Dynamics and Stability Summary

1. Basic concepts

There are quite some advanced techniques in the Dynamics and Stability course. Before we're going to examine those, we need to make sure we have a good foundation. We will examine some basic concepts. You will have seen most of them before. So we'll go through them quickly.

1.1 Using coordinate systems

1.1.1 Coordinate systems

Kinematics is the study concerned with describing motion. To describe motion, we need to have **coordinate systems**. Examples are **normal**, **cylindrical** and **spherical** coordinate systems. In these coordinates systems, we denote the **position** of a particle by the vector $\mathbf{r}$.

Every vector has components. We can, for example, write $\mathbf{r}$ in the normal coordinate system as

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}. \tag{1.1.1}$$

Here, $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are **unit vectors**. (They per definition have length 1. This is also indicated by the hats.) x, y and z are the components. They are different in different coordinate systems. To transform vectors between coordinate systems, we use **transformation matrices** $[T]$. To transform a vector $\mathbf{r_1}$ in coordinate system 1 to the same vector $\mathbf{r_2}$ in coordinate system 2, we can use

$$\mathbf{r_2} = [T]_1^2 \mathbf{r_1}. \tag{1.1.2}$$

Transformation matrices are **orthogonal**. This implies that $[T]^{-1} = [T]^T$. The inverse equals the transpose.

1.1.2 Changing vectors

Next to the velocity, also the **velocity v** and the **acceleration a** are vectors. They satisfy

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}. \tag{1.1.3}$$

We see that the time rate of change is important. Now let's examine a vector $\mathbf{r} = r\hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a unit vector in some coordinate system 1. We now examine $\mathbf{r}$ from another coordinate system 2. Coordinate system 1 has a **rotation vector** ω with respect to system 2. The time derivative of $\mathbf{r}$ is now given by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}} + r\dot{\hat{\mathbf{e}}}. \tag{1.1.4}$$

It can be shown that $\dot{\hat{\mathbf{e}}} = \omega \times \hat{\mathbf{e}}$. This implies that

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}} + r\left(\omega \times \hat{\mathbf{e}}\right). \tag{1.1.5}$$

1.1.3 Non-inertial reference frames

The basis of dynamics and stability are Newton's laws. But sadly, Newton's laws are not valid in a non-inertial reference frame. To find the differences, we examine a non-inertial reference frame F_B with

origin B. We do this with respect to an inertial reference frame F_O with origin O. The position vector $\mathbf{r}$ of a particle P can be expressed as

$$\mathbf{r_O} = \mathbf{r_B} + \mathbf{r_{rel}}. \tag{1.1.6}$$

In this equation, $\mathbf{r_B}$ is the position vector of B, with respect to O. Also, $\mathbf{r_{rel}}$ is the position vector of P with respect to B. It is important to see that $\mathbf{r_O}$ and $\mathbf{r_B}$ are given with respect to F_O, while $\mathbf{r_{rel}}$ is given with respect to F_B.

We can differentiate the above equation, with respect to time. However, as time changes, also the coordinate system of F_B changes. To take that into account, we have to use equation (1.1.5). It then follows that

$$\mathbf{v_O} = \mathbf{v_B} + (\omega \times \mathbf{r_{rel}}) + \mathbf{v_{rel}}, \tag{1.1.7}$$

where ω is the rotational velocity of F_B with respect to F_O. Also, $\mathbf{v_{rel}}$ is the velocity of P, as seen from F_B.

We have now expressed the velocity of point P in F_O as a function of the velocity of point P in F_B. We can go even further. Differentiating again will give us the acceleration. Doing this, and working out the results, will give

$$\mathbf{a_O} = \mathbf{a_B} + \alpha \times \mathbf{r_{rel}} + \omega \times (\omega \times \mathbf{r_{rel}}) + 2\,(\omega \times \mathbf{v_{rel}}) + \mathbf{a_{rel}}. \tag{1.1.8}$$

The vector $\mathbf{a_{rel}}$ is the acceleration of P as seen from F_B. Also, $\alpha = \dot{\omega}$ is the **angular acceleration** of F_B, with respect to F_O. The term $\omega \times (\omega \times \mathbf{r_{rel}})$ is known as the **centrifugal** acceleration. The term $2\,(\omega \times \mathbf{v_{rel}})$ is called the **Coriolis acceleration**.

Now let's suppose a force $\mathbf{F}$ is acting on particle P. Newton's second law implies that $\mathbf{F} = m\mathbf{a_O}$. This implies that

$$m\mathbf{a_{rel}} = \mathbf{F} - m\mathbf{a_B} - m\alpha \times \mathbf{r_{rel}} - m\omega \times (\omega \times \mathbf{r_{rel}}) - 2m\,(\omega \times \mathbf{v_{rel}}) = \mathbf{F} + \mathbf{F_{fict}}. \tag{1.1.9}$$

Note that, in F_B, we do not have $\mathbf{F} = m\mathbf{a_{rel}}$. Newton's second law therefore does not hold in F_B. Instead, it appears as if some **fictitious force** $\mathbf{F_{fict}}$ is present. This fictitious force strangely effects the motion of the particle in point P. It is important to remember that $\mathbf{F_{fict}}$ is not a real force. It's only a force that appears to be present, if an observer forgets that he's in a non-inertial reference frame.

1.2 Momentum, work, energy and potential functions

1.2.1 Momentum

The **linear momentum p** of a particle is defined as

$$\mathbf{p} = m\mathbf{v}, \tag{1.2.1}$$

where m is the mass of the particle. The time rate of change $\dot{\mathbf{p}}$ equals the **resultant force** $\sum \mathbf{F}$ acting on the particle. From this follows that

$$\mathbf{p_1} + \int_{t_1}^{t_2} \sum \mathbf{F}\, dt = \mathbf{p_2}. \tag{1.2.2}$$

The change in linear momentum $\mathbf{p_2} - \mathbf{p_1}$ is called the **impulse**. The above equation is also known as the **Principle of Impulse and Momentum**.

In a similar way, we can define the **angular momentum** $\mathbf{L_O}$ of a particle about some point O as

$$\mathbf{L_O} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}. \tag{1.2.3}$$

The time rate of change $\dot{\mathbf{L}}_\mathbf{O}$ now equals the **resultant moment** $\sum \mathbf{r} \times \mathbf{F}$. This knowledge gives us the rotational version of the Principle of Impulse and Momentum. It is given by

$$\mathbf{L_1} + \int_{t_1}^{t_2} \sum (\mathbf{r} \times \mathbf{F})\ dt = \mathbf{L_2}. \tag{1.2.4}$$

1.2.2 Work and energy

Let's examine a particle moving from position $\mathbf{r_1}$ to position $\mathbf{r_2}$. The **work** done on the particle, by a force $\mathbf{F}$, is given by

$$W_{1\to 2} = \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} \cdot d\mathbf{r}. \tag{1.2.5}$$

It can be shown that $W_{1\to 2}$ also equals

$$W_{1\to 2} = T_2 - T_1 = \Delta T, \tag{1.2.6}$$

where $T = \frac{1}{2}mv^2$ is the **kinetic energy** of the particle. The above equation is also known as the **Principle of Work and Energy**.

