$$b_1 x + 2^2 b_2\, x^2 + \sum_{n=3}^{\infty}\big(n^2 b_n + b_{n-2}\big)x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n(2n)x^{2n}}{2^{2n}(n!)^2}\,.$$

(c). Equating the coefficients on both sides of the equation, we find that

$$b_1 = b_3 = \cdots = b_{2m+1} = \cdots = 0\,.$$

Also, with $n = 1$, $2^2 b_2 = 1/(1!)^2$, that is, $b_2 = 1/\big[2^2(1!)^2\big]$. Furthermore, for $m \geq 2$,

$$(2m)^2 b_{2m} + b_{2m-2} = -2\frac{(-1)^m(2m)}{2^{2m}(m!)^2}\,.$$

More explicitly,

$$\begin{aligned} b_4 &= -\frac{1}{2^2\,4^2}\left(1+\frac{1}{2}\right) \\ b_6 &= \frac{1}{2^2\,4^2\,6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) \\ &\vdots \end{aligned}$$

It can be shown, in general, that

$$b_{2m} = (-1)^{m+1}\frac{H_m}{2^{2m}(m!)^2}\,.$$

11. Bessel's equation of *order one* is

$$x^2 y'' + x\, y' + \big(x^2 - 1\big)y = 0\,.$$

Based on Problem 9, the roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\, x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\, x^{r+n} + \\ + \sum_{n=0}^{\infty}a_n x^{r+n+2} - \sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the *second-to-last* series, we obtain

$$a_0[r(r-1)+r-1]x^r + a_1[(r+1)r+(r+1)-1]+$$
$$+\sum_{n=2}^{\infty}[(r+n)(r+n-1)a_n+(r+n)a_n-a_n+\ a_{n-2}]x^{r+n}=0.$$

Setting the coefficients equal to *zero*, we find that $a_1 = 0$, and

$$\begin{aligned} a_n(r) &= \frac{-1}{(r+n)^2-1}\,a_{n-2}(r) \\ &= \frac{-1}{(n+r+1)(n+r-1)}\,a_{n-2}(r)\,, \end{aligned}$$

for $n \geq 2$. It follows that $a_3 = a_5 = \cdots = a_{2m+1} = \cdots = 0$. Solving the recurrence relation,

$$a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)(2m+r-1)^2\cdots(r+3)^2(r+1)}\,a_0.$$

With $r = r_1 = 1$,

$$a_{2m}(1) = \frac{(-1)^m}{2^{2m}(m+1)!\,m!}\,a_0.$$

For a *second* linearly independent solution, we follow the discussion in Section 5.7 . Since $r_1 - r_2 = N = 2$, we find that

$$a_2(r) = -\frac{1}{(r+3)(r+1)}\,,$$

with $a_0 = 1$. Hence the leading coefficient in the solution is

$$a = \lim_{r\to -1}(r+1)\,a_2(r) = -\frac{1}{2}\,.$$

Further,

$$(r+1)\,a_{2m}(r) = \frac{(-1)^m}{(2m+r+1)\,[(2m+r-1)\cdots(3+r)]^2}\,.$$

Let $A_n(r) = (r+1)\,a_n(r)$. It follows that

$$\frac{A'_{2m}(r)}{A_{2m}(r)} = -\frac{1}{2m+r+1} - 2\left[\frac{1}{2m+r-1}+\cdots+\frac{1}{3+r}\right].$$

Setting $r = r_2 = -1$, we calculate

$$\begin{aligned} c_{2m}(-1) &= -\frac{1}{2}(H_m + H_{m-1})A_{2m}(-1) \\ &= -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2m\,[(2m-2)\cdots 2]^2} \\ &= -\frac{1}{2}(H_m + H_{m-1})\frac{(-1)^m}{2^{2m-1}\,m!(m-1)!}\,. \end{aligned}$$

Note that $a_{2m+1}(r) = 0$ implies that $A_{2m+1}(r) = 0$, so

$$c_{2m+1}(-1) = \left[\frac{d}{dr}A_{2m+1}(r)\right]_{r=r_2} = 0\,.$$

Therefore,

$$y_2(x) = -\frac{1}{2}\left[x\sum_{m=0}^{\infty}\frac{(-1)^m}{(m+1)!\,m!}\left(\frac{x}{2}\right)^{2m}\right] ln\,x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty}\frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!}\left(\frac{x}{2}\right)^{2m}\right].$$

Based on the definition of $J_1(x)$,

$$y_2(x) = -J_1(x)\,ln\,x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty}\frac{(-1)^m(H_m + H_{m-1})}{m!(m-1)!}\left(\frac{x}{2}\right)^{2m}\right].$$

12. Consider a solution of the form

$$y(x) = \sqrt{x}\;f\big(\alpha x^{\beta}\big).$$

Then

$$y' = \frac{df}{d\xi}\cdot\frac{\alpha\beta\,x^{\beta}}{\sqrt{x}} + \frac{f(\xi)}{2\sqrt{x}}$$

in which $\xi = \alpha x^{\beta}$. Hence

$$y'' = \frac{d^2 f}{d\xi^2}\cdot\frac{\alpha^2\beta^2\,x^{2\beta}}{x\sqrt{x}} + \frac{df}{d\xi}\cdot\frac{\alpha\beta^2\,x^{\beta}}{x\sqrt{x}} - \frac{f(\xi)}{4x\sqrt{x}}\,,$$

and

$$x^2\,y'' = \alpha^2\beta^2\,x^{2\beta}\sqrt{x}\,\frac{d^2 f}{d\xi^2} + \alpha\beta^2\,x^{\beta}\sqrt{x}\,\frac{df}{d\xi} - \frac{1}{4}\sqrt{x}\,f(\xi)\,.$$

Substitution into the ODE results in

$$\alpha^2\beta^2\,x^{2\beta}\,\frac{d^2 f}{d\xi^2} + \alpha\beta^2\,x^{\beta}\,\frac{df}{d\xi} - \frac{1}{4}\,f(\xi) + \left(\alpha^2\beta^2\,x^{2\beta} + \frac{1}{4} - \nu^2\beta^2\right)f(\xi) = 0\,.$$

Simplifying, and setting $\xi = \alpha x^{\beta}$, we find that

$$\xi^2 \frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} + \left(\xi^2 - \nu^2\right) f(\xi) = 0\,, \quad (*)$$

which is a *Bessel* equation of *order* ν. Therefore, the general solution of the given ODE is

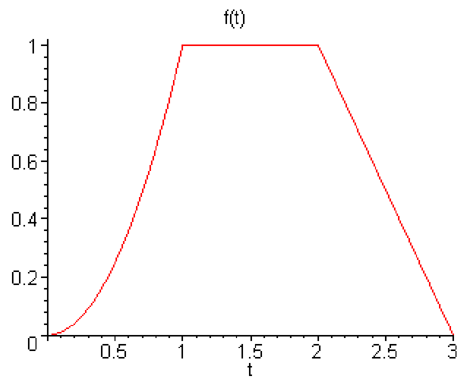
$$y(x) = \sqrt{x}\,\left[c_1\, f_1\!\left(\alpha x^\beta\right) + c_2\, f_2\!\left(\alpha x^\beta\right)\right],$$

in which $f_1(\xi)$ and $f_2(\xi)$ are the linearly independent solutions of $(*)$.

