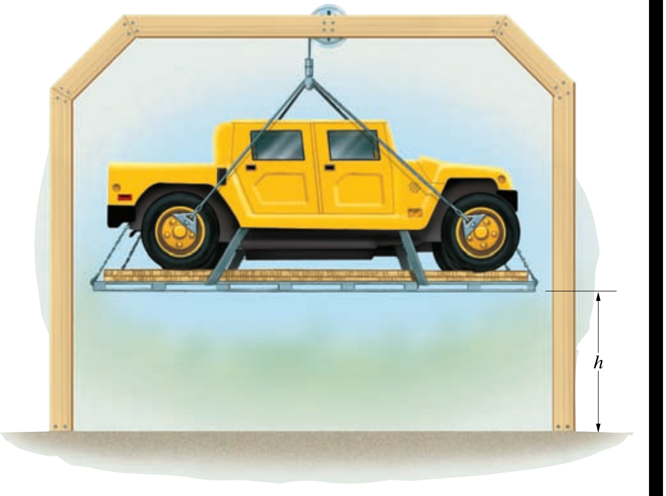
Problem 13.1 In Example 13.2, suppose that the vehicle is dropped from a height $h = 6$m. (a) What is the downward velocity 1 s after it is released? (b) What is its downward velocity just before it reaches the ground?



Solution: The equations that govern the motion are:

$a = -g = -9.81 \text{ m/s}^2$

$v = -gt$

$s = -\frac{1}{2}gt^2 + h$

(a) $v = -gt = -(9.81 \text{ m/s}^2)(1 \text{ s}) = -9.81 \text{ m/s}$.

The downward velocity is $\boxed{9.81 \text{ m/s.}}$

(b) We need to first determine the time at which the vehicle hits the ground

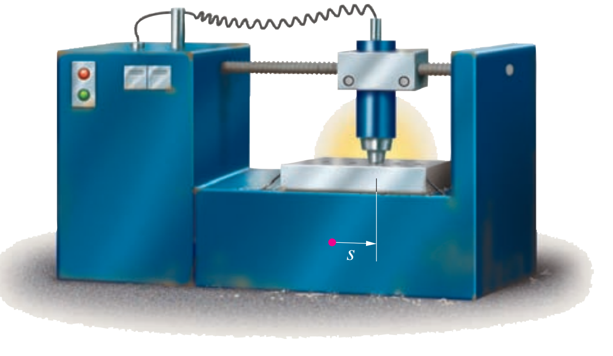
$$s = 0 = -\frac{1}{2}gt^2 + h \Rightarrow t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(6 \text{ m})}{9.81 \text{ m/s}^2}} = 1.106 \text{ s}$$

Now we can solve for the velocity

$$v = -gt = -(9.81 \text{ m/s}^2)(1.106 \text{ s}) = -10.8 \text{ m/s}.$$

The downward velocity is $\boxed{10.8 \text{ m/s.}}$

Problem 13.2 The milling machine is programmed so that during the interval of time from $t = 0$ to $t = 2$ s, the position of its head (in inches) is given as a function of time by $s = 4t - 2t^3$. What are the velocity (in in/s) and acceleration (in in/s^2) of the head at $t = 1$ s?



Solution: The motion is governed by the equations

$s = (4 \text{ in/s})t - (2 \text{ in/s}^2)t^2,$

$v = (4 \text{ in/s}) - 2(2 \text{ in/s}^2)t,$

$a = -2(2 \text{ in/s}^2).$

At $t = 1$ s, we have $\boxed{v = 0, a = -4 \text{ in/s}^2.}$

Problem 13.3 In an experiment to estimate the acceleration due to gravity, a student drops a ball at a distance of 1 m above the floor. His lab partner measures the time it takes to fall and obtains an estimate of 0.46 s.

(a) What do they estimate the acceleration due to gravity to be?
(b) Let s be the ball's position relative to the floor. Using the value of the acceleration due to gravity that they obtained, and assuming that the ball is released at $t = 0$, determine s (in m) as a function of time.

Solution: The governing equations are

$$a = -g$$

$$v = -gt$$

$$s = -\tfrac{1}{2}gt^2 + h$$

(a) When the ball hits the floor we have

$$0 = -\tfrac{1}{2}gt^2 + h \Rightarrow g = \frac{2h}{t^2} = \frac{2(1\ \text{m})}{(0.46\ \text{s})^2} = 9.45\ \text{m/s}^2$$

$$\boxed{g = 9.45\ \text{m/s}^2}$$

(b) The distance s is then given by

$$s = -\tfrac{1}{2}(9.45\ \text{m/s}^2) + 1\ \text{m}. \quad \boxed{s = -(4.73\ \text{m/s}^2)t^2 + 1.0\ \text{m}.}$$

Problem 13.4 The boat's position during the interval of time from $t = 2$ s to $t = 10$ s is given by $s = 4t + 1.6t^2 - 0.08t^3$ m.

(a) Determine the boat's velocity and acceleration at $t = 4$ s.
(b) What is the boat's maximum velocity during this interval of time, and when does it occur?

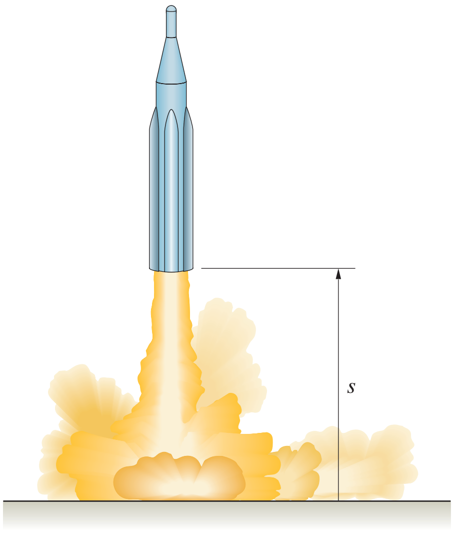
Solution:

$$s = 4t + 1.6t^2 - 0.08t^3$$

$$v = \frac{ds}{dt} = 4 + 3.2t - 0.24t^2 \Rightarrow \boxed{\begin{array}{l} a)\ v(4s) = 12.96\ \text{m/s}^2 \\ \quad a(4s) = 1.28\ \text{m/s}^2 \\ b)\ a = 3.2 - 0.48t = 0 \Rightarrow t = 6.67s \\ \quad v(6.67s) = 14.67\ \text{m/s} \end{array}}$$

$$a = \frac{dv}{dt} = 3.2 - 0.48t$$

Problem 13.5 The rocket starts from rest at $t = 0$ and travels straight up. Its height above the ground as a function of time can be approximated by $s = bt^2 + ct^3$, where b and c are constants. At $t = 10$ s, the rocket's velocity and acceleration are $v = 229$ m/s and $a = 28.2$ m/s^2. Determine the time at which the rocket reaches supersonic speed (325 m/s). What is its altitude when that occurs?



Solution: The governing equations are

$$s = bt^2 + ct^3,$$

$$v = 2bt + 3ct^2,$$

$$a = 2b + 6ct.$$

Using the information that we have allows us to solve for the constants b and c.

$$(229 \text{ m/s}) = 2b(10 \text{ s}) + 3c(10 \text{ s})^2,$$

$$(28.2 \text{ m/s}^2) = 2b + 6c(10 \text{ s}).$$

Solving these two equations, we find $b = 8.80$ m/s^2, $c = 0.177$ m/s^3.

When the rocket hits supersonic speed we have

$$(325 \text{ m/s}) = 2(8.80 \text{ m/s}^2)t + 3(0.177 \text{ m/s}^3)t^2 \Rightarrow \boxed{t = 13.2 \text{ s.}}$$

The altitude at this time is

$$s = (8.80 \text{ m/s}^2)(13.2 \text{ s})^2 + (0.177 \text{ m/s}^3)(13.2 \text{ s})^3 \quad \boxed{s = 1940 \text{ m.}}$$

Problem 13.6 The position of a point during the interval of time from $t = 0$ to $t = 6$ s is given by $s = -\frac{1}{2}t^3 + 6t^2 + 4t$ m.

(a) What is the maximum velocity during this interval of time, and at what time does it occur?
(b) What is the acceleration when the velocity is a maximum?

Solution:

$$s = -\tfrac{1}{2}t^3 + 6t^2 + 4t \text{ m}$$

$$v = -\tfrac{3}{2}t^2 + 12t + 4 \text{ m/s}$$

$$a = -3t + 12 \text{ m/s}^2$$

Maximum velocity occurs where $a = \dfrac{dv}{dt} = 0$ (it could be a minimum)

This occurs at $t = 4$ s. At this point $\dfrac{da}{dt} = -3$ so we have a maximum.

(a) Max velocity is at $t = 4$ s. where $\underline{v = 28 \text{ m/s}}$ and
(b) $\underline{a = 0 \text{ m/s}^2}$

Problem 13.7 The position of a point during the interval of time from $t = 0$ to $t = 3$ seconds is $s = 12 + 5t^2 - t^3$ m.

(a) What is the maximum velocity during this interval of time, and at what time does it occur?
(b) What is the acceleration when the velocity is a maximum?

Solution:

(a) The velocity is $\dfrac{ds}{dt} = 10t - 3t^2$. The maximum occurs when $\dfrac{dv}{dt} = 10 - 6t = 0$, from which

$\boxed{t = \dfrac{10}{6} = 1.667}$ seconds.

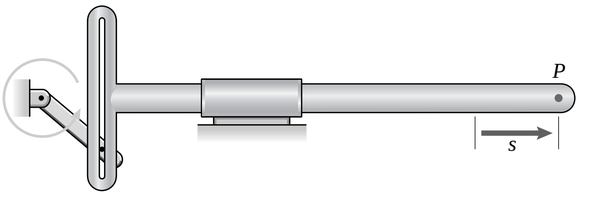
This is indeed a maximum, since $\dfrac{d^2v}{dt^2} = -6 < 0$. The maximum velocity is

$$\boxed{v = \left[10t - 3t^2\right]_{t=1.667} = 8.33 \text{ m/s}}$$

(b) The acceleration is $\boxed{\dfrac{dv}{dt} = 0}$ when the velocity is a maximum.

Problem 13.8 The rotating crank causes the position of point P as a function of time to be $s = 0.4\sin(2\pi t)$ m.

(a) Determine the velocity and acceleration of P at $t = 0.375$ s.
(b) What is the maximum magnitude of the velocity of P?
(c) When the magnitude of the velocity of P is a maximum, what is the acceleration of P?



Solution:

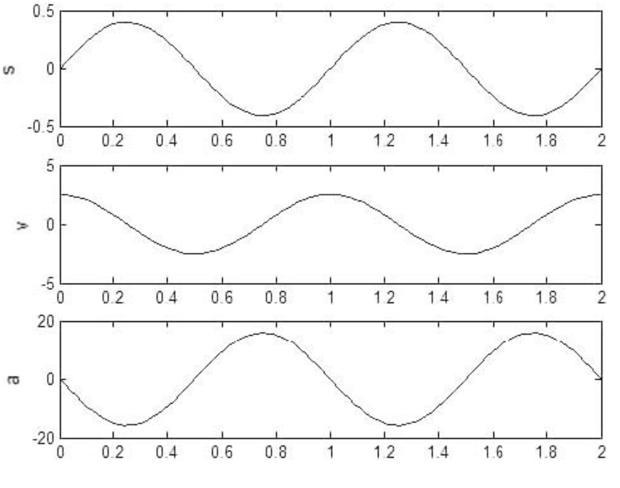
$s = 0.4\sin(2\pi t)$

$$v = \frac{ds}{dt} = 0.8\pi\cos(2\pi t) \quad \Rightarrow \quad \boxed{\begin{array}{l} a)\ v(0.375s) = -1.777 \text{ m/s} \\ \quad a(0.375) = -11.2 \text{ m/s}^2 \\ b)\ v_{max} = 0.8\pi = 2.513 \text{ m/s}^2 \\ c)\ v_{max} \Rightarrow t = 0, n\pi \Rightarrow a = 0 \end{array}}$$

$$a = \frac{dv}{dt} = -1.6\pi^2\sin(2\pi t)$$

Problem 13.9 For the mechanism in Problem 13.8, draw graphs of the position s, velocity v, and acceleration a of point P as functions of time for $0 \le t \le 2$ s. Using your graphs, confirm that the slope of the graph of s is zero at times for which v is zero, and the slope of the graph of v is zero at times for which a is zero.

Solution:



Problem 13.10 A seismograph measures the horizontal motion of the ground during an earthquake. An engineer analyzing the data determines that for a 10-s interval of time beginning at $t = 0$, the position is approximated by $s = 100\cos(2\pi t)$ mm. What are (a) the maximum velocity and (b) maximum acceleration of the ground during the 10-s interval?

Solution:

(a) The velocity is

$$\frac{ds}{dt} = -(2\pi)100\sin(2\pi t) \text{ mm/s} = -0.2\pi \sin(2\pi t) \text{ m/s}.$$

The velocity maxima occur at

$$\frac{dv}{dt} = -0.4\pi^2\cos(2\pi t) = 0,$$

from which

$$2\pi t = \frac{(2n-1)\pi}{2}, \text{ or } t = \frac{(2n-1)}{4},$$

$$n = 1, 2, 3, \ldots M, \quad \text{where} \quad \frac{(2M-1)}{4} \le 10 \text{ seconds.}$$

These velocity maxima have the absolute value

$$\left|\frac{ds}{dt}\right|_{t=\frac{(2n-1)}{4}} = [0.2\pi] = \underline{0.628 \text{ m/s}}.$$

(b) The acceleration is

$$\frac{d^2s}{dt^2} = -0.4\pi^2\cos(2\pi t).$$

The acceleration maxima occur at

$$\frac{d^3s}{dt^3} = \frac{d^2v}{dt^2} = 0.8\pi^3\sin(2\pi t) = 0,$$

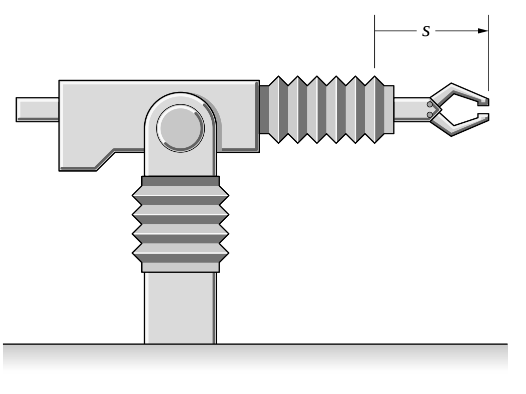
from which $2\pi t = n\pi$, or $t = \frac{n}{2}$, $n = 0, 1, 2, \ldots K$, where

$$\frac{K}{2} \le 10 \text{ seconds.}$$

These acceleration maxima have the absolute value

$$\left|\frac{dv}{dt}\right|_{t=\frac{n\pi}{2}} = 0.4\pi^2 = \underline{3.95 \text{ m/s}^2}.$$

Problem 13.11 In an assembly operation, the robot's arm moves along a straight horizontal line. During an interval of time from $t = 0$ to $t = 1$ s, the position of the arm is given by $s = 30t^2 - 20t^3$ mm. (a) Determine the maximum velocity during this interval of time. (b) What are the position and acceleration when the velocity is a maximum?



Solution:

$s = 30t^2 - 20t^3$ mm

$v = 60t - 60t^2$ mm/s

$a = 60 - 120t$ mm/s^2

$$\frac{da}{dt} = -120 \text{ mm/s}^3$$

(a) Maximum velocity occurs when $\frac{dv}{dt} = a = 0$. This occurs at $0 = 60 - 120t$ or $t = 1/2$ second. (since $da/dt < 0$, we have a maximum). The velocity at this time is

$$v = (60)\left(\frac{1}{2}\right) - 60\left(\frac{1}{4}\right) \text{ mm/s}$$

$v = 15$ mm/s

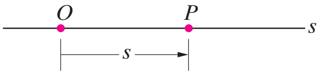
(b) The position and acceleration at this time are

$s = 7.5 - 2.5$ mm

$\underline{s = 5 \text{ mm}}$

$\underline{a = 0 \text{ mm/s}^2}$

Problem 13.12 In Active Example 13.1, the acceleration (in m/s^2) of point P relative to point O is given as a function of time by $a = 3t^2$. Suppose that at $t = 0$ the position and velocity of P are $s = 5$ m and $v = 2$ m/s. Determine the position and velocity of P at $t = 4$ s.



Solution: The governing equations are

$$a = (3 \text{ m/s}^4)t^2$$

$$v = \frac{1}{3}(3 \text{ m/s}^4)t^3 + (2 \text{ m/s})$$

$$s = \frac{1}{12}(3 \text{ m/s}^4)t^4 + (2 \text{ m/s})t + (5 \text{ m})$$

At $t = 4$ s, we have $\boxed{s = 77 \text{ m}, v = 66 \text{ m/s}.}$

Problem 13.13 The Porsche starts from rest at time $t = 0$. During the first 10 seconds of its motion, its velocity in km/h is given as a function of time by $v = 22.8t - 0.88t^2$, where t is in seconds. (a) What is the car's maximum acceleration in m/s^2, and when does it occur? (b) What distance in km does the car travel during the 10 seconds?

Solution: First convert the numbers into meters and seconds

$$22.8\frac{\text{km}}{\text{hr}}\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1\text{hr}}{3600 \text{ s}}\right) = 6.33 \text{ m/s}$$

$$0.88\frac{\text{km}}{\text{hr}}\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ hr}}{3600 \text{ s}}\right) = 0.244 \text{ m/s}$$

The governing equations are then

$$s = \frac{1}{2}(6.33 \text{ m/s})(t^2/\text{s}) - \frac{1}{3}(0.88 \text{ m/s})(t^3/\text{s}^2),$$

$$v = (6.33 \text{ m/s})(t/\text{s}) - (0.88 \text{ m/s})(t^2/\text{s}^2),$$

$$a = (6.33 \text{ m/s}^2) - 2(0.88 \text{ m/s})(t/\text{s}^2),$$

$$\frac{d}{dt}a = -2(0.88 \text{ m/s})(1/\text{s}^2) = -1.76 \text{ m/s}^3.$$

The maximum acceleration occurs at $t = 0$ (and decreases linearly from its initial value).

$$\boxed{a_{\max} = 6.33 \text{ m/s}^2 \ @ \ t = 0}$$

In the first 10 seconds the car travels a distance

$$s = \left[\frac{1}{2}\left(22.8\frac{\text{km}}{\text{hr}}\right)\frac{(10 \text{ s})^2}{s} - \frac{1}{3}\left(0.88\frac{\text{km}}{\text{hr}}\right)\frac{(10 \text{ s})^3}{\text{s}^2}\right]\left(\frac{1 \text{ hr}}{3600 \text{ s}}\right)$$

$$\boxed{s = 0.235 \text{ km.}}$$

Problem 13.14 The acceleration of a point is $a = 20t$ m/s^2. When $t = 0, s = 40$ m and $v = -10$ m/s. What are the position and velocity at $t = 3$ s?

Solution: The velocity is

$$v = \int a\,dt + C_1,$$

where C_1 is the constant of integration. Thus

$$v = \int 20t\,dt + C_1 = 10t^2 + C_1.$$

At $t = 0$, $v = -10$ m/s, hence $C_1 = -10$ and the velocity is $v = 10t^2 - 10$ m/s. The position is

$$s = \int v\,dt + C_2,$$

where C_2 is the constant of integration.

$$s = \int (10t^2 - 10)\,dt + C_2 = \left(\frac{10}{3}\right)t^3 - 10t + C_2.$$

At $t = 0$, $s = 40$ m, thus $C_2 = 40$. The position is

$$s = \left(\frac{10}{3}\right)t^3 - 10t + 40 \text{ m}.$$

At $t = 3$ seconds,

$$\boxed{s = \left[\frac{10}{3}t^3 - 10t + 40\right]_{t=3} = 100 \text{ m.}}$$

The velocity at $t = 3$ seconds is

$$\boxed{v = \left[10t^2 - 10\right]_{t=3} = 80 \text{ m/s}}.$$

Problem 13.15 The acceleration of a point is $a = 60t - 36t^2$ m/s^2. When $t = 0, s = 0$ and $v = 20$ m/s. What are position and velocity as a function of time?

Solution: The velocity is

$$v = \int a\,dt + C_1 = \int (60t - 36t^2) + C_1 = 30t^2 - 12t^3 + C_1.$$

At $t = 0$, $v = 20$ m/s, hence $C_1 = 20$, and the velocity as a function of time is

$$\boxed{v = 30t^2 - 12t^3 + 20 \text{ m/s}}.$$

The position is

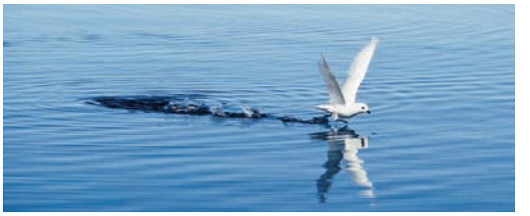
$$s = \int v\,dt + C_2 = \int (30t^2 - 12t^3 + 20) + C_2$$

$$= 10t^3 - 3t^4 + 20t + C_2.$$

At $t = 0$, $s = 0$, hence $C_2 = 0$, and the position is

$$\boxed{s = 10t^3 - 3t^4 + 20t \text{ m}}$$

Problem 13.16 As a first approximation, a bioengineer studying the mechanics of bird flight assumes that the snow petrel takes off with constant acceleration. Video measurements indicate that a bird requires a distance of 4.3 m to take off and is moving at 6.1 m/s when it does. What is its acceleration?



Solution: The governing equations are

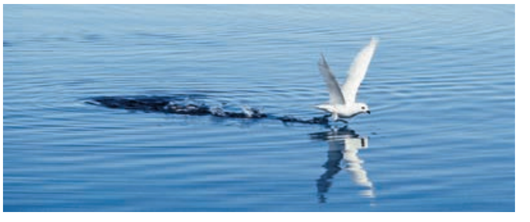
$$a = \text{constant}, \quad v = at, \quad s = \frac{1}{2}at^2.$$

Using the information given, we have

$$6.1 \text{ m/s} = at, \quad 4.3 \text{ m} = \frac{1}{2}at^2.$$

Solving these two equations, we find $t = 1.41$ s and $\boxed{a = 4.33 \text{ m/s}^2.}$

Problem 13.17 Progressively developing a more realistic model, the bioengineer next models the acceleration of the snow petrel by an equation of the form $a = C(1 + \sin \omega t)$, where C and ω are constants. From video measurements of a bird taking off, he estimates that $\omega = 18/\text{s}$ and determines that the bird requires 1.42 s to take off and is moving at 6.1 m/s when it does. What is the constant C?



Solution: We find an expression for the velocity by integrating the acceleration

$$a = C(1 + \sin \omega t),$$

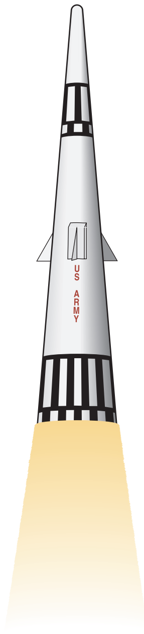
$$v = Ct + \frac{C}{\omega}(1 - \cos \omega t) = C\left(t + \frac{1}{\omega} - \frac{1}{\omega}\cos \omega t\right).$$

Using the information given, we have

$$6.1 \text{ m/s} = C\left(1.42 \text{ s} + \frac{\text{s}}{18} - \frac{\text{s}}{18}\cos[18(1.42)]\right)$$

Solving this equation, we find $\boxed{C = 4.28 \text{ m/s}^2.}$

Problem 13.18 Missiles designed for defense against ballistic missiles have attained accelerations in excess of 100 g's, or 100 times the acceleration due to gravity. Suppose that the missile shown lifts off from the ground and has a constant acceleration of 100 g's. How long does it take to reach an altitude of 3000 m? How fast is it going when it ranches that altitude?



Solution: The governing equations are

$$a = \text{constant}, \quad v = at, \quad s = \frac{1}{2}at^2$$

Using the given information we have

$$3000 \text{ m} = \frac{1}{2}100(9.81 \text{ m/s}^2)t^2 \Rightarrow \quad \boxed{t = 2.47 \text{ s}}$$

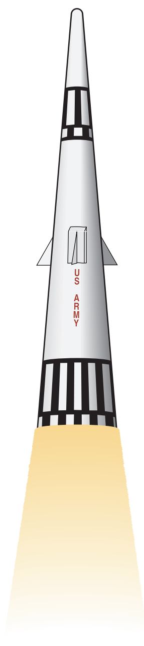
The velocity at that time is

$$v = 100(9.81 \text{ m/s}^2)(2.47 \text{ s}) = 243 \text{ m/s}. \quad \boxed{v = 2430 \text{ m/s.}}$$

Problem 13.19 Suppose that the missile shown lifts off from the ground and, because it becomes lighter as its fuel is expended, its acceleration (in g's) is given as a function of time in seconds by

$$a = \frac{100}{1 - 0.2t}.$$

What is the missile's velocity in kilometres per hour 1 s after liftoff?



Solution: We find an expression for the velocity by integrating the acceleration (valid only for $0 < t < 5$s).

$$v = \int_0^t a\,dt = \int_0^t \frac{100g}{1 - 0.2(t/s)}dt = \frac{100gs}{0.2}\ln\left(\frac{1}{1 - 0.2[t/s]}\right)$$

At time $t = 1$ s, we have

$$v = \frac{100(9.81 \text{ m/s}^2)(1 \text{ s})}{0.2}\ln\left(\frac{1}{1 - 0.2}\right) = 1095 \text{ m/s}.$$

$v = 3940$ km/h.

Problem 13.20 The airplane releases its drag parachute at time $t = 0$. Its velocity is given as a function of time by

$$v = \frac{80}{1 + 0.32t} \text{ m/s.}$$

What is the airplane's acceleration at $t = 3$ s?

Solution:

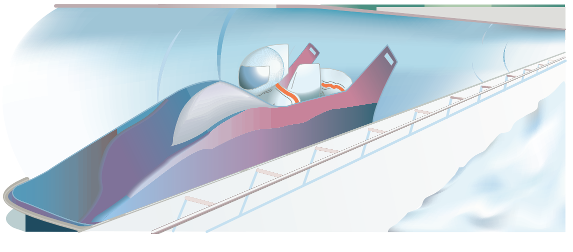
$$v = \frac{80}{1 + 0.32t}; \; a = \frac{dv}{dt} = \frac{-25.6}{(1 + 0.32t)^2} \Rightarrow \boxed{a(3 \text{ s}) = -6.66 \text{ m/s}^2}$$

Problem 13.21 How far does the airplane in Problem 13.20 travel during the interval of time from $t = 0$ to $t = 10$ s?

Solution:

$$\boxed{v = \frac{80}{1 + 0.32t}; \; s = \int_0^{10\text{ s}} \frac{80}{1 + 0.32t} dt = 250 \ln\left(\frac{1 + 3.2}{1}\right) = 359 \text{ m}}$$

Problem 13.22 The velocity of a bobsled is $v = 10t$ m/s. When $t = 2$ s, the position is $s = 25$ m. What is its position at $t = 10$ s?



Solution: The equation for straight line displacement under constant acceleration is

$$s = \frac{a(t - t_0)^2}{2} + v(t_0)(t - t_0) + s(t_0).$$

Choose $t_0 = 0$. At $t = 2$, the acceleration is

$$a = \left[\frac{dv(t)}{dt}\right]_{t=2} = 10 \text{ m/s}^2,$$

the velocity is $v(t_0) = 10(2) = 20$ m/s, and the initial displacement is $s(t_0) = 25$ m. At $t = 10$ seconds, the displacement is

$$\boxed{s = \frac{10}{2}(10 - 2)^2 + 20(10 - 2) + 25 = 505 \text{ m}}$$

Problem 13.23 In September, 2003, Tony Schumacher started from rest and drove 402 km in 4.498 seconds in a National Hot Rod Association race. His speed as he crossed the finish line was 528 km/h. Assume that the car's acceleration can be expressed by a linear function of time $a = b + ct$.

(a) Determine the constants b and c.
(b) What was the car's speed 2 s after the start of the race?

Solution:

$$a = b + ct, \quad v = bt + \frac{ct^2}{2}, \quad s = \frac{bt^2}{2} + \frac{ct^3}{6}$$

Both constants of integration are zero.

(a) $528 \text{ km/h} = b(4.498 \text{ s}) + \frac{c}{2}(4.498 \text{ s})^2$

$$402 \text{ km} = \frac{b}{2}(4.498 \text{ s})^2 + \frac{c}{6}(4.498 \text{ s})^3$$

$$\Rightarrow \boxed{\begin{array}{l} b = 54 \text{ m/s}^2 \\ c = -9.5 \text{ m/s}^3 \end{array}}$$

(b) $\boxed{v = b(2 \text{ s}) + \frac{c}{2}(2 \text{ s})^2 = 89 \text{ m/s}}$

Problem 13.24 The velocity of an object is $v = 200 - 2t^2$ m/s. When $t = 3$ seconds, its position is $s = 600$ m. What are the position and acceleration of the object at $t = 6$ s?

Solution: The acceleration is

$$\frac{dv(t)}{dt} = -4t \text{ m/s}^2.$$

At $t = 6$ seconds, the acceleration is $a = -24$ m/s^2. Choose the initial conditions at $t_0 = 3$ seconds. The position is obtained from the velocity:

$$\boxed{s(t - t_0) = \int_3^6 v(t)\, dt + s(t_0) = \left[200t - \frac{2}{3}t^3\right]_3^6 + 600 = 1070 \text{ m}}.$$

Problem 13.25 An inertial navigation system measures the acceleration of a vehicle from $t = 0$ to $t = 6$ s and determines it to be $a = 2 + 0.1t$ m/s^2. At $t = 0$, the vehicle's position and velocity are $s = 240$ m, $v = 42$ m/s, respectively. What are the vehicle's position and velocity at $t = 6$ s?

Solution:

$a = 2 + 0.1t$ m/s^2

$v_0 = 42$ m/s $\quad s_0 = 240$ m

Integrating

$v = v_0 + 2t + 0.1t^2/2$

$s = v_0 t + t^2 + 0.1t^3/6 + s_0$

Substituting the known values at $t = 6$ s, we get

$\underline{v = 55.8 \text{ m/s}}$

$\underline{s = 531.6 \text{ m}}$

Problem 13.26 In Example 13.3, suppose that the cheetah's acceleration is constant and it reaches its top speed of 120 km/h in 5 s. What distance can it cover in 10 s?

Solution: The governing equations while accelerating are

$$a = \text{constant}, \quad v = at, \quad s = \frac{1}{2}at^2.$$

Using the information supplied, we have

$$\left(\frac{120 \times 1000 \text{ m}}{3600 \text{ s}}\right) = a(5 \text{ s}) \Rightarrow a = 6.67 \text{ m/s}^2$$

The distance that he travels in the first 10 s (5 seconds accelerating and then the last 5 seconds traveling at top speed) is

$$s = \frac{1}{2}(6.67 \text{ m/s}^2)(5 \text{ s})^2 + \left(\frac{120 \times 1000 \text{ m}}{3600 \text{ s}}\right)(10 \text{ s} - 5 \text{ s}) = 250 \text{ m}.$$

$$\boxed{s = 250 \text{ m}.}$$

Problem 13.27 The graph shows the airplane's acceleration during its takeoff. What is the airplane's velocity when it rotates (lifts off) at $t = 30$ s?

Solution: Velocity = Area under the curve

$$\boxed{v = \frac{1}{2}(3 \text{ m/s}^2 + 9 \text{ m/s}^2)(5 \text{ s}) + (9 \text{ m/s}^2)(25 \text{ s}) = 255 \text{ m/s}}$$

Problem 13.28 Determine the distance traveled during its takeoff by the airplane in Problem 13.27.

Solution: for $0 \le t \le 5$ s

$$a = \left(\frac{6 \text{ m/s}^2}{5 \text{ s}}\right)t + (3 \text{ m/s}^2), \quad v = \left(\frac{6 \text{ m/s}^2}{5 \text{ s}}\right)\frac{t^2}{2} + (3 \text{ m/s}^2)t$$

$$s = \left(\frac{6 \text{ m/s}^2}{5 \text{ s}}\right)\frac{t^3}{6} + (3 \text{ m/s}^2)\frac{t^2}{2}$$

$v(5 \text{ s}) = 30$ m/s, $s(5 \text{ s}) = 62.5$ m

for $5 \text{ s} \le t \le 30$ s

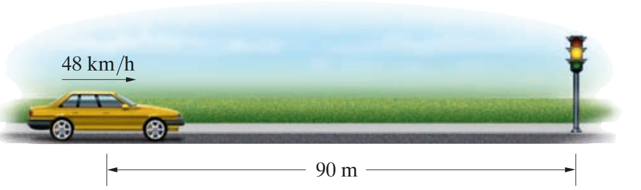
$a = 9 \text{ m/s}^2$, $v = (9 \text{ m/s}^2)(t - 5 \text{ s}) + 30$ m/s,

$$s = (9 \text{ m/s}^2)\frac{(t - 5 \text{ s})^2}{2} + (30 \text{ m/s})(t - 5 \text{ s}) + 62.5 \text{ m}$$

$$\Rightarrow \boxed{s(30 \text{ s}) = 3625 \text{ m}}$$

Problem 13.29 The car is traveling at 48 km/h when the traffic light 90 m ahead turns yellow. The driver takes one second to react before he applies the brakes.

(a) After he applies the brakes, what constant rate of deceleration will cause the car to come to a stop just as it reaches the light?
(b) How long does it take the car to travel the 90 m?



Solution: for $0 \le t \le 1$ s

$a = 0,\ v = 48\text{ km/h} = 13.33\text{ m/s},\ s = (13.33\text{ m/s})t$

$s(1\text{ s}) = 13.33\text{ m}$

for $t > 1$ s

$a = -c\text{ (constant)},\ v = -ct + 13.33\text{ m/s},\ s = -c\dfrac{t^2}{2} + (13.33\text{ m/s})t + 13.33\text{ m}$

At the stop we have

$$90\text{ m} = -c\frac{t^2}{2} + (13.33\text{ m/s})t + 13.33\text{ m}$$

$$0 = -ct + 13.33\text{ m/s}$$

$$\Rightarrow \boxed{\begin{array}{ll} a) & c = 1.17\text{ m/s}^2 \\ b) & t = 11.41\text{ s} \end{array}}$$

Problem 13.30 The car is traveling at 48 km/h when the traffic light 90 m ahead turns yellow. The driver takes 1 s to react before he applies the accelerator. If the car has a constant acceleration of 2 m/s^2 and the light remains yellow for 5 s, will the car reach the light before it turns red? How fast is the car moving when it reaches the light?

Solution: First, convert the initial speed into m/s.

$$48\text{ km/h} = 13.33\text{ m/s}.$$

At the end of the 5 s, the car will have traveled a distance

$$d = (13.33\text{ m/s})(1\text{ s}) + \left[\frac{1}{2}(2\text{ m/s}^2)(5\text{ s} - 1\text{ s})^2 + (13.33\text{ m/s})(5\text{ s} - 1\text{ s})\right] = 82.65\text{ m}.$$

When the light turns red, the driver will still be 7.35 m from the light. $\boxed{\text{No.}}$

To find the time at which the car does reach the light, we solve

$$90\text{ m} = (13.33\text{ m/s})(1\text{ s}) + \left[\frac{1}{2}(2\text{ m/s}^2)(t - 1\text{ s})^2 + (13.33\text{ m/s})(t - 1\text{ s})\right]$$

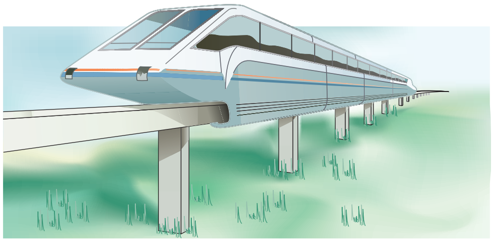
$$\Rightarrow t = 5.34\text{ s}.$$

The speed at this time is

$$v = 13.33\text{ m/s} + (2\text{ m/s}^2)(5.34\text{ s} - 1\text{ s}) = 22.01\text{ m/s}.$$

$$\boxed{v = 79.2\text{ km/h.}}$$

Problem 13.31 A high-speed rail transportation system has a top speed of 100 m/s. For the comfort of the passengers, the magnitude of the acceleration and deceleration is limited to 2 m/s^2. Determine the time required for a trip of 100 km.



Strategy: A graphical approach can help you solve this problem. Recall that the change in the position from an initial time t_0 to a time t is equal to the area defined by the graph of the velocity as a function of time from t_0 to t.

Solution: Divide the time of travel into three intervals: The time required to reach a top speed of 100 m/s, the time traveling at top speed, and the time required to decelerate from top speed to zero. From symmetry, the first and last time intervals are equal, and the distances traveled during these intervals are equal. The initial time is obtained from $v(t_1) = at_1$, from which $t_1 = 100/2 = 50$ s. The distance traveled during this time is $s(t_1) = at_1^2/2$ from which $s(t_1) = 2(50)^2/2 = 2500$ m. The third time interval is given by $v(t_3) = -at_3 + 100 = 0$, from which $t_3 = 100/2 = 50$ s. *Check*. The distance traveled is $s(t_3) = -\frac{a}{2}t_3^2 + 100t_3$, from which $s(t_3) = 2500$ m. *Check*. The distance traveled at top speed is $s(t_2) = 100000 - 2500 - 2500 = 95000$ m $= 95$ km. The time of travel is obtained from the distance traveled at zero acceleration: $s(t_2) = 95000 = 100t_2$, from which $t_2 = 950$. The total time of travel is $t_{\text{total}} = t_1 + t_2 + t_3 = 50 + 950 + 50 = 1050$ s

$\boxed{= 17.5 \text{ minutes}}$.

A plot of velocity versus time can be made and the area under the curve will be the distance traveled. The length of the constant speed section of the trip can be adjusted to force the length of the trip to be the required 100 km.

Problem 13.32 The nearest star, Proxima Centauri, is 4.22 light years from the Earth. Ignoring relative motion between the solar system and Proxima Centauri, suppose that a spacecraft accelerates from the vicinity of the Earth at 0.01 g (0.01 times the acceleration due to gravity at sea level) until it reaches one-tenth the speed of light, coasts until it is time to decelerate, then decelerates at 0.01 g until it comes to rest in the vicinity of Proxima Centauri. How long does the trip take? (Light travels at 3×10^8 m/s.)

Solution: The distance to Proxima Centauri is

$$d = (4.22 \text{ light - year})(3 \times 10^8 \text{ m/s})(365.2422 \text{ day})\left(\frac{86400 \text{ s}}{1 \text{ day}}\right)$$

$$= 3.995 \times 10^{16} \text{ m}.$$

Divide the time of flight into the three intervals. The time required to reach 0.1 times the speed of light is

$$t_1 = \frac{v}{a} = \frac{3 \times 10^7 \text{ m/s}}{0.0981 \text{ m/s}^2} = 3.0581 \times 10^8 \text{ seconds}.$$

The distance traveled is

$$s(t_1) = \frac{a}{2}t_1^2 + v(0)t + s(0),$$

where $v(0) = 0$ and $s(0) = 0$ (from the conditions in the problem), from which $s(t_1) = 4.587 \times 10^{15}$ m. From symmetry, $t_3 = t_1$, and $s(t_1) = s(t_3)$. The length of the middle interval is $s(t_2) = d - s(t_1) - s(t_3) = 3.0777 \times 10^{16}$ m. The time of flight at constant velocity is

$$t_2 = \frac{3.0777 \times 10^{16} \text{ m}}{3 \times 10^7} = 1.026 \times 10^9 \text{ seconds}.$$

The total time of flight is $t_{\text{total}} = t_1 + t_2 + t_3 = 1.63751 \times 10^9$ seconds. In solar years:

$$t_{\text{total}} = \left(1.63751 \times 10^9 \text{ sec}\right)\left(\frac{1 \text{ solar years}}{365.2422 \text{ days}}\right)\left(\frac{1 \text{ days}}{86400 \text{ sec}}\right)$$

$$= 51.9 \text{ solar years}$$

Problem 13.33 A race car starts from rest and accelerates at $a = 5 + 2t$ m/s^2 for 10 seconds. The brakes are then applied, and the car has a constant acceleration $a = -30$ m/s^2 until it comes to rest. Determine (a) the maximum velocity, (b) the total distance traveled; (c) the total time of travel.

Solution:

(a) For the first interval, the velocity is

$$v(t) = \int (5 + 2t)\,dt + v(0) = 5t + t^2$$

since $v(0) = 0$. The velocity is an increasing monotone function; hence the maximum occurs at the end of the interval, $t = 10$ s, from which

$\boxed{v_{\max} = 150 \text{ m/s}}$.

(b) The distance traveled in the first interval is

$$s(10) = \textstyle\int_0^{10} (5t + t^2)\,dt = \left[\dfrac{5}{2}t^2 + \dfrac{1}{3}t^3\right]_0^{10} = 583.33 \text{ m}.$$

The time of travel in the second interval is

$$v(t_2 - 10) = 0 = a(t_2 - 10) + v(10),\, t_2 \geq 10 \text{ s},$$

from which

$$(t_2 - 10) = -\frac{150}{-30} = 5, \quad \text{and}$$

(c) the total time of travel is $t_2 = 15$. The total distance traveled is

$$s(t_2 - 10) = \frac{a}{2}(t_2 - 10)^2 + v(10)(t_2 - 10) + s(10),$$

from which (b)

$$\boxed{s(5) = \frac{-30}{2}5^2 + 150(5) + 583.33 = 958.33 \text{ m}}$$

Problem 13.34 When $t = 0$, the position of a point is $s = 6$ m and its velocity is $v = 2$ m/s. From $t = 0$ to $t = 6$ s, the acceleration of the point is $a = 2 + 2t^2$ m/s^2. From $t = 6$ s until it comes to rest, its acceleration is $a = -4$ m/s^2.

(a) What is the total time of travel?
(b) What total distance does the point move?

Solution: For the first interval the velocity is

$$v(t) = \int (2 + 2t^2)\,dt + v(0) = \left[2t + \frac{2}{3}t^3\right] + 2 \text{ m/s}.$$

The velocity at the end of the interval is $v(6) = 158$ m/s. The displacement in the first interval is

$$s(t) = \int \left(2t + \frac{2}{3}t^3 + 2\right)\,dt + 6 = \left[t^2 + \frac{1}{6}t^4 + 2t\right] + 6.$$

The displacement at the end of the interval is $s(6) = 270$ m. For the second interval, the velocity is $v(t - 6) = a(t - 6) + v(6) = 0$, $t \geq 6$, from which

$$(t - 6) = -\frac{v(6)}{a} = -\frac{158}{-4} = 39.5.$$

The total time of travel is

(a) $\boxed{t_{\text{total}} = 39.5 + 6 = 45.5}$ seconds.

(b) The distance traveled is

$$s(t - 6) = \frac{-4}{2}(t - 6)^2 + v(6)(t - 6) + s(6)$$

$$= -2(39.5)^2 + 158(39.5) + 270,$$

from which the total distance is $\boxed{s_{\text{total}} = 3390 \text{ m}}$

Problem 13.35 Zoologists studying the ecology of the Serengeti Plain estimate that the average adult cheetah can run 100 km/h and that the average springbuck can run 65 km/h. If the animals run along the same straight line, start at the same time, and are each assumed to have constant acceleration and reach top speed in 4 s, how close must the a cheetah be when the chase begins to catch a springbuck in 15 s?

Solution: The top speeds are $V_c = 100$ km/h $= 27.78$ m/s for the cheetah, and $V_s = 65$ km/h $= 18.06$ m/s. The acceleration is $a_c = \dfrac{V_c}{4} = 6.94$ m/s^2 for the cheetah, and $a_s = \dfrac{V_s}{4} = 4.513$ m/s^2 for the springbuck. Divide the intervals into the acceleration phase and the chase phase. For the cheetah, the distance traveled in the first is $s_c(t) = \dfrac{6.94}{2}(4)^2 = 55.56$ m. The total distance traveled at the end of the second phase is $s_{\text{total}} = V_c(11) + 55.56 = 361.1$ m. For the springbuck, the distance traveled during the acceleration phase is $s_s(t) = \dfrac{4.513}{2}(4)^2 = 36.11$ m. The distance traveled at the end of the second phase is $s_s(t) = 18.06(11) + 36.1 = 234.7$ m. The permissible separation between the two at the beginning for a successful chase is

$$\boxed{d = s_c(15) - s_s(15) = 361.1 - 234.7 = 126.4 \text{ m}}$$

Problem 13.36 Suppose that a person unwisely drives 120 km/h in a 88 km/h zone and passes a police car going 88 km/h in the same direction. If the police officers begin constant acceleration at the instant they are passed and increase their speed to 129 km/h in 4 s, how long does it take them to be even with the pursued car?

Solution: The conversion from mi/h to m/s is

$$\frac{\text{km}}{\text{h}} = \frac{1000 \text{ m}}{3600 \text{ s}} = 0.278 \text{ m/s}$$

The acceleration of the police car is

$$a = \frac{(129-88)(0.278) \text{ m/s}}{4 \text{ s}} = 2.85 \text{ m/s}^2.$$

The distance traveled during acceleration is

$$s(t_1) = \frac{2.85}{2}(4)^2 + 88(0.278)(4) = 121 \text{ m}.$$

The distance traveled by the pursued car during this acceleration is

$$s_c(t_1) = 120(0.278)\,t_1 = 33.36(4) = 133.4 \text{ m}.$$

The separation between the two cars at 4 seconds is

$$d = 133.4 - 121 = 12.4 \text{ m}.$$

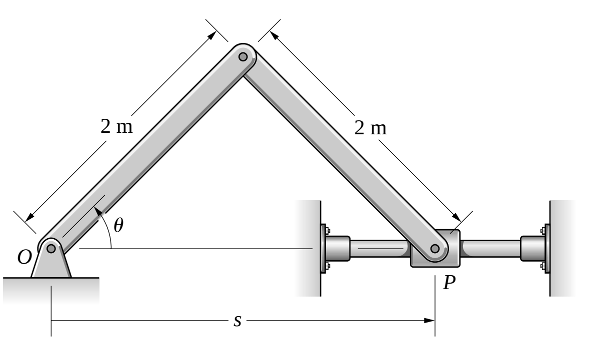
This distance is traversed in the time

$$t_2 = \frac{12.4}{(129-88)(0.278)} = 1.09.$$

The total time is $\boxed{t_{\text{total}} = 1.09 + 4 = 5.09}$ seconds.

Problem 13.37 If $\theta = 1$ rad and $\dfrac{d\theta}{dt} = 1$ rad/s, what is the velocity of P relative to O?

Strategy: You can write the position of P relative to O as $s = (2 \text{ m})\cos\theta + (2 \text{ m})\cos\theta$ and then take the derivative of this expression with respect to time to determine the velocity.



Solution: The distance s from point O is

$$s = (2 \text{ m})\cos\theta + (2 \text{ m})\cos\theta.$$

The derivative is

$$\frac{ds}{dt} = -4\sin\theta\frac{d\theta}{dt}.$$

For $\theta = 1$ radian and $\dfrac{d\theta}{dt} = 1$ radian/second,

$$\boxed{\frac{ds}{dt} = v(t) = -4(\sin(1 \text{ rad})) = -4(0.841) = -3.37 \text{ m/s}}$$

Problem 13.38 In Problem 13.37, if $\theta = 1$ rad, $d\theta/dt = -2$ rad/s and $d^2\theta/dt^2 = 0$, what are the velocity and acceleration of P relative to O?

Solution: The velocity is

$$\boxed{\frac{ds}{dt} = -4\sin\theta\frac{d\theta}{dt} = -4(\sin(1\text{ rad}))(-2) = 6.73\text{ m/s}}.$$

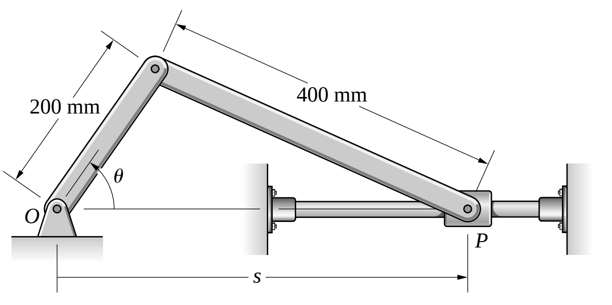
The acceleration is

$$\frac{d^2s}{dt^2} = -4\cos\theta\left(\frac{d\theta}{dt}\right)^2 - 4\sin\theta\left(\frac{d^2\theta}{dt^2}\right),$$

from which

$$\boxed{\frac{d^2s}{dt^2} = a = -4\cos(1\text{ rad})(4) = -8.64\text{ m/s}^2}$$

Problem 13.39 If $\theta = 1$ rad and $\dfrac{d\theta}{dt} = 1$ rad/s, what is the velocity of P relative to O?



Solution: The acute angle formed by the 400 mm arm with the horizontal is given by the sine law:

$$\frac{200}{\sin\alpha} = \frac{400}{\sin\theta},$$

from which

$$\sin\alpha = \left(\frac{200}{400}\right)\sin\theta.$$

For $\theta = 1$ radian, $\alpha = 0.4343$ radians. The position relative to O is.

$$s = 200\cos\theta + 400\cos\alpha.$$

The velocity is

$$\frac{ds}{dt} = v(t) = -200\sin\theta\left(\frac{d\theta}{dt}\right) - 400\sin\alpha\left(\frac{d\alpha}{dt}\right).$$

From the expression for the angle

$$\alpha, \cos\alpha\left(\frac{d\alpha}{dt}\right) = 0.5\cos\theta\left(\frac{d\theta}{dt}\right),$$

from which the velocity is

$$v(t) = (-200\sin\theta - 200\tan\alpha\cos\theta)\left(\frac{d\theta}{dt}\right).$$

Substitute: $\boxed{v(t) = -218.4\text{ mm/s}}$.

Problem 13.40 In Active Example 13.4, determine the time required for the plane's velocity to decrease from 50 m/s to 10 m/s.

Solution: From Active Example 13.4 we know that the acceleration is given by

$$a = -(0.004/\text{m})v^2.$$

We can find an expression for the velocity as a function of time by integrating

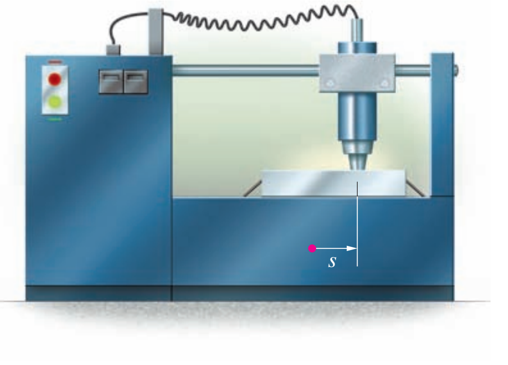
$$a = \frac{dv}{dt} = -(0.004/\text{m})v^2 \Rightarrow \frac{dv}{v^2} = -(0.004/\text{m})t$$

$$\int_{50\ \text{m/s}}^{10\ \text{m/s}} \frac{dv}{v^2} = \left(-\frac{1}{v}\right)_{50\ \text{m/s}}^{10\ \text{m/s}} = \left(-\frac{1\ \text{s}}{10\ \text{m}} + \frac{1\ \text{s}}{50\ \text{m}}\right) = \left(-\frac{4\ \text{s}}{50\ \text{m}}\right) = -(0.004/\text{m})t$$

$$t = \left(-\frac{4\ \text{s}}{50\ \text{m}}\right)\left(\frac{-1\ \text{m}}{0.004}\right) = 20\ \text{s}.$$

$$\boxed{t = 20\ \text{s}.}$$

Problem 13.41 An engineer designing a system to control a router for a machining process models the system so that the router's acceleration (in cm/s^2) during an interval of time is given by $a = -0.4v$, where v is the velocity of the router in cm/s. When $t = 0$, the position is $s = 0$ and the velocity is $v = 2$ cm/s. What is the position at $t = 3$ s?



Solution: We will first find the velocity at t = 3.

$$a = \frac{dv}{dt} = -\left(\frac{0.4}{\text{s}}\right)v,$$

$$\int_{2\ \text{cm/s}}^{v_2} \frac{dv}{v} = \int_0^{3\ s} \left(\frac{-0.4}{\text{s}}\right) dt,$$

$$\ln\left(\frac{v_2}{2\ \text{cm/s}}\right) = \left(\frac{-0.4}{\text{s}}\right)(3\ \text{s}) = -1.2$$

$$v_2 = (2\ \text{cm/s})e^{-1.2} = 0.602\ \text{cm/s}.$$

Now we can find the position

$$a = \frac{v\,dv}{ds} = \left(\frac{-0.4}{\text{s}}\right)v,$$

$$\int_{2\ \text{cm/s}}^{0.602\ \text{cm/s}} \frac{v\,dv}{v} = \left(\frac{-0.4}{\text{s}}\right)\int_0^{s_2} ds$$

$$(0.602\ \text{cm/s}) - (2\ \text{cm/s}) = \left(\frac{-0.4}{s}\right)s_2$$

$$s_2 = \frac{-1.398\ \text{cm/s}}{-0.4/\ \text{s}} = 3.49\ \text{cm}$$

$$\boxed{s_2 = 3.49\ \text{cm}}$$

Problem 13.42 The boat is moving at 10 m/s when its engine is shut down. Due to hydrodynamic drag, its subsequent acceleration is $a = -0.05v^2$ m/s^2, where v is the velocity of the boat in m/s. What is the boat's velocity 4 s after the engine is shut down?

Solution:

$$a = \frac{dv}{dt} = -(0.05\ \text{m}^{-1})v^2$$

$$\int_{10\,\text{m/s}}^{v} \frac{dv}{v^2} = -(0.05\ \text{m}^{-1})\int_0^t dt \Rightarrow -\left.\frac{1}{v}\right|_{10\,\text{m/s}}^{v} = -(0.05\ \text{m}^{-1})t$$

$$v = \frac{10\ \text{m/s}}{1 + (0.5\ \text{s}^{-1})t}$$

$$\boxed{v(4\ \text{s}) = 3.33\ \text{m/s}}$$

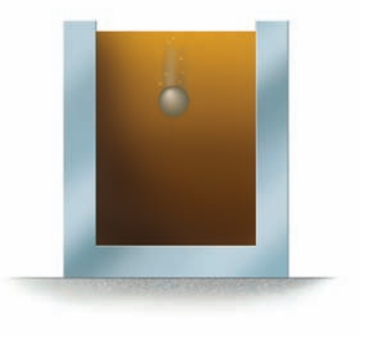
Problem 13.43 In Problem 13.42, what distance does the boat move in the 4 s following the shutdown of its engine?

Solution: From Problem 13.42 we know

$$v = \frac{ds}{dt} = \frac{10\ \text{m/s}}{1 + (0.5\ \text{s}^{-1})t} \Rightarrow s(4\ \text{s}) = \int_0^{4\ \text{s}} \frac{10\ \text{m/s}}{1 + (0.5\ \text{s}^{-1})t}dt$$

$$\boxed{s(4\ \text{s}) = (20\ \text{m})\ln\left[\frac{2 + (1\ \text{s}^{-1})(4\ \text{s})}{2}\right] = 21.97\ \text{m}}$$

Problem 13.44 A steel ball is released from rest in a container of oil. Its downward acceleration is $a = 2.4 - 0.6v$ cm/s^2, where v is the ball's velocity in cm/s. What is the ball's downward velocity 2 s after it is released?



Solution:

$$a = \frac{dv}{dt} = (2.4\ \text{cm/s}) - (0.6\ \text{s}^{-1})v$$

$$\int_0^v \frac{dv}{(2.4\ \text{cm /s}) - (0.6\ \text{s}^{-1})v} = \int_0^t dt$$

$$-\frac{5}{3}\ln\left(\frac{v + 4\ \text{cm/s}}{4\ \text{cm/s}}\right) = t \Rightarrow v = (4\ \text{cm/s})\left(1 - e^{-(0.6\ \text{s}^{-1})t}\right)$$

$$\boxed{v(2\ \text{s}) = 2.795\ \text{cm/s}}$$

Problem 13.45 In Problem 13.44, what distance does the ball fall in the first 2 s after its release?

Solution: From 13.44 we know

$$v = \frac{ds}{dt} = (4\ \text{cm/s})\left(1 - e^{-(0.6\ \text{s}^{-1})t}\right)$$

$$s(2\ \text{s}) = \int_0^t (4\ \text{cm/s})\left(1 - e^{-(0.6\ \text{s}^{-1})t}\right)dt$$

$$= \frac{20\ \text{cm}}{3}\left(e^{(-0.6\ \text{s}^{-1})t} - 1\right) + (4\ \text{cm/s})t$$

$$\boxed{s(2\ \text{s}) = 3.34\ \text{cm.}}$$

Problem 13.46 The greatest ocean depth yet discovered is the Marianas Trench in the western Pacific Ocean. A steel ball released at the surface requires 64 minutes to reach the bottom. The ball's downward acceleration is $a = 0.9g - cv$, where $g = 9.81$ m/s^2 and the constant $c = 3.02$ s^{-1}. What is the depth of the Marianas Trench in kilometers?

Solution:

$$a = \frac{dv}{dt} = 0.9g - cv.$$

Separating variables and integrating,

$$\int_0^v \frac{dv}{0.9g - cv} = \int_0^t dt = t.$$

Integrating and solving for v,

$$v = \frac{ds}{dt} = \frac{0.9g}{c}(1 - e^{-ct}).$$

Integrating,

$$\int_0^s ds = \int_0^t \frac{0.9g}{c}(1 - e^{-ct})\, dt.$$

We obtain

$$s = \frac{0.9g}{c}\left(t + \frac{e^{-ct}}{c} - \frac{1}{c}\right).$$

At $t = (64)(60) = 3840$ s, we obtain

$\underline{s = 11{,}225 \text{ m}}$.

Problem 13.47 The acceleration of a regional airliner during its takeoff run is $a = 14 - 0.0003v^2$ m/s^2, where v is its velocity in m/s. How long does it take the airliner to reach its takeoff speed of 200 m/s?

Solution:

$$a = \frac{dv}{dt} = (14 \text{ m/s}^2) - (0.0003 \text{ m}^{-1})v^2$$

$$\int_0^{200 \text{ m/s}} \frac{dv}{(14 \text{ m/s}^2) - (0.0003 \text{ m}^{-1})v^2} = \int_0^t dt$$

$$\boxed{t = 25.1 \text{ s}}$$

Problem 13.48 In Problem 13.47, what distance does the airliner require to take off?

Solution:

$$a = v\frac{dv}{ds} = (14 \text{ m/s}^2) - (0.0003 \text{ m}^{-1})v^2$$

$$\int_0^{200 \text{ m/s}} \frac{v\,dv}{(14 \text{ m/s}^2) - (0.0003 \text{ m}^{-1})v^2} = \int_0^s ds$$

$$\boxed{s = 3243 \text{ m}}$$

Problem 13.49 A sky diver jumps from a helicopter and is falling straight down at 30 m/s when her parachute opens. From then on, her downward acceleration is approximately $a = g - cv^2$, where $g = 9.81$ m/s^2 and c is a constant. After an initial "transient" period she descends at a nearly constant velocity of 5 m/s.

(a) What is the value of c, and what are its SI units?
(b) What maximum deceleration is the sky diver subjected to?
(c) What is her downward velocity when she has fallen 2 meters from the point at which her parachute opens?

Solution: Assume $c > 0$.

(a) After the initial transient, she falls at a constant velocity, so that the acceleration is zero and $cv^2 = g$, from which

$$\boxed{c = \frac{g}{v^2} = \frac{9.81 \text{ m/s}^2}{(5)^2 \text{ m}^2/\text{s}^2} = 0.3924 \text{ m}^{-1}}$$

(b) The maximum acceleration (in absolute value) occurs when the parachute first opens, when the velocity is highest:

$$\boxed{a_{\max} = |g - cv^2| = |g - c(30)^2| = 343.4 \text{ m/s}^2}$$

(c) Choose coordinates such that distance is measured positive downward. The velocity is related to position by the chain rule:

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = a,$$

from which

$$\frac{v\,dv}{g - cv^2} = ds.$$

Integrate:

$$\left(-\frac{1}{2c}\right) \ln|g - cv^2| = s + C.$$

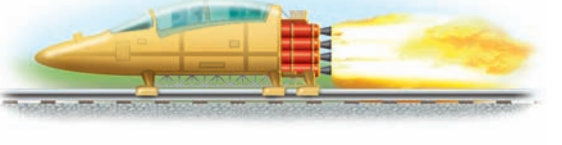
When the parachute opens $s = 0$ and $v = 30$ m/s, from which

$$C = -\left(\frac{1}{2c}\right) \ln|g - 900c| = -7.4398.$$

The velocity as a function of distance is $\ln|g - cv^2| = -2c(s + C)$. For $s = 2$ m,

$$\boxed{v = 14.4 \text{ m/s}}$$

Problem 13.50 The rocket sled starts from rest and accelerates at $a = 30 + 2t$ m/s^2 until its velocity is 400 m/s. It then hits a water brake and its acceleration is $a = -0.003v^2$ m/s^2 until its velocity decreases to 100 m/s. What total distance does the sled travel?



Solution: Acceleration Phase

$$a = 30 + 2t \text{ m/s}^2$$

$$v = 30t + t^2 \text{ m/s}$$

$$s = 15t^2 + t^3/3 \text{ m}$$

When $v = 400$ m/s, acceleration ends. At this point, $t = 10$ s and $s = 1833$ m. Deceleration Phase starts at $s_1 = 1833$ m, $v_1 = 400$ m/s. Let us start a new clock for the deceleration phase. $v_f = 100$ m/s

$$a = v\frac{dv}{ds} = -0.003v^2$$

$$\int_{s_1}^{s_f} ds = -\frac{1}{(0.003)} \int_{v_1}^{v_f} \frac{v\,dv}{v^2}$$

$$s_f - 1833 \text{ m} = -\frac{1}{(0.003)}[\ln(100) - \ln(400)]$$

$$\underline{s_f = 2300 \text{ m}}$$

Problem 13.51 In Problem 13.50, what is the sled's total time of travel?

Solution: From the solution to Problem 13.50, the acceleration takes 10 s. At $t = 10$ s, the velocity is 400 m/s. We need to find out how long it takes to decelerate from 400 m/s to 100 m/s and add this to the 10 s required for acceleration. The deceleration is given as

$$a = \frac{dv}{dt} = -0.003v^2 \text{ m/s}^2$$

$$-0.003\int_0^{t_d} dt = \int_{400}^{100} \frac{dv}{v^2}$$

$$-0.003t_d = -\frac{1}{v}\bigg|_{400}^{100} = -\left(\frac{1}{100} - \frac{1}{400}\right)$$

$$0.003t_d = \frac{3}{400}$$

$$t_d = 2.5 \text{ s}$$

$$\underline{t = 10 + t_d = 12.5 \text{ s}}$$

Problem 13.52 A car's acceleration is related to its position by $a = 0.01s$ m/s^2. When $s = 100$ m, the car is moving at 12 m/s. How fast is the car moving when $s = 420$ m?

Solution:

$$a = v\frac{dv}{ds} = 0.01s \text{ m/s}^2$$

$$\int_{12}^{v_f} v\,dv = 0.01\int_{100}^{420} s\,ds$$

$$\left[\frac{v^2}{2}\right]_{12 \text{ m/s}}^{v_f} = 0.01\left[\frac{s^2}{2}\right]_{100 \text{ m}}^{420 \text{ m}}$$

$$\frac{v_f^2}{2} = \frac{12^2}{2} + 0.01\frac{(420^2 - 100^2)}{2}$$

$$v_f = 42.5 \text{ m/s}$$

Problem 13.53 Engineers analyzing the motion of a linkage determine that the velocity of an attachment point is given by $v = A + 4s^2$ m/s, where A is a constant. When $s = 2$ m, its acceleration is measured and determined to be $a = 320$ m/s^2. What is its velocity of the point when $s = 2$ m?

Solution: The velocity as a function of the distance is

$$v\frac{dv}{ds} = a.$$

Solve for a and carry out the differentiation.

$$a = v\frac{dv}{ds} = (A + 4s^2)(8s).$$

When $s = 2$ m, $a = 320$ m/s^2, from which $A = 4$.

The velocity at $s = 2$ m is

$$\boxed{v = 4 + 4(2^2) = 20 \text{ m/s}}$$

Problem 13.54 The acceleration of an object is given as a function of its position in feet by $a = 2\ s^2 (\text{m/s}^2)$. When $s = 0$, its velocity is $v = 1$ m/s. What is the velocity of the object when $s = 2$ m?

Solution: We are given

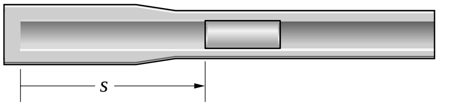
$$a = \frac{vdv}{ds} = \left(\frac{2}{\text{m-s}^2}\right) s^2,$$

$$\int_{1\ \text{m/s}}^{v} vdv = \left(\frac{2}{\text{m-s}^2}\right) \int_0^{2\text{m}} s^2 ds$$

$$\frac{v^2}{2} - \frac{(1\ \text{m/s})^2}{2} = \left(\frac{2}{\text{m-s}^2}\right) \frac{(2\ \text{m})^3}{3}$$

$$\boxed{v = 3.42\ \text{m/s.}}$$

Problem 13.55 Gas guns are used to investigate the properties of materials subjected to high-velocity impacts. A projectile is accelerated through the barrel of the gun by gas at high pressure. Assume that the acceleration of the projectile is given by $a = c/s$, where s is the position of the projectile in the barrel in meters and c is a constant that depends on the initial gas pressure behind the projectile. The projectile starts from rest at $s = 1.5$ m and accelerates until it reaches the end of the barrel at $s = 3$ m. Determine the value of the constant c necessary for the projectile to leave the barrel with a velocity of 200 m/s.