A force $\mathbf{F}$ is **conservative** if $\nabla \times \mathbf{F} = \mathbf{0}$. In this case, there is a certain **potential function V** satisfying $\mathbf{F} = -\nabla V$. (The minus sign is present by convention.) V is also known as the **potential energy**. It can be determined up to a constant. The point where $V = 0$ is called the **datum**. Usually, first a datum is set. Then, from the datum, the arbitrary constant is derived.

Now let's again examine the work done by force $\mathbf{F}$. It can be shown that

$$W_{1\to 2} = -(V_2 - V_1) = -\Delta V. \tag{1.2.7}$$

So the work done by a conservative force is independent of the path of the particle. It only depends on the initial and final position. Now let's examine all the forces acting on a particle. The total work done $W_{1\to 2}$ can be split up into two parts: The work done by conservative forces W_{cons} and the work done by non-conservative forces W_{nc}. It follows that

$$W_{nc} = \Delta T + \Delta V, \tag{1.2.8}$$

where V is the potential function of the conservative forces. If there are no non-conservative forces, then $W_{nc} = 0$. In this case, we have

$$T_1 + V_1 = T_2 + V_2 = E, \tag{1.2.9}$$

where E is the **total (mechanical) energy** of the particle. It is constant. The above statement is known as the **Conservation of Energy** relation.

1.2.3 Basic ideas of stability

For simplicity, let's reduce our problem to a one-dimensional problem. We can then rewrite the conservation of energy relation as

$$\frac{1}{2}mv^2 + V(x) = E. \tag{1.2.10}$$

So, given the total energy E, the magnitude of v only depends on the position x. This means that we can make a plot of v versus x. The resulting curves are called **phase curves**. If we plot multiple phase curves (for different values of E), then we get a **phase plane**, also known as a **phase portrait**.

We can examine a few interesting points in the phase portrait. A **turning point** is a point where the particle changes direction. At such points, we must have $v = 0$ for a brief moment. Similarly, an **equilibrium point** is a point where the resultant force is zero. In other words, $\mathbf{F} = -dV/dx = 0$.

This means that, in an equilibrium point, the potential function V must be either at a minimum or at a maximum.

Whether an equilibrium point is a minimum or a maximum of V is, in fact, rather important. It determines whether the equilibrium point is stable. Let's suppose that we give the particle a small deviation from an equilibrium position. If the point is a minimum of V (and thus $d^2V/dx^2 > 0$), then the resulting force will point towards the equilibrium point. (The force is attractive.) The point will thus be **stable**. If, however, the point is a maximum of V (and thus $d^2V/dx^2 < 0$), then the resulting force will point away from the equilibrium point. (The force is repulsive.) The point is therefore **unstable**. We will later go more into depth on the stability of equilibrium points.

2. Lagrangian dynamics

Newton examined forces. From this, he derived accelerations and such. Lagrange did something different. He examined energy, by using generalized coordinates. And that's what we'll do in this chapter as well.

2.1 System configurations and coordinates

2.1.1 Degrees of freedom

Let's consider a particle in a three-dimensional space. We need three numbers to fully describe this particle. We thus say that the particle has three **degrees of freedom**. On the other hand, a rigid body has six degrees of freedom. (Three due to its position, and three due to its orientation.) For a general system, the number of degrees of freedom is denoted by NDOF. We usually thus need NDOF numbers (called **coordinates**) to describe the system.

2.1.2 System configurations

Describing the **configuration** of a system can be done in many ways. (We could use many kinds of coordinate systems.) However, we want to be able to work with any description of the system. To accomplish this, we define **generalized coordinates** q_i. The collection of all possible sets of coordinates $(q_1, q_2, \ldots, q_n)$ is called the **configuration space**. By the way, the formulation of dynamics problems in terms of generalized coordinates is known as **Lagrangian dynamics**.

2.1.3 Transforming coordinates

Once a problem is described in certain generalized coordiantes, it can also be described in other coordinate systems. For this, we use **coordinate transformations**, like

$$q_i = q_i(x_1, x_2, \ldots, x_n, t) \qquad \text{and similarly} \qquad x_i = x_i(q_1, q_2, \ldots, q_n, t). \tag{2.1.1}$$

The latter part of the above equation is known as the **inverse transformation**.

2.2 Generalizing energy, momentum and forces

We have generalized coordinates. It would be nice if we could generalize other parameters as well. That's what we'll do in this part.

2.2.1 Kinetic energy

Let's examine a system with generalized coordinates $\mathbf{q} = (q_1, q_2, \ldots, q_n)$. The **generalized velocities** $\dot{q}_i$ of the system are the time derivatives of the coordinates. In other words,

$$\dot{q}_i = \frac{dq_i}{dt}. \tag{2.2.1}$$

From the generalized velocities, the **kinetic energy**, in terms of the generalized coordinates, can be derived. It can be shown that

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n}\beta_i\dot{q}_i + \gamma = T_2 + T_1 + T_0. \tag{2.2.2}$$

In this equation, T_2 is the collection of terms with the α_{ij} coefficients, T_1 with the β_i coefficients and $T_0 = \gamma$. If the transformations from the generalized coordinates to the actual coordinates do not depend on time, then $T_1 = T_0 = 0$. (This is the case if $x_i = x_i(q_1, q_2, \ldots, q_n)$, or, equivalently, $\partial x_i/\partial t = 0$ for all i.) In this case, we call the system a **natural system**.

Sadly, the above relation isn't a very easy one. The coefficients α_{ij}, β_j and γ depend on the system and the way in which the generalized coordinates are defined. Finding them requires some skill and work.

2.2.2 The generalized momentum

The **generalized momentum** in q_i direction p_i can be derived from the kinetic energy. It is defined as

$$p_i = \frac{\partial T}{\partial \dot{q}_i}. \tag{2.2.3}$$

When calculating this partial derivative, all other variables (including the time t) remain fixed.

2.2.3 Generalized forces

Now let's examine **generalized forces Q**. To do this, we examine work. We can remember that work is force times displacement in the corresponding direction. We thus have

$$\delta W = \sum_{i=1}^{n} F_i \, \delta x_i. \tag{2.2.4}$$

By using the transformations to generalized coordinates, we can rewrite this to

$$\delta W = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} F_j \frac{\partial x_j}{\partial q_i} \right) \delta q_i = \sum_{i=1}^{n} Q_i \, \delta q_i. \tag{2.2.5}$$

In this equation, Q_i is the generalized force in the direction of the coordinate q_i. It is given by

$$Q_i = \sum_{j=1}^{n} F_j \frac{\partial x_j}{\partial q_i}. \tag{2.2.6}$$

It may be important to note that the amount of work performed does not depend on changes in time.