Chapter Six

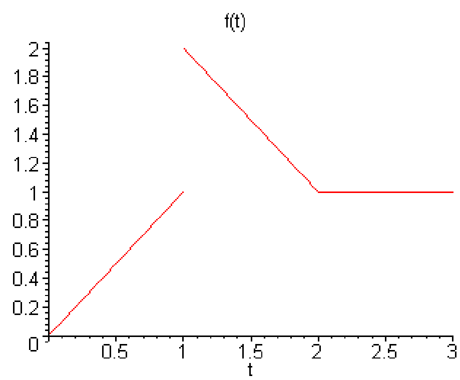
Section 6.1

3.



The function $f(t)$ is *continuous*.

4.



The function $f(t)$ has a *jump discontinuity* at $t = 1$.

7. Integration is a linear operation. It follows that

$$\begin{aligned}\int_0^A cosh\, bt \cdot e^{-st} dt &= \frac{1}{2}\int_0^A e^{bt} \cdot e^{-st} dt + \frac{1}{2}\int_0^A e^{-bt} \cdot e^{-st} dt \\ &= \frac{1}{2}\int_0^A e^{(b-s)t} dt + \frac{1}{2}\int_0^A e^{-(b+s)t} dt\,.\end{aligned}$$

Hence

$$\int_0^A cosh\, bt \cdot e^{-st} dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] + \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a *limit*, as $A \rightarrow \infty$,

$$\int_0^\infty cosh\, bt \cdot e^{-st}dt \;= \frac{1}{2}\left[\frac{1}{s-b}\right] + \frac{1}{2}\left[\frac{1}{s+b}\right] \\ = \frac{s}{s^2-b^2}\,.$$

Note that the above is valid for $s > |b|$.

8. Proceeding as in Prob. 7,

$$\int_0^A sinh\, bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1-e^{(b-s)A}}{s-b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b+s)A}}{s+b}\right].$$

Taking a *limit*, as $A \rightarrow \infty$,

$$\int_0^\infty sinh\, bt \cdot e^{-st}dt \;= \frac{1}{2}\left[\frac{1}{s-b}\right] - \frac{1}{2}\left[\frac{1}{s+b}\right] \\ = \frac{b}{s^2-b^2}\,.$$

The limit exists as long as $s > |b|$.

10. Observe that $e^{at}sinh\, bt = \left(e^{(a+b)t} - e^{(a-b)t}\right)/2$. It follows that

$$\int_0^A e^{at}\, sinh\, bt \cdot e^{-st}dt = \frac{1}{2}\left[\frac{1-e^{(a+b-s)A}}{s-a+b}\right] - \frac{1}{2}\left[\frac{1-e^{-(b-a+s)A}}{s+b-a}\right].$$

Taking a *limit*, as $A \rightarrow \infty$,

$$\int_0^\infty e^{at}\, sinh\, bt \cdot e^{-st}dt \;= \frac{1}{2}\left[\frac{1}{s-a+b}\right] - \frac{1}{2}\left[\frac{1}{s+b-a}\right] \\ = \frac{b}{(s-a)^2-b^2}\,.$$

The limit exists as long as $s - a > |b|$.

11. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\, sin\, bt] = \frac{1}{2i}\mathcal{L}\left[e^{ibt}\right] - \frac{1}{2i}\mathcal{L}\left[e^{-ibt}\right].$$

Since

$$\int_0^\infty e^{(a+ib)t}e^{-st}dt = \frac{1}{s-a-ib}\,,$$

we have

$$\int_0^\infty e^{\pm\, ibt}\, e^{-st} dt = \frac{1}{s \mp ib}\,.$$

Therefore

$$\begin{aligned}\mathcal{L}[sin\, bt] \ &= \frac{1}{2i}\left[\frac{1}{s\,-\,ib} - \frac{1}{s\,+\,ib}\right] \\ &= \frac{b}{s^2+b^2}\,.\end{aligned}$$

12. Using the *linearity* of the Laplace transform,

$$\mathcal{L}[\, cos\, bt] = \frac{1}{2}\mathcal{L}\left[e^{ibt}\right] + \frac{1}{2}\mathcal{L}\left[e^{-ibt}\right].$$

From Prob. 11 , we have

$$\int_0^\infty e^{\pm\, ibt}\, e^{-st} dt = \frac{1}{s \mp ib}\,.$$

Therefore

$$\begin{aligned}\mathcal{L}[cos\, bt] \ &= \frac{1}{2}\left[\frac{1}{s\,-\,ib} + \frac{1}{s\,+\,ib}\right] \\ &= \frac{s}{s^2+b^2}\,.\end{aligned}$$

14. Using the *linearity* of the Laplace transform,

$$\mathcal{L}\left[\, e^{at} cos\, bt\right] = \frac{1}{2}\mathcal{L}\left[e^{(a+ib)t}\right] + \frac{1}{2}\mathcal{L}\left[e^{(a-ib)t}\right].$$

Based on the integration in Prob. 11 ,

$$\int_0^\infty e^{(a\,\pm\, ib)t} e^{-st} dt = \frac{1}{s-a \mp ib}\,.$$

Therefore

$$\begin{aligned}\mathcal{L}\left[e^{at} cos\, bt\right] \ &= \frac{1}{2}\left[\frac{1}{s\,-a-\,ib} + \frac{1}{s\,-a+\,ib}\right] \\ &= \frac{s-a}{(s-a)^2+b^2}\,.\end{aligned}$$

The above is valid for $s > a$.

15. Integrating *by parts*,

$$\int_0^A te^{at}\cdot e^{-st}dt \;=\; -\left.\frac{t\,e^{(a-s)t}}{s-a}\right|_0^A + \int_0^A \frac{1}{s-a}e^{(a-s)t}dt$$
$$= \frac{1-e^{A(a-s)}+A(a-s)e^{A(a-s)}}{(s-a)^2}\,.$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty te^{at}\cdot e^{-st}dt = \frac{1}{(s-a)^2}\,.$$

Note that the limit exists as long as $s > a$.