Solution:

$$a = v\frac{dv}{ds} = \frac{c}{s}, \Rightarrow \int_0^{200\,\text{m/s}} vdv = \int_{1.5\,\text{m}}^{3\,\text{m}} \frac{c}{s} ds$$

$$\Rightarrow \frac{(200\ \text{m/s})^2}{2} = c \ln\left(\frac{3\ \text{m}}{1.5\ \text{m}}\right)$$

$$\boxed{c = 28.85 \times 10^3\ \text{m}^2/\text{s}^2}$$

Problem 13.56 If the propelling gas in the gas gun described in Problem 13.55 is air, a more accurate modeling of the acceleration of the projectile is obtained by assuming that the acceleration of the projectile is given by $a = c/s^\gamma$, where $\gamma = 1.4$ is the ratio of specific heats for air. (This means that an isentropic expansion process is assumed instead of the isothermal process assumed in Problem 13.55.) Determine the value of the constant c necessary for the projectile to leave the barrel with a velocity of 200 m/s.

Solution:

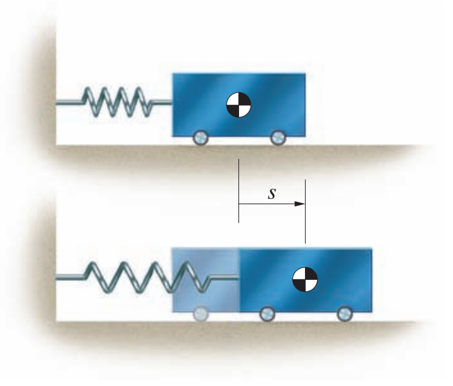
$$a = v\frac{dv}{ds} = \frac{c}{s^{1.4}}, \ \Rightarrow \int_0^{200\,\text{m/s}} vdv = \int_{1.5\,\text{m}}^{3\,\text{m}} \frac{c}{s^{1.4}} ds$$

$$\frac{(200\ \text{m/s})^2}{2} = -2.5c\left((3m)^{-0.4} - (1.5m)^{-0.4}\right)$$

$$\boxed{c = 38.86 \times 10^3\ \text{m}^{2.4}/\text{s}^2}$$

Problem 13.57 A spring-mass oscillator consists of a mass and a spring connected as shown. The coordinate s measures the displacement of the mass relative to its position when the spring is unstretched. If the spring is linear, the mass is subjected to a deceleration proportional to s. Suppose that $a = -4s$ m/s^2, and that you give the mass a velocity $v = 1$ m/s in the position $s = 0$.

(a) How far will the mass move to the right before the spring brings it to a stop?
(b) What will be the velocity of the mass when it has returned to the position $s = 0$?



Solution: The velocity of the mass as a function of its position is given by $v\,dv/ds = a$. Substitute the given acceleration, separate variables, and integrate: $v\,dv = -4s\,ds$, from which $v^2/2 = -2s^2 + C$. The initial velocity $v(0) = 1$ m/s at $s = 0$, from which $C = 1/2$. The velocity is $v^2/2 = -2s^2 + 1/2$.

(a) The velocity is zero at the position given by

$$0 = -2(s_1)^2 + \frac{1}{2},$$

from which $s_1 = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$ m.

Since the displacement has the same sign as the velocity,

$\boxed{s_1 = +1/2 \text{ m}}$

is the distance traveled before the spring brings it to a stop.

(b) At the return to $s = 0$, the velocity is $v = \pm\sqrt{\frac{2}{2}} = \pm 1$ m/s. From the physical situation, the velocity on the first return is negative (opposite the sign of the initial displacement),

$\boxed{v = -1 \text{ m/s}}$.

Problem 13.58 In Problem 13.57, suppose that at $t = 0$ you release the mass from rest in the position $s = 1$ m. Determine the velocity of the mass as a function of s as it moves from the initial position to $s = 0$.

Solution: From the solution to Problem 13.57, the velocity as a function of position is given by

$$\frac{v^2}{2} = -2s^2 + C.$$

At $t = 0$, $v = 0$ and $s = 1$ m, from which $C = 2(1)^2 = 2$. The velocity is given by

$$v(s) = \pm(-4s^2 + 4)^{\frac{1}{2}} = \pm 2\sqrt{1 - s^2} \text{ m/s}.$$

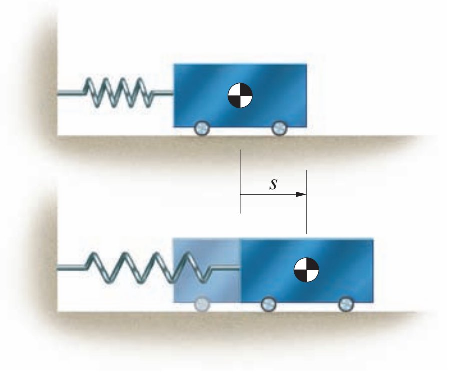
From the physical situation, the velocity is negative (opposite the sign of the initial displacement):

$\boxed{v = -2\sqrt{1 - s^2} \text{ m/s}}$.

[*Note*: From the initial conditions, $s^2 \leq 1$ always.]

Problem 13.59 A spring-mass oscillator consists of a mass and a spring connected as shown. The coordinate s measures the displacement of the mass relative to its position when the spring is unstretched. Suppose that the nonlinear spring subjects the mass to an acceleration $a = -4s - 2s^3$ m/s^2 and that you give the mass a velocity $v = 1$ m/s in the position $s = 0$.

(a) How far will the mass move to the right before the springs brings it to a stop?
(b) What will be the velocity of the mass when it has returned to the position $s = 0$?



Solution:

(a) Find the distance when the velocity is zero.

$$a = \frac{v\,dv}{ds} = \left(\frac{-4}{\text{s}^2}\right)s + \left(\frac{-2}{\text{m}^2\text{s}^2}\right)s^3$$

$$\int_{1\text{ m/s}}^{0} v\,dv = \int_0^d \left[\left(\frac{-4}{\text{s}^2}\right)s + \left(\frac{-2}{\text{m}^2\text{s}^2}\right)s^3\right]ds$$

$$-\frac{(1\text{ m/s})^2}{2} = -\left(\frac{4}{\text{s}^2}\right)\frac{d^2}{2} - \left(\frac{2}{\text{m}^2\text{s}^2}\right)\frac{d^4}{4}$$

Solving for d we find $\boxed{d = 0.486\ m.}$

(b) When the cart returns to the position $s = 0$, we have

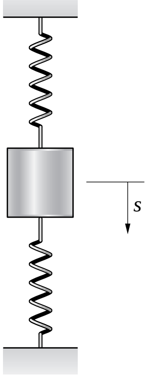
$$a = \frac{v\,dv}{ds} = \left(\frac{-4}{\text{s}^2}\right)s + \left(\frac{-2}{\text{m}^2\text{s}^2}\right)s^3$$

$$\int_{1\text{ m/s}}^{v} v\,dv = \int_0^0 \left[\left(\frac{-4}{\text{s}^2}\right)s + \left(\frac{-2}{\text{m}^2\text{s}^2}\right)s^3\right]ds = 0$$

$$\frac{v^2}{2} - \frac{(1\text{ m/s})^2}{2} = 0, \Rightarrow v = \pm 1\text{ m/s}.$$

The cart will be moving to the left, so we choose $\boxed{v = -1\text{ m/s.}}$

Problem 13.60 The mass is released from rest with the springs unstretched. Its downward acceleration is $a = 32.2 - 50s$ m/s^2, where s is the position of the mass measured from the position in which it is released. (a) How far does the mass fall? (b) What is the maximum velocity of the mass as it falls?



Solution: The acceleration is given by

$$a = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds} = 32.2 - 50s\ \text{m/s}^2.$$

Integrating, we get

$$\int_0^v v\,dv = \int_0^s (32.2 - 50s)\,ds \quad \text{or} \quad \frac{v^2}{2} = 32.2s - 25s^2.$$

(a) The mass falls until $v = 0$. Setting $v = 0$, we get $0 = (32.2 - 25s)s$. We find $v = 0$ at $s = 0$ and at $s = 1.288$ m. Thus, the mass falls 1.288 m before coming to rest.

(b) From the integration of the equation of motion, we have $v^2 = 2(32.2s - 25s^2)$. The maximum velocity occurs where $\frac{dv}{ds} = 0$. From the original equation for acceleration, we have $a = v\frac{dv}{ds} = (32.2 - 50s)$ m/s^2. Since we want maximum velocity, we can assume that $v \neq 0$ at this point. Thus, $0 = (32.2 - 50s)$, or $s = (32.2/50)$ m when $v = v_{\text{MAX}}$. Substituting this value for s into the equation for v, we get

$$v_{\text{MAX}}^2 = 2\left(\frac{(32.2)^2}{50} - \frac{(25)(32.2)^2}{50^2}\right),$$

or $\quad v_{\text{MAX}} = 4.55$ m/s

Problem 13.61 Suppose that the mass in Problem 13.60 is in the position $s = 0$ and is given a downward velocity of 10 m/s.

(a) How far does the mass fall?
(b) What is the maximum velocity of the mass as it falls?

Solution:

$$a = v\frac{dv}{ds} = (32.2 \text{ m/s}^2) - (50 \text{ s}^{-2})s$$

$$\int_{10 \text{ m/s}}^{v} v\,dv = \int_0^s [(32.2 \text{ m/s}^2) - (50 \text{ s}^{-2})s]ds$$

$$\frac{v^2}{2} - \frac{(10 \text{ m/s})^2}{2} = (32.2 \text{ m/s}^2)\, s - (50 \text{ s}^{-2})\frac{s^2}{2}$$

$$v^2 = (10 \text{ m/s})^2 + (64.4 \text{ m/s}^2)s - (50 \text{ s}^{-2})s^2$$

(a) The mass falls until $v = 0$

$$\boxed{0 = (10 \text{ m/s})^2 + (64.4 \text{ m/s}^2)s - (50 \text{ s}^{-2})s^2 \Rightarrow s = 2.20 \text{ m}}$$

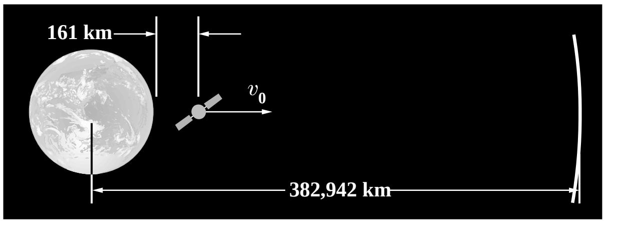
(b) The maximum velocity occurs when $a = 0$

$$0 = (32.2 \text{ m/s}^2) - (50 \text{ s}^{-2})s \Rightarrow s = 0.644 \text{ m}$$

$$v^2 = (10 \text{ m/s})^2 + (64.4 \text{ m/s}^2)(0.644 \text{ m}) - (50 \text{ s}^{-2})(0.644 \text{ m})^2$$

$$\boxed{v = 10.99 \text{ m/s}}$$

Problem 13.62 If a spacecraft is 161 km above the surface of the earth, what initial velocity v_0 straight away from the earth would be required for the vehicle to reach the moon's orbit 382,942 km from the center of the earth? The radius of the earth is 6372 km. Neglect the effect of the moon's gravity. (See Example 13.5.)



Solution: For computational convenience, convert the acceleration due to Earth's gravity into the units given in the problem, namely miles and hours:

$$g = \left(\frac{9.81 \text{ m}}{1\ s^2}\right)\left(\frac{1 \text{ km}}{1000 \text{ m}}\right)\left(\frac{3600^2 \text{ s}^2}{1 \text{ h}^2}\right) = 127137.6 \text{ km/h}^2.$$

The velocity as a function of position is given by

$$v\frac{dv}{ds} = a = -\frac{gR_E^2}{s^2}.$$

Separate variables,

$$v\,dv = -gR_E^2\frac{ds}{s^2}.$$

Integrate:

$$v^2 = -2gR_E^2\left(-\frac{1}{2}\right) + C.$$

Suppose that the velocity at the distance of the Moon's orbit is zero. Then

$$0 = 2(127137.6)\left(\frac{6372^2}{382942}\right) + C,$$

from which $C = -26960164 \text{ km}^2/\text{h}^2$. At the 161 km altitude, the equation for the velocity is

$$v_0^2 = 2\,g\left(\frac{R_E^2}{R_E + 161}\right) + C.$$

From which

$$\boxed{v_0 = \sqrt{1553351991} = 39{,}413 \text{ km/h}}$$

Converting:

$$\boxed{v_0 = \left(\frac{39413 \text{ km}}{1 \text{ h}}\right)\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 10{,}948 \text{ m/s}}.$$

Check: Use the result of Example 13.5

$$v_0 = \sqrt{2gR_E^2\left(\frac{1}{s_0} - \frac{1}{\text{H}}\right)},$$

(where $\text{H} > s_0$ always), and $\text{H} = 382{,}942$,

from which $v_0 = 39{,}413$ km/h. *check*.

Problem 13.63 The moon's radius is 1738 km. The magnitude of the acceleration due to gravity of the moon at a distance s from the center of the moon is

$$\frac{4.89 \times 10^{12}}{s^2} \text{ m/s}^2.$$

Suppose that a spacecraft is launched straight up from the moon's surface with a velocity of 2000 m/s.

(a) What will the magnitude of its velocity be when it is 1000 km above the surface of the moon?
(b) What maximum height above the moon's surface will it reach?

Solution: Set $G = 4.89 \times 10^{12}$ m^3/s^2, $r_0 = 1.738 \times 10^6$ m, $v_0 = 2000$ m/s

$$a = v\frac{dv}{ds} = -\frac{G}{s^2} \Rightarrow \int_{v_0}^{v} v\,dv = -\int_{r_0}^{r} \frac{G}{s^2}ds \Rightarrow \frac{v^2}{2} - \frac{{v_0}^2}{2} = G\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$$v^2 = {v_0}^2 + 2G\left(\frac{r_0 - r}{r r_0}\right)$$

(a) $\boxed{v(r_0 + 1.0 \times 10^6 \text{ m}) = 1395 \text{ m/s}}$

(b) The maximum velocity occurs when $v = 0$

$$r = \frac{2G}{2G - r_0 {v_0}^2} = 6010 \text{ km} \quad \Rightarrow \boxed{h = r - r_0 = 4272 \text{ km}}$$

Problem 13.64* The velocity of an object subjected only to the earth's gravitational field is

$$v = \left[v_0^2 + 2gR_{\text{E}}^2\left(\frac{1}{s} - \frac{1}{s_0}\right)\right]^{1/2},$$

where s is the object's position relative to the center of the earth, v_0 is the velocity at position s_0, and R_{E} is the earth's radius. Using this equation, show that the object's acceleration is given as a function of s by $a = -gR_{\text{E}}^2/s^2$.

Solution:

$$v = v_0^2 + 2gR_{\text{E}}^2\left(\frac{1}{s} - \frac{1}{s_0}\right)^{1/2}$$

$$a = \frac{dv}{dt} = v\frac{dv}{ds}$$

Rewrite the equation given as $v^2 = v_0^2 + \dfrac{2gR_{\text{E}}^2}{s} - \dfrac{2gR_{\text{E}}^2}{s_0}$

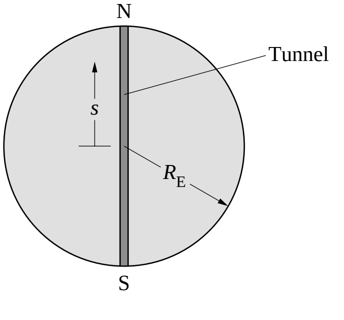
Take the derivative with respect to s.

$$2v\frac{dv}{ds} = -\frac{2gR_{\text{E}}^2}{s^2}$$

Thus

$$\underline{a = v\frac{dv}{ds} - \frac{gR_{\text{E}}^2}{s^2}}$$

Problem 13.65 Suppose that a tunnel could be drilled straight through the earth from the North Pole to the South Pole and the air was evacuated. An object dropped from the surface would fall with the acceleration $a = -gs/R_{\text{E}}$, where g is the acceleration of gravity at sea level, R_{E} is radius of the earth, and s is the distance of the object from the center of the earth. (The acceleration due to gravitation is equal to zero at the center of the earth and increases linearly with the distance from the center.) What is the magnitude of the velocity of the dropped object when it reaches the center of the earth?



Solution: The velocity as a function of position is given by

$$v\frac{dv}{ds} = -\frac{gs}{R_{\text{E}}}.$$

Separate variables and integrate:

$$v^2 = -\left(\frac{g}{R_{\text{E}}}\right)s^2 + C.$$

At $s = R_{\text{E}}$, $v = 0$, from which $C = gR_{\text{E}}$. Combine and reduce:

$$v^2 = gR_{\text{E}}\left(1 - \frac{s^2}{R_{\text{E}}^2}\right)$$

At the center of the earth $s = 0$, and the velocity is $\boxed{v = \sqrt{gR_{\text{E}}}}$

Problem 13.66 Determine the time in seconds required for the object in Problem 13.65 to fall from the surface of the earth to the center. The earth's radius is 6370 km.

Solution: From Problem 13.65, the acceleration is

$$a = v\frac{dv}{ds} = -\frac{g}{R_E}s$$

$$\int_0^v v\,du = -\int_{R_E}^s \left(\frac{g}{R_E}\right) s\,ds$$

$$v^2 = \left(\frac{g}{R_E}\right)(R_E^2 - s^2)$$

Recall that $v = ds/dt$

$$v = \frac{ds}{dt} = \pm\sqrt{\frac{g}{R_E}}\sqrt{R_E^2 - s^2}$$

$$\int_{R_E}^0 \frac{ds}{\sqrt{R_E^2 - s^2}} = \pm\sqrt{\frac{g}{R_E}}\int_0^{t_f} dt$$

$$\sqrt{\frac{g}{R_E}}t_f = \pm\sin^{-1}\left(\frac{s}{R_E}\right)\Bigg|_{R_E}^0 = \pm\sin^{-1}(1)$$

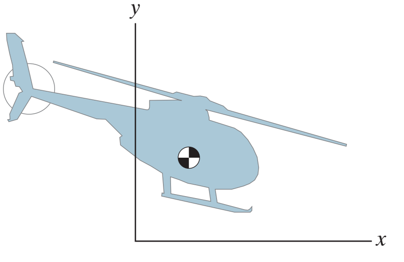
$$t_f = \pm\sqrt{\frac{R_E}{g}}\frac{\pi}{2} = \pm 1266 \text{ s} = \pm 21.1 \text{ min}$$

Problem 13.67 In a second test, the coordinates of the position (in m) of the helicopter in Active Example 13.6 are given as functions of time by

$x = 4 + 2t,$

$y = 4 + 4t + t^2.$

(a) What is the magnitude of the helicopter's velocity at $t = 3$ s?
(b) What is the magnitude of the helicopter's acceleration at $t = 3$ s?



Solution: We have

$$x = (4 \text{ m}) + (2 \text{ m/s})t, \quad y = (4 \text{ m}) + (4 \text{ m/s})t + (1 \text{ m/s}^2)t^2,$$

$$v_x = (2 \text{ m/s}), \quad v_y = (4 \text{ m/s}) + 2(1 \text{ m/s}^2)t,$$

$$a_x = 0, \quad a_y = 2(1 \text{ m/s}^2).$$

At $t = 3$ s, we have

$$x = 10 \text{ m}, \quad v_x = 2 \text{ m/s}, \quad a_x = 0,$$

$$y = 25 \text{ m}, \quad v_y = 10 \text{ m/s}, \quad a_y = 2 \text{ m/s}^2.$$

Thus

$$v = \sqrt{v_x^2 + v_y^2} = 10.2 \text{ m/s} \quad \boxed{v = 10.2 \text{ m/s.}}$$

$$a = \sqrt{0^2 + (2 \text{ m/s}^2)^2} = 2 \text{ m/s}^2 \quad \boxed{a = 2 \text{ m/s}^2.}$$

Problem 13.68 In terms of a particular reference frame, the position of the center of mass of the F-14 at the time shown ($t = 0$) is $\mathbf{r} = 10\mathbf{i} + 6\mathbf{j} + 22\mathbf{k}$ (m). The velocity from $t = 0$ to $t = 4$ s is $\mathbf{v} = (52 + 6t)\mathbf{i} + (12 + t^2)\mathbf{j} - (4 + 2t^2)\mathbf{k}$ (m/s). What is the position of the center of mass of the plane at $t = 4$ s?

Solution:

$$\mathbf{r}_0 = 10\mathbf{i} + 6\mathbf{j} + 22\mathbf{k} \text{ m}$$

$$\mathbf{v} = (52 + 6t)\mathbf{i} + (12 + t^2)\mathbf{j} - (4 + 2t^2)\mathbf{k} \text{ m/s}$$

$$x_4 = \int_0^4 v_x\,dt = 52t + 3t^2 + x_0$$

$$x_4 = (52)(4) + 3(4)^2 + 10 \text{ m} = 266.0 \text{ m}$$

$$y_4 = \int_0^4 v_y\,dt = 12t + t^3/3 + y_0$$

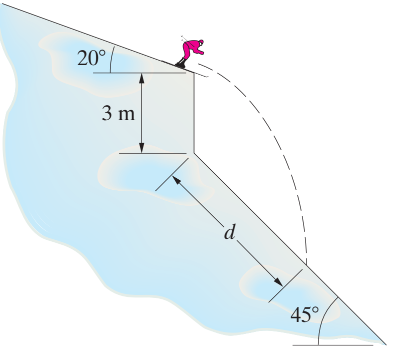
$$y_4 = 12(4) + (4)^3/3 + 6 \text{ m} = 75.3 \text{ m}$$

$$z_4 = \int_0^4 v_z\,dt = -(4t + 2t^3/3) + z_0$$

$$z_4 = -4(4) - 2(4)^3/3 + 22 = -36.7 \text{ m}$$

$$\mathbf{r}|_{t=4s} = 266\mathbf{i} + 75.3\mathbf{j} - 36.7\mathbf{k} \text{ (m)}$$

Problem 13.69 In Example 13.7, suppose that the angle between the horizontal and the slope on which the skier lands is 30° instead of 45°. Determine the distance d to the point where he lands.



Solution: The skier leaves the 20° surface at 10 m/s.

The equations are

$a_x = 0,\quad a_y = -9.81 \text{ m/s}^2,$

$v_x = (10 \text{ m/s})\cos 20^\circ,\quad v_y = -(9.81 \text{ m/s}^2)t - (10 \text{ m/s})\sin 20^\circ$

$s_x = (10 \text{ m/s})\cos 20^\circ t,\quad s_y = -\frac{1}{2}(9.81 \text{ m/s}^2)t^2 - (10 \text{ m/s})\sin 20^\circ t$

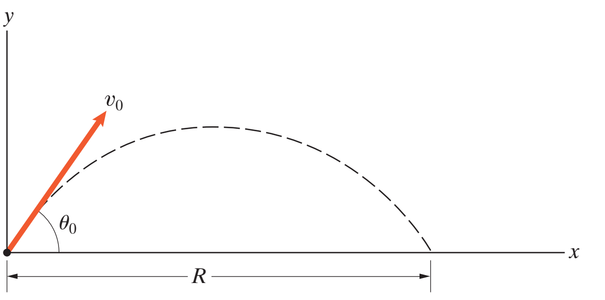
When he hits the slope, we have

$s_x = d\cos 30^\circ = (10 \text{ m/s})\cos 20^\circ t,$

$s_y = (-3 \text{ m}) - d\sin 30^\circ = -\frac{1}{2}(9.81 \text{ m/s}^2)t^2 - (10 \text{ m/s})\sin 20^\circ t$

Solving these two equations together, we find $t = 1.01$ s, $\boxed{d = 11.0 \text{ m.}}$

Problem 13.70 A projectile is launched from ground level with initial velocity $v_0 = 20$ m/s. Determine its range R if (a) $\theta_0 = 30^\circ$; (b) $\theta_0 = 45^\circ$ (c) $\theta_0 = 60^\circ$.



Solution: Set $g = 9.81\ \text{m/s}^2,\ v_0 = 20\ \text{m/s}$

$$a_y = -g,\ v_y = -gt + v_0 \sin\theta_0,\ s_y = -\frac{1}{2}gt^2 + v_0 \sin\theta_0 t$$

$$a_x = 0,\ v_x = v_0 \cos\theta_0,\ s_x = v_0 \cos\theta_0 t$$

When it hits the ground, we have

$$0 = -\frac{1}{2}gt^2 + v_0 \sin\theta_0 t \Rightarrow t = \frac{2v_0 \sin\theta_0}{g}$$

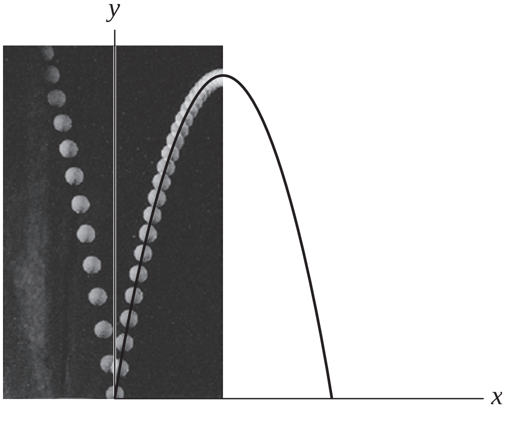
$$R = v_0 \cos\theta_0 t \Rightarrow R = \frac{v_0{}^2 \sin 2\theta_0}{g}$$

$$\Rightarrow \boxed{\begin{array}{ll} a) & \theta_0 = 30^\circ \Rightarrow R = 35.3\ \text{m} \\ b) & \theta_0 = 45^\circ \Rightarrow R = 40.8\ \text{m} \\ c) & \theta_0 = 60^\circ \Rightarrow R = 35.3\ \text{m} \end{array}}$$

Problem 13.71 Immediately after the bouncing golf ball leaves the floor, its components of velocity are $v_x = 0.662$ m/s and $v_y = 3.66$ m/s.

(a) Determine the horizontal distance from the point where the ball left the floor to the point where it hits the floor again.

(b) The ball leaves the floor at $x = 0$, $y = 0$. Determine the ball's y coordinate as a function of x. (The parabolic function you obtain is shown superimposed on the photograph of the ball.)



Solution: The governing equations are

$$a_x = 0, \quad a_y = -g,$$

$$v_x = v_{x0}, \quad v_y = -gt + v_{y0},$$

$$x = v_{x0}t, \quad y = -\frac{1}{2}gt^2 + v_{y0}t$$

(a) When it hits the ground again

$$0 = -\frac{1}{2}gt^2 + v_{y0}t \Rightarrow t = \frac{2v_{y0}}{g} \Rightarrow x = v_{x0}t = v_{x0}\left(\frac{2y_0}{g}\right) = \frac{2v_{x0}v_{y0}}{g}$$

$$x = \frac{2(0.662\ \text{m/s})(3.66\ \text{m/s})}{9.81\ \text{m/s}^2} \quad \boxed{x = 0.494\ \text{m.}}$$

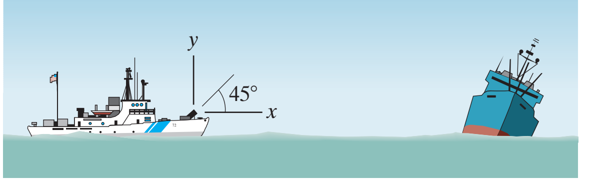
(b) At any point of the flight we have

$$t = \frac{x}{v_{x0}},\ y = -\frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2 + v_{y0}\left(\frac{x}{v_{x0}}\right)$$

$$y = -\frac{1}{2}\left(\frac{9.81\ \text{m/s}^2}{[0.662\ \text{m/s}]^2}\right)x^2 + \frac{3.66\ \text{m/s}}{0.662\ \text{m/s}}x$$

$$\boxed{y = -\left(\frac{11.2}{\text{m}}\right)x^2 + 5.53x.}$$

Problem 13.72 Suppose that you are designing a mortar to launch a rescue line from coast guard vessel to ships in distress. The light line is attached to a weight fired by the mortar. Neglect aerodynamic drag and the weight of the line for your preliminary analysis. If you want the line to be able to reach a ship 91 m away when the mortar is fired at 45° above the horizontal, what muzzle velocity is required?



Solution: From 13.70 we know that

$$R = \frac{{v_0}^2 \sin 2\theta_0}{g} \Rightarrow v_0 = \sqrt{\frac{Rg}{\sin 2\theta_0}}$$

$$\boxed{v_0 = \sqrt{\frac{(91)(9.81 \text{ m/s}^2)}{\sin(90^\circ)}} = 29.9 \text{ m/s}}$$

Problem 13.73 In Problem 13.72, what maximum height above the point where it was fired is reached by the weight?

Solution: From Problem 13.70 we have

$$v_y = -gt + v_0 \sin\theta_0, \quad s_y = -\frac{1}{2}gt^2 + v_0 \sin\theta_0 t$$

When we reach the maximum height,

$$0 = -gt + v_0 \sin\theta_0 \Rightarrow t = \frac{v_0 \sin\theta_0}{g}$$

$$h = -\frac{1}{2}gt^2 + v_0 \sin\theta_0 t \Rightarrow h = -\frac{1}{2}g\left(\frac{v_0 \sin\theta_0}{g}\right)^2 + v_0 \sin\theta_0 \left(\frac{v_0 \sin\theta_0}{g}\right)$$

$$h = \frac{{v_0}^2 \sin^2\theta_0}{2g}$$

Putting in the numbers we have

$$\boxed{h = \frac{(29.9 \text{ m/s})^2 \sin^2(45^\circ)}{2(9.81 \text{ m/s}^2)} = 22.78 \text{ m}}$$

Problem 13.74 When the athlete releases the shot, it is 1.82 m above the ground and its initial velocity is $v_0 = 13.6$ m/s. Determine the horizontal distance the shot travels from the point of release to the point where it hits the ground.

Solution: The governing equations are

$$a_x = 0,$$

$$v_x = v_0 \cos 30^\circ,$$

$$s_x = v_0 \cos 30^\circ t,$$

$$a_y = -g$$

$$v_y = -gt + v_0 \sin 30^\circ$$

$$s_y = -\frac{1}{2}gt^2 + v_0 \sin 30^\circ t + h$$

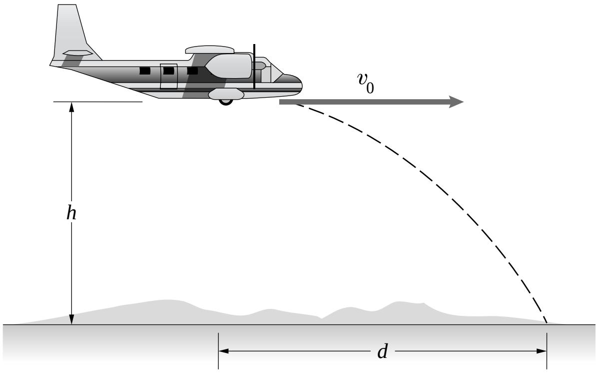
When it hits the ground, we have

$$s_x = (13.6 \text{ m/s}) \cos 30^\circ t,$$

$$s_y = 0 = -\frac{1}{2}(9.81 \text{ m/s}^2)t^2 + (13.6 \text{ m/s}) \sin 30^\circ t + 1.82 \text{ m}.$$

Solving these two equations, we find $t = 1.62$ s, $\boxed{s_x = 19.0 \text{ m.}}$

Problem 13.75 A pilot wants to drop survey markers at remote locations in the Australian outback. If he flies at a constant velocity $v_0 = 40$ m/s at altitude $h = 30$ m and the marker is released with zero velocity relative to the plane, at what horizontal d from the desired impact point should the marker be released?



Solution: We want to find the horizontal distance traveled by the marker before it strikes the ground (y goes to zero for $t > 0$.)

$$a_x = 0 \qquad a_y = -g$$

$$v_x = v_{x0} \qquad v_y = v_{y0} - gt$$

$$x = x_0 + v_{x0}t \quad y = y_0 + v_{y0}t - gt^2/2$$

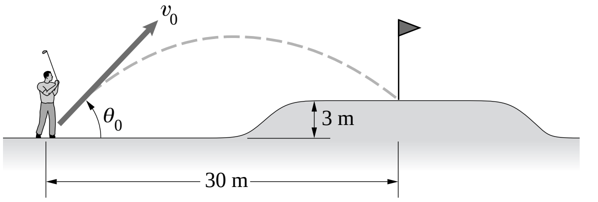
From the problem statement, $x_0 = 0$, $v_{y0} = 0$, $v_{x0} = 40$ m/s, and $y_0 =$ 30 m The equation for y becomes

$$y = 30 - (9.81)t^2/2$$

Solving with $y = 0$, we get $t_f = 2.47$ s.
Substituting this into the equation for x, we get

$$\underline{x_f = 40t_f = 98.9 \text{ m}}$$

Problem 13.76 If the pitching wedge the golfer is using gives the ball an initial angle $\theta_0 = 50^\circ$, what range of velocities v_0 will cause the ball to land within 3 m of the hole? (Assume the hole lies in the plane of the ball's trajectory).



Solution: Set the coordinate origin at the point where the golfer strikes the ball. The motion in the horizontal (x) direction is given by $a_x = 0,\ V_x = V_0 \cos\theta_0,\ x = (V_0 \cos\theta_0)t$. The motion in the vertical (y) direction is given by

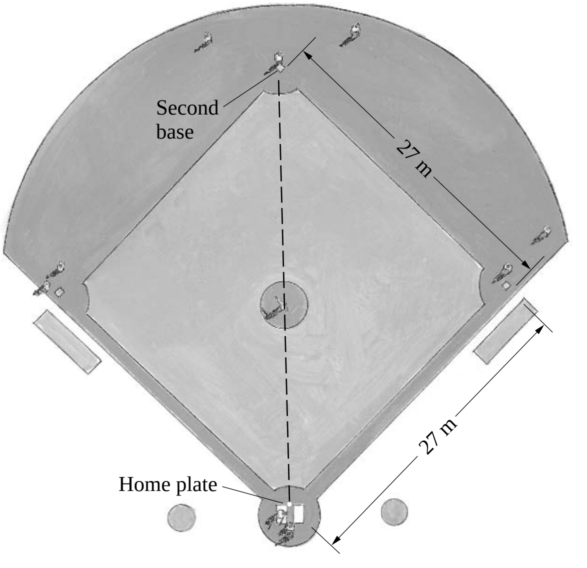
$$a_y = -g, \quad V_y = V_0 \sin\theta_0 - gt, \quad y = (V_0 \sin\theta_0)t - \frac{gt^2}{2}.$$

From the x equation, we can find the time at which the ball reaches the required value of x (27 or 33 metres). This time is

$$t_f = x_f/(V_0 \cos\theta_0).$$

We can substitute this information the equation for Y with $Y_f = 3\,\text{m}$ and solve for V_0. The results are: For hitting (27,3) metre, $V_0 = 31.2$ m/s. For hitting (33,3) metre, $V_0 = 34.2$ m/s.

Problem 13.77 A batter strikes a baseball 0.9 m above home plate and pops it up. The second baseman catches it 1.8 m above second base 3.68 s after it was hit. What was the ball's initial velocity, and what was the angle between the ball's initial velocity vector and the horizontal?



Solution: The equations of motion $g = 9.81$ m/s^2

$$a_x = 0 \qquad a_y = -g$$

$$v_x = v_0 \cos\theta_0 \qquad v_y = -gt + v_0 \sin\theta_0$$

$$s_x = v_0 \cos\theta_0 t \qquad s_y = -\frac{1}{2}gt^2 + v_0 \sin\theta_0 t + 0.9\text{ m}$$

When the second baseman catches the ball we have

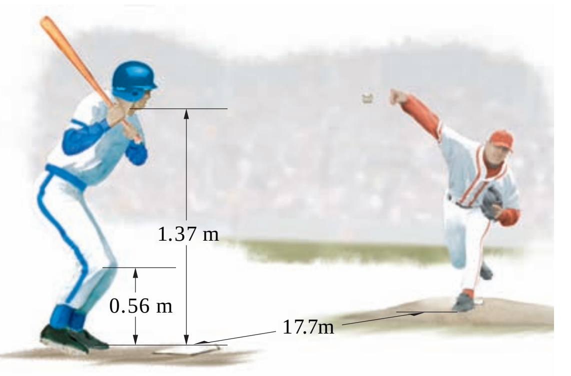
$$38.2\text{ m} = v_0 \cos\theta_0 (3.68\text{ s})$$

$$1.8\text{ m} = \frac{1}{2}-(9.81\text{ m/s}^2)(3.68\text{ s})^2 + v_0 \sin\theta_0 (3.68\text{ s})$$

Solving simultaneously we find

$$\boxed{v_0 = 21\,\text{m/s}, \qquad \theta_0 = 60.4^\circ}$$

Problem 13.78 A baseball pitcher releases a fastball with an initial velocity $v_0 = 144.8$ km/h. Let θ be the initial angle of the ball's velocity vector above the horizontal. When it is released, the ball is 1.83 m above the ground and 17.7 m from the batter's plate. The batter's strike zone extends from 0.56 m above the ground to 1.37 m above the ground. Neglecting aerodynamic effects, determine whether the ball will hit the strike zone (a) if $\theta = 1^\circ$; (b) if $\theta = 2^\circ$.



Solution: The initial velocity is $v_0 = 144.8$ km/h $= 40.3$ m/s. The velocity equations are

(1) $\dfrac{dv_x}{dt} = 0$, from which $v_x = v_0 \cos\theta$.

(2) $\dfrac{dv_y}{dt} = -g$, from which $v_y = -gt + v_0 \sin\theta$.

(3) $\dfrac{dx}{dt} = v_0 \cos\theta$, from which $x(t) = v_0 \cos\theta t$, since the initial position is zero.

(4) $\dfrac{dy}{dt} = -gt + v_0 \sin\theta$, from which

$$y(t) = -\frac{g}{2}t^2 + v_0 \sin\theta t + 1.83,$$

since the initial position is $y(0) = 1.83$ m. At a distance $d = 17.7$ m, the height is h. The time of passage across the home plate is $x(t_p) = d = v_0 \cos\theta t_p$, from which

$$t_p = \frac{d}{v_0 \cos\theta}.$$

Substitute:

$$y(t_p) = h = -\frac{g}{2}\left(\frac{d}{v_0 \cos\theta}\right)^2 + d\tan\theta + 1.83.$$

For $\theta = 1^\circ$, $\boxed{h = 1.2 \text{ m}}$, Yes, the pitcher hits the strike zone.

For $\theta = 2^\circ$, $\boxed{h = 1.5 \text{ m}}$ No, the pitcher misses the strike zone.

Problem 13.79 In Problem 13.78, assume that the pitcher releases the ball at an angle $\theta = 1^\circ$ above the horizontal, and determine the range of velocities v_0 (in m/s) within which he must release the ball to hit the strike zone.

Solution: From the solution to Problem 13.78,

$$h = -\frac{g}{2}\left(\frac{d}{v_0 \cos\theta}\right)^2 + d\tan\theta + 1.83,$$

where $d = 17.7$ m, and $1.37 \geq h \geq 0.56$ m. Solve for the initial velocity:

$$v_0 = \sqrt{\frac{gd^2}{2\cos^2\theta(d\tan\theta + 1.83 - h)}}.$$

For $h = 1.37$, $v_0 = 44.74$ m/s. For $h = 0.56$ m, $v_0 = 31.15$ m/s. The pitcher will hit the strike zone for velocities of release of

$$\boxed{31.15 \leq v_0 \leq 44.74 \text{ m/s}},$$

and a release angle of $\theta = 1^\circ$. *Check*: The range of velocities in miles per hour is 112.1 km/h $\leq v_0 \leq 161.1$ km/h, which is within the range of major league pitchers, although the 160.9 km upper value is achievable only by a talented few (Nolan Ryan, while with the Houston Astros, would occasionally in a game throw a 168.9 km fast ball, as measured by hand held radar from behind the plate).

Problem 13.80 A zoology student is provided with a bow and an arrow tipped with a syringe of sedative and is assigned to measure the temperature of a black rhinoceros (*Diceros bicornis*). The range of his bow when it is fully drawn and aimed 45° above the horizontal is 100 m. A truculent rhino charges straight toward him at 30 km/h. If he fully draws his bow and aims 20° above the horizontal, how far away should the rhino be when the student releases the arrow?

Solution: The strategy is (a) to determine the range and flight time of the arrow when aimed 20° above the horizontal, (b) to determine the distance traveled by the rhino during this flight time, and then (c) to add this distance to the range of the arrow. Neglect aerodynamic drag on the arrow. The equations for the trajectory are: Denote the constants of integration by V_x, V_y, C_x, C_y, and the velocity of the arrow by V_A.

(1) $\dfrac{dv_x}{dt} = 0$, from which $v_x = V_x$. At $t = 0$, $V_x = V_A \cos\theta$.

(2) $\dfrac{dv_y}{dt} = -g$, from which $v_y = -gt + V_y$. At $t = 0$, $V_y = V_A \sin\theta$.

(3) $\dfrac{dx}{dt} = v_x = V_A \cos\theta$, from which $x(t) = V_A \cos\theta t + C_x$. At $t = 0$, $x(0) = 0$, from which $C_x = 0$.

(4) $\dfrac{dy}{dt} = v_y = -gt + V_A \sin\theta$, from which

$$y = -\frac{g}{2}t^2 + V_A \sin\theta t + C_y.$$

At $t = 0$, $y = 0$, from which $C_y = 0$. The time of flight is given by

$$y(t_{\text{flight}}) = 0 = \left(-\frac{g}{2}t_{\text{flight}} + V_A \sin\theta\right) t_{\text{flight}},$$

from which

$$t_{\text{flight}} = \frac{2V_A \sin\theta}{g}.$$

The range is given by

$$x(t_{\text{flight}}) = R = V_A \cos\theta t_{\text{flight}} = \frac{2V_A^2 \cos\theta \sin\theta}{g}.$$

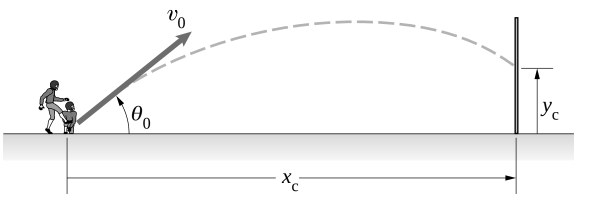
The maximum range (100 meters) occurs when the arrow is aimed 45° above the horizon. Solve for the arrow velocity: $V_A = \sqrt{gR_{\max}} = 31.3$ m/s. The time of flight when the angle is 20° is

$$t_{\text{flight}} = \frac{2V_A \sin\theta}{g} = 2.18 \text{ s},$$

and the range is $R = V_A \cos\theta t_{\text{flight}} = 64.3$ m. The speed of the rhino is 30 km/h = 8.33 m/s. The rhino travels a distance $d = 8.33(2.18) = 18.2$ m. The required range when the arrow is released is

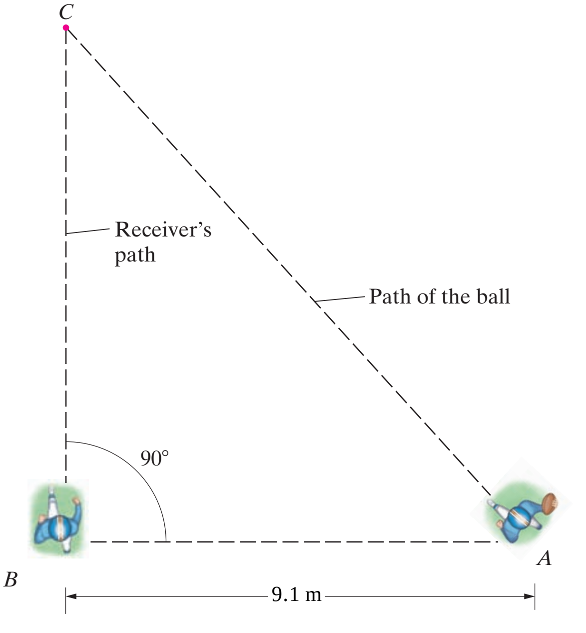
$$\boxed{d + R = 82.5 \text{ m}}.$$

Problem 13.81 The crossbar of the goalposts in American football is $y_c = 3.05$ m above the ground. To kick a field goal, the ball must make the ball go between the two uprights supporting the crossbar and be above the crossbar when it does so. Suppose that the kicker attempts a 36.58 m field goal, and kicks the ball with an initial velocity $v_0 = 21.3$ m/s and $\theta_0 = 40°$. By what vertical distance does the ball clear the crossbar?



Solution: Set the coordinate origin at the point where the ball is kicked. The x (horizontal) motion of the ball is given by $a_x = 0$, $V_x = V_0 \cos\theta_0$, $x = (V_0 \cos\theta_0)t$. The y motion is given by $a_y = -g$, $V_y = V_0 \sin\theta_0 - gt$, $y = (V_0 \sin\theta_0)t - \dfrac{gt^2}{2}$. Set $x = x_c = 36.58$ m and find the time t_c at which the ball crossed the plane of the goal posts. Substitute this time into the y equation to find the y coordinate Y_B of the ball as it passes over the crossbar. Substituting in the numbers ($g = 9.81$ m/s^2), we get $t_c = 2.24$ s and $y_B = 6.11$ m. Thus, the ball clears the crossbar by 3.07 m.

Problem 13.82 An American football quarterback stands at A. At the instant the quarterback throws the football, the receiver is at B running at 6.1 m/s toward C, where he catches the ball. The ball is thrown at an angle of 45° above the horizontal, and it is thrown and caught at the same height above the ground. Determine the magnitude of the ball's initial velocity and the length of time it is in the air.



Solution: Set x as the horizontal motion of the football, y as the vertical motion of the football and z as the horizontal motion of the receiver. Set $g = 9.81$ m/s^2, $\theta_0 = 45°$. We have

$$a_z = 0,\ \ v_z = 6.1 \text{ m/s},\ \ s_z = (6.1 \text{ m/s})t$$

$$a_y = -g,\ \ v_y = -gt + v_0 \sin\theta_0,\ \ s_y = -\frac{1}{2}gt^2 + v_0 \sin\theta_0 t$$

$$a_x = 0,\ \ v_x = v_0 \cos\theta_0,\ \ s_x = v_0 \cos\theta_0 t$$

When the ball is caught we have

$$s_z = (6.1 \text{ m/s})t$$

$$0 = -\frac{1}{2}gt^2 + v_0 \sin\theta_0 t$$

$$s_x = v_0 \cos\theta_0 t$$

$$s_x{}^2 = s_z{}^2 + (9.1 \text{ m})^2$$

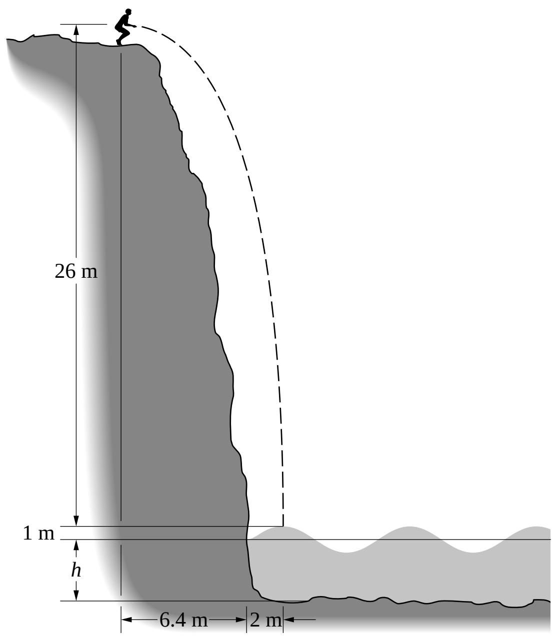
We can solve these four equations for the four unknowns s_x, s_z, v_0, t

We find $\boxed{t = 1.67 \text{ s},\quad v_0 = 11.6 \text{ m/s}}$

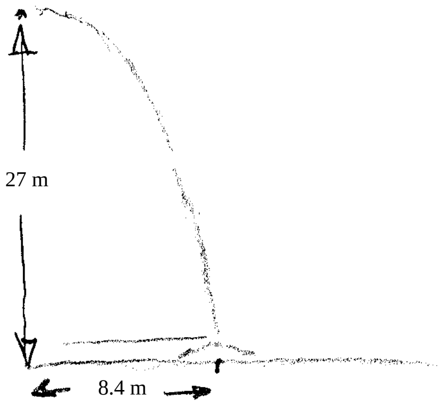
Problem 13.83 The cliff divers of Acapulco, Mexico must time their dives that they enter the water at the crest (high point) of a wave. The crests of the waves are 1 m above the mean water depth $h = 4$ m. The horizontal velocity of the waves is equal to $\sqrt{gh}$. The diver's aiming point is 2 m out from the base of the cliff. Assume that his velocity is horizontal when he begins the dive.

(a) What is the magnitude of the driver's velocity when he enters the water?

(b) How far from his aiming point must a wave crest be when he dives in order for him to enter the water at the crest?



Solution:



$t = 0,\ v_{y\Delta} = 0,\ y = 27$ m, $x_0 = 0$

$a_y = -g = -9.81$ m/s

$V_y = \cancel{V_{y_0}}^{0} - gt$

$y = y_0 - gt^{2/2}$

$y = 1$ m at t_{IMPACT}

for an ideal dive to hit the crest of the wave

$\underline{t_1 = t_{\text{IMPACT}} = 2.30 \text{ s}}$

$\underline{V_y(t_1) = 22.59 \text{ m/s}}$

$a_x = 0$

$V_x = V_{x_0}$

$X_I = V_{x_0} t_1 + X_0$

At impact $X_I = 8.4$ m.

For impact to occur as planned, then

$V_x = 8.4/t_1 = 3.65$ m/s = constant

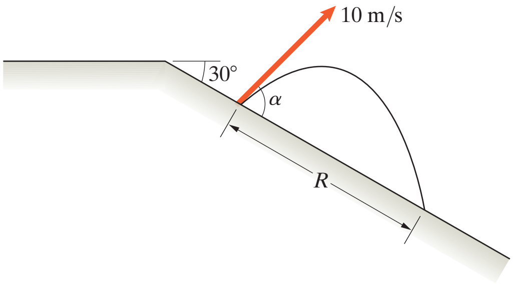
The velocity at impact is

$(a)\ |V| = \sqrt{(V_x)^2 + [V_y(t_1)]^2} = 22.9$ m/s

The wave moves at $\sqrt{gh} = 6.26$ m/s.

The wave crest travels 2.30 seconds while the diver is in their $s = \sqrt{gh}t_1 = 14.4$ m.

Problem 13.84 A projectile is launched at 10 m/s from a sloping surface. The angle $\alpha = 80^\circ$. Determine the range R.



Solution: Set $g = 9.81\ \text{m/s}^2$, $v_0 = 10$ m/s.

The equations of motion are

$$a_x = 0,\ \ v_x = v_0 \cos(80^\circ - 30^\circ),\ \ s_x = v_0 \cos 50^\circ t$$

$$a_y = -g,\ \ v_y = -gt + v_0 \sin(80^\circ - 30^\circ)t,\ \ s_y = -\frac{1}{2}gt^2 + v_0 \sin 50^\circ t$$

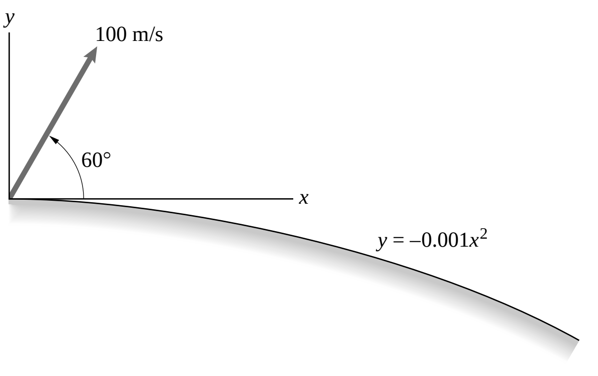
When the projectile hits we have

$$R\cos 30^\circ = v_0 \cos 50^\circ t$$

$$-R\sin 30^\circ = -\frac{1}{2}gt^2 + v_0 \sin 50^\circ t$$

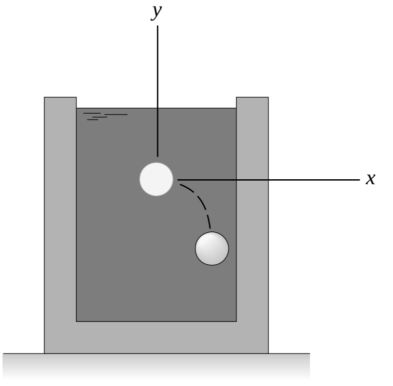
$$\Rightarrow \boxed{t = 2.32\ \text{s},\quad R = 17.21\ \text{m}}$$

Problem 13.85 A projectile is launched at 100 m/s at 60° above the horizontal. The surface on which it lands is described by the equation shown. Determine the point of impact.



Solution: The motion in the x direction is $a_x = 0$, $v_x = V_0 \cos\theta_0$, $x = (V_0 \cos\theta_0)t$, and the motion in the y direction is given by $a_y = -g$, $v_y = (V_0 \sin\theta_0) - gt$, $y = (V_0 \sin\theta_0)t - gt^2/2$. We know that $V_0 = 100$ m/s and $\theta_0 = 60^\circ$. The equation of the surface upon which the projectile impacts is $y = -0.001x^2$. Thus, the time of impact, t_I, can be determined by substituting the values of x and y from the motion equations into the equation for the surface. Hence, we get $(V_0 \sin\theta_0)t_I - g\dfrac{t_I^2}{2} = -0.001(V_0\cos\theta_0)^2 t_I^2$. Evaluating with the known values, we get $t_I = 6.37$ s Substituting this value into the motion equations reveals that impact occurs at $(x, y) = (318.4, -101.4)$ m.

Problem 13.86 At $t = 0$, a steel ball in a tank of oil is given a horizontal velocity $\mathbf{v} = 2\mathbf{i}$ (m/s). The components of the ball's acceleration in m/s^2 are $a_x = -1.2v_x$, $a_y = -8 - 1.2v_y$, $a_z = -1.2v_z$. What is the velocity of the ball at $t = 1$ s?



Solution: Assume that the effect of gravity is included in the given accelerations. The equations for the path are obtained from:

(1) $\dfrac{dv_x}{dt} = a_x = -1.2v_x$. Separate variables and integrate:

$$\frac{dv_x}{v_x} = -1.2\,dt,$$

from which $\ln(v_x) = -1.2t + V_x$. At $t = 0$, $v_x(0) = 2$, from which

$$\ln\left(\frac{v_x}{2}\right) = -1.2t.$$

Inverting: $v_x(t) = 2e^{-1.2t}$.

(2) $\dfrac{dv_y}{dt} = a_y = -8 - 1.2v_y$. Separate variables and integrate:

$$\frac{dv_y}{\dfrac{8}{1.2} + v_y} = -1.2\,dt,$$

from which

$$\ln\left(\frac{8}{1.2} + v_y\right) = -1.2t + V_y.$$

At $t = 0$, $v_y(0) = 0$, from

$$\ln\left(1 + \frac{1.2}{8}v_y\right) = -1.2t.$$

Inverting: $v_y(t) = \dfrac{8}{1.2}(e^{-1.2t} - 1)$.

(3) $\dfrac{dv_z}{dt} = a_z = -1.2v_z$, from which $\ln(v_z) = -1.2t + V_z$. Invert to obtain $v_z(t) = V_z e^{-1.2t}$. At $t = 0$, $v_z(0) = 0$, hence $V_z = 0$ and $v_z(t) = 0$. At $t = 1$ second,

$\boxed{v_x(1) = 2e^{-1.2} = 0.6024 \text{ m/s}}$, and

$\boxed{v_y(1) = -\left(\dfrac{8}{12}\right)(1 - e^{-1.2}) = -4.66 \text{ m/s}}$, or

$\boxed{\mathbf{v} = 0.602\mathbf{i} - 4.66\mathbf{j} \text{ (m/s)}}$.

Problem 13.87 In Problem 13.86, what is the position of the ball at $t = 1$ s relative to its position at $t = 0$?

Solution: Use the solution for the velocity components from Problem 13.86. The equations for the coordinates:

(1) $\dfrac{dx}{dt} = v_x = 2e^{-1.2t}$, from which

$$x(t) = -\left(\frac{2}{1.2}\right) e^{-1.2t} + C_x.$$

At $t = 0$, $x(0) = 0$, from which

$$x(t) = \left(\frac{2}{1.2}\right)(1 - e^{-1.2t}).$$

(2) $\dfrac{dy}{dt} = \left(\dfrac{8}{1.2}\right)(e^{-1.2t} - 1)$, from which

$$y(t) = -\left(\frac{8}{1.2}\right)\left(\frac{e^{-1.2t}}{1.2} + t\right) + C_y.$$

At $t = 0$, $y(0) = 0$, from which

$$y(t) = -\left(\frac{8}{1.2}\right)\left(\frac{e^{-1.2t}}{1.2} + t - \frac{1}{1.2}\right).$$

(3) Since $v_z(0) = 0$ and $z(0) = 0$, then $z(t) = 0$. At $t = 1$,

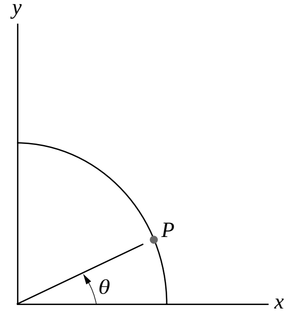
$$\boxed{x(1) = \left(\frac{2}{1.2}\right)(1 - e^{-1.2}) = 1.165 \text{ m}}.$$

$$\boxed{y(1) = -\left(\frac{8}{1.2}\right)\left(\frac{e^{-1.2}}{1.2} + 1 - \frac{1}{1.2}\right) = -2.784 \text{ m}}, \quad \text{or}$$

$$\boxed{\mathbf{r} = 1.165\mathbf{i} - 2.784\mathbf{j} \text{ (m)}}.$$

Problem 13.88 The point P moves along a circular path with radius R. Show that the magnitude of its velocity is $|v| = R|d\theta/dt|$.

Strategy: Use Eqs. (13.23).



Solution:

$$x = R\cos\theta$$

$$y = R\sin\theta$$

$$v_x = -R\sin\theta\left(\frac{d\theta}{dt}\right)$$

$$v_y = R\cos\theta\left(\frac{d\theta}{dt}\right)$$

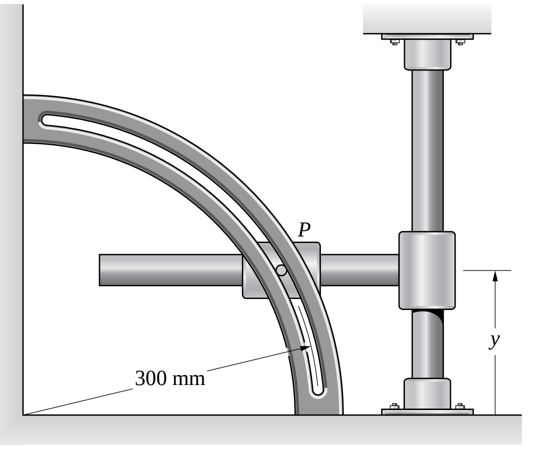
$$|\mathbf{V}| = \sqrt{V_x^2 + V_y^2}$$

$$|\mathbf{V}| = \sqrt{R^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + R^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2}$$

$$|\mathbf{V}| = \sqrt{R^2\left(\frac{d\theta}{dt}\right)^2(\sin^2\theta + \cos^2\theta)}$$

$$|\mathbf{V}| = R\left|\frac{d\theta}{dt}\right|$$

Problem 13.89 If $y = 150$ mm, $\dfrac{dy}{dt} = 300$ mm/s, and $\dfrac{d^2y}{dt^2} = 0$, what are the magnitudes of the velocity and acceleration of point P?



Solution: The equation for the location of the point P is $R^2 = x^2 + y^2$, from which $x = (R^2 - y^2)^{\frac{1}{2}} = 0.2598$ m, and

$$\frac{dx}{dt} = -\left(\frac{y}{x}\right)\left(\frac{dy}{dt}\right) = -0.1732 \text{ m/s},$$

$$\frac{d^2x}{dt^2} = -\frac{1}{x}\left(\frac{dy}{dt}\right)^2 + \frac{y}{x^2}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) - \left(\frac{y}{x}\right)\left(\frac{d^2y}{dt^2}\right)$$

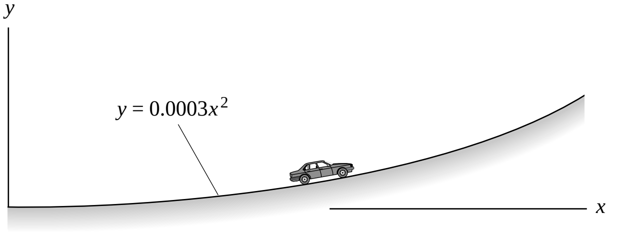
$$= -0.4619 \text{ m/s}^2.$$

The magnitudes are:

$$\boxed{|v_P| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 0.3464 \text{ m/s}}$$

$$\boxed{|a_p| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = 0.4619 \text{ m/s}^2}$$

Problem 13.90 A car travels at a constant speed of 100 km/h on a straight road of increasing grade whose vertical profile can be approximated by the equation shown. When the car's horizontal coordinate is $x = 400$ m, what is the car's acceleration?



Solution: Denote $C = 0.0003$ and $V = 100$ km/h $= 27.78$ m/s. The magnitude of the constant velocity is

$$V = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$$

The equation for the road is $y = Cx^2$ from which

$$\frac{dy}{dt} = 2Cx\left(\frac{dx}{dt}\right).$$

Substitute and solve:

$$\left|\frac{dx}{dt}\right| = \frac{V}{\sqrt{(2Cx)^2 + 1}} = 27.01 \text{ m/s}.$$

$\dfrac{dx}{dt}$ is positive (car is moving to right in sketch). The acceleration is

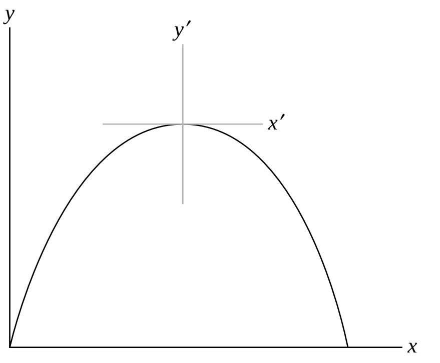
$$\boxed{\begin{aligned}\frac{d^2x}{dt^2} &= \frac{d}{dt}\left(\frac{V}{\sqrt{(2Cx)^2+1}}\right) = \frac{-4C^2Vx}{((2Cx)^2+1)^{\frac{3}{2}}}\left(\frac{dx}{dt}\right) \\ &= -0.0993 \text{ m/s}^2\end{aligned}}.$$

$$\boxed{\frac{d^2y}{dt^2} = 2C\left(\frac{dx}{dt}\right)^2 + 2Cx\left(\frac{d^2x}{dt^2}\right) = 0.4139 \text{ m/s}^2}, \quad \text{or}$$

$$\boxed{\mathbf{a} = -0.099\mathbf{i} + 0.414\mathbf{j} (\text{m/s}^2)}$$

Problem 13.91 Suppose that a projectile has the initial conditions shown in Fig. 13.12. Show that in terms of the $x'y'$ coordinate system with its origin at the highest point of the trajectory, the equation describing the trajectory is

$$y' = -\frac{g}{2v_0^2\cos^2\theta_0}(x')^2.$$



Solution: The initial conditions are $t = 0$, $x(0) = 0$, $y(0) = 0$, $v_x(0) = v_0\cos\theta_0$, and $v_y(0) = v_0\sin\theta_0$. The accelerations are $a_x(t) = 0$, $a_y(t) = -g$. The path of the projectile in the x, y system is obtained by solving the differential equations subject to the initial conditions:

$$x(t) = (v_0\cos\theta_0)t,\ y(t) = -\frac{g}{2}t^2 + (v_0\sin\theta_0)t$$

Eliminate t from the equations by substituting

$$t = \frac{x}{v_0\cos\theta_0}$$

to obtain

$$y(x) = -\frac{gx^2}{2v_0^2\cos^2\theta_0} + x\tan\theta_0.$$

At the peak,

$$\left|\frac{dy}{dx}\right|_{\text{peak}} = 0,$$

from which

$$x_p = \frac{v_0^2\cos\theta_0\sin\theta_0}{g},$$

and $\quad y_p = \dfrac{v_0^2\sin^2\theta_0}{2g}.$

The primed coordinates: $y' = y - y_p \quad x' = x - x_p$. Substitute and reduce:

$$y' = -\frac{g(x' + x_p)^2}{2v_0^2\cos^2\theta_0} + (x' + x_p)\tan\theta_0 - y_p.$$

$$y' = -\frac{g}{2v_0^2\cos^2\theta_0}((x')^2 + x_p^2 + 2x'x_p) + (x' + x_p)\tan\theta_0 - \frac{v_0^2\sin^2\theta_0}{2g}.$$

Substitute $x_p = \dfrac{v_0^2\cos\theta_0\sin\theta_0}{g}$,

$$y' = \frac{-g(x')^2}{2v_0^2\cos^2\theta_0} - \frac{v_0^2\sin^2\theta_0}{2g} - x'\tan\theta_0 + x'\tan\theta_0 + \frac{v_0^2\sin^2\theta_0}{g} - \frac{v_0^2\sin^2\theta_0}{2g}.$$

$$\boxed{y' = -\frac{g}{2v_0^2\cos^2\theta_0}(x')^2}$$

Problem 13.92 The acceleration components of a point are $a_x = -4\cos 2t$, $a_y = -4\sin 2t$, $a_z = 0$. At $t = 0$, its position and velocity are $\mathbf{r} = \mathbf{i}$, $\mathbf{v} = 2\mathbf{j}$. Show that (a) the magnitude of the velocity is constant; (b) the velocity and acceleration vectors are perpendicular; (c) the magnitude of the acceleration is constant and points toward the origin; (d) the trajectory of a point is a circle with its center at the origin.