Let's examine a force that is conservative in the real physical word. The corresponding potential function is V. (We thus have $\mathbf{F} = -\nabla V$.) In this case, it can be shown that the corresponding generalized force is also conservative. In fact, we have

$$Q_i = -\frac{\partial V}{\partial q_i} \qquad \text{or, equivalently,} \qquad \mathbf{Q} = -\nabla V. \tag{2.2.7}$$

Note that, to use this function, we first have to transform V. First, V was a function of the physical coordinates. (So $V = V(x_1, x_2, \ldots, x_n)$.) But, to use the above equation, V has to be a function of the generalized coordinates. (Thus $V = V(q_1, q_2, \ldots, q_n)$.)

2.3 The generalized equations of motion

2.3.1 Finding the generalized equations of motion

Newton once stated that $\mathbf{F} = m\ddot{\mathbf{r}}$ or, equivalently, $\mathbf{F} = \dot{\mathbf{p}}$. This relation doesn't always work with generalized forces. This is because, in generalized coordinates, force is not the time derivative of momentum.

(So $\dot{\mathbf{p}} \neq \mathbf{Q}$.) Instead, we have

$$\dot{\mathbf{p}} = \mathbf{Q} + \nabla T, \qquad \text{or, in components,} \qquad \dot{p}_i = Q_i + \frac{\partial T}{\partial q_i}. \tag{2.3.1}$$

We can also recall that $\dot{p}_i = \frac{dp_i}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)$. Putting things together gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i. \tag{2.3.2}$$

This very important relation is known as **Lagrange's Equations of Motions**. (It holds for every coordinate q_i.)

2.3.2 The Lagrangian function

In some cases, we can simplify Lagrange's equations of motion. Let's suppose that the force Q_i acting on the system is conservative. We thus have a **conservative system**. So there is a function $V(\mathbf{q})$ such that $Q_i = -\partial V/\partial q_k$. If this is the case, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \tag{2.3.3}$$

where $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q})$ is known as the **Lagrangian function**.

Sometimes, however, we can't find a potential function $V(\mathbf{q})$ such that $Q_i = -\partial V/\partial q_i$. But, we may be able to find a function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$ such that

$$Q_i = \frac{d}{dt}\left(\frac{\partial V}{d\dot{q}_i}\right) - \frac{\partial V}{\partial q_i}. \tag{2.3.4}$$

The function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$ is then known as the **generalized potential**. If there is such a generalized potential, then the system is called a **Lagrangian system**. And, if we again define $L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, \dot{\mathbf{q}}, t)$, then equation (2.3.3) still holds.

2.3.3 Lagrangian systems

Energy is not always conserved in a Lagrangian system. However, there may be another quantity that is conserved. We define the **Jacobi energy function** h as

$$h(\mathbf{q}, \dot{\mathbf{q}}, t) = -L + \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}. \tag{2.3.5}$$

It can now be shown that

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}. \tag{2.3.6}$$

So, if L does not explicitly depend on the time t, then h is constant. It is thus conserved. If this is indeed the case, then we have

$$h = T_2 - T_0 + V. \tag{2.3.7}$$

For natural systems, we have $T_1 = T_0 = 0$ and thus $h = T + V$. In this case, the mechanical energy $T + V$ is thus conserved as well.

2.3.4 Ignorable coordinates

Let's consider a Lagrangian system with n degrees of freedom. We suppose that there are m generalized coordinates $q_{n-m+1}, \ldots, q_n$ that do not appear in the Lagrangian L. These coordinates are called **inactive** or **ignorable coordinates**. (We will soon see why.) For these coordinates, we have $\partial L/\partial q_i = 0$, where $n - m < i \leq n$. This implies that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \qquad \text{or, equivalently,} \qquad \frac{\partial L}{\partial \dot{q}_i} = C_i, \tag{2.3.8}$$

where again $n - m < i \leq n$. The above equation is known as an **integral of motion** (a quantity that stays constant during the motion). The constant C_i is related to the momentum corresponding to coordinate q_i.

We use the above relation, when defining the Routhian function. The **Routhian function** R is defined as

$$R = -L + \sum_{i=n-m+1}^{n} C_i \dot{q}_i. \tag{2.3.9}$$

The ignorable coordinates do not appear in the Routhian function. Instead, the Routhian function contains the constants C_i. By using the Routhian function, we can rewrite the equations of motion for the nonignorable coordinates to

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0, \tag{2.3.10}$$

where this time $1 \leq i \leq n - m$. We now only have $n - m$ equations of motion left. We therefore 'ignore' the equations of motion corresponding to the ignorable coordinates. Once the equations of motion have been solved, the ignorable coordinates can be found using

$$\dot{q}_i = \frac{\partial R}{\partial C_i} \qquad \text{or, equivalently,} \qquad q_i(t) = \int_{t_0}^{t} \frac{\partial R}{\partial C_i} d\tau, \tag{2.3.11}$$

where $n - m < i \leq n$. By the way, we can also express the Jacobi energy function h as a function of R. We then find that

$$h = R - \sum_{i=1}^{n-m} \dot{q}_i \frac{\partial R}{\partial \dot{q}_k}. \tag{2.3.12}$$

2.3.5 Steady motion

A special type of motion is steady motion. In a steady motion, all the nonignorable coordinates are constant. We thus have $\dot{q}_i = \ddot{q}_i = 0$ for $1 \leq i \leq n - m$. Because of this, the equations of motion reduce to

$$\frac{\partial R}{\partial q_i} = 0, \tag{2.3.13}$$

for $1 \leq i \leq n - m$. On the other hand, the ignorable coordinates have a constant velocity. We thus have $\dot{q}_i = v_i = \text{constant}$ for $n - m < i \leq n$.

2.3.6 Disippative systems

Let's examine the forces Q_i acting on a non-Lagrangian system. Part of these forces Q_i^{lagr} can be derived from a generalized potential function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$, according to equation (2.3.4). However, another part can not be derived from such a potential function.

Examples of forces that can't be derived from a potential are frictional forces. These forces can not be connected to a potential, since energy is dissipated. In real (physical) coordinates, such forces are usually described by functions like $F_i^{fr} = -c_i \dot{x}_i$. (This relation holds for all i.) From this, we can derive that the frictional generalized forces Q_i^{fr} are given by

$$Q_i^{fr} = -\frac{1}{2}\frac{\partial}{\partial \dot{q}_i}\sum_{i=1}^{n} c_i \dot{x}_i^2 = -\frac{\partial D}{\partial \dot{q}_i}. \tag{2.3.14}$$

The parameter D is known as **Rayleigh's Dissipation Function**. It is defined as

$$D = \frac{1}{2}\sum_{i=1}^{n} c_i \dot{x}_i^2. \tag{2.3.15}$$

By using this function, we can rewrite the equations of motion to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i^*. \tag{2.3.16}$$

In this equation, Q_i^* denotes the part of the force Q_i that is not derivable from a potential function or a dissipation function.

2.4 Constraints

2.4.1 Holonomic constraints

Systems can have certain **constraints**. Constraints reduce the number of degrees of freedom. Let's examine a system normally having n degrees of freedom. If this system is given c constraints, then the remaining number of degrees of freedom is NDOF $= n - c$.