17. Observe that $t\,cosh\,at = (t\,e^{at} + t\,e^{-at})/2$. For any value of c,

$$\int_0^A t\,e^{ct}\cdot e^{-st}dt \;=\; -\left.\frac{t\,e^{(c-s)t}}{s-c}\right|_0^A + \int_0^A \frac{1}{s-c}e^{(c-s)t}dt$$
$$= \frac{1-e^{A(c-s)}+A(c-s)e^{A(c-s)}}{(s-c)^2}\,.$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty te^{ct}\cdot e^{-st}dt = \frac{1}{(s-c)^2}\,.$$

Note that the limit exists as long as $s > |c|$. Therefore,

$$\int_0^\infty t\,cosh\,at\cdot e^{-st}dt \;=\; \frac{1}{2}\left[\frac{1}{(s-a)^2}+\frac{1}{(s+a)^2}\right]$$
$$= \frac{s^2+a^2}{(s-a)^2(s+a)^2}\,.$$

18. Integrating *by parts*,

$$\int_0^A t^n e^{at}\cdot e^{-st}dt \;=\; -\left.\frac{t^n\,e^{(a-s)t}}{s-a}\right|_0^A + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt$$
$$= -\,\frac{A^n e^{-(s-a)A}}{s-a} + \int_0^A \frac{n}{s-a}\,t^{n-1}e^{(a-s)t}dt\,.$$

Continuing to integrate by parts, it follows that

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = -\frac{A^n e^{(a-s)A}}{s-a} - \frac{nA^{n-1}e^{(a-s)A}}{(s-a)^2} - \\ - \frac{n!Ae^{(a-s)A}}{(n-2)!(s-a)^3} - \cdots - \frac{n!\left(e^{(a-s)A}-1\right)}{(s-a)^{n+1}}.$$

That is,

$$\int_0^A t^n e^{at} \cdot e^{-st} dt = p_n(A) \cdot e^{(a-s)A} + \frac{n!}{(s-a)^{n+1}},$$

in which $p_n(\xi)$ is a *polynomial* of degree n. For any *given* polynomial,

$$\lim_{A\to\infty} p_n(A) \cdot e^{-(s-a)A} = 0,$$

as long as $s > a$. Therefore,

$$\int_0^\infty t^n e^{at} \cdot e^{-st} dt = \frac{n!}{(s-a)^{n+1}}.$$

20. Observe that $t^2 sinh\, at = (t^2 e^{at} - t^2 e^{-at})/2$. Using the result in Prob. 18,

$$\int_0^\infty t^2 sinh\, at \cdot e^{-st} dt = \frac{1}{2}\left[\frac{2!}{(s-a)^3} - \frac{2!}{(s+a)^3}\right] \\ = \frac{2a(3s^2+a^2)}{(s^2-a^2)^3}.$$

The above is valid for $s > |a|$.

22. Integrating *by parts*,

$$\int_0^A t\, e^{-t} dt = -t\, e^{-t}\Big|_0^A + \int_0^A e^{-t} dt \\ = 1 - e^{-A} - Ae^{-A}.$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty t\, e^{-t} dt = 1 - e^{-A}.$$

Hence the integral *converges*.

23. Based on a series expansion, note that for $t > 0$,

$$e^t > 1 + t + t^2/2 > t^2/2.$$

It follows that for $t > 0$,

$$t^{-2}e^{t} > \frac{1}{2}.$$

Hence for any finite $A > 1$,

$$\int_1^A t^{-2}e^{t}dt > \frac{A-1}{2}.$$

It is evident that the limit as $A \to \infty$ does not exist.

24. Using the fact that $|cos\, t| \leq 1$, and the fact that

$$\int_0^\infty e^{-t}dt = 1,$$

it follows that the given integral *converges*.

25(a). Let $p > 0$. Integrating *by parts*,

$$\begin{aligned}\int_0^A e^{-x}x^{p}dx &= -e^{-x}x^{p}\Big|_0^A + p\int_0^A e^{-x}x^{p-1}dx \\ &= -A^{p}e^{-A} + p\int_0^A e^{-x}x^{p-1}dx.\end{aligned}$$

Taking a *limit*, as $A \to \infty$,

$$\int_0^\infty e^{-x}x^{p}dx = p\int_0^\infty e^{-x}x^{p-1}dx.$$

That is, $\Gamma(p+1) = p\,\Gamma(p)$.

(b). Setting $p = 0$,

$$\Gamma(1) = \int_0^\infty e^{-x}dx = 1.$$

(c). Let $p = n$. Using the result in Part (b),

$$\begin{aligned}\Gamma(n+1) &= n\,\Gamma(n) \\ &= n(n-1)\Gamma(n-1) \\ &\;\;\vdots \\ &= n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1).\end{aligned}$$

Since $\Gamma(1) = 1$, $\Gamma(n+1) = n!$.

(d). Using the result in Part (b),

$$
\begin{aligned}
\Gamma(p+n) &= (p+n-1)\,\Gamma(p+n-1) \\
&= (p+n-1)(p+n-2)\Gamma(p+n-2) \\
&\;\vdots \\
&= (p+n-1)(p+n-2)\cdots(p+1)p\,\Gamma(p)\,.
\end{aligned}
$$

Hence

$$
\frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)(p+1)\cdots(p+n-1)\,.
$$

Given that $\Gamma(1/2) = \sqrt{\pi}$, it follows that

$$
\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}
$$

and

$$
\Gamma\left(\frac{11}{2}\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{945\sqrt{\pi}}{32}\,.
$$

Section 6.2

1. Write the function as

$$\frac{3}{s^2+4} = \frac{3}{2}\,\frac{2}{s^2+4}\,.$$

Hence $\mathcal{L}^{-1}[Y(s)] = \frac{3}{2}\,sin\,2t\,.$

3. Using *partial fractions*,

$$\frac{2}{s^2+3s-4} = \frac{2}{5}\left[\frac{1}{s-1} - \frac{1}{s+4}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)] = \frac{2}{5}\left(e^t - e^{-4t}\right)$.

5. Note that the denominator s^2+2s+5 is *irreducible* over the reals. Completing the square, $s^2+2s+5 = (s+1)^2+4$. Now convert the function to a *rational function* of the variable $\xi = s+1$. That is,

$$\frac{2s+2}{s^2+2s+5} = \frac{2(s+1)}{(s+1)^2+4}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4}\right] = 2\,cos\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s+2}{s^2+2s+5}\right] = 2e^{-t}cos\,2t\,.$$

6. Using *partial fractions*,

$$\frac{2s-3}{s^2-4} = \frac{1}{4}\left[\frac{1}{s-2} + \frac{7}{s+2}\right].$$

Hence $\mathcal{L}^{-1}[Y(s)] = \frac{1}{4}(e^{2t} + 7e^{-2t})$. Note that we can also write

$$\frac{2s-3}{s^2-4} = 2\,\frac{s}{s^2-4} - \frac{3}{2}\,\frac{2}{s^2-4}.$$

8. Using *partial fractions*,

$$\frac{8s^2-4s+12}{s(s^2+4)} = 3\,\frac{1}{s} + 5\,\frac{s}{s^2+4} - 2\,\frac{2}{s^2+4}.$$

Hence $\mathcal{L}^{-1}[Y(s)] = 3 + 5\,cos\,2t - 2\,sin\,2t$.