Solution: The equations for the path are

(1) $\dfrac{dv_x}{dt} = a_x = -4\cos(2t)$, from which $v_x(t) = -2\sin(2t) + V_x$. At $t = 0$, $v_x(0) = 0$, from which $V_x = 0$. $\dfrac{dx}{dt} = v_x = -2\sin(2t)$, from which $x(t) = \cos(2t) + C_x$. At $t = 0$, $x(0) = 1$, from which $C_x = 0$.

(2) $\dfrac{dv_y}{dt} = a_y = -4\sin(2t)$, from which $v_y(t) = 2\cos(2t) + V_y$. At $t = 0$, $v_y(0) = 2$, from which $V_y = 0$. $\dfrac{dy}{dt} = v_y = 2\cos(2t)$, from which $y(t) = \sin(2t) + C_y$. At $t = 0$, $y(0) = 0$, from which $C_y = 0$.

(3) For $a_z = 0$ and zero initial conditions, it follows that $v_z(t) = 0$ and $z(t) = 0$.

(a) The magnitude of the velocity is

$$|\mathbf{v}| = \sqrt{(-2\sin(2t))^2 + (2\cos(2t))^2} = 2 = \text{const.}$$

(b) The velocity is $\mathbf{v}(t) = -\mathbf{i}2\sin(2\mathbf{t}) + \mathbf{j}2\cos(2t)$. The acceleration is $\mathbf{a}(t) = -\mathbf{i}4\cos(2t) - \mathbf{j}4\sin(2t)$. If the two are perpendicular, the dot product should vanish: $\mathbf{a}(t)\cdot\mathbf{v}(t) = (-2\sin(2t))(-4\cos(2t)) + (2\cos(2t))(-4\sin(2t)) = 0$, and it does

(c) The magnitude of the acceleration:

$$\boxed{|\mathbf{a}| = \sqrt{(-4\cos(2t))^2 + (-4\sin(2t))^2} = 4 = \text{const}}.$$

The unit vector parallel to the acceleration is

$$\mathbf{e} = \frac{\mathbf{a}}{|\mathbf{a}|} = -\mathbf{i}\cos(2t) - \mathbf{j}\sin(2\mathbf{t}),$$

which always points to the origin.

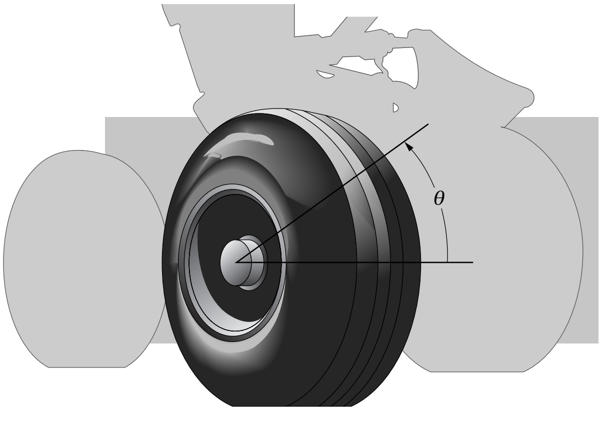
(d) The trajectory path is $x(t) = \cos(2t)$ and $y(t) = \sin(2t)$. These satisfy the condition for a circle of radius 1:

$$\boxed{1 = x^2 + y^2}$$

Problem 13.93 When an airplane touches down at $t = 0$, a stationary wheel is subjected to a constant angular acceleration $\alpha = 110\ \text{rad/s}^2$ until $t = 1$ s.

(a) What is the wheel's angular velocity at $t = 1$ s?

(b) At $t = 0$, the angle $\theta = 0$. Determine θ in radians and in revolutions at $t = 1$ s.



Solution:

$\alpha = 110\ \text{rad/s}^2$

$\omega = \alpha t + \omega_0$

$\theta = (\frac{1}{2}\alpha t^2) + \omega_0 t + \theta_0$

From the problem statement, $\omega_0 = \theta_0 = 0$

(a) At $t = 1$ s,

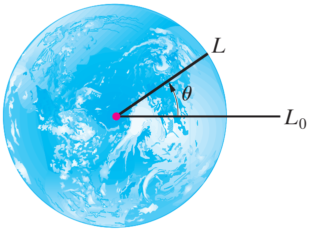
$\underline{\omega = (110)(1) + 0 = 110\ \text{rad/s}}$

(b) At $t = 1$ s,

$\underline{\theta = 110(1)^2/2 = 55\ \text{radians}}$ (8.75 revolutions)

Problem 13.94 Let L be a line from the center of the earth to a fixed point on the equator, and let L_0 be a fixed reference direction. The figure views the earth from above the north pole.

(a) Is $d\theta/dt$ positive or negative? (Remember that the sun rises in the east.)
(b) Determine the approximate value of $d\theta/dt$ in rad/s and use it to calculate the angle through which the earth rotates in one hour.



Solution:

(a) $\boxed{\dfrac{d\theta}{dt} > 0.}$

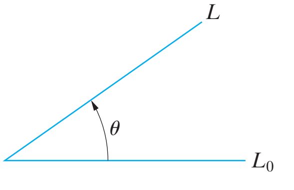
(b) $\dfrac{d\theta}{dt} \approx \dfrac{2\pi \text{ rad}}{24(3600)\ s} = 7.27 \times 10^{-5}$ rad/s.

In one hour $\Delta\theta \approx (7.27 \times 10^{-5} \text{ rad/s})(1 \text{ hr})\left(\dfrac{3600 \text{ s}}{1 \text{ hr}}\right)$

$= 0.262$ rad $= 15°$

$$\boxed{\frac{d\theta}{dt} \approx 7.27 \times 10^{-5} \text{ rad/s}, \quad \Delta\theta \approx 15°.}$$

Problem 13.95 The angular acceleration of the line L relative to the line L_0 is given as a function of time by $\alpha = 2.5 - 1.2t$ rad/s^2. At $t = 0$, $\theta = 0$ and the angular velocity of L relative to L_0 is $\omega = 5$ rad/s. Determine θ and ω at $t = 3$ s.



Solution:

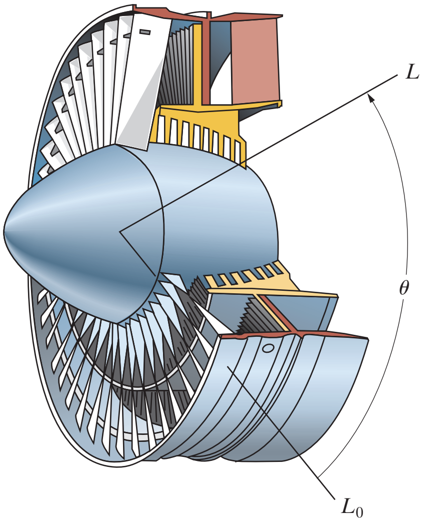
$\alpha = 2.5 - 1.2t$

$\omega = 2.5t - 0.6t^2 + 5$

$\theta = 1.25t^2 - 0.2t^3 + 5t$

$$\Rightarrow \boxed{\begin{aligned} \theta(3) &= 1.25(3)^2 - 0.2(3)^3 + 5(3) = 20.85 \text{ rad} \\ \omega(3) &= 2.5(3) - 0.6(3)^2 + 5 = 7.1 \text{ rad/s} \end{aligned}}$$

Problem 13.96 In Active Example 13.8, suppose that the angular acceleration of the rotor is $\alpha = -0.00002\omega^2$, where ω is the angular velocity of the rotor in rad/s. How long does it take the rotor to slow from 10,000 rpm to 1000 rpm?



Solution: Let $\alpha = k\omega^2$, where $k = -0.0002$. Then

$$a = \frac{d\omega}{dt} = k\omega^2,$$

$$\int_{\omega_1}^{\omega_2} \frac{d\omega}{\omega^2} = \int_0^t k\,dt,$$

$$-\frac{1}{\omega_2} + \frac{1}{\omega_1} = kt \Rightarrow t = \frac{1}{k}\left(\frac{1}{\omega_1} - \frac{1}{\omega_2}\right)$$

Convert the numbers to rad/s

$$\omega_1 = 10000 \text{ rpm}\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = 1047 \text{ rad/s}$$

$$\omega_2 = 1000 \text{ rpm}\left(\frac{2\pi \text{rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = 104.7 \text{ rad/s}$$

Using the numbers in the problem

$$t = \frac{1}{-0.0002}\left(\frac{1}{1047 \text{ rad/s}} - \frac{1}{104.7 \text{ rad/s}}\right)$$

$$\boxed{t = 430 \text{ s.}}$$

Problem 13.97 The astronaut is not rotating. He has an orientation control system that can subject him to a constant angular acceleration of 0.1 rad/s^2 about the vertical axis is either direction. If he wants to rotate 180° about the vertical axis (that is, rotate so that he is facing toward the left) and not be rotating in his new orientation, what is the minimum time in which he could achieve the new orientation?

Solution: He could achieve the rotation in minimum time by accelerating until he has turned 90° and then decelerating for the same time while he rotates the final 90°.

Thus the time needed to turn 90° ($\pi/2$ rad) is

$\alpha = 0.1$ rad/s^2,

$\omega = \alpha t$

$$\theta = \frac{1}{2}\alpha t^2 \Rightarrow t = \sqrt{\frac{2\theta}{\alpha}} = \sqrt{\frac{2(\pi/2)}{0.1 \text{ rad/s}^2}} = 5.605 \text{ s.}$$

The total maneuver time is $2t$. $\boxed{2t = 11.2 \text{ s.}}$

Problem 13.98 The astronaut is not rotating. He has an orientation control system that can subject him to a constant angular acceleration of 0.1 rad/s^2 about the vertical axis in either direction. Refer to problem 13.97. For safety, the control system will not allow his angular velocity to exceed 15° per second. If he wants to rotate 180° about the vertical axis (that is, rotate so that he is facing toward the left) and not be rotating in his new orientation, what is the minimum time in which he could achieve the new orientation?

Solution: He could achieve the rotation in minimum time by accelerating until he has reached the angular velocity limit, then coasting until he has turned 90°. He would then continue to coast until he needed to decelerate to a stop at the 180° position. The maneuver is symmetric in the spin up and the spin down phases.

We will first find the time needed to reach the angular velocity limit and also find the angle through which he has rotated in this time.

$\alpha = 0.1 \text{ rad/s}^2$

$$\omega = \alpha t_1 \Rightarrow t_1 = \frac{\omega}{\alpha} = \frac{(15/180)\pi \text{ rad/s}}{0.1 \text{ rad/s}^2} = 2.62 \text{ s.}$$

$$\theta_1 = \frac{1}{2}\alpha t^2 = \frac{1}{2}(0.1 \text{ rad/s}^2)(2.62 \text{ s})^2 = 0.343 \text{ rad.}$$

Now we need to find the coast time (constant angular velocity)

$$\frac{\pi}{2} \text{ rad} - 0.343 \text{ rad} = \left(\frac{15°}{180°}\right)\pi \text{ rad/s } t_2 \Rightarrow t_2 = 4.69 \text{ s}$$

Thus the total time to complete a 90° turn is $t = t_1 + t_2 = 2.62 \text{ s} + 4.69 \text{ s} = 7.31 \text{ s}$.

The time for the full 180° turn is $2t$ $\boxed{2t = 14.6 \text{ s.}}$

Problem 13.99 The rotor of an electric generator is rotating at 200 rpm when the motor is turned off. Due to frictional effects, the angular acceleration of the rotor after the motor is turned off is $\alpha = -0.01\omega$ rad/s^2, where ω is the angular velocity in rad/s.

(a) What is the rotor's angular velocity one minute after the motor is turned off?
(b) After the motor is turned off, how many revolutions does the rotor turn before it comes to rest?

Strategy: To do part (b), use the chain rule to write the angular acceleration as

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega.$$

Solution: Let $\alpha = k\omega$, where $k = -0.01 \text{ s}^{-1}$. Note that 200 rpm = 20.9 rad/s.

(a) One minute after the motor is turned off

$$\alpha = \frac{d\omega}{dt} = k\omega \Rightarrow \int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = \int_0^t k\,dt \Rightarrow \ln\left(\frac{\omega}{\omega_0}\right) = kt$$

$$\omega = \omega_0 e^{kt} = (20.9 \text{ rad/s})e^{(-0.01/\text{s})(60 \text{ s})} = 11.5 \text{ rad/s.}$$

$\boxed{\omega = 11.5 \text{ rad/s (110 rpm).}}$

(b) When the rotor comes to rest

$$\alpha = \omega\frac{d\omega}{d\theta} = k\omega \Rightarrow \int_{\omega_0}^{0} d\omega = \int_0^{\theta} k\,d\theta \Rightarrow -\omega_0 = k\theta$$

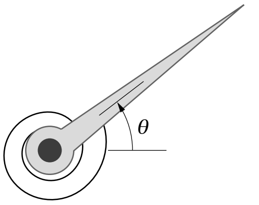
$$\theta = \frac{1}{k}(-\omega_0) = \frac{1}{-0.01 \text{ s}^{-1}}(-20.9 \text{ rad/s})$$

$$\theta = 2094 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 333 \text{ rev.}$$

$\boxed{\theta = 333 \text{ rev.}}$

Problem 13.100 The needle of a measuring instrument is connected to a *torsional spring* that gives it an angular acceleration $\alpha = -4\theta$ rad/s^2, where θ is the needle's angular position in radians relative to a reference direction. The needle is given an angular velocity $\omega = 2$ rad/s in the position $\theta = 0$.

(a) What is the magnitude of the needle's angular velocity when $\theta = 30°$?
(b) What maximum angle θ does the needle reach before it rebounds?



Solution:

$$\alpha = \omega\frac{d\omega}{d\theta} = -4\theta \Rightarrow \int_2^\omega \omega d\omega = -4\int_0^\theta \theta d\theta \Rightarrow \frac{\omega^2}{2} - \frac{2^2}{2} = -2\theta^2$$

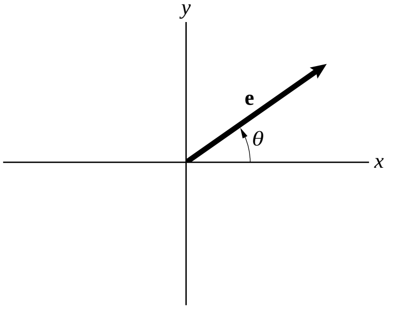
$$\omega = 2\sqrt{1-\theta^2}$$

(a) $\boxed{\omega = 2\sqrt{1-(\pi/6)^2} = 1.704 \text{ rad/s}}$

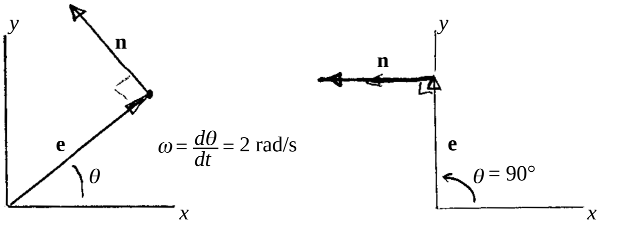
(b) Maximum angle means $\omega = 0$. $\boxed{\theta = 1 \text{ rad} = 57.3°}$

Problem 13.101 The angle θ measures the direction of the unit vector **e** relative to the x axis. The angular velocity of **e** is $\omega = d\theta/dt = 2$ rad/s, constant. Determine the derivative $d\mathbf{e}/dt$ when $\theta = 90°$ in two ways:

(a) Use Eq. (13.33).
(b) Express the vector **e** in terms of its x and y components and take the time derivative of **e**.



Solution:



(a)
$$\frac{d\mathbf{e}}{dt} = \frac{d\theta}{dt}\mathbf{n} = \omega\mathbf{n}$$

when $\theta = 90°, n = -\mathbf{i}$

$$\frac{d\mathbf{e}}{dt} = -2\mathbf{i} \text{ rad/s when } \theta = 90°$$

(b)
$$\mathbf{e} = (1)\cos\theta\mathbf{i} + (1)\sin\theta\mathbf{j}$$

$$\frac{d\mathbf{e}}{dt} = -\sin\theta\left(\frac{d\theta}{dt}\right)\mathbf{i} + \cos\theta\left(\frac{d\theta}{dt}\right)\mathbf{j}$$

Evaluating at $\theta = 90°$

$$\frac{d\mathbf{e}}{dt} = -\frac{d\theta}{dt}\mathbf{i} = -2\mathbf{i} \text{ rad/s}$$

Problem 13.102 The angle θ measures the direction of the unit vector $\mathbf{e}$ relative to the x axis. The angle θ is given as a function of time by $\theta = 2t^2$ rad. What is the vector $d\mathbf{e}/dt$ at $t = 4$ s?

Solution: By definition:

$$\frac{d\mathbf{e}}{dt} = \left(\frac{d\theta}{dt}\right)\mathbf{n},$$

where

$$\mathbf{n} = \mathbf{i}\cos\left(\theta + \frac{\pi}{2}\right) + \mathbf{j}\sin\left(\theta + \frac{\pi}{2}\right)$$

is a unit vector in the direction of positive θ. The angular rate of change is

$$\left[\frac{d\theta}{dt}\right]_{t=4} = [4t]_{t=4} = 16 \text{ rad/s}.$$

The angle is $\theta = [\mathrm{mod}(2t^2, 2\pi)]_{t=4} = \mathrm{mod}(32, 2\pi) = 0.5841$ rad, where $\mathrm{mod}(x, y)$ ("modulus") is a standard function that returns the remainder of division of the first argument by the second. From which,

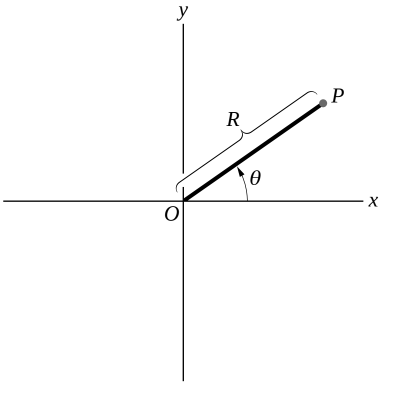
$$\boxed{\begin{aligned}\left[\frac{d\mathbf{e}}{dt}\right]_{t=4} &= 16\left(\mathbf{i}\cos\left(0.5841 + \frac{\pi}{2}\right) + \mathbf{j}\sin\left(0.5841 + \frac{\pi}{2}\right)\right)\\ &= -8.823\mathbf{i} + 13.35\mathbf{j}\end{aligned}}$$

Problem 13.103 The line OP is of constant length R. The angle $\theta = \omega_0 t$, where ω_0 is a constant.

(a) Use the relations $v_x = \dfrac{dx}{dt}$ and $v_y = \dfrac{dy}{dt}$ to determine the velocity of P relative to O.

(b) Use Eq. (13.33) to determine the velocity of P relative to O, and confirm that your result agrees with the result of (a).

Strategy: In part (b), write the position vector of P relative to O as $\mathbf{r} = R\mathbf{e}$ where $\mathbf{e}$ is a unit vector that points from O toward P.



Solution:

(a) The point P is described by $\mathbf{P} = \mathbf{i}x + \mathbf{j}y$. Take the derivative:

$$\frac{d\mathbf{P}}{dt} = \mathbf{i}\left(\frac{dx}{dt}\right) + \mathbf{j}\left(\frac{dy}{dt}\right).$$

The coordinates are related to the angle θ by $x = R\cos\theta$, $y = R\sin\theta$. Take the derivative and note that R is a constant and $\theta = \omega_0 t$, so that

$$\frac{d\theta}{dt} = \omega_0 : \frac{dx}{dt} = -R\sin\theta\left(\frac{d\theta}{dt}\right),$$

$$\frac{dy}{dt} = R\cos\theta\left(\frac{d\theta}{dt}\right).$$

Substitute into the derivative of the vector $\mathbf{P}$,

$$\boxed{\begin{aligned}\frac{d\mathbf{P}}{dt} &= R\left(\frac{d\theta}{dt}\right)(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta)\\ &= R\omega_0(-\mathbf{i}\sin(\omega_0 t) + \mathbf{j}\cos(\omega_0 t))\end{aligned}}$$

which is the velocity of the point P relative to the origin O.

(b) Note that $\mathbf{P} = R\mathbf{e}$, and $\dfrac{d\mathbf{P}}{dt} = R\dfrac{d\mathbf{e}}{dt}$ when R is constant. Use the definition (Eq. (13.33)),

$$\frac{d\mathbf{e}}{dt} = \left(\frac{d\theta}{dt}\right)\mathbf{n},$$

where $\mathbf{n}$ is a unit vector in the direction of positive θ, (i.e., perpendicular to $\mathbf{e}$). Thus

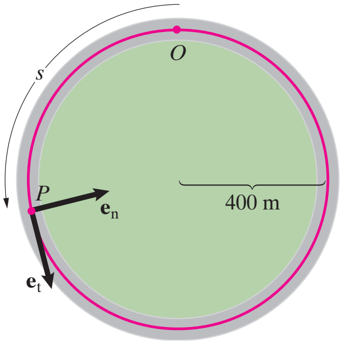
$$\mathbf{n} = \mathbf{i}\cos\left(\theta + \frac{\pi}{2}\right) + \mathbf{j}\sin\left(\theta + \frac{\pi}{2}\right).$$

Use the trigonometric sum-of-angles identities to obtain: $\mathbf{n} = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta$. Substitute,

$$\boxed{\frac{d\mathbf{P}}{dt} = R\omega_0(-\mathbf{i}\sin(\omega_0 t) + \mathbf{j}\cos(\omega_0 t))}$$

The results are the same.

Problem 13.104 In Active Example 13.9, determine the motorcycle's velocity and acceleration in terms of normal and tangential components at $t = 5$s.



Solution: We are given the tangential acceleration

$$a_t = (2 \text{ m/s}^2) + (0.2 \text{ m/s}^3)t$$

We can find the velocity at $t = 5$ s.

$$v = \int_0^{5 \text{ s}} ([2 \text{ m/s}^2] + [0.2 \text{ m/s}^3]t)dt$$

$$= [2 \text{ m/s}^2][5 \text{ s}] + \frac{1}{2}[0.2 \text{ m/s}^3][5 \text{ s}]^2$$

$$= 1.2 \text{ m/s}.$$

The velocity is then $\boxed{\mathbf{v} = (12.5 \text{ m/s})\ \mathbf{e}_t}$

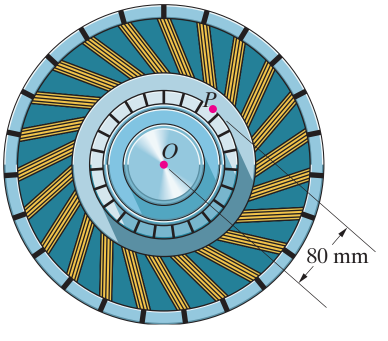
The accelerations are

$$a_t = (2 \text{ m/s}^2) + (0.2 \text{ m/s}^3)(5 \text{ s}) = 3 \text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} = \frac{(12.5 \text{ m/s})^2}{400 \text{ m}} = 0.391 \text{ m/s}^2.$$

$$\boxed{\mathbf{a} = (3 \text{ m/s}^2)\mathbf{e}_t + (0.391 \text{ m/s}^2)\mathbf{e}_n}$$

Problem 13.105 The armature starts from rest at $t = 0$ and has constant angular acceleration $\alpha = 2$ rad/s^2. At $t = 4$ s, what are the velocity and acceleration of point P relative to point O in terms of normal and tangential components?



Solution: We can find the angular velocity at $t = 4$s.

$$\alpha = 2 \text{ rad/s}^2$$

$$\omega = (2 \text{ rad/s}^2)(4 \text{ s}) = 8 \text{ rad/s}.$$

Then

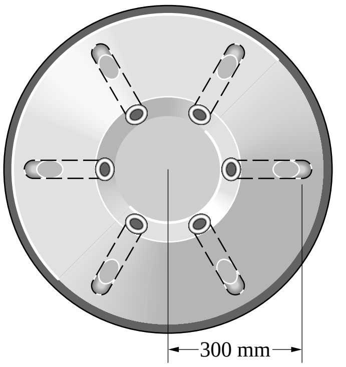
$$v = r\omega = (0.08 \text{ m})\ (8 \text{ rad/s}) = 0.64 \text{ m/s},$$

$$a_t = r\alpha = (0.08 \text{ m})\ (2 \text{ rad/s}^2) = 0.16 \text{ m/s}^2,$$

$$a_n = \frac{v^2}{r} = \frac{(0.64 \text{ m/s})^2}{0.08 \text{ m}} = 5.12 \text{ m/s}^2$$

$$\boxed{\begin{array}{l} \mathbf{v} = (0.64 \text{ m/s})\mathbf{e}_t, \\ \mathbf{a} = (0.16 \text{ m/s}^2)\mathbf{e}_t + (5.12 \text{ m/s}^2)\mathbf{e}_n. \end{array}}$$

Problem 13.106 Suppose you want to design a medical centrifuge to subject samples to normal accelerations of 1000 g's. (a) If the distance from the center of the centrifuge to the sample is 300 mm, what speed of rotation in rpm is necessary? (b) If you want the centrifuge to reach its design rpm in 1 min, what constant angular acceleration is necessary?



Solution:

(a) The normal acceleration at a constant rotation rate is $a_n = R\omega^2$, giving

$$\omega = \sqrt{\frac{a_n}{R}} = \sqrt{\frac{(1000)9.81}{0.3}} = 180.83 \text{ rad/s.}$$

The speed in rpm is

$$\boxed{N = \omega\left(\frac{\text{rad}}{\text{s}}\right)\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 1730 \text{ rpm}}.$$

(b) The angular acceleration is

$$\boxed{\alpha = \frac{\omega}{t} = \frac{180.83}{60} = 3.01 \text{ rad/s}^2}$$

Problem 13.107 The medical centrifuge shown in Problem 13.106 starts from rest at $t = 0$ and is subjected to a constant angular acceleration $\alpha = 3 \text{ rad/s}^2$. What is the magnitude of the total acceleration to which the samples are subjected at $t = 1$ s?

Solution: $\alpha = 3, \quad \omega = 3t, \quad \theta = 1.5t^2$

$$a_t = (0.3 \text{ m})(3 \text{ rad/s}^2) = 0.9 \text{ m/s}^2$$

$$a_n = (0.3 \text{ m})(3 \text{ rad/s})^2 = 2.7 \text{ m/s}^2$$

$$\boxed{a = \sqrt{(0.9)^2 + (2.7)^2} \text{ m/s}^2 = 2.85 \text{ m/s}^2}$$

Problem 13.108 A centrifuge used to subject engineering components to high acceleration has a radius of 8 m. It starts from rest at $t = 0$, and during its two-minute acceleration phase it is programmed so that its angular acceleration is given as a function of time in seconds by $\alpha = 0.192 - 0.0016t$ rad/s^2. At $t = 120$ s, what is the magnitude of the acceleration a component is subjected to?

Solution: We will first calculate the angular velocity

$$\omega = \int_0^{120\text{ s}} ([0.192 \text{ rad/s}^2] - [0.0016 \text{ rad/s}^3]t)\, dt$$

$$= [0.192 \text{ rad/s}][120 \text{ s}] - \frac{1}{2}[0.0016 \text{ rad/s}^3][120 \text{ s}]^2$$

$$= 11.52 \text{ rad/s}$$

The normal and tangential components of acceleration are

$$a_t = r\alpha = (8 \text{ m})([0.192 \text{ rad/s}^2] - [0.0016 \text{ rad/s}^3][120 \text{ s}]) = 0$$

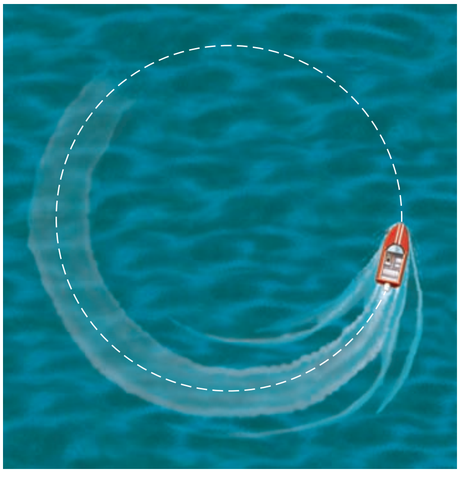
$$a_n = r\omega^2 = (8 \text{ m})(11.52 \text{ rad/s})^2 = 1060 \text{ m/s}^2$$

Since the tangential component is zero, then the total acceleration is the same as the normal acceleration

$$\boxed{a = 1060 \text{ m/s}^2 (108g\text{'s}).}$$

Problem 13.109 A powerboat being tested for maneuverability is started from rest at $t = 0$ and driven in a circular path 12 m in radius. The tangential component of the boat's acceleration as a function of time is $a_t = 0.4t$ m/s^2.

(a) What are the boat's velocity and acceleration in terms of normal and tangential components at $t = 4$ s?
(b) What distance does the boat move along its circular path from $t = 0$ to $t = 4$ s?



Solution:

(a) $a_t = 0.4t$ m/s^2 $\quad a_n = +v^2/r$

$v = 0.2t^2$ m/s

At $t = 4$ s,

$\mathbf{a} = 0.4t\mathbf{e}_t + v^2/r\mathbf{e}_n$

$\mathbf{a} = 1.6\mathbf{e}_t + 0.853\mathbf{e}_n$

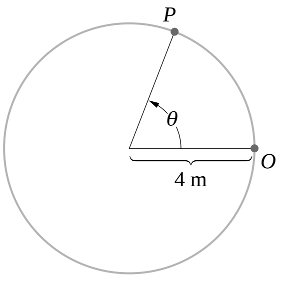
$\mathbf{v} = 3.2\mathbf{e}_t$ m/s

(b) $s = 0.2t^3/3$

$\underline{s|_{4s} = 4.27 \text{ m}}$

Problem 13.110 The angle $\theta = 2t^2$ rad.

(a) What are the velocity and acceleration of point P in terms of normal and tangential components at $t = 1$ s?
(b) What distance along the circular path does point P move from $t = 0$ to $t = 1$ s?



Solution:

$$\theta = 2t^2$$

$$\frac{d\theta}{dt} = 4t = \omega$$

$$\frac{d^2\theta}{dt^2} = 4\frac{\text{rad}}{\text{s}^2} = \alpha$$

$$s = r\theta = 4\theta = 8t^2$$

$$v_t = 16t \text{ m/s}$$

$$v = r\omega = 4(4t) = 16t$$

$$a_t = \frac{dv}{dt} = 16 \text{ m/s}^2$$

(a) $\underline{\mathbf{v} = 16(1)\mathbf{e}_t \text{ m/s} = 16\ e_t \text{ (m/s)}}$

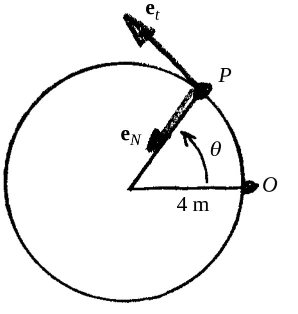
$$\mathbf{a} = R\alpha\mathbf{e}_t + R\omega^2\mathbf{e}_N$$

$$\mathbf{a} = (4)(4)\mathbf{e}_t + (4)(4^2)\mathbf{e}_N \text{ (m/s}^2)$$

$$\underline{\mathbf{a} = 16\mathbf{e}_t + 64\mathbf{e}_N \text{ (m/s}^2)}$$

(b) $s = R\theta = 8t^2$

$$= 8(1)^2 = 8 \text{ m}$$



Problem 13.111 The angle $\theta = 2t^2$ rad. What are the velocity and acceleration of point P in terms of normal and tangential components when P has gone one revolution around the circular path starting at $t = 0$?

Solution: From the solution to Problem 13.110,

$$\theta = 2t^2 \text{ rad}$$

$$\omega = 4t \text{ rad/s}$$

$$\alpha = 4 \text{ rad/s}^2$$

$$s = 8t^2 \text{ m}$$

$$v_t = 16t \text{ m/s}$$

$$a_t = 16 \text{ m/s}^2$$

We want to know $\mathbf{v}$ and $\mathbf{a}$ when $\theta = 2\pi$. Substituting into the first eqn, we find that $\theta = 2\pi$ when $t = t_1 = 1.77$ seconds. From the solution to Problem 13.110,

$\mathbf{v}_t = 16t\mathbf{e}_t$ and

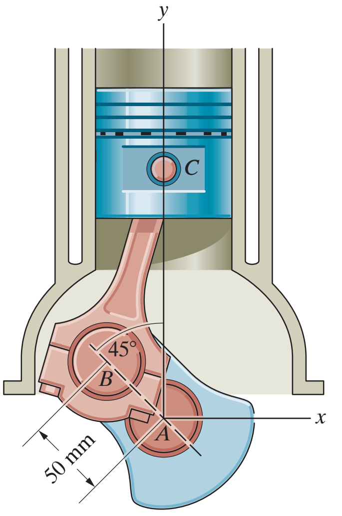
$$\mathbf{a} = R\alpha\mathbf{e}_t + R\omega^2\mathbf{e}_N$$

Substituting in the time t_1, we get

$$\underline{\mathbf{v}_t = 28.4\mathbf{e}_t \text{ (m/s)}}$$

$$\underline{\mathbf{a} = 16\mathbf{e}_t + 201.1\mathbf{e}_N \text{ (m/s}^2)}$$

Problem 13.112 At the instant shown, the crank AB is rotating with a constant counterclockwise angular velocity of 5000 rpm. Determine the velocity of point B (a) in terms of normal and tangential components; (b) in terms of cartesian components.



Solution:

$$\omega = (5000 \text{ rpm})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{\text{min}}{60 \text{ sec}}\right) = 524 \text{ rad/s}$$

(a) $\boxed{\mathbf{V}_B = 0.05 \text{ m } (524 \text{ rad/s})\mathbf{e}_t = (26.2\,\mathbf{e}_t) \text{ m/s}}$

(b) $\boxed{\mathbf{V}_B = (26.2 \text{ m/s})(-\cos 45^\circ\,\mathbf{i} - \sin 45^\circ\,\mathbf{j}) = (-18.53\mathbf{i} - 18.53\mathbf{j}) \text{ m/s}}$

Problem 13.113 The crank AB in Problem 13.112 is rotating with a constant counterclockwise angular velocity of 5000 rpm. Determine the acceleration of point B (a) in terms of normal and tangential components; (b) in terms of cartesian components.

Solution:

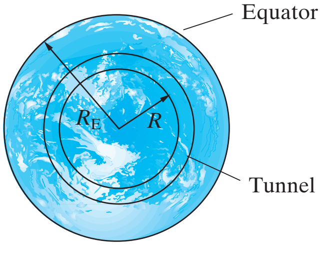
$$\omega = (5000 \text{ rpm})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{\text{min}}{60 \text{ sec}}\right) = 524 \text{ rad/s}$$

$$a_t = 0, \; a_n = 0.05 \text{ m } (524 \text{ rad/s})^2 = 13728.8 \approx 13729$$

(a) $\boxed{\mathbf{a}_P = (13729\,\mathbf{e}_n) \text{ m/s}^2}$

(b) $\boxed{\begin{aligned}\mathbf{a}_p &= (13729 \text{ m/s}^2)(\cos 45^\circ\,\mathbf{i} - \sin 45^\circ\,\mathbf{j}) \\ &= (9708\mathbf{i} - 9708\mathbf{j}) \text{ m/s}^2\end{aligned}}$

Problem 13.114 Suppose that a circular tunnel of radius R could be dug beneath the equator. In principle, a satellite could be placed in orbit about the center of the earth within the tunnel. The acceleration due to gravity in the tunnel would be gR/R_E, where g is the acceleration due to gravity at sea level and R_E is the earth's radius. Determine the velocity of the satellite and show that the time required to complete one orbit is independent of the radius R. (See Example 13.10.)



Solution: To be in orbit we must have

$$a_n = \frac{v^2}{R} = \frac{gR}{R_E} \Rightarrow \boxed{v = R\sqrt{\frac{g}{R_E}}}$$

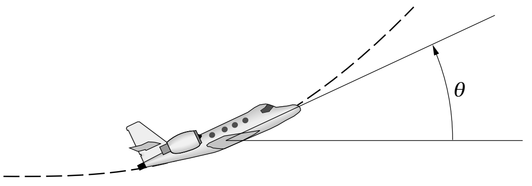
The velocity of the satellite is given by the distance it travels in one orbit divided by the time needed to complete that orbit.

$$v = \frac{2\pi R}{t} \Rightarrow t = \frac{v}{2\pi R} = \frac{R\sqrt{\dfrac{g}{R_E}}}{R} = \sqrt{\frac{g}{R_E}}.$$

Notice that the time does not depend on the radius R.

Problem 13.115 At the instant shown, the magnitude of the airplane's velocity is 130 m/s, its tangential component of acceleration is $a_t = -4\ \text{m/s}^2$, and the rate of change of its path angle is $d\theta/dt = 5°/\text{s}$.

(a) What are the airplane's velocity and acceleration in terms of normal and tangential components?
(b) What is the instantaneous radius of curvature of the airplane's path?



Solution:

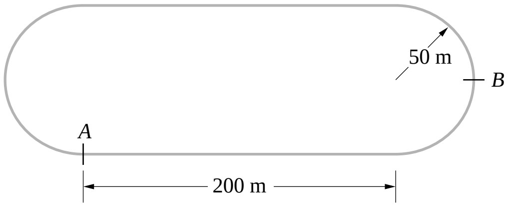
$$\omega = (5°/\text{s})\left(\frac{\pi \text{ rad}}{180°}\right) = \left(\frac{\pi}{36}\right) \text{rad/s}$$

$a_{Pt} = -4\ \text{m/s}^2$, $a_n = (130\ \text{m/s})\omega = 11.34\ \text{m/s}^2$

(a) $$\boxed{\begin{aligned}\mathbf{v}_p &= (130\mathbf{e}_t)\ \text{m/s}\\ \mathbf{a}_p &= (-4\mathbf{e}_t + 11.34\mathbf{e}_n)\ \text{m/s}^2\end{aligned}}$$

(b) $$\boxed{\rho = \frac{v^2}{a_n} = \frac{(130\ \text{m/s})^2}{11.34\ \text{m/s}^2} = 1490\ \text{m}}$$

Problem 13.116 In the preliminary design of a sun-powered car, a group of engineering students estimates that the car's acceleration will be $0.6\ \text{m/s}^2$. Suppose that the car starts from rest at A and the tangential component of its acceleration is $a_t = 0.6\ \text{m/s}^2$. What are the car's velocity and acceleration in terms of normal and tangential components when it reaches B?



Solution:

$$a_t = v\frac{dv}{ds} = 0.6\ \text{m/s}^2 \Rightarrow \int_0^v v\,dv = \int_0^s (0.6\ \text{m/s}^2)ds$$

$$v^2 = 2(0.6\ \text{m/s}^2)s$$

At point B

$$S_B = \left(200 + \frac{50\pi}{2}\right)m \Rightarrow v_B = 18.28\ \text{m/s}, \quad a_{Bn} = \frac{v_B^2}{50\ \text{m}} = 6.68\ \text{m/s}^2$$

Thus

$$\boxed{\begin{aligned}\mathbf{v}_B &= (18.28\mathbf{e}_t)\ \text{m/s}\\ \mathbf{a}_B &= (0.6\mathbf{e}_t + 6.68\mathbf{e}_n)\ \text{m/s}^2\end{aligned}}$$

Problem 13.117 After subjecting the car design described in Problem 13.116 to wind tunnel testing, the students estimate that the tangential component of the car's acceleration will be $a_t = 0.6 - 0.002v^2\ \text{m/s}^2$, where v is the car's velocity in m/s. If the car starts from rest at A, what are its velocity and acceleration in terms of normal and tangential components when it reaches B?

Solution: At point B $S_B = \left(200 + \frac{50\pi}{2}\right)$ m

$$a_t = v\frac{dv}{ds} = 0.6 - 0.002v^2 \Rightarrow \int_0^{v_B} \frac{v\,dv}{0.6 - 0.002v^2} = \int_0^{s_B} ds$$

$$v_B = 14.20\ \text{m/s}, \quad a_{Bn} = \frac{v_B^2}{50\ \text{m}} = 4.03\ \text{m/s}^2$$

$$a_t = 0.6 - 0.002(14.20\ \text{m/s})^2 = 0.197\ \text{m/s}^2$$

Thus

$$\boxed{\begin{aligned}\mathbf{v}_B &= (14.20\mathbf{e}_t)\ \text{m/s}\\ \mathbf{a}_B &= (0.197\mathbf{e}_t + 4.03\mathbf{e}_n)\ \text{m/s}^2\end{aligned}}$$

Problem 13.118 Suppose that the tangential component of acceleration of the car described in Problem 13.117 is given in terms of the car's position by $a_t = 0.4 - 0.001s$ m/s^2, where s is the distance the car travels along the track from point A. What are the car's velocity and acceleration in terms of normal and tangential components at point B?

Solution:

$$a_t = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = 0.4 - 0.001s \text{ m/s}^2$$

$$v\frac{dv}{ds} = 0.4 - 0.001s$$

$$\int_0^v v\,dv = \int_0^{S_B} (0.4 - 0.001s)\,ds$$

$$\frac{v^2}{2} = \left[0.4s - \frac{0.001s^2}{2}\right]\Bigg|_0^{S_B}$$

From Fig. P13.116, $S_B = 200 + 2\pi\rho/4$ where $\rho = 50$ m, so $S_B = 278.5$ m

Solving for v,

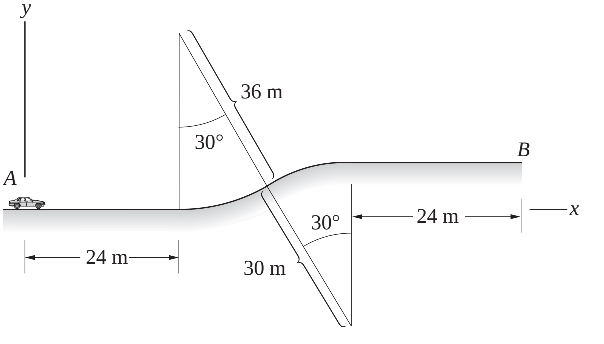
$v = 12.05$ m/s

$\underline{\mathbf{v} = 12.05\mathbf{e}_t \text{ (m/s)}}$

$\mathbf{a} = (0.4 - 0.001s_B)\mathbf{e}_t + v^2/\rho\mathbf{e}_N$ (m/s^2)

Solving, $\underline{\mathbf{a} = 0.121\mathbf{e}_t + 2.905\mathbf{e}_N \text{ (m/s}^2)}$

Problem 13.119 A car increases its speed at a constant rate from 64 km/h at A to 96 km/h at B. What is the magnitude of its acceleration 2 s after the car passes point A?



Solution: Use the chain rule to obtain

$$v\frac{dv}{ds} = a,$$

where a is constant. Separate variables and integrate: $v^2 = 2as + C$. At

$$s = 0, \quad v(0) = 64\left(\frac{1000}{3600}\right) = 17.78 \text{ m/s},$$

from which $C = 316.13$. The acceleration is

$$a = \frac{v^2 - C}{2s}$$

The distance traveled from A to B is

$$s = 2(24) + (30)\left(\frac{\pi}{180}\right)(36 + 30) = 82.6 \text{ m},$$

and the speed in

$$[v(s)]_{s=82.6} = 96\left(\frac{1000}{3600}\right) = 26.67 \text{ m/s},$$

from the constant acceleration is

$$a = \frac{(26.67)^2 - 316.13}{2(82.6)} = 2.39 \text{ m/s}^2.$$

The velocity is as a function of time is $v(t) = v(0) + at = 17.78 + 2.39t$ m/s. The distance from A is

$$s(t) = 17.78t + \frac{2.39}{2}t^2.$$

At a point 2 seconds past A, the distance is $s(2) = 40.34$ m, and the velocity is $v(2) = 22.56$ m/s. The first part of the hill ends at 43, so that at this point the car is still in the first part of the hill. The tangential acceleration is $a_t = 2.39$ m/s^2. The normal acceleration is

$$a_n = \frac{v^2}{R} = \frac{(22.56)^2}{36} = 14.1 \text{ m/s}^2.$$

The magnitude of the acceleration is

$$\boxed{|\mathbf{a}| = \sqrt{2.39^2 + 14.1^2} = 14.3 \text{ m/s}^2}$$

Note: This is a large acceleration–the driver (and passengers) would no doubt be uncomfortable.

Problem 13.120 The car increases its speed at a constant rate from 64 km/h at A to 96 km/h at B. Determine the magnitude of its acceleration when it has traveled along the road a distance (a) 36 m from A and (b) 48 m from A.

Solution: Use the solution in Problem 13.119.

(a) The velocity at a distance 36 m from A is

$$v(36) = \sqrt{2as + C} = \sqrt{(2)(2.39)(36) + 316.13}$$

$$= 22.1 \text{ m/s}.$$

At 36 m the car is in the first part of the hill. The tangential acceleration is $a_t = 2.39 \text{ m/s}^2$ from Problem 13.119. The normal acceleration is

$$a_n = \frac{(v(36))^2}{R} = \frac{(22.1)^2}{36} = 13.6 \text{ m/s}^2.$$

The magnitude of the acceleration is

$$\boxed{|\mathbf{a}| = \sqrt{2.39^2 + 13.6^2} = 13.81 \text{ m/s}^2}$$

(b) The velocity at distance 48 m from A is

$$v(48) = \sqrt{2(2.39)(48) + C} = 23.4 \text{ m/s}.$$

At 48 m the car is on the second part of the hill. The tangential acceleration is unchanged: $a_t = 2.39 \text{ m/s}^2$. The normal acceleration is

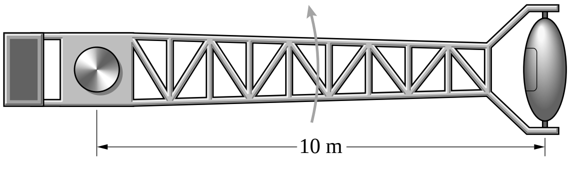
$$a_n = \frac{(v(48))^2}{R} = \frac{23.42}{30} = 18.3 \text{ m/s}^2.$$

The magnitude of the acceleration is

$$\boxed{|\mathbf{a}| = \sqrt{2.392^2 + 18.32^2} = 18.5 \text{ m/s}^2}$$

[*Note*: The car will "lift off" from the road.]

Problem 13.121 Astronaut candidates are to be tested in a centrifuge with 10-m radius that rotates in the horizontal plane. Test engineers want to subject the candidates to an acceleration of 5 g's, or five times the acceleration due to gravity. Earth's gravity effectively exerts an acceleration of 1 g in the vertical direction. Determine the angular velocity of the centrifuge in revolutions per second so that the magnitude of the total acceleration is 5 g's.



Solution:

$$a_n{}^2 + g^2 = (5g)^2 \Rightarrow a_n = \sqrt{24}g$$

$$a_n = r\omega^2 \Rightarrow \omega = \sqrt{a_n/r}$$

$$\boxed{\omega = \sqrt{\frac{\sqrt{24}(9.81 \text{ m/s}^2)}{10\text{m}}} = 2.19 \text{ rad/s}}$$

Problem 13.122 In Example 13.11, what is the helicopter's velocity in turns of normal and tangential components at $t = 4$ s?

Solution: In Example 13.11 we find the x and y components of acceleration and velocity at $t = 4$ s.

$a_x = 2.4 \text{ m/s}^2$, $a_y = 0.36 \text{ m/s}^2$,

$v_x = 4.80$ m/s, $v_y = 4.32$ m/s

The total velocity of the helicopter is

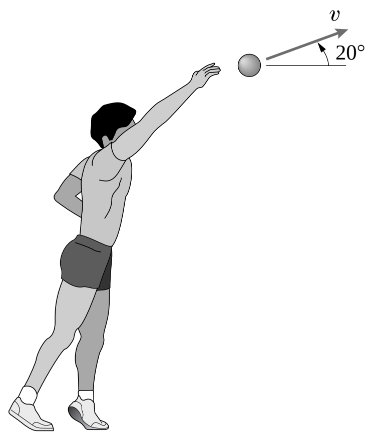
$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(4.80 \text{ m/s})^2 + (4.32 \text{ m/s})^2} = 6.46 \text{ m/s}.$$

By definition, the velocity is in the tangential direction, therefore

$$\boxed{\mathbf{v} = 6.46 \text{ m/s})\mathbf{e}_t.}$$

Problem 13.123 The athlete releases the shot with velocity $v = 16$ m/s at 20° above the horizontal.

(a) What are the velocity and acceleration of the shot in terms of normal and tangential components when it is at the highest point of its trajectory?
(b) What is the instantaneous radius of curvature of the shot's path when it is at the highest point of its trajectory?



Solution:

$a_x = 0$

$v_x = v_{x0} = 16\cos 20^\circ$, $v_{y0} = 16\sin 20^\circ$

$a_y = -9.81$ m/s^2

$v_y = v_{y0} - 9.81t = 5.47 - 9.81t$

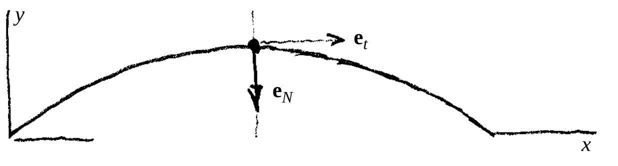
At highest point, $v_y = 0$

(a) $\underline{\mathbf{v} = 16\cos 20^\circ \mathbf{e}_t = 15.0\mathbf{e}_t \text{ (m/s)}}$

(b) $\underline{\mathbf{a} = 9.81\mathbf{e}_n \text{ (m/s}^2)}$

(c) $|\mathbf{a}_N| = v^2/\rho, \quad \rho = v^2/|a_n|$

$\rho = (15)^2/9.81 = 23.0$ m



Problem 13.124 At $t = 0$, the athlete releases the shot with velocity $v = 16$ m/s.

(a) What are the velocity and acceleration of the shot in terms of normal and tangential components at $t = 0.3$ s?
(b) Use the relation $a_n = v^2/\rho$ to determine the instantaneous radius of curvature of the shot's path at $t = 0.3$ s.

Solution: From the solution to Problem 13.123,

$v_x = 15.0$ m/s $\quad v_y = 5.47 - 9.81t$ m/s

$v = \sqrt{v_x^2 + v_y^2}$

At $t = 0.3$ s, $\quad v = 15.2$ m/s

$\underline{\mathbf{v} = 15.2\mathbf{e}_t \text{ (m/s)}}$

We have the following geometry
From the diagram

$\tan r = v_y/v_x \quad r = 9.55^\circ$

$|\mathbf{a}_n| = 9.81\cos r = 9.67$ m/s^2

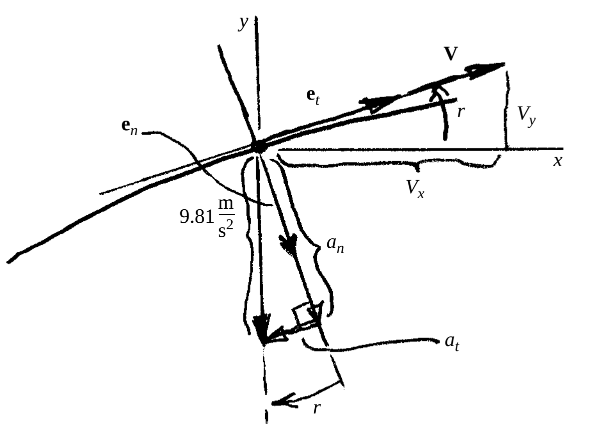
$|\mathbf{a}_t| = 9.81\sin r = 1.63$ m/s^2

$\underline{\mathbf{a} = -1{,}63\mathbf{e}_t + 9.67\mathbf{e}_n \text{ (m/s}^2)}$

$|a_n| = v^2/\rho$

$\rho = v^2/|a_n| = (15.2)^2/9.67$

$\underline{\rho = 24.0 \text{ m}}$



Problem 13.125 At $t = 0$, the athlete releases the shot with velocity $v = 16$ m/s. Use Eq. (13.42) to determine the instantaneous radius of curvature of the shot's path at $t = 0.3$ s.

Solution:

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$

From the solution to 13.123,

$v_y = v_{y0} - 9.81t$, hence

$y = y_0 + v_{y0}t - 9.8/(t^2/2), \quad y_0 \equiv 0$

Also, $v_x = v_{x0}$

$x = x_0 + v_{x0} \quad t x_0 \equiv 0$

Hence $t = x/v_{x0}$ and

$$y = v_{y0}\left(\frac{x}{v_{x0}}\right) - \frac{9.81}{2}\left(\frac{x}{v_{x0}}\right)^2$$

We now have $y(x)$

$$\frac{dy}{dx} = \frac{v_{y0}}{v_{x0}} - 9.81\left(\frac{x}{v_{x0}^2}\right)$$

$$\frac{d^2y}{dx^2} = -9.81/v_{x0}^2$$

We also know $v_0 = 16$ m/s and

$v_{y0} = v_0 \sin 20^\circ = 5.47$ m/s

$v_{x0} = v_0 \cos 20^\circ = 15.04$ m/s

At $t = 0.3$ s, $x = 4.5$ m,

$$\frac{dy}{dx} = 0.168$$

$$\frac{d^2y}{dx^2} = -0.0434$$

and $\underline{\rho = 24.0 \text{ m}}$

Problem 13.126 The cartesian coordinates of a point moving in the xy-plane are $x = 20 + 4t^2$ m, $y = 10 - t^3$ m. What is the instantaneous radius of curvature of the path of the point at $t = 3$ s?

Solution: The components of the velocity: $\mathbf{v} = 8t\mathbf{i} - (3t^2)\mathbf{j}$. At $t = 3$ seconds, the magnitude of the velocity is $|\mathbf{v}|_{t=3} = \sqrt{(8t)^2 + (-3t^2)^2} = 36.12$ m/s. The components of the acceleration are $\mathbf{a} = 8\mathbf{i} - (6t)\mathbf{j}$. The instantaneous path angle is

$$\tan\beta = \frac{v_y}{v_x} = \frac{-3t^2}{8t}.$$

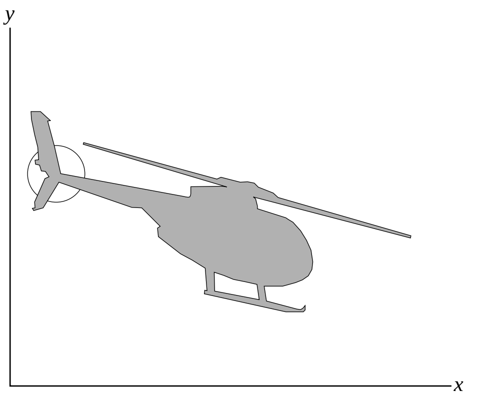
At $t = 3$ seconds, $\beta = -0.8442$ rad. The unit vector parallel to the path is $\mathbf{e}_t = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta$. The unit vector normal to the path pointing toward the instantaneous radial center is

$$\mathbf{e}_n = \mathbf{i}\cos\left(\beta - \frac{\pi}{2}\right) + \mathbf{j}\sin\left(\beta - \frac{\pi}{2}\right) = \mathbf{i}\sin\beta - \mathbf{j}\cos\beta.$$

The normal acceleration is the component of acceleration in the direction of $\mathbf{e}_n$. Thus, $a_n = \mathbf{e}_n \cdot \mathbf{a}$ or $a_n = 8\sin\beta + (6t)\cos\beta$. At $t = 3$ seconds, $a_n = 5.98$ m/s^2. The radius of curvature at $t = 3$ seconds is

$$\boxed{\rho = \frac{|\mathbf{v}|^2}{a_n} = 218 \text{ m}}$$

Problem 13.127 The helicopter starts from rest at $t = 0$. The cartesian components of its acceleration are $a_x = 0.6t\ \text{m/s}^2$ and $a_y = 1.8 - 0.36t\ \text{m/s}^2$. Determine the tangential and normal components of its acceleration at $t = 6$ s.



Solution: The solution will follow that of Example 13.11, with the time changed to $t = 6$ s. The helicopter starts from rest $(v_x, v_y) = (0, 0)$ at $t = 0$. Assume that motion starts at the origin (0, 0). The equations for the motion in the x direction are $a_x = 0.6t\ \text{m/s}^2$, $v_x = 0.3t^2$ m/s, $x = 0.1t^3$ m, and the equations for motion in the y direction are $a_y = 1.8 - 0.36t\ \text{m/s}^2$, $v_y = 1.8t - 0.18t^2$ m/s, and $y = 0.9t^2 - 0.06t^3$ m. At $t = 6$ s, the variables have the values $a_x = 3.6\ \text{m/s}^2$, $a_y = -0.36\ \text{m/s}^2$, $v_x = 10.8$ m/s, $v_y = 4.32$ m/s, $x = 21.6$ m, and $y = 19.44$ m. The magnitude of the velocity is given by

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = 11.63 \text{ m/s}.$$

The unit vector in the tangential direction is given by

$$\mathbf{e}_T = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{v_x\mathbf{i} + v_y\mathbf{j}}{|\mathbf{v}|} = 0.928\mathbf{i} + 0.371\mathbf{j}.$$

The tangential acceleration component is given by $a_T = \mathbf{a}\cdot\mathbf{e}_T = 0.928a_x + 0.371a_y = 3.21\ \text{m/s}^2$.

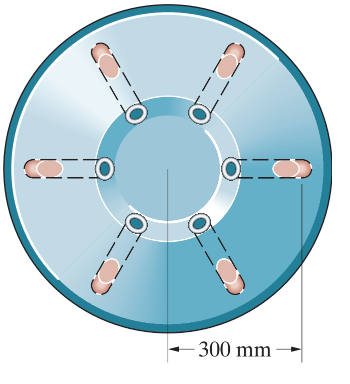
The magnitude of the acceleration is given by

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2} = 3.62 \text{ m/s}^2.$$

The normal acceleration component is given by

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^2} = 1.67 \text{ m/s}^2$$

Problem 13.128 Suppose that when the centrifuge in Example 13.12 is turned on, its motor and control system give it an angular acceleration (in rad/s^2) $\alpha = 12 - 0.02\omega$, where ω is the centrifuge's angular velocity. Determine the tangential and normal components of the acceleration of the sample at $t = 0.2$ s.



Solution: We will first integrate to find the angular velocity at t = 0.2 s.

$$\alpha = \frac{d\omega}{dt} = ([12 \text{ rad/s}^2] - [0.02 \text{ s}^{-1}]\omega)$$

$$\int_0^{\omega} \frac{d\omega}{[12 \text{ rad/s}^2] - [0.02 \text{ s}^{-1}]\omega} = \int_0^{0.2 \text{ s}} dt$$

$$\frac{-1 \text{ s}}{0.02}\ln\left(\frac{[12 \text{ rad/s}^2] - [0.02 \text{ s}^{-1}]\omega}{[12 \text{ rad/s}^2]}\right) = 0.2 \text{ s}$$

$$[12 \text{ rad/s}^2] - [0.02 \text{ s}^{-1}]\omega = [12 \text{ rad/s}^2]e^{-0.004}$$

$$\omega = \frac{1 \text{ s}}{0.02}(12 \text{ rad/s}^2)(1 - e^{-0.004}) = 2.40 \text{ rad/s}$$

At this time, the angular acceleration is

$$\alpha = (12 \text{ rad/s}^2) - (0.02 \text{ s}^{-1})(2.40 \text{ rad/s}) = 11.95 \text{ rad/s}^2.$$

The components of acceleration are

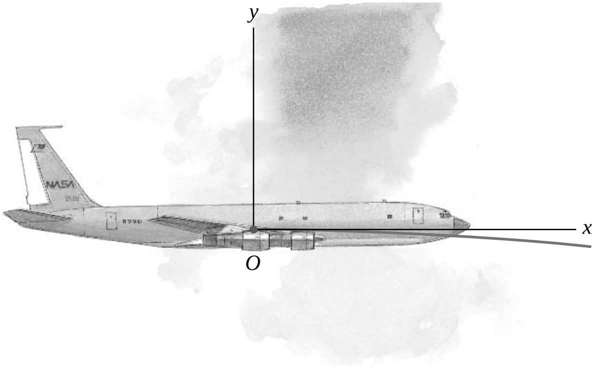
$$a_t = r\alpha = (0.3 \text{ m})(11.95 \text{ rad/s}^2) = 3.59 \text{ m/s}^2$$

$$\alpha_n = r\omega^2 = (0.3 \text{ m})(2.40 \text{ rad/s})^2 = 1.72 \text{ m/s}^2.$$

$$\boxed{\mathbf{a} = (3.59\mathbf{e}_t + 1.72\mathbf{e}_n) \text{ m/s}^2.}$$

Problem 13.129* For astronaut training, the airplane shown is to achieve "weightlessness" for a short period of time by flying along a path such that its acceleration is $a_x = 0$ and $a_y = -g$. If the velocity of the plane at O at time $t = 0$ is $\mathbf{v} = v_0\mathbf{i}$, show that the autopilot must fly the airplane so that its tangential component of the acceleration as a function of time is

$$a_t = g\frac{\left(\dfrac{gt}{v_0}\right)}{\sqrt{1+\left(\dfrac{gt}{v_0}\right)^2}}.$$



Solution: The velocity of the path is $\mathbf{v}(t) = v_0\mathbf{i} - gt\mathbf{j}$. The path angle is

$$\beta : \tan\beta = \frac{v_y}{v_x} = \frac{-gt}{v_0},$$

$$\sin\beta = \frac{-gt}{\sqrt{v_0^2 + (gt)^2}}.$$

The unit vector parallel to the velocity vector is $\mathbf{e} = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta$. The acceleration vector is $\mathbf{a} = -\mathbf{j}g$. The component of the acceleration tangent to the flight path is $a_t = -g\sin\beta$., from which

$$a_t = g\frac{gt}{\sqrt{v_0^2 + (gt)^2}}.$$

Divide by v_0,

$$\boxed{a_t = g\left[1+\left(\frac{gt}{v_0}\right)^2\right]^{-\frac{1}{2}}\left(\frac{gt}{v_0}\right)}$$

Problem 13.130* In Problem 13.129, what is the airplane's normal component of acceleration as a function of time?

Solution: From Problem 13.129, the velocity is $\mathbf{v}(t) = v_0\mathbf{i} - gt\mathbf{j}$. The flight path angle is β, from which

$$\cos\beta = \frac{v_0}{\sqrt{v_0^2 + (gt)^2}}.$$

The unit vector parallel to the flight path is $\mathbf{e} = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta$. The unit vector normal to $\mathbf{e}$ is

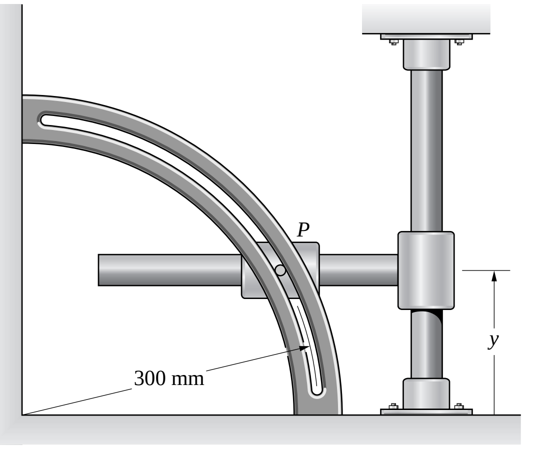
$$\mathbf{e}_n = \mathbf{i}\cos\left(\beta - \frac{\pi}{2}\right) + \mathbf{j}\sin\left(\beta - \frac{\pi}{2}\right)$$

$$= \mathbf{i}\sin\beta - \mathbf{j}\cos\beta,$$

pointing toward the instantaneous radial center of the path. The acceleration is $\mathbf{a} = -\mathbf{j}g$. The component parallel to the normal component is $a_n = g\cos\beta$, from which

$$\boxed{a_n = g\frac{v_0}{\sqrt{v_0^2 + (gt)^2}} = g\left[1+\left(\frac{gt}{v_0}\right)^2\right]^{-\frac{1}{2}}}$$

Problem 13.131 If $y = 100$ mm, $\dfrac{dy}{dt} = 200$ mm/s, and $\dfrac{d^2y}{dt^2} = 0$, what are the velocity and acceleration of P in terms of normal and tangential components?



Solution: The equation for the circular guide is $R^2 = x^2 + y^2$, from which $x = \sqrt{R^2 - y^2} = 0.283$ m, and

$$\frac{dx}{dt} = -\left(\frac{y}{x}\right)\frac{dy}{dt} = v_x = -0.0707 \text{ m/s}.$$

The velocity of point **P** is $\mathbf{v}_p = \mathbf{i}v_x + \mathbf{j}v_y$, from which the velocity is $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = 0.212$ m/s. The angular velocity

$$\omega = \frac{|\mathbf{v}|}{R} = 0.7071 \text{ rad/s}.$$

The angle is

$$\beta = \tan^{-1}\left(\frac{y}{x}\right) = 19.5^\circ$$

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}\left(-\frac{y}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x}\left(\frac{dy}{dt}\right)^2 + \frac{y}{x^2}\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) - \left(\frac{y}{x}\right)\left(\frac{d^2y}{dt^2}\right)$$

$$= -0.1591 \text{ m/s}^2$$

The unit vector tangent to the path (normal to the radius vector *for a circle*) is $\mathbf{e}_p = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta$, from which

$$\boxed{a_t = -a_x \sin\beta = 53.0 \text{ mm/s}^2}$$

since $a_y = 0$

$$\boxed{a_n = -R\omega^2 = -0.150 \text{ m/s}^2}.$$

Check: $a_n = a_x \cos\beta = -0.15 \text{ m/s}^2$ *check*.

Problem 13.132* Suppose that the point P in Problem 13.131 moves upward in the slot with velocity $\mathbf{v} = 300\mathbf{e}_t$ (mm/s). When $y = 150$ mm, what are $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$?

