Summary AE3-914 2008-2009

Course Dynamics & Stability

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I Introduction

The course is based on "Analytical Mechanics with an introduction to dynamical systems" by J.S.Torok, which can be used for background material and exercises.

Laws of Newton

In the second law of Newton momentum is referred to as motion and equals $\underline{p} = m\underline{v}$, such that $\sum \underline{F} = \dot{\underline{p}} = \frac{d}{dt} m\underline{v}$. The third law yields $\underline{F}_{ij} = -\underline{F}_{ji}$. All motion is defined with respect to an inertial reference system, so $V_{xyz} =$ constant and $\omega_{xyz} = 0$.

Relative motion

$$\underline{r}_{rel} = x\underline{i} + y\underline{j} + z\underline{k}$$
$$\underline{v}_{rel} = \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k}$$
$$\underline{a}_{rel} = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$$

True motion

The position in the xyz-frame with respect to the inertial reference frame can be found using:

$$\underline{r}_p = \underline{r}_{xyz} + \underline{r}_{rel}$$
$$\underline{v}_p = \frac{d}{dt}\underline{r}_p = \frac{d\underline{r}_{xyz}}{dt} + \frac{d\underline{r}_{rel}}{dt} = \underline{v}_{xyz} + \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k} + x\frac{d\underline{i}}{dt} + y\frac{d\underline{j}}{dt} + z\frac{d\underline{k}}{dt}$$
$$= \underline{v}_{xyz} + \underline{v}_{rel} + x\omega_{xyz}\underline{i} + y\omega_{xyz}\underline{j} + z\omega_{xyz}\underline{k}$$
$$= \underline{v}_{xyz} + \underline{v}_{rel} + \omega_{xyz} \times \underline{r}_{rel}$$

From which may be concluded that $\dot{\underline{u}} = \dot{u}\underline{e} + \omega \times \underline{u}$, such that:

$$\underline{a}_p = \frac{d}{dt}\underline{v}_p =$$
$$= \dot{\underline{v}}_{xyz} + \dot{\omega} \times \underline{r}_{rel} + \omega \times \dot{\underline{r}}_{rel} \qquad + \dot{\underline{v}}_{rel}$$
$$= \underline{a}_{xyz} + \alpha \times \underline{r}_{rel} + \omega \times \left(\underline{v}_{rel} + \omega \times \underline{r}_{rel}\right) + \underline{a}_{rel} + \omega \times \underline{v}_{rel}$$
$$= \underline{a}_{xyz} + \alpha \times \underline{r}_{rel} + \omega \times \left(\omega \times \underline{r}_{rel}\right) + 2\omega \times \underline{v}_{rel} + \underline{a}_{rel}$$

Where $2\omega \times \underline{v}_{rel}$ is referred to as the Coriolis-acceleration.

Fictious forces

$$m\underline{a}_{rel} = \sum \underline{F} - m\left(\underline{a}_{xyz} + \alpha \times \underline{r}_{rel} + \omega \times \left(\omega \times \underline{r}_{rel}\right) + 2\omega \times \underline{v}_{rel}\right)$$

Now $\sum \underline{F} = m\underline{a}_p$ or $m\underline{a}_{rel} = \sum \underline{F} - \underline{F}_{fict}$.

Example: double deck

II. Qualitative Analysis

Work is defined as $W = \int_{r_1}^{r_2} F \cdot dr = \int_{t_1}^{t_2} F \cdot \dot{r} dt$. In a conservative force field a potential can be defined such that $\underline{F} = -\nabla V = -\frac{\partial V}{\partial x}\underline{i} - \frac{\partial V}{\partial y}\underline{j} - \frac{\partial V}{\partial z}\underline{k}$ and $E = T + V =$ constant.

Example:

$$F_x = -\frac{dV}{dx} = m\ddot{x} \quad \text{or} \quad m\ddot{x} + \frac{dV}{dx} = 0$$

$$\dot{x}\left(m\ddot{x} + \frac{dV}{dx}\right) = m\dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) \quad \text{or} \quad \frac{1}{2}m\dot{x}^2 + V(x) = \text{constant.}$$

Kinetic energy

Kinetic energy is defined as $dT = \frac{1}{2}v \cdot vdm$ or $T = \frac{1}{2}\int v \cdot vdm$.

With $v = v_0 + \omega \times r$:

$$T = \frac{1}{2}\int (v_0 + \omega \times r) \cdot (v_0 + \omega \times r) dm = \frac{1}{2}mv_0^2 + v_0 \cdot \omega \times \int rdm + \frac{1}{2}\int (\omega \times r)(\omega \times r) dm$$

$$= \frac{1}{2}mv_0^2 + \frac{1}{2}\omega^T I \omega$$

Inertia Tensor

The inertia tensor $I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$ can be defined for two specific cases: I_0 wrt a fixed point and I_G wrt the mass center of a body. Now fixed point rotation yields $T = \frac{1}{2}\omega^T I_0 \omega$ and general motion $T = \frac{1}{2}mv_G^2 + \frac{1}{2}\omega^T I_G \omega$.

Using principal axes such as symmetry axes gives a diagonal matrix.

For a xy-planar body $I_{zz} = I_{xx} + I_{yy}$ and $I_{xz} = I_{yz} = 0$.

For a body with at least three symmetry axes in the xy-plane: $I_{xx} = I_{yy}$ and $I_{xy} = 0$.