9. The denominator $s^2 + 4s + 5$ is *irreducible* over the reals. Completing the square, $s^2 + 4s + 5 = (s+2)^2 + 1$. Now convert the function to a *rational function* of the variable $\xi = s + 2$. That is,

$$\frac{1-2s}{s^2+4s+5} = \frac{5-2(s+2)}{(s+2)^2+1} \,.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{5}{\xi^2+1} - \frac{2\,\xi}{\xi^2+1}\right] = 5\,sin\,t - 2\,cos\,t \,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1-2s}{s^2+4s+5}\right] = e^{-2t}(5\,sin\,t - 2\,cos\,t) \,.$$

10. Note that the denominator $s^2 + 2s + 10$ is *irreducible* over the reals. Completing the square, $s^2 + 2s + 10 = (s+1)^2 + 9$. Now convert the function to a *rational function* of the variable $\xi = s + 1$. That is,

$$\frac{2s-3}{s^2+2s+10} = \frac{2(s+1)-5}{(s+1)^2+9} \,.$$

We find that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+9} - \frac{5}{\xi^2+9}\right] = 2\,cos\,3t - \frac{5}{3}\,sin\,3t \,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2+2s+10}\right] = e^{-t}\left(2\,cos\,3t - \frac{5}{3}\,sin\,3t\right) .$$

12. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 3[s\,Y(s) - y(0)] + 2\,Y(s) = 0 \,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) - s - 3 = 0 \,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s+3}{s^2+3s+2} \,.$$

Using *partial fractions*,

$$\frac{s+3}{s^2+3s+2} = \frac{2}{s+1} - \frac{1}{s+2}.$$

Hence $y(t) = \mathcal{L}^{-1}[Y(s)] = 2\,e^{-t} - e^{-2t}$.

13. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - 1 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{1}{s^2-2s+2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. That is,

$$\frac{1}{s^2-2s+2} = \frac{1}{(s-1)^2+1}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{1}{\xi^2+1}\right] = sin\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+2}\right] = e^t sin\,t\,.$$

Hence $y(t) = e^t sin\,t$.

15. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] - 2\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) - 2\,Y(s) - 2s + 4 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s-4}{s^2-2s-2}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s-1$. Completing the square,

$$\frac{2s-4}{s^2-2s-2} = \frac{2(s-1)-2}{(s-1)^2-3}.$$

First note that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2-3} - \frac{2}{\xi^2-3}\right] = 2\,cosh\,\sqrt{3}\,t - \frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s-4}{s^2-2s-2}\right] = e^t\left(2\,cosh\,\sqrt{3}\,t - \frac{2}{\sqrt{3}}\,sinh\,\sqrt{3}\,t\right).$$

16. Taking the Laplace transform of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + 5\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 2s\,Y(s) + 5\,Y(s) - 2s - 3 = 0\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{2s+3}{s^2+2s+5}\,.$$

Since the denominator is *irreducible*, write the transform as a function of $\xi = s+1$. That is,

$$\frac{2s+3}{s^2+2s+5} = \frac{2(s+1)+1}{(s+1)^2+4}.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+4} + \frac{1}{\xi^2+4}\right] = 2\,cos\,2t + \frac{1}{2}\,sin\,2t\,.$$

Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$, the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{2s+3}{s^2+2s+5}\right] = e^{-t}\left(2\,cos\,2t + \frac{1}{2}\,sin\,2t\right).$$

17. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\left[s^3Y(s) - s^2\,y(0) - s\,y'(0) - y''(0)\right] +$$
$$+\,6\left[s^2\,Y(s) - s\,y(0) - y'(0)\right] - 4[s\,Y(s) - y(0)] + Y(s) = 0$$

Applying the *initial conditions*,

$$s^4Y(s) - 4s^3Y(s) + 6s^2\,Y(s) - 4s\,Y(s) + \,Y(s) - s^2 + 4s - 7 = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2\, - 4s\, + 1} = \frac{s^2 - 4s + 7}{(s-1)^4}\,.$$

Using *partial fractions*,

$$\frac{s^2 - 4s + 7}{(s-1)^4} = \frac{4}{(s-1)^4} - \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2}.$$

Note that $\mathcal{L}[\,t^n\,] = (n!)/s^{n+1}$ and $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s \to s-a}$. Hence the solution of the IVP is

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2 - 4s + 7}{(s-1)^4}\right] = \frac{2}{3}\,t^3e^t - t^2e^t + t\,e^t.$$

18. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - \,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,Y(s) - s^3 - s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 - 1}\,.$$

By inspection, it follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right] = cosh\,t\,.$

19. Taking the Laplace transform of the ODE, we obtain

$$s^4\,Y(s) - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - 4\,Y(s) = 0\,.$$

Applying the *initial conditions*,

$$s^4Y(s) - \,4Y(s) - s^3 + 2s = 0\,.$$

Solving for the transform of the solution,

$$Y(s) = \frac{s}{s^2 + 2}\,.$$

It follows that $y(t) = \mathcal{L}^{-1}\left[\frac{s}{s^2+2}\right] = cos\,\sqrt{2}\;t\,.$

20. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + \omega^2\,Y(s) = \frac{s}{s^2+4}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + \omega^2\,Y(s) - s = \frac{s}{s^2+4}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{1}{4-\omega^2}\left[\frac{s}{s^2+\omega^2} - \frac{s}{s^2+4}\right].$$

First note that

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = cos\,\omega t \quad \text{and} \quad \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = cos\,2t\,.$$

Hence the solution of the IVP is

$$\begin{aligned} y(t) &= \frac{1}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t + cos\,\omega t \\ &= \frac{5-\omega^2}{4-\omega^2}\,cos\,\omega t - \frac{1}{4-\omega^2}\,cos\,2t\,. \end{aligned}$$

21. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) - 2[s\,Y(s) - y(0)] + 2\,Y(s) = \frac{s}{s^2+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) - 2s\,Y(s) + 2\,Y(s) - s + 2 = \frac{s}{s^2+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{s}{(s^2-2s+2)(s^2+1)} + \frac{s-2}{s^2-2s+2}\,.$$

Using *partial fractions* on the first term,

$$\frac{s}{(s^2-2s+2)(s^2+1)} = \frac{1}{5}\left[\frac{s-2}{s^2+1} - \frac{s-4}{s^2-2s+2}\right].$$

Thus we can write

$$Y(s) = \frac{1}{5}\,\frac{s}{s^2+1} - \frac{2}{5}\,\frac{1}{s^2+1} + \frac{2}{5}\,\frac{2s-3}{s^2-2s+2}\,.$$

For the *last term*, we note that $s^2 - 2s + 2 = (s-1)^2 + 1$. So that

$$\frac{2s-3}{s^2-2s+2} = \frac{2(s-1)-1}{(s-1)^2+1}\,.$$

We know that

$$\mathcal{L}^{-1}\left[\frac{2\,\xi}{\xi^2+1} - \frac{1}{\xi^2+1}\right] = 2\,cos\,t - sin\,t\,.$$