Solution: The position in the guide slot is $y = R\sin\theta$, from which

$$\theta = \sin^{-1}\left(\frac{y}{R}\right) = \sin^{-1}(0.5) = 30^\circ.$$

$$x = R\cos\theta = 259.8 \text{ mm}.$$

From the solution to Problem 13.131,

$$v_x = -\left(\frac{y}{x}\right)\frac{dy}{dt} = -\left(\frac{y}{x}\right)v_y.$$

The velocity is $|\mathbf{v}| = 300 = \sqrt{v_x^2 + v_y^2} = v_y\sqrt{\left(\frac{y}{x}\right)^2 + 1}$, from which

$$\boxed{v_y = 300\left(\left(\frac{y}{x}\right)^2 + 1\right)^{-\frac{1}{2}} = 259.8 \text{ mm/s}}$$

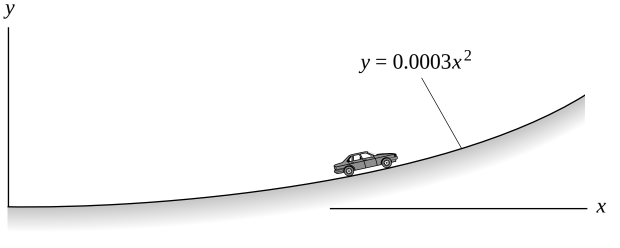
and $v_x = -150$ mm/s (Since the point is moving upward in the slot, v_y is positive.). The velocity along the path in the guide slot is assumed constant, hence $a_t = 0$. The normal acceleration is

$$a_n = \frac{|\mathbf{v}|^2}{R} = 300 \text{ mm/s}^2$$

directed toward the radius center, from which

$$\boxed{\frac{d^2y}{dt^2} = -a_n\sin\theta = -150 \text{ mm/s}^2}$$

Problem 13.133* A car travels at 100 km/h on a straight road of increasing grade whose vertical profile can be approximated by the equation shown. When $x = 400$ m, what are the tangential and normal components of the car's acceleration?



Solution: The strategy is to use the acceleration in cartesian coordinates found in the solution to Problem 13.90, find the angle with respect to the x-axis,

$$\theta = \tan^{-1}\left(\frac{dy}{dx}\right),$$

and use this angle to transform the accelerations to tangential and normal components. From the solution to Problem 13.90 the accelerations are $\mathbf{a} = -0.0993\mathbf{i} + 0.4139\mathbf{j}$ (m/s^2). The angle at

$$\theta = \tan^{-1}\left(\frac{d}{dx}Cx^2\right)_{x=400} = \tan^{-1}(6x \times 10^{-4})_{x=400} = 13.5^\circ.$$

From trigonometry (see figure) the transformation is $a_t = a_x\cos\theta + a_y\sin\theta$, $a_n = -a_x\sin\theta + a_y\cos\theta$, from which

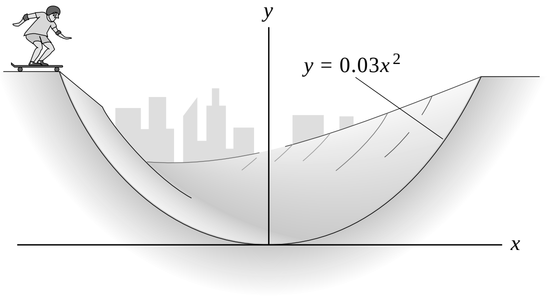
$\boxed{a_t = 0.000035\ldots = 0}$,

$\boxed{a_n = 0.4256 \text{ m/s}^2}$

Check: The velocity is constant along the path, so the tangential component of the acceleration is zero, $a_t = \dfrac{dv}{dt} = 0$, *check*.

Problem 13.134 A boy rides a skateboard on the concrete surface of an empty drainage canal described by the equation shown. He starts at $y = 20$ m, and the magnitude of his velocity is approximated by $v = \sqrt{2(9.81)(20 - y)}$ m/s.

(a) Use Equation (13.42) to determine the instantaneous radius of curvature of the boy's path when he reaches the bottom.
(b) What is the normal component of his acceleration when he reaches the bottom?



Solution:

(a) $y = 0.03x^2$, so

$$\frac{dy}{dx} = 0.06x \text{ and } \frac{d^2y}{dx^2} = 0.06.$$

From Eq (13.42),

$$\rho = \frac{[1 + (0.06x)^2]^{3/2}}{0.06} \text{ m.}$$

At $x = 0$, $\underline{\rho = 16.7 \text{ m}}$.

(b) The magnitude of the velocity is

$$\sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} = v = K(20 - y)^{\frac{1}{2}} = K(20 - Cx^2)^{\frac{1}{2}},$$

where $K = 8.025$, $C = 0.03$. From $y = Cx^2$,

$$\frac{dy}{dt} = 2Cx\left(\frac{dx}{dt}\right),$$

$$\frac{d^2y}{dt^2} = 2C\left(\frac{dx}{dt}\right)^2 + 2Cx\left(\frac{d^2x}{dt^2}\right).$$

Substitute:

$$\left|\frac{dx}{dt}\right| = \frac{K(20 - Cx^2)^{\frac{1}{2}}}{(4C^2x^2 + 1)^{\frac{1}{2}}}.$$

Since the boy is moving the right,

$$\frac{dx}{dt} > 0, \quad \text{and} \quad \left|\frac{dx}{dt}\right| = \frac{dx}{dt}.$$

The acceleration is

$$\frac{d^2x}{dt^2} = \frac{-KCx}{(20 - Cx^2)^{\frac{1}{2}}(4C^2x^2 + 1)^{\frac{1}{2}}}\left(\frac{dx}{dt}\right)$$

$$- \frac{K(4C^2x)(20 - Cx^2)^{\frac{1}{2}}}{(4C^2x^2 + 1)^{\frac{3}{2}}}\left(\frac{dx}{dt}\right).$$

At the bottom of the canal the values are

$$\left(\frac{dx}{dt}\right)_{x=0} = K\sqrt{20} = 35.89 \text{ m/s.}$$

$$\left(\frac{dy}{dt}\right)_{x=0} = 0, \left(\frac{d^2x}{dt^2}\right)_{x=0} = 0,$$

$$\left(\frac{d^2y}{dt^2}\right)_{x=0} = 2C\left(\frac{dx}{dt}\right)^2\bigg|_{x=0} = 77.28 \text{ m/s}^2.$$

The angle with respect to the x axis at the bottom of the canal is

$$\theta = \tan^{-1}\left(\frac{dy}{dx}\right)_{x=0} = 0.$$

From the solution to Problem 2.133, the tangential and normal accelerations are $a_t = a_x \cos\theta + a_y \sin\theta$, $a_n = -a_x \sin\theta + a_y \cos\theta$, from which

$\boxed{a_t = 0}$, and $\boxed{a_n = 77.28 \text{ m/s}^2}$.

Check: The velocity is constant along the path, so the tangential component of the acceleration is zero, $a_t = \frac{dv}{dt} = 0$. *check*. By inspection, the normal acceleration at the bottom of the canal is identical to the y component of the acceleration. *check*.

Problem 13.135 In Problem 13.134, what is the normal component of the boy's acceleration when he has passed the bottom and reached $y = 10$ m?

Solution: Use the results of the solutions to Problems 13.133 and 13.134. From the solution to Problem 13.134, at $y = 10$ m,

$$x = \left(\sqrt{\frac{y}{C}}\right) = 18.257 \text{ m}, \quad \text{from which}$$

$$\left(\frac{dx}{dt}\right)_{y=10} = \left(K(20 - Cx^2)^{\frac{1}{2}}(4C^2x^2 + 1)^{-\frac{1}{2}}\right)_{y=10} = 17.11 \text{ m/s}.$$

$$\left(\frac{d^2x}{dt^2}\right)_{y=10} = -K\left(\frac{dx}{dt}\right)_{y=10}\left(\frac{Cx}{(20 - Cx^2)^{\frac{1}{2}}(4C^2x^2 + 1)^{\frac{1}{2}}}\right.$$

$$\left. + \frac{(4C^2x)(20 - Cx^2)^{\frac{1}{2}}}{(4C^2x^2 + 1)^{\frac{3}{2}}}\right)_{y=10}$$

$$= -24.78 \text{ m/s}.$$

$$\left(\frac{d^2y}{dt^2}\right)_{y=10} = \left(2C\left(\frac{dx}{dt}\right)^2 + 2Cx\left(\frac{d^2x}{dt^2}\right)\right)_{y=10} = -9.58 \text{ m/s}^2.$$

The angle is $\theta = \tan^{-1}\left(\dfrac{dy}{dx}\right)_{y=10} = 47.61°$.

From the solution to Problem 13.133,

$$a_t = a_x \cos\theta + a_y \sin\theta, \; a_n = -a_x \sin\theta + a_y \cos\theta,$$

from which

$$\boxed{a_t = -23.78 \text{ m/s}^2}, \; \boxed{a_n = 11.84 \text{ m/s}^2}$$

Problem 13.136* Using Eqs (13.41): (a) Show that the relations between the cartesian unit vectors and the unit vectors $\mathbf{e}_t$ and $\mathbf{e}_n$ are

$$\mathbf{i} = \cos\theta\mathbf{e}_t - \sin\theta\mathbf{e}_n$$

and $\quad \mathbf{j} = \sin\theta\mathbf{e}_t + \cos\theta\mathbf{e}_n$

(b) Show that

$d\mathbf{e}_t/dt = d\theta/dt\mathbf{e}_n$ and $d\mathbf{e}_n/dt = -d\theta/dt\mathbf{e}_t$.

Solution: Equations (13.41) are $\mathbf{e}_t = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{e}_n = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

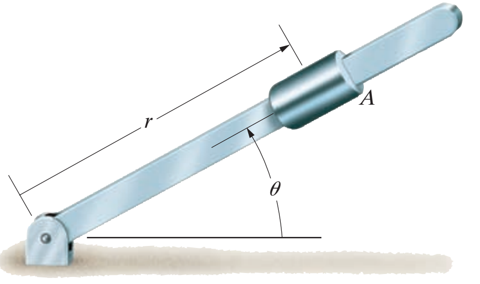
(a) Multiplying the equation for $\mathbf{e}_t$ by $\cos\theta$ and the equation for $\mathbf{e}_n$ by $(-\sin\theta)$ and adding the two equations, we get $\mathbf{i} = \cos\theta\mathbf{e}_t - \sin\theta\mathbf{e}_n$. Similarly, by multiplying the equation for $\mathbf{e}_t$ by $\sin\theta$ and the equation for $\mathbf{e}_n$ by $\cos\theta$ and adding, we get $\mathbf{j} = \sin\theta\mathbf{e}_t + \cos\theta\mathbf{e}_n$.

(b) Taking the derivative of $\mathbf{e}_t = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, we get $\dfrac{d\mathbf{e}_t}{dt} = (-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\dfrac{d\theta}{dt} = \mathbf{e}_n\dfrac{d\theta}{dt}$.
Similarly, taking the derivative of $\mathbf{e}_n = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, we get $d\mathbf{e}_n/dt = -(d\theta/dt)\mathbf{e}_t$

Problem 13.137 The polar coordinates of the collar A are given as functions of time in seconds by

$$r = 1 + 0.2t^2 \text{ m and } \theta = 2t \text{ rad.}$$

What are the magnitudes of the velocity and acceleration of the collar at $t = 2$ s?



Solution: We have

$r = (1 \text{ ft}) + (0.2 \text{ m/s}^2)t^2, \quad \theta = (2 \text{ rad/s})t,$

$\dfrac{dr}{dt} = (0.4 \text{ ft/s}^2)t, \qquad \dfrac{d\theta}{dt} = 2 \text{ rad/s},$

$\dfrac{d^2r}{dt^2} = 0.4 \text{ m/s}^2, \qquad \dfrac{d^2\theta}{dt^2} = 0$

At time $t = 2$ s, we have

$r = 1.8 \text{ ft}, \quad \dfrac{dr}{dt} = 0.8 \text{ m/s}, \qquad \dfrac{d^2r}{dt^2} = 0.4 \text{ m/s}^2,$

$\theta = 8 \text{ rad}, \quad \dfrac{d\theta}{dt} = 2 \text{ rad/s}, \qquad \dfrac{d^2\theta}{dt^2} = 0$

The components of the velocity and acceleration are

$v_r = \dfrac{dr}{dt} = 0.8 \text{ m/s}, v_\theta = r\dfrac{d\theta}{dt} = (1.8 \text{ m})(2 \text{ rad/s}) = 3.6 \text{ m/s},$

$a_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = (0.4 \text{ m/s}^2) - (1.8 \text{ m})(2 \text{ rad/s}^2) = -\ 6.8 \text{ m/s}^2,$

$a_\theta = r\dfrac{d^2\theta}{dt^2} + 2\dfrac{dr}{dt}\dfrac{d\theta}{dt} = 0 + 2(0.8 \text{ m/s})(2 \text{ rad/s}) = 3.2 \text{ m/s}^2.$

The magnitudes are

$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(0.8 \text{ m/s})^2 + (3.6 \text{ m/s})^2} = 3.69 \text{ m/s},$

$a = \sqrt{a_r^2 + a_\theta^2} = \sqrt{(-6.8 \text{ m/s}^2)^2 + (3.2 \text{ m/s}^2)^2} = 7.52 \text{ m/s}^2.$

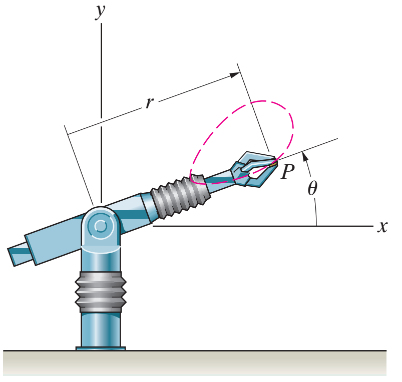
$\boxed{v = 3.69 \text{ m/s}, a = 7.52 \text{ m/s}^2.}$

Problem 13.138 In Active Example 13.13, suppose that the robot arm is reprogrammed so that the point P traverses the path described by

$r = 1 - 0.5 \sin 2\pi t$ m,

$\theta = 0.5 - 0.2 \cos 2\pi t$ rad.

What is the velocity of P in terms of polar coordinates at $t = 0.8$ s?



Solution: We have

$$r = \left[1 - 0.5 \sin\left(2\pi \frac{t}{s}\right)\right] \text{ m}, \qquad \theta = \left[0.5 - 0.2 \cos\left(2\pi \frac{t}{s}\right)\right] \text{rad},$$

$$\frac{dr}{dt} = -\pi \cos\left(2\pi \frac{t}{s}\right) \text{m/s}, \qquad \frac{d\theta}{dt} = 0.4\pi \sin\left(2\pi \frac{t}{s}\right) \text{rad/s}.$$

At time t = 0.8 s,

$$r = 1.48 \text{ m}, \frac{dr}{dt} = -0.971 \text{ m/s}, \ \theta = 0.438 \text{ rad}, \frac{d\theta}{dt} = -1.20 \text{ rad/s}.$$

The velocity is

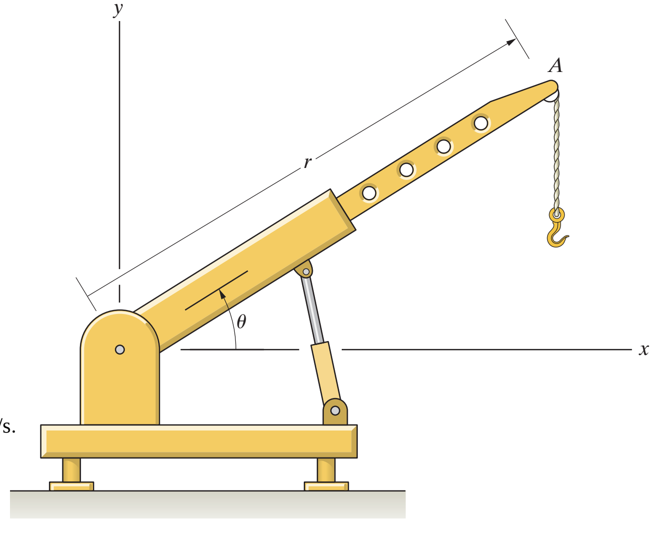
$$v_r = \frac{dr}{dt} = -0.971 \text{ m/s}, v_\theta = r\frac{d\theta}{dt} = (1.48 \text{ m})(-1.20 \text{ rad/s}) = -1.76 \text{ m/s}.$$

$$\boxed{\mathbf{v} = (-0.971\mathbf{e}_r - 1.76\mathbf{e}_\theta) \text{ m/s.}}$$

Problem 13.139 At the instant shown, $r = 3$ m and $\theta = 30^\circ$. The cartesian components of the velocity of point A are $v_x = 2$ m/s and $v_y = 8$ m/s.

(a) Determine the velocity of point A in terms of polar coordinates.

(b) What is the angular velocity $d\theta/dt$ of the crane at the instant shown?



Solution: To transform to polar coordinates we have

$$v_r = v_x \cos\theta + v_y \sin\theta = (2 \text{ m/s}) \cos 30^\circ + (8 \text{ m/s}) \sin 30^\circ = 5.73 \text{ m/s}.$$

$$v_\theta = -v_x \sin\theta + v_y \cos\theta = -(2 \text{ m/s}) \sin 30^\circ + (8 \text{ m/s}) \cos 30^\circ = 5.93 \text{ m/s}.$$

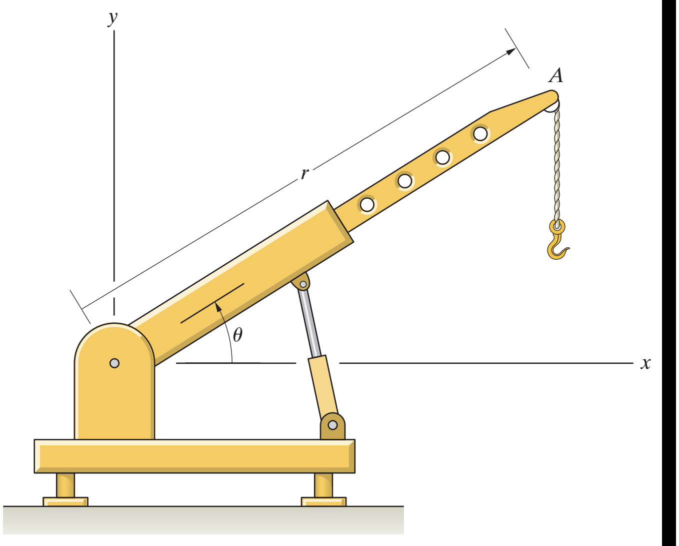
$$\boxed{\mathbf{v}_A = (5.73\mathbf{e}_r + 5.93\mathbf{e}_\theta)\text{m/s.}}$$

The angular velocity is found

$$v_\theta = r\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{v_\theta}{r} = \frac{5.93 \text{ m/s}}{3\text{m}} = 1.98 \text{ rad/s}.$$

$$\boxed{\frac{d\theta}{dt} = 1.98 \text{ rad/s.}}$$

Problem 13.140 The polar coordinates of point A of the crane are given as functions of time in seconds by $r = 3 + 0.2t^2$ m and $\theta = 0.02t^2$ rad. Determine the acceleration of point A in terms of polar coordinates at $t = 3s$.



Solution: We have

$$r = (3\text{ m}) + (0.2\text{ m/s}^2)t^2,$$

$$\frac{dr}{dt} = (0.4\text{ m/s}^2)t,$$

$$\frac{d^2r}{dt^2} = 0.4\text{ m/s}^2,$$

$$\theta = (0.02)t^2\text{ rad/s}^2.$$

$$\frac{d\theta}{dt} = (0.04)t\text{ rad/s}^2,$$

$$\frac{d^2\theta}{dt^2} = 0.04\text{ rad/s}^2$$

At $t = 3$ s,

$$r = 4.8\text{ m},\quad \frac{dr}{dt} = 1.2\text{ m/s},\quad \frac{d^2r}{dt^2} = 0.4\text{ m/s}^2,$$

$$\theta = 0.18\text{ rad},\quad \frac{d\theta}{dt} = 0.12\text{ rad/s},\quad \frac{d^2\theta}{dt^2} = 0.04\text{ rad/s}^2.$$

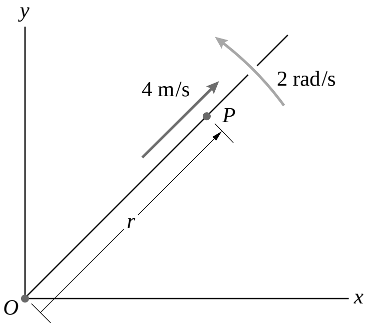
The acceleration components are

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = (0.4\text{ m/s}^2) - (4.8\text{ m})(0.12\text{ rad/s})^2,$$

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = (4.8\text{ m})(0.04\text{ rad/s}^2) + 2(1.2\text{ m/s})(0.12\text{ rad/s})$$

$$\boxed{\mathbf{a}_A = (0.331\mathbf{e}_r + 0.480\mathbf{e}_\theta)\text{ m/s}^2.}$$

Problem 13.141 The radial line rotates with a constant angular velocity of 2 rad/s. Point P moves along the line at a constant speed of 4 m/s. Determine the magnitude of the velocity and acceleration of P when $r = 2$ m. (See Example 13.14.)



Solution: The angular velocity of the line is

$$\frac{d\theta}{dt} = \omega = 2 \text{ rad/s},$$

from which $\dfrac{d^2\theta}{dt^2} = 0.$

The radial velocity of the point is

$$\frac{dr}{dt} = 4 \text{ m/s},$$

from which $\dfrac{d^2 r}{dt^2} = 0.$

The vector velocity is

$$\mathbf{v} = \left(\frac{dr}{dt}\right)\mathbf{e}_r + r\left(\frac{d\theta}{dt}\right)\mathbf{e}_\theta = 4\mathbf{e}_r + 4\mathbf{e}_\theta \text{ (m/s)}.$$

The magnitude is

$$\boxed{|\mathbf{v}| = \sqrt{4^2 + 4^2} = 5.66 \text{ m/s}}.$$

The acceleration is

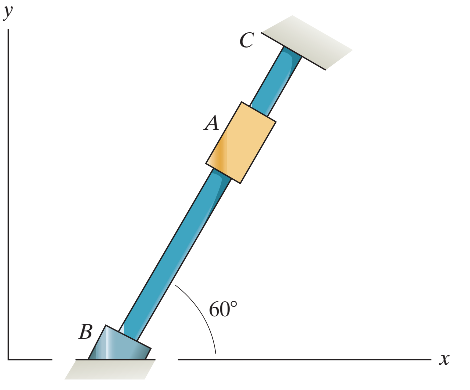
$$\mathbf{a} = [-2(4)]\mathbf{e}_r + [2(4)(2)]\mathbf{e}_\theta = -8\mathbf{e}_r + 16\mathbf{e}_\theta \text{ (m/s}^2\text{)}.$$

The magnitude is

$$\boxed{|\mathbf{a}| = \sqrt{8^2 + 16^2} = 17.89 \text{ m/s}^2}$$

Problem 13.142 At the instant shown, the coordinates of the collar A are $x = 2.3$ m, $y = 1.9$ m. The collar is sliding on the bar from B toward C at a constant speed of 4 m/s.

(a) What is the velocity of the collar in terms of polar coordinates?
(b) Use the answer to part (a) to determine the angular velocity of the radial line from the origin to the collar A at the instant shown.



Solution: We will write the velocity in terms of cartesian coordinates.

$v_x = (4 \text{ m/s}) \cos 60^\circ = 2 \text{ m/s},$

$v_y = (4 \text{ m/s}) \sin 60^\circ = 3.46 \text{ m/s}.$

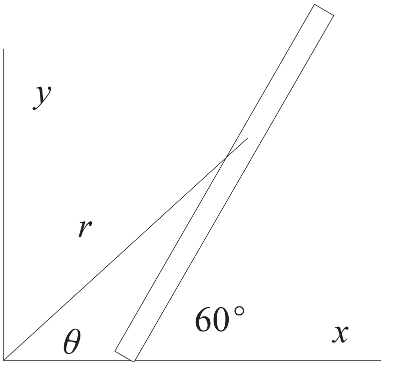
The angle θ is

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1.9 \text{ m}}{2.3 \text{ m}}\right) = 39.6^\circ.$$

Now we can convert to polar coordinates

$v_r = v_x \cos\theta + v_y \sin\theta = (2 \text{ m/s}) \cos 39.6^\circ + (3.46 \text{ m/s}) \sin 39.6^\circ,$

$v_\theta = -v_x \sin\theta + v_y \cos\theta = -(2 \text{ m/s}) \sin 39.6^\circ + (3.46 \text{ m/s}) \cos 39.6^\circ.$



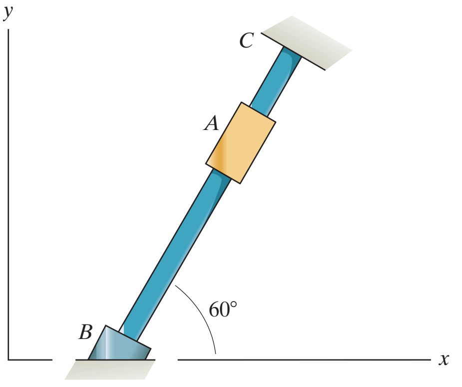
(a) $\boxed{\mathbf{v} = (3.75\mathbf{e}_r + 1.40\mathbf{e}_\theta) \text{ m/s.}}$

(b) $\dfrac{d\theta}{dt} = \dfrac{v_\theta}{r} = \dfrac{1.40 \text{ m/s}}{\sqrt{(2.3 \text{ m})^2 + (1.9 \text{ m})^2}} = 0.468 \text{ rad/s.}$ $\boxed{\dfrac{d\theta}{dt} = 0.468 \text{ rad/s.}}$

Problem 13.143 At the instant shown, the coordinates of the collar A are $x = 2.3$ m, $y = 1.9$ m. The collar is sliding on the bar from B toward C at a constant speed of 4 m/s.

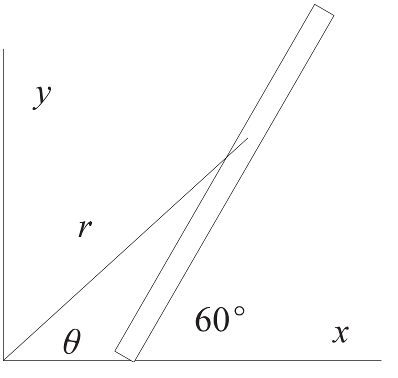
(a) What is the acceleration of the collar in terms of polar coordinates?

(b) Use the answer to part (a) to determine the angular acceleration of the radial line from the origin to the collar A at the instant shown.



Solution: The velocity is constant, so the acceleration is zero.

(a) $\boxed{\mathbf{a}_A = 0.}$



Form Problem 13.141 we know that

$$r = \sqrt{(2.3\text{ m})^2 + (1.9\text{ m})^2} = 2.98\text{ m},$$

$$\frac{dr}{dt} = v_r = 3.75\text{ m/s},$$

$$\frac{d\theta}{dt} = 0.468\text{ rad/s}.$$

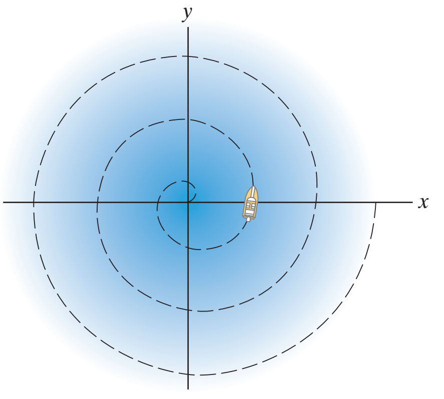
Using the θ component of the acceleration we can solve for the angular acceleration.

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

$$\frac{d^2\theta}{dt^2} = -\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = -\frac{2}{2.98\text{ m}}(3.75\text{ m/s})(0.486\text{ rad/s})$$

$$\boxed{\frac{d^2\theta}{dt^2} = -1.18\text{ rad/s}^2.}$$

Problem 13.144 A boat searching for underwater archaeological sites in the Aegean Sea moves at 2.06 m/s and follows the path $r = 10\theta$ m, where θ is in radians. When $\theta = 2\pi$ rad, determine the boat's velocity (a) in terms of polar coordinates and (b) in terms of cartesian coordinates.



Solution: The velocity along the path is

$$v = 2.06 \text{ m/s}.$$

(a) The path is $r = 10\theta$. The velocity

$$v_r = \frac{dr}{dt} = \frac{d}{dt}(10\theta) = 10\frac{d\theta}{dt} \text{ m/s}.$$

The velocity along the path is related to the components by

$$v^2 = v_r^2 + v_\theta^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = 2.06^2.$$

At $\theta = 2\pi$, $r = 10(2\pi) = 62.8$ m. Substitute:

$$2.06^2 = \left(10\frac{d\theta}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = (100 + 62.8^2)\left(\frac{d\theta}{dt}\right)^2,$$

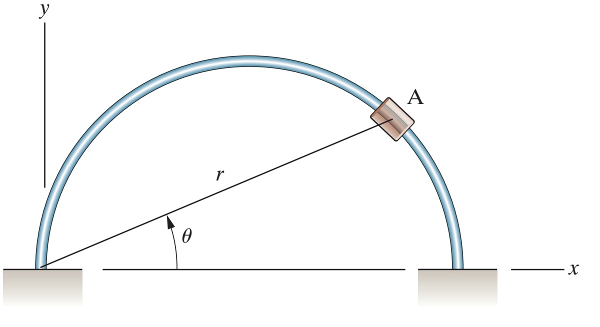
from which $\frac{d\theta}{dt} = 0.0323$ rad/s,

$$\boxed{v_r = 10\frac{d\theta}{dt} = 0.323 \text{ m/s}}, \quad \boxed{v_\theta = r\frac{d\theta}{dt} = 2.032 \text{ m/s}}$$

(b) From geometry, the cartesian components are $v_x = v_r\cos\theta + v_\theta\sin\theta$, and $v_y = v_r\sin\theta + v_\theta\cos\theta$. At $\theta = 2\pi$,

$$\boxed{v_x = v_r}, \quad \text{and} \quad \boxed{v_y = v_\theta}$$

Problem 13.145 The collar A slides on the circular bar. The radial position of A (in meters) is given as a function of θ by $r = 2\cos\theta$. At the instant shown, $\theta = 25^\circ$ and $d\theta/dt = 4$ rad/s. Determine the velocity of A in terms of polar coordinates.



Solution:

$r = 2\cos\theta$, $\dot{r} = -2\sin\theta\dot{\theta}$, $\ddot{r} = -2\sin\theta\ddot{\theta} - 2\cos\theta\dot{\theta}^2$

Using the given data we have

$\theta = 25^\circ$, $\dot{\theta} = 4$, $\ddot{\theta} = 0$

$r = 1.813$, $\dot{r} = -3.381$, $\ddot{r} = -29.00$

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta = (-3.381\mathbf{e}_r + 7.25\mathbf{e}_\theta) \text{ m/s}}$$

Problem 13.146 In Problem 13.145, $d^2\theta/dt^2 = 0$ at the instant shown. Determine the acceleration of A in terms of polar coordinates.

Solution: See Problem 13.145

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta = (-58.0\mathbf{e}_r - 27.0\mathbf{e}_\theta)\ \text{m/s}^2}$$

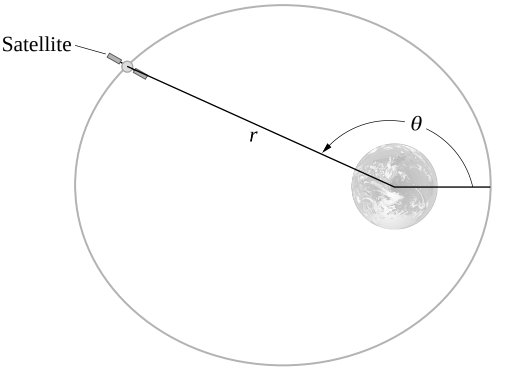
Problem 13.147 The radial coordinate of the earth satellite is related to its angular position θ by

$$r = \frac{1.91 \times 10^7}{1 + 0.5\cos\theta}\ \text{m}.$$

The product of the radial position and the transverse component of the velocity is

$$rv_\theta = 8.72 \times 10^{10}\ \text{m}^2/\text{s}.$$

What is the satellite's velocity in terms of polar coordinates when $\theta = 90^\circ$?



Solution:

At $\theta = 90^\circ$, $r = 1.91 \times 10^7\ \text{m} = p$

$$r = \frac{p}{1 + 0.5\cos\theta},$$

$$\dot{r} = \frac{(-p)(0.5)(-\sin\theta)\dot{\theta}}{(1 + 0.5\cos\theta)^2}$$

We also know that

$rv_\theta = 8.72 \times 10^{10}\ \text{m}^2/\text{s}$

However $v_\theta = r\dot{\theta}$, hence

$r^2\dot{\theta} = 8.72 \times 10^{10}\ \text{m}^2/\text{s}$

Solving for $\dot{\theta}$, we get

$\dot{\theta} = 0.000239$ rad/s

and $\dot{r} = 2283$ m/s and from above

$v_\theta = 4565$ m/s

$\underline{\mathbf{v} = 2283\mathbf{e}_r + 4565\mathbf{e}_\theta\ (\text{m/s})}$

Problem 13.148* In Problem 13.147, what is the satellite's acceleration in terms of polar coordinates when $\theta = 90^\circ$?

Solution: Set $A = 1.91 \times 10^7$ m, $B = 8.72 \times 10^{10}$ m^2/s

$$r = \frac{A}{1 + 0.5\cos\theta}, \quad r v_\theta = r(r\dot{\theta}) = B$$

$$\dot{\theta} = \frac{B}{r^2} = \left(\frac{B}{A^2}\right)(1 + 0.5\cos\theta)^2$$

$$\ddot{\theta} = -\left(\frac{B}{A^2}\right)(1 + 0.5\cos\theta)\sin\theta\dot{\theta} = -\left(\frac{B^2}{A^4}\right)(1 + 0.5\cos\theta)^3\sin\theta$$

$$\dot{r} = \frac{0.5A\sin\theta}{(1 + 0.5\cos\theta)^2}\dot{\theta} = \frac{0.5B\sin\theta}{A}$$

$$\ddot{r} = \frac{0.5B\cos\theta}{A}\dot{\theta} = 0.5\left(\frac{B^2}{A^3}\right)\cos\theta(1 + 0.5\cos\theta)^2$$

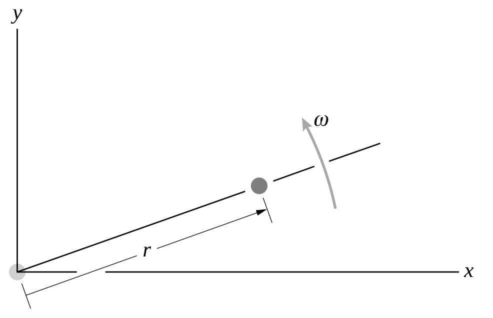
When $\theta = 90^\circ$ we have

$$r = A, \; \dot{r} = \frac{B}{2A}, \; \ddot{r} = 0, \; \dot{\theta} = \frac{B}{A^2}, \; \ddot{\theta} = -\frac{B^2}{A^4}$$

Thus

$$\boxed{\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta = (-1.091\mathbf{e}_r) \text{ m/s}^2}$$

Problem 13.149 A bead slides along a wire that rotates in the xy-plane with constant angular velocity ω_0. The radial component of the bead's acceleration is zero. The radial component of its velocity is v_0 when $r = r_0$. Determine the polar components of the bead's velocity as a function of r.



Strategy: The radial component of the bead's velocity is $v_r = \dfrac{dr}{dt}$, and the radial component of its acceleration is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{dv_r}{dt}\right) - r\omega_0^2.$$

By using the chain rule,

$$\frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = \frac{dv_r}{dr}v_r.$$

you can express the radial component of the acceleration in the form $a_r = \dfrac{dv_r}{dr}v_r - r\omega_0^2$.

Solution: From the strategy:

$$a_r = 0 = v_r\frac{dv_r}{dr} - \omega_0^2 r.$$

Separate variables and integrate: $v_r dv_r = \omega_0^2 r\,dr$, from which

$$\frac{v_r^2}{2} = \omega_0^2\frac{r^2}{2} + C.$$

At $r = r_0$, $v_r = v_0$, from which

$$C = \frac{v_0^2 - \omega_0^2 r_0^2}{2}, \quad \text{and}$$

$$v_r = \sqrt{v_0^2 + \omega_0^2(r^2 - r_0^2)}.$$

The transverse component is

$$v_\theta = r\left(\frac{d\theta}{dt}\right) = r\omega_0, \text{ from which}$$

$$\boxed{\mathbf{v} = \sqrt{v_0^2 + \omega_0^2(r^2 - r_0^2)}\mathbf{e}_r + r\omega_0\mathbf{e}_\theta}$$

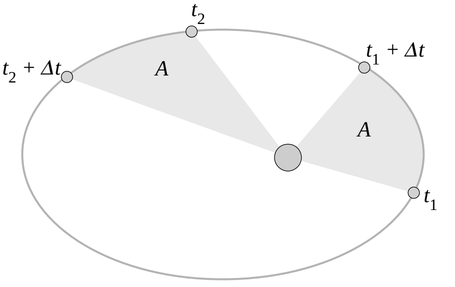
Problem 13.150 If the motion of a point in the x–y-plane is such that its transverse component of acceleration a_θ is zero, show that the product of its radial position and its transverse velocity is constant: $rv_\theta = \text{constant}$.

Solution: We are given that $a_\theta = r\alpha + 2v_r\omega = 0$. Multiply the entire relationship by r. We get

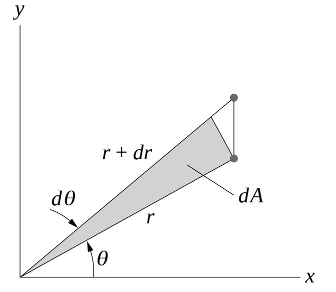
$$0 = (r^2\alpha + 2rv_r\omega) = \left(r^2\left(\frac{d\omega}{dt}\right) + 2r\left(\frac{dr}{dt}\right)_r\omega\right) = \frac{d}{dt}(r^2\omega).$$

Note that if $\dfrac{d}{dt}(r^2\omega) = 0$, then $r^2\omega = \text{constant}$. Now note that $v_\theta = r\omega$. We have $r^2\omega = r(r\omega) = rv_\theta = \text{constant}$. This was what we needed to prove.

Problem 13.151* From astronomical data, Kepler deduced that the line from the sun to a planet traces out equal areas in equal times (Fig. a). Show that this result follows from the fact that the transverse component a_θ of the planet's acceleration is zero. [When r changes by an amount dr and θ changes by an amount $d\theta$ (Fig. b), the resulting differential element of area is $dA = \frac{1}{2}r(r\,d\theta)$].



(a)



(b)

Solution: From the solution to Problem 13.150, $a_\theta = 0$ implies that

$$r^2\omega = r^2\frac{d\theta}{dt} = \text{constant}.$$

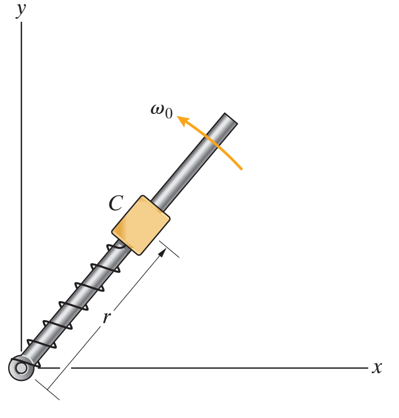
The element of area is

$$dA = \frac{1}{2}r(r\,d\theta),$$

or $$\frac{dA}{dt} = \frac{1}{2}r\left(r\frac{d\theta}{dt}\right) = \frac{1}{2}r^2\omega = \text{constant}.$$

Thus, if $\dfrac{dA}{dt} = \text{constant}$, then equal areas are swept out in equal times.

Problem 13.152 The bar rotates in the $x-y$ plane with constant angular velocity $\omega_0 = 12$ rad/s. The radial component of acceleration of the collar C (in m/s^2) is given as a function of the radial position in meters by $a_r = -8r$. When $r = 1$ m, the radial component of velocity of C is $v_r = 2$ m/s. Determine the velocity of C in terms of polar coordinates when $r = 1.5$ m.



Strategy: Use the chain rule to write the first term in the radial component of the acceleration as

$$\frac{d^2r}{dt^2} = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = \frac{dv_r}{dr}v_r$$

Solution: We have

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -(8\text{ rad/s}^2)r,$$

$$\frac{d^2r}{dt^2} = \left(\left[\frac{d\theta}{dt}\right]^2 - (8\text{ rad/s}^2)\right)r = ([12\text{ rad/s}]^2 - [8\text{ rad/s}^2])r = ([136\text{ rad/s}^2)r$$

Using the supplied strategy we can solve for the radial velocity

$$\frac{d^2r}{dt^2} = v_r\frac{dv_r}{dr} = (136\text{ rad/s}^2)r$$

$$\int_{2\text{ m/s}}^{v_r} v_r\,dv_r = (136\text{ rad/s}^2)\int_{1\text{ m}}^{1.5\text{ m}} r\,dr$$

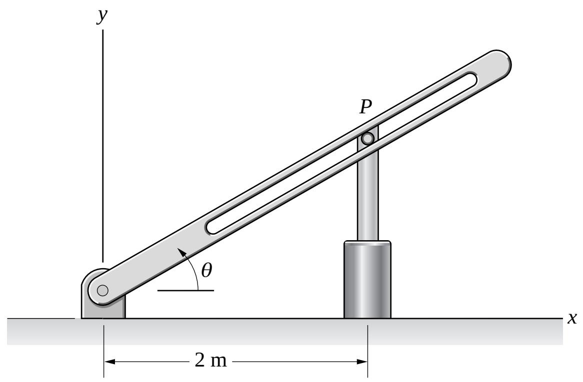
$$\frac{v_r^2}{2} - \frac{(2\text{ m/s})^2}{2} = (136\text{ rad/s}^2)\left(\frac{[1.5\text{ m}]^2}{2} - \frac{[1\text{ m}]^2}{2}\right)$$

Solving we find $v_r = 13.2$ m/s.

We also have $v_\theta = r\dfrac{d\theta}{dt} = (1.5\text{ m})(12\text{ rad/s}) = 18$ m/s.

Thus $\boxed{\mathbf{v} = (13.2\mathbf{e}_r + 18\mathbf{e}_\theta)\text{ m/s.}}$

Problem 13.153 The hydraulic actuator moves the pin P upward with constant velocity $\mathbf{v} = 2\mathbf{j}$ (m/s). Determine the velocity of the pin in terms of polar coordinates and the angular velocity of the slotted bar when $\theta = 35^\circ$.



Solution:

$$\mathbf{v}_P = 2\mathbf{j} \text{ (m/s)}$$

$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

Also, $\mathbf{r} = 2\mathbf{i} + y\mathbf{j}$ (m)

$$\mathbf{v} = \dot{y}\mathbf{j} \text{ (m/s)} = 2\mathbf{j} \text{ (m/s)}$$

$$\dot{r} = \dot{y}\sin\theta \quad \tan\theta = \frac{y}{x}$$

$$r\dot{\theta} = \dot{y}\cos\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = 35^\circ,$$

Solving, we get $y = 1.40$ m,

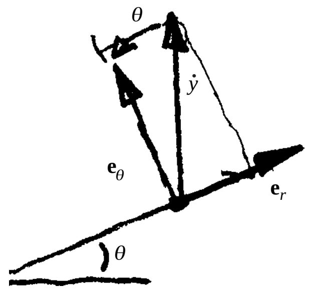
$$\dot{r} = 1.15 \text{ m/s},$$

$$r = 2.44 \text{ m},$$

$$\dot{\theta} = 0.671 \ \frac{\text{rad}}{\text{s}}$$

Hence $\mathbf{V} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$

$$\underline{\mathbf{V} = 1.15\mathbf{e}_r + 1.64\overline{e}_\theta \text{ (m/s)}}$$



Problem 13.154 The hydraulic actuator moves the pin P upward with constant velocity $\mathbf{v} = 2\mathbf{j}$ (m/s). Determine the acceleration of the pin in terms of polar coordinates and the angular acceleration of the slotted bar when $\theta = 35^\circ$.

Solution: From Problem 13.153

$\mathbf{V} = 2\mathbf{j}$ m/s, constant

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \equiv 0 \qquad \begin{array}{l} \dot{\theta} = \omega = 0.671 \text{ rad/s} \\ \theta = 35^\circ \end{array}$$

$$\dot{\theta} = \omega \quad \dot{y} = 2 \text{ m/s}$$

$$\ddot{\theta} = \alpha \quad \ddot{y} = 0$$

$$\tan\theta = \frac{y}{x} = \frac{y}{2}$$

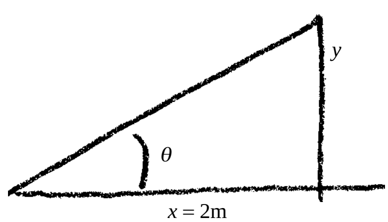
$$y = 2\tan\theta$$

$$\dot{y} = 2\sec^2\theta\dot{\theta}$$

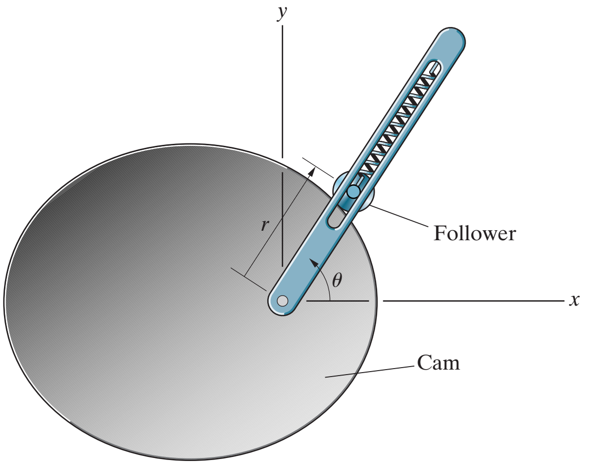
$$\ddot{y} = 2(2\sec\theta)(\sec\theta\tan\theta)\dot{\theta}^2 + 2\sec^2\theta\ddot{\theta}$$

$$\ddot{\theta} = \frac{[-2\sec\theta\tan\theta](\dot{\theta})^2}{\sec\theta}$$

$$\ddot{\theta} = 0.631 \text{ rad/s}^2$$



Problem 13.155 In Example 13.15, determine the velocity of the cam follower when $\theta = 135^\circ$ (a) in terms of polar coordinates and (b) in terms of cartesian coordinates.



Solution:

(a) $\theta = 135^\circ$, $\omega = d\theta/dt = 4 rad/s$, and $\alpha = 0$.

$$r = 0.15(1 + 0.5\cos\theta)^{-1}$$

$$= 0.232 \text{ m.}$$

$$\frac{dr}{dt} = 0.075\frac{d\theta}{dt}\sin\theta(1 + 0.5\cos\theta)^{-2}$$

$$= 0.508 \text{ m/s.}$$

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

$$= 0.508\mathbf{e}_r + 0.928\mathbf{e}_\theta \text{ (m/s).}$$

(b) $v_x = v_r\cos\theta - v_\theta\sin\theta$

$$= -1.015 \text{ m/s.}$$

$$v_y = v_r\sin\theta + v_\theta\cos\theta$$

$$= -0.297 \text{ m/s.}$$

Problem 13.156* In Example 13.15, determine the acceleration of the cam follower when $\theta = 135^\circ$ (a) in terms of polar coordinates and (b) in terms of cartesian coordinates.

Solution: See the solution of Problem 13.155.

(a) $$\frac{d^2r}{dt^2} = 0.075\left(\frac{d\theta}{dt}\right)^2\cos\theta(1 + 0.5\cos\theta)^{-2}$$

$$+ 0.075\left(\frac{d\theta}{dt}\right)^2\sin^2\theta(1 + 0.5\cos\theta)^{-3}$$

$$= 0.1905 \text{ m/s}^2.$$

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{e}_\theta$$

$$= -3.52\mathbf{e}_r + 4.06\mathbf{e}_\theta \text{ (m/s}^2\text{).}$$

(b) $a_x = a_r\cos\theta - a_\theta\sin\theta$

$$= -0.381 \text{ m/s}^2$$

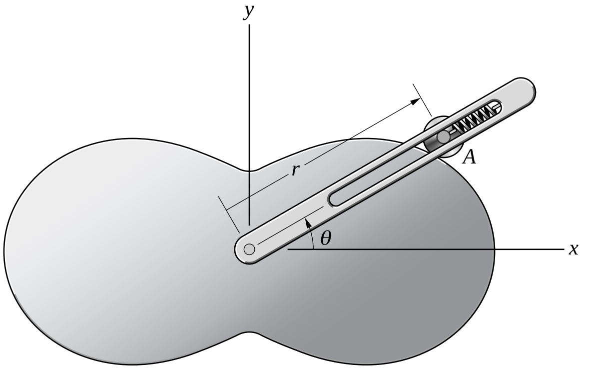
$$a_y = a_r\sin\theta + a_\theta\cos\theta$$

$$= -5.362 \text{ m/s}^2.$$

Problem 13.157 In the cam-follower mechanism, the slotted bar rotates with constant angular velocity $\omega = 10$ rad/s and the radial position of the follower A is determined by the profile of the stationary cam. The path of the follower is described by the polar equation

$$r = 1 + 0.5\cos(2\theta) \text{ m}.$$

Determine the velocity of the cam follower when $\theta = 30^\circ$ (a) in terms of polar coordinates and (b) in terms of cartesian coordinates.



Solution:

(a) $\theta = 30^\circ$, $\omega = d\theta/dt = 10$ rad/s, and $\alpha = 0$.

$$r = 1 + 0.5\cos 2\theta$$

$$= 1.25 \text{ m}.$$

$$\frac{dr}{dt} = -\frac{d\theta}{dt}\sin 2\theta$$

$$= -8.66 \text{ m/s}.$$

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$$

$$= -8.66\mathbf{e}_r + 12.5\mathbf{e}_\theta \text{ (m/s)}.$$

(b) $v_x = v_r \cos\theta - v_\theta \sin\theta$

$$= -13.75 \text{ m/s},$$

$$v_y = v_r \sin\theta + v_\theta \cos\theta$$

$$= 6.50 \text{ m/s}.$$

Problem 13.158* In Problem 13.157, determine the acceleration of the cam follower when $\theta = 30^\circ$ (a) in terms of polar coordinates and (b) in terms of cartesian coordinates.

Solution: See the solution of Problem 13.157.

(a) $\dfrac{d^2r}{dt^2} = -2\theta^2\cos 2\theta$

$$= -100 \text{ m/s}^2.$$

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{e}_\theta.$$

$$= -225\mathbf{e}_r - 173\mathbf{e}_\theta \text{ (m/s}^2\text{)}.$$

(b) $a_x = a_r\cos\theta - a_\theta\sin\theta$

$$= -108 \text{ m/s}^2,$$

$$a_y = a_r\sin\theta + a_\theta\cos\theta$$

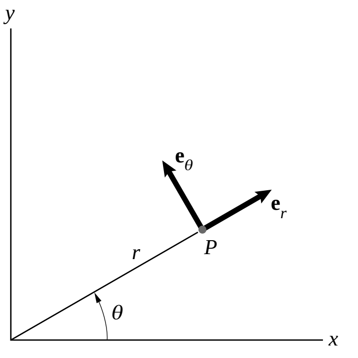
$$= -263 \text{ m/s}^2.$$

Problem 13.159 The cartesian coordinates of a point P in the $x-y$ plane are related to its polar coordinates of the point by the equations $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Show that the unit vectors $\mathbf{i}$, $\mathbf{j}$ are related to the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ by $\mathbf{i} = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta$ and $\mathbf{j} = \mathbf{e}_r\sin\theta + \mathbf{e}_\theta\cos\theta$.

(b) Beginning with the expression for the position vector of P in terms of cartesian coordinates, $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, derive Eq. (13.52) for the position vector in terms of polar coordinates.

(c) By taking the time derivative of the position vector of point P expressed in terms of cartesian coordinates, derive Eq. (13.47) for the velocity in terms of polar coordinates.



Solution:

(a) From geometry (see Figure), the radial unit vector is $\mathbf{e}_r = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$, and since the transverse unit vector is at right angles:

$$\mathbf{e}_\theta = \mathbf{i}\cos\left(\theta + \frac{\pi}{2}\right) + \mathbf{j}\sin\left(\theta + \frac{\pi}{2}\right) = -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta.$$

Solve for $\mathbf{i}$ by multiplying $\mathbf{e}_r$ by $\cos\theta$, $\mathbf{e}_\theta$ by $\sin\theta$, and subtracting the resulting equations:

$$\boxed{\mathbf{i} = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta}.$$

Solve for $\mathbf{j}$ by multiplying $\mathbf{e}_r$ by $\sin\theta$, and $\mathbf{e}_\theta$ by $\cos\theta$, and the results:

$$\boxed{\mathbf{j} = \mathbf{e}_r\sin\theta + \mathbf{e}_\theta\cos\theta}$$

(b) The position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} = r(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$. Use the results of Part (a) expressing $\mathbf{i}$, $\mathbf{j}$ in terms of $\mathbf{e}_r$, $\mathbf{e}_\theta$:

$$\boxed{\begin{aligned}\mathbf{r} &= r(\mathbf{e}_r\cos^2\theta - \mathbf{e}_\theta\cos\theta\sin\theta + \mathbf{e}_r\sin^2\theta + \mathbf{e}_\theta\sin\theta\cos\theta)\\ &= r\mathbf{e}_r\end{aligned}}$$

(c) The time derivatives are:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \mathbf{i}\left(\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)$$

$$+\mathbf{j}\left(\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right),$$

from which

$$\mathbf{v} = \frac{dr}{dt}(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) + r\frac{d\theta}{dt}(-\mathbf{i}\sin\theta + \mathbf{j}\cos\theta).$$

Substitute the results of Part (a)

$$\boxed{\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta = \frac{dr}{dt}\mathbf{e}_r + r\omega\mathbf{e}_\theta}$$

Problem 13.160 The airplane flies in a straight line at 643.6 km/h. The radius of its propellor is 1.524 m, and the propeller turns at 2000 rpm in the counterclockwise direction when seen from the front of the airplane. Determine the velocity and acceleration of a point on the tip of the propeller in terms of cylindrical coordinates. (Let the z-axis be oriented as shown in the figure.)

Solution: The speed is

$$v = 643.6 \times 1000/3600 = 178.8 \text{ m/s}$$

The angular velocity is

$$\omega = 2000\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 209.4 \text{ rad/s}.$$

The radial velocity at the propeller tip is zero. The transverse velocity is $v_\theta = \omega r = 319.2$ m/s. The velocity vector in cylindrical coordinates is

$$\boxed{\mathbf{v} = 319.2\mathbf{e}_\theta + 178.8\mathbf{e}_z \text{ (m/s)}}.$$

The radial acceleration is

$$a_r = -r\omega^2 = -1.524\,(209.4)^2 = -66825 \text{ m/s}^2.$$

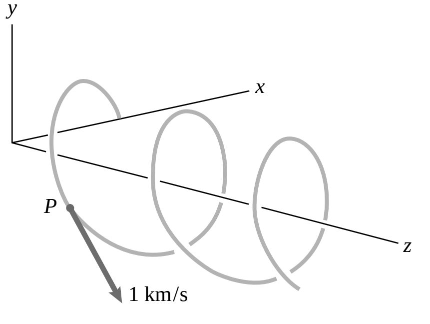
The transverse acceleration is

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) = 0,$$

since the propeller rotates at a constant angular velocity. The acceleration $a_z = 0$, since the airplane travels at constant speed. Thus

$$\mathbf{a} = -66825\mathbf{e}_r \text{ (m/s}^2\text{)}$$

Problem 13.161 A charged particle P in a magnetic field moves along the spiral path described by $r = 1$ m, $\theta = 2z$ rad, where z is in meters. The particle moves along the path in the direction shown with constant speed $|\mathbf{v}| = 1$ km/s. What is the velocity of the particle in terms of cylindrical coordinates?



Solution: The radial velocity is zero, since the path has a constant radius. The magnitude of the velocity is

$$v = \sqrt{r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = 1000 \text{ m/s}.$$

The angular velocity is $\dfrac{d\theta}{dt} = 2\dfrac{dz}{dt}$.

Substitute: $v = \sqrt{r^2\left(\dfrac{d\theta}{dt}\right)^2 + \dfrac{1}{4}\left(\dfrac{d\theta}{dt}\right)^2}$

$$= \left(\frac{d\theta}{dt}\right)\sqrt{r^2 + \frac{1}{4}} = \sqrt{1.25},$$

from which $\dfrac{d\theta}{dt} = \dfrac{1000}{\sqrt{1.25}} = 894.4$ rad/s,

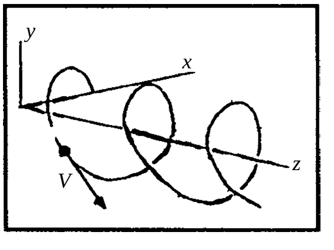
from which the transverse velocity is

$$v_\theta = r\left(\frac{d\theta}{dt}\right) = 894.4 \text{ m/s}.$$

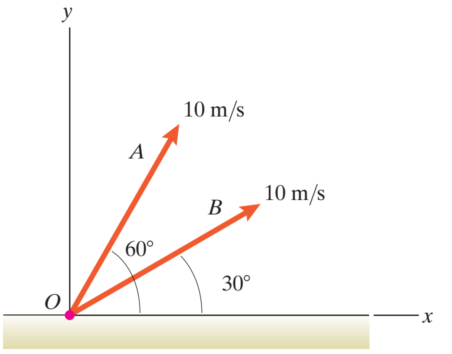
The velocity along the cylindrical axis is

$$\frac{dz}{dt} = \frac{1}{2}\left(\frac{d\theta}{dt}\right) = 447.2 \text{ m/s}.$$

The velocity vector: $\boxed{\mathbf{v} = 894.4\mathbf{e}_\theta + 447.2\mathbf{e}_z}$



Problem 13.162 At $t = 0$, two projectiles A and B are simultaneously launched from O with the initial velocities and elevation angles shown. Determine the velocity of projectile A relative to projectile B (a) at $t = 0.5$ s and (b) at $t = 1$ s.



Solution:

$$\mathbf{v}_A = -(9.81 \text{ m/s}^2\mathbf{j})t + (10 \text{ m/s})(\cos 60^\circ \mathbf{i} + \sin 60^\circ j)$$

$$\mathbf{v}_B = -(9.81 \text{ m/s}^2\mathbf{j})t + (10 \text{ m/s})(\cos 30^\circ \mathbf{i} + \sin 30^\circ j)$$

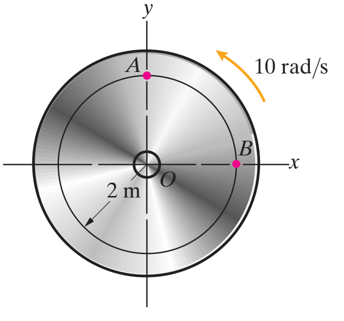
$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B = (10 \text{ m/s})(-0.366\mathbf{i} + 0.366\mathbf{j})$$

$$\mathbf{v}_{A/B} = (-3.66\mathbf{i} + 3.66\mathbf{j}) \text{ m/s}$$

Since $\mathbf{v}_{A/B}$ doesn't depend on time, the answer is the same for both times

$$\boxed{\mathbf{v}_{A/B} = (-3.66\mathbf{i} + 3.66\mathbf{j}) \text{ m/s}}$$

Problem 13.163 Relative to the earth-fixed coordinate system, the disk rotates about the fixed point O at 10 rad/s. What is the velocity of point A relative to point B at the instant shown?



Solution:

$$\mathbf{v}_A = -(10 \text{ rad/s})(2 \text{ m})\mathbf{i} = -(20 \text{ m/s})\mathbf{i}$$

$$\mathbf{v}_B = (10 \text{ rad/s})(2 \text{ m})\mathbf{j} = (20 \text{ m/s})\mathbf{j}$$

$$\boxed{\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B = (-20\mathbf{i} - 20\mathbf{j}) \text{ m/s}}$$

Problem 13.164 Relative to the earth-fixed coordinate system, the disk rotates about the fixed point O with a constant angular velocity of 10 rad/s. What is the acceleration of point A relative to point B at the instant shown?

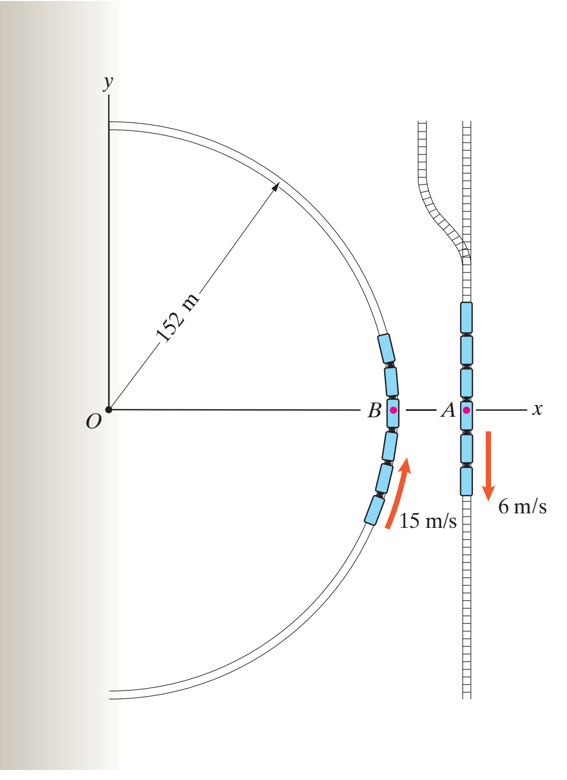
Solution:

$$\mathbf{a}_A = -(10 \text{ rad/s})^2(2 \text{ m})\mathbf{j} = -(200 \text{ m/s}^2)\mathbf{j}$$

$$\mathbf{a}_B = -(10 \text{ rad/s})^2(2 \text{ m})\mathbf{i} = -(200 \text{ m/s}^2)\mathbf{i}$$

$$\boxed{\mathbf{a}_{A/B} = \mathbf{a}_A - \mathbf{a}_B = (200\mathbf{i} - 200\mathbf{j}) \text{ m/s}^2}$$

Problem 13.165 The train on the circular track is traveling at 15 m/s. The train on the straight track is traveling at 6 m/s. In terms of the earth-fixed coordinate system shown, what is the velocity of passenger A relative to passenger B?



Solution:

$\mathbf{v}_A = (-6\mathbf{j})$ m/s, $\mathbf{v}_B = (15\mathbf{j})$ m/s

$\boxed{\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B = (-21\mathbf{j}) \text{ m/s}}$

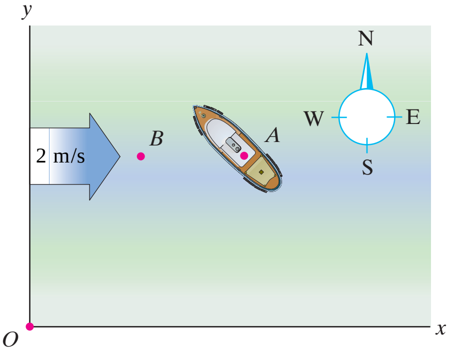
Problem 13.166 The train on the circular track is traveling at a constant speed of 15 m/s. The train on the straight track is traveling at 6 m/s and is increasing its speed at 0.6 m/s^2. In terms of the earth-fixed coordinate system shown, what is the acceleration of passenger A relative to passenger B?

Solution:

$\mathbf{a}_A = (-0.6\mathbf{j})$ m/s^2, $\mathbf{a}_B = -\dfrac{(15\text{ m/s})^2}{152\text{ m}}\mathbf{i} = (-1.48)$ m/s^2

$\boxed{\mathbf{a}_{A/B} = \mathbf{a}_A - \mathbf{a}_B = (1.48\mathbf{i} - 0.6\mathbf{j}) \text{ m/s}^2}$

Problem 13.167 In Active Example 13.16, suppose that the velocity of the current increases to 3 m/s flowing east. If the helmsman wants to travel northwest relative to the earth, what direction must he point the ship? What is the resulting magnitude of the ships' velocity relative to the earth?



Solution: The ship is moving at 5 m/s relative to the water. Use relative velocity concepts

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$v_A(-\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}) = (3 \text{ m/s})\mathbf{i} + (5 \text{ m/s})(-\cos\theta \mathbf{i} + \sin\theta \mathbf{j})$$

Breaking into components we have

$$-v_A \cos 45^\circ = (3 \text{ m/s}) - (5 \text{ m/s})\cos\theta,$$

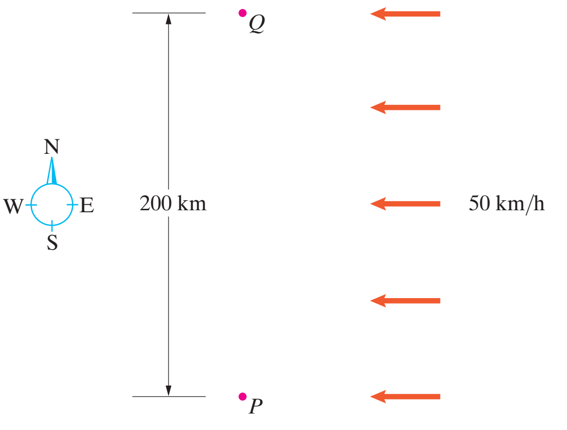
$$v_A \sin 45^\circ = (5 \text{ m/s})\sin\theta.$$

Solving these equations we find

$$\theta = 19.9^\circ, \quad v_A = 2.41 \text{ m/s}.$$

$$\boxed{70.1^\circ \text{ west of north}, \quad 2.41 \text{ m/s}.}$$

Problem 13.168 A private pilot wishes to fly from a city P to a city Q that is 200 km directly north of city P. The airplane will fly with an airspeed of 290 km/h. At the altitude at which the airplane will be flying, there is an east wind (that is, the wind's direction is west) with a speed of 50 km/h. What direction should the pilot point the airplane to fly directly from city P to city Q? How long will the trip take?