There are many types of constraints. So-called **holonomic constraints** can be written as

$$f(q_1, q_2, \ldots, q_n) = \text{constant} \qquad \text{or} \qquad f(q_1, q_2, \ldots, q_n, t) = \text{constant}. \tag{2.4.1}$$

If the constraint does not depend on time (as in the first relation), then the constraint is **scleronomic**. Otherwise, it is **rheonomic**. We can put holonomic constraints in the so-called **differential form**. To do this, we have to use the chain rule. We then find that

$$df = \frac{\partial f}{\partial q_1}dq_1 + \frac{\partial f}{\partial q_2}dq_2 + \ldots + \frac{\partial f}{\partial q_n}dq_n + \frac{\partial f}{\partial t}dt = 0. \tag{2.4.2}$$

2.4.2 Nonholonomic constraints

Some constraints can not be written in the form shown in equation (2.4.1). These constraints are known as **nonholonomic constraints**. There are two important types of nonhomolic constraints. These are **inequalities** and **nonintegrable differential expressions**. Their forms are, respectively,

$$g(q_1, q_2, \ldots, q_n, t) \geq 0 \qquad \text{and} \qquad A_1 dq_1 + A_2 dq_2 + \ldots + A_n dq_n + A_0 dt = 0, \tag{2.4.3}$$

where the differential expression (the one on the right) is nonintegrable. By the way, the coefficients A_i don't have to be constant. They can depend on time. Nonholonomic constraints are usually a bit more difficult to deal with than holonomic constraints. Luckily, they appear less frequently too.

2.4.3 Forces caused by constraints

Let's examine a system. This system has a set of J constraints that can be written as

$$\mathbf{A_j} \cdot \delta\mathbf{q} = A_{j1}\,\delta q_1 + A_{j2}\,\delta q_2 + \ldots + A_{jn}\delta q_n = 0, \tag{2.4.4}$$

where the index j is between 1 and J. Let's consider one of these constraints. This constraint demands that the position of the system (in the configuration space) moves along a certain $n-1$-dimensional plane. To keep the position of the system in this plane, a reacting **constraint force** R_j acts on the system. The only effect/goal of this force is to keep the configuration of the system in the plane. It thus acts perpendicular to the plane.

Now let's examine a movement $\delta\mathbf{q}$ of the system. Due to the constraint, this movement $\delta\mathbf{q}$ must be performed in the $n-1$-dimensional plane. Since $\mathbf{A_j} \cdot \delta\mathbf{q} = 0$, the vector $\mathbf{A_j}$ must be perpendicular to the plane. This implies that $\mathbf{R_j}$ and $\mathbf{A_j}$ have the same direction. We can thus write $\mathbf{R_j} = \lambda_j \mathbf{A_j}$, where the **Lagrange multiplier** λ_j is (at the moment) an unknown number.

In the equations of motion, we need to take the reaction forces R_j into account. We thus rewrite these equations to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{J} R_{ji} = Q_i + \sum_{j=1}^{J} \lambda_j A_{ji}. \tag{2.4.5}$$

We now have $n + J$ equations, being n equations of motion, and J constraint equations. We also have $n + J$ unknowns, being the n coordinates q_i and the J Lagrange multipliers λ_j. To find the unknowns, all the equations have to be solved simultaneously.

3. Stability theory

Some systems are stable, while others are not. In this chapter, we'll investigate the effects of the equations of motion and the potential function on stability.

3.1 Definitions

3.1.1 Equilibrium points

Let's consider a system with a **state x**. This system is described by the **evolution equations**

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t). \tag{3.1.1}$$

Once an **initial condition** $\mathbf{x^0} = \mathbf{x}(t_0)$ is specified, then the **solution** $\mathbf{x}(t)$ is uniquely determined.

A point $\mathbf{x}^*$ is called an **equilibrium point** or a **singular point** if

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}^*, t) = \mathbf{0}. \tag{3.1.2}$$

In other words, if $\mathbf{x^0} = \mathbf{x}^*$, then the system state remains constant.

3.1.2 Definition of stability

Let's examine an equilibrium point $\mathbf{x}^*$. We want to know whether this point is **stable**. To find this out, we take an initial point $\mathbf{x^0}$ close to $\mathbf{x}^*$. With close, we mean that it is less than a distance δ away:

$$|\mathbf{x^0} - \mathbf{x}^*| < \delta. \tag{3.1.3}$$

Now we examine the solution $\mathbf{x}(t)$. If it remains close to the equilibrium point $\mathbf{x}^*$, then the equilibrium point is called **stable**. With close, we again mean that it stays within a certain bound ε:

$$|\mathbf{x}(t) - \mathbf{x}^*| < \varepsilon. \tag{3.1.4}$$

(If there is no such ε, then the equilibrium point is called **unstable**.) We can examine the above difference more closely. We say the equilibrium point is **asymptotically stable** if this difference converges to zero:

$$\lim_{t\to\infty} |\mathbf{x}(t) - \mathbf{x}^*| = 0. \tag{3.1.5}$$

3.2 Examining matrices

3.2.1 Linear systems

We say that a system is linear, if we can write the evolution equations as

$$\dot{\mathbf{x}} = A(t)\mathbf{x}. \tag{3.2.1}$$

It directly follows that $\mathbf{x} = \mathbf{0}$ is an equilibrium point. Often, the matrix A does not depend on time. In this case, it is not very hard to determine the stability of the system. For this, we assume $\mathbf{x}$ has a solution of the form

$$\mathbf{x}(t) = \sum \mathbf{c_i} e^{\lambda_i t}. \tag{3.2.2}$$

It follows that λ_i must be an **eigenvalue** of A, with $\mathbf{c_i}$ the corresponding **eigenvector**. The real part of these eigenvalues determine the stability of the system. If $\mathrm{Re}(\lambda_i) \leq 0$, for all eigenvalues λ_i, then the system is **stable**. (If also $\mathrm{Re}(\lambda_i) < 0$, for all eigenvalues λ_i, then the system is **asymptotically stable**.) If there, however, is an eigenvalue with a real part bigger than zero (so $\mathrm{Re}(\lambda_i) > 0$), then the system is **unstable**.

3.2.2 Nonlinear systems

Of course not all systems are linear. And not all systems have $\mathbf{x} = \mathbf{0}$ as an equilibrium point either. Let's examine a nonlinear system of equations $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$ with equilibrium point $\mathbf{x}^*$. Now let's take an initial point $\mathbf{x^0}$ close to $\mathbf{x}^*$. The corresponding solution is $\mathbf{x}(t)$. We now define the **perturbation vector** $\eta(t)$ such that

$$\mathbf{x}(t) = \mathbf{x}^* + \eta(t). \tag{3.2.3}$$

Let's consider this perturbation $\eta(t)$. By applying a linearization about the equilibrium point, we can find that

$$\frac{d\eta}{dt} = DF(\mathbf{x}^*, t)\eta(t) + \mathbf{g}(\eta, \mathbf{x}^*, t). \tag{3.2.4}$$

In this equation, $\mathbf{g}(\eta, \mathbf{x}^*, t)$ is the part taking into account nonlinear terms. When examining stability, it is usually neglected. Also, $DF(\mathbf{x}^*, t)$ is the **derivative** of $\mathbf{F}$ at $\mathbf{x}^*$. It is a matrix with components

$$DF_{ij}(t) = \left.\frac{\partial F_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}^*}. \tag{3.2.5}$$

If the nonlinear system does not depend on time, then we have reduced our system to $\dot{\eta} = DF(\mathbf{x}^*)\eta$. The stability of the system thus depends on the eigenvalues of $DF(\mathbf{x}^*)$, just like in the previous paragraph.