Based on the *translation property* of the Laplace transform,

$$\mathcal{L}^{-1}\left[\frac{2s-3}{s^2-2s+2}\right] = e^t(2\,cos\,t - sin\,t).$$

Combining the above, the solution of the IVP is

$$y(t) = \frac{1}{5}\,cos\,t - \frac{2}{5}\,sin\,t + \frac{2}{5}\,e^t(2\,cos\,t - sin\,t)\,.$$

23. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + 2[s\,Y(s) - y(0)] + Y(s) = \frac{4}{s+1}\,.$$

Applying the *initial conditions*,

$$s^2\,Y(s) + 2s\,Y(s) + Y(s) - 2s - 3 = \frac{4}{s+1}\,.$$

Solving for $Y(s)$, the transform of the solution is

$$Y(s) = \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2}\,.$$

First write

$$\frac{2s+3}{(s+1)^2} = \frac{2(s+1)+1}{(s+1)^2} = \frac{2}{s+1} + \frac{1}{(s+1)^2}\,.$$

We note that

$$\mathcal{L}^{-1}\left[\frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2}\right] = 2\,t^2 + 2 + t\,.$$

So based on the *translation property* of the Laplace transform, the solution of the IVP is

$$y(t) = 2\,t^2 e^{-t} + t\,e^{-t} + 2\,e^{-t}.$$

25. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform
of both sides of the ODE, we obtain

$$s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\,Y(s) + Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f(t)] &= \int_0^\infty f(t)\,e^{-st}dt \\ &= \int_0^1 t\,e^{-st}dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}\,.\end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\frac{s+1}{s^2(s^2+1)}\,.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

and

$$\frac{s}{s^2(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}\,.$$

We find, by inspection, that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = t - sin\,t\,.$$

Referring to *Line* 13 , in Table 6.2.1 ,

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Let

$$\mathcal{L}[\,g(t)\,] = \frac{s+1}{s^2(s^2+1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\,.$$

Then $g(t) = 1 + t - cos\, t - sin\, t$. It follows, therefore, that

$$\mathcal{L}^{-1}\left[e^{-s} \cdot \frac{s+1}{s^2(s^2+1)}\right] = u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,]\,.$$

Combining the above, the solution of the IVP is

$$y(t) = t - sin\, t - u_1(t)[\,1 + (t-1) - cos(t-1) - sin(t-1)\,]\,.$$

26. Let $f(t)$ be the *forcing function* on the right-hand-side. Taking the Laplace transform
of both sides of the ODE, we obtain

$$s^2\, Y(s) - s\, y(0) - y'(0) + 4\, Y(s) = \mathcal{L}[f(t)].$$

Applying the *initial conditions*,

$$s^2\, Y(s) + \ 4\, Y(s) = \mathcal{L}[f(t)].$$

Based on the definition of the Laplace transform,

$$\begin{aligned}\mathcal{L}[f(t)] \ &= \int_0^\infty f(t)\, e^{-st} dt \\ &= \int_0^1 t\, e^{-st} dt + \int_1^\infty e^{-st} dt \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}\,.\end{aligned}$$

Solving for the transform,

$$Y(s) = \frac{1}{s^2(s^2+4)} - e^{-s}\frac{1}{s^2(s^2+4)}\,.$$

Using *partial fractions*,

$$\frac{1}{s^2(s^2+4)} = \frac{1}{4}\left[\frac{1}{s^2} - \frac{1}{s^2+4}\right].$$

We find that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+4)}\right] = \frac{1}{4}\, t - \frac{1}{8}\, sin\, t\,.$$

Referring to *Line* 13 , in Table 6.2.1 ,

$$\mathcal{L}[\, u_c(t) f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

It follows that

$$\mathcal{L}^{-1}\left[e^{-s}\cdot\frac{1}{s^2(s^2+4)}\right]=u_1(t)\left[\frac{1}{4}(t-1)-\frac{1}{8}\,sin(t-1)\right].$$

Combining the above, the solution of the IVP is

$$y(t)=\frac{1}{4}\,t-\frac{1}{8}\,sin\,t-u_1(t)\left[\frac{1}{4}(t-1)-\frac{1}{8}\,sin(t-1)\right].$$

$28(a)$. Assuming that the conditions of Theorem 6.2.1 are satisfied,

$$\begin{aligned}F'(s)&=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt\\&=\int_0^\infty\frac{\partial}{\partial s}\big[e^{-st}f(t)\big]dt\\&=\int_0^\infty\big[-t\,e^{-st}f(t)\big]dt\\&=\int_0^\infty e^{-st}[-tf(t)]dt\,.\end{aligned}$$

(b). Using *mathematical induction*, suppose that for some $k\geq 1$,

$$F^{(k)}(s)=\int_0^\infty e^{-st}\Big[(-t)^k f(t)\Big]dt\,.$$

Differentiating both sides,

$$\begin{aligned}F^{(k+1)}(s)&=\frac{d}{ds}\int_0^\infty e^{-st}\Big[(-t)^k f(t)\Big]dt\\&=\int_0^\infty\frac{\partial}{\partial s}\Big[e^{-st}(-t)^k f(t)\Big]dt\\&=\int_0^\infty\Big[-t\,e^{-st}(-t)^k f(t)\Big]dt\\&=\int_0^\infty e^{-st}\Big[(-t)^{k+1}f(t)\Big]dt\,.\end{aligned}$$

29. We know that

$$\mathcal{L}\big[e^{at}\big]=\frac{1}{s-a}\,.$$

Based on Prob. 28,

$$\mathcal{L}\big[-t\,e^{at}\big]=\frac{d}{ds}\left[\frac{1}{s-a}\right].$$

Therefore,

$$\mathcal{L}\big[t\,e^{at}\big] = \frac{1}{(s-a)^2}.$$

31. Based on Prob. 28,

$$\begin{aligned}\mathcal{L}[(-t)^n] &= \frac{d^n}{ds^n}\mathcal{L}[1] \\ &= \frac{d^n}{ds^n}\left[\frac{1}{s}\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{L}[t^n] &= (-1)^n\frac{(-1)^n n!}{s^{n+1}} \\ &= \frac{n!}{s^{n+1}}.\end{aligned}$$

33. Using the *translation property* of the Laplace transform,

$$\mathcal{L}\big[e^{at}\,sin\,bt\big] = \frac{b}{(s-a)^2+b^2}.$$

Therefore,

$$\begin{aligned}\mathcal{L}\big[t\,e^{at}\,sin\,bt\big] &= -\frac{d}{ds}\left[\frac{b}{(s-a)^2+b^2}\right] \\ &= \frac{2b(s-a)}{(s^2-2as+a^2+b^2)^2}.\end{aligned}$$