Virtual work

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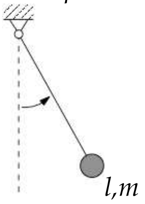
III. Generalised coordinates

Example: Pendulum



$$m\ddot{\theta}+m\frac{g}{l}\sin\theta=0$$

$$\ddot{\theta}+\frac{g}{l}\sin\theta=0$$

which is conservative!

$$\dot{\theta}\ddot{\theta}+\dot{\theta}\frac{g}{l}\sin\theta=\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2-\frac{g}{l}\cos\theta\right)=0$$

$\frac{1}{2}\dot{\theta}^2-\frac{g}{l}\cos\theta$ *=constant.*

This is however not expressed in the right coordinates and units and therefore can not be energy.

The generalised coordinates $\{\underline{q}_1,\underline{q}_2,\ldots,\underline{q}_n\}$ are defined as $\underline{q}_i=q_i(x_1,x_2,\ldots,x_n,t)$ or $x_i=x_i(\underline{q}_1,\underline{q}_2,\ldots\underline{q}_n,t)$ with $i=1\ldots n$ and n is the number of degrees of freedom of the system.

Degrees of freedom

The number of degrees of freedom are the number of values that need to be fixed to determine the state of the system. In general:

- 2D particle: n=2,
- 2D body: n=3,
- 3D particle: n=3,
- 3D body n= 6.

N particles or bodies have $N\cdot n$ degrees of freedom.

Holonomic constraints are defined as $f(\underline{q}_1,\underline{q}_2,\ldots\underline{q}_n)$ or $f(\underline{q}_1,\underline{q}_2,\ldots\underline{q}_n,t)$ being constant, such that $n_{dof}=n_{coord's}-n_{constr's}$.

This way a body can be interpreted as a collection of particles with fixed relative distances.

Generalized velocities

When $x_i=x_i(\underline{q}_1,\underline{q}_2,\ldots\underline{q}_n,t)$, $\dot{x}_i=\frac{\partial x_i}{\partial t}\sum_{j=1}^{n}\frac{\partial x_i}{\partial q_i}$.

Now $T=\frac{1}{2}m\sum_{i=1}^{3}\dot{x}_i^2=\frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}\alpha_{jk}\dot{q}_j\dot{q}_k+\sum_{j=1}^{3}\beta_j\dot{q}_j+\gamma$ or $T=T_2+T_1+T_0$, with $\alpha_{jk}=\alpha_{jk}(\mathbf{q},t);\ \beta_j=\beta_j(\mathbf{q},t);\ \gamma=\gamma(\mathbf{q},t)$, such that $T=T(\mathbf{q},\dot{\mathbf{q}},t)$.

Example: Moving pendulum with spring(no gravity)

Determine $x = s + l\sin\theta$ *and* $y = -l\cos\theta$, *such that* $\dot{x} = \dot{s} + l\cos\theta\dot{\theta}$ *and* $\dot{y} = l \sin \theta \dot{\theta}$.
Now $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\left(\dot{s} + l\cos\theta\dot{\theta}\right)^2 + \left(l\sin\theta\dot{\theta}\right)^2\right) = \frac{1}{2}m\left(l^2\dot{\theta}^2 + 2l\dot{\theta}\dot{s}\cos\theta + \dot{s}^2\right)$ *which is defined as* T_2.

Generalized momenta

Furthermore the generalized momentum is defined as $p_i = \frac{\partial T}{\partial q_i}$.

Example:
$T = \frac{1}{2}m\dot{x}^2$; $p = \frac{\partial T}{\partial x} = m\dot{x}$ *(linear momentum)*
Pendulum: $T = \frac{1}{2}ml^2\dot{\theta}^2$; $p_\theta = \frac{\partial T}{\partial \theta} = ml^2\theta^2$ *(angular momentum)*

Generalized work

Defining $\mathbf{F}_i = F_{ix}\mathbf{i} + F_{iy}\mathbf{j} + F_{iz}\mathbf{k}$ with $\delta x_i = \sum_{j=1}^{ndof} \frac{\partial x_i}{\partial q_j}\delta q_j \quad \delta y_i = \sum_{j=1}^{ndof} \frac{\partial y_i}{\partial q_j}\delta q_j \quad \delta x_i = \sum_{j=1}^{ndof} \frac{\partial z_i}{\partial q_j}\delta q_j$ generalized and virtual work gives the generalized force Q via $\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^{ndof} Q_j \delta q_j = \mathbf{Q} \cdot \delta \mathbf{q}$

Example: Pendulum
$r = l\cos\theta\underline{i} + l\sin\theta\underline{j}$; $F = mg\underline{i}$
$\partial r_\theta = -l\sin\theta\partial\theta\underline{i} + l\cos\theta\partial\theta\underline{j}$
$\partial W = -mgl\sin\theta\partial\theta$, *such that* $Q = -mgl\sin\theta$.

The generalized force can also be determined directly $Q_j = \sum_{i=1}^{n}\left(F_{ix}\frac{\partial x_i}{\partial q_j} + F_{iy}\frac{\partial y_i}{\partial q_j} + F_{iz}\frac{\partial z_i}{\partial q_j}\right)$ or for a conservative system $Q_j = -\frac{\partial V}{\partial q_j}$.