3.3 Stability in Lagrangian systems

3.3.1 Stability and the potential function

Let's examine a Lagrangian system with generalized coordiantes q_i and a potential function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$. The forces Q_i acting on the system are given by $Q_i = -\partial V/\partial q_i$. We have an **equilibrium configuration** $\mathbf{q}^*$ if the forces acting on the system are zero. So, an equilibrium configuration must satisfy

$$\left.\frac{\partial V}{\partial q_i}\right|_{q_i=q_i^*} = 0 \quad \text{for all } i. \tag{3.3.1}$$

Now let's examine the stability of an equilibrium configuration $\mathbf{q}^*$. We give the system a small deviation from the equilibrium configuration. If all forces Q_i push the configuration back, then the system is **stable**. This is the case if

$$\left.\frac{\partial^2 V}{\partial q_i^2}\right|_{q_i=q_i^*} > 0 \quad \text{for all } i. \tag{3.3.2}$$

In other words, the potential function is at a **minimum**. If, however, we have $\partial^2 V/\partial q_i^2 < 0$ at $q_i = q_i^*$ for at least one i, then the system is **unstable**. (By the way, if $\partial^2 V/\partial q_i^2 < 0$ for all i, then the potential function V is at a **maximum**. If $\partial^2 V/\partial q_i^2 > 0$ for some i, and $\partial^2 V/\partial q_i^2 < 0$ for other i, then V has a **saddle point**. In both cases, the system is unstable.)

3.3.2 Stability and the Jacobi energy function

Now let's examine a Lagrangian system with a Jacobi energy function h. We know that $h = T_2 - T_0 + V =$ constant. In an equilibrium configuration, we have $\dot{q}_i = 0$ for all i. This means that also $T_2 = 0$, and thus $h = V - T_0 =$ constant. This trick also works the other way around. If $V - T_0$ is constant, and thus

$$\left.\frac{\partial\,(V - T_0)}{\partial q_i}\right|_{q_i=q_i^*} = 0 \tag{3.3.3}$$

for all i, then $\mathbf{q}^*$ is an equilibrium configuration.

We can combine this rule with what we've seen in the previous paragraph. To do this, we define the **effective potential** $V_{eff} = V - T_0$. It follows that, if $\partial V_{eff}/\partial q_i = 0$, for all i, then we have an equilibrium configuration. This means that, if we use the effective potential, all the rules of the previous paragraph still hold.

4. Rigid bodies

Rigid bodies occur quite often in daily life. For that reason, it's worth while examining them. What relations apply to them? That's what we'll find out in this chapter.

4.1 Basic ideas of rigid bodies

4.1.1 The center of gravity

Previously, we have only examined particles. We will now examine rigid bodies. A **rigid body** is a collection of particles. The distance between the particles does not change.

Let's examine a rigid body. The position $\mathbf{r_C}$ of the **center of gravity** (CG) is defined as

$$\mathbf{r_C} = \frac{1}{m}\sum_{i=1}^{N} m_i \mathbf{r_i}. \tag{4.1.1}$$

In this equation, m_i denotes the mass of a certain particle i and $\mathbf{r_i}$ its position. Also, $m = \sum_{i=1}^{N} m_i$ is the **total mass** of the body. It can be shown that the acceleration $\mathbf{a_C} = \ddot{\mathbf{r}}_\mathbf{C}$ of the CG satisfies

$$m\mathbf{a_C} = \sum_{i=1}^{N} \mathbf{F_i} = \mathbf{F_R}. \tag{4.1.2}$$

So, by using the **resultant force** $\mathbf{F_R}$, we can examine the translation of a body just as if it was just a particle. Sadly, rotation isn't so easy. And that is exactly the reason why this chapter is so long.

4.1.2 Rigid body kinematics

Let's suppose we know the position $\mathbf{r_C}$ of the CG, with respect to some inertial reference system F_O. The position $\mathbf{r_O}$ of any point P on the body, with respect to F_O, now satisfies

$$\mathbf{r_O} = \mathbf{r_C} + \mathbf{r_{rel}}. \tag{4.1.3}$$

In this equation, $\mathbf{r_{rel}}$ is the position of P with respect to the CG of the body. Let's now use a coordinate system with as origin the CG of the body. It also rotates in the same way the body rotates. Using the basic concepts shown in the first chapter of this summary, we find that

$$\mathbf{v_O} = \mathbf{v_C} + \omega \times \mathbf{r_{rel}}, \tag{4.1.4}$$

$$\mathbf{a_O} = \mathbf{a_C} + \alpha \times \mathbf{r_{rel}} + \omega \times (\omega \times \mathbf{r_{rel}}). \tag{4.1.5}$$

Note that, since the body is rigid, we must have $\mathbf{v_{rel}} = \mathbf{a_{rel}} = \mathbf{0}$.

4.2 The inertia tensor

4.2.1 The definition of the inertia tensor

Let's examine a body. The distribution of mass in this body is quite important. To quantify this, we use the **inertia tensor** $[I_O]$ with respect to some point O. It is defined as

$$[I_O] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} = \begin{bmatrix} \int (y^2+z^2)\,dm & -\int xy\,dm & -\int xz\,dm \\ -\int xy\,dm & \int (x^2+z^2)\,dm & -\int yz\,dm \\ -\int xz\,dm & -\int yz\,dm & \int (x^2+y^2)\,dm \end{bmatrix}. \tag{4.2.1}$$

The diagonal terms are called the **mass moments of inertia**. The non-diagonal terms are known as the **products of inertia**. By the way, in the above equation, all distance are measured with respect to point O.

4.2.2 The Parallel Axis Theorem

Let's suppose we know the value of I_{xx} or I_{xy} with respect to some axis, but want to know it with respect to another axis. In that case, we can use the **Parallel Axis Theorem**. It states that

$$I_{xx} = I_{xx_C} + m\left(d_y^2 + d_z^2\right) \qquad \text{and} \qquad I_{xy} = I_{xy_C} + m d_x d_y. \tag{4.2.2}$$

In this equation, I_{xx_C} and I_{xy_C} indicate the moment/product of inertia with respect to the CG. Also, the parameters d indicate the distance over which the axis is shifted. (You can apply the same trick for the other moments/products of inertia as well. You then just have to adjust the subscripts.)