34. Using the *translation property* of the Laplace transform,

$$\mathcal{L}\big[e^{at}\,cos\,bt\big] = \frac{s-a}{(s-a)^2+b^2}.$$

Therefore,

$$\begin{aligned}\mathcal{L}\big[t\,e^{at}\,cos\,bt\big] &= -\frac{d}{ds}\left[\frac{s-a}{(s-a)^2+b^2}\right] \\ &= \frac{(s-a)^2-b^2}{(s^2-2as+a^2+b^2)^2}.\end{aligned}$$

35(a). Taking the Laplace transform of the given *Bessel equation*,

$$\mathcal{L}[\,t\,y''] + \mathcal{L}[\;y'] + \mathcal{L}[\,t\,y\,] = 0\,.$$

Using the *differentiation property* of the transform,

$$-\frac{d}{ds}\mathcal{L}[\,y''] + \mathcal{L}[\;y'] - \frac{d}{ds}\mathcal{L}[\,y\,] = 0\,.$$

That is,

$$-\frac{d}{ds}\big[s^2Y(s) - s\,y(0) - y'(0)\big] + s\,Y(s) -\; y(0) - \frac{d}{ds}Y(s) = 0\,.$$

It follows that

$$\big(1+s^2\big)Y'(s) + s\,Y(s) = 0\,.$$

(b). We obtain a *first-order linear* ODE in $Y(s)$:

$$Y'(s) + \frac{s}{s^2+1}Y(s) = 0\,,$$

with *integrating factor*

$$\mu(s) = exp\bigg(\int\frac{s}{s^2+1}ds\bigg) = \sqrt{s^2+1}\;.$$

The first-order ODE can be written as

$$\frac{d}{ds}\Big[\sqrt{s^2+1}\cdot Y(s)\Big] = 0\,,$$

with solution

$$Y(s) = \frac{c}{\sqrt{s^2+1}}\;.$$

(c). In order to obtain *negative* powers of s, first write

$$\frac{1}{\sqrt{s^2+1}} = \frac{1}{s}\left[1+\frac{1}{s^2}\right]^{-1/2}.$$

Expanding $\left(1+\frac{1}{s^2}\right)^{-1/2}$ in a *binomial series*,

$$\frac{1}{\sqrt{1+(1/s^2)}} = 1 - \frac{1}{2}\,s^{-2} + \frac{1\cdot 3}{2\cdot 4}\,s^{-4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,s^{-6} + \cdots,$$

valid for $s^{-2} < 1$. Hence, we can formally express $Y(s)$ as

$$Y(s) = c\left[\frac{1}{s} - \frac{1}{2}\,\frac{1}{s^3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{1}{s^5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{s^7} + \cdots\right].$$

Assuming that *term-by-term* inversion is valid,

$$\begin{aligned} y(t) &= c\left[1 - \frac{1}{2}\frac{t^2}{2!} + \frac{1\cdot 3}{2\cdot 4}\frac{t^4}{4!} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{t^6}{6!} + \cdots\right] \\ &= c\left[1 - \frac{2!}{2^2}\frac{t^2}{2!} + \frac{4!}{2^2\cdot 4^2}\frac{t^4}{4!} - \frac{6!}{2^2\cdot 4^2\cdot 6^2}\frac{t^6}{6!} + \cdots\right]. \end{aligned}$$

It follows that

$$\begin{aligned} y(t) &= c\left[1 - \frac{1}{2^2}t^2 + \frac{1}{2^2\cdot 4^2}t^4 - \frac{1}{2^2\cdot 4^2\cdot 6^2}t^6 + \cdots\right] \\ &= c\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}t^{2n}. \end{aligned}$$

The series is evidently the expansion, about $x = 0$, of $J_0(t)$.

36(b). Taking the Laplace transform of the given *Legendre equation*,

$$\mathcal{L}[\,y''] - \mathcal{L}\left[t^2 y''\right] - 2\,\mathcal{L}[t\,y'] + \alpha(\alpha+1)\mathcal{L}[y] = 0\,.$$

Using the *differentiation property* of the transform,

$$\mathcal{L}[\,y''] - \frac{d^2}{ds^2}\mathcal{L}[y''] + 2\frac{d}{ds}\mathcal{L}[\,y'] + \alpha(\alpha+1)\mathcal{L}[y] = 0\,.$$

That is,

$$\begin{aligned} &\left[s^2Y(s) - s\,y(0) - y'(0)\right] - \frac{d^2}{ds^2}\left[s^2Y(s) - s\,y(0) - y'(0)\right] + \\ &\qquad + 2\frac{d}{ds}[s\,Y(s) - y(0)] + \alpha(\alpha+1)Y(s) = 0\,. \end{aligned}$$

Invoking the *initial conditions*, we have

$$s^2Y(s) - 1 - \frac{d^2}{ds^2}\left[s^2Y(s) - 1\right] + 2\frac{d}{ds}[s\,Y(s)] + \alpha(\alpha+1)Y(s) = 0\,.$$

After carrying out the differentiation, the equation simplifies to

$$\frac{d^2}{ds^2}\left[s^2Y(s)\right] - 2\frac{d}{ds}[s\,Y(s)] - \left[s^2 + \alpha(\alpha+1)\right]Y(s) = -1\,.$$

That is,

$$s^2\frac{d^2}{ds^2}Y(s) + 2s\,\frac{d}{ds}Y(s) - \left[s^2 + \alpha(\alpha+1)\right]Y(s) = -1\,.$$

37. By definition of the Laplace transform, given the appropriate conditions,

$$\begin{aligned}\mathcal{L}[\,g(t)] &= \int_0^\infty e^{-st}\left[\int_0^t f(\tau)d\tau\right]dt \\ &= \int_0^\infty \int_0^t e^{-st} f(\tau)d\tau dt\,.\end{aligned}$$

Assuming that the order of integration can be exchanged,

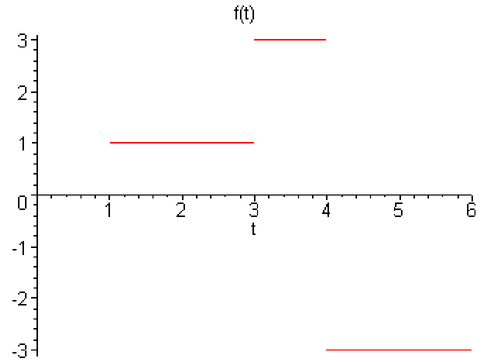
$$\begin{aligned}\mathcal{L}[\,g(t)] &= \int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st}dt\right]d\tau \\ &= \int_0^\infty f(\tau)\left[\frac{e^{-s\tau}}{s}\right]d\tau.\end{aligned}$$

[Note the *region* of integration is the area between the lines $\tau(t) = t$ and $\tau(t) = 0$.]
Hence