Example: Moving pendulum with spring(with gravity)
In handwritten notes!

IV. Lagrangian Dynamics

$\underline{F}_i = -\frac{d\underline{p}_i}{dt}$ only if $\underline{p}_i = m_i\underline{\dot{r}}_i$

Because $\dot{x}_i = \frac{\partial x_i}{\partial t}\sum_{j=1}^{n}\frac{\partial x_i}{\partial q_i} = \dot{x}_i(q,\dot{q}) \neq f(t)$ it can be derived that $\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}$, such that:

Example: Rotating spring

Finally, it can be said that $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k$ with k all degrees of freedom. This gives a set of differential equations that describe the behaviour of the system, the equations of motion. This method is faster than Newtonian mechanics.

Q can be found via $Q_j = -\frac{\partial V}{\partial q_j}$ for a conservative system and else via virtual work.

Example: Rotating spring

In practical cases of conservative systems the potential V is only a function of position, such that

$$\frac{\partial V}{\partial \dot{q}_j} = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T-V)\right) - \left(\frac{\partial}{\partial q_j}(T-V)\right) = 0$$

Defining the Lagrangian L=T-V this can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}L\right) - \frac{\partial L}{\partial q_j} = 0$$

And for a non-conservative system $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{nc}$

V.

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VI. Ignorables and constraints

When $L = L(\dot{q}) \neq L(q)$ the general coordinate q is ignorable and $\frac{\partial L}{\partial q} = 0$, which yields $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$=0 or $\frac{\partial L}{\partial \dot{q}_j}$=constant=$C_q$. The generalised momentum of q is the integral of motion, because $\frac{\partial L}{\partial \dot{q}} = \frac{\partial T - V}{\partial \dot{q}} = -\frac{\partial T}{\partial \dot{q}} = p$.

Example: Satellite

The Rothian is defined as $R=\sum_{i=n-m+1}^{n} c_i\dot{q}_i - L$ and is equivalent to the Lagrangian without ignorables, because for non-ignorables: $\frac{\partial R}{\partial \dot{q}}=-\frac{\partial L}{\partial \dot{q}}$ and $\frac{\partial R}{\partial q}=-\frac{\partial L}{\partial q}$, for ignorables: $\frac{\partial R}{\partial \dot{q}}=0$ and $\frac{\partial R}{\partial q}=0$. Now the Lagrangian equation for non-ignorables becomes: $\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_k}-\frac{\partial R}{\partial q_k}=0$.

In practice one can find the equations of motion by:

1. Setting up the Lagrangian
2. Determining the ignorable(s)
3. Find integrals of motion: $\frac{\partial L}{\partial \dot{q}_{ign}}=C_{q_{ign}} \Rightarrow \dot{q}_{ign}$
4. Now $R=\sum_{i=n-m+1}^{n} C_i\dot{q}_i - L$.
5. Set up the Rothian equation: $\frac{d}{dt}\frac{\partial R}{\partial \dot{q}_k}-\frac{\partial R}{\partial q_k}=0$

Example: Satellite
1+2:
The same way as in the previous example the Lagrangian $L=\frac{1}{2}m\left(\dot{r}^2-r^2\dot{\theta}^2\right)+\frac{km}{r}$*, such that* θ *is the ignorable coordinate.*
3:

$$\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}=C_\theta \Rightarrow \dot{\theta}=\frac{C_\theta}{mr^2}$$

4:

$$R=C_\theta\dot{\theta}-L=C_\theta\frac{C_\theta}{mr^2}-\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-\frac{km}{r}=\frac{{C_\theta}^2}{mr^2}-\frac{1}{2}m\left(\dot{r}^2+r^2\frac{{C_\theta}^2}{m^2r^4}\right)-\frac{km}{r}$$

$$=\frac{{C_\theta}^2}{2mr^2}-\frac{1}{2}m\dot{r}^2-\frac{km}{r}$$

5:

$$\frac{\partial R}{\partial \dot{r}}=-m\dot{r}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}}=-m\ddot{r}$$

$$\frac{\partial R}{\partial r}=-\frac{2{C_\theta}^2}{2mr^3}+\frac{km}{r^2}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}}-\frac{\partial R}{\partial r}=-m\ddot{r}+\frac{{C_\theta}^2}{mr^3}-\frac{km}{r^2}=0 \textit{ or } -\ddot{r}+\frac{{C_\theta}^2}{m^2r^3}-\frac{k}{r^2}=0.$$

Jacobi energy integral

The Jacobi energy integral is defined as $h = R - \sum_{k=1}^{m-n} \dot{q}_k \frac{\partial R}{\partial \dot{q}_k}$, where the second term represents the non-ignorable coordinates.

> *Example: Satellite*
> *With the Rothian as defined in the previous example, the Jacobi energy integral*
> $h = R - \frac{\partial R}{\partial \dot{r}} \dot{r} = \frac{C_\theta^{\ 2}}{2mr^2} - \frac{1}{2} m\dot{r}^2 - \frac{km}{r} - m\dot{r}^2 = \frac{C_\theta^{\ 2}}{2mr^2} - \frac{3}{2} m\dot{r}^2 - \frac{km}{r}$.

Steady motion

A motion is steady (constant in time) when all $\dot{q}_k = 0$ and $\dot{p}_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$, such that $\frac{\partial R}{\partial q_k} = 0$.