Solution: Assume an angle θ, measured ccw from the east.

$$\mathbf{V}_{Plane/Ground} = \mathbf{V}_{Plane/Air} + \mathbf{V}_{Air/Ground}$$

$$\mathbf{V}_{Plane/Air} = (290 \text{ km/h})(\cos\theta \mathbf{i} + \sin\theta \mathbf{j})$$

$$\mathbf{V}_{Air/Ground} = -(50 \text{ km/h})\mathbf{i}$$

$$\mathbf{V}_{Plane/Ground} = [(290\cos\theta - 50)\mathbf{i} + (290\sin\theta)\mathbf{j}] \text{ km/h}$$

We want the airplane to travel due north therefore

$$290\cos\theta - 50 = 0 \Rightarrow \theta = \cos^{-1}\left(\frac{50}{290}\right) = 80.07^\circ$$

Thus the heading is

$$\boxed{90^\circ - 80.07^\circ = 9.93^\circ \text{ east of north}}$$

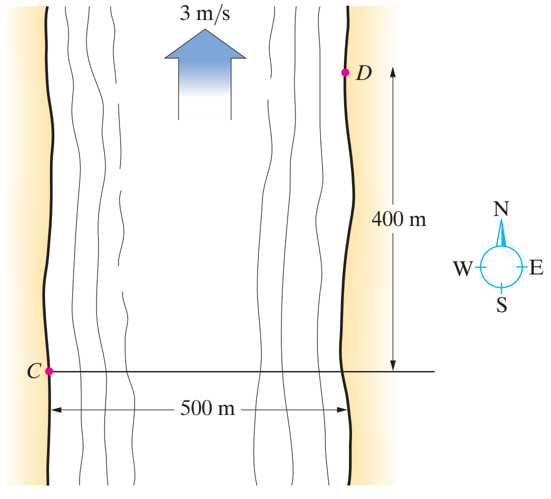
The ground speed is now

$$v = (290 \text{ km/h})\sin(80.1^\circ) = 285.6 \text{ km/h}$$

The time is

$$\boxed{t = \frac{d}{v} = \frac{200 \text{ km}}{285.6 \text{ km/h}} = 0.700 \text{ h} = 42.0 \text{ min}}$$

Problem 13.169 The river flows north at 3 m/s. (Assume that the current is uniform.) If you want to travel in a straight line from point C to point D in a boat that moves at a constant speed of 10 m/s relative to the water, in what direction should you point the boat? How long does it take to make the crossing?



Solution: Assume an angle θ, measured ccw from the east.

$$\mathbf{V}_{Boat/Ground} = \mathbf{V}_{Boat/Water} + \mathbf{V}_{Water/Ground}$$

$$\mathbf{V}_{Boat/Water} = (10\text{ m/s})(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$\mathbf{V}_{Water/Ground} = (3\text{ m/s})\,j$$

$$\mathbf{V}_{Boat/Ground} = [(10\cos\theta)\mathbf{i} + (3 + 10\sin\theta)\mathbf{j}]\text{ m/s}$$

We want the boat to travel at an angle $\tan\phi = \dfrac{400}{500}$

Therefore

$$\frac{3 + 10\sin\theta}{10\cos\theta} = \frac{400}{500} \Rightarrow \theta = 25.11^\circ$$

Thus the heading is

$$\boxed{25.11^\circ \text{ north of east}}$$

The ground speed is now

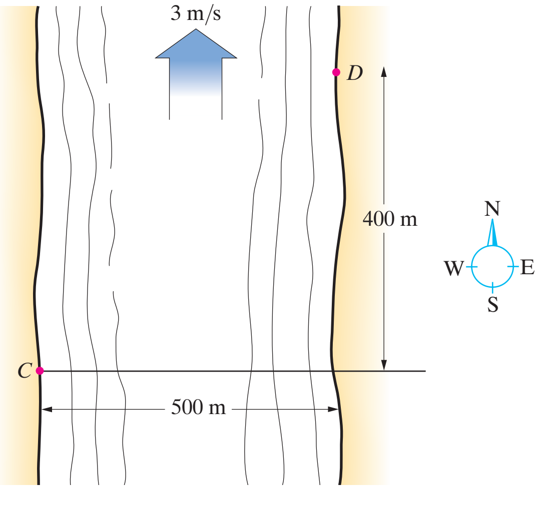
$$v = \sqrt{(10\cos\theta)^2 + (3 + 10\sin\theta)^2} = 11.60\text{ m/s}$$

The time is

$$\boxed{t = \frac{d}{v} = \frac{\sqrt{500^2 + 400^2}\text{ m}}{11.60\text{ m/s}} = 55.2\text{ s}}$$

Problem 13.170 The river flows north at 3 m/s (Assume that the current is uniform.) What minimum speed must a boat have relative to the water in order to travel in a straight line form point C to point D? How long does it take to make the crossing?

Strategy: Draw a vector diagram showing the relationships of the velocity of the river relative to the earth, the velocity of the boat relative to the river, and the velocity of the boat relative to the earth. See which direction of the velocity of the boat relative to the river causes it magnitude to be a minimum.



Solution: The minimum velocity occurs when the velocity of the boat relative to the water is 90° from the velocity of the boat relative to the earth.

$$\mathbf{v}_B = \mathbf{v}_W + \mathbf{v}_{B/W}.$$

The angle of travel is

$$\theta = \tan^{-1}\left(\frac{400\text{ m}}{500\text{ m}}\right) = 38.7^\circ.$$

Using the triangle that is drawn we have

$$v_{B/W} = (3\text{ m/s})\cos 38.7^\circ = 2.34\text{ m/s},$$

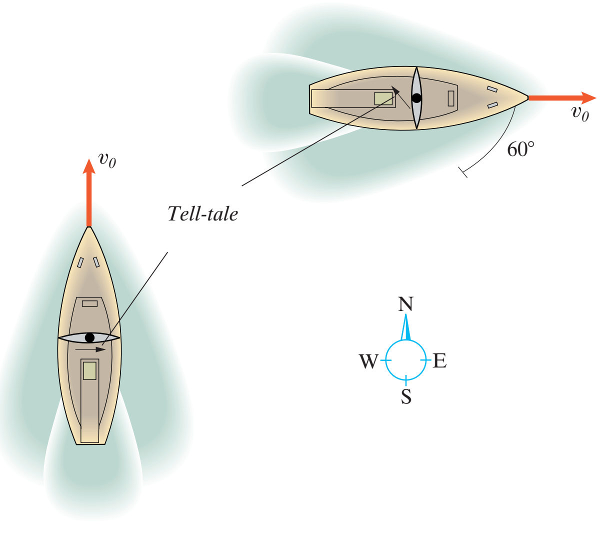
$$v_B = (3\text{ m/s})\sin 38.7^\circ = 1.87\text{ m/s}.$$

The time is given by

$$t = \frac{\sqrt{(400\text{ m})^2 + (500\text{ m})^2}}{v_B} = 342\text{ s}.$$

$$\boxed{v_{B/W} = 2.34\text{ m/s}.\ t = 342\text{ s}.}$$

Problem 13.171 Relative to the earth, the sailboat sails north with speed $v_0 = 6$ m/s and then sails east at the same speed. The tell-tale indicates the direction of the wind *relative to the boat*. Determine the direction and magnitude of the wind's velocity (in m/s) relative to the earth.



Solution:

$$\mathbf{v}_{wind/ground} = \mathbf{v}_{wind/boat} + \mathbf{v}_{boat/ground}$$

In position one we have

$$\mathbf{v}_{wind/ground} = v_{wind/boat1}\,\mathbf{i} + (6\text{ m/s})\mathbf{j}$$

In position two we have

$$\mathbf{v}_{wind/ground} = v_{wind/boat2}(-\cos 60^\circ\mathbf{i} + \sin 60^\circ\mathbf{j}) + (6\text{ m/s})\mathbf{i}$$

Since the wind has not changed these two expressions must be the same. Therefore

$$\left.\begin{aligned} v_{wind/boat1} &= -v_{wind/boat2}\cos 60^\circ + 6\text{ m/s} \\ 6\text{ m/s} &= v_{wind/boat2}\sin 60^\circ \end{aligned}\right\}$$

$$\Rightarrow \left\{\begin{aligned} v_{wind/boat1} &= 2.536\text{ m/s} \\ v_{wind/boat2} &= 6.928\text{ m/s} \end{aligned}\right.$$

Using either position one or position two we have

$$\boxed{\begin{aligned} \mathbf{v}_{wind/ground} &= (2.536\mathbf{i} + 6\mathbf{j})\text{ m/s} \\ v_{wind/ground} &= \sqrt{(2.536)^2 + (6)^2}\text{ m/s} = 6.51\text{ m/s} \\ \text{direction} &= \tan^{-1}\left(\frac{2.536}{6}\right) = 22.91^\circ\text{east of north} \end{aligned}}$$

Problem 13.172 Suppose you throw a ball straight up at 10 m/s and release it at 2 m above the ground. (a) What maximum height above the ground does the ball reach? (b) How long after release it does the ball hit the ground? (c) What is the magnitude of its velocity just before it hits the ground?

Solution: The equations of motion for the ball are

$$a_y = -g = -9.81\text{ m/s}^2,$$

$$v_y = v_{y0} - gt = 10 - 9.81t\text{ (m/s), and}$$

$$y = y_0 + v_{y0}t - gt^2/2 = 2 + 10t - 9.81t^2/2\text{ (m)}.$$

(a) The maximum height occurs when the velocity is zero. Call this time $t = t_1$. It can be obtained by setting velocity to zero, i.e., $v_y = 0 = 10 - 9.81t_1$ (m/s). Solving, we get $t_1 = 1.02$ s. Substituting this time into the y equation, we get a maximum height of $y_{MAX} = 7.10$ m.

(b) The ball hits the ground when $y = 0$ m. To find out when this occurs, we set $y = 0$ m into the y equation and solve for the time(s) when this occurs. There will be two times, one positive and one negative. Only the positive time has meaning for us. Let this time be $t = t_2$. The equation for t_2 is $y = 0 = 2 + 10t_2 - 9.81{t_2}^2/2$ (m). Solving, we get $t_2 = 2.22$ s.

(c) The velocity at impact is determined by substituting $t_2 = 2.22$ s into the equation for v_y. Doing this, we find that at impact, $v_y = -11.8$ m/s

Problem 13.173 Suppose that you must determine the duration of the yellow light at a highway intersection. Assume that cars will be approaching the intersection traveling as fast as 104.6 km/h, that the drivers' reaction times are as long as 0.5 s, and that cars can safely achieve a deceleration of at least $0.4g$.

(a) How long must the light remain yellow to allow drivers to come to a stop safely before the light turns red?
(b) What is the minimum distance cars must be from the intersection when the light turns yellow to come to a stop safely at the intersection?

Solution: The speed-time equation from initial speed to stop is given by integrating the equation $\dfrac{d^2s}{dt^2} = -0.4g$. From which

$$\frac{ds}{dt} = -0.4gt + V_0, \text{ and } s(t) = -0.2gt^2 + V_0 t,$$

where V_0 is the initial speed and the distance is referenced from the point where the brakes are applied. The initial speed is:

$$V_0 = 104.6 \times 1000/3600 = 29.05 \text{ m/s}.$$

(a) The time required to come to a full stop

$$\frac{ds(t_0)}{dt} = 0 \text{ is } t_0 = \frac{V_0}{0.4g} = \frac{29.05}{(0.4)(9.81)} = 7.40 \text{ s}.$$

The driver's reaction time increases this by 0.5 second, hence the total time to stop after observing the yellow light is $T = t_0 + 0.5 = 7.90$ s

(b) The distance traveled after brake application is traveled from brake application to full stop is given by

$$s(t)_0 = -0.2gt_0^2 + V_0 t_0, \text{ from which } s(t_0) = 107.6 \text{ m}.$$

The distance traveled during the reaction time is

$$d = V_0(0.5) = 29.05(0.5) = 14.53 \text{ m},$$

from which the total distance is

$$d_t = 29.05 + 14.53 = 43.58 \text{ m}$$

Problem 13.174 The acceleration of a point moving along a straight line is $a = 4t + 2$ m/s^2. When $t = 2$ s, its position is $s = 36$ m, and when $t = 4$ seconds, its position is $s = 90$ meters. What is its velocity when $t = 4$ s?

Solution: The position-time equation is given by integrating

$$\frac{d^2s}{dt^2} = 4t + 2, \text{ from which } \frac{ds}{dt} = 2t^2 + 2t + V_0, \text{ and}$$

$$s(t) = \left(\frac{2}{3}\right) t^3 + t^2 + V_0 t + d_0,$$

where V_0, d_0 are the initial velocity and position. From the problem conditions:

$$s(2) = \left(\frac{2}{3}\right) 2^3 + (2^2) + V_0(2) + d_0 = 36,$$

from which

$$(1) \quad 2V_0 + d_0 = \left(\frac{80}{3}\right). s(4) = \left(\frac{2}{3}\right) 4^3 + (4^2) + V_0(4) + d_0 = 90,$$

from which

$$(2) \quad 4V_0 + d_0 = \left(\frac{94}{3}\right).$$

Subtract (1) from (2) to obtain

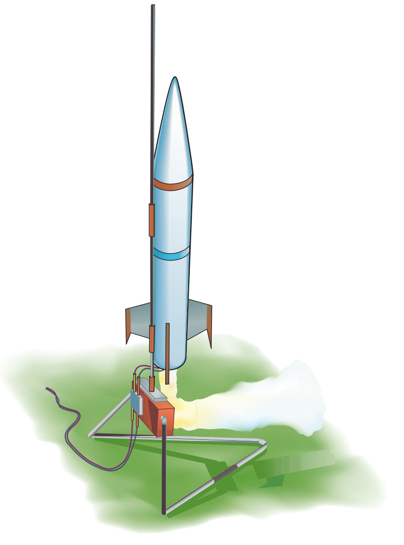
$$V_0 = \left(\frac{94 - 80}{6}\right) = 2.33 \text{ m/s}.$$

The velocity at $t = 4$ seconds is

$$\left[\frac{ds(t)}{dt}\right]_{t=4} = [2t^2 + 2t + V_0]_{t=4} = 32 + 8 + 2.33 = 42.33 \text{ m/s}$$

Problem 13.175 A model rocket takes off straight up. Its acceleration during the 2 s its motor burns is 25 m/s^2. Neglect aerodynamic drag, and determine

(a) the maximum velocity of the rocket during the flight and
(b) the maximum altitude the rocket reaches.



Solution: The strategy is to solve the equations of motion for the two phases of the flight: during burn $0 \le t \le 2$ s seconds, and after burnout: $t > 2$ s.

Phase 1: The acceleration is:

$$\frac{d^2s}{dt^2} = 25, \text{ from which } \frac{ds}{dt} = 25t, \text{ and } s(t) = 12.5t^2,$$

since the initial velocity and position are zero. The velocity at burnout is $V_{\text{burnout}} = (25)(2) = 50$ m/s. The altitude at burnout is $h_{\text{burnout}} = (12.5)(4) = 50$ m.

Phase 2. The acceleration is:

$$\frac{d^2s}{dt^2} = -g, \text{ from which } \frac{ds}{dt} = -g(t-2) + V_{\text{burnout}}(t \ge 2), \text{ and}$$

$$s(t) = -g(t-2)^2/2 + V_{\text{burnout}}(t-2) + h_{\text{burnout}}, (t \ge 2).$$

The velocity during phase 1 is constantly increasing because of the rocket's positive acceleration. Maximum occurs at burnout because after burnout, the rocket has negative acceleration and velocity constantly decreases until it reaches zero at maximum altitude. The velocity from maximum altitude to impact must be constantly increasing since the rocket is falling straight down under the action of gravity. Thus the maximum velocity during phase 2 occurs when the rocket impacts the ground. The issue of maximum velocity becomes this: is the velocity at burnout greater or less than the velocity at ground impact? The time of flight is given by $0 = -g(t_{\text{flight}} - 2)^2/2 + V_{\text{burnout}}(t_{\text{flight}} - 2) + h_{\text{burnout}}$, from which, in canonical form:

$$(t_{\text{flight}} - 2)^2 + 2b(t_{\text{flight}} - 2) + c = 0,$$

where $b = -(V_{\text{burnout}}/g)$ and $c = -(2h_{\text{burnout}}/g)$.

The solution $(t_{\text{flight}} - 2) = -b \pm \sqrt{b^2 - c} = 11.11, = -0.92$ s. Since the negative time is not allowed, the time of flight is $t_{\text{flight}} = 13.11$ s.

The velocity at impact is

$$V_{\text{impact}} = -g(t_{\text{flight}} - 2) + V_{\text{burnout}} = -59 \text{ m/s}$$

which is <u>higher in magnitude</u> than the velocity at burnout. The time of maximum altitude is given by

$$\frac{ds}{dt} = 0 = -g(t_{\max alt} - 2) + V_{\text{burnout}}, \quad \text{from which}$$

$$t_{\max alt} - 2 = \frac{V_{\text{burnout}}}{g} = 5.1 \text{ s}, \quad \text{from which}$$

$$t_{\max alt} = 7.1 \text{ s}.$$

The maximum altitude is

$$h_{\max} = -\frac{g}{2}(t_{\max alt} - 2)^2 + V_{\text{burnout}}(t_{\max alt} - 2) + h_{\text{burnout}} = 177.42 \text{ m}$$

Problem 13.176 In Problem 13.175, if the rocket's parachute fails to open, what is the total time of flight from takeoff until the rocket hits the ground?

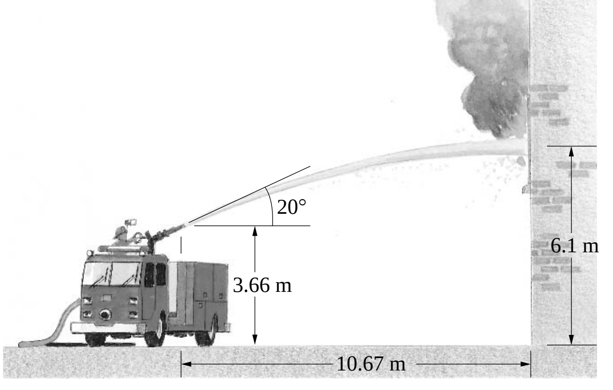
Solution: The solution to Problem 13.175 was (serendipitously) posed in a manner to yield the time of flight as a peripheral answer. The time of flight is given there as $t_{\text{flight}} = 13.11$ s

Problem 13.177 The acceleration of a point moving along a straight line is $a = -cv^3$, where c is a constant. If the velocity of the point is v_0, what distance does the point move before its velocity decreases to $\dfrac{v_0}{2}$?

Solution: The acceleration is $\dfrac{dv}{dt} = -cv^3$. Using the chain rule, $\dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = v\dfrac{dv}{ds} = -cv^3$. Separating variables and integrating: $\dfrac{dv}{v^2} = -c\,ds$, from which $-\dfrac{1}{v} = -cs + C$. At $s = 0$, $v = v_0$, from which $-\dfrac{1}{v} = -cs - \dfrac{1}{v_0}$, and $v = \dfrac{v_0}{1 + v_0 cs}$. Invert: $v_0 cs = \dfrac{v_0}{v} - 1$. When $v = \dfrac{v_0}{2}$, $\boxed{s = \left(\dfrac{1}{cv_0}\right)}$

Problem 13.178 Water leaves the nozzle at 20° above the horizontal and strikes the wall at the point indicated. What was the velocity of the water as it left the nozzle?

Strategy: Determine the motion of the water by treating each particle of water as a projectile.



Solution: Denote $\theta = 20^\circ$. The path is obtained by integrating the equations:

$$\frac{dv_y}{dt} = -g \text{ and } \frac{dv_x}{dt} = 0, \quad \text{from which}$$

$$\frac{dy}{dt} = -gt + V_n \sin\theta, \; \frac{dx}{dt} = V_n \cos\theta.$$

$$y = -\frac{g}{2}t^2 + (V_n \sin\theta)t + y_0.$$

$$x = (V_n \cos\theta)t + x_0.$$

Choose the origin at the nozzle so that $y_0 = 0$, and $x_0 = 0$. When the stream is $y(t_{\text{impact}}) = 6.1 - 3.66 = 2.44$ m, the time is

$$0 = -\frac{g}{2}(t_{\text{impact}})^2 + (V_n \sin\theta)t_{\text{impact}} - 2.44.$$

At this same time the horizontal distance is

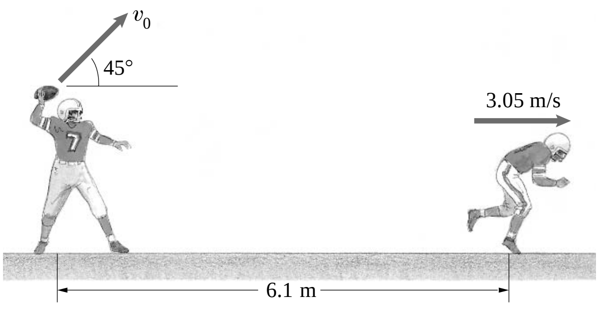
$$x(t_{\text{impact}}) = 10.67 = (V_n \cos\theta)t_{\text{impact}}, \text{ from which } t_{\text{impact}} = \frac{10.67}{V_n \cos\theta}.$$

Substitute:

$$0 = -\frac{g}{2}\left(\frac{10.67}{V_n \cos\theta}\right)^2 + 10.67 \tan\theta - 2.44,$$

from which $\boxed{V_n = \left(\dfrac{10.67}{\cos\theta}\right)\sqrt{\dfrac{g}{2(10.67\tan\theta - 2.44)}} = 20.9 \text{ m/s}}$

Problem 13.179 In practice, a quarterback throws the football with a velocity v_0 at 45° above the horizontal. At the same instant, a receiver standing 6.1 m in front of him starts running straight down field at 3.05 m/s and catches the ball. Assume that the ball is thrown and caught at the same height above the ground. What is the velocity v_0?



Solution: Denote $\theta = 45°$. The path is determined by integrating the equations;

$$\frac{d^2y}{dt^2} = -g, \; \frac{d^2x}{dt^2} = 0, \quad \text{from which}$$

$$\frac{dy}{dt} = -gt + v_0 \sin\theta, \; \frac{dx}{dt} = v_0 \cos\theta.$$

$$y = -\frac{g}{2}t^2 + (v_0 \sin\theta)t,$$

$$x = (v_0 \cos\theta)t,$$

where the origin is taken at the point where the ball leaves the quarterback's hand.

When the ball reaches the receiver's hands,

$$y = 0, \text{ from which } t_{\text{flight}} = \sqrt{\frac{2v_0 \sin\theta}{g}}.$$

At this time the distance down field is the distance to the receiver:

$$x = 3.05\, t_{\text{flight}} + 6.1. \text{ But also}$$

$$x = (v_0 \cos\theta)t_{\text{flight}}, \quad \text{from which}$$

$$t_{\text{flight}} = \frac{6.1}{(v_0 \cos\theta - 3.05)}.$$

Substitute:

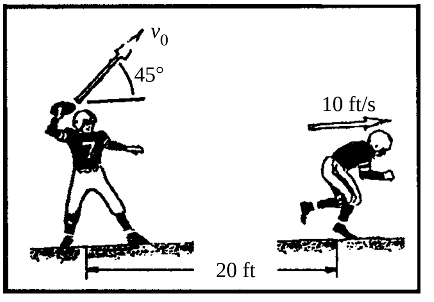
$$\frac{6.1}{(v_0 \cos\theta - 3.05)} = \sqrt{\frac{2v_0 \sin\theta}{g}}, \quad \text{from which}$$

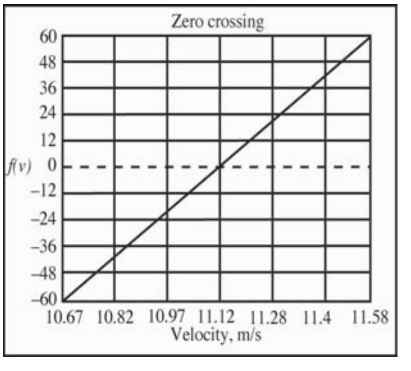
$$37.21 \text{ g} = 2v_0 \sin\theta (v_0 \cos\theta - 3.05)^2.$$

The function

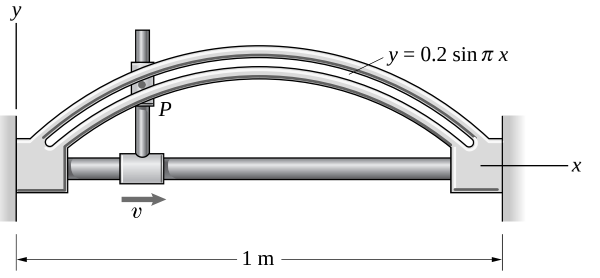
$$f(v_0) = 2v_0 \sin\theta(v_0 \cos\theta - 3.05)^2 - 37.21 \text{ g}$$

was graphed to find the zero crossing, and the result refined by iteration: $\boxed{v_0 = 11.12 \text{ m/s}}$. *Check*: The time of flight is $t = 1.27$ s and the distance down field that the quarterback throws the ball is $d = 3.87 + 6.1 = 10$ m, which seem reasonable for a short, "lob" pass. *check*.





Problem 13.180 The constant velocity $v = 2$ m/s. What are the magnitudes of the velocity and acceleration of point P when $x = 0.25$ m?



Solution: Let $x = 2t$ m/s. Then $x = 0.25$ m at $t = 0.125$ s. We know that $v_x = 2$ m/s and $a_x = 0$.

From

$$y = 0.2\sin(2\pi t), \quad \text{we obtain}$$

$$\frac{dy}{dt} = 0.4\pi\cos(2\pi t) \quad \text{and}$$

$$\frac{d^2y}{dt^2} = -0.8\pi^2\sin(2\pi t).$$

At $t = 0.125$ s,

$$y = 0.141 \text{ m and}$$

$$\frac{dy}{dt} = v_y = 0.889 \text{ m/s and}$$

$$\frac{d^2y}{dt^2} = a_y = -5.58 \text{ m/s.}$$

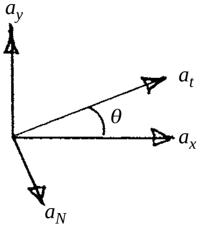
Therefore

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2} = 2.19 \text{ m/s},$$

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2} = 5.58 \text{ m/s}^2.$$

Problem 13.181 The constant velocity $v = 2$ m/s. What is the acceleration of point P in terms of normal and tangential components when $x = 0.25$ m?

Solution: See the solution of Problem 13.180. The angle θ between the x axis and the path is



$$\theta = \arctan\left(\frac{v_y}{v_x}\right) = \arctan\left(\frac{0.889}{2}\right) = 24.0^\circ. \quad \text{Then}$$

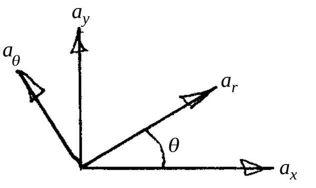
$$a_t = a_x\cos\theta + a_y\sin\theta = 0 + (-5.58)\sin 24.0^\circ = -2.27 \text{ m/s}^2,$$

$$a_N = a_x\sin\theta - a_y\cos\theta = 0 - (-5.58)\cos 24.0^\circ = 5.10 \text{ m/s}^2.$$

The instantaneous radius is

$$\rho = \frac{v_x^2 + v_y^2}{a_N} = \frac{(2)^2 + (0.889)^2}{5.10} = 0.939 \text{ m}.$$

Problem 13.182 The constant velocity $v = 2$ m/s. What is the acceleration of point P in terms of polar coordinates when $x = 0.25$ m?



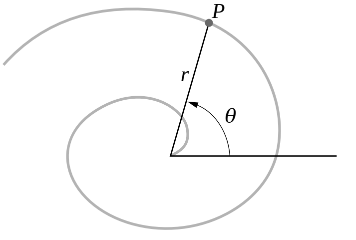
Solution: See the solution of Problem 13.192. The polar angle θ is

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{0.141}{0.25}\right) = 29.5^\circ. \quad \text{Then}$$

$$a_r = a_x \cos\theta + a_y \sin\theta = 0 + (-5.58)\sin 29.5^\circ = -2.75 \text{ m/s}^2,$$

$$a_\theta = -a_x \sin\theta + a_y \cos\theta = 0 + (-5.58)\cos 29.5^\circ = -4.86 \text{ m/s}^2.$$

Problem 13.183 A point P moves along the spiral path $r = (0.1)\theta$ m, where θ is in radians. The angular position $\theta = 2t$ rad, where t is in seconds, and $r = 0$ at $t = 0$. Determine the magnitudes of the velocity and acceleration of P at $t = 1$ s.



Solution: The path $r = 0.2t$ m, $\theta = 2t$ rad. The velocity components are

$$v_r = \frac{dr}{dt} = 0.2 \text{ m/s}, v_\theta = r\frac{d\theta}{dt} = (0.2t)2 = 0.4t.$$

At $t = 1$ seconds the magnitude of the velocity is

$$\boxed{|\mathbf{v}| = \sqrt{v_r^2 + v_\theta^2} = \sqrt{0.2^2 + 0.4^2} = 0.447 \text{ m/s}}$$

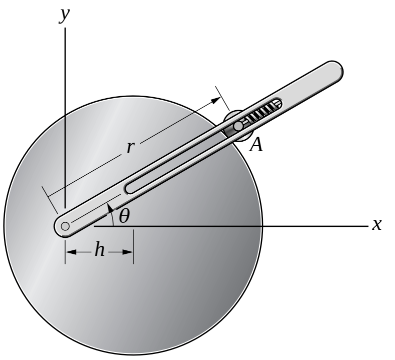
The acceleration components are:

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -(0.2t)(2^2) \text{ m/s}^2,$$

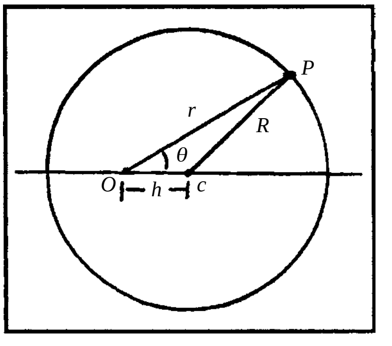
$$a_\theta = r\left(\frac{d^2\theta}{dt^2}\right) + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right) = 2(0.2)(2) = 0.8 \text{ m/s}^2.$$

The magnitude of the acceleration is $\boxed{|\mathbf{a}| = \sqrt{a_r^2 + a_\theta^2} = 1.13 \text{ m/s}^2}$

Problem 13.184 In the cam-follower mechanism, the slotted bar rotates with a constant angular velocity $\omega = 12$ rad/s, and the radial position of the follower A is determined by the profile of the stationary cam. The slotted bar is pinned a distance $h = 0.2$ m to the left of the center of the circular cam. The follower moves in a circular path of 0.42 m radius. Determine the velocity of the follower when $\theta = 40^\circ$ (a) in terms of polar coordinates, and (b) in terms of cartesian coordinates.



Solution:



(a) The first step is to get an equation for the path of the follower in terms of the angle θ. This can be most easily done by referring to the diagram at the right. Using the law of cosines, we can write $R^2 = h^2 + r^2 - 2hr\cos\theta$. This can be rewritten as $r^2 - 2hr\cos\theta + (h^2 - R^2) = 0$. We need to find the components of the velocity. These are $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$. We can differentiate the relation derived from the law of cosines to get $\dot{r}$. Carrying out this differentiation, we get $2r\dot{r} - 2h\dot{r}\cos\theta + 2hr\dot{\theta}\sin\theta = 0$. Solving for $\dot{r}$, we get

$$\dot{r} = \frac{hr\dot{\theta}\sin\theta}{(h\cos\theta - r)}.$$

Recalling that $\omega = \dot{\theta}$ and substituting in the numerical values, i.e., $R = 0.42$ m, $h = 0.2$ m, $\omega = 12$ rad/s, and $\theta = 40^\circ$, we get $r = 0.553$ m, $v_r = -2.13$ m/s, and $v_\theta = 6.64$ m/s

(b) The transformation to cartesian coordinates can be derived from $\mathbf{e_r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, and $\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$. Substituting these into $\mathbf{v} = v_r\mathbf{e_r} + v_\theta\mathbf{e}_\theta$, we get $\mathbf{v} = (v_r\cos\theta - v_\theta\sin\theta)\mathbf{i} + (v_r\sin\theta + v_\theta\cos\theta)\mathbf{j}$. Substituting in the numbers, $\mathbf{v} = -5.90\mathbf{i} + 3.71\mathbf{j}$ (m/s)

Problem 13.185* In Problem 13.184, determine the acceleration of the follower when $\theta = 40^\circ$ (a) in terms of polar coordinates and (b) in terms of cartesian coordinates.

Solution:

(a) Information from the solution to Problem 13.184 will be used in this solution. In order to determine the components of the acceleration in polar coordinates, we need to be able to determine all of the variables in the right hand sides of $a_r = \ddot{r} - r\dot{\theta}^2$ and that $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$. We already know everything except $\ddot{r}$ and $\ddot{\theta}$. Since ω is constant, $\ddot{\theta} = \dot{\omega} = 0$. We need only to find the value for r and the value for $\ddot{r}$ at $\theta = 40^\circ$. Substituting into the original equation for r, we find that $r = 0.553$ m at this position on the cam. To find $\ddot{r}$, we start with $\dot{r} = v_r$. Taking a derivative, we start with $r\dot{r} - h\dot{r}\cos\theta + hr\dot{\theta}\sin\theta = 0$ from Problem 13.184 (we divided through by 2). Taking a derivative with respect to time, we get

$$\ddot{r} = \frac{\dot{r}^2 + 2hr\dot{\theta}\sin\theta + hr\dot{\theta}^2\cos\theta + hr\ddot{\theta}\sin\theta}{(h\cos\theta - r)}.$$

Evaluating, we get $\ddot{r} = -46.17$ m/s^2. Substituting this into the equation for a_r and evaluating a_n, we get $a_r = -125.81$ m/s^2 and $a_\theta = -51.2$ m/s^2

(b) The transformation of cartesian coordinates can be derived from $\mathbf{e_r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$, and $\mathbf{e}_\theta = \sin\theta\mathbf{i} + \cos\theta\mathbf{j}$. Substituting these into $\mathbf{a} = a_r\mathbf{e_r} + a_e\mathbf{e}_\theta$, we get $\mathbf{a} = a_r\mathbf{e_r} + a_\theta\mathbf{e}_\theta$, we get $\mathbf{a} = (a_r\cos\theta - a_\theta\sin\theta)\mathbf{i} + (a_r\sin\theta + a_\theta\cos\theta)\mathbf{j}$. Substituting in the numbers, we get $\mathbf{a} = -63.46\ \mathbf{i} - 120.1\ \mathbf{j}$(m/s^2).

Problem 12.1 The value of π is 3.1415962654.... If C is the circumference of a circle and r is its radius, determine the value of r/C to four significant digits.

Solution:

$$C = 2\pi r \Rightarrow \frac{r}{C} = \frac{1}{2\pi} = 0.159154943.$$

To four significant digits we have $\boxed{\frac{r}{C} = 0.1592}$

Problem 12.2 The base of natural logarithms is $e = 2.718281828\ldots$

(a) Express e to five significant digits.
(b) Determine the value of e^2 to five significant digits.
(c) Use the value of e you obtained in part (a) to determine the value of e^2 to five significant digits.

[Part (c) demonstrates the hazard of using rounded-off values in calculations.]

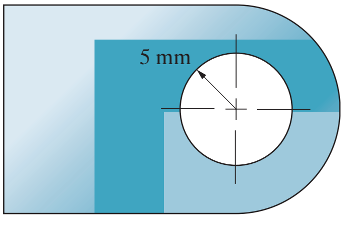
Solution: The value of e is: $e = 2.718281828$

(a) To five significant figures $\boxed{e = 2.7183}$

(b) e^2 to five significant figures is $\boxed{e^2 = 7.3891}$

(c) Using the value from part (a) we find $\boxed{e^2 = 7.3892}$ which is not correct in the fifth digit.

Problem 12.3 A machinist drills a circular hole in a panel with a nominal radius $r = 5$ mm. The actual radius of the hole is in the range $r = 5 \pm 0.01$ mm. (a) To what number of significant digits can you express the radius? (b) To what number of significant digits can you express the area of the hole?



Solution:

(a) The radius is in the range $r_1 = 4.99$ mm to $r_2 = 5.01$ mm. These numbers are not equal at the level of three significant digits, but they are equal if they are rounded off to two significant digits.

$\boxed{\text{Two: } r = 5.0 \text{ mm}}$

(b) The area of the hole is in the range from $A_1 = \pi r_1^2 = 78.226$ m^2 to $A_2 = \pi r_2^2 = 78.854$ m^2. These numbers are equal only if rounded to one significant digit:

$\boxed{\text{One: } A = 80 \text{ mm}^2}$

Problem 12.4 The opening in the soccer goal is 25 ft wide and 8 ft high, so its area is 24 ft $\times$ 8 ft = 192 ft^2. What is its area in m^2 to three significant digits?

Solution:

$$A = 192 \text{ ft}^2 \left(\frac{1 \text{ m}}{3.281 \text{ ft}}\right)^2 = 17.8 \text{ m}^2$$

$\boxed{A = 17.8 \text{ m}^2}$

Problem 12.5 The Burj Dubai, scheduled for completion in 2008, will be the world's tallest building with a height of 705 m. The area of its ground footprint will be 8000 m^2. Convert its height and footprint area to U.S. customary units to three significant digits.

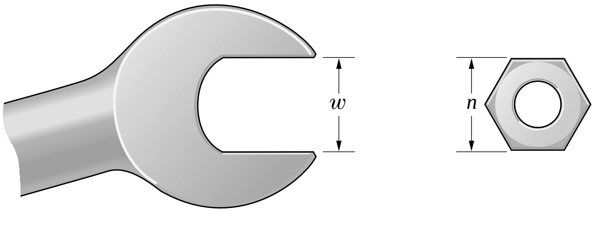
Solution:

$$h = 705\ \text{m}\left(\frac{3.281\ \text{ft}}{1\ \text{m}}\right) = 2.31 \times 10^3\ \text{ft}$$

$$A = 8000\ \text{m}^2\left(\frac{3.218\ \text{ft}}{1\ \text{m}}\right)^2 = 8.61 \times 10^4\ \text{ft}^2$$

$$\boxed{h = 2.31 \times 10^3\ \text{ft}, \quad A = 8.61 \times 10^4\ \text{ft}^2}$$

Problem 12.6 Suppose that you have just purchased a Ferrari F355 coupe and you want to know whether you can use your set of SAE (U.S. Customary Units) wrenches to work on it. You have wrenches with widths $w = 1/4$ in, 1/2 in, 3/4 in, and 1 in, and the car has nuts with dimensions $n = 5$ mm, 10 mm, 15 mm, 20 mm, and 25 mm. Defining a wrench to fit if w is no more than 2% larger than n, which of your wrenches can you use?



Solution: Convert the metric size n to inches, and compute the percentage difference between the metric sized nut and the SAE wrench. The results are:

$$5\ \text{mm}\left(\frac{1\ \text{inch}}{25.4\ \text{mm}}\right) = 0.19685..\ \text{in},\ \left(\frac{0.19685 - 0.25}{0.19685}\right)100$$

$$= -27.0\%$$

$$10\ \text{mm}\left(\frac{1\ \text{inch}}{25.4\ \text{mm}}\right) = 0.3937..\ \text{in},\ \left(\frac{0.3937 - 0.5}{0.3937}\right)100 = -27.0\%$$

$$15\ \text{mm}\left(\frac{1\ \text{inch}}{25.4\ \text{mm}}\right) = 0.5905..\ \text{in},\ \left(\frac{0.5905 - 0.5}{0.5905}\right)100 = +15.3\%$$

$$20\ \text{mm}\left(\frac{1\ \text{inch}}{25.4\ \text{mm}}\right) = 0.7874..\ \text{in},\ \left(\frac{0.7874 - 0.75}{0.7874}\right)100 = +4.7\%$$

$$25\ \text{mm}\left(\frac{1\ \text{inch}}{25.4\ \text{mm}}\right) = 0.9843..\ \text{in},\ \left(\frac{0.9843 - 1.0}{0.9843}\right)100 = -1.6\%$$

A negative percentage implies that the metric nut is smaller than the SAE wrench; a positive percentage means that the nut is larger then the wrench. Thus within the definition of the 2% fit, the 1 in wrench will fit the 25 mm nut. **The other wrenches cannot be used.**

Problem 12.7 Suppose that the height of Mt. Everest is known to be between 29,032 ft and 29,034 ft. Based on this information, to how many significant digits can you express the height (a) in feet? (b) in meters?

Solution:

(a) $h_1 = 29032$ ft

$h_2 = 29034$ ft

The two heights are equal if rounded off to four significant digits. The fifth digit is not meaningful.

$\boxed{\text{Four: } h = 29{,}030 \text{ ft}}$

(b) In meters we have

$$h_1 = 29032 \text{ ft}\left(\frac{1 \text{ m}}{3.281 \text{ ft}}\right) = 8848.52 \text{ m}$$

$$h_2 = 29034 \text{ ft}\left(\frac{1 \text{ m}}{3.281 \text{ ft}}\right) = 8849.13 \text{ m}$$

These two heights are equal if rounded off to three significant digits. The fourth digit is not meaningful.

$\boxed{\text{Three: } h = 8850 \text{ m}}$

Problem 12.8 The maglev (magnetic levitation) train from Shanghai to the airport at Pudong reaches a speed of 430 km/h. Determine its speed (a) in mi/h; (b) ft/s.

Solution:

(a) $v = 430\dfrac{\text{km}}{\text{h}}\left(\dfrac{0.6214 \text{ mi}}{1 \text{ km}}\right) = 267 \text{ mi/h}$ $\boxed{v = 267 \text{ mi/h}}$

(b) $v = 430\dfrac{\text{km}}{\text{h}}\left(\dfrac{1000 \text{ m}}{1 \text{ km}}\right)\left(\dfrac{1 \text{ ft}}{0.3048 \text{ m}}\right)\left(\dfrac{1 \text{ h}}{3600 \text{ s}}\right) = 392 \text{ ft/s}$

$\boxed{v = 392 \text{ ft/s}}$

Problem 12.9 In the 2006 Winter Olympics, the men's 15-km cross-country skiing race was won by Andrus Veerpalu of Estonia in a time of 38 minutes, 1.3 seconds. Determine his average speed (the distance traveled divided by the time required) to three significant digits (a) in km/h; (b) in mi/h.

Solution:

(a) $v = \dfrac{15 \text{ km}}{\left(38 + \dfrac{1.3}{60}\right) \text{min}}\left(\dfrac{60 \text{ min}}{1 \text{ h}}\right) = 23.7 \text{ km/h}$ $\boxed{v = 23.7 \text{ km/h}}$

(b) $v = (23.7 \text{ km/h})\left(\dfrac{1 \text{ mi}}{1.609 \text{ km}}\right) = 14.7 \text{ mi/h}$ $\boxed{v = 14.7 \text{ mi/h}}$

Problem 12.10 The Porsche's engine exerts 229 ft-lb (foot-pounds) of torque at 4600 rpm. Determine the value of the torque in N-m (Newton-meters).

Solution:

$$T = 229 \text{ ft-lb}\left(\frac{1 \text{ N}}{0.2248 \text{ lb}}\right)\left(\frac{1 \text{ m}}{3.281 \text{ ft}}\right) = 310 \text{ N-m} \qquad \boxed{T = 310 \text{ N-m}}$$

Problem 12.11 The *kinetic energy* of the man in Active Example 12.1 is defined by $\frac{1}{2}mv^2$, where m is his mass and v is his velocity. The man's mass is 68 kg and he is moving at 6 m/s, so his kinetic energy is $\frac{1}{2}(68 \text{ kg})(6 \text{ m/s})^2 = 1224 \text{ kg-m}^2/\text{s}^2$. What is his kinetic energy in U.S. Customary units?

Solution:

$$T = 1224 \text{ kg-m}^2/\text{s}^2\left(\frac{1 \text{ slug}}{14.59 \text{ kg}}\right)\left(\frac{1 \text{ ft}}{0.3048 \text{ m}}\right)^2 = 903 \text{ slug-ft}^2/\text{s}$$

$$\boxed{T = 903 \text{ slug-ft}^2/\text{s}}$$

Problem 12.12 The acceleration due to gravity at sea level in SI units is $g = 9.81 \text{ m/s}^2$. By converting units, use this value to determine the acceleration due to gravity at sea level in U.S. Customary units.

Solution: Use Table 1.2. The result is:

$$g = 9.81\left(\frac{\text{m}}{\text{s}^2}\right)\left(\frac{1 \text{ ft}}{0.3048 \text{ m}}\right) = 32.185\ldots\left(\frac{\text{ft}}{\text{s}^2}\right) = 32.2\left(\frac{\text{ft}}{\text{s}^2}\right)$$

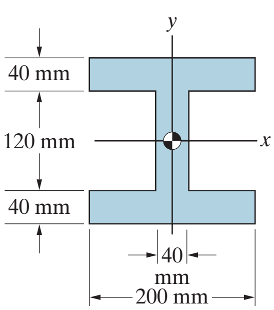
Problem 12.13 A *furlong per fortnight* is a facetious unit of velocity, perhaps made up by a student as a satirical comment on the bewildering variety of units engineers must deal with. A furlong is 660 ft (1/8 mile). A fortnight is 2 weeks (14 nights). If you walk to class at 2 m/s, what is your speed in furlongs per fortnight to three significant digits?

Solution:

$$v = 2 \text{ m/s}\left(\frac{1 \text{ ft}}{0.3048 \text{ m}}\right)\left(\frac{1 \text{ furlong}}{660 \text{ ft}}\right)\left(\frac{3600 \text{ s}}{\text{hr}}\right)\left(\frac{24 \text{ hr}}{1 \text{ day}}\right)\left(\frac{14 \text{ day}}{1 \text{ fortnight}}\right)$$

$$\boxed{v = 12{,}000\frac{\text{furlongs}}{\text{fortnight}}}$$

Problem 12.14 Determine the cross-sectional area of the beam (a) in m^2; (b) in in^2.



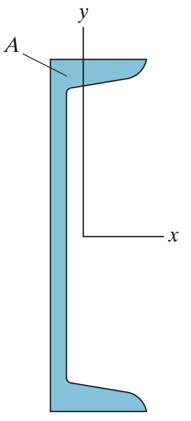
Solution:

$$A = (200 \text{ mm})^2 - 2(80 \text{ mm})(120 \text{ mm}) = 20800 \text{ mm}^2$$

(a) $A = 20800 \text{ mm}^2\left(\frac{1 \text{ m}}{1000 \text{ mm}}\right)^2 = 0.0208 \text{ m}^2 \qquad \boxed{A = 0.0208 \text{ m}^2}$

(b) $A = 20800 \text{ mm}^2\left(\frac{1 \text{ in}}{25.4 \text{ mm}}\right)^2 = 32.2 \text{ in}^2 \qquad \boxed{A = 32.2 \text{ in}^2}$

Problem 12.15 The cross-sectional area of the C12×30 American Standard Channel steel beam is $A = 8.81\ \text{in}^2$. What is its cross-sectional area in mm^2?



Solution:

$$\boxed{A = 8.81\ \text{in}^2 \left(\frac{25.4\ \text{mm}}{1\ \text{in}}\right)^2 = 5680\ \text{mm}^2}$$

Problem 12.16 A pressure transducer measures a value of $300\ \text{lb/in}^2$. Determine the value of the pressure in pascals. A pascal (Pa) is one newton per meter squared.

Solution: Convert the units using Table 12.2 and the definition of the Pascal unit. The result:

$$300\left(\frac{\text{lb}}{\text{in}^2}\right)\left(\frac{4.448\ \text{N}}{1\ \text{lb}}\right)\left(\frac{12\ \text{in}}{1\ \text{ft}}\right)^2\left(\frac{1\ \text{ft}}{0.3048\ \text{m}}\right)^2$$

$$= 2.0683\ldots(10^6)\left(\frac{\text{N}}{\text{m}^2}\right) = 2.07(10^6)\ \text{Pa}$$

Problem 12.17 A horsepower is 550 ft-lb/s. A watt is 1 N-m/s. Determine how many watts are generated by the engines of the passenger jet if they are producing 7000 horsepower.

Solution:

$$P = 7000\ \text{hp}\left(\frac{550\ \text{ft-lb/s}}{1\ \text{hp}}\right)\left(\frac{1\ \text{m}}{3.281\ \text{ft}}\right)\left(\frac{1\ \text{N}}{0.2248\ \text{lb}}\right) = 5.22\times10^6\ \text{W}$$

$$\boxed{P = 5.22\times10^6\ \text{W}}$$

Problem 12.18 Distributed loads on beams are expressed in units of force per unit length. If the value of a distributed load is 400 N/m, what is its value in lb/ft?.

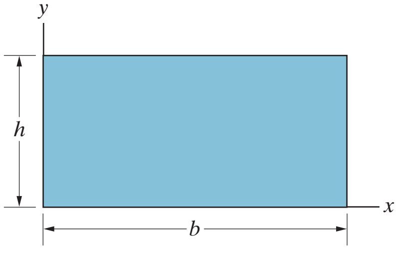
Solution:

$$w = 400\ \text{N/m}\left(\frac{0.2248\ \text{lb}}{1\ \text{N}}\right)\left(\frac{1\ \text{m}}{3.281\ \text{ft}}\right) = 27.4\ \text{lb/ft} \quad \boxed{w = 27.4\ \text{lb/ft}}$$

Problem 12.19 The moment of inertia of the rectangular area about the x axis is given by the equation

$$I = \frac{1}{3}bh^3.$$

The dimensions of the area are $b = 200$ mm and $h = 100$ mm. Determine the value of I to four significant digits in terms of (a) mm^4; (b) m^4; (c) in^4.



Solution:

(a) $\boxed{I = \frac{1}{3}(200\text{ mm})(100\text{ mm})^3 = 66.7 \times 10^6\text{ mm}^4}$

(b) $\boxed{I = 66.7 \times 10^6\text{ mm}^4 \left(\frac{1\text{ m}}{1000\text{ mm}}\right)^4 = 66.7 \times 10^{-6}\text{ m}^4}$

(c) $\boxed{I = 66.7 \times 10^6\text{ mm}^4 \left(\frac{1\text{ in}}{25.4\text{ mm}}\right)^4 = 160\text{ in}^4}$

Problem 12.20 In Example 12.3, instead of Einstein's equation consider the equation $L = mc$, where the mass m is in kilograms and the velocity of light c is in meters per second. (a) What are the SI units of L? (b) If the value of L in SI units is 12, what is the value in U.S. Customay base units?

Solution:

(a) $L = mc \Rightarrow \boxed{\text{Units}(L) = \text{kg-m/s}}$

(b) $L = 12\text{ kg-m/s}\left(\frac{0.0685\text{ slug}}{1\text{ kg}}\right)\left(\frac{3.281\text{ ft}}{1\text{ m}}\right) = 2.70\text{ slug-ft/s}$

$\boxed{L = 2.70\text{ slug-ft/s}}$

Problem 12.21 The equation

$$\sigma = \frac{My}{I}$$

is used in the mechanics of materials to determine normal stresses in beams.

(a) When this equation is expressed in terms of SI base units, M is in newton-meters (N-m), y is in meters (m), and I is in meters to the fourth power (m^4). What are the SI units of σ?

(b) If $M = 2000$ N-m, $y = 0.1$ m, and $I = 7 \times 10^{-5}\text{ m}^4$, what is the value of σ in U.S. Customary base units?

Solution:

(a) $\boxed{\sigma = \frac{My}{I} = \frac{(\text{N-m})\text{m}}{\text{m}^4} = \frac{\text{N}}{\text{m}^2}}$

(b) $$\boxed{\begin{aligned}\sigma &= \frac{My}{I} = \frac{(2000\text{ N-m})(0.1\text{ m})}{7 \times 10^{-5}\text{ m}^4}\left(\frac{1\text{ lb}}{4.448\text{ N}}\right)\left(\frac{0.3048\text{ m}}{\text{ft}}\right)^2 \\ &= 59{,}700\frac{\text{lb}}{\text{ft}^2}\end{aligned}}$$

Problem 12.22 The acceleration due to gravity on the surface of the moon is 1.62 m/s^2. (a) What would be the mass of the C-clamp in Active Example 12.4 be on the surface of the moon? (b) What would the weight of the C-clamp in newtons be on the surface of the moon?

Solution:

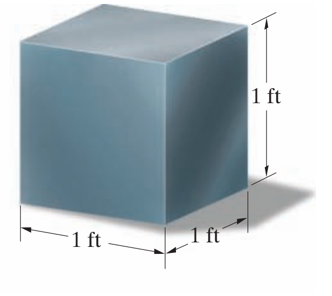
(a) The mass does not depend on location. The mass in kg is

$0.0272\text{ slug}\left(\frac{14.59\text{ kg}}{1\text{ slug}}\right) = 0.397\text{ kg}$ $\boxed{\text{mass} = 0.397\text{ kg}}$

(b) The weight on the surface of the moon is

$W = mg = (0.397\text{ kg})(1.62\text{ m/s}^2) = 0.643\text{ N}$ $\boxed{W = 0.643\text{ N}}$

Problem 12.23 The 1 ft × 1 ft × 1 ft cube of iron weighs 490 lb at sea level. Determine the weight in newtons of a 1 m × 1 m × 1 m cube of the same material at sea level.



Solution: The weight density is $\gamma = \dfrac{490\text{ lb}}{1\text{ ft}^3}$

The weight of the 1 m^3 cube is:

$$\boxed{W = \gamma V = \left(\frac{490\text{ lb}}{1\text{ ft}^3}\right)(1\text{ m})^3\left(\frac{1\text{ ft}}{0.3048\text{ m}}\right)^3\left(\frac{1\text{ N}}{0.2248\text{ lb}}\right) = 77.0\text{ kN}}$$

Problem 12.24 The area of the Pacific Ocean is 64,186,000 square miles and its average depth is 12,925 ft. Assume that the weight per unit volume of ocean water is 64 lb/ft^3. Determine the mass of the Pacific Ocean (a) in slugs; (b) in kilograms.

Solution: The volume of the ocean is

$$V = (64{,}186{,}000\text{ mi}^2)(12{,}925\text{ ft})\left(\frac{5{,}280\text{ ft}}{1\text{ mi}}\right)^2 = 2.312\times10^{19}\text{ ft}^3$$

(a) $$\boxed{m = \rho V = \left(\frac{64\text{ lb/ft}^3}{32.2\text{ ft/s}^2}\right)(2.312\times10^{19}\,ft^3) = 4.60\times10^{19}\text{ slugs}}$$

(b) $$\boxed{m = (4.60\times10^{19}\text{ slugs})\left(\frac{14.59\text{ kg}}{1\text{ slug}}\right) = 6.71\times10^{20}\text{ kg}}$$

Problem 12.25 The acceleration due to gravity at sea level is $g = 9.81\text{ m/s}^2$. The radius of the earth is 6370 km. The universal gravitational constant is $G = 6.67\times10^{-11}\text{ N-m}^2/\text{kg}^2$. Use this information to determine the mass of the earth.

Solution: Use Eq. (12.3) $a = \dfrac{Gm_E}{R^2}$. Solve for the mass,

$$m_E = \frac{gR^2}{G} = \frac{(9.81\text{ m/s}^2)(6370\text{ km})^2\left(10^3\dfrac{\text{m}}{\text{km}}\right)^2}{6.67(10^{-11})\left(\dfrac{\text{N-m}^2}{\text{kg}^2}\right)}$$

$$= 5.9679\ldots(10^{24})\text{ kg} = 5.97(10^{24})\text{ kg}$$

Problem 12.26 A person weighs 800 N sea level. The radius of the earth is 6372 km. What force is exerted on the person by the gravitational attraction of the earth if he is in a space station in orbit 322 km above the surface of the earth?

Solution: Use Eq. (12.5).

$$W = mg\left(\frac{R_E}{r}\right)^2 = \left(\frac{W_E}{g}\right)g\left(\frac{R_E}{R_E+H}\right)^2 = W_E\left(\frac{6372}{6372+322}\right)^2$$

$$= (800)(0.9519) = 262\text{ N}$$

Problem 12.27 The acceleration due to gravity on the surface of the moon is 1.62 m/s^2. The moon's radius is $R_M = 1738$ km.

(a) What is the weight in newtons on the surface of the moon of an object that has a mass of 10 kg?
(b) Using the approach described in Example 12.5, determine the force exerted on the object by the gravity of the moon if the object is located 1738 km above the moon's surface.

Solution:

(a) $W = mg_M = (10\text{ kg})(1.26\text{ m/s}^2) = 12.6\text{ N}$ $\boxed{W = 12.6\text{ N}}$

(b) Adapting equation 1.4 we have $a_M = g_M\left(\dfrac{R_M}{r}\right)^2$. The force is then

$$F = ma_M = (10\text{ kg})(1.62\text{m/s}^2)\left(\frac{1738\text{ km}}{1738\text{ km}+1738\text{ km}}\right)^2 = 4.05\text{ N}$$

$\boxed{F = 4.05\text{ N}}$

Problem 12.28 If an object is near the surface of the earth, the variation of its weight with distance from the center of the earth can often be neglected. The acceleration due to gravity at sea level is $g = 9.81\ \text{m/s}^2$. The radius of the earth is 6370 km. The weight of an object at sea level is mg, where m is its mass. At what height above the earth does the weight of the object decrease to 0.99 mg?

Solution: Use a variation of Eq. (12.5).

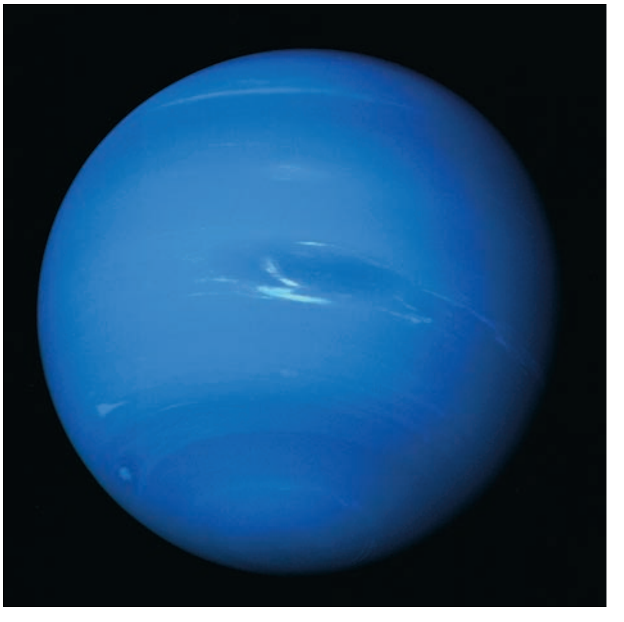
$$W = mg\left(\frac{R_E}{R_E + h}\right)^2 = 0.99\ \text{mg}$$

Solve for the radial height,

$$h = R_E\left(\frac{1}{\sqrt{0.99}} - 1\right) = (6370)(1.0050378 - 1.0)$$

$$= 32.09\ldots\text{km} = 32{,}100\ \text{m} = 32.1\ \text{km}$$

Problem 12.29 The planet Neptune has an equatorial diameter of 49,532 km and its mass is 1.0247×10^{26} kg. If the planet is modeled as a homogeneous sphere, what is the acceleration due to gravity at its surface? (The universal gravitational constant is $G = 6.67 \times 10^{-11}\ \text{h-m}^2/\text{kg}^2$.)



Solution:

We have: $W = G\dfrac{m_N m}{r_N^2} = \left(G\dfrac{m_N}{r^2}\right)m \Rightarrow g_N = G\dfrac{m_N}{r_N^2}$

Note that the radius of Neptune is $r_N = \frac{1}{2}(49{,}532\ \text{km}) = 24{,}766\ \text{km}$

Thus $$g_N = \left(6.67 \times 10^{-11}\frac{\text{N-m}^2}{\text{kg}^2}\right)\left(\frac{1.0247 \times 10^{26}\ \text{kg}}{(24766\ \text{km})^2}\right)$$

$$\times\left(\frac{1\ \text{km}}{1000\ \text{m}}\right)^2 = 11.1\ \text{m/s}^2$$

$$\boxed{g_N = 11.1\ \text{m/s}^2}$$

Problem 12.30 At a point between the earth and the moon, the magnitude of the force exerted on an object by the earth's gravity equals the magnitude of the force exerted on the object by the moon's gravity. What is the distance from the center of the earth to that point to three significant digits? The distance from the center of the earth to the center of the moon is 383,000 km, and the radius of the earth is 6370 km. The radius of the moon is 1738 km, and the acceleration due to gravity at its surface is 1.62 m/s^2.

Solution: Let r_{Ep} be the distance from the Earth to the point where the gravitational accelerations are the same and let r_{Mp} be the distance from the Moon to that point. Then, $r_{Ep} + r_{\text{Mp}} = r_{EM} = 383{,}000$ km. The fact that the gravitational attractions by the Earth and the Moon at this point are equal leads to the equation

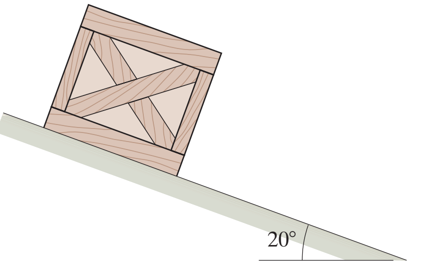
$$g_E\left(\frac{R_E}{r_{Ep}}\right)^2 = g_M\left(\frac{R_M}{r_{\text{Mp}}}\right)^2,$$

where $r_{EM} = 383{,}000$ km. Substituting the correct numerical values leads to the equation

$$9.81\left(\frac{\text{m}}{\text{s}^2}\right)\left(\frac{6370\ \text{km}}{r_{Ep}}\right)^2 = 1.62\left(\frac{\text{m}}{\text{s}^2}\right)\left(\frac{1738\ \text{km}}{r_{EM} - r_{Ep}}\right)^2,$$

where r_{Ep} is the only unknown. Solving, we get $r_{Ep} = 344{,}770$ km $= 345{,}000$ km.

Problem 14.1 In Active Example 14.1, suppose that the coefficient of kinetic friction between the crate and the inclined surface is $\mu_k = 0.12$. Determine the distance the crate has moved down the inclined surface at $t = 1$ s.



Solution: There are three unknowns in this problem: N, f, and a. We will first assume that the crate does not slip. The governing equations are

$$\Sigma F \searrow: (445\ \text{N}) \sin 20^\circ - f = \left(\frac{445\ \text{N}}{9.81\ \text{m/s}^2}\right) a$$

$$\Sigma F \nearrow: N - (445\ \text{N}) \cos 20^\circ = 0$$

No slip: $a = 0$

Solving, we find that $N = 418.2$ N, $f = 152.2$ N, $a = 0$.

To check the no slip assumption, we calculate the maximum friction force

$$f_{\max} = \mu_s N = 0.2(418.2\ \text{N}) = 83.64\ \text{N}.$$

Since $f > f_{\max}$, we conclude that our no slip assumption is false. The governing equations are now known to be

$$\Sigma F \searrow: (445\ \text{N}) \sin 20^\circ - f = \left(\frac{445\ \text{N}}{9.81\ \text{m/s}^2}\right) a$$

$$\Sigma F \nearrow: N - (445\ \text{N}) \cos 20^\circ = 0$$

Slip: $f = 0.12\ N$

Solving we have $N = 418.2$ N, $f = 50.2$ N, $a = 2.25$ m/s^2.

To find the distance we have $d = \frac{1}{2}at^2 = \frac{1}{2}(2.25\ \text{m/s}^2)\ (1\ \text{s})^2 = 1.13$ m.

$$\boxed{d = 1.13\ \text{m.}}$$

Problem 14.2 The mass of the Sikorsky UH-60A helicopter is 9300 kg. It takes off vertically with its rotor exerting a constant upward thrust of 112 kN.

(a) How fast is the helicopter rising 3 s after it takes off?
(b) How high has it risen 3 s after it takes off?

Strategy: Be sure to draw the free-body diagram of the helicopter.

Solution: The equation of motion is

$$\Sigma F : 112\text{ kN} - 9.3(9.81)\text{ kN}$$

$$= (9{,}300\text{ kg})a$$

Solving, we find that

$$a = 2.23\text{ m/s}^2.$$

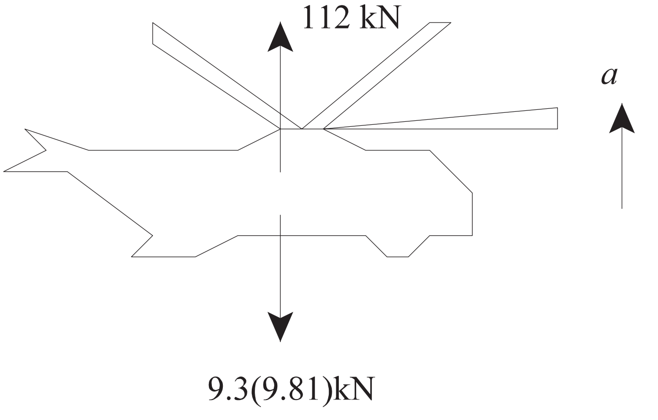
Using kinematics we can answer the questions

$$a = 2.23\text{ m/s}^2,$$

$$v = at = (2.23\text{ m/s}^2)(3\text{ s}) = 6.70\text{ m/s},$$

$$h = \frac{1}{2}at^2 = \frac{1}{2}(2.23\text{ m/s}^2)(3\text{ s})^2 = 10.0\text{ m}.$$

$$\boxed{(a)\ 6.70\text{ m/s},\quad (b)\ 10.0\text{ m}.}$$



Problem 14.3 The mass of the Sikorsky UH-60A helicopter is 9,300 kg. It takes off vertically at $t = 0$. The pilot advances the throttle so that the upward thrust of its engine (in kN) is given as a function of time in seconds by $T = 100 + 2t^2$.

(a) How fast is the helicopter rising 3 s after it takes off?
(b) How high has it risen 3 s after it takes off?