4.2.3 The moment of inertia with respect to a line

Sometimes, we want to find the moment of inertia I_{ll} with respect to a line l. This moment of inertia is now defined as

$$I_{ll} = \int h^2 \, dm, \tag{4.2.3}$$

where h denotes the distance from the corresponding point to the line l. Let's suppose that the line is denoted by a unit vector $\hat{\mathbf{e}}$. The moment of inertia with respect to the line l is then given by

$$I_{ll} = \hat{\mathbf{e}}^T [I_O] \hat{\mathbf{e}}. \tag{4.2.4}$$

4.2.4 Principal axes

What happens to the inertia tensor, when we change the axes? In this case, also the inertia tensor changes. We can change the axes such that the inertia tensor becomes a diagonal matrix. The corresponding axes are called the **principal axes**. (The directions of the principal axes are equal to the directions of the eigenvectors of the original inertia tensor.) We denote these axes by the subscripts 1, 2 and 3. We thus have

$$[I_O] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}. \tag{4.2.5}$$

Getting rid of the nondiagonal components of the inertia tensor often comes in handy, as we will soon see.

4.3 Euler's equations of motion

4.3.1 Linear and angular momentum

We will now examine the linear and angular momentum of a body. We start with the linear momentum. This is, in fact, quite easy. It can be shown that the linear momentum of the body $\mathbf{p}$ is given by $\mathbf{p} = m\mathbf{v_c}$. So, to find the linear momentum, we only have to consider the velocity of the CG of the body.

Now let's find the angular momentum $\mathbf{L_O}$ with respect to some point O. (Don't confuse the angular momentum $\mathbf{L}$ with the Lagrangian function L.) It can be shown that

$$\mathbf{L_O} = \mathbf{r_C} \times m\mathbf{v_O} + \int \mathbf{r} \times (\omega \times \mathbf{r}) \, dm. \tag{4.3.1}$$

(In the above equation, all the position vectors $\mathbf{r}$ are with respect to point O.) We usually pick O such that the first term of this relation is zero. To accomplish this, we can do two things. We can pick a point that is fixed in space, such that $\mathbf{v_O} = \mathbf{0}$. We can also pick the CG, implying that $\mathbf{r_C} = \mathbf{0}$. That leaves us only with the right side. Using the inertia tensor, we can also rewrite the right side. We then remain with

$$\mathbf{L_O} = [I_O]\omega. \tag{4.3.2}$$

Remember that this equation only holds with respect to a fixed point, or with respect to the CG of the body.

4.3.2 Equations of motion for a rigid body

We know that the resultant force $\mathbf{F_R}$ effects the linear momentum of a body. Similarly, the **resultant moment N** effects the angular momentum. It does this according to

$$\mathbf{N_C} = \sum \mathbf{M_{C_i}} = \frac{d\mathbf{L_C}}{dt}, \qquad \text{or} \qquad \mathbf{N_O} = \sum \mathbf{M_{O_i}} = \frac{d\mathbf{L_O}}{dt}, \tag{4.3.3}$$

where O is a certain reference point. (Again, the above equation only holds if O is a fixed point, or O is the CG.) Let's examine the time derivative of the angular momentum. It is given by

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\left(L_x\mathbf{i} + L_y\mathbf{j} + L_z\mathbf{k}\right) = \dot{L}_x\mathbf{i} + \dot{L}_y\mathbf{j} + \dot{L}_z\mathbf{k} + \omega \times \mathbf{L}. \tag{4.3.4}$$

The last term in the above equation comes from the fact that the unit vectors may change too. We can combine the above two equations with the knowledge that $\mathbf{L_O} = [I_O]\omega$. This will then give us the equations of motion for a rotating rigid body. These equations will, however, be rather big. But we can use a trick to make a lot of terms disappear out of these equations. We can use the principal axes of the object. If we do that, it follows that

$$N_1 = I_1\dot{\omega_1} - (I_2 - I_3)\,\omega_2\omega_3, \tag{4.3.5}$$

$$N_2 = I_2\dot{\omega_2} - (I_3 - I_1)\,\omega_3\omega_1, \tag{4.3.6}$$

$$N_3 = I_3\dot{\omega_3} - (I_1 - I_2)\,\omega_1\omega_2. \tag{4.3.7}$$

The above equations are known as **Euler's equations of motion**. They only hold with respect to the principal axes of the body.

4.3.3 Analyzing Euler's equations of motion

We can derive some interesting facts from Euler's equations of motion. First, we can see that the angular acceleration $\dot{\omega}$ doesn't only depend on the moment applied. It also depends on the rotation rates about the other two principal axes. This is sometimes important to keep in mind.

We can also ask ourselves, when is a **steady rotation** possible, without any outside moments acting on the system? (So when is $\dot{\omega_1} = \dot{\omega_2} = \dot{\omega_3} = 0$ possible, while also $N_1 = N_2 = N_3 = 0$?) We can now see that, to ensure a steady rotation, 2 out of the 3 terms ω_1, ω_2 and ω_3 must be zero. In other words, a steady rotation is only possible about one of the principal axes.

Now let's assume that we're trying to rotate our body about one of the principal axes. For example, we try to rotate it about the third principal axis. In the real world, small perturbations are always present. So although ω_1 and ω_2 are much smaller than ω_3, they are not exactly zero. Sometimes, these perturbations damp out. (The rotation is stable.) In other cases, these perturbations grow. (The rotation is unstable.) It can be derived that the **condition for instability** is

$$I_1 > I_3 > I_2, \qquad \text{or} \qquad I_2 > I_3 > I_1. \tag{4.3.8}$$

A steady rotation is thus **stable** if it is performed about the principal axis with either the smallest or the largest moment of inertia. It is **unstable** if it is performed about the principal axis with the moment of inertia which is between the values of the other two.

4.4 Lagrange's equations of motion

4.4.1 Kinetic energy

There is another way to analyze rotating bodies. We then have to make use of the kinetic energy of the body. But how do we find the kinetic energy? We can do that, using the general equation

$$T = \frac{1}{2}mv_O^2 + \mathbf{v_O} \cdot \omega \times \int \mathbf{r}\, dm + \frac{1}{2}\int (\omega \times \mathbf{r}) \cdot (\omega \times \mathbf{r})\, dm. \tag{4.4.1}$$

This equation works for any point O on the body. But it's kind of a horrific equation. To simplify it, we usually pick O such that the middle term disappears. Again, we can do this in two ways. Either O is fixed ($\mathbf{v_O} = \mathbf{0}$) or O is the CG (the integral is zero). If we do the first (and thus have O fixed), then we can rewrite the above equation to

$$T = \frac{1}{2}\omega^T [I_O]\omega. \tag{4.4.2}$$

If, on the other hand, we choose point O to be the CG, then we have

$$T = \frac{1}{2}mv_C^2 + \frac{1}{2}\omega^T [I_C]\omega. \tag{4.4.3}$$

4.4.2 Euler angles

We have previously only looked at rotations, as seen from the rotating object itself. We haven't examined rotations from an inertial reference frame. When doing this, we need to know something about the orientation of the body, with respect to the inertial reference frame. For this, **Euler angles** come in handy. They allow us to express orientations.

Let's suppose we start at an inertial coordinate system, having axes X, Y and Z. We want to transform this to a coordinate system xyz, connected to a body. First, we rotate the inertial coordinate system by an angle ϕ about the Z axis. The new X axis is known as the **line of nodes** ξ. Second, we rotate the coordinate system about the line of nodes (the current X axis) by an angle θ. We do this to change the Z axis into the z axis. Finally, we rotate the system by an angle ψ about the z axis. We have now arrived at the xyz coordinate system.