> *Example: Satellite*
> $\dot{r} = 0$
> $p_r = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} = 0$ *(Speed and acceleration both are zero)*
> $\frac{\partial R}{\partial r} = 0$
>
> $\dot{r} = 0$ *means that r is constant and thus that* $\dot{\theta} = \frac{C_\theta}{mr^2}$ *=constant, more specific:*
> $\frac{\partial R}{\partial r} = -\frac{2C_\theta^{\ 2}}{2mr^3} + \frac{km}{r^2}$ *or* $\frac{km}{r^2} = \frac{C_\theta^{\ 2}}{mr^3} = \frac{\left(mr^2\dot{\theta}\right)^2}{mr^3} = mr\dot{\theta}^2$ *or* $\dot{\theta} = \sqrt{\frac{k}{r^3}}$.

Raleigh dissipation function

For a non-conservative system dissipative forces like $F_x = -c_x \dot{x}$, $F_y = -c_y \dot{y}$ and $F_z = -c_z \dot{z}$ may play a role. Those can be taken into account via the Raleigh dissipation function:
$D = \frac{1}{2}\left(c_x \dot{x}^2 + c_y \dot{y}^2 + c_z \dot{z}^2\right)$. Now $Q = -\frac{\partial D}{\partial \dot{q}}$, such that via virtual work the Lagrangian equation for k number of freedom becomes: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k^*$.

Lagrange multipliers

For a system with n variables ($\{q_1 \ldots q_n\}$), m constraints ($f_j(q_1 \ldots q_n) = 0$) introduce reaction forces $R = \lambda \nabla f \ \ (= \lambda \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$ for the 3D-case) perpendicular to f. The Lagrangian multiplier λ is an extra unknown and can be used to model contact between two flexible bodies.

Via virtual work the Lagrangian equation now becomes: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_k}$.

Example: Pendulum (with gravity)

From earlier problems it is known that $L = \frac{1}{2} m\left(\dot{r}^2 - r^2\dot{\theta}^2\right) + mgr\cos\theta$ *(no ignorables!)*

The constraint in this case $r = l$ *or* $f(r) = r - l = 0$.

1. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda \frac{\partial f}{\partial r}$
2. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta}$
3. $f(r) = 0$

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1. $m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda$
2. $2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + mgr\sin\theta = 0$
3. $r - l = 0$

Equation 3 yields r=l, $\dot{r}$ *=0 and* $\ddot{r}$ *=0, such that:*

1. $-ml\dot{\theta}^2 - mg\cos\theta = \lambda$
2. $ml^2\ddot{\theta} + mgl\sin\theta = 0$ *or* $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

VII. Dynamic systems

A first order differential equation can describe a dynamical system like: $\dot{x} = F(x,t)$ with $x(0) = x_0$.
x^* is an equilibrium point when $F(x^*, t) = 0$. The stability of x^* can be judged on by a phase diagram.
By definition x^* is stable if and only if $\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \,|\, \left\| x_0 - x^* \right\| < \delta \Rightarrow \left\| x(t) - x^* \right\| < \varepsilon$ or the solution x(t) is always inside the area of the initial condition x_0 around an equilibrium point x^*.
This represents the effect of a small deviation in initial condition.

Example: pendulum (with gravity)

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$ *is a second order differential equation, which can be converted to a first order d.e. via* $\dot{\theta} = \omega$.

Then $\dot{\omega} = -\frac{g}{l}\sin\theta$ *, such that* $\underline{x} = [\theta \quad \omega]^T$ *. The system is in equilibrium when* $\dot{\underline{x}} = 0$ *or when* $\dot{\theta} = 0$ *and* $\dot{\omega} = 0$ *, which is the fact for* $\theta = \pm k\pi$ *or when the pendulum is vertical.*

Linear dynamic systems

When the function F is strictly linear the system can be described by $\dot{\underline{x}} = A\underline{x}$ with A constant. The equilibrium point(s) can once more be found because of $\dot{\underline{x}} = 0$.

This yields a solution $x(t) = \sum k_i c_i e^{\lambda_i t}$ with $Ac = \lambda c$. The solution is stable when:

- $\lambda_i \in R$ and $\lambda_i < 0$,
- $\lambda_i \in C$ and $\mathrm{Re}(\lambda_i) \le 0$,
- or combinations of those.

> *Example: Trolley with horizontal spring*
>
> $\ddot{x} + \frac{k}{m} x = 0$, *which can be converted to a 1st order d.e. with* $\dot{x} = y$ *and* $\dot{y} = -\frac{k}{m} x$.
>
> $$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$
>
> *In an equilibrium point* $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$*=0, which yields x*=0.???*
>
> $\begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\lambda \end{vmatrix} = \lambda^2 + \frac{k}{m} = 0$ *or* $\lambda = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}$ *and* $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{i\sqrt{\frac{k}{m}}} + c_2 e^{-i\sqrt{\frac{k}{m}}}$*, this solution is stable because both Eigen values are complex with a zero real part.*

Linearization

Non-linear dynamic systems can be linearized about equilibrium points by a Taylor expansion:

$$F(\underline{x}) = F(\underline{x}_0) + \nabla F(\underline{x}_0)(\underline{x} - \underline{x}_0)$$

> *Example: pendulum (with gravity)*
>
> $$\dot{\theta} = \omega$$
>
> $$\dot{\omega} = -\frac{g}{l}\sin\theta$$
>
> *This systems equilibrium points are 0,0 (I) and* π*,0 (II).*
>
> *Linearization about point I gives:*
>
> $$F(x_0) = 0$$
>
> $$\nabla F(x_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}\Bigg|_{x=x_0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & 0 \end{bmatrix}\Bigg|_{x=x_0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$$
>
> $$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta - 0 \\ \omega - 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{g}{l} & -\lambda \end{vmatrix} = \lambda^2 + \frac{g}{l} = 0 \rightarrow \lambda = \pm i\sqrt{\frac{g}{l}}$$

, such that λ is complex with $\text{Re}(\lambda)=0$, thus equilibrium point I is stable.