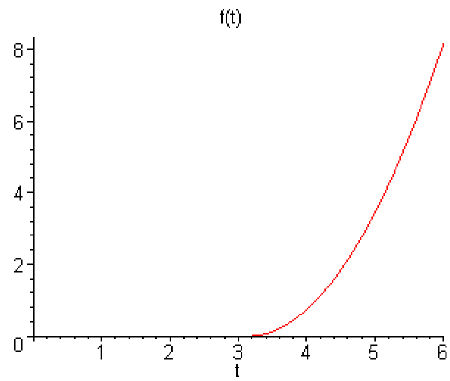
$$\begin{aligned}\mathcal{L}[\,g(t)] &= \frac{1}{s}\int_0^\infty f(\tau)\, e^{-s\tau}d\tau \\ &= \frac{1}{s}\,\mathcal{L}[\,f(t)].\end{aligned}$$

Section 6.3

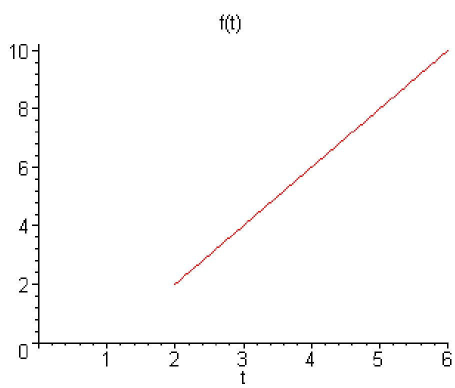
1.



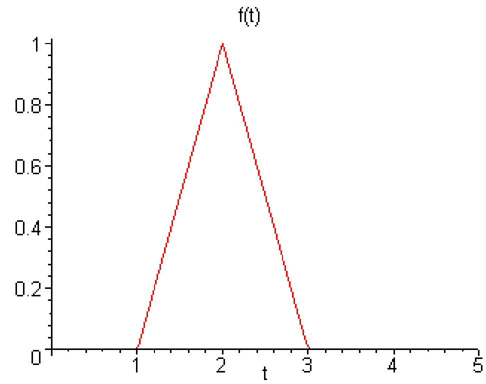
3.



5.



6.



7. Using the Heaviside function, we can write

$$f(t) = (t-2)^2\, u_2(t)\,.$$

The Laplace transform has the property that

$$\mathcal{L}[\,u_c(t)f(t-c)] = e^{-cs}\mathcal{L}[f(t)].$$

Hence

$$\mathcal{L}\big[\,(t-2)^2\,u_2(t)\big] = \frac{2\,e^{-2s}}{s^2}.$$

9. The function can be expressed as

$$f(t) = (t-\pi)[u_\pi(t) - u_{2\pi}(t)].$$

Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-\pi)\,u_\pi(t) - (t-2\pi)\,u_{2\pi}(t) - \pi\,u_{2\pi}(t)\,.$$

It follows that

$$\mathcal{L}[\,f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s}.$$

10. It follows directly from the *translation property* of the transform that

$$\mathcal{L}[\,f(t)] = \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s}.$$

11. Before invoking the *translation property* of the transform, write the function as

$$f(t) = (t-2)\,u_2(t) - \;u_2(t) - (t-3)\,u_3(t) - u_3(t)\,.$$

It follows that

$$\mathcal{L}[\,f(t)] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}.$$

12. It follows directly from the $translation\,property$ of the transform that

$$\mathcal{L}[\,f(t)] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

13. Using the fact that $\mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]_{s\to s-a}$,

$$\mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3 e^{2t}\,.$$

15. First consider the function

$$G(s) = \frac{2(s-1)}{s^2-2s+2}\,.$$

Completing the square in the denominator,

$$G(s) = \frac{2(s-1)}{(s-1)^2+1}\,.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = 2\,e^t\,cos\,t\,.$$

Hence

$$\mathcal{L}^{-1}\left[e^{-2s}G(s)\right] = 2\,e^{(t-2)}cos\,(t-2)\,u_2(t)\,.$$

16. The *inverse transform* of the function $2/(s^2-4)$ is $f(t) = sinh\,2t$. Using the $translation\,property$ of the transform,

$$\mathcal{L}^{-1}\left[\frac{2\,e^{-2s}}{s^2-4}\right] = sinh\,2(t-2)\cdot\,u_2(t)\,.$$

17. First consider the function

$$G(s) = \frac{(s-2)}{s^2-4s+3}\,.$$

Completing the square in the denominator,

$$G(s) = \frac{(s-2)}{(s-2)^2 - 1}\,.$$

It follows that

$$\mathcal{L}^{-1}[G(s)] = e^{2t}\,cosh\,t\,.$$

Hence

$$\mathcal{L}^{-1}\left[\frac{(s-2)e^{-s}}{s^2-4s+3}\right] = e^{2(t-1)}cosh\,(t-1)\,u_1(t)\,.$$

18. Write the function as

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}\,.$$

It follows from the $translation\;property$ of the transform, that

$$\mathcal{L}^{-1}\left[\frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}\right] = u_1(t)+u_2(t)-u_3(t)-u_4(t)\,.$$

19(a). By definition of the Laplace transform,

$$\mathcal{L}[\,f(ct)] = \int_0^\infty e^{-st}f(ct)dt\,.$$

Making a change of variable, $\tau = ct$, we have

$$\begin{aligned}\mathcal{L}[\,f(ct)] &= \frac{1}{c}\int_0^\infty e^{-s(\tau/c)}f(\tau)d\tau \\ &= \frac{1}{c}\int_0^\infty e^{-(s/c)\tau}f(\tau)d\tau\,.\end{aligned}$$

Hence $\mathcal{L}[\,f(ct)] = \frac{1}{c}\,F\left(\frac{s}{c}\right)$, where $s/c > a$.

(b). Using the result in Part (a),

$$\mathcal{L}\left[\,f\left(\frac{t}{k}\right)\right] = k\,F(ks).$$

Hence

$$\mathcal{L}^{-1}[F(ks)] = \frac{1}{k}\,f\left(\frac{t}{k}\right).$$

(c). From Part (b),

$$\mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Note that $as + b = a(s + b/a)$. Using the fact that $\mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)]_{s \to s-c}$,

$$\mathcal{L}^{-1}[F(as+b)] = e^{-bt/a} \, \frac{1}{a} f\left(\frac{t}{a}\right).$$

20. First write

$$F(s) = \frac{n!}{\left(\frac{s}{2}\right)^{n+1}}.$$

Let $G(s) = n!/s^{n+1}$. Based on the results in Prob. 19,

$$\frac{1}{2} \mathcal{L}^{-1}\left[G\left(\frac{s}{2}\right)\right] = g(2t),$$

in which $g(t) = t^n$. Hence

$$\mathcal{L}^{-1}[F(s)] = 2\,(2t)^n = 2^{n+1} t^n.$$

23. First write

$$F(s) = \frac{e^{-4(s-1/2)}}{2(s-1/2)}.$$

Now consider

$$G(s) = \frac{e^{-2s}}{s}.$$

Using the result in Prob. $19(b)$,

$$\mathcal{L}^{-1}[G(2s)] = \frac{1}{2} g\left(\frac{t}{2}\right),$$

in which $g(t) = u_2(t)$. Hence $\mathcal{L}^{-1}[G(2s)] = \frac{1}{2}\, u_2(t/2) = \frac{1}{2}\, u_4(t)$. It follows that