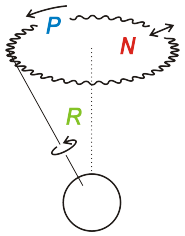

Figure 4.1: Clarification of the meanings of precession, nutation and rotation.

The angular velocity ω of the body can be expressed in the Euler angles. To be more specific, it can be

expressed as a function of the **precession** $\dot{\phi}$, the **nutation** $\dot{\theta}$ and the **spin/rotation** $\dot{\psi}$, according to

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{4.4.4}$$

The precise meaning of the precession, the nutation and the rotation can be seen in figure 4.1

4.4.3 The equations of motion for a rotating object

We can also express the kinetic energy due to rotation as a function of the Euler angles. To do this, we use $T_{rot} = \frac{1}{2}\omega^T[I]\omega$. Doing so, while using the principal axes, will give

$$T_{rot} = \frac{1}{2}\left(I_1\left(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + I_2\left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + I_3\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\right). \tag{4.4.5}$$

We can now find the moments Q_ϕ, Q_θ and Q_ψ, by using Lagrange's equations of motion. These equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i. \tag{4.4.6}$$

In this equation, q_i can be replaced by ϕ, θ or ψ. Q_i then has to be replaced by the moment Q_ϕ, Q_θ or Q_ψ, respectively.

4.4.4 Gyrodynamics

The equations of motion that we just derived are usually quite difficult to solve. But for some bodies, like a gyroscope or a top, we can simplify them. We will now examine such a body. Let's suppose that we have a body with the I_3 axis as an axis of symmetry. We thus have $I_1 = I_2 = I$ and $I_3 = I_s$. It follows that the kinetic energy, due to rotation, is given by

$$T_{rot} = \frac{1}{2}\left(I\left(\dot{\phi}^2\sin^2\theta + \dot{\theta}^2\right) + I_s\left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\right). \tag{4.4.7}$$

It often also occurs that the potential function V does not depend on ϕ or ψ. Since $L = T - V$, also the Lagrangian function L then does not depend on ϕ or ψ. This means that these two coordinates are ignorable coordinates. We thus also have two integrals of motion, being

$$C_\phi = \frac{dL}{d\dot{\phi}} = I\dot{\phi}\sin^2\theta + I_s\left(\dot{\phi}\cos\theta + \dot{\psi}\right)\cos\theta \qquad \text{and} \qquad C_\psi = \frac{dL}{d\dot{\psi}} = I_s\left(\dot{\phi}\cos\theta + \dot{\psi}\right). \tag{4.4.8}$$

The second of these equations can also be written as $C_\psi = I_s\omega_3$. In other words, the spin rate ω_3 is constant too. By using the above equations, we can derive the Routhian R and the energy E. They are

$$R = -\frac{1}{2}I\dot{\theta}^2 + \frac{(C_\phi - C_\psi\cos\theta)^2}{2I\sin^2\theta} + \frac{C_\psi^2}{2I_s} + V(\theta), \tag{4.4.9}$$

$$E = \frac{1}{2}I\dot{\theta}^2 + V_{eff}(\theta), \qquad \text{where} \qquad V_{eff}(\theta) = V(\theta) + \frac{(C_\phi - C_\psi\cos\theta)^2}{2I\sin^2\theta} + \frac{C_\psi^2}{2I_s}. \tag{4.4.10}$$

4.4.5 Steady precession

Now let's suppose we want to have a steady rotation, without nutation. We thus want to have θ constant. This is equivalent to $\dot{\theta} = 0$ or $\frac{dV_{eff}}{d\theta} = 0$. If we apply this, and perform a lot of mathematical rewriting, we can find that

$$-I\dot{\phi}^2 \sin\theta\cos\theta + I_s\dot{\phi}\omega_3 \sin\theta + \frac{dV}{d\theta} = 0. \tag{4.4.11}$$

From this equation, the precession rate $\dot{\phi}$ necessary to maintain equilibrium can be found. In fact, there are two solutions. They are known as the **slow** and the **fast precession**.

5. Calculus of variations

The calculus of variations has turned out to be a handy tool for solving dynamic problems. What is the calculus of variations? And how can we apply it to dynamic problems? That's what we'll look at now.

5.1 The basic principles of calculus of variations

5.1.1 The definition of a functional

A **function** (like $y(x)$) takes one or more numbers as input. The output is also a number. A generalization of a function is a functional. A **functional** (like $I[y(x)]$) takes one or more functions as input. The output is again a number. A specific kind of functional is the **integral functional**. The general form of such a functional is

$$I[y(x)] = \int_{x_1}^{x_2} F\left(x, y(x), y'(x)\right) dx. \tag{5.1.1}$$

The integrand F is called the **Lagrangian** of the integral function. An example of such an integral functional (for the so-called **brachistochrone problem**) is

$$I[y(x)] = \int_{x_1}^{x_2} \frac{\sqrt{1 + [y'(x)]}^2}{\sqrt{2gy(x)}} dx. \tag{5.1.2}$$

An **admissible function** is a function $y(x)$ that satisfies certain conditions. These conditions are usually differentiability conditions (we, for example, want continuously differentiable functions) and boundary conditions (we, for example, want to have $y(x_1) = y_1$ and $y(x_2) = y_2$). The full set of all admissible functions is called the **domain** of the functional.

5.1.2 The variational problem

Let's suppose we have a certain functional $I[y(x)]$. It then often occurs that we want to find the function $y(x)$ which minimizes or maximizes the functional $I[y(x)]$. This kind of problem is called a **variational problem**. The resulting function $y(x)$ is called an **extremal**.

Finding an extremum of a function is easy: you simply set the derivative to zero. But functionals don't really have a derivative. For that reason, we examine

$$I[y^*(x) + \varepsilon\eta(x)] = \int_{x_1}^{x_2} F(x, y^*(x) + \varepsilon\eta(x), y^{*\prime}(x) + \varepsilon\eta'(x))dx, \tag{5.1.3}$$

where $y^*(x)$ is a certain admissible function. Also, the **perturbation** $\eta(x)$ is an arbitrary function, such that $y^*(x) + \varepsilon\eta(x)$ is still an admissible function. The term $\varepsilon\eta(x)$ is known as the **variation**. It is often also written as $\delta y^*(x)$, with δ being the **variational operator**. We then also write $I[\varepsilon\eta] = I[\delta y^*] = \delta I$.

The above function now only depends on one variable, being ε. So, to find the extremum, we simply have to find the derivative with respect to ε and set it to zero. This will give us

$$\frac{dI}{d\varepsilon}[y^* + \varepsilon\eta] = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) dx = 0. \tag{5.1.4}$$

We can now apply integration by parts to rewrite the term $\frac{\partial F}{\partial y'}\frac{d\eta}{dx}$. The result will be

$$\frac{dI}{d\varepsilon}[y^* + \varepsilon\eta] = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right) \eta\, dx + \left.\frac{\partial F}{\partial y'}\eta\right|_{x_1}^{x_2} = 0. \tag{5.1.5}$$

This is a rather interesting equation. From it, we can derive conditions which any extremal $y(x)$ must satisfy. We will examine these conditions in the upcoming paragraph.