Solution: The equation of motion is

$$\Sigma F : 100\text{ kN} + 2\text{ kN}\left(\frac{t}{s}\right)^2$$

$$-9.3(9.81)\text{ kN} = (9,300\text{ kg})a$$

Solving, we find that

$$a = (0.943\text{ m/s}^2) + (0.215\text{ m/s}^4)t^2.$$

Using kinematics we can answer the questions

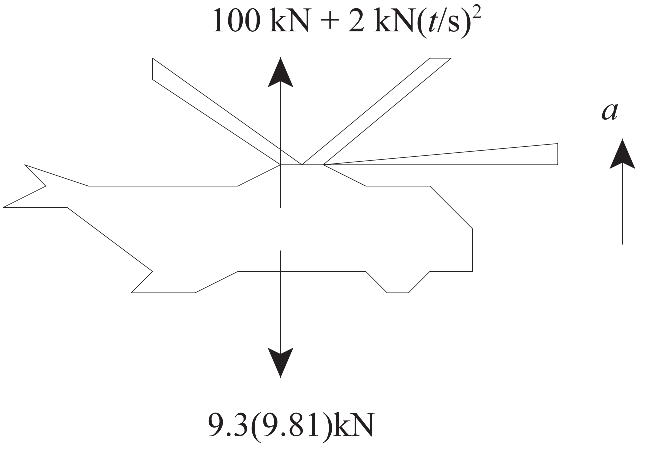
$$a = (0.943\text{ m/s}^2) + (0.215\text{ m/s}^4)t^2$$

$$v = (0.943\text{ m/s}^2)t + \frac{1}{3}(0.215\text{ m/s}^4)t^3$$

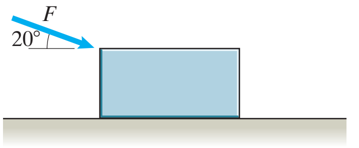
$$h = \frac{1}{2}(0.943\text{ m/s}^2)t^2 + \frac{1}{12}(0.215\text{ m/s}^4)t^4$$

Evaluating these expressions at $t = 3$ s,

$$\boxed{(a)\ v = 4.76\text{ m/s},\quad (b)\ d = 5.69\text{ m}.}$$



Problem 14.4 The horizontal surface is smooth. The 30-N box is at rest when the constant force F is applied. Two seconds later, the box is moving to the right at 20 m/s. Determine F.

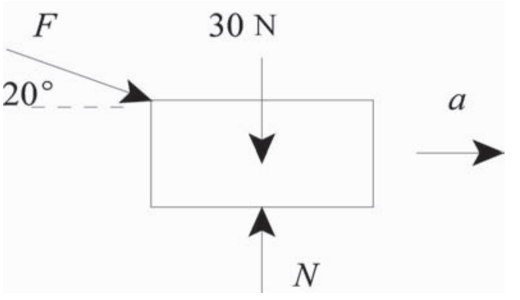


Solution: We use one governing equation and one kinematic relation

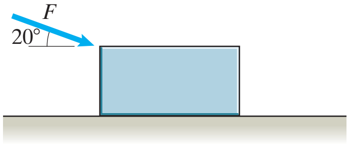
$$\Sigma F_x : F\cos 20^\circ = \left(\frac{30\text{ N}}{9.81\text{ m/s}^2}\right)a,$$

$$v = (20\text{ m/s}) = a(2\text{ s}).$$

Solving, we find $a = 10\text{ m/s}^2$, $\boxed{F = 32.5\text{ N.}}$



Problem 14.5 The coefficient of kinetic friction between the 30-N box and the horizontal surface is $\mu_k = 0.1$. The box is at rest when the constant force F is applied. Two seconds later, the box is moving to the right at 20 m/s. Determine F.



Solution: We use two governing equations, one slip equation, and one kinematic relation

$$\Sigma F_x : F\cos 20^\circ - f = \left(\frac{30\text{ N}}{9.81\text{ m/s}^2}\right)a,$$

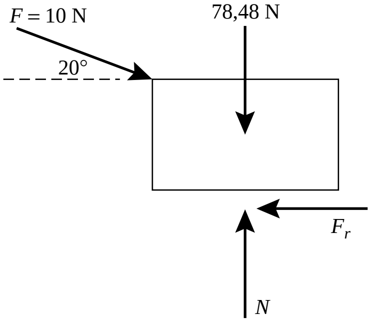
$$\Sigma F_y : N - F\sin 20^\circ - 30\text{ N} = 0,$$

$$f = (0.1)N,$$

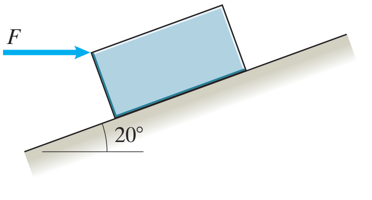
$$v = (20\text{ m/s}) = a(2\text{ s}).$$

Solving, we find

$$a = 10\text{ m/s}^2,\ N = 42.7\text{ N},\ f = 4.27\text{ N},\ \boxed{F = 37.1\text{ N.}}$$



Problem 14.6 The inclined surface is smooth. The velocity of the 114-kg box is zero when it is subjected to a constant horizontal force $F = 20$ N. What is the velocity of the box two seconds later?



Solution: From the governing equation we can find the acceleration of the box (assumed to be down the incline).

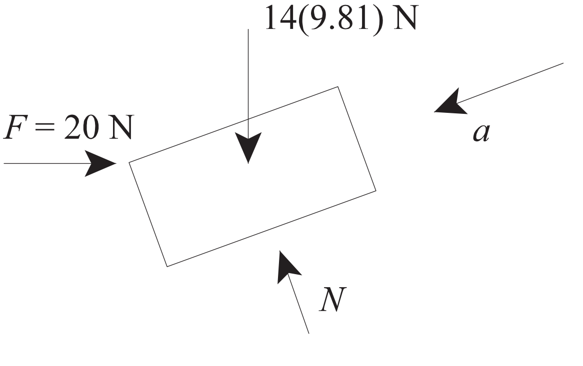
$$\Sigma F \swarrow: 14(9.81)\text{ N }\sin 20^\circ$$

$$-(20\text{ N})\cos 20^\circ = (14\text{ kg})a$$

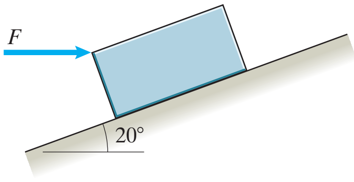
Solving, we have $a = 2.01\text{ m/s}^2$.
Since $a > 0$, our assumption is correct.
Using kinematics,

$$v = at = (2.01\text{ m/s}^2)(2\text{ s}) = 4.03\text{ m/s}.$$

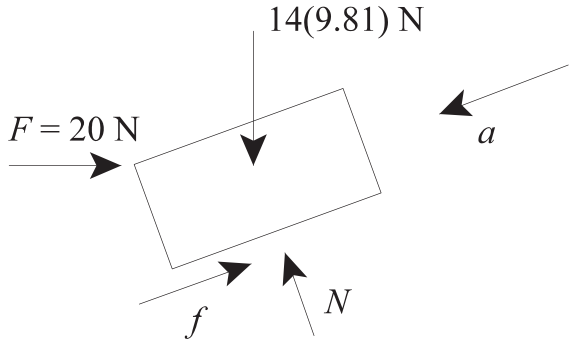
$$\boxed{v = 4.03\text{ m/s down the incline.}}$$



Problem 14.7 The coefficient of kinetic friction between the 14-kg box and the inclined surface is $\mu_k = 0.1$. The velocity of the box is zero when it is subjected to a constant horizontal force $F = 20$ N. What is the velocity of the box two seconds later?



Solution: From the governing equations and the slip equation, we can find the acceleration of the box (assumed to be down the incline).



$$\Sigma F \swarrow: 14(9.81)\ \text{N} \sin 20^\circ - f$$

$$-(20\ \text{N}) \cos 20^\circ = (14\ \text{kg})a,$$

$$\Sigma F \nwarrow: N - 14(9.81)\ \text{N} \cos 20^\circ$$

$$-(20\ \text{N}) \sin 20^\circ = 0.$$

$$\text{Slip: } f = (0.1)N.$$

Solving, we have

$$a = 1.04\ \text{m/s}^2,\ N = 136\ \text{N},\ f = 13.6\ \text{N}.$$

Since $a > 0$, our assumption is correct.

Using kinematics,

$$v = at = (1.04\ \text{m/s}^2)(2\ \text{s}) = 2.08\ \text{m/s}.$$

$$\boxed{v = 2.08\ \text{m/s down the incline.}}$$

Problem 14.8 The 700 N skier is schussing on a 25° slope. At the instant shown, he is moving at 20 m/s. The kinetic coefficient of friction between his skis and the snow is $\mu_k = 0.08$. If he makes no attempt to check his speed, how long does it take for it to increase to 30 m/s?

Solution: The governing equations and the slip equation are used to find the acceleration

$$\Sigma F \nearrow: N - (700\text{ N})\cos 25^\circ = 0,$$

$$\Sigma F \searrow: (700\text{ N})\sin 25^\circ - f$$

$$= \left(\frac{700\text{ N}}{9.81\text{ m/s}^2}\right)a.$$

$$\text{Slip: } f = (0.08)N.$$

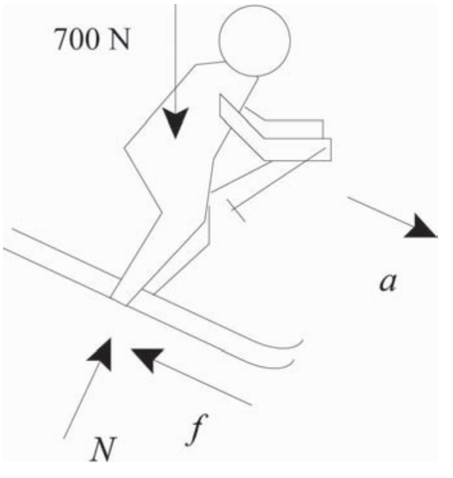
Solving yields

$$a = 3.43\text{ m/s}^2,\ N = 634.4\text{ N},\ f = 50.8\text{ N}.$$

Using kinematics, we find

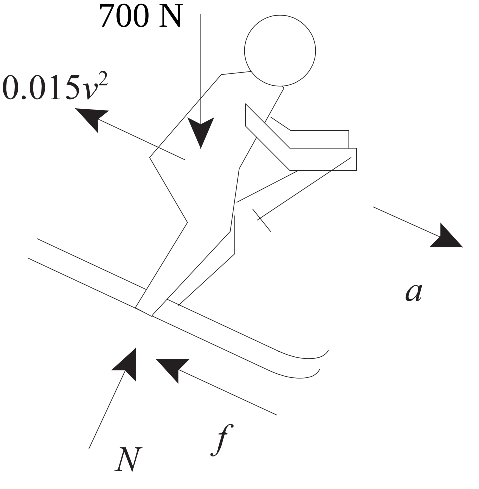
$$30\text{ m/s} = 20\text{ m/s} + (3.43\text{ m/s}^2)t$$

Solving yields $\boxed{t = 2.92\text{ s.}}$



Problem 14.9 The 700 N skier is schussing on a 25° slope. At the instant shown, he is moving at 20 m/s. The kinetic coefficient of friction between his skis and the snow is $\mu_k = 0.08$. Aerodynamic drag exerts a resisting force on him of magnitude $0.015v^2$, where v is the magnitude of his velocity. If he makes no attempt to check his speed, how long does it take for it to increase to 60 m/s?

Solution: The governing equations and the slip equation are used to find the acceleration



$$\Sigma F \nearrow: N - (700 \text{ N}) \cos 25^\circ = 0,$$

$$\Sigma F \searrow: (700 \text{ N}) \sin 25^\circ - f$$

$$- (0.015)v^2$$

$$= \left(\frac{700 \text{ N}}{9.81 \text{ m/s}^2}\right) a.$$

Slip: $f = (0.08)N$.

Solving yields

$$N = 634.4 \text{ N}, \ f = 50.8 \text{ N},$$

$$a = (3.43 \text{ m/s}^2) - (0.00021 \text{ m}^{-1})v^2.$$

Using kinematics, we find

$$a = \frac{dv}{dt} = (3.43 \text{ m/s}^2) - (0.00021 \text{ m}^{-1})v^2$$

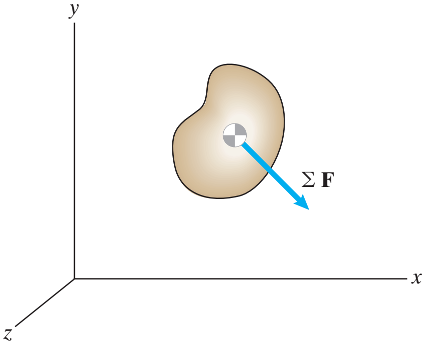
$$\int_{20 \text{ m/s}}^{60 \text{ m/s}} \frac{dv}{(3.43 \text{ m/s}^2) - (0.00021 \text{ m}^{-1})v^2} = \int_0^t dt = t$$

Performing the integration, we find

$$t = \frac{1}{2(0.0268)} \ln \left[\frac{3.43 + x(0.0268)}{3.43 - x(0.0268)}\right]_{20}^{60}$$

Solving yields $\boxed{t = 13.1 \text{ s.}}$

Problem 14.10 The total external force on the 10-kg object is constant and equal to $\Sigma \mathbf{F} = 90\mathbf{i} - 60\mathbf{j} + 20\mathbf{k}$ (N). At time $t = 0$, its velocity is $\mathbf{v} = -14\mathbf{i} + 26\mathbf{j} + 32\mathbf{k}$ (m/s). What is its velocity at $t = 4$ s? (See Active Example 14.2.)



Solution:

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{(90\mathbf{i} - 60\mathbf{j} + 20\mathbf{k}) \text{ N}}{10 \text{ kg}} = (9\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) \text{ m/s}^2.$$

$$\mathbf{v} = \mathbf{a}t + \mathbf{v}_0 = [(9\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}) \text{ m/s}^2](4\text{s}) + (-14\mathbf{i} + 26\mathbf{j} + 32\mathbf{k}) \text{ m/s}.$$

$$\boxed{\mathbf{v} = (22\mathbf{i} + 2\mathbf{j} + 40\mathbf{k}) \text{ m/s.}}$$

Problem 14.11 The total external force on the 10-kg object shown in Problem 14.10 is given as a function of time by $\Sigma\mathbf{F} = (-20t + 90)\mathbf{i} - 60\mathbf{j} + (10t + 40)\mathbf{k}$ (N). At time $t = 0$, its position is $\mathbf{r} = 40\mathbf{i} + 30\mathbf{j} - 360\mathbf{k}$ (m) and its velocity is $\mathbf{v} = -14\mathbf{i} + 26\mathbf{j} + 32\mathbf{k}$ (m/s). What is its position at $t = 4$ s?

Solution:

$$\mathbf{a} = \frac{1}{(10 \text{ kg})}[(-20t + 90)\mathbf{i} - 60\mathbf{j} + (10t + 40)\mathbf{k}]N$$

$$\mathbf{a} = [(-2t + 9)\mathbf{i} - 6\mathbf{j} + (t + 4)\mathbf{k}] \text{ m/s}^2$$

Integrate to get the velocity

$$\mathbf{v} = \int \mathbf{a}\, dt + \mathbf{v}_0$$

$$\mathbf{v} = \left[(-t^2 + 9t - 14)\mathbf{i} + (-6t + 26)\mathbf{j} + \left(\frac{1}{2}t^2 + 4t + 32\right)\mathbf{k}\right] \text{ m/s}$$

Integrate again to get the position

$$\mathbf{r} = \int \mathbf{v}\, dt + \mathbf{r}_0$$

$$\mathbf{r} = \left[\left(-\frac{1}{3}t^3 + \frac{9}{2}t^2 - 14t + 40\right)\mathbf{i} + (-3t^2 + 26t + 30)\mathbf{j}\right.$$

$$\left. + \left(\frac{1}{6}t^3 + 2t^2 + 32t - 360\right)\mathbf{k}\right] \text{ m}$$

At the time indicated ($t = 4$ s) we have

$$\boxed{\mathbf{r} = [34.7\mathbf{i} + 86\mathbf{j} - 189.3\mathbf{k}] \text{ m}}$$

Problem 14.12 The position of the 10-kg object shown in Problem 14.10 is given as a function of time by $\mathbf{r} = (20t^3 - 300)\mathbf{i} + 60t^2\mathbf{j} + (6t^4 - 40t^2)\mathbf{k}$ (m). What is the total external force on the object at $t = 2$ s?

Solution:

$$\mathbf{r} = [(20t^3 - 300)\mathbf{i} + (60t^2)\mathbf{j} + (6t^4 - 40t^2)\mathbf{k}] \text{ m}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = [(60t^2)\mathbf{i} + (120t)\mathbf{j} + (24t^3 - 80t)\mathbf{k}] \text{ m/s}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = [(120t)\mathbf{i} + (120)\mathbf{j} + (72t^2 - 80)\mathbf{k}] \text{ m/s}^2$$

$$\mathbf{F} = m\mathbf{a} = (10 \text{ kg})[(120t)\mathbf{i} + (120)\mathbf{j} + (72t^2 - 80)\mathbf{k}] \text{ m/s}^2$$

$$\mathbf{F} = [(1200t)\mathbf{i} + (1200)\mathbf{j} + (720t^2 - 800)\mathbf{k}] \text{ N}$$

At the time $t = 2$ s,

$$\boxed{\mathbf{F} = [2.40\mathbf{i} + 1.20\mathbf{j} + 2.08\mathbf{k}] \text{ kN}}$$

Problem 14.13 The total force exerted on the 80,000-N launch vehicle by the thrust of its engine, its weight, and aerodynamic forces during the interval of time from $t = 2$ s to $t = 4$ s is given as a function of time by $\Sigma\mathbf{F} = (2000 - 400t^2)\mathbf{i} + (5200 + 440t)\mathbf{j} + (800 + 60t^2)\mathbf{k}$ (N). At $t = 2$ s, its velocity is $\mathbf{v} = 12\mathbf{i} + 220\mathbf{j} - 30\mathbf{k}$ (m/s). What is its velocity at $t = 4$ s?

Solution: Working in components we have

$$a_x = \frac{F_x}{m} = \frac{(2000 - 400t^2)\text{ N}}{\left(\dfrac{80{,}000\text{ N}}{9.81\text{ m/s}^2}\right)} = (0.245 - 0.05t^2)\text{ m/s}^2,$$

$$a_y = \frac{F_y}{m} = \frac{(5200 + 440t)\text{ N}}{\left(\dfrac{80{,}000\text{ N}}{9.81\text{ m/s}^2}\right)} = (0.638 + 0.054t)\text{ m/s}^2,$$

$$a_z = \frac{F_z}{m} = \frac{(800 + 60t^2)\text{ N}}{\left(\dfrac{80{,}000\text{ N}}{9.81\text{ m/s}^2}\right)} = (0.098 + 0.0074t^2)\text{ m/s}^2$$

We find the velocity at $t = 4$ s, by integrating: $\mathbf{v} = \int_{2\text{ s}}^{4\text{ s}} \mathbf{a}\,dt + \mathbf{v}_0$. In components this is

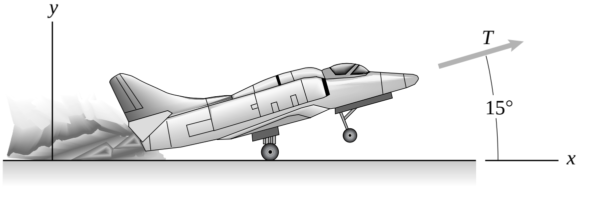
$$v_x = \left([0.245][4-2] - \frac{1}{3}[0.05][4^3 - 2^3] + 12\right)\text{ m/s} = 11.56\text{ m/s},$$

$$v_y = \left([0.638][4-2] + \frac{1}{2}[0.054][4^2 - 2^2] + 220\right)\text{ m/s} = 222.8\text{ m/s},$$

$$v_z = \left([0.098][4-2] + \frac{1}{3}[0.0074][4^3 - 2^3] - 30\right)\text{ m/s} = -29.7\text{ m/s},$$

Thus $\boxed{\mathbf{v} = (11.56\mathbf{i} + 222.8\mathbf{j} - 29.7\mathbf{k})\text{ m/s.}}$

Problem 14.14 At the instant shown, the horizontal component of acceleration of the 115.6 kN airplane due to the sum of the external forces acting on it is 14 m/s^2. If the pilot suddenly increases the magnitude of the thrust T by 17.8 kN, what is the horizontal component of the plane's acceleration immediately afterward?



Solution: Before

$$\sum F_x : F_x = \left(\frac{115600\text{ N}}{9.81\text{ m/s}^2}\right)(14\text{ m/s}^2) = 164975\text{ N}$$

After

$$\sum F_x : 164975\text{ N} + (17800\text{ N})\cos 15^\circ = \left(\frac{11560\text{ N}}{9.81\text{ m/s}^2}\right)a$$

$$\Rightarrow \boxed{a = 15.46\text{ m/s}^2}$$

Problem 14.15 At the instant shown, the rocket is traveling straight up at 100 m/s. Its mass is 90,000 kg and the thrust of its engine is 2400 kN. Aerodynamic drag exerts a resisting force (in newtons) of magnitude $0.8v^2$, where v is the magnitude of the velocity. How long does it take for the rocket's velocity to increase to 200 m/s?

Solution: The equation of motion is

$$\Sigma F: (2400\ \text{kN}) - (90{,}000\ \text{kg})(9.81\ \text{m/s}^2)$$

$$-(0.8\ \text{kg/m})v^2 = (90{,}000\ \text{kg})a$$

Solving for the acceleration we have

$$a = \frac{dv}{dt} = (16.9\ \text{m/s}^2) - (8.89\times10^{-6}\ \text{m}^{-1})v^2$$

$$\int_{100\ \text{m/s}}^{200\ \text{m/s}} \frac{dv}{(16.9\ \text{m/s}^2) - (8.89\times10^{-6}\ \text{m}^{-1})v^2} = \int_0^t dt = t$$

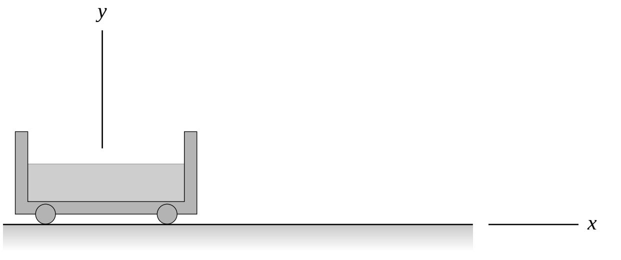
Carrying out the integration, we find

$$t = (81.7\ \text{s})(\tanh^{-1}[(0.000726)(200)] - \tanh^{-1}[(0.000726)(100)])$$

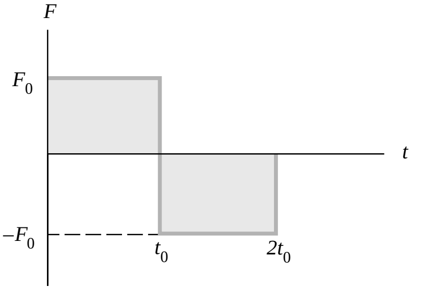
$$\boxed{t = 6.01\ \text{s}}$$

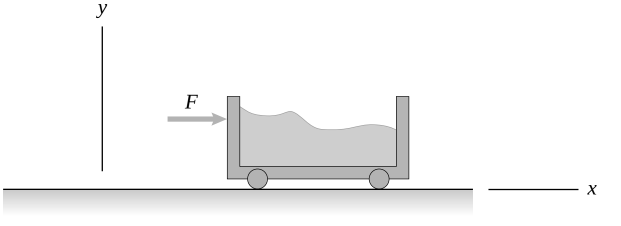
Problem 14.16 A 2-kg cart containing 8 kg of water is initially stationary (Fig. a). The center of mass of the "object" consisting of the cart and water is at $x = 0$. The cart is subjected to the time-dependent force shown in Fig. b, where $F_0 = 5$ N and $t_0 = 2$ s. Assume that no water spills out of the cart and that the horizontal forces exerted on the wheels by the floor are negligible.

(a) Do you know the acceleration of the cart during the period $0 < t < t_0$?
(b) Do you know the acceleration of the center of mass of the "object" consisting of the cart and water during the period $0 < t < t_0$?
(c) What is the x-coordinate of the center of mass of the "object" when $t > 2t_0$?



(a)





(b)

Solution:

(a) No, the internal dynamics make it impossible to determine the acceleration of just the cart.
(b) Yes, the entire system (cart + water) obeys Newton's 2nd Law.

$$\boxed{\sum F : (5\text{ N}) = (10\text{ kg})a \;\Rightarrow\; a = \frac{5\text{ N}}{10\text{ kg}} = 0.5\text{ m/s}^2}$$

(c) The center of mass moves as a "super particle".

For $0 < t < t_0$

$$5\text{ N} = (10\text{ kg})a \Rightarrow a = \frac{5\text{ N}}{10\text{ kg}} = 0.5\text{ m/s}^2$$

$v = (0.5\text{ m/s}^2)t, \quad s = (0.25\text{ m/s}^2)t^2$

At $t = t_0 = 2$ s, $v = 1.0$ m/s, $s = 1.0$ m

For $t_0 < t < 2t_0$,

$-5\text{ N} = (10\text{ kg})a, \; a = -0.5\text{ m/s}^2, \; v = -(0.5\text{ m/s}^2)(t - t_0) + 1.0\text{ m/s}$

$s = -(0.25\text{ m/s}^2)(t - t_0)^2 + (1.0\text{ m/s})(t - t_0) + 1.0\text{ m}$

For $t \geq 2t_0$, $\boxed{a = v = 0, \; s = 2.0\text{ m}}$

Problem 14.17 The combined weight of the motorcycle and rider is 1601 N. The coefficient of kinetic friction between the tires and the road is $\mu_k = 0.8$. The rider starts from rest, spinning the rear wheel. Neglect the horizontal force exerted on the front wheel by the road. In two seconds, the motorcycle moves 10.67 m. What was the normal force between the rear wheel and the road?

Solution: Kinematics

$a =$ constant, $v = at$, $s = \frac{1}{2}at^2$, $10.67 \text{ m} = \frac{1}{2}a(2 \text{ s})^2 \Rightarrow a = 5.33 \text{ m/s}^2$

Dynamics: $F_r = \left(\frac{1601 \text{ N}}{9.81 \text{ m/s}^2}\right)(5.33 \text{ m/s}^2) = 870 \text{ N}$

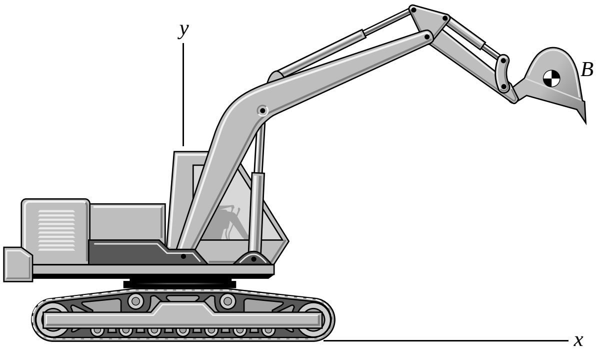
Friction: $\boxed{F_r = (0.8)N \Rightarrow N = \frac{870 \text{ N}}{0.8} = 1089.8 \text{ N}}$

Problem 14.18 The mass of the bucket B is 180 kg. From $t = 0$ to $t = 2$ s, the x and y coordinates of the center of mass of the bucket are

$x = -0.2t^3 + 0.05t^2 + 10$ m,

$y = 0.1t^2 + 0.4t + 6$ m.

Determine the x and y components of the force exerted on the bucket by its supports at $t = 1$ s.



Solution:

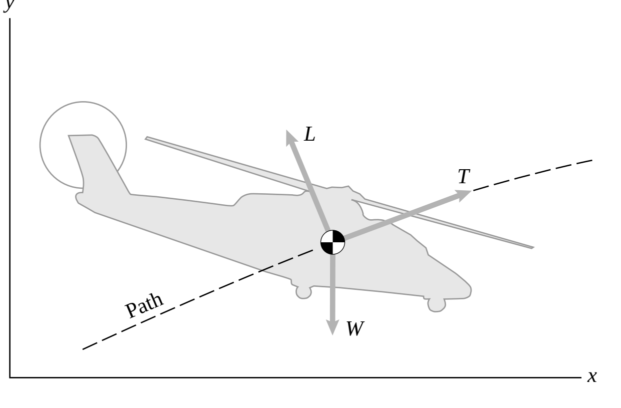
$x = -0.2t^3 + 0.05t^2 + 10, \quad y = 0.1t^2 + 0.4t + 6$

$v_x = -0.6t^2 + 0.1t, \quad v_y = 0.2t + 0.4$

$a_x = -1.2t + 0.1, \quad a_y = 0.2$

$$\boxed{\begin{aligned} F_x &= ma_x = (180 \text{ kg})(-1.2[1 \text{ s}] + 0.1) \text{ m/s}^2 = -198 \text{ N} \\ F_y &= ma_y + mg = (180 \text{ kg})(0.2) \text{ m/s}^2 + (180 \text{ kg})(9.81 \text{ m/s}^2) \\ &= 1800 \text{ N} \end{aligned}}$$

Problem 14.19 During a test flight in which a 9000-kg helicopter starts from rest at $t = 0$, the acceleration of its center of mass from $t = 0$ to $t = 10$ s is $\mathbf{a} = (0.6t)\mathbf{i} + (1.8 - 0.36t)\mathbf{j}$ m/s^2. What is the magnitude of the total external force on the helicopter (including its weight) at $t = 6$ s?



Solution: From Newton's second law: $\sum \mathbf{F} = m\mathbf{a}$. The sum of the external forces is $\sum \mathbf{F} = \mathbf{F} - \mathbf{W} = 9000[(0.6t)\mathbf{i} + (1.8 - 0.36t)\mathbf{j}]_{t=6} = 32400\mathbf{i} - 3240\mathbf{j}$, from which the magnitude is

$$\boxed{\left|\sum \mathbf{F}\right| = \sqrt{32400^2 + 3240^2} = 32562 \text{ (N)}.}$$

Problem 14.20 The engineers conducting the test described in Problem 14.19 want to express the total force on the helicopter at $t = 6$ s in terms of three forces: the weight W, a component T tangent to the path, and a component L normal to the path. What are the values of W, T, and L?

Solution: Integrate the acceleration: $\mathbf{v} = (0.3t^2)\mathbf{i} + (1.8t - 0.18t^2)\mathbf{j}$, since the helicopter starts from rest. The instantaneous flight path angle is $\tan\beta = \dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right)\left(\dfrac{dx}{dt}\right)^{-1} = \dfrac{(1.8t - 0.18t^2)}{(0.3t^2)}$. At $t = 6$ s, $\beta_{t=6} = \tan^{-1}\left(\dfrac{(1.8(6) - 0.18(6)^2)}{0.3(6)^2}\right) = 21.8^\circ$. A unit vector tangent to this path is $\mathbf{e}_t = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta$. A unit vector normal to this path $\mathbf{e}_n = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta$. The weight acts downward:

$$\boxed{\mathbf{W} = -\mathbf{j}(9000)(9.81) = -88.29\mathbf{j}\ (\text{kN}).}$$

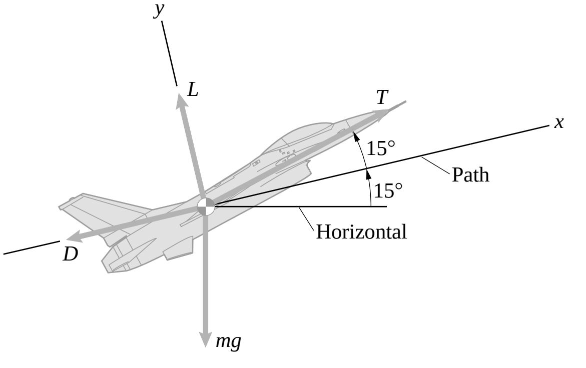
From Newton's second law, $\mathbf{F} - \mathbf{W} = m\mathbf{a}$, from which $\mathbf{F} = \mathbf{W} + m\mathbf{a} = 32400\mathbf{i} + 85050\mathbf{j}$ (N). The component tangent to the path is

$$\boxed{T = \mathbf{F}\cdot\mathbf{e}_t = 32400\cos\beta + 85050\sin\beta = 61669.4\ (\text{N})}$$

The component normal to the path is

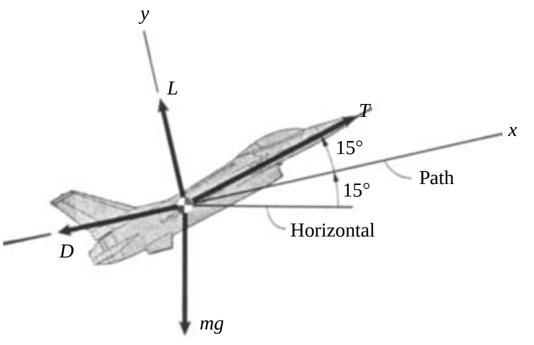
$$\boxed{L = \mathbf{F}\cdot\mathbf{e}_n = -32400\sin\beta + 85050\cos\beta = 66934\ (\text{N})}$$

Problem 14.21 At the instant shown, the 11,000-kg airplane's velocity is $\mathbf{v} = 270\ \mathbf{i}$ m/s. The forces acting on the plane are its weight, the thrust $T = 110$ kN, the lift $L = 260$ kN, and the drag $D = 34$ kN. (The x-axis is parallel to the airplane's path.) Determine the magnitude of the airplane's acceleration.



Solution: Let us sum forces and write the acceleration components along the x and y axes as shown. After the acceleration components are known, we can determine its magnitude. The equations of motion, in the coordinate directions, are $\sum F_x = T\cos 15^\circ - D - W\sin 15^\circ = ma_x$, and $\sum F_y = L + T\sin 15^\circ - W\cos 15^\circ = ma_y$. Substituting in the given values for the force magnitudes, we get $a_x = 4.03$ m/s^2 and $a_y = 16.75$ m/s^2. The magnitude of the acceleration is $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2} = 17.23$ m/s^2

Problem 14.22 At the instant shown, the 11,000-kg airplane's velocity is $\mathbf{v} = 300\mathbf{i}$ (m/s). The rate of change of the magnitude of the velocity is $dv/dt = 5$ m/s^2. The radius of curvature of the airplane's path is 4500 m, and the y axis points toward the concave side of the path. The thrust is $T = 120{,}000$ N. Determine the lift L and drag D.



Solution:

$$a_x = 5 \text{ m/s}^2$$

$$m = 11000 \text{ kg}$$

$$g = 9.81 \text{ m/s}^2$$

$$|\mathbf{V}| = 300 \text{ m/s}$$

$$T = 120000 \text{ N}$$

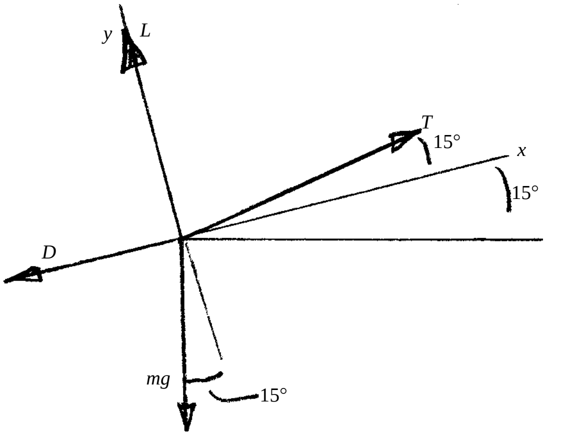
$$\rho = 4500 \text{ m}$$

$$\sum F_x: \quad T\cos 15^\circ - D - mg\sin 15^\circ = ma_x$$

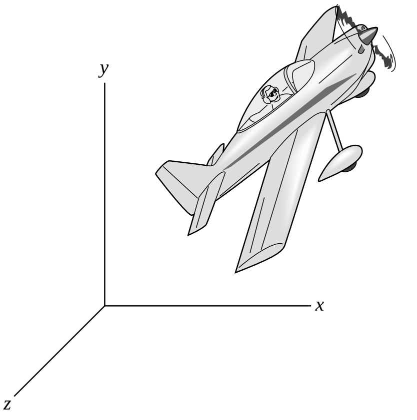
$$\sum F_y: \quad L + T\sin 15^\circ - mg\cos 15^\circ = ma_y$$

$$a_y = V^2/\rho$$

Solving, $\underline{D = 33.0 \text{ kN}, \quad L = 293 \text{ kN}}$



Problem 14.23 The coordinates in meters of the 360-kg sport plane's center of mass relative to an earth-fixed reference frame during an interval of time are $x = 20t - 1.63t^2$, $y = 35t - 0.15t^3$, and $z = -20t + 1.38t^2$, where t is the time in seconds. The y- axis points upward. The forces exerted on the plane are its weight, the thrust vector $\mathbf{T}$ exerted by its engine, the lift force vector $\mathbf{L}$, and the drag force vector $\mathbf{D}$. At $t = 4$ s, determine $\mathbf{T} + \mathbf{L} + \mathbf{D}$.



Solution: There are four forces acting on the airplane. Newton's second law, in vector form, given $\mathbf{T} + \mathbf{L} + \mathbf{D} + \mathbf{W} = (\mathbf{T} + \mathbf{L} + \mathbf{D}) - mg\mathbf{j} = m\mathbf{a}$. Since we know the weight of the airplane and can evaluate the total acceleration of the airplane from the information given, we can evaluate the $(\mathbf{T} + \mathbf{L} + \mathbf{D})$ (but we cannot evaluate these forces separately without more information. Differentiating the position equations twice and evaluating at $t = 4.0$ s, we get $a_X = -3.26$ m/s^2, $a_Y = -3.60$ m/s^2, and $a_Z = 2.76$ m/s^2. (Note that the acceleration components are constant over this time interval. Substituting into the equation for acceleration, we get $(\mathbf{T} + \mathbf{D} + \mathbf{L}) = m\mathbf{a} + mg\mathbf{j}$. The mass of the airplane is 360 kg. Thus, $(\mathbf{T} + \mathbf{D} + \mathbf{L}) = -1174\mathbf{i} + 2236\mathbf{j} + 994\mathbf{k}$ (N)

Problem 14.24 The force in newtons exerted on the 360-kg sport plane in Problem 14.23 by its engine, the lift force, and the drag force during an interval of time is $\mathbf{T} + \mathbf{L} + \mathbf{D} = (-1000 + 280t)\,\mathbf{i} + (4000 - 430t)\,\mathbf{j} + (720 + 200t)\,\mathbf{k}$, where t is the time in seconds. If the coordinates of the plane's center of mass are (0, 0, 0) and its velocity is $20\mathbf{i} + 35\mathbf{j} - 20\mathbf{k}$ (m/s) at $t = 0$, what are the coordinates of the center of mass at $t = 4$ s?

Solution: Since we are working in nonrotating rectangular Cartesian coordinates, we can consider the motion in each axis separately. From Problem 14.23, we have $(\mathbf{T} + \mathbf{D} + \mathbf{L}) = m\mathbf{a} + mg\mathbf{j}$. Separating the information for each axis, we have $ma_X = -1000 + 280t$, $ma_Y = 4000 - 430t - mg$, and $ma_Z = 720 + 200t$

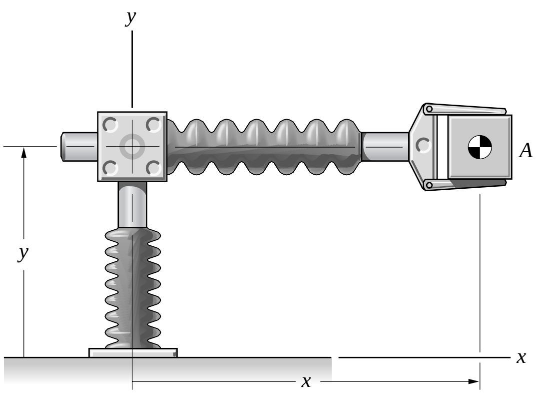
Integrating the x equation, we get $v_x = v_{x0} + (1/m)(-1000t + 280t^2/2)$ and $x = v_{X0}t + (1/m)(-1000t^2/2 + 280t^3/6)$.

Integrating the y equation, we get $v_Y = v_{Y0} + (1/m)((4000 - mg)t - 430t^2/2)$ and $y = v_{Y0}t + (1/m)((4000 - mg)t^2/2 - 430t^3/6)$

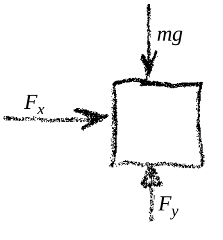
Integrating the z equation, we get $v_Z = v_{Z0} + (1/m)(720t + 200t^2/2)$ and $z = v_{Z0}t + (1/m)(720t^2/2 + 200t^3/6)$.

Evaluating at $t = 4$ s we find the aircraft at $(66.1, 137.7, -58.1)m$ relative to its initial position at $t = 0$.

Problem 14.25 The robot manipulator is programmed so that $x = 40 + 24t^2$ mm, $y = 4t^3$ mm, and $z = 0$ during the interval of time from $t = 0$ to $t = 4$ s. The y axis points upward. What are the x and y components of the total force exerted by the jaws of the manipulator on the 2-kg widget A at $t = 3$ s?



Solution:



$x = 40 + 24t^2$ mm $\qquad y = 4t^3$ mm

$v_x = 48t$ mm/s $\qquad v_y = 12t^2$ mm/s

$a_x = 48$ mm/s^2 $\qquad a_y = 24t$ mm/s^2

At $t = 3$ s

$a_x = 48 \times 10^{-3}$ m/s^2, $\qquad a_y = 72 \times 10^{-3}$ m/s^2

$F_x = ma_x \qquad m = 2$ kg

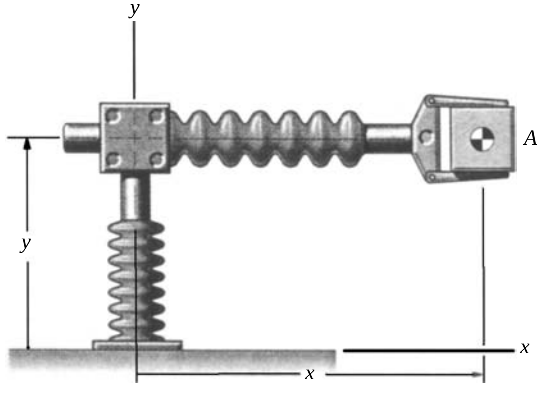
$F_y - mg = ma_y$

Solving,

$\underline{F_x = 0.096 \text{ N}}$

$\underline{F_y = 19.764 \text{ N}}$

Problem 14.26 The robot manipulator described in Problem 14.25 is reprogrammed so that it is stationary at $t = 0$ and the components of its acceleration are $a_x = 400 - 0.8v_x$ mm/s^2, $a_y = 200 - 0.4\ v_y$ mm/s^2 from $t = 0$ to $t = 2$ s, where v_x and v_y are the components of robot's velocity in mm/s. The y axis points upward. What are the x and y components of the total force exerted by the jaws of the manipulator on the 2-kg widget A at $t = 1$ s?



Solution:

$$a_x = \frac{dv_x}{dt} = 400 - 0.8v_x$$

$$\int_0^t dt = \int_0^{v_x} \frac{dv_x}{(400 - 0.8v_x)}$$

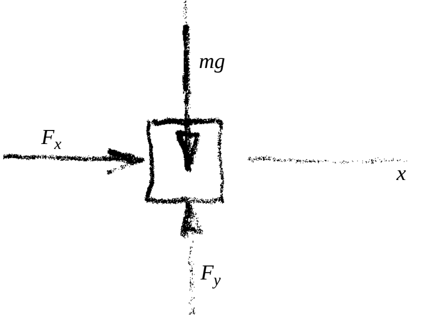
$$t = \frac{1}{(-0.8)} \ln(400 - 0.8v_x)\Big|_0^{v_x}$$

$$(-0.8t) = \ln\left(\frac{400 - 0.8v_x}{400}\right)$$

or $400 - 0.8v_x = 400e^{-0.8t}$

$$v_x = \frac{1}{(0.8)}(400)(1 - e^{-0.8t})$$

At $t = 1$ s, $\quad v_x = 275.3$ mm/s



A similar analysis for v_y yields

$v_y = 164.8$ mm/s at $t = 1$ s.

At $t = 1$ s,

$a_x = 400 - 0.8\ v_x = 179.7$ mm/s^2

$a_y = 200 - 0.4\ v_y = 134.1$ mm/s^2

$m = 2$ Kg

$g = 9.81$ m/s^2

$a_x = 0.180$ m/s^2

$a_y = 0.134$ m/s^2

$\sum F_x$: $\quad F_x = ma_x$

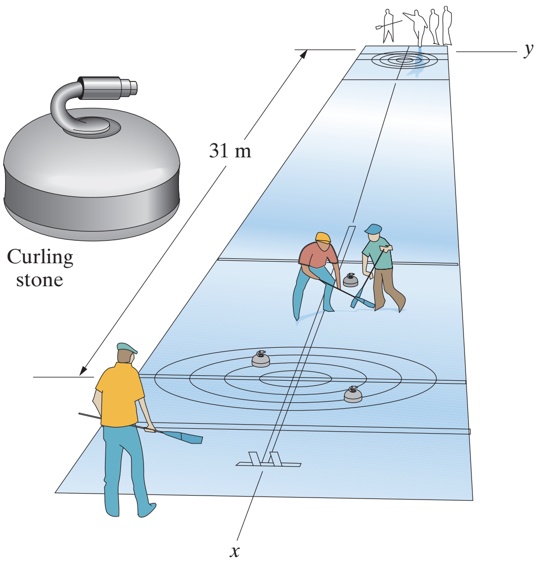
$\sum F_y$: $\quad F_y - mg = ma_y$

Solving,

$\underline{F_x = 0.359 \text{ N}}$

$\underline{F_y = 19.89 \text{ N}}$

Problem 14.27 In the sport of curling, the object is to slide a "stone" weighting 44 N into the center of a target located 31 m from the point of release. In terms of the coordinate system shown, the point of release is at $x = 0$, $y = 0$. Suppose that a shot comes to rest at $x = 31$ m, $y = 1$ m. Assume that the coefficient of kinetic friction is constant and equal to $\mu_k = 0.01$. What were the x and y components of the stone's velocity at release?



Solution: The stone travels at an angle relative to the x axis.

$$\theta = \tan^{-1}\left(\frac{1 \text{ m}}{31 \text{ m}}\right) = 1.85^\circ$$

The accelerations and distances are related as

$$a_x = v_x \frac{dv_x}{dx} = -(0.01)(9.81 \text{ m/s}^2)\cos(1.85^\circ) = -0.098 \text{ m/s}^2$$

$$\int_{v_{x0}}^{0} v_x dv_x = \int_0^{31 \text{ m}} (-0.098 \text{ m/s}^2)dx,$$

$$0 - \frac{v_{x0}^2}{2} = -(0.098 \text{ m/s}^2)(31 \text{ m}) \Rightarrow v_{x0} = 3.04 \text{ m/s}.$$

$$a_y = v_y \frac{dv_y}{dy} = -(0.01)(0.098 \text{ m/s}^2)\sin(1.85^\circ) = -0.00316 \text{ m/s}^2$$

$$\int_{v_{y0}}^{0} v_y dv_y = \int_0^{1 \text{ m}} (-0.00316 \text{ m/s}^2)dx,$$

$$0 - \frac{v_{y0}^2}{2} = -(0.00316 \text{ m/s}^2)(1 \text{ m}) \Rightarrow v_{y0} = 0.00316 \text{ m/s}^2.$$

$$\boxed{v_{x0} = 3.04 \text{ m/s}, v_{y0} = 0.00316 \text{ m/s}^2.}$$

Problem 14.28 The two masses are released from rest. How fast are they moving at $t = 0.5$ s? (See Example 14.3.)

Solution: The free-body diagrams are shown. The governing equations are

$$\Sigma F_{y \text{ left}} : T - 2(9.81)\text{ N} = (2\text{ kg})a$$

$$\Sigma F_{y \text{ right}} : T - 5(9.81)\text{ N} = -(5\text{ kg})a$$

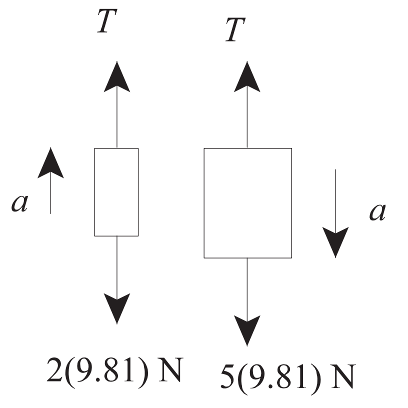
Solving, we find

$$T = 28.0\text{ N}, a = 4.20\text{ m/s}^2.$$

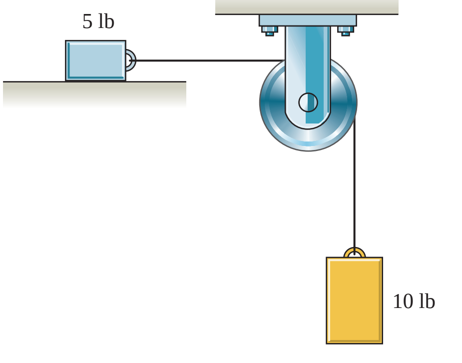
To find the velocity, we integrate the acceleration

$$v = at = (4.20\text{ m/s})(0.5\text{ s}) = 2.10\text{ m/s}.$$

$$\boxed{v = 2.10\text{ m/s.}}$$



Problem 14.29 The two weights are released from rest. The horizontal surface is smooth. (a) What is the tension in the cable after the weights are released? (b) How fast are the weights moving one second after they are released?



Solution: The free-body diagrams are shown. The governing equations are

$$\Sigma F_{xA} : T = \left(\frac{5\text{ lb}}{32.2\text{ ft/s}^2}\right)a$$

$$\Sigma F_{yB} : T - (10\text{ lb}) = -\left(\frac{10\text{ lb}}{32.2\text{ ft/s}^2}\right)a$$

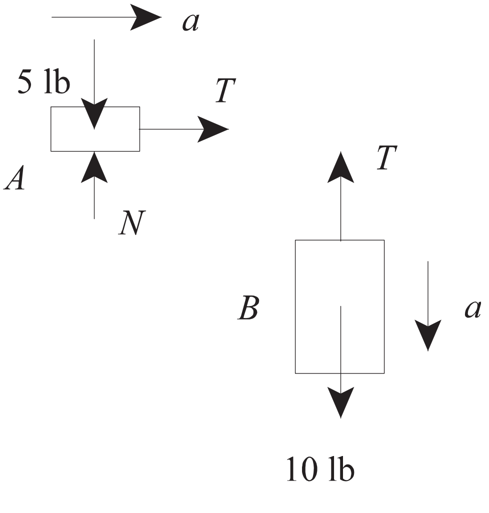
Solving, we find

$$T = 3.33\text{ lb}, a = 21.5\text{ ft/s}^2.$$

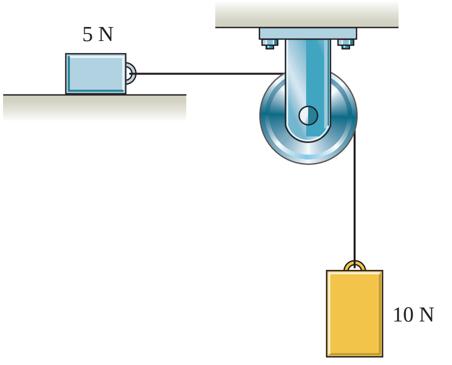
To find the velocity we integrate the acceleration

$$v = at = (21.5\text{ ft/s}^2)(1\text{ s}) = 21.5\text{ ft/s}.$$

$$\boxed{(a)\ T = 3.33\text{ lb},\quad (b)\ v = 21.5\text{ ft/s.}}$$



Problem 14.30 The two weights are released from rest. The coefficient of kinetic friction between the horizontal surface and the 5-N weight is $\mu_k = 0.18$. (a) What is the tension in the cable after the weights are released? (b) How fast are the weights moving one second after they are released?



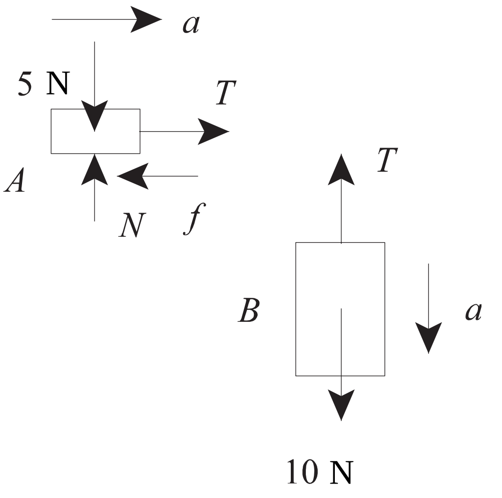
Solution: The free-body diagrams are shown. The governing equations are

$$\Sigma F_{xA} : T - f = \left(\frac{5\ \text{N}}{9.81\ \text{m/s}^2}\right) a,$$

$$\Sigma F_{yA} : N - 5\ \text{N} = 0,$$

$$\Sigma F_{yB} : T - (10\ \text{N}) = -\left(\frac{10\ \text{N}}{9.81\ \text{m/s}^2}\right) a,$$

$$f = (0.18)N.$$



Solving, we find

$$T = 3.93\ \text{N}, a = 5.95\ \text{m/s}^2,$$

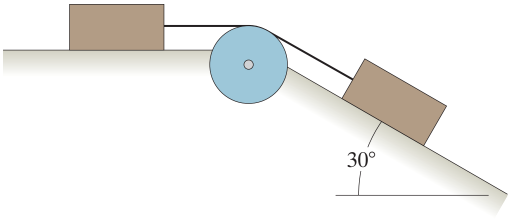
$$N = 5\ \text{N}, f = 0.9\ \text{N}.$$

To find the velocity we integrate the acceleration

$$v = at = (5.95\text{m/s}^2)(1\ \text{s}) = 5.95\ \text{m/s}.$$

$$\boxed{(a)\ T = 3.93\ \text{N} \quad (b)\ v = 5.95\ \text{m/s}.}$$

Problem 14.31 The mass of each box is 14 kg. One second after they are released from rest, they have moved 0.3 m from their initial positions. What is the coefficient of kinetic friction between the boxes and the surface?



Solution: We will first use the kinematic information to find the acceleration a

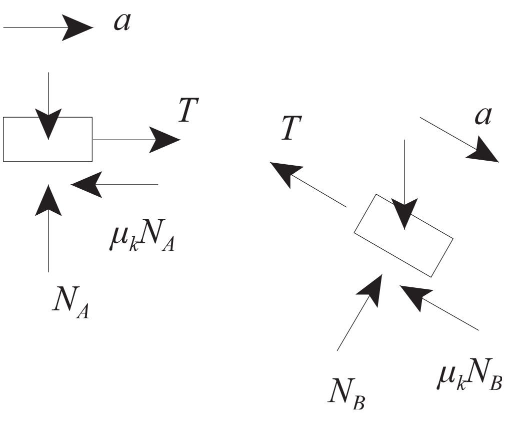
$$a = \text{constant},$$

$$v = at,$$

$$d = \frac{1}{2}at^2$$

$$0.3 \text{ m} = \frac{1}{2}a(1 \text{ s})^2$$

$$a = 0.6 \text{ m/s}^2.$$



From the free-body diagrams we have four equations of motion:

$$\Sigma F_{xA} : T - \mu_k N_A = (14 \text{ kg})a,$$

$$\Sigma F_{yA} : N_A - (14 \text{ kg})(9.18 \text{ m/s}^2) = 0$$

$$\Sigma F_{\searrow B} : (14 \text{ kg})(9.81 \text{ m/s}^2)\sin 30^\circ - T - \mu_k N_B = (14 \text{ kg})a,$$

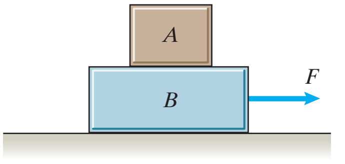
$$\Sigma F_{\nearrow B} : N_B - (14 \text{ kg})(9.81 \text{ m/s}^2)\cos 30^\circ = 0$$

Solving these equations, we find

$$T = 36.2 \text{ N}, N_A = 137 \text{ N}, N_B = 119 \text{ N}, \mu_k = 0.202.$$

$$\boxed{\mu_k = 0.202.}$$

Problem 14.32 The masses $m_A = 15$ kg and $m_B = 30$ kg, and the coefficients of friction between all of the surfaces are $\mu_s = 0.4$ and $\mu_k = 0.35$. The blocks are stationary when the constant force F is applied. Determine the resulting acceleration of block B if (a) $F = 200$ N; (b) $F = 400$ N.

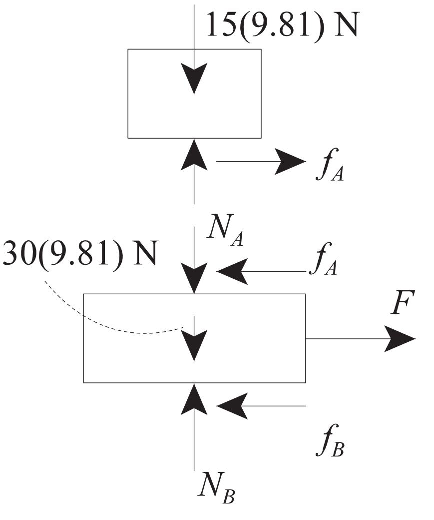


Solution: Assume that no motion occurs anywhere. Then

$$N_B = (45 \text{ kg})(9.81 \text{ m/s}^2) = 441 \text{ N}$$

$$f_{B\,\max} = \mu_s N = 0.4(441 \text{ N}) = 177 \text{ N}.$$

The blocks will slip as long as $F > 177$ N. Assume that blocks A and B move together with the common acceleration a.



$$\Sigma F_{xA} : f_A = (15 \text{ kg})a$$

$$\Sigma F_{yA} : N_A - (15 \text{ kg})(9.81 \text{ m/s}^2) = 0$$

$$\Sigma F_{xB} : F - f_A - f_B = (30 \text{ kg})a$$

$$\Sigma F_{yB} : N_B - (30 \text{ kg})(9.81 \text{ m/s}^2) - N_A = 0$$

$$\text{Slip at } B : f_B = (0.35)N_B$$

$$f_{A\,\max} = (0.4)N_A$$

(a) $F = 200$ N. Solving we find that $a = 1.01$ m/s^2, $f_A = 15.2$ N, $f_{A\,\max} = 58.9$ N. Since $f_A < f_{A\,\max}$, we know that our assumption is correct (the blocks move together).

(b) $F = 400$ N. Solving we find that $a = 5.46$ m/s^2, $f_A = 81.8$ N, $f_{A\,\max} = 58.9$ N. Since $f_A > f_{A\,\max}$, we know that our assumption is wrong (the blocks will not move together, but slip will occur at all surfaces). The equations are now

$$\Sigma F_{xA} : (0.35)N_A = (15 \text{ kg})a_A$$

$$\Sigma F_{yA} : N_A - (15 \text{ kg})(9.81 \text{ m/s}^2) = 0$$

$$\Sigma F_{xB} : F - (0.35)(N_A + N_B) = (30 \text{ kg})a_B$$

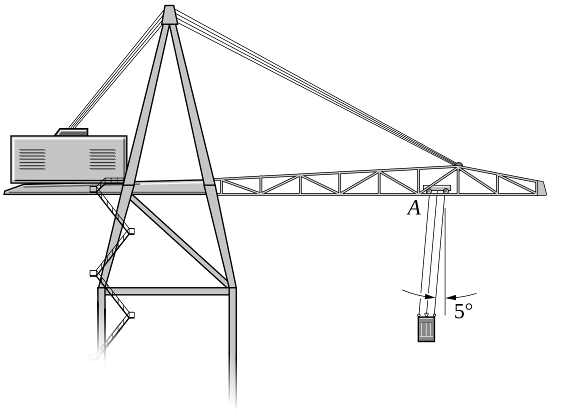
$$\Sigma F_{yB} : N_B - (30 \text{ kg})(9.81 \text{ m/s}^2) - N_A = 0$$

Solving we find that $a_A = 3.43$ m/s^2, $a_B = 6.47$ m/s^2.

$$\boxed{(a)\ a_B = 1.01 \text{ m/s}^2, \quad (b)\ a_B = 6.47 \text{ m/s}^2.}$$

Problem 14.33 The crane's trolley at A moves to the right with constant acceleration, and the 800-kg load moves without swinging.

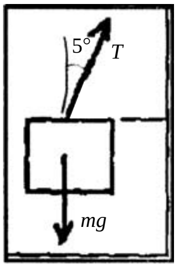
(a) What is the acceleration of the trolley and load?
(b) What is the sum of the tensions in the parallel cables supporting the load?



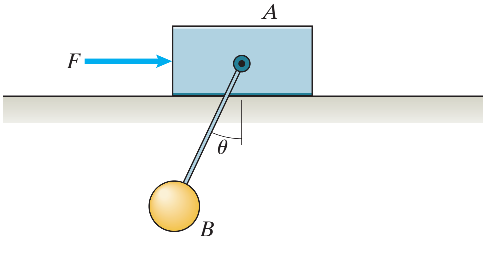
Solution: (a) From Newton's second law, $T\sin\theta = ma_x$, and $T\cos\theta - mg = 0$. Solve $a_x = \dfrac{T\sin\theta}{m}$, $T = \dfrac{mg}{\cos\theta}$, from which

$$\boxed{a = g\tan\theta = 9.81(\tan 5^\circ) = 0.858\ \text{m/s}^2}$$

(b) $$\boxed{T = \frac{800(9.81)}{\cos 5^\circ} = 7878\ \text{N}}$$



Problem 14.34 The mass of A is 30 kg and the mass of B is 5 kg. The horizontal surface is smooth. The constant force F causes the system to accelerate. The angle $\theta = 20^\circ$ is constant. Determine F.



Solution: We have four unknowns (F, T, N, a) and four equations of motion:

$$\Sigma F_{xA}: F - T\sin\theta = (30\ \text{kg})a,$$

$$\Sigma F_{yA}: N - T\cos\theta - (30\ \text{kg})(9.81\ \text{m/s}^2) = 0,$$

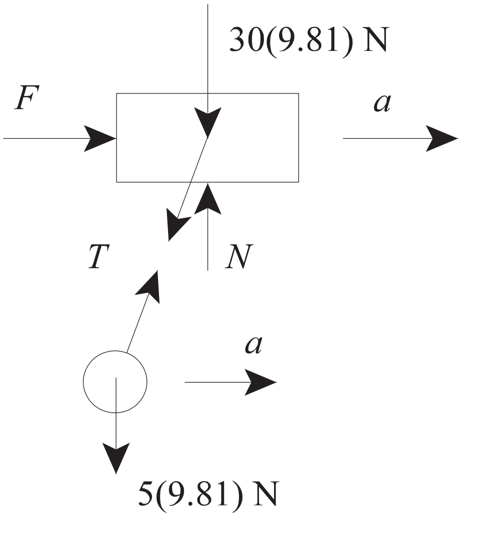
$$\Sigma F_{xB}: T\sin\theta = (5\ \text{kg})a,$$

$$\Sigma F_{yB}: T\cos\theta - (5\ \text{kg})(9.81\ \text{m/s}^2) = 0.$$

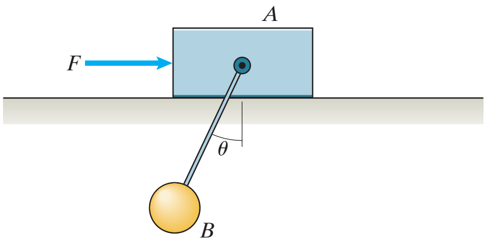
Solving, we find

$$T = 52.2\ \text{N},\ a = 3.57\ \text{m/s}^2,\ N = 343\ \text{N},$$

$$\boxed{F = 125\ \text{N.}}$$



Problem 14.35 The mass of A is 30 kg and the mass of B is 5 kg. The coefficient of kinetic friction between A and the horizontal surface is $\mu_k = 0.2$. The constant force F causes the system to accelerate. The angle $\theta = 20^\circ$ is constant. Determine F.



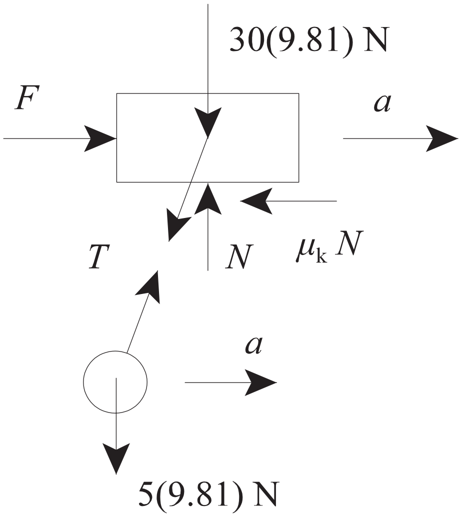
Solution: We have four unknowns (F, T, N, a) and four equations of motion:

$$\Sigma F_{xA} : F - T\sin\theta - \mu_k N = (30\ \text{kg})a,$$

$$\Sigma F_{yA} : N - T\cos\theta - (30\ \text{kg})(9.81\ \text{m/s}^2) = 0,$$

$$\Sigma F_{xB} : T\sin\theta = (5\ \text{kg})a,$$

$$\Sigma F_{yB} : T\cos\theta - (5\ \text{kg})(9.81\ \text{m/s}^2) = 0.$$

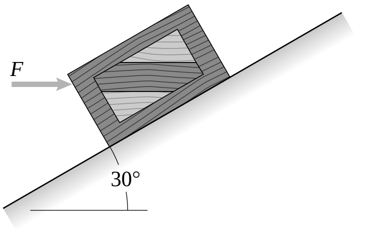


Solving, we find

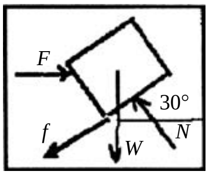
$$T = 52.2\ \text{N},\ a = 3.57\ \text{m/s}^2,\ N = 343\ \text{N},$$

$$\boxed{F = 194\ \text{N}.}$$

Problem 14.36 The 445 N crate is initially stationary. The coefficients of friction between the crate and the inclined surface are $\mu_s = 0.2$ and $\mu_k = 0.16$. Determine how far the crate moves from its initial position in 2 s if the horizontal force $F = 400$ N.