Linearization about equilibrium point II gives:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta & 0 \end{bmatrix}\Bigg|_{x=x_0} \begin{bmatrix} \theta-\pi \\ \omega-0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta-\pi \\ \omega \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ \frac{g}{l} & -\lambda \end{vmatrix} = \lambda^2 - \frac{g}{l} = 0 \rightarrow \lambda = \pm\sqrt{\frac{g}{l}}$$

, such that λ is real with one of λ>0, thus the equilibrium point II is unstable.

In general the stability of any system can be checked by:

1. Setting up the 2nd order differential equation (equation of motion) by one of the methods from previous chapters (in general via the Lagrangian equation)
2. Deriving the 1st order system of differential equations
3. Finding the equilibrium points
4. Linearizing about those points
5. Analysing the Eigen values.

Example: vertical rod with springand force acting on it
Example: satellite
Elaborations of this example can be found in handwritten notes.

VIII. Stability of conservative systems

Sometimes it is hard to set up the Lagrangian equation (kinetic and/or potential energy). Because <u>in a conservative system</u> the applied forces $Q = -\frac{\partial V}{\partial q}$ and in an equilibrium position $\dot{q} = 0$ and $\ddot{q} = 0$,

$$\frac{\partial V}{\partial q}\bigg|_{q=q^*} = 0 .$$

Around a maximum of V(q), Q is opposite to the disturbance in q and around a minimum Q is in the same direction. This means that:

When $\frac{\partial^2 V}{\partial q^2}\bigg|_{q=q^*} > 0$, V(q*) is a minimum and therefore stable.

When $\frac{\partial^2 V}{\partial q^2}\bigg|_{q=q^*} < 0$, V(q*) is a maximum and therefore unstable.

Example: Pendulum without gravity

$V = -mgl\cos\theta\,;\dfrac{\partial V}{\partial q} = mgl\sin\theta = 0 \rightarrow \theta = k\pi\,;\dfrac{\partial^2 V}{\partial q^2} = mgl\cos\theta$, which is
> 0 and thus stable for $\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$
< 0 and thus unstable for $\theta = 0, \pm 2\pi, \pm 4\pi, \ldots$

Example: Bar in triangle

Stability of Lagrangian systems

When all applied forces can be expressed in terms of the generalised potential V,the same procedure can be used as for conservative systems.
Because the Jacobian energy integral becomes $h = T_2 - T_0 + V$ =constant and for equilibrium all $\dot{q} = 0$ (such that T_2 =0) $-T_0 + V = h$ =constant.
The effective potential is defined as $V_{eff} = V - T_0$ and the equilibrium condition becomes:
$\dfrac{\partial V_{eff}}{\partial q} = 0$. Again for a minimum of V_{eff} the system is stable and for a minimum of V_{eff} it is unstable.

Examples

IX. Dynamics of rotating bodies: Gemist

X. Lagrangian dynamics

The equations of motion of a solid body are based on its angular velocity, therefore rotational transformations are needed. Because this transformation of coordinate systems depends on sequence, the rotation angles about coordinate axes are no suitable general coordinates.

Euler angles

The transformation from X,Y,Z to x,y,z can be performed via the Euler angles φ about the Z-axis, θ about the X′ axis and ψ about the z axis. There angular velocities are respectively $\underline{\omega}_\varphi = \dot{\varphi}\underline{K}$, $\underline{\omega}_\theta = \dot{\theta}\underline{e}_{x'}$ and $\underline{\omega}_\psi = \dot{\psi}\underline{k}$, with $\dot{\varphi}$ =precession, $\dot{\theta}$ =nutation and $\dot{\psi}$ =spin. These vectors do obviously not form an orthogonal basis: $\underline{\omega} = \underline{\omega}_\varphi + \underline{\omega}_\theta + \underline{\omega}_\psi$.

The actual transformation can be done via matrices or by projection. The latter case yields:

$$\underline{\omega}_x = \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\cos\psi$$
$$\underline{\omega}_y = -\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\sin\psi$$
$$\underline{\omega}_z = \dot{\psi} + \dot{\varphi}\cos\theta$$

Now $T = \dfrac{1}{2}\begin{pmatrix}\omega_x & \omega_y & \omega_z\end{pmatrix}\begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}\begin{pmatrix}\omega_x \\ \omega_y \\ \omega_z\end{pmatrix}$

Rotation symmetric bodies

For a body that is rotation symmetric about the z-axis I_z=I_s (axial moment of inertia) and I_x=I_y=I (transverse moment of inertia), such that $T=\frac{1}{2}\left(I\left(\dot{\varphi}^2\sin^2\theta+\dot{\theta}^2\right)+I_S\left(\dot{\varphi}\cos\theta+\dot{\psi}\right)^2\right)$, such that ψ and φ are ignorable coordinates.