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2} e^{t/2}\, u_4(t).$$

24. By definition of the Laplace transform,

$$\mathcal{L}[\,f(t)] = \int_0^\infty e^{-st} u_1(t)dt\,.$$

That is,

$$\begin{aligned}\mathcal{L}[\,f(t)] &= \int_0^1 e^{-st}\,dt \\ &= \frac{1-e^{-s}}{s}\,.\end{aligned}$$

25. First write the function as $f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$. It follows that

$$\mathcal{L}[\,f(t)] = \int_0^1 e^{-st}dt + \int_2^3 e^{-st}dt\,.$$

That is,

$$\begin{aligned}\mathcal{L}[\,f(t)] &= \frac{1-e^{-s}}{s} + \frac{e^{-2s}-e^{-3s}}{s} \\ &= \frac{1-e^{-s}+e^{-2s}-e^{-3s}}{s}\,.\end{aligned}$$

26. The transform may be computed directly. On the other hand, using the *translation property* of the transform,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{1}{s} + \sum_{k=1}^{2n+1}(\,-1)^k\,\frac{e^{-ks}}{s} \\ &= \frac{1}{s}\left[\sum_{k=0}^{2n+1}(\,-e^{-s})^k\right] \\ &= \frac{1}{s}\,\frac{1-(\,-e^{-s})^{2n+2}}{1+e^{-s}}\,.\end{aligned}$$

That is,

$$\mathcal{L}[f(t)] = \frac{1-(e^{-2s})^{n+1}}{s(1+e^{-s})}\,.$$

29. The given function is *periodic*, with $T = 2$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt = \frac{1}{1-e^{-2s}}\int_0^1 e^{-st}dt\,.$$

That is,

$$\mathcal{L}[f(t)] = \frac{1-e^{-s}}{s(1-e^{-2s})} = \frac{1}{s(1+e^{-s})}\,.$$

31. The function is *periodic*, with $T = 1$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-s}}\int_0^1 t\,e^{-st}dt\,.$$

It follows that

$$\mathcal{L}[f(t)] = \frac{1-e^{-s}(1+s)}{s^2(1-e^{-s})}\,.$$

32. The function is *periodic*, with $T = \pi$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-\pi s}}\int_0^\pi sin\,t\cdot e^{-st}dt\,.$$

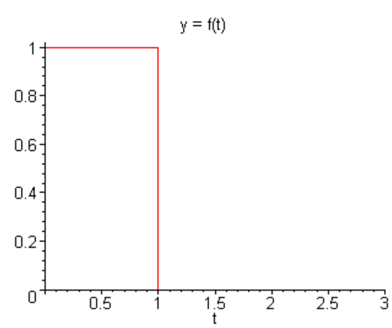
We first calculate

$$\int_0^\pi sin\,t\cdot e^{-st}dt = \frac{1+e^{-\pi s}}{1+s^2}\,.$$

Hence

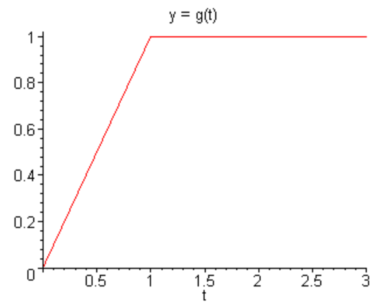
$$\mathcal{L}[f(t)] = \frac{1+e^{-\pi s}}{(1-e^{-\pi s})(1+s^2)}\,.$$

33(a).



$$\mathcal{L}[f(t)] = \mathcal{L}[1] - \mathcal{L}[u_1(t)] = \frac{1}{s} - \frac{e^{-s}}{s}\,.$$

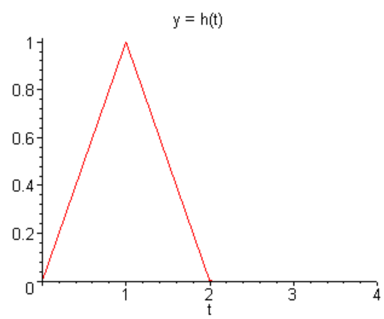
(b).



Let $F(s) = \mathcal{L}[1 - u_1(t)]$. Then

$$\mathcal{L}\left[\int_0^t [1 - u_1(\tau)]d\tau\right] = \frac{1}{s}\,F(s) = \frac{1 - e^{-s}}{s^2}\,.$$

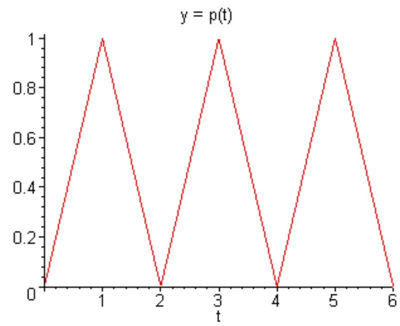
(c).



Let $G(s) = \mathcal{L}[g(t)]$. Then

$$\begin{aligned}\mathcal{L}[h(t)] &= G(s) - e^{-s}\,G(s) \\ &= \frac{1 - e^{-s}}{s^2} - e^{-s}\,\frac{1 - e^{-s}}{s^2} \\ &= \frac{(1 - e^{-s})^2}{s^2}\,.\end{aligned}$$

$34(a)$.



(b). The given function is *periodic*, with $T = 2$. Using the result of Prob. 28,

$$\mathcal{L}[f(t)] = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} p(t) dt \, .$$

Based on the piecewise definition of $p(t)$,

$$\begin{aligned} \int_0^2 e^{-st} p(t) dt &= \int_0^1 t \, e^{-st} dt + \int_1^2 (2 - t) e^{-st} dt \\ &= \frac{1}{s^2} (1 - e^{-s})^2 . \end{aligned}$$

Hence

$$\mathcal{L}[p(t)] = \frac{(1 - e^{-s})}{s^2 (1 + e^{-s})} \, .$$

(c). Since $p(t)$ satisfies the hypotheses of Theorem 6.2.1 ,

$$\mathcal{L}[p'(t)] = s \, \mathcal{L}[p(t)] - p(0) \, .$$

Using the result of Prob. 30,

$$\mathcal{L}[p'(t)] = \frac{(1 - e^{-s})}{s(1 + e^{-s})} \, .$$

We note the $p(0) = 0$, hence

$$\mathcal{L}[p(t)] = \frac{1}{s} \left[\frac{(1 - e^{-s})}{s(1 + e^{-s})} \right] .$$