Solution: Denote $W = 445$ N, $g = 9.81$ m/s^2, $F = 400$ N, and $\theta = 30^\circ$. Choose a coordinate system with the positive x axis parallel to the inclined surface. (See free body diagram next page.) The normal force exerted by the surface on the box is $N = F\sin\theta + W\cos\theta = 585.4$ N. The sum of the non-friction forces acting to move the box is $F_c = F\cos\theta - W\sin\theta = 124.1$ N. Slip only occurs if $|F_c| \geq |N\mu_s|$ which $124.1 > 117.1$ (N), *so slip occurs*.



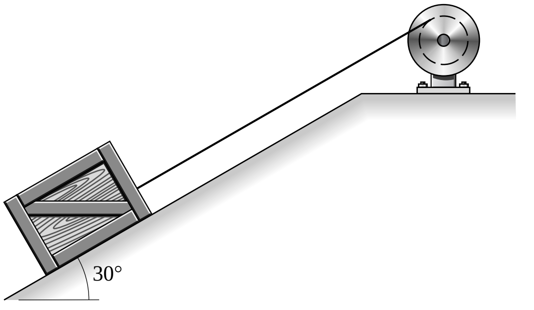
The direction of slip is determined from the sign of the sum of the non friction forces: $F_c > 0$, implies that the box slips up the surface, and $F_c < 0$ implies that the box slips down the surface (if the condition for slip is met). Since $F_c > 0$ *the box slips up the surface*. After the box slips, the sum of the forces on the box parallel to the surface is $\sum F_x = F_c - \text{sgn}(F_c)\mu_k N$, where $\text{sgn}(F_c) = \dfrac{F_c}{|F_c|}$. From Newton's second law, $\sum F_x = \left(\dfrac{W}{g}\right)a$, from which $a = \dfrac{g}{W}(F_c - \text{sgn}(F_c)\mu_k N) = 0.65$ m/s^2. The velocity is $v(t) = at$ m/s, since $v(0) = 0$. The displacement is $s = \dfrac{a}{2}t^2 ft$, since $s(0) = 0$. The position after 2 s is $\boxed{s(2) = 1.35\ \text{m}}$ up the inclined surface.

Problem 14.37 In Problem 14.36, determine how far the crate moves from its initial position in 2 s if the horizontal force $F = 133.4$ N.

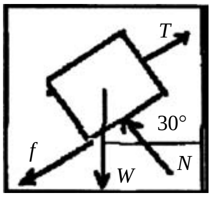
Solution: Use the definitions of terms given in the solution to Problem 14.36. For $F = 133.4$ N, $N = F\sin\theta + W\cos\theta = 452$ N, and $F_c = F\cos\theta - W\sin\theta = -106.8$ N from which, $|F_c| = 106.8 > |\mu_s N| = 90.4$, so slip occurs. Since $F_c < 0$, *the box will slip down the surface*. From the solution to Problem 14.36, after slip occurs, $a = \left(\frac{g}{W}\right)(F_c - \operatorname{sgn}(F_c)\mu_k N) = -0.761$ m/s^2. The position is $s(t) = \frac{a}{2}t^2$. At 2 seconds, $\boxed{s(2) = -1.52 \text{ m}}$ down the surface.

Problem 14.38 The crate has a mass of 120 kg, and the coefficients of friction between it and the sloping dock are $\mu_s = 0.6$ and $\mu_k = 0.5$.

(a) What tension must the winch exert on the cable to start the stationary crate sliding up the dock?
(b) If the tension is maintained at the value determined in part (a), what is the magnitude of the crate's velocity when it has moved 2 m up the dock?



Solution: Choose a coordinate system with the x axis parallel to the surface. Denote $\theta = 30°$.



(a) The normal force exerted by the surface on the crate is $N = W\cos\theta = 120(9.81)(0.866) = 1019.5$ N. The force tending to move the crate is $F_c = T - W\sin\theta$, from which the tension required to start slip is $\boxed{T = W(\sin\theta) + \mu_s N = 1200.3 \text{ N.}}$

(b) After slip begins, the force acting to move the crate is $F = T - W\sin\theta - \mu_k N = 101.95$ N. From Newton's second law, $F = ma$, from which $a = \left(\frac{F}{m}\right) = \frac{101.95}{120} = 0.8496$ m/s^2. The velocity is $v(t) = at = 0.8496t$ m/s, since $v(0) = 0$. The position is $s(t) = \frac{a}{2}t^2$, since $s(0) = 0$. When the crate has moved 2 m up the slope, $t_{10} = \sqrt{\frac{2(2)}{a}} = 2.17$ s and the velocity is $\boxed{v = a(2.17) = 1.84 \text{ m/s.}}$

Problem 14.39 The coefficients of friction between the load A and the bed of the utility vehicle are $\mu_s = 0.4$ and $\mu_k = 0.36$. If the floor is level ($\theta = 0$), what is the largest acceleration (in m/s^2) of the vehicle for which the load will not slide on the bed?

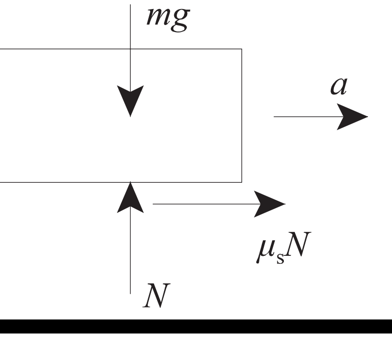
Solution: The load is on the verge of slipping. There are two unknowns (a and N). The equations are

$$\Sigma F_x : \mu_s N = ma, \quad \Sigma F_y : N - mg = 0$$

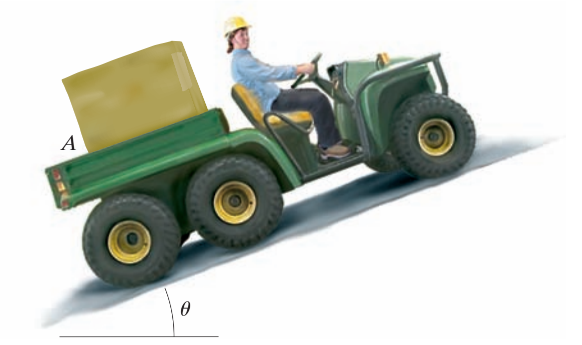
Solving we find

$$a = \mu_s g = (0.4)(9.81 \text{ m/s}^2) = 3.92 \text{ m/s}^2.$$

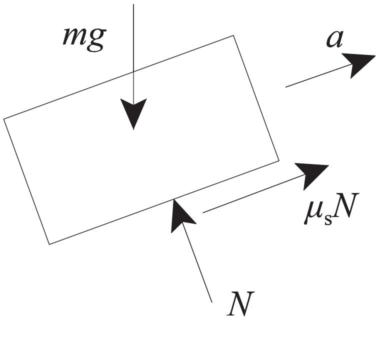
$$\boxed{a = 3.92 \text{ m/s}^2.}$$



Problem 14.40 The coefficients of friction between the load A and the bed of the utility vehicle are $\mu_s = 0.4$ and $\mu_k = 0.36$. The angle $\theta = 20^\circ$. Determine the largest forward and rearward acceleration of the vehicle for which the load will not slide on the bed.



Solution: The load is on the verge of slipping. There are two unknowns (a and N).



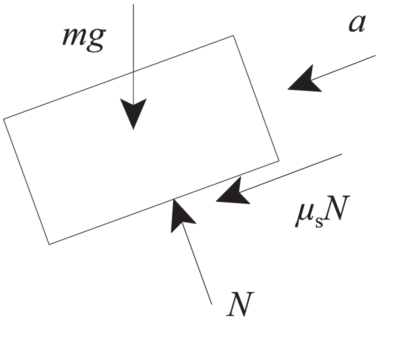
Forward: The equations are

$$\Sigma F_x : \mu_s N - mg \sin\theta = ma,$$

$$\Sigma F_y : N - mg \cos\theta = 0$$

Solving we find

$$a = g(\mu_s \cos\theta - \sin\theta)$$
$$= (9.81 \text{ m/s}^2)([0.4]\cos 20^\circ - \sin 20^\circ)$$
$$\boxed{a = 0.332 \text{ m/s}^2.}$$



Rearward: The equations are

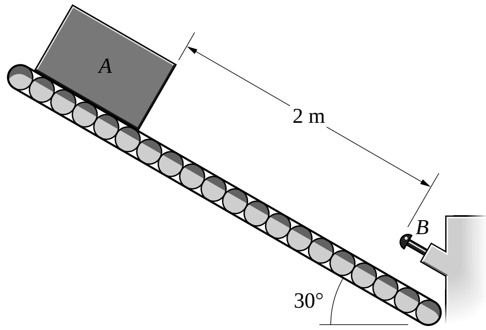
$$\Sigma F_x : -\mu_s N - mg \sin\theta = -ma,$$

$$\Sigma F_y : N - mg \cos\theta = 0$$

Solving we find

$$a = g(\mu_s \cos\theta + \sin\theta)$$
$$= (9.81 \text{ m/s}^2)([0.4]\cos 20^\circ + \sin 20^\circ)$$
$$\boxed{a = 7.04 \text{ m/s}^2.}$$

Problem 14.41 The package starts from rest and slides down the smooth ramp. The hydraulic device B exerts a constant 2000-N force and brings the package to rest in a distance of 100 mm from the point where it makes contact. What is the mass of the package?

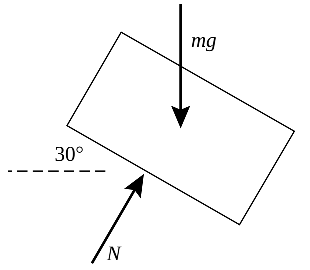


Solution: Set $g = 9.81$ m/s^2

First analyze the motion before it gets to point B.

$$\sum F_{\searrow} : mg \sin 30^\circ = ma$$

$$a = g\sin 30^\circ, \quad v = (g \sin 30^\circ)t, \quad s = (g\sin 30^\circ)\frac{t^2}{2}$$



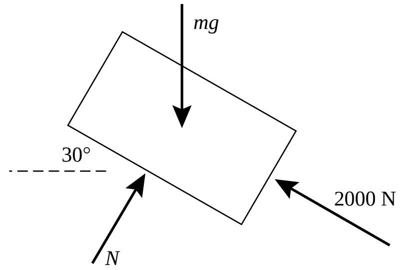
When it gets to B we have

$$s = 2 \text{ m} = (g\sin 30^\circ)\frac{t^2}{2} \Rightarrow t = 0.903 \text{ s}$$

$$v = (g\sin 30^\circ)(0.903 \text{ s}) = 4.43 \text{ m/s}$$

Now analyze the motion after it hits point B.

$$\sum F_{\searrow} : mg\sin 30^\circ - 2000 \text{ N} = ma$$



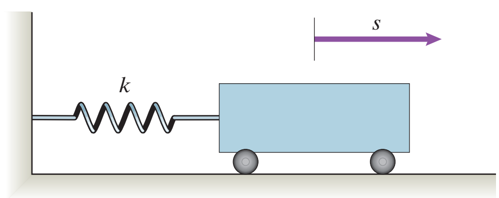
$$a = v\frac{dv}{ds} = g\sin 30^\circ - \frac{2000 \text{ N}}{m}$$

$$\int_{4.43\,\text{m/s}}^{0} v\,dv = \int_0^{0.1\,\text{m}} \left(g\sin 30^\circ - \frac{2000\text{ N}}{m}\right) ds$$

$$0 - \frac{(4.43 \text{ m/s})^2}{2} = \left(g\sin 30^\circ - \frac{2000 \text{ N}}{m}\right)(0.1 \text{ m})$$

Solving the last equation we find $\boxed{m = 19.4 \text{ kg}}$

Problem 14.42 The force exerted on the 10-kg mass by the linear spring is $F = -ks$, where k is the spring constant and s is the displacement of the mass relative to its position when the spring is unstretched. The value of k is 40 N/m. The mass is in the position $s = 0$ and is given an initial velocity of 4 m/s toward the right. Determine the velocity of the mass as a function of s.

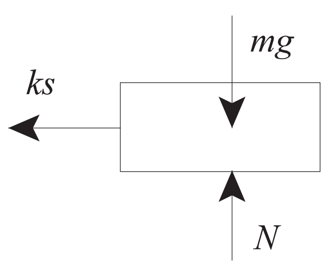


Strategy: : Use the chain rule to write the acceleration as

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v.$$

Solution: The equation of motion is $-ks = ma$

$$a = v\frac{dv}{ds} = -\frac{k}{m}s \Rightarrow \int_{v_0}^{v} v dv = -\int_0^s \frac{k}{m} ds \Rightarrow \frac{v^2}{2} - \frac{v_0^2}{2} = -\frac{k}{m}\frac{s^2}{2}$$



Thus

$$v = \pm\sqrt{v_0^2 - \frac{k}{m}s^2} = \pm\sqrt{(4 \text{ m/s})^2 - \frac{40 \text{ N/m}}{10 \text{ kg}}s^2}$$

$$\boxed{v = \pm 2\sqrt{4 - s^2} \text{ m/s}}$$

Problem 14.43 The 450-kg boat is moving at 10 m/s when its engine is shut down. The magnitude of the hydrodynamic drag force (in newtons) is $40v^2$, where v is the magnitude of the velocity in m/s. When the boat's velocity has decreased to 1 m/s, what distance has it moved from its position when the engine was shut down?

Solution: The equation of motion is

$$F = -40v^2 = (450\text{ kg})a \Rightarrow a = -\frac{40}{450}v^2$$

To integrate, we write the acceleration as

$$a = v\frac{dv}{ds} = -\frac{40}{450}v^2 \Rightarrow \int_{10\text{ m/s}}^{1\text{ m/s}} \frac{dv}{v} = -\frac{40}{450}\int_0^s ds \Rightarrow \ln\left(\frac{1\text{ m/s}}{10\text{ m/s}}\right) = -\frac{40}{450}s$$

$$s = -\frac{450}{40}\ln(0.1) = 25.9\text{ m}.$$

$$\boxed{s = 25.9\text{ m.}}$$

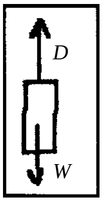
Problem 14.44 A sky diver and his parachute weigh 890 N. He is falling vertically at 30.5 m/s when his parachute opens. With the parachute open, the magnitude of the drag force (in Newton) is $0.5v^2$. (a) What is the magnitude of the sky diver's acceleration at the instant the parachute opens? (b) What is the magnitude of his velocity when he has descended 6.1 m from the point where his parachute opens?

Solution: Choose a coordinate system with s positive downward. For brevity denote $C_d = 0.5$, $W = 890$ N, $g = 9.81$ m/s^2. From Newton's second law $W - D = \left(\frac{W}{g}\right)\left(\frac{dv}{dt}\right)$, where $D = 0.5v^2$. Use the chain rule to write $v\frac{dv}{ds} = -\frac{0.5v^2 g}{W} + g = g\left(1 - \frac{C_d v^2}{W}\right)$.

(a) At the instant of full opening, the initial velocity has not decreased, and the magnitude of the acceleration is

$$\boxed{|a_{\text{init}}| = \left|g\left(1 - \frac{C_d}{W}v^2\right)\right| = 24\text{ g} = 235.3\text{ m/s}^2.}$$

(b) Separate variables and integrate: $\dfrac{v\,dv}{1 - \dfrac{C_d v^2}{W}} = g\,ds$, from which



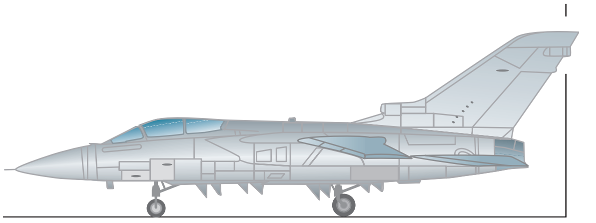
$$\ln\left(1 - \frac{C_d v^2}{W}\right) = -\left(\frac{2C_d g}{W}\right)s + C.$$ Invert and solve:

$$v^2 = \left(\frac{W}{C_d}\right)\left(1 - Ce^{-\frac{2C_d g}{W}s}\right).$$ At $s = 0$, $v(0) = 30.5$ m/s, from which $C = 1 - \dfrac{C_d 7.75^4}{W} = -1.03$, and

$$v^2 = \left(\frac{W}{C_d}\right)\left(1 + 1.03e^{-\frac{2C_d g}{W}s}\right).$$ At $s = 6.1$ m the velocity is

$$\boxed{v(s = 6.1) = \sqrt{2W\left(1 + 1.03e^{-\frac{g}{W}(6.1)}\right)} = 8.53\text{ m/s}}$$

Problem 14.45 The Panavia Tornado, with a mass of 18,000 kg, lands at a speed of 213 km/h. The decelerating force (in newtons) exerted on it by its thrust reversers and aerodynamic drag is $80{,}000 + 2.5v^2$, where v is the airplane's velocity in m/s. What is the length of the airplane's landing roll? (See Example 14.4.)



Solution:

$$v_0 = 213 \text{ km/h}\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 59.2 \text{ m/s}$$

$$\Sigma F = -(80{,}000 + 2.5v^2) = (18{,}000 \text{ kg})a$$

$$a = v\frac{dv}{ds} = -\frac{80{,}000 + 2.5v^2}{18{,}000}$$

$$\int_{v_0}^{0} \frac{v\,dv}{80{,}000 + 2.5v^2} = -\int_0^s \frac{ds}{18{,}000}$$

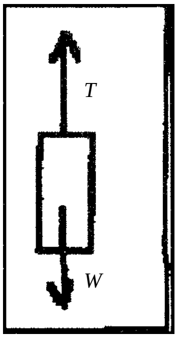
$$\frac{1}{5}\ln\left(\frac{80{,}000}{80{,}000 + 2.5v_0^2}\right) = -\frac{s}{18{,}000}$$

$$\boxed{x = 374 \text{ m.}}$$

Problem 14.46 A 890 N bungee jumper jumps from a bridge 39.6 m above a river. The bungee cord has an unstretched length of 18.3 m and has a spring constant $k = 204$ N/m. (a) How far above the river is the jumper when the cord brings him to a stop? (b) What maximum force does the cord exert on him?

Solution: Choose a coordinate system with s positive downward. Divide the fall into two parts: (1) the free fall until the bungee unstretched length is reached, (2) the fall to the full extension of the bungee cord. For Part (1): From Newton's second law $\frac{ds}{dt} = g$. Use the chain rule to write: $v\frac{dv}{ds} = g$. Separate variables and integrate:

$v^2(s) = 2$ gs, since $v(0) = 0$. At $s = 18.3$ m, $v(s = 18.3) = \sqrt{2\text{ gs}} = 18.93$ m/s. For Part (2):
From Newton's second law



$$W - T = \frac{W}{g}\left(\frac{dv}{dt}\right), \text{ where } T = k(s - 18.3).$$

Use the chain rule to write:

$$v\frac{dv}{ds} = g - \frac{gk}{W}(s - 18.3) = g\left(1 - \frac{k}{W}(s - 18.3)\right)\ (s \geq 18.3\text{ m}).$$

Separate variables and integrate:

$$v^2(s) = 2\,gs\left(1 - \frac{k}{2W}(s - 39.6)\right) + C. \text{ At } s = 18.3,$$

$$v^2(s = 18.3) = [2\text{ gs}]_{s=18.3} = 39.6\text{ g},$$

from which $C = -\frac{gk}{W}(18.3^2) = -753$. The velocity is

$$v^2(s) = -\frac{gk}{W}s^2 + 2g\left(1 + \frac{18.3\,k}{W}\right)s - \frac{gk}{W}(18.3^2).$$

When the jumper is brought to a stop, $v(s_{stop}) = 0$, from which $s^2 + 2bs + c = 0$, where $b = -\left(\frac{W}{k} + 18.3\right)$, and $c = 18.3^2$. The solution:

$$s_{\text{stop}} = -b \pm \sqrt{b^2 - c} = 36, = 9.29\text{ m}.$$

(a) The first value represents the maximum distance on the first excursion, from which

$$\boxed{h = 39.6 - 36 = 3.6\text{ m}}$$

is the height above the river at which he comes to a stop. (b) The maximum force exerted by the bungee cord is

$$\boxed{F = k(s - 18.3) = 204(36 - 18.3) = 3610.8\text{ N}}$$

Problem 14.47 A helicopter weighs 91.2 kN. It takes off vertically from sea level, and its upward velocity is given as a function of its altitude h in metre by $v = 66 - 0.01\ h$ m/s

(a) How long does it take the helicopter to climb to an altitude of 1219 m?
(b) What is the sum of the vertical forces on the helicopter when its altitude is 610 m?

Solution:

(a) $v = \dfrac{dh}{dt} = 66 - 0.01h, \Rightarrow \int_0^t dt = \int_0^{1219} \dfrac{dh}{66 - 0.01h}$

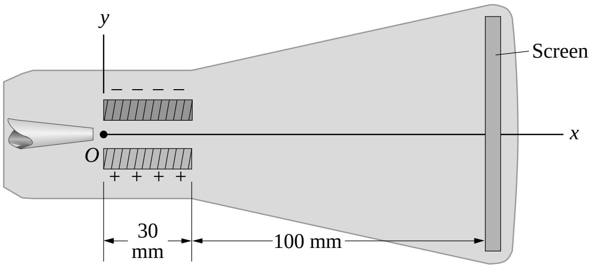
$\Rightarrow \boxed{t = 20.5\ \text{s}}$

(b) $a = v\dfrac{dv}{dh} = (66 - 0.01h)(-0.01) = -0.66 + 0.0001h$

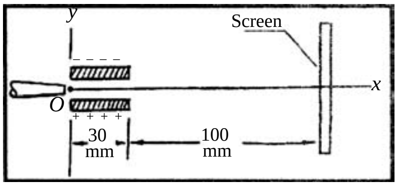
At $h = 610$ m we have

$a = -0.6\ \text{m/s}^2 \Rightarrow \boxed{F = \left(\dfrac{91200\ \text{N}}{9.81\ \text{m/s}^2}\right)(-0.6\ \text{m/s}^2) = -5578\ \text{N}}$

Problem 14.48 In a cathode-ray tube, an electron (mass $= 9.11 \times 10^{-31}$ kg) is projected at O with velocity $\mathbf{v} = (2.2 \times 10^7)\mathbf{i}$ (m/s). While the electron is between the charged plates, the electric field generated by the plates subjects it to a force $\mathbf{F} = -eE\mathbf{j}$, where the charge of the electron $e = 1.6 \times 10^{-19}$ C (coulombs) and the electric field strength $E = 15$ kN/C. External forces on the electron are negligible when it is not between the plates. Where does the electron strike the screen?



Solution: For brevity denote $L = 0.03$ m, $D = 0.1$ m. The time spent between the charged plates is $t_p = \dfrac{L}{V} = \dfrac{3 \times 10^{-2}\ \text{m}}{2.2 \times 10^7\ \text{m/s}} = 1.3636 \times 10^{-9}$ s. From Newton's second law, $\mathbf{F} = m_e\mathbf{a}_p$. The acceleration due to the charged plates is



$$\mathbf{a}_p = \frac{-eE}{m_e}\mathbf{j} = -\frac{(1.6 \times 10^{-19})(15 \times 10^3)}{9.11 \times 10^{-31}}\mathbf{j} = \mathbf{j}2.6345 \times 10^{15}\ \text{m/s}^2.$$

The velocity is $\mathbf{v}_y = -\mathbf{a}_p t$ and the displacement is $\mathbf{y} = \dfrac{\mathbf{a}_p}{2}t^2$. At the exit from the plates the displacement is $\mathbf{y}_p = -\dfrac{\mathbf{a}_p t_p^2}{2} = -\mathbf{j}2.4494 \times 10^{-3}$ (m). The velocity is $\mathbf{v}_{yp} = -\mathbf{a}_p t = -\mathbf{j}3.59246 \times 10^6$ m/s. The time spent in traversing the distance between the plates and the screen is $t_{ps} = \dfrac{D}{V} = \dfrac{10^{-1}\ \text{m}}{2.2 \times 10^7\ \text{m/s}} = 4.5455 \times 10^{-9}$ s. The vertical displacement at the screen is

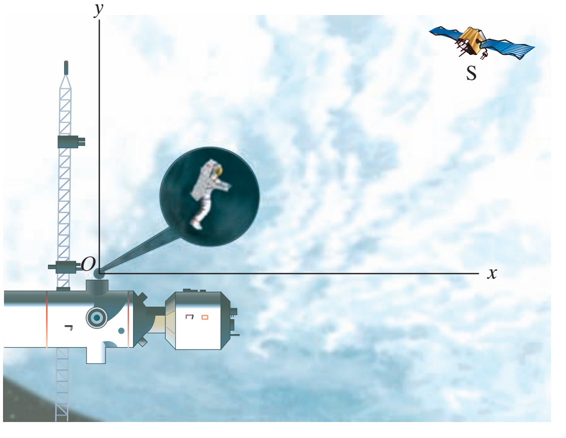
$$\boxed{\begin{aligned}\mathbf{y}_s = \mathbf{v}_{yp}t_{ps} + \mathbf{y}_p &= -\mathbf{j}(3.592456 \times 10^6)(4.5455 \times 10^{-9}) \\ &- \mathbf{j}2.4494 \times 10^{-3} = -18.8\mathbf{j}\ \text{(mm)}\end{aligned}}$$

Problem 14.49 In Problem 14.48, determine where the electron strikes the screen if the electric field strength is $E = 15\sin(\omega t)$ kN/C, where the frequency $\omega = 2 \times 10^9\ \text{s}^{-1}$.

Solution: Use the solution to Problem 14.48. Assume that the electron enters the space between the charged plates at $t = 0$, so that at that instant the electric field strength is zero. The acceleration due to the charged plates is $\mathbf{a} = -\dfrac{eE}{m_e}\mathbf{j} = -\dfrac{(1.6 \times 10^{-19}\ \text{C})(15000\sin\omega t\ \text{N/C})}{9.11 \times 10^{-31}\ \text{kg}}\mathbf{j} = -\mathbf{j}(2.6345 \times 10^{15})$ $\sin\omega t\,(\text{m/s}^2)$. The velocity is $\mathbf{v}_y = \mathbf{j}\dfrac{(2.6345 \times 10^{15})}{\omega}\cos\omega t + \mathbf{C}$. Since $\mathbf{v}_y(0) = 0$ $\mathbf{C} = -\dfrac{2.6345 \times 10^{15}}{2 \times 10^9}\mathbf{j} = -\mathbf{j}1.3172 \times 10^6$. The displacement is $\mathbf{y} = \mathbf{j}\dfrac{(2.6345 \times 10^{15})}{\omega^2}\sin\omega t + \mathbf{C}t$, since $\mathbf{y}(0) = 0$. The time spent between the charged plates is (see Problem 14.48) $t_p = 1.3636 \times 10^{-9}$ s, from which $\omega t_p = 2.7273$ rad. At exit from the plates, the vertical velocity is $\mathbf{v}_{yp} = \mathbf{j}\dfrac{2.6345 \times 10^{15}}{2 \times 10^9}\cos(\omega t_p) + \mathbf{C} = -\mathbf{j}2.523 \times 10^6$ (m/s).

The displacement is $\mathbf{y}_p = \mathbf{j}\dfrac{2.6345 \times 10^{15}}{4 \times 10^{18}}\sin(\omega t_p) + \mathbf{C}t_p = -\mathbf{j}1.531 \times 10^{-3}$ (m). The time spent between the plates and the screen is $t_{ps} = 4.5455 \times 10^{-9}$ s. The vertical deflection at the screen is $\boxed{\mathbf{y}_s = \mathbf{y}_p + \mathbf{v}_{yp}t_{ps} = -13\mathbf{j}(\text{mm})}$

Problem 14.50 An astronaut wants to travel from a space station to a satellites S that needs repair. She departs the space station at O. A spring-loaded launching device gives her maneuvering unit an initial velocity of 1 m/s (relative to the space station) in the y direction. At that instant, the position of the satellite is $x = 70$ m, $y = 50$ m, $z = 0$, and it is drifting at 2 m/s (relative to the station) in the x direction. The astronaut intercepts the satellite by applying a constant thrust parallel to the x axis. The total mass of the astronaut and her maneuvering unit is 300 kg. (a) How long does it take the astronaut to reach the satellite? (b) What is the magnitude of the thrust she must apply to make the intercept? (c) What is the astronaut's velocity *relative to the satellite* when she reaches it?

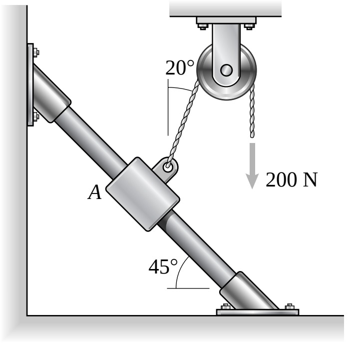


Solution: The path of the satellite relative to the space station is $x_s(t) = 2t + 70$ m, $y_s(t) = 50$ m. From Newton's second law, $T = ma_x$, $0 = ma_y$. Integrate to obtain the path of the astronaut, using the initial conditions $v_x = 0$, $v_y = 1$ m/s, $x = 0$, $y = 0$. $y_a(t) = t$, $x_a(t) = \dfrac{T}{2m}t^2$. (a) When the astronaut intercepts the x path of the satellite, $y_a(t_{\text{int}}) = y_s(t_{\text{int}})$, from which $\boxed{t_{\text{int}} = 50\ \text{s}}$. (b) The intercept of the y-axis path occurs when $x_a(t_{\text{int}}) = x_s(t_{\text{int}})$, from which $\dfrac{T}{2m}t_{\text{int}}^2 = 2t_{\text{int}} + 70$, from which

$$\boxed{T = (2m)\left(\frac{2t_{\text{int}} + 70}{t_{\text{int}}^2}\right) = 2(300)\left(\frac{170}{2500}\right) = 40.8\ \text{N}.}$$

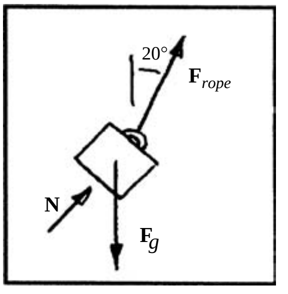
(c) The velocity of the astronaut relative to the space station is $\mathbf{v} = \mathbf{i}\left(\dfrac{T}{m}\right)t_{\text{int}} + \mathbf{j} = 6.8\mathbf{i} + \mathbf{j}$. The velocity of the satellite relative to the space station is $\mathbf{v}_s = 2\mathbf{i}$. The velocity of the astronaut relative to the satellite is $\boxed{\mathbf{v}_{a/s} = \mathbf{i}(6.8 - 2) + \mathbf{j} = 4.8\mathbf{i} + \mathbf{j}\ (\text{m/s})}$

Problem 14.51 What is the acceleration of the 8-kg collar A relative to the smooth bar?



Solution: For brevity, denote $\theta = 20^\circ$, $\alpha = 45^\circ$, $F = 200$ N, $m = 8$ kg. The force exerted by the rope on the collar is $\mathbf{F}_{rope} = 200(\mathbf{i}\sin\theta + \mathbf{j}\cos\theta) = 68.4\mathbf{i} + 187.9\mathbf{j}$ (N). The force due to gravity is $\mathbf{F}_g = -gm\mathbf{j} = -78.5\mathbf{j}$ N. The unit vector parallel to the bar, positive upward, is $\mathbf{e}_B = -\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha$. The sum of the forces acting to move the collar is $\sum F = F_c = \mathbf{e}_B \cdot \mathbf{F}_{rope} + \mathbf{e}_B \cdot \mathbf{F}_g = |\mathbf{F}_{rope}|\sin(\alpha - \theta) - gm\sin\alpha = 29.03$ N. The collar tends to slide up the bar since $F_c > 0$. From Newton's second law, the acceleration is

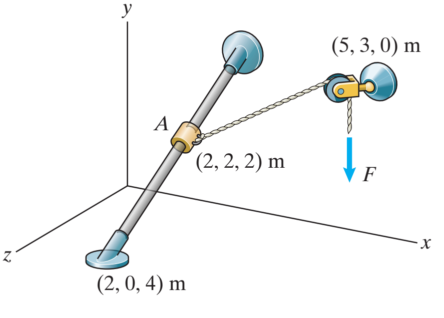
$$\boxed{a = \frac{F_c}{m} = 3.63 \text{ m/s}^2.}$$



Problem 14.52 In Problem 14.51, determine the acceleration of the 8-kg collar A relative to the bar if the coefficient of kinetic friction between the collar and the bar is $\mu_k = 0.1$.

Solution: Use the solution to Problem 14.51. $F_c = |\mathbf{F}_{rope}|\sin(\alpha - \theta) - gm\sin\alpha = 29.03$ N. The normal force is perpendicular to the bar, with the unit vector $\mathbf{e}_N = \mathbf{i}\sin\alpha + \mathbf{j}\cos\alpha$. The normal force is $N = \mathbf{e}_N \cdot \mathbf{F}_{rope} + \mathbf{e}_N \cdot \mathbf{F}_g = |\mathbf{F}_{rope}|\cos(\alpha - \theta) - gm\cos\alpha = 125.77$ N. The collar tends to slide up the bar since $F_c > 0$. The friction force opposes the motion, so that the sum of the forces on the collar is $\sum F = F_c - \mu_k N = 16.45$ N. From Newton's second law, the acceleration of the collar is $\boxed{a = \frac{16.45}{8} = 2.06 \text{ m/s}^2}$ up the bar.

Problem 14.53 The force $F = 50$ N. What is the magnitude of the acceleration of the 20-N collar A along the smooth bar at the instant shown?



Solution: The force in the rope is

$$\mathbf{F}_{rope} = (50 \text{ N})\frac{(5-2)\mathbf{i} + (3-2)\mathbf{j} + (0-2)\mathbf{k}}{\sqrt{(5-2)^2 + (3-2)^2 + (0-2)^2}}$$

The force of gravity is $\mathbf{F}_{grav} = -(20 \text{ N})\mathbf{j}$

The unit vector along the bar is

$$\mathbf{e}_{bar} = \frac{(2-2)\mathbf{i} + (2-0)\mathbf{j} + (2-4)\mathbf{k}}{\sqrt{(2-2)^2 + (2-0)^2 + (2-4)^2}}$$

The component of the total force along the bar is

$$F_{bar} = \mathbf{F}\bullet\mathbf{e}_{bar} = 14.2 \text{ N}$$

Thus $\boxed{14.2 \text{ N} = \left(\frac{20 \text{ N}}{9.81 \text{ m/s}^2}\right) a \Rightarrow a = 6.96 \text{ m/s}^2}$

Problem 14.54* In Problem 14.53, determine the magnitude of the acceleration of the 20-N collar A along the bar at the instant shown if the coefficient of static friction between the collar and the bar is $\mu_k = 0.2$.

Solution: Use the results from Problem 14.53.

The magnitude of the total force exerted on the bar is

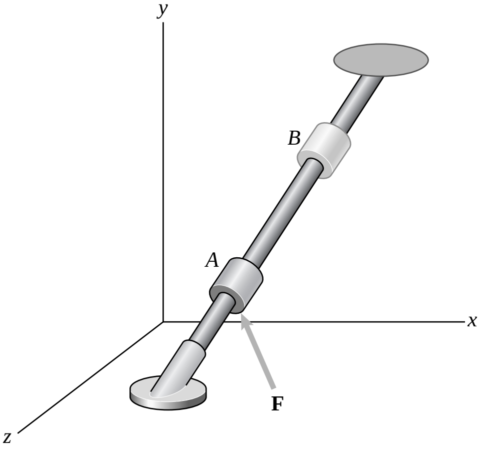
$$F = |\mathbf{F}_{rope} + \mathbf{F}_{grav}| = 48.6 \text{ N}$$

The normal force is $N = \sqrt{F^2 - F_{bar}{}^2} = 46.5$ N

The total force along the bar is now $F_{bar} - 0.2N = 4.90$ N

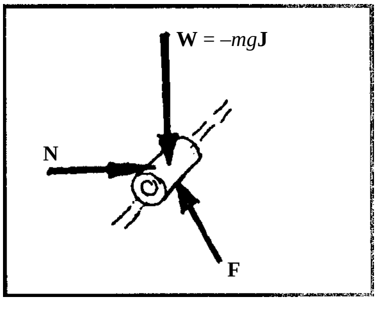
Thus $\boxed{4.90 \text{ N} = \left(\dfrac{20 \text{ N}}{9.81 \text{ m/s}^2}\right) a \Rightarrow a = 2.4 \text{ m/s}^2}$

Problem 14.55 The 6-kg collar starts from rest at position A, where the coordinates of its center of mass are (400, 200, 200) mm, and slides up the smooth bar to position B, where the coordinates of its center of mass are (500, 400, 0) mm under the action of a constant force $\mathbf{F} = -40\mathbf{i} + 70\mathbf{j} - 40\mathbf{k}$ (N). How long does it take to go from A to B?



Strategy: There are several ways to work this problem. One of the most straightforward ways is to note that the motion is along the straight line from A to B and that only the force components parallel to line AB cause acceleration. Thus, a good plan would be to find a unit vector from A toward B and to project all of the forces acting on the collar onto line AB. The resulting constant force (tangent to the path), will cause the acceleration of the collar. We then only need to find the distance from A to B to be able to analyze the motion of the collar.

Solution: The unit vector from A toward B is $\mathbf{e}_{AB} = \mathbf{e}_t = 0.333\mathbf{i} + 0.667\mathbf{j} - 0.667\mathbf{k}$ and the distance from A to B is 0.3 m. The free body diagram of the collar is shown at the right. There are three forces acting on the collar. These are the applied force $\mathbf{F}$, the weight force $\mathbf{W} = -mg\mathbf{j} = -58.86\mathbf{j}$ (N), and the force $\mathbf{N}$ which acts normal to the smooth bar. Note that $\mathbf{N}$, by its definition, will have no component tangent to the bar. Thus, we need only consider $\mathbf{F}$ and $\mathbf{W}$ when finding force components tangent to the bar. Also note that $\mathbf{N}$ is the force that the bar exerts on the collar to keep it in line with the bar. This will be important in the next problem.



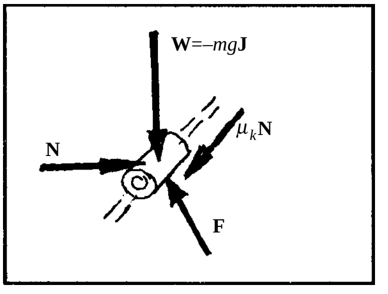
The equation of motion for the collar is $\sum \mathbf{F}_{\text{collar}} = \mathbf{F} + \mathbf{W} + \mathbf{N} = m\mathbf{a}$. In the direction tangent to the bar, the equation is $(\mathbf{F} + \mathbf{W})\cdot\mathbf{e}_{AB} = ma_t$.

The projection of $(\mathbf{F} + \mathbf{W})$ onto line AB yields a force along AB which is $|\mathbf{F}_{AB}| = 20.76$ N. The acceleration of the 6-kg collar caused by this force is $a_t = 3.46$ m/s^2. We now only need to know how long it takes the collar to move a distance of 0.3 m, starting from rest, with this acceleration. The kinematic equations are $v_t = a_t t$, and $s_t = a_t t^2/2$. We set $s_t = 0.3$ m and solve for the time. The time required is $t = 0.416$ s

Problem 14.56* In Problem 14.55, how long does the collar take to go from A to B if the coefficient of kinetic friction between the collar and the bar is $\mu_k = 0.2$?

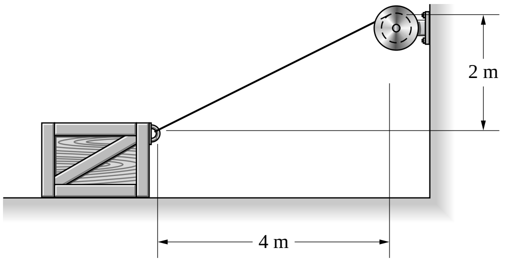
Strategy: This problem is almost the same as problem 14.55. The major difference is that now we must calculate the magnitude of the normal force, **N**, and then must add a term $\mu_k|\mathbf{N}|$ to the forces tangent to the bar (in the direction from B toward A — opposing the motion). This will give us a new acceleration, which will result in a longer time for the collar to go from A to B.

Solution: We use the unit vector $\mathbf{e}_{AB}$ from Problem 14.55. The free body diagram for the collar is shown at the right. There are four forces acting on the collar. These are the applied force **F**, the weight force $\mathbf{W} = -mg\mathbf{j} = -58.86\ \mathbf{j}$ (N), the force **N** which acts normal to the smooth bar, and the friction force $\mathbf{f} = -\mu_k|\mathbf{N}|\mathbf{e}_{AB}$. The normal force must be equal and opposite to the components of the forces **F** and **W** which are perpendicular (not parallel) to AB. The friction force is parallel to AB. The magnitude of $|\mathbf{F}+\mathbf{W}|$ is calculate by adding these two known forces and then finding the magnitude of the sum. The result is that $|\mathbf{F}+\mathbf{W}| = 57.66$ N. From Problem 14.55, we know that the component of $|\mathbf{F}+\mathbf{W}|$ tangent to the bar is $|\mathbf{F}_{AB}| = 20.76$ N. Hence, knowing the total force and its component tangent to the bar, we can find the magnitude of its component normal to the bar. Thus, the magnitude of the component of $|\mathbf{F}+\mathbf{W}|$ normal to the bar is 53.79 N. This is also the magnitude of the normal force **N**. The equation of motion for the collar is $\sum \mathbf{F}_{\text{collar}} = \mathbf{F}+\mathbf{W}+\mathbf{N}-\mu_k|\mathbf{N}|\mathbf{e}_{AB} = m\mathbf{a}$. In the direction tangent to the bar, the equation is $(\mathbf{F}+\mathbf{W})\cdot\mathbf{e}_{AB} - \mu_k|\mathbf{N}| = ma_t$.

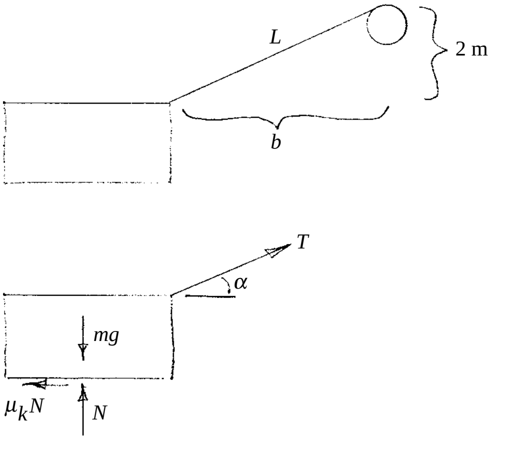


The force tangent to the bar is $F_{AB} = (\mathbf{F}+\mathbf{W})\cdot\mathbf{e}_{AB} - \mu_k|\mathbf{N}| = 10.00$ N. The acceleration of the 6-kg collar caused by this force is $a_t = 1.667$ m/s^2. We now only need to know how long it takes the collar to move a distance of 0.3 m, starting from rest, with this acceleration. The kinematic equations are $v_t = a_t t$, and $s_t = a_t t^2/2$. We set $s_t = 0.3$ m and solve for the time. The time required is $t = 0.600$ s

Problem 14.57 The crate is drawn across the floor by a winch that retracts the cable at a constant rate of 0.2 m/s. The crate's mass is 120 kg, and the coefficient of kinetic friction between the crate and the floor is $\mu_k = 0.24$. (a) At the instant shown, what is the tension in the cable? (b) Obtain a "quasi-static" solution for the tension in the cable by ignoring the crate's acceleration. Compare this solution with your result in (a).



Solution:



(a) Note that $b^2 + (2)^2 = L^2$, so $b\dfrac{db}{dt} = L\dfrac{dL}{dt}$ and $b\dfrac{d^2b}{dt^2} + \left(\dfrac{db}{dt}\right)^2 = L\dfrac{d^2L}{dt^2} + \left(\dfrac{dL}{dt}\right)^2$.

Setting $b = 4$ m, $dL/dt = -0.2$ m/s and $d^2L/dt^2 = 0$, we obtain $d^2b/dt^2 = -0.0025$ m/s^2. The crate's acceleration toward the right is $a = 0.0025$ m/s^2.

From the free-body diagram,

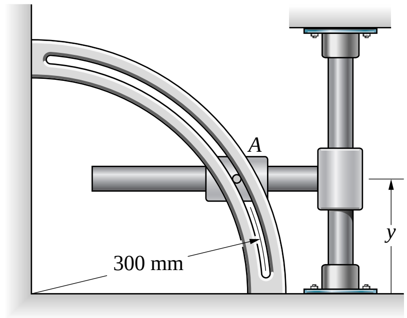
$$T\sin\alpha + N - mg = 0, \quad (1)$$

$$T\cos\alpha - \mu_k N = ma, \quad (2)$$

where $\alpha = \arctan(2/4) = 26.6^\circ$. Solving Eqs (1) and (2), we obtain $\underline{T = 282.3\text{ N}}$.

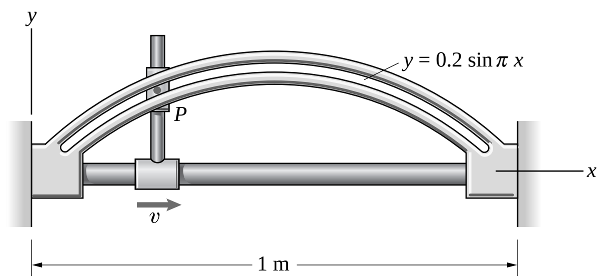
(b) Solving Eqs (1) and (2) with $a = 0$, we obtain $\underline{T = 282.0\text{ N}}$.

Problem 14.58 If $y = 100$ mm, $\dfrac{dy}{dt} = 600$ mm/s, and $\dfrac{d^2y}{dt^2} = -200$ mm/s^2, what horizontal force is exerted on the 0.4 kg slider A by the smooth circular slot?



Solution: The horizontal displacement is $x^2 = R^2 - y^2$. Differentiate twice with respect to time: $x\dfrac{dx}{dt} = -y\dfrac{dy}{dt}$, $\left(\dfrac{dx}{dt}\right)^2 + x\dfrac{d^2x}{dt^2} = -\left(\dfrac{dy}{dt}\right)^2 - y\left(\dfrac{d^2y}{dt^2}\right)$, from which. $\dfrac{d^2x}{dt^2} = -\left(\dfrac{1}{x}\right)\left(\left(\dfrac{y}{x}\right)^2 + 1\right)\left(\dfrac{dy}{dt}\right)^2 - \left(\dfrac{y}{x}\right)\dfrac{d^2y}{dt^2}$. Substitute: $\dfrac{d^2x}{dt^2} = -1.3612$ m/s^2. From Newton's second law, $\boxed{F_h = ma_x = -1.361(0.4) = -0.544 \text{ N}}$

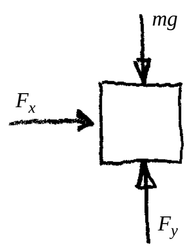
Problem 14.59 The 1-kg collar P slides on the vertical bar and has a pin that slides in the curved slot. The vertical bar moves with constant velocity $v = 2$ m/s. The y axis is vertical. What are the x and y components of the total force exerted on the collar by the vertical bar and the slotted bar when $x = 0.25$ m?



Solution:

$$v_x = \frac{dx}{dt} = 2 \text{ m/s, constant}$$

$$a_x = 0 = \frac{d^2x}{dt^2}$$



$$y = 0.2\sin(\pi x)$$

$$v_y = 0.2\pi\cos(\pi x)\frac{dx}{dt}$$

$$a_y = -0.2\pi^2\sin(\pi x)\left(\frac{dx}{dt}\right)^2 + 0.2\pi\cos(\pi x)\frac{d^2x}{dt^2}$$

when $x = 0.25$ m,

$$a_x = 0$$

$$a_y = -0.2\pi^2\sin\left(\frac{\pi}{4}\right)(2)^2 \text{ m/s}$$

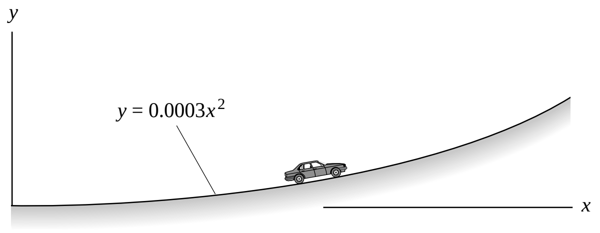
$$a_y = -5.58 \text{ m/s}^2$$

$$\sum F_x: \quad F_x = m(0)$$

$$\sum F_y: \quad F_y - mg = ma_y$$

Solving, $\underline{F_x = 0,\ F_y = 4.23 \text{ N}}$

Problem 14.60* The 1360-kg car travels along a straight road of increasing grade whose vertical profile is given by the equation shown. The magnitude of the car's velocity is a constant 100 km/h. When $x = 200$ m, what are the x and y components of the total force acting on the car (including its weight)?



Strategy: You know that the tangential component of the car's acceleration is zero. You can use this condition together with the equation for the profile of the road to determine the x and y components of the car's acceleration.

Solution:

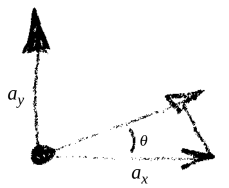
(1) $v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 100 \text{ km/h} = 27.78 \text{ m/s, const.}$

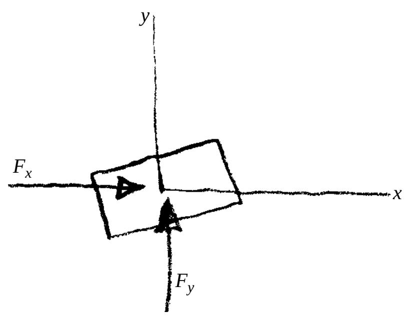
(2) $y = 0.0003x^2$

(3) $\frac{dy}{dt} = 0.0006x\frac{dx}{dt}$

(4) $\frac{d^2y}{dt^2} = 0.0006\left(\frac{dx}{dt}\right)^2 + 0.0006x\frac{d^2x}{dt^2}$

The component of acceleration parallel to the path is zero.





$\tan\theta = \frac{dy}{dx} = 0.0006\ x$

At $x = 200$ m, $\theta = 0.1194$ rad

$\theta = 6.84^\circ$

(5) $a_x \cos\theta + a_y \sin\theta = 0$

Solving eqns (1) through (5) simultaneously, we get

$a_x = -0.054 \text{ m/s}^2, \quad v_x = 27.6 \text{ m/s}$

$a_y = 0.450 \text{ m/s}^2, \quad v_y = 3.31 \text{ m/s}$

$m = 1360$ kg

$\sum F_x: \quad F_x = ma_x$

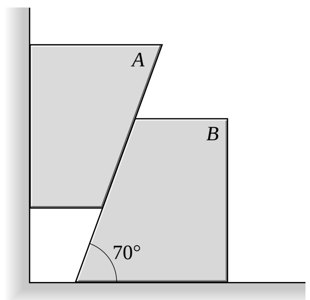
$\sum F_y: \quad F_y = ma_y$

Solving, $\underline{F_x = -73.4 \text{ N}}$

$\underline{F_y = 612 \text{ N}}$

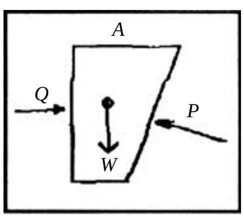
Problem 14.61* The two 445 N blocks are released from rest. Determine the magnitudes of their accelerations if friction at all contacting surfaces is negligible.

Strategy: Use the fact the components of the accelerations of the blocks perpendicular to their mutual interface must be equal.



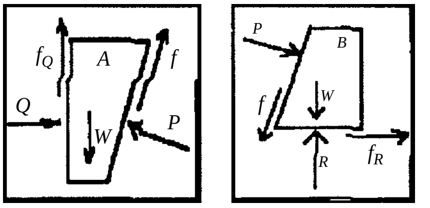
Solution: The relative motion of the blocks is constrained by the surface separating the blocks. The equation of the line separating the blocks is $y = x \tan 70^\circ$, where y is positive upward and x is positive to the right. A positive displacement of block A results in a negative displacement of B (as contact is maintained) from which $s_A = -s_B \tan 70^\circ$, and from which $\frac{d^2s_A}{dt^2} = -\frac{d^2s_B}{dt^2}\tan 70^\circ$. Thus (1) $a_A = -a_B \tan 70^\circ$.

From Newton's second law: for block A, (2) $\sum F_y = -W + P\cos 70^\circ = ma_A$, for block B, (3) $\sum F_x = P\sin 70^\circ = ma_B$, from which $a_A = -\frac{W}{m} + \frac{a_B}{\tan 70^\circ}$ m/s^2. Use (1) to obtain $a_A = -\frac{g}{1 + \cot^2 70^\circ} = -8.66 \text{ m/s}^2$ and $a_B = -\frac{a_A}{\tan 70^\circ} = 3.15 \text{ m/s}^2$, where a_A is positive upward and a_B is positive to the right.



Problem 14.62* The two 445 N blocks are released from rest. The coefficient of kinetic friction between all contacting surfaces is $\mu_k = 0.1$. How long does it take block A to fall 0.305 m?

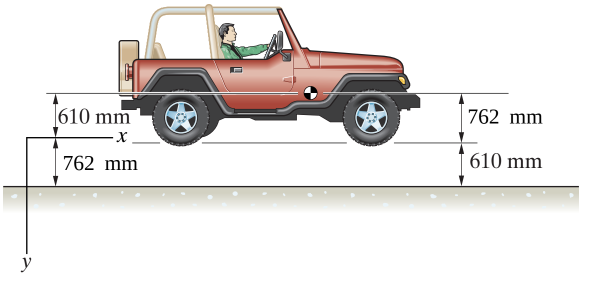
Solution: Use the results of the solution to Problem 14.61. Denote by Q the normal force at the wall, and by P the normal force at the contacting surface, and R the normal force exerted by the floor on block B. For a_A positive upward and a_B positive to the right, (1) $a_A = -a_B \tan 70^\circ$ so long as contact is maintained. From Newton's second law for block A, (2) $\sum F_x = Q - P \sin 70^\circ + f \cos 70^\circ = 0$, (3) $\sum F_y = -W + f_Q + f \cos 70^\circ + P \cos 70^\circ = ma_A$. For block B: (4) $\sum F_x = P \sin 70^\circ - f \cos 70^\circ - f_R = ma_B$, (5) $\sum F_y = -W + R - P \cos 70^\circ - f \sin 70^\circ = 0$. In addition: (6) $f = \mu_k P$, (7) $f_R = \mu_k R$, (8) $f_q = \mu_k Q$. Solve these eight equations by iteration: $a_A = -7.53$ m/s^2, $a_B = 2.74$ m/s^2. *Check*: (1) The effect of friction should reduce the downward acceleration of A in Problem 3.61, and (2) for $\mu_k = 0$, this should reduce to the solution to Problem 14.61. *check*.



The displacement is $y = \dfrac{a_A}{2} t^2$ m, from which, for $y = -0.305$ m,

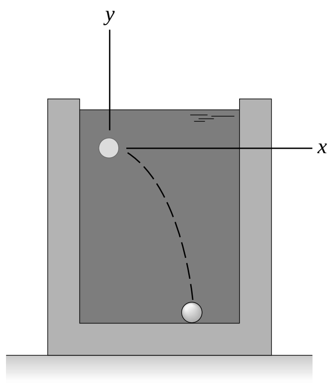
$t = \sqrt{-\dfrac{2}{a_A}} = 0.284$ s

Problem 14.63 The 3000-N vehicle has left the ground after driving over a rise. At the instant shown, it is moving horizontally at 30 km/h and the bottoms of its tires are 610 mm above the (approximately) level ground. The earth-fixed coordinate system is placed with its origin 762 mm above the ground, at the height of the vehicle's center of mass when the tires first contact the ground (Assume that the vehicle remains horizontal.) When that occurs, the vehicle's center of mass initially continues moving downward and then rebounds upward due to the flexure of the suspension system. While the tires are in contact with the ground, the force exerted on them by the ground is $-2400\mathbf{i} - 18000y\mathbf{j}$ (N), where y is the vertical position of the center of mass in metre. When the vehicle rebounds what is the vertical component of the velocity, of the center of mass at the instant the wheels leave the ground? (The wheels leave the ground when the center of mass is at $y = 0$.)



Solution: This analysis follows that of Example 14.3. The equation for velocity used to determine how far down the vehicle compresses its springs also applies as the vehicle rebounds. From Example 14.3, we know that the vehicle comes to rest with $v_Y = 0$ and $y = 0.305$ m. Following the Example, the velocity on the rebound is given by $\int_0^{v_Y} v_Y \, dv_Y = \int_{0.305}^{0} 9.81(0.305 - 15y)\, dy$. Evaluation, we get $v_Y = 3.44$ m/s. ($+y$ is down). Note that the vertical velocity component on rebound is the negative of the vertical velocity of impact.

Problem 14.64* A steel sphere in a tank of oil is given an initial velocity $\mathbf{v} = 2\mathbf{i}$ (m/s) at the origin of the coordinate system shown. The radius of the sphere is 15 mm. The density of the steel is 8000 kg/m^3 and the density of the oil is 980 kg/m^3. If V is the sphere's volume, the (upward) buoyancy force on the sphere is equal to the weight of a volume V of oil. The magnitude of the hydrodynamic drag force $\mathbf{D}$ on the sphere as it falls is $|\mathbf{D}| = 1.6|\mathbf{v}|$ N, where $|\mathbf{v}|$ is the magnitude of the sphere's velocity in m/s. What are the x and y components of the sphere's velocity at $t = 0.1$ s?



Solution:

$$B = \rho_{\text{oil}} V_g$$

$$W = \rho_{\text{STEEL}} V_g$$

$$V = \frac{4}{3}\pi r^3$$

$$\sum F_x: \quad m_s \frac{dv_x}{dt} = -d_x = -1.6v_x$$

$$\sum F_y: \quad m_s \frac{dv_y}{dt} = B - W - d_y$$

$$m_s \frac{dv_y}{dt} = (\rho_{\text{oil}} - \rho_{\text{STEEL}})V_g - 1.6v_y$$

Rewriting the equations, we get

$$\sum F_x: \quad \frac{dv_x}{dt} = -\frac{1.6}{m_s}v_x$$

$$\int_{v_{xo}}^{v_x} \frac{dv_x}{v_x} = -\frac{1.6}{m_s}\int_0^t dt$$

$$\ln(v_x)|_{v_{x0}}^{v_x} = -\frac{1.6}{m_s}t|_0^t$$

$$v_x = v_{x0} e^{-\frac{1.6}{m_s}t}$$

Substituting, we have $m_s = 0.114$ kg

$v_{x0} = 2$ m/s. At $t = 0.1$ s,

$\underline{v_x = 0.486 \text{ m/s}}$

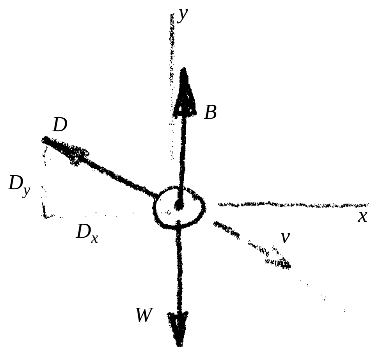
$$\sum F_y: \quad \frac{dv_y}{dt} = \frac{(\rho_{\text{oil}} - \rho_{\text{STEEL}})V_g}{m_s} - \frac{1.6}{m_s}v_y$$

Let $a = (\rho_{\text{oil}} - \rho_{\text{STEEL}})V_{g/m_s} = -8.608$

$$b = -\frac{1.6}{m_s} = -14.147$$

$$\frac{dv_y}{dt} = a + bv_y$$

$$\int_0^{v_y} \frac{dy}{a + bv_y} = \int_0^t dt$$



Integrating, we get

$$\frac{1}{b}\ln(a + bv_y)|_0^{v_y} = t$$

$$\ln\left(\frac{a + bv_y}{a}\right) = bt$$

$$a + bv_y = ae^{bt}$$

$$v_y = \frac{a}{b}(e^{bt} - 1)$$

Substituting numerical values for a and b, and setting $t = 0.1$ s

$\underline{v_y = -0.461 \text{ m/s}}$

Problem 14.65* In Problem 14.64, what are the x and y coordinates of the sphere at $t = 0.1$ s?

Solution: From the solution to Problem 14.64, $\frac{dx}{dt} = v_x = v_{x0} e^{-\frac{1.6}{m_s} t} = v_{x0} e^{bt}$ where $v_{x0} = 2$ m/s and $m_s = 0.114$ kg.

Also, $\frac{dy}{dt} = v_y = \frac{a}{b}(e^{bt} - 1)$

where $a = (\rho_{\text{oil}} - \rho_{\text{STEEL}}) V_{g/m} = -8.608$

$$b = -\frac{1.6}{m_s} = -14.147$$

Integrating the v_x and v_y eqns, noting that $x = 0$, $y = 0$, at $t = 0$, we get

$$x = \left(\frac{v_{xo}}{b}\right)(e^{bt} - 1)$$

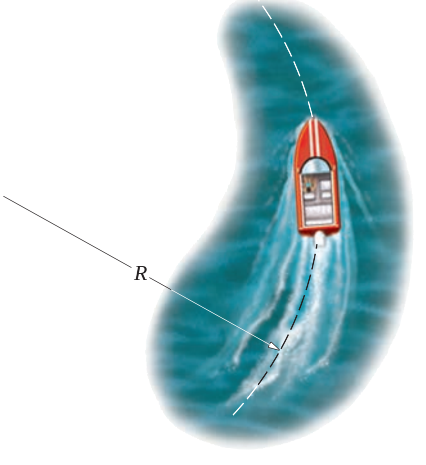
$$y = \frac{a}{b^2}(e^{bt} - 1) - \frac{a}{b} t$$

Solving at $t = 0.1$ s,

$\underline{x = 0.1070 \text{ m} = 107.0 \text{ mm}}$

$\underline{y = -0.0283 \text{ m} = -28.3 \text{ mm}}$

Problem 14.66 The boat in Active Example 14.5 weighs 1200 N with its passengers. Suppose that the boat is moving at a constant speed of 20 m/s in a circular path with radius $R = 40$ m. Determine the tangential and normal components of force acting on the boat.



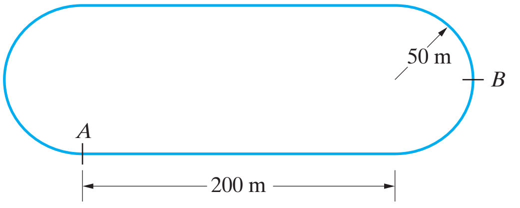
Solution: Since the speed is constant, the tangential acceleration is zero. We have

$F_t = ma_t = 0$,

$$F_n = ma_n = m\frac{v^2}{R} = \left(\frac{1200 \text{ N}}{9.81 \text{ m/s}^2}\right)\frac{(20 \text{ m/s})^2}{40 \text{ m}} = 1223 \text{ N}.$$

$\boxed{F_t = 0, F_n = 1223 \text{ N}.}$

Problem 14.67 In preliminary design studies for a sun-powered car, it is estimated that the mass of the car and driver will be 100 kg and the torque produced by the engine will result in a 60-N tangential force on the car. Suppose that the car starts from rest on the track at A and is subjected to a constant 60-N tangential force. Determine the magnitude of the car's velocity and the normal component of force on the car when it reaches B.



Solution: We first find the tangential acceleration and use that to find the velocity at B.

$F_t = ma_t \Rightarrow 60\text{ N} = (100\text{ kg})\,a_t \Rightarrow a_t = 0.6\text{ m/s}^2,$

$$a_t = v\frac{dv}{ds} \Rightarrow \int_0^v v\,dv = \int_0^s a_t\,ds \Rightarrow \frac{v^2}{2} = a_t s,$$

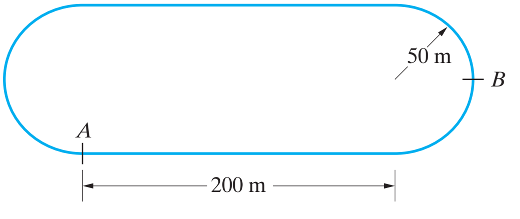
$$v_B = \sqrt{2a_t s_B} = \sqrt{2(0.6\text{ m/s}^2)\left(200\text{ m} + \frac{\pi}{2}[50\text{ m}]\right)} = 18.3\text{ m/s}.$$

The normal component of the force is

$$F_n = ma_n = m\frac{v^2}{R} = (100\text{ kg})\frac{(18.3\text{ m/s})^2}{50\text{ m}} = 668\text{ N}.$$

$$\boxed{v_B = 18.3\text{ m/s},\quad F_n = 668\text{ N}.}$$

Problem 14.68 In a test of a sun-powered car, the mass of the car and driver is 100 kg. The car starts from rest on the track at A, moving toward the right. The tangential force exerted on the car (in newtons) is given as a function of time by $\Sigma F_t = 20 + 1.2t$. Determine the magnitude of the car's velocity and the normal component of force on the car at $t = 40$ s.



Solution: We first find the tangential acceleration and use that to find the velocity v and distance s as functions of time.

$$\Sigma F_t = (20 + 1.2t)N = (100\text{ kg})\,a_t$$

$$a_t = \frac{dv}{dt} = 0.2 + 0.012t$$

$$v = 0.2t + 0.006t^2$$

$$s = 0.1t^2 + 0.002t^3$$

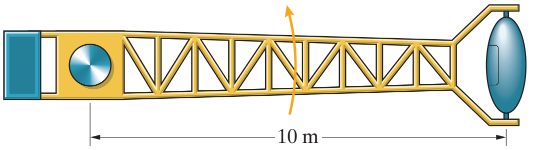
At $t = 40$ s, we have $v = 17.6$ m/s, $s = 288$ m.