Because $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}=0$ for non ignorable coordinates and $\frac{\partial T}{\partial \dot{q}}=C_q$ for ignorable coordinates, this results in three expressions. For steady motion $\dot{\theta}=\ddot{\theta}=0$, which yields $\sin\theta=0$ or $\dot{\varphi}=\frac{I_S\dot{\psi}}{\left(I_S-I\right)\cos\theta}$, the relation between precession and spin for steady motion of an rotation symmetric body.

When $I_S<I$, $I-I_S>0$, then: $\dot{\varphi}>0\rightarrow\dot{\psi}>0$ evv. Rotation about the Z-axis (spin) causes motion of the Z-axis itself (precession) in the same direction, this effect is called *direct precession*.

When $I_S>I$, $I-I_S<0$, then: $\dot{\varphi}>0\rightarrow\dot{\psi}<0$ evv. Rotation about the Z-axis (spin) causes motion of the Z-axis itself (precession) in the opposite direction, this effect is called *retrograde precession*.

Rotation non-symmetric bodies

In general for an arbitrary rotating body: $\begin{aligned}M_x&=I_x\dot{\omega}_x-\left(I_y-I_z\right)\omega_y\omega_z\\M_y&=I_y\dot{\omega}_y-\left(I_z-I_x\right)\omega_z\omega_x\\M_z&=I_y\dot{\omega}_y-\left(I_x-I_y\right)\omega_x\omega_y\end{aligned}$.

Without applying any forces ($M_x=M_y=M_z=0$), this yields $\begin{aligned}\dot{\omega}_x&=\frac{I_y-I_z}{I_x}\omega_y\omega_z\\\dot{\omega}_y&=\frac{I_z-I_x}{I_y}\omega_z\omega_x\end{aligned}$.

This can be rewritten as the linear dynamic system: $\begin{bmatrix}\dot{\omega}_x\\\dot{\omega}_y\end{bmatrix}=\begin{bmatrix}0&\frac{I_y-I_z}{I_x}\omega_z\\\frac{I_z-I_x}{I_y}\omega_z&0\end{bmatrix}\begin{bmatrix}\omega_x\\\omega_y\end{bmatrix}$, with its eigenvalues via $\begin{vmatrix}-\lambda&\frac{I_y-I_z}{I_x}\omega_z\\\frac{I_z-I_x}{I_y}\omega_z&-\lambda\end{vmatrix}=\lambda^2-\frac{\left(I_y-I_z\right)\left(I_z-I_x\right)}{I_xI_y}\omega_z^2$. For stability λ should be purely imaginary, such that $\left(I_y-I_z\right)\left(I_z-I_x\right)<0$: $I_z>I_x,I_y$ or $I_z<I_x,I_y$, the rotation axis should have the largest or smallest moment of inertia.

XI. Functionals

A functional makes a real value from a function, in contrary to a function that makes a real value of a real value. Finding the function (extremal) that maximizes or minimizes a functional is called *calculus of variations*.

Example: The length of an arbitrary path between position a and b $s_{ab} = \int_{x_a}^{x_b} \sqrt{1+y'(x)^2}\,dx$ *, which is a functional. Minimalizing this one finds the shortest path y(x).*

The travel time between position a and b of a particle influenced by gravity can be described by $dt = \frac{ds}{v}$ *,*

$v = \sqrt{2gh} = \sqrt{2gy(x)}$ *, such that* $t_{ab} = \int_{x_a}^{x_b} \sqrt{\frac{1+y'(x)^2}{2gy(x)}}\,dx$ *.*

For a function f(x) the extremum x^* can be found by $f'(x^*) = 0$. Then for a minimum $f(x^* + dx) > f(x)$, for every *dx*.

This way the extremal $y^*(x)$ can be determined by variation of y(x). $y(x) = y^*(x) + \varepsilon\eta(x)$, with the arbitrary function $\eta(x)$. For now $\eta(x_a) = \eta(x_b) = 0$.

Defining the functional $I(y)$, y^* is a minimum of I when $I(y^* + \varepsilon\eta) \geq I(y^*)$ for any $\eta(x)$.

Now $I(y^* + \varepsilon\eta) = I(\varepsilon)$ for a given $\eta(x)$. The extremal can be found for $\frac{d}{d\varepsilon} I(\varepsilon) = 0$.

Euler-Lagrange equation

Because x is integrated, y* is the anwer and $\eta(x)$ is fixed. $I(y) = \int_{x_a}^{x_b} F(x, y, y')dx$ can be written as

$$I(y^* + \varepsilon\eta) = \int_{x_a}^{x_b} F(x, y^* + \varepsilon\eta, y'^* + \varepsilon\eta')dx = I(\varepsilon).$$

$$\frac{dI}{d\varepsilon} \overset{\text{chain rule}}{=} \int_{x_a}^{x_b} \frac{\partial F}{\partial y}\frac{\partial y}{\partial \varepsilon} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \varepsilon} dx = \int_{x_a}^{x_b} \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' dx$$

$$\overset{\text{partial integration}}{=} \int_{x_a}^{x_b} \frac{\partial F}{\partial y}\eta dx + \left[\frac{\partial F}{\partial y}\eta\right]_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{d}{dx}\frac{\partial F}{\partial y'}\eta dx$$

$$= \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta dx + \left(\frac{\partial F}{\partial y}\right]_{x_b} \not\eta(x_b) - \left(\frac{\partial F}{\partial y}\right]_{x_a} \not\eta(x_a)$$

$$= \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\eta dx = 0 \text{ for all } \eta.$$

The fundamental lemma now yields $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$, which is called the *Euler-Lagrange equation*.