5.1.3 Extremum conditions

Let's examine the last equation of the previous paragraph. It must hold for any function $\eta(x)$. For this reason, the part within brackets has to be zero for every x. In other words, we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right). \tag{5.1.6}$$

This equation is known as the **Euler-Lagrange equation** for the functional $I[y(x)]$. It is the equation we have to use to find the extremal $y(x)$.

Now let's look at the term $\frac{\partial F}{\partial y'}\eta\Big|_{x_1}^{x_2}$. This term also has to be zero. However, it is a bit difficult to analyze. This is because some problems have so-called **essential boundary conditions** (EBCs) $y(x_1) = y_1$ and $y(x_2) = y_2$. We know that both $y^*(x)$ and $y^*(x) + \varepsilon\eta(x)$ have to satisfy these conditions. (They are both admissible functions.) This implies that $\eta(x_1) = \eta(x_2) = 0$. So, in this case, the term $\frac{\partial F}{\partial y'}\eta\Big|_{x_1}^{x_2}$ is automatically zero. We can't derive any conditions from it. (That is, other than the EBCs that were already present.)

However, other problems only have one or zero EBCs. For example, let's examine a problem having only the EBC $y(x_1) = y_1$. We thus have $\eta(x_1) = 0$. But we don't necessarily have $\eta(x_2) = 0$. In this case, we therefore must have

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_2} = 0. \tag{5.1.7}$$

If we don't have the EBC $y(x_1) = y_1$ as well, then we should also have

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_1} = 0. \tag{5.1.8}$$

The above two boundary conditions are called **natural boundary conditions** (NBCs). So we see that every end always has a boundary condition. Sometimes an EBC has been specified. If this is not the case, an NBC will automatically be present.

5.1.4 Generalizing the variational problem

We can make the variational problem a bit more general. We can, for example, examine a Lagrangian function with several more input functions $F(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n)$. In this case, the resulting Euler-Lagrange equation will be

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0, \tag{5.1.9}$$

for every i from 1 to n. We thus have n conditions, resulting in n (possibly coupled) differential equations. Solving this set of equations is usually quite difficult.

Sometimes, we might also have to deal with a function $y(x_1, \ldots, x_n)$ of multiple variables. In this case, the Lagrangian is $F(x_1, \ldots, x_n, y, y_{x_1}, \ldots, y_{x_n})$, where y_{x_i} means the derivative $\frac{\partial y}{\partial x_i}$. If we want to find the extremal of the functional $I[y]$, then we now have to use the condition

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial y_{x_1}}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial y_{x_2}}\right) - \ldots - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial y_{x_n}}\right) = 0. \tag{5.1.10}$$

This time, we wind up with a partial differential equation. It's usually rather hard to solve as well.

We could also have to deal with higher derivatives of a function $y(x)$. In this case, the Lagrangian is given by $F(x, y, y', y'', \ldots, y^{(n)})$, where $y^{(i)}$ stands for the i^{th} derivative of the function y. The Euler-Lagrange

equation now becomes

$$\frac{\partial F}{\partial y} + \sum_{i=1}^{n} (-1)^i \frac{d^i}{dx^i} \left(\frac{\partial F}{\partial y^{(i)}} \right) = 0. \tag{5.1.11}$$

We now have a higher-order differential equation. Again, solving it can be rather difficult.

Finally, it can also occur that multiple of the above cases occur simultaneously. In that case, you would have to combine the above conditions. But we won't go into detail on this.

5.2 Applying the calculus of variations

5.2.1 Hamilton's principle

It is time to apply the calculus of variations to dynamic systems. For this, we use **Hamilton's principle**. It states that

$$\int_{t_1}^{t_2} (\delta T + \delta W)\, dt = 0. \tag{5.2.1}$$

For conservative systems, we have $\delta W = -\delta V$. It now follows that the integral

$$I = \int_{t_1}^{t_2} (T - V)\, dt. \tag{5.2.2}$$

should be at an extremum. In fact, it has to be at a minimum. The above integral is called the **action** of the system. We can rewrite $T - V$ as $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. In this case, it follows that we should have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0, \tag{5.2.3}$$

for every i from 1 to n. So we see that, from Hamilton's principle, and from the calculus of variations, we have derived Lagrange's equations of motion.

Sometimes, we don't have a conservative system. In this case, the nonconservative forces Q_i^{nc} also have to be taken into account. Rewriting equation (5.2.1) gives

$$\int_{t_1}^{t_2} \left(\delta T - \delta V + \sum_{i=1}^{n} Q_i^{nc}\, \delta q_i \right) dt = \int_{t_1}^{t_2} \left(\delta L + \sum_{i=1}^{n} Q_i^{nc}\, \delta q_i \right) dt = 0. \tag{5.2.4}$$

Using the calculus of variations in dynamic problems has a big advantage. This method also works when there are an infinite amount of degrees of freedom. So, when examining **continuous mechanical systems**, like a vibrating string or a bending bar, this method is most likely to be used.

5.2.2 The Ritz method

Let's consider the variational problem

$$I[y(x)] = \int_{x_1}^{x_2} F\left(x, y(x), y'(x)\right) dx, \qquad \text{with } y(x_1) = y_1 \text{ and } y(x_2) = y_2. \tag{5.2.5}$$

The **Ritz method** is a method of finding an approximate solution $\bar{y}(x)$ to this problem. To start, we take n linearly independent known functions $h_i(x)$, called **shape functions**. (Deciding which functions to take can be rather difficult. It is something that requires quite some experience. But often $h_1(x) =$

$1, h_2(x) = x, h_3(x) = x^2, \ldots$ works quite well.) We then assume that our approximate solution can be written as

$$\bar{y}(x) = \sum_{i=1}^{n} a_i h_i(x). \tag{5.2.6}$$

The unknown coefficients a_i are called **degrees of freedom**. It's our job to find them. To do this, we insert the above relation into equation (5.2.5). This reduces the functional $I[y(x)]$ to a function of n variables, being

$$\Phi(a_1, a_2, \ldots, a_n) = \int_{x_1}^{x_2} F\left(x, \sum_{i=1}^{n} a_i h_i(x), \sum_{i=1}^{n} a_i h_i'(x)\right) dx. \tag{5.2.7}$$

We can now also apply the boundary conditions $\bar{y}(x_1) = y_1$ and $\bar{y}(x_2) = y_2$. This further reduces the function Φ to a function of $n - 2$ variables.

We want to find the extremums of Φ. To do this, we simply set $\frac{\partial \Phi}{\partial a_i} = 0$ for all remaining a_i. This gives us $n - 2$ equations and $n - 2$ unknowns. It can thus be solved, giving us all the coefficients a_i. Once these coefficients are known, also our approximate solution $\bar{y}(x)$ is known. And it has turned out that $\bar{y}(x)$ is often a rather good approximation of the real solution.