Example: s_{ab}*,* t_{ab}*, min resistance*

XII. Variational operator

Now $y(x) = y^*(x) + \varepsilon\eta(x) = y^*(x) + \delta y^*(x)$, where δ is the variational operator.
This yields $I(y+\delta y) = I(y) + \delta I$, such that y is an extremal when $\delta I = 0$.

δI is the difference in shape of function y and thus the variation of a function, comparable to the differential dx, which is the variation of a x-value.

$$\delta I = \int_{x_a}^{x_b} \delta F(x,y,y')dx \overset{\text{chain rule}}{=} \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right) dx$$

$$\overset{\text{partial integration}}{=} \int_{x_a}^{x_b} \frac{\partial F}{\partial y}\delta y dx + \frac{\partial F}{\partial y'}\delta y \Big|_{x_a}^{x_b} - \int_{x_a}^{x_b} \frac{d}{dx}\frac{\partial F}{\partial y}\delta y dx$$

$$= \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right)\delta y dx + \frac{\partial F}{\partial y'}\Big|_{x_b} \cancel{\delta y(x_b)}^{=0} - \frac{\partial F}{\partial y'}\Big|_{x_a} \delta\, \cancel{y(x_a)}^{=0} = 0$$

for every δy , such that $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$.

$\delta y(x_a) = 0$ means that $y(x_a)$ has a fixed value y_a , which is called an essential *boundary condition.*
Non-fixed boundaries lead to *natural boundary conditions*:

$y(x_a) = y_a \quad \delta y(x_a) = 0$

$y(x_b) = ? \quad \delta y(x_b) \neq 0$

$\delta I = \ldots = \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right)\delta y dx + \frac{\partial F}{\partial y'}\Big|_{x_b} \delta y(x_b) - \frac{\partial F}{\partial y'}\Big|_{x_a} \delta\, \cancel{y(x_a)}^{=0} = 0$ for every $\delta y(x_b)$ yields the natural boundary condition $\frac{\partial F}{\partial y'}\Big|_{x_b} = 0$.

Example: Shortest distance

$$s_{ab} = \int_{x_a}^{x_b} \sqrt{1+y'^2}\,dx$$

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{\partial F}{\partial y'}\Big|_{x_b} = \frac{y'(x_b)}{\sqrt{1+y'(x_b)^2}} \to y'(x_b) = 0$$

In part XI it is shown that Euler-Lagrange equation yields a straight line and the natural boundary condition now yields the slope being 0.

Generalisation

It can be shown that in general for $F(x,y,y',\ldots,y^{(n)})$ the Euler-Lagrange equation becomes $\frac{\partial F}{\partial y} + \sum_{i=1}^{n}(-1)^i \frac{d^i}{dx^i}\frac{\partial F}{\partial y^{(i)}} = 0$.

The function y can also depend on more variables. For example $y = y(x,t)$, such that $I(y) = \int_{t_a}^{t_b}\int_{x_a}^{x_b} F(x,t,y,y_x,y_t)dxdt$, with $y_x = \frac{\partial y}{\partial x}$ and $y_t = \frac{\partial y}{\partial t}$.

Now

$$\delta I = \int_{t_a}^{t_b}\int_{x_a}^{x_b}\left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y_x}\delta y_x + \frac{\partial F}{\partial y_x}\delta y_x\right)dxdt$$

$$\overset{\text{partial integration}}{=} \int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial F}{\partial y}\delta y dxdt + \int_{t_a}^{t_b}\frac{\partial F}{\partial y_x}\delta y\Big|_{x_a}^{x_b} dt - \int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial}{\partial x}\frac{\partial F}{\partial y_x}\delta y dxdt$$

$$+\int_{x_a}^{x_b}\frac{\partial F}{\partial y_t}\delta y\Big|_{t_a}^{t_b} dx - \int_{t_a}^{t_b}\int_{x_a}^{x_b}\frac{\partial}{\partial t}\frac{\partial F}{\partial y_t}\delta y dxdt$$

$$= \int_{t_a}^{t_b}\int_{x_a}^{x_b}\left(\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\frac{\partial F}{\partial y_x} - \frac{\partial}{\partial t}\frac{\partial F}{\partial y_t}\right)\delta y dxdt + \int_{t_a}^{t_b}\frac{\partial F}{\partial y_x}\delta y\Big|_{x_a}^{x_b} dt + \int_{x_a}^{x_b}\frac{\partial F}{\partial y_t}\delta y\Big|_{t_a}^{t_b} dx = 0$$

for every δy, yields $\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\frac{\partial F}{\partial y_x} - \frac{\partial}{\partial t}\frac{\partial F}{\partial y_t} = 0$ and boundary conditions.

Example: $F(x,t,u_t,u_x,u_{xx})$ *can be found in sample problems on Blackboard.*

XIII. Hamilton's principle

For a path in space $\underline{r} = \underline{r}(t)$, the force working along this path is $F(\underline{r}) = m\underline{\ddot{r}}$ and $\delta W = F\cdot\delta r = m\ddot{r}\cdot\delta r$. (Note: vector notation is dropped)

Use $\frac{d}{dt}\left(m\dot{r}\cdot\delta r\right) = m\ddot{r}\cdot\delta r + m\dot{r}\cdot\delta\dot{r} = m\ddot{r}\cdot\delta r + \frac{1}{2}m\delta\left(\dot{r}\cdot\dot{r}\right) = \delta W + \frac{1}{2}m\delta\left(v^2\right) = \delta W + \delta T$

or $m\dot{r}\cdot\delta r\Big|_{t_a}^{t_b} = \int_{t_a}^{t_b}\delta W + \delta T$.

Defining the essential boundary conditions $\begin{matrix} r(t_a) = r_a \\ r(t_b) = r_b \end{matrix} \rightarrow \begin{matrix} \delta(r_a) = 0 \\ \delta(r_b) = 0 \end{matrix}$ yields $\int_{t_a}^{t_b}\delta W + \delta T = 0$.

For conservative forces $Q = -\frac{\partial V}{\partial q}$ and $\delta W = Q\delta q = -\frac{\partial V}{\partial q}\delta q = -\delta V$, such that

$$\int_{t_a}^{t_b}\left(\delta T - \delta V\right)dt = \int_{t_a}^{t_b}\delta\left(T - V\right)dt = \int_{t_a}^{t_b}\delta L dt = 0$$

Now for $I(\underline{r}) = \int_{t_a}^{t_b} L dt$, which is defined as *action,* $\delta I = 0$ gives the minimum for the motion of a Lagrangian system. This is called *Hamilton's principle*.

Example: beam with distributed properties

For dissipative forces $\delta I = \int_{t_a}^{t_b} \delta L + \sum Q_i^{np} \delta q_i dt = 0$, where $Q_i^{np} \delta q_i = \delta W^{np}$ is virtual work.

In statics the system is steady in time and $T = 0$, such that $\delta V = 0$, which is called *the principle of stationary potential energy*.

Example: Aero-elasticity/wing

XIV. Ritz method

Earlier solutions of the Euler-Lagrange equations and natural boundary conditions were mainly differential equations; therefore an approximation can be used, called Ritz-method.

Assume an approximate solution as a linear combination of a finite number of linearly independent functions: $\bar{y}(x) = \sum_{i=1}^{n} a_i h_i(x)$. The coefficients a_i are the *degrees of freedom* and the functions $h_i(x)$ the *coordinate-* or *shape functions*. Translate the boundary conditions to $\sum_{i=1}^{n} a_i h_i(x_a) = y_a$ and $\sum_{i=1}^{n} a_i h_i(x_b) = y_b$.

Example:

Try to find the solution of $I(y) = \int_0^1 y^2 + y'^2 dx$, *with* $y(0) = y(1) = 1$

$$h_1(x) = 1; h_2(x) = x; h_3(x) = x^2$$

$$\bar{y} = a_1 h_1 + a_2 h_2 + a_3 h_3 = \alpha + \beta x + \gamma x^2$$

$$I(\bar{y}) = \int_0^1 \bar{y}^2 + \bar{y}'^2 \, dx = \int_0^1 \left(\alpha + \beta x + \gamma x^2\right)^2 + \left(\beta + 2\gamma x\right)^2 dx = \Phi(\alpha, \beta, \gamma)$$

The functional now became a function and has a minimum when all partial derivatives are zero:

$$\frac{\partial \Phi}{\partial \alpha} = 0; \frac{\partial \Phi}{\partial \beta} = 0; \frac{\partial \Phi}{\partial \gamma} = 0.$$

The boundary conditions are $\bar{y}(0) = \alpha + 0 + 0 = 1 \rightarrow \alpha = 1$ *and* $\bar{y}(1) = \alpha + \beta + \gamma = 1 \rightarrow \beta = -\gamma$.

Now $I(\bar{y}) = \int_0^1 \left(1 + \beta x - \beta x^2\right)^2 + \left(\beta - 2\beta x\right)^2 dx = 1 + \frac{1}{3}\beta + \frac{11}{30}\beta^2 = \Phi(\beta)$

$\frac{\partial \Phi}{\partial \beta} = \frac{1}{3} + \frac{22}{30}\beta = 0 \rightarrow \beta = \frac{-5}{11}$, *such that the approximate solution becomes*

$$\bar{y} = \alpha + \beta x + \gamma x^2 = 1 - \frac{5}{11}x + \frac{5}{11}x^2$$

This is only 0.05% of w.r.t. the real solution $y(x) = \frac{\sinh(x) + \sinh(1-x)}{\sinh(1)}$.

Ritz method is commonly used in computerized computational mechanics like finite element method, using non-continuous shape functions are chosen.

The choice of shape functions is rather arbitrarily, as long as they are linear independent:

$\{\underline{v}_1, \ldots, \underline{v}_n\}$ is linear independent $\Leftrightarrow a_1 v_1 + \ldots + a_n v_n = 0 \Rightarrow a_1 = \ldots = a_n = 0$

For computational reasons polynomials should be chosen.

Example: Dynamics of a beam