For this distance the car will be on the curved part of the track. The normal component of the force is

$$F_n = ma_n = m\frac{v^2}{R} = (100\text{ kg})\frac{(17.6\text{ m/s})^2}{50\text{ m}} = 620\text{ N}.$$

$$\boxed{v_B = 17.6\text{ m/s},\quad F_n = 620\text{ N}.}$$

Problem 14.69 An astronaut candidate with a mass of 72 kg is tested in a centrifuge with a radius of 10 m. The centrifuge rotates in the horizontal plane. It starts from rest at time $t = 0$ and has a constant angular acceleration of 0.2 rad/s^2. Determine the magnitude of the horizontal force exerted on him by the centrifuge (a) at $t = 0$; (b) at $t = 10$ s.



Solution: The accelerations are

$$a_t = r\alpha = (10 \text{ m})\ (0.2 \text{ rad/s}^2) = 2 \text{ m/s}^2$$

$$a_n = r\omega^2 = r(\alpha t)^2 = (10 \text{ m})\ (0.2 \text{ rad/s}^2)^2 t^2 = (0.4 \text{ m/s}^4)t^2$$

(a)

At $t = 0$

$$F_t = ma_t = (72 \text{ kg})(2 \text{ m/s}^2) = 144 \text{ N},\ F_n = ma_n = 0$$

$$F = \sqrt{F_t^2 + F_n^2} = 144 \text{ N}.$$

(b)

At $t = 10$ s

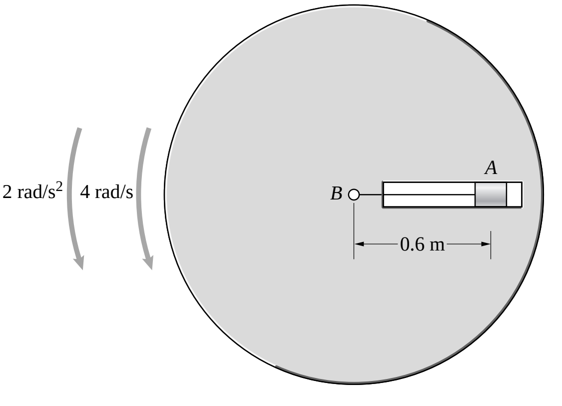
$$F_t = ma_t = (72 \text{ kg})(2 \text{ m/s}^2) = 144 \text{ N},$$

$$F_n = ma_n = (72 \text{ kg})(0.4 \text{ m/s}^4)(10 \text{ s})^2 = 2880 \text{ N}$$

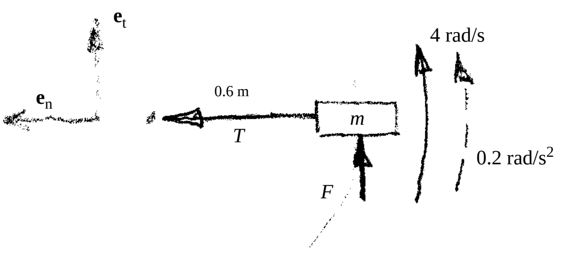
$$F = \sqrt{F_t^2 + F_n^2} = 2880 \text{ N}.$$

$$\boxed{(a)\ F = 144 \text{ N},\quad (b)\ F = 2880 \text{ N}.}$$

Problem 14.70 The circular disk lies *in the horizontal plane*. At the instant shown, the disk rotates with a counterclockwise angular velocity of 4 rad/s and a counterclockwise angular acceleration of 2 rad/s^2. The 0.5-kg slider A is supported horizontally by the smooth slot and the string attached at B. Determine the tension in the string and the magnitude of the horizontal force exerted on the slider by the slot.



Solution:



$\omega = 4$ rad/s

$\alpha = 2$ rad/s^2

$R = 0.6$ m

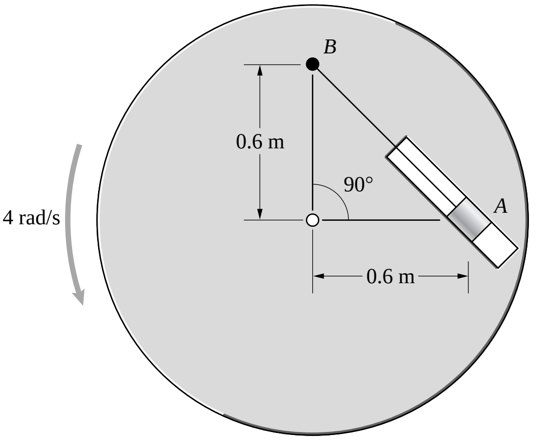
$m = 0.5$ kg

$\sum F_n : \quad T = mR\omega^2$

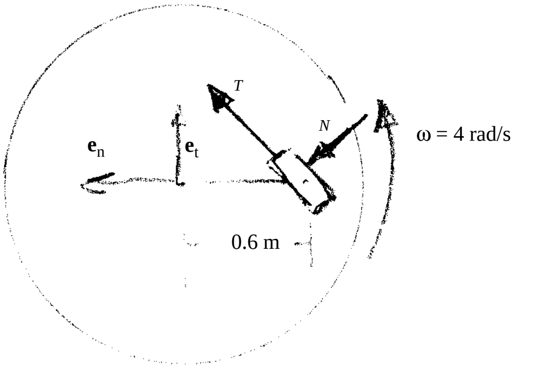
$\sum F_t : \quad F = mR\alpha$

Solving, $\underline{T = 4.8 \text{ N},\ F = 0.6 \text{ N}}$

Problem 14.71 The circular disk lies *in the horizontal plane* and rotates with a constant counterclockwise angular velocity of 4 rad/s. The 0.5-kg slider A is supported horizontally by the smooth slot and the string attached at B. Determine the tension in the string and the magnitude of the horizontal force exerted on the slider by the slot.



Solution:



$R = 0.6$ m

$\omega = 4$ rad/s

$m = 0.5$ kg

$\alpha = 0$

$\sum F_n$: $N\cos 45^\circ + T\sin 45^\circ = mR\omega^2$

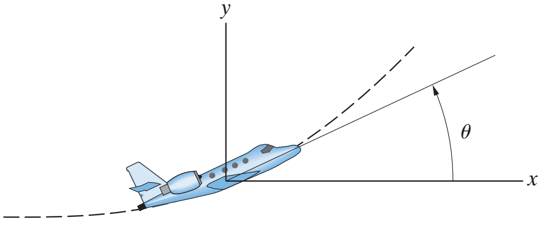
$\sum F_t$: $-N\sin 45^\circ + T\cos 45^\circ = mR\alpha = 0$

Solving, $\underline{N = T = 3.39 \text{ N}}$

Problem 14.72 The 142 kN airplane is flying in the vertical plane at 128 m/s. At the instant shown the angle $\theta = 30^\circ$ and the cartesian components of the plane's acceleration are $a_x = -1.83$ m/s^2, $a_y = 9.1$ m/s^2.

(a) What are the tangential and normal components of the total force acting on the airplane (including its weight)?

(b) What is $d\theta/dt$ in degrees per second?



Solution:

$$\mathbf{F} = \left(\frac{142000 \text{ N}}{9.81 \text{ m/s}^2}\right)(-1.83\mathbf{i} + 9.1\mathbf{j}) \text{ m/s}^2 = (-26523\mathbf{i} + 132613\mathbf{j}) \text{ N}$$

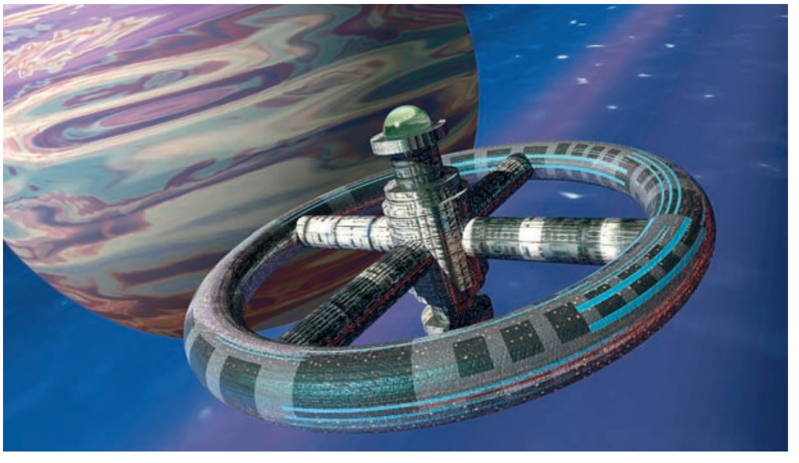
(a)
$$\boxed{\begin{aligned} F_t &= \mathbf{F}\cdot(\cos 30^\circ\mathbf{i} + \sin 30^\circ\mathbf{j}) = 43324 \text{ N} \\ F_n &= \mathbf{F}\cdot(-\sin 30^\circ\mathbf{i} + \cos 30^\circ\mathbf{j}) = 128102 \text{ N}\end{aligned}}$$

(b)
$$a_n = \frac{128102 \text{ N}}{(142000 \text{ N})/9.81 \text{ m/s}^2} = 8.83 \text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} \Rightarrow \rho = \frac{v^2}{a_n} = \frac{(128 \text{ m/s})^2}{(8.83 \text{ m/s}^2)} = 1855.3 \text{ m}$$

$$v = \rho\dot{\theta} \Rightarrow \boxed{\dot{\theta} = \frac{128 \text{ m/s}}{1855.3 \text{ m}} = 0.0690 \text{ rad/s}\left(\frac{180\ ^\circ}{\pi \text{ rad}}\right) = 3.95^\circ/\text{s}}$$

Problem 14.73 Consider a person with a mass of 72 kg who is in the space station described in Example 14.7. When he is in the occupied outer ring, his simulated weight in newtons is $\frac{1}{2}(72\ \text{kg})(9.81\ \text{m/s}^2) = 353$ N. Suppose that he climbs to a position in one of the radial tunnels that leads to the center of the station. Let r be his distance in meters from the center of the station. (a) Determine his simulated weight in his new position in terms of r. (b) What would his simulated weight be when he reaches the center of the station?



Solution: The distance to the outer ring is 100 m.

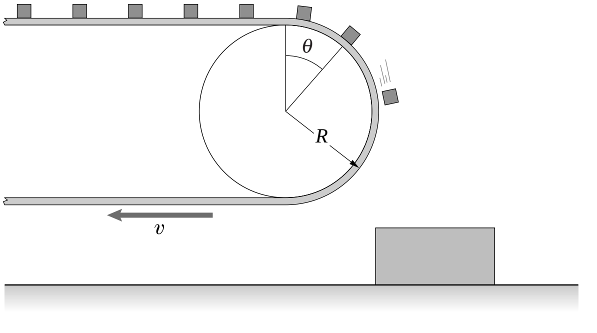
(a) At a distance r the weight would be $W = \dfrac{r}{100\ \text{m}}(353\ \text{N}) = (3.53\ \text{N/m})r$

$\boxed{W = (3.53\ \text{N/m})r.}$

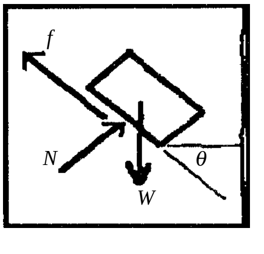
(b) At the center, $r = 0$

$\boxed{W = 0.}$

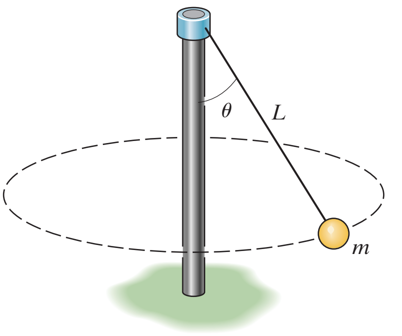
Problem 14.74 Small parts on a conveyor belt moving with constant velocity v are allowed to drop into a bin. Show that the angle θ at which the parts start sliding on the belt satisfies the equation $\cos\theta - \dfrac{1}{\mu_s}\sin\theta = \dfrac{v^2}{gR}$, where μ_s is the coefficient of static friction between the parts and the belt.



Solution: The condition for sliding is $\sum F_t = -mg\sin\theta + f = 0$, where $-mg\sin\theta$ is the component of weight acting tangentially to the belt, and $f = \mu_s$ N is the friction force tangential to the belt. From Newton's second law the force perpendicular to the belt is $N - mg\cos\theta = -m\dfrac{v^2}{R}$, from which the condition for slip is $-mg\sin\theta + \mu_s mg\cos\theta - \mu_s m\dfrac{v^2}{R} = 0$. Solve: $\cos\theta - \dfrac{1}{\mu_s}\sin\theta = \dfrac{v^2}{gR}$

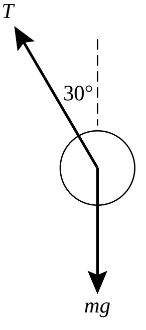


Problem 14.75 The 1-kg mass m rotates around the vertical pole in a horizontal circular path. The angle $\alpha = 30^\circ$ and the length of the string is $L = 1.22$ m. What is the magnitude of the velocity of the mass?



Strategy: Notice that the vertical acceleration of the mass is zero. Draw the free-body diagram of the mass and write Newton's second law in terms of tangential and normal components.

Solution:



$$\sum F_\uparrow : T\cos 30^\circ - mg = 0$$

$$\sum F_n : T\sin 30^\circ = m\frac{v^2}{\rho} = m\frac{v^2}{L\sin 30^\circ}$$

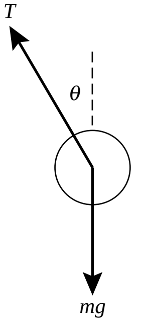
Solving we have

$$T = \frac{mg}{\cos 30^\circ},\ \ v^2 = g(L\sin 30^\circ)\tan 30^\circ$$

$$\boxed{v = \sqrt{\frac{(9.81\ \text{m/s}^2)(1.22\ \text{m})\sin^2 30^\circ}{\cos 30^\circ}} = 1.86\ \text{m/s}}$$

Problem 14.76 In Problem 14.75, determine the magnitude of the velocity of the mass and the angle θ if the tension in the string is 50 N.

Solution:



$$\sum F_\uparrow : T\cos\theta - mg = 0$$

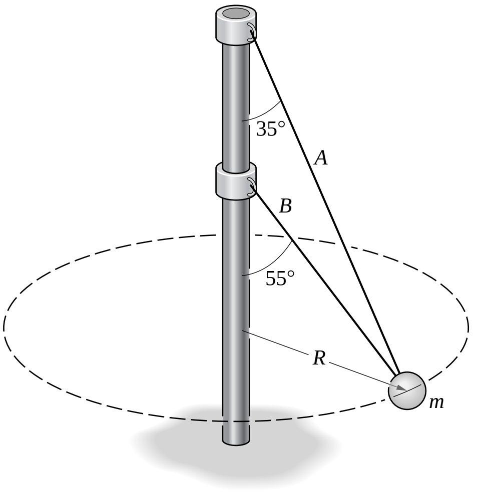
$$\sum F_n : T\sin\theta = m\frac{v^2}{L\sin\theta}$$

Solving we find $\theta = \cos^{-1}\left(\frac{mg}{T}\right),\ \ v = \sqrt{\frac{(T^2 - m^2g^2)L}{Tm}}$

Using the problem numbers we have

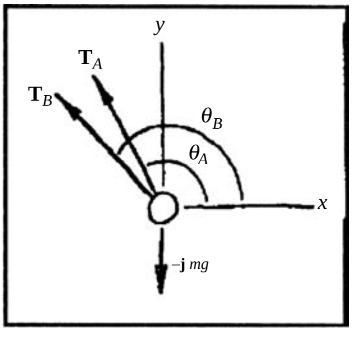
$$\boxed{\begin{aligned} \theta &= \cos^{-1}\left(\frac{1\ \text{kg}\ 9.81\ \text{m/s}^2}{50\ \text{N}}\right) = 78.68^\circ \\ v &= \sqrt{\frac{[(50\ \text{N})^2 - (1\ \text{kg}\ 9.81\ \text{m/s}^2)^2]1.22\ \text{m}}{(50\ \text{N})(1\ \text{kg})}} = 7.66\ \text{m/s} \end{aligned}}$$

Problem 14.77 The 10-kg mass m rotates around the vertical pole in a horizontal circular path of radius $R = 1$ m. If the magnitude of the velocity is $v = 3$ m/s, what are the tensions in the strings A and B?



Solution: Choose a Cartesian coordinate system in the vertical plane that rotates with the mass. The weight of the mass is $\mathbf{W} = -\mathbf{j}mg = -\mathbf{j}98.1$ N. The radial acceleration is by definition directed inward:

$\mathbf{a}_n = -\mathbf{i}\left(\frac{v^2}{R}\right) = -9\mathbf{i}$ m/s^2. The angles from the horizontal are $\theta_A = 90^\circ + 35^\circ = 125^\circ$, $\theta_B = 90^\circ + 55^\circ = 145^\circ$. The unit vectors parallel to the strings, from the pole to the mass, are: $\mathbf{e}_A = +\mathbf{i}\cos\theta_A + \mathbf{j}\sin\theta_A$. $\mathbf{e}_B = +\mathbf{i}\cos\theta_B + \mathbf{j}\sin\theta_B$. From Newton's second law for the mass, $\mathbf{T} - \mathbf{W} = m\mathbf{a}_n$, from which $|\mathbf{T}_A|\mathbf{e}_A + |\mathbf{T}_B|\mathbf{e}_B - \mathbf{j}mg = -\mathbf{i}\left(m\frac{v^2}{R}\right)$. Separate components to obtain the two simultaneous equations: $|\mathbf{T}_A|\cos 125^\circ + |\mathbf{T}_B|\cos 145^\circ = -90$ N$|\mathbf{T}_A|\sin 55^\circ + |\mathbf{T}_B|\sin 35^\circ = 98.1$ N. Solve:



$\boxed{|\mathbf{T}_A| = 84 \text{ N.}}$ $\boxed{|\mathbf{T}_B| = 51 \text{ N}}$

Problem 14.78 The 10-kg mass m rotates around the vertical pole in a horizontal circular path of radius $R = 1$ m. For what range of values of the velocity v of the mass will the mass remain in the circular path described?

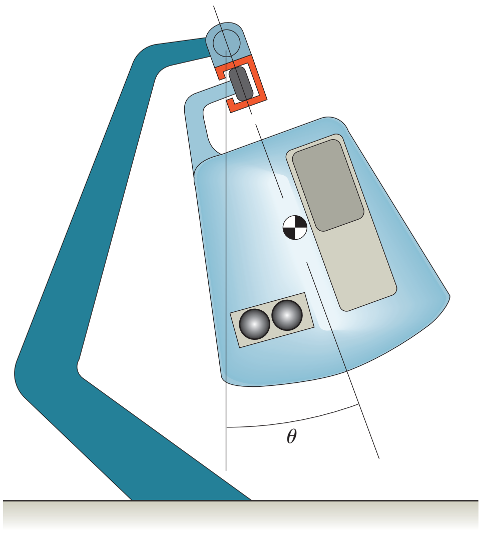
Solution: The minimum value of v will occur when the string B is impending zero, and the maximum will occur when string A is impending zero. From the solution to Problem 14.77,

$$|\mathbf{T}_A|\cos 125^\circ + |\mathbf{T}_B|\cos 145^\circ = -m\left(\frac{v^2}{R}\right),$$

$$|\mathbf{T}_A|\sin 125^\circ + |\mathbf{T}_B|\sin 145^\circ = \text{mg}.$$

These equations are to be solved for the velocity when one of the string tensions is set equal to zero. For $|\mathbf{T}_A| = 0$, $v = 3.743$ m/s. For $|\mathbf{T}_B| = 0$, $v = 2.621$ m/s. The range: $\boxed{2.62 \le v \le 3.74 \text{ m/s}}$

Problem 14.79 Suppose you are designing a monorail transportation system that will travel at 50 m/s, and you decide that the angle θ that the cars swing out from the vertical when they go through a turn must not be larger than 20°. If the turns in the track consist of circular arcs of constant radius R, what is the minimum allowable value of R? (See Active Example 14.6)



Solution: The equations of motion are

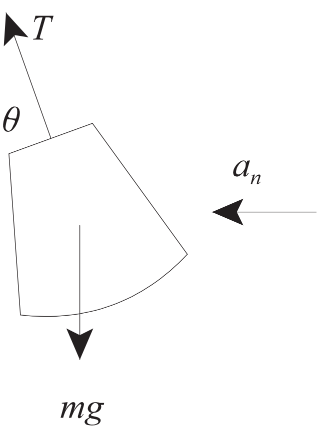
$$\Sigma F_y : T\cos\theta - mg = 0$$

$$\Sigma F_n : T\sin\theta = ma_n = m\frac{v^2}{R}$$

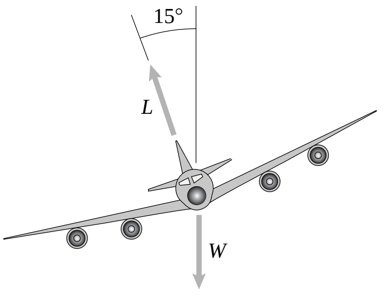
Solving we have

$$R = \frac{v^2}{g\tan\theta} = \frac{(50\text{ m/s})^2}{(9.81\text{ m/s}^2)\tan 20^\circ} = 700\text{ m}$$

$$\boxed{R = 700\text{ m.}}$$



Problem 14.80 An airplane of weight $W = 890$ kN makes a turn at constant altitude and at constant velocity $v = 183$ m/s. The bank angle is 15°. (a) Determine the lift force L. (b) What is the radius of curvature of the plane's path?



Solution: The weight is $\mathbf{W} = -\mathbf{j}W = -\mathbf{j}(890 \times 10^3)$ N. The normal acceleration is $\mathbf{a}_n = \mathbf{i}\left(\frac{v^2}{\rho}\right)$. The lift is $\mathbf{L} = |\mathbf{L}|(\mathbf{i}\cos 105^\circ + \mathbf{j}\sin 105^\circ) = |\mathbf{L}|(-0.2588\mathbf{i} + 0.9659\mathbf{j})$.

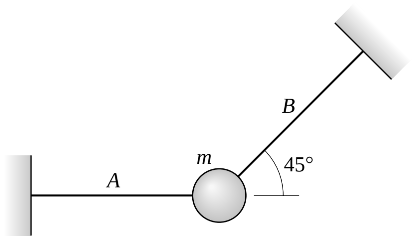
(a) From Newton's second law, $\sum \mathbf{F} = \mathbf{L} + \mathbf{W}_n = m\mathbf{a}_n$, from which, substituting values and separating the $\mathbf{j}$ components: $|\mathbf{L}|(0.9659) = 890 \times 10^3$, $\boxed{|\mathbf{L}| = \frac{890 \times 10^3}{0.9659} = 921420\text{ N}}$

(b) The radius of curvature is obtained from Newton's law: $|\mathbf{L}|(-0.2588) = -m\left(\frac{v^2}{\rho}\right)$, from which

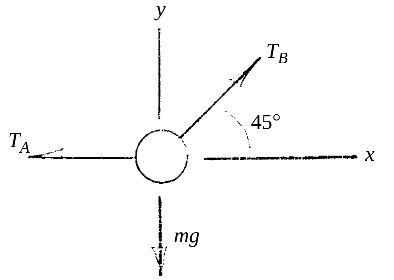
$$\boxed{\rho = \left(\frac{W}{g}\right)\left(\frac{v^2}{|\mathbf{L}|(0.2588)}\right) = 12729.6\text{ m.}}$$

Problem 14.81 The suspended 2-kg mass m is stationary.

(a) What are the tensions in the strings A and B?
(b) If string A is cut, what is the tension in string B immediately afterward?



Solution:



(a) $\sum F_x = T_B \cos 45^\circ - T_A = 0,$

$\sum F_y = T_B \sin 45^\circ - mg = 0.$

Solving yields $T_A = 19.6$ N, $T_B = 27.7$ N.

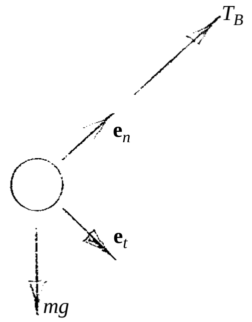
(b) Use Normal and tangential components.

$\sum F_{\text{n}} = ma_{\text{n}}$:

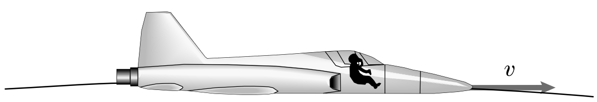
$$T_B - mg\cos 45^\circ = m\frac{v^2}{\rho}.$$

But $v = 0$ at the instant of release, so

$T_B = mg\cos 45^\circ = 13.9$ N.

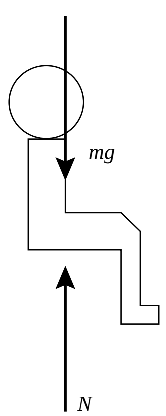


Problem 14.82 The airplane flies with constant velocity v along a circular path in the vertical plane. The radius of the airplane's circular path is 2000 m. The mass of the pilot is 72 kg.



(a) The pilot will experience "weightlessness" at the top of the circular path if the airplane exerts no net force on him at that point. Draw a free-body diagram of the pilot and use Newton's second law to determine the velocity v necessary to achieve this condition.
(b) Suppose that you don't want the force exerted on the pilot by the airplane to exceed four times his weight. If he performs this maneuver at $v =$ 200 m/s, what is the minimum acceptable radius of the circular path?

Solution: The FBD assumes that the seat pushes up on the pilot. If the seat (or shoulder straps) pushes down, we will us a negative sign for N.



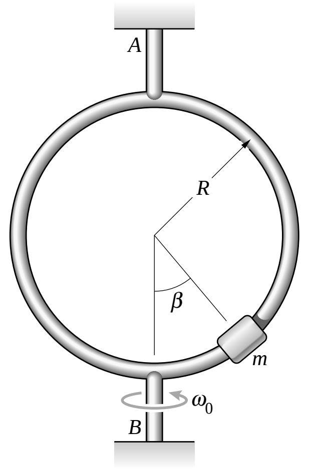
Dynamics: $\sum F_\uparrow : N - mg = -m\dfrac{v^2}{\rho}$

(a) $\boxed{N = 0 \Rightarrow v = \sqrt{g\rho} = \sqrt{(9.81\text{ m/s}^2)(2000\text{ m})} = 140\text{ m/s}}$

(b) The force will push down on the pilot

$$N = -4mg \Rightarrow -5mg = -m\frac{v^2}{\rho} \Rightarrow \rho = \frac{v^2}{5g} \quad \boxed{\rho = 815\text{ m}}$$

Problem 14.83 The smooth circular bar rotates with constant angular velocity ω_0 about the vertical axis AB. The radius $R = 0.5$ m. The mass m remains stationary relative to the circular bar at $\beta = 40^\circ$. Determine ω_0.



Solution:

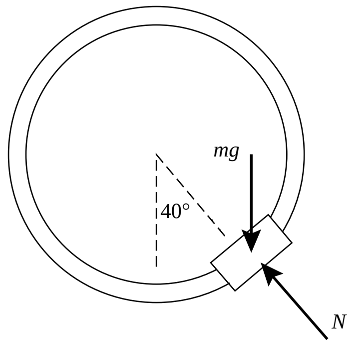
$$\sum F_\uparrow : N\cos 40^\circ - mg = 0$$

$$\sum F_\leftarrow : N\sin 40^\circ = m\frac{v^2}{\rho} = m\frac{(R\sin 40^\circ \omega_0)^2}{R\sin 40^\circ}$$

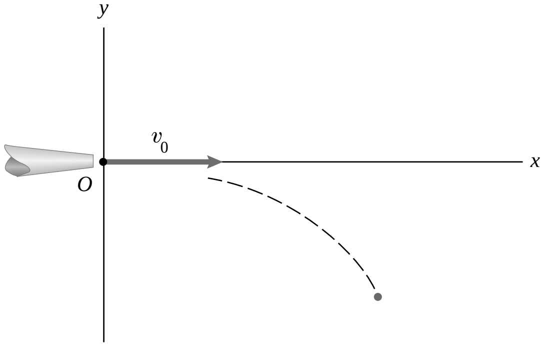
Solving we find

$$N = \frac{mg}{\cos 40^\circ}, \quad \omega_0 = \sqrt{\frac{g}{R\cos 40^\circ}}$$

$$\Rightarrow \boxed{\omega_0 = \sqrt{\frac{9.81 \text{ m/s}^2}{0.5 \text{ m}\cos 40^\circ}} = 5.06 \text{ rad/s}}$$



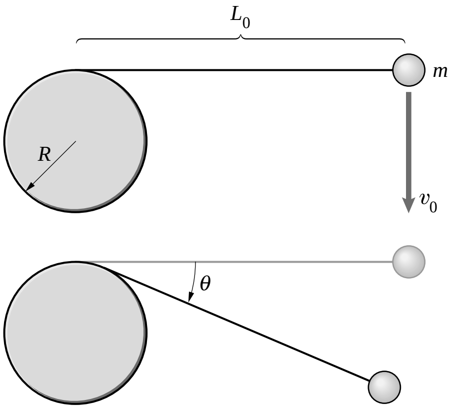
Problem 14.84 The force exerted on a charged particle by a magnetic field is $\mathbf{F} = q\mathbf{v}\times\mathbf{B}$, where q and $\mathbf{v}$ are the charge and velocity of the particle, and $\mathbf{B}$ is the magnetic field vector. A particle of mass m and positive charge q is projected at O with velocity $\mathbf{v} = v_0\mathbf{i}$ into a uniform magnetic field $\mathbf{B} = B_0\mathbf{k}$. Using normal and tangential components, show that (a) the magnitude of the particle's velocity is constant and (b) the particle's path is a circle of radius $m\dfrac{v_0}{qB_0}$.



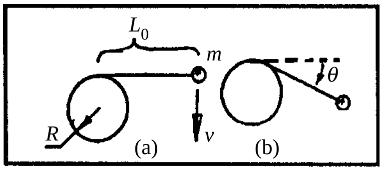
Solution: (a) The force $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$ is everywhere normal to the velocity and the magnetic field vector on the particle path. Therefore the tangential component of the force is zero, hence from Newton's second law the tangential component of the acceleration $\dfrac{dv}{dt} = 0$, from which $\boxed{v(t) = C = v_0}$, and the velocity is a constant. Since there is no component of force in the z-direction, and no initial z-component of the velocity, the motion remains in the *x-y* plane. The unit vector $\mathbf{k}$ is positive out of the paper in the figure; by application of the right hand rule the cross product $\mathbf{v}\times\mathbf{B}$ is directed along a unit vector toward the instantaneous center of the path at every instant, from which $\mathbf{F} = -|\mathbf{F}|\mathbf{e}_\text{n}$, where $\mathbf{e}_\text{n}$ is a unit vector normal to the path. The normal component of the acceleration is $\mathbf{a}_n = -(v_0^2/\rho)\mathbf{e}_\text{n}$, where ρ is the radius of curvature of the path. From Newton's second law, $\mathbf{F} = m\mathbf{a}_\text{n}$, from which $-|\mathbf{F}| = -m(v_0^2/\rho)$. The magnitude of the cross product can be written as $|\mathbf{v}\times\mathbf{B}| = v_0B_0\sin\theta = v_0B_0$, since $\theta = 90^\circ$ is the angle between $\mathbf{v}$ and $\mathbf{B}$. From which: $qv_0B_0 = m\dfrac{v_0^2}{\rho}$, from which the radius of curvature is $\boxed{\rho = \dfrac{mv_0}{qB_0}}$. Since the term on the right is a constant, the radius of curvature is a constant, and the path is a circle with radius $\dfrac{mv_0}{qB_0}$.

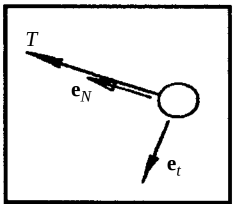
Problem 14.85 The mass m is attached to a string that is wrapped around a fixed post of radius R. At $t = 0$, the object is given a velocity v_0 as shown. Neglect external forces on m other that the force exerted by the string. Determine the tension in the string as a function of the angle θ.

Strategy: The velocity vector of the mass is perpendicular to the string. Express Newton's second law in terms of normal and tangential components.



Solution: Make a hypothetical cut in the string and denote the tension in the part connected to the mass by T. The acceleration normal to the path is $\dfrac{v^2}{\rho}$. The instantaneous radius of the path is $\rho = L_0 - R\theta$, from which by Newton's second law, $\sum F_n = T = m\dfrac{v_0^2}{L_0 - R\theta}$, from which $\boxed{T = m\dfrac{v_0^2}{L_0 - R\theta}}$



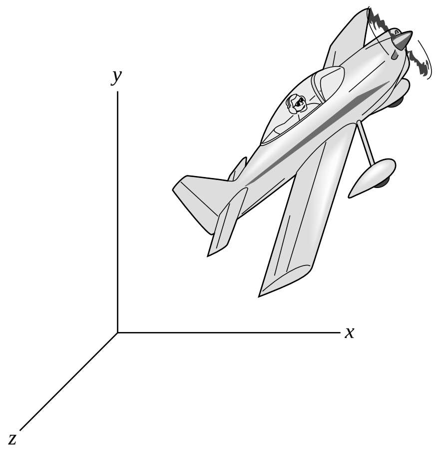


Problem 14.86 The mass m is attached to a string that is wrapped around the fixed post of radius R. At $t = 0$, the mass is given a velocity v_0 as shown. Neglect external forces on m other than the force exerted by the string. Determine the angle θ as a function of time.

Solution: Use the solution to Problem 14.85. The angular velocity is $\dfrac{d\theta}{dt} = \dfrac{v_0}{\rho}$. From Problem 14.85, $\rho = L_0 - R\theta$, from which $\dfrac{d\theta}{dt} = \dfrac{v_0}{(L_0 - R\theta)}$. Separate variables: $(L_0 - R\theta)d\theta = v_0 dt$. Integrate: $L_0\theta - \dfrac{R}{2}\theta^2 = v_0 t$, since $\theta(0) = 0$. In canonical form $\theta^2 + 2b\theta + c = 0$, where $b = -\dfrac{L}{R}$, and $c = \dfrac{2v_0 t}{R}$. The solution: $\theta = -b \pm \sqrt{b^2 - c} = \dfrac{L_0}{R} \pm \sqrt{\left(\dfrac{L_0}{R}\right)^2 - \dfrac{2v_0 t}{R}}$. The angle increases with time, so the negative sign applies. Reduce: $\boxed{\theta = \dfrac{L_0}{R}\left(1 - \sqrt{1 - \dfrac{2Rv_0 t}{L_0^2}}\right)}$

Check: When $R\theta = L_0$, the string has been fully wrapped around the post. Substitute to obtain: $\sqrt{1 - \dfrac{2Rv_0 t}{L_0^2}} = 0$, from which $\dfrac{2Rv_0 t}{L_0^2} = 1$, which is the value for impending failure as t increases because of the imaginary square root. Thus the solution behaves as expected. *check*.

Problem 14.87 The sum of the forces in newtons exerted on the 360-kg sport plane (including its weight) during an interval of time is $(-1000 + 280t)\mathbf{i} + (480 - 430t)\mathbf{j} + (720 + 200t)\mathbf{k}$, where t is the time in seconds. At $t = 0$, the velocity of the plane's center of gravity is $20\mathbf{i} + 35\mathbf{j} - 20\mathbf{k}$ (m/s). If you resolve the sum of the forces on the plane into components tangent and normal to the plane's path at $t = 2$ s, what are their values of $\sum F_t$ and $\sum F_n$?



Solution: This problem has several steps. First, we must write Newton's second law and find the acceleration of the aircraft. We then integrate the components of the acceleration (separately) to find the velocity components as functions of time. Then we evaluate the velocity of the aircraft and the force acting on the aircraft at $t = 2s$. Next, we find a unit vector along the velocity vector of the aircraft and project the total force acting on the aircraft onto this direction. Finally, we find the magnitude of the total force acting on the aircraft and the force component normal to the direction of flight. We have $a_X = (1/m)(-1000 + 280t)$, $a_Y = (1/m)(480 - 430t)$, and $a_Z = (1/m)(720 + 200t)$. Integrating and inserting the known initial velocities, we obtain the relations $v_X = v_{X0} + (1/m)(-1000t + 280t^2/2)$ (m/s) $= 20 + (1/m)(-1000t + 140t^2)$ (m/s), $v_Y = 35 + (1/m)(480t - 215t^2)$ (m/s), and $v_Z = -20 + (1/m)(720t + 100t^2)$ (m/s). The velocity at $t = 2s$ is $\mathbf{v} = 16\mathbf{i} + 35.3\mathbf{j} - 14.9\mathbf{k}$ (m/s) and the unit vector parallel to $\mathbf{v}$ is $\mathbf{e_v} = 0.386\mathbf{i} + 0.850\mathbf{j} - 0.359\mathbf{k}$. The total force acting on the aircraft at $t = 2s$ is $\mathbf{F} = -440\mathbf{i} - 380\mathbf{j} + 1120\mathbf{k}$ N. The component of $\mathbf{F}$ tangent to the direction of flight is $\sum F_t = \mathbf{F} \cdot \mathbf{e_v} = -894.5$ N. The magnitude of the total force acting on the aircraft is $|\mathbf{F}| = 1261.9$ N. The component of $\mathbf{F}$ normal to the direction of flight is given by $\sum F_n = \sqrt{|\mathbf{F}|^2 - (\sum F_t)^2} = 890.1$ N.

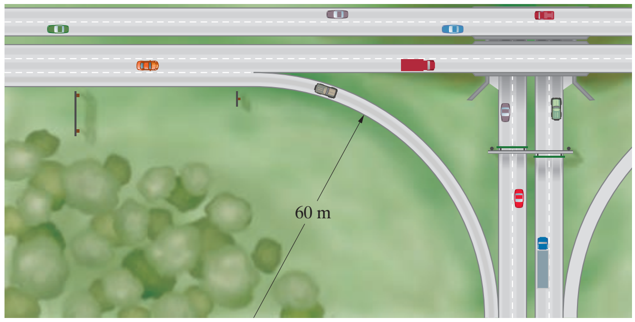
Problem 14.88 In Problem 14.87, what is the instantaneous radius of curvature of the plane's path at $t = 2$ s? The vector components of the sum of the forces in the directions tangenial and normal to the path lie in the osculating plane. Determine the components of a unit vector perpendicular to the osculating plane at $t = 2$ s.

Strategy: From the solution to problem 14.87, we know the total force vector and acceleration vector acting on the plane. We also know the direction of the velocity vector. From the velocity and the magnitude of the normal acceleration, we can determine the radius of curvature of the path. The cross product of the velocity vector and the total force vector will give a vector perpendicular to the plane containing the velocity vector and the total force vector. This vector is perpendicular to the plane of the osculating path. We need then only find a unit vector in the direction of this vector.

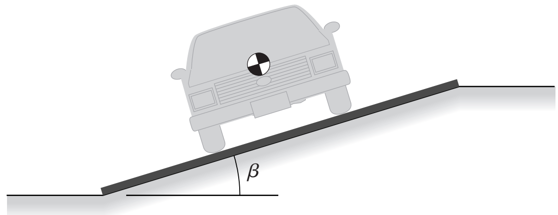
Solution: From Problem 14.87, we know at $t = 2$ s, that $a_n = \sum F_n/m = 2.47$ m/s^2. We can find the magnitude of the velocity $|\mathbf{v}| = 41.5$ m/s at this time. The radius of curvature of the path can then be found from $\rho = |\mathbf{v}|^2/a_n = 696.5$ m.

The cross product yields the desired unit vector, i.e., $\mathbf{e} = (\mathbf{F} \times \mathbf{v})/|\mathbf{F} \times \mathbf{v}| = -0.916\mathbf{i} + 0.308\mathbf{j} - 0.256\mathbf{k}$

Problem 14.89 The freeway off-ramp is circular with 60-m radius (Fig. a). The off-ramp has a slope $\beta = 15^\circ$ (Fig. b). If the coefficient of static friction between the tires of a car and the road is $\mu_s = 0.4$, what is the maximum speed at which it can enter the ramp without losing traction? (See Example 14.18.)



(a)



(b)

Solution:

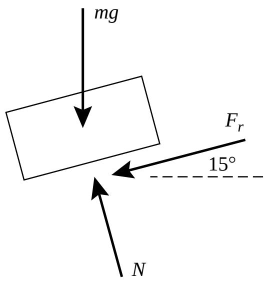
$\rho = 60$ m, $\quad g = 9.81$ m/s^2

$$\sum F_\uparrow : N\cos 15^\circ - F_r \sin 15^\circ - mg = 0$$

$$\sum F_\leftarrow : N\sin 15^\circ + F_r \cos 15^\circ = m\frac{v^2}{\rho}$$

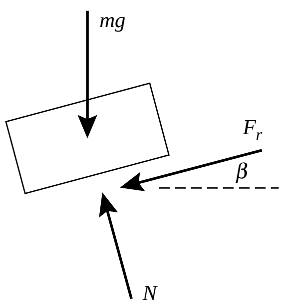
$F_r = 0.4N$

Solving we have $\boxed{v = 21.0 \text{ m/s}}$



Problem 14.90* The freeway off-ramp is circular with 60-m radius (Fig. a). The off-ramp has a slope β (Fig. b). If the coefficient of static friction between the tires of a car and the road is $\mu_s = 0.4$ what minimum slope β is needed so that the car could (in theory) enter the off-ramp at any speed without losing traction? (See Example 14.8.)

Solution:



$$\sum F_\uparrow : N\cos\beta - F_r \sin\beta - mg = 0$$

$$\sum F_\leftarrow : N\sin\beta + F_r \cos\beta = m\frac{v^2}{\rho}$$

$F = \mu N$

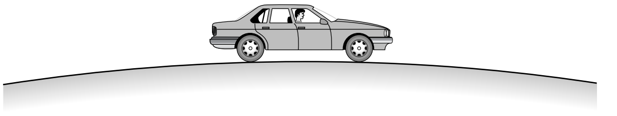
Solving we have $F_r = \dfrac{\mu m g}{\cos\beta - \mu\sin\beta}$.

If we set the denominator equal to zero, then we will always have enough friction to prevent sliding.

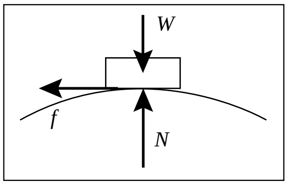
Thus $\boxed{\beta = \tan^{-1}\left(\frac{1}{\mu}\right) = \tan^{-1}\left(\frac{1}{0.4}\right) = 68.2^\circ}$

We would also need to check the low-speed case, where the car might slip down the ramp.

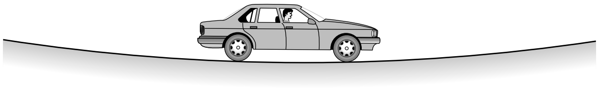
Problem 14.91 A car traveling at 30 m/s is at the top of a hill. The coefficient of kinetic friction between the tires and the road is $\mu_k = 0.8$. The instantaneous radius of curvature of the car's path is 200 m. If the driver applies the brakes and the car's wheels lock, what is the resulting deceleration of the car tangent to its path?



Solution: From Newton's second law; $N - W = -m\dfrac{v^2}{R}$ from which $N = m\left(g - \dfrac{v^2}{R}\right)$. The acceleration tangent to the path is $\dfrac{dv}{dt}$, from which $\dfrac{dv}{dt} = -\dfrac{\mu_k N}{m}$, and $\boxed{\dfrac{dv}{dt} = -\mu_k\left(g - \dfrac{v^2}{R}\right) = 4.25 \text{ m/s}^2}$



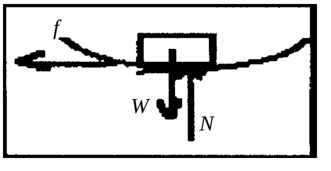
Problem 14.92 A car traveling at 30 m/s is at the bottom of a depression. The coefficient of kinetic friction between the tires and the road is $\mu_{\mathrm{k}} = 0.8$. The instantaneous radius of curvature of the car's path is 200 m. If the driver applies the brakes and the car's wheel lock, what is the resulting deceleration of the car in the direction tangential to its path? Compare your answer to that of Problem 14.91.



Solution: Use the solution to Problem 14.91: $\dfrac{dv}{dt} = -\dfrac{\mu_k N}{m}$. From Newton's second law, $N - W = m\left(\dfrac{v^2}{R}\right)$, from which

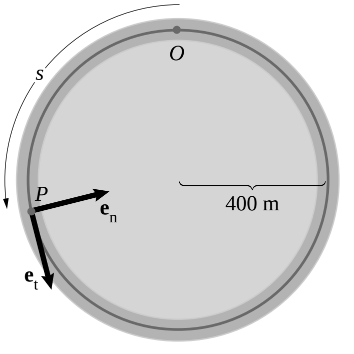
$$N = m\left(g + \left(\frac{v^2}{R}\right)\right),$$

and $\boxed{\dfrac{dv}{dt} = -\mu_k\left(g + \dfrac{v^2}{R}\right) = -11.45 \text{ m/s}^2}$



Problem 14.93 The combined mass of the motorcycle and rider is 160 kg. The motorcycle starts from rest at $t = 0$ and moves along a circular track with 400-m radius. The tangential component of acceleration as a function of time is $a_t = 2 + 0.2t$ m/s^2. The coefficient of static friction between the tires and the track is $\mu_s = 0.8$. How long after it starts does the motorcycle reach the limit of adhesion, which means its tires are on the verge of slipping? How fast is the motorcycle moving when that occurs?

Strategy: Draw a free-body diagram showing the tangential and normal components of force acting on the motorcycle.



Solution:

$m = 160$ kg

$R = 400$ m

Along track motion:

$a_t = 2 + 0.2t$ m/s^2

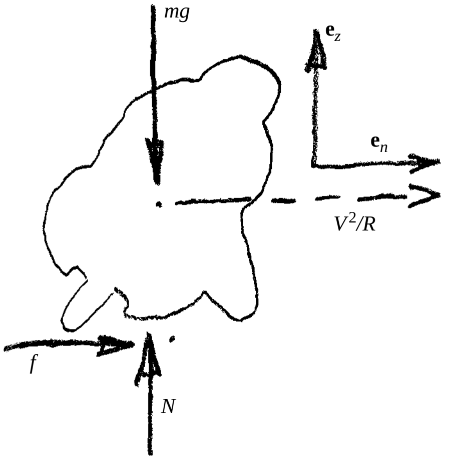
$V_t = V = 2t + 0.1t^2$ m/s

$s = t^2 + 0.1t^3/3$ m

Forces at impending slip

$|\mathbf{F} + \mathbf{f}| = \mu_k N$ at impending slip

$|\mathbf{F} + \mathbf{f}| = \sqrt{F^2 + f^2}$ since $\mathbf{f} \perp \mathbf{F}$



Force eqns.

$\sum F_t : F = ma_t$ $\quad$ $R = 400$ m

$\sum F_n : f = mv^2/R$ $\quad$ $m = 160$ kg

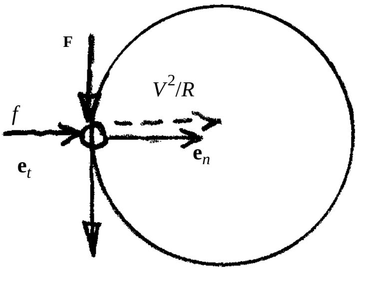
$\sum F_z : N - mg = 0$ $\quad$ $g = 9.81$ m/s^2

$\mu_s = 0.8$

$\sqrt{F^2 + f^2} = \mu_s N$

$a_t = 2 + 0.2t$

$v = 2t + 0.1t^2$



Six eqns, six unknowns (F, f, v, a_t, N, t)

Solving, we have

$\underline{t = 14.4 \text{ s}}$ $\quad$ $F = 781$ N

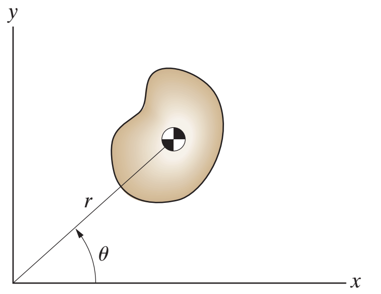
$\underline{v = 49.6 \text{ m/s}}$ $\quad$ $f = 983$ N

$N = 1570$ N

$a_t = 4.88$ m/s^2

(at $t = 14.4$ s)

Problem 14.94 The center of mass of the 12-kg object moves in the $x-y$ plane. Its polar coordinates are given as functions of time by $r = 12 - 0.4t^2$ m, $\theta = 0.02t^3$ rad. Determine the polar components of the total force acting on the object at $t = 2$ s.



Solution:

$r = 12 - 0.4t^2, \quad \theta = 0.02t^3$

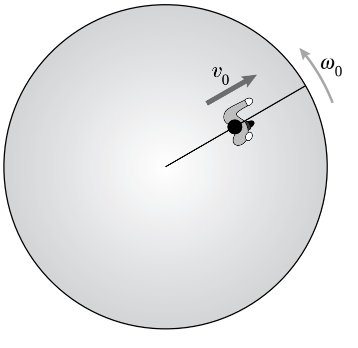
$\dot{r} = -0.8t, \quad \dot{\theta} = 0.06t^2$

$\ddot{r} = -0.8, \quad \ddot{\theta} = 0.12t$

Set t = 2 s

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = (12\text{ kg})(-0.8 - [10.4][0.24]^2)\text{ m/s}^2$$
$$= -16.8\text{ N}$$
$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = (12\text{ kg})([10.4][0.24] + 2[-1.6][0.24])\text{ m/s}^2$$
$$= 20.7\text{ N}$$

Problem 14.95 A 445 N person walks on a large disk that rotates with constant angular velocity $\omega_0 = 0.3$ rad/s. He walks at a constant speed $v_0 = 1.52$ m/s along a straight radial line painted on the disk. Determine the polar components of the horizontal force exerted on him when he is 1.83 m from the center of the disk. (How are these forces exerted on him?)



Solution:

$r = 1.83$ m, $\dot{r} = 1.52$ m/s, $\ddot{r} = 0$, $\dot{\theta} = 0.3$ rad/s, $\ddot{\theta} = 0$

$$F_r = m(\ddot{r} - r\dot{\theta}^2) = \left(\frac{445\text{ N}}{9.81\text{ m/s}^2}\right)(0 - [1.83\text{ m}][0.3\text{ rad/s}]^2)$$
$$= -7.47\text{ N}$$
$$F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \left(\frac{445\text{ N}}{9.81\text{ m/s}^2}\right)(0 + 2[1.52\text{ m/s}][0.3\text{ rad/s}])$$
$$= 41.5\text{ N}$$

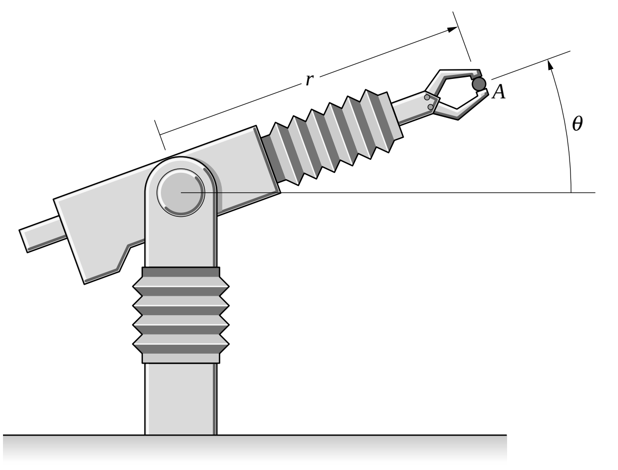
The forces are exerted as friction between the disk and the man's feet.

Problem 14.96 The robot is programmed so that the 0.4-kg part A describes the path

$$r = 1 - 0.5\cos(2\pi t)\ \text{m},$$

$$\theta = 0.5 - 0.2\sin(2\pi t)\ \text{rad}.$$

Determine the radial and transverse components of the force exerted on A by the robot's jaws at $t = 2$ s.



Solution: The radial component of the acceleration is

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2.$$

The derivatives:

$$\frac{dr}{dt} = \frac{d}{dt}(1 - 0.5\cos 2\pi t) = \pi\sin 2\pi t,$$

$$\frac{d^2r}{dt^2} = \frac{d}{dt}(\pi\sin 2\pi t) = 2\pi^2\cos 2\pi t;$$

$$\frac{d\theta}{dt} = \frac{d}{dt}(0.5 - 0.2\sin 2\pi t) = -0.4\pi\cos 2\pi t.$$

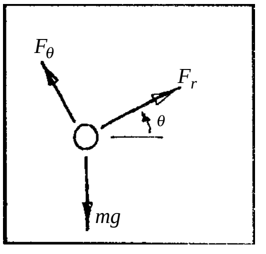
$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}(-0.4\pi\cos 2\pi t) = 0.8\pi^2\sin 2\pi t.$$

From which

$$[a_r]_{t=2} = 2\pi^2\cos 4\pi - (1 - 0.5\cos 4\pi)(-0.4\pi\cos 4\pi)^2,$$

$$= 2\pi^2 - 0.08\pi^2 = 18.95\ \text{m/s}^2;$$

$\theta(t = 2) = 0.5$ rad.



From Newton's second law, $F_r - mg\sin\theta = ma_r$, and $F_\theta - mg\cos\theta = ma_\theta$, from which

$$\boxed{F_r = 0.4a_r + 0.4g\sin\theta = 9.46\ \text{N}.}$$

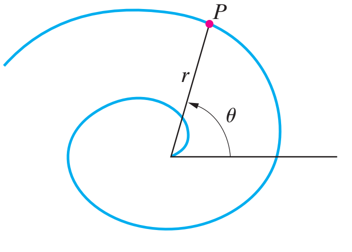
The transverse component of the acceleration is

$$a_\theta = r\left(\frac{d^2\theta}{dt^2}\right) + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right),$$

from which $[a_\theta]_{t=2} = (1 - 0.5\cos 4\pi)(0.8\pi^2\sin 4\pi) + 2(\pi\sin 4\pi)(-0.4\pi\sin 4\pi) = 0$, and

$$\boxed{F_\theta = 3.44\ \text{N}}$$

Problem 14.97 A 50-N object P moves along the spiral path $r = (0.1)\theta$ m, where θ is in radians. Its angular position is given as a function of time by $\theta = 2t$ rad, and $r = 0$ at $t = 0$. Determine the polar components of the total force acting on the object at $t = 4$ s.



Solution:

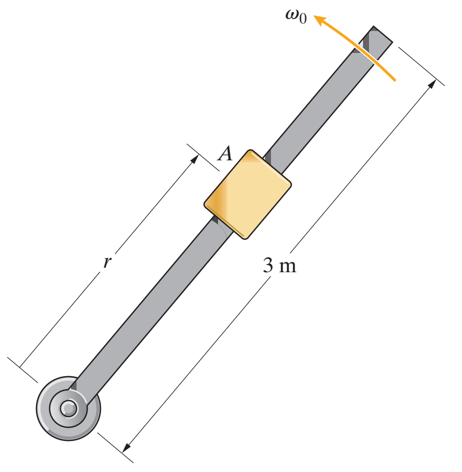
$\theta = 2t$, $\dot\theta = 2$, $\ddot\theta = 0$, $r = 0.1\theta = 0.2t$, $\dot r = 0.2$, $\ddot r = 0$

At $t = 4$ s, $\theta = 8$, $\dot\theta = 2$, $\ddot\theta = 0$, $r = 0.8$, $\dot r = 0.2$, $\ddot r = 0$

Thus

$$\boxed{\begin{aligned} F_r &= m(\ddot r - r\dot\theta^2) = \left(\frac{50\ \text{N}}{9.81\ \text{m/s}^2}\right)(0 - [0.8\ \text{m}][2\ \text{rad/s}]^2) \\ &= -16.3\ \text{N} \\ F_\theta &= m(r\ddot\theta + 2\dot r\dot\theta) = \left(\frac{50\ \text{N}}{9.81\ \text{m/s}^2}\right)(0 + 2[0.2\ \text{m/s}][2\ \text{rad/s}]) \\ &= 4.08\ \text{N} \end{aligned}}$$

Problem 14.98 The smooth bar rotates *in the horizontal plane* with constant angular velocity $\omega_0 = 60$ rpm. If the radial velocity of the 1-kg collar A is $v_r = 10$ m/s when its radial position is $r = 1$ m, what is its radial velocity when $r = 2$ m? (See Active Example 14.9).



Solution: Notice that no radial force acts on the collar, so the radial acceleration is zero. Write the term

$$\frac{d^2r}{dt^2} = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

The angular velocity is

$$\omega = 60 \text{ rpm}\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 6.28 \text{ rad/s}.$$

Then

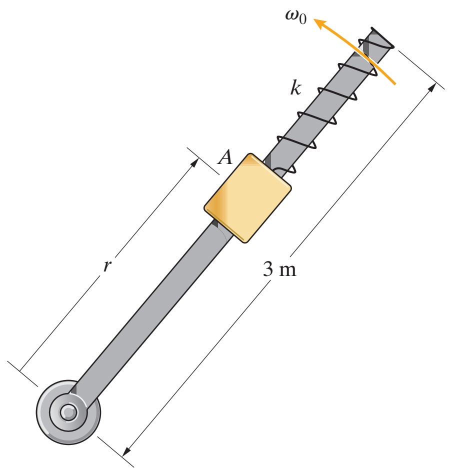
$$a_r = \frac{d^2r}{dt^2} - r\omega^2 = 0 \Rightarrow \frac{d^2r}{dt^2} = r\omega^2$$

$$\frac{d^2r}{dt^2} = v_r\frac{dv_r}{dr} = r\omega^2 \Rightarrow \int_{10\text{m/s}}^{v_r} v_r\,dv_r = \int_{1\text{m}}^{2\text{m}} \omega^2 r\,dr$$

$$\frac{v_r^2}{2} - \frac{(10 \text{ m/s})^2}{2} = (6.28 \text{ rad/s})^2\left(\frac{[2 \text{ m}]^2}{2} - \frac{[1 \text{ m}]^2}{2}\right)$$

$$\boxed{v_r = 14.8 \text{ m/s.}}$$

Problem 14.99 The smooth bar rotates *in the horizontal plane* with constant angular velocity $\omega_0 = 60$ rpm. The spring constant is $k = 20$ N/m and the unstretched length of the spring is 3 m. If the radial velocity of the 1-kg collar A is $v_r = 10$ m/s when its radial position is $r = 1$ m, what is its radial velocity when $r = 2$ m? (See Active Example 14.9.)



Solution: Notice that the only radial force comes from the spring. Write the term

$$\frac{d^2r}{dt^2} = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

The angular velocity is

$$\omega = 60 \text{ rpm}\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 6.28 \text{ rad/s}.$$

The equation of motion in the radial direction is

$$\Sigma F_{rm} : -kr = ma_r \Rightarrow a_r = -\frac{k}{m}r$$

Then

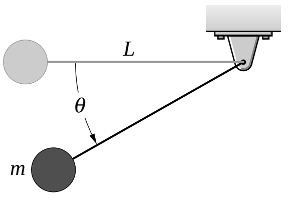
$$a_r = \frac{d^2r}{dt^2} - r\omega^2 = -\frac{k}{m}r \Rightarrow \frac{d^2r}{dt^2} = r\left(\omega^2 - \frac{k}{m}\right)$$

$$\frac{d^2r}{dt^2} = v_r\frac{dv_r}{dr} = r\left(\omega^2 - \frac{k}{m}\right) \Rightarrow \int_{10 \text{ m/s}}^{v_r} v_r\,dv_r = \int_{1 \text{ m}}^{2 \text{ m}} \left(\omega^2 - \frac{k}{m}\right) r\,dr$$

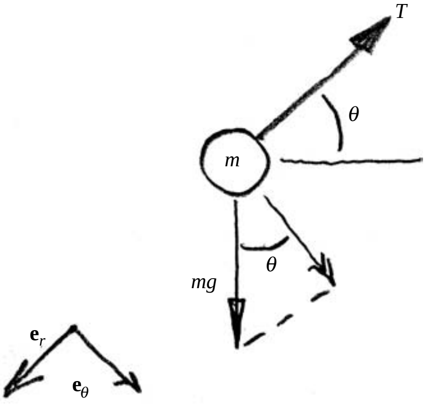
$$\frac{v_r^2}{2} - \frac{(10 \text{ m/s})^2}{2} = \left[(6.28 \text{ rad/s})^2 - \frac{20 \text{ N/m}}{1 \text{ kg}}\right]\left(\frac{[2 \text{ m}]^2}{2} - \frac{[1 \text{ m}]^2}{2}\right)$$

$$\boxed{v_r = 12.6 \text{ m/s.}}$$

Problem 14.100 The 2-kg mass m is released from rest with the string horizontal. The length of the string is $L = 0.6$ m. By using Newton's second law in terms of polar coordinates, determine the magnitude of the velocity of the mass and the tension in the string when $\theta = 45^\circ$.



Solution:



$L = 0.6$ m

$m = 2$ kg

$F_r = ma_r$

$$-T + mg\sin\theta = m\left(\frac{d^2L}{dt^2} - Lw^2\right)$$

$$\sum F_\theta = ma_\theta$$

$$mg\cos\theta = m\left(2\frac{\mathrm{d}L}{\mathrm{d}t}w + L\alpha\right)$$

However $\dfrac{\mathrm{d}L}{\mathrm{d}t} = \dfrac{\mathrm{d}^2L}{\mathrm{d}t^2} = 0$

$$L\alpha = L\frac{dw}{d\theta}w = g\cos\theta$$

$$\int_0^w w\,dw = \frac{g}{L}\int_0^{\pi/4}\cos\theta\, d\theta$$

$$\frac{w^2}{2} = \frac{g}{L}\sin\theta\Big|_0^{\pi/4} = \frac{g}{L}\sin\frac{\pi}{4}$$

$w = 4.81$ rad/s

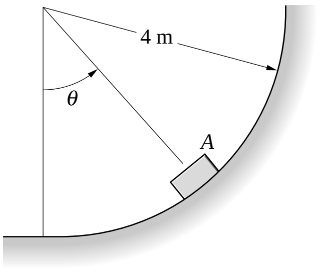
$\underline{|v| = Lw = 2.89 \text{ m/s}}$

$-T + mg\sin\theta = -mLw^2$

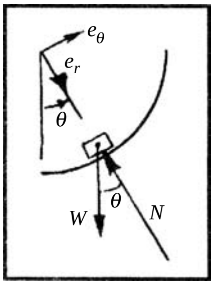
$T = m(g\sin\theta + Lw^2)$

$\underline{T = 41.6 \text{ N}}$

Problem 14.101 The 1-N block A is given an initial velocity $v_0 = 14$ m/s to the right when it is in the position $\theta = 0$, causing it to slide up the smooth circular surface. By using Newton's second law in terms of polar coordinates, determine the magnitude of the velocity of the block when $\theta = 60^\circ$.



Solution: For this problem, Newton's second law in polar coordinates states



$\sum F_r = mg\cos\theta - N = m(d^2r/dt^2 - r\omega^2)$ and

$\sum F_\theta = -mg\sin\theta = m(r\alpha + 2\omega(dr/dt))$.

In this problem, r is constant. Thus $(dr/dt) = (d^2r/dt^2) = 0$, and the equations reduce to $N = mr\omega^2 + mg\cos\theta$ and $r\alpha = -g\sin\theta$. The first equation gives us a way to evaluate the normal force while the second can be integrated to give $\omega(\theta)$. We rewrite the second equation as

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} = -\left(\frac{g}{r}\right)\sin\theta$$

and then integrate $\int_{\omega_0}^{\omega_{60}} \omega\, d\omega = -\left(\frac{g}{r}\right)\int_0^{60^\circ} \sin\theta\, d\theta$. Carrying out the integration, we get

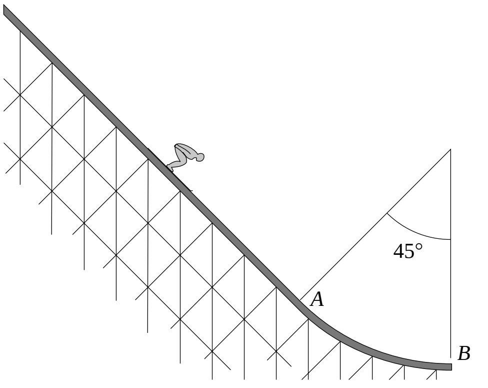
$$\frac{\omega_{60}^2}{2} - \frac{\omega_0^2}{2} = -\left(\frac{g}{r}\right)(-\cos\theta)\Big|_0^{60^\circ} = -\left(\frac{g}{r}\right)(1 - \cos 60^\circ).$$

Noting that $\omega_0 = v_0/R = 3.5$ rad/s, we can solve for $\omega_{60} = 2.05$ rad/s and $v_{60} = R\omega_{60} = 8.20$ m/s.

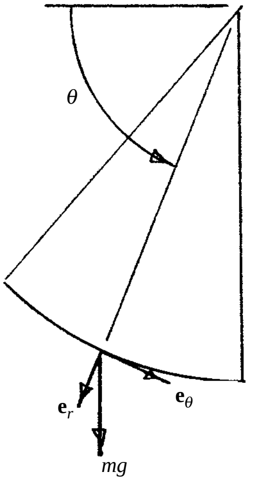
Problem 14.102 The 1-N block is given an initial velocity $v_0 = 14$ m/s to the right when it is in the position $\theta = 0$, causing it to slide up the smooth circular surface. Determine the normal force exerted on the block by the surface when $\theta = 60^\circ$.

Solution: From the solution to Problem 14.101, we have $N_{60} = mr\omega_{60}^2 + mg\cos 60^\circ$ or $N = 2.21$ N.

Problem 14.103 The skier passes point A going 17 m/s. From A to B, the radius of his circular path is 6 m. By using Newton's second law in terms of polar coordinates, determine the magnitude of the skier's velocity as he leaves the jump at B. Neglect tangential forces other than the tangential component of his weight.



Solution: In terms of the angle θ shown, the transverse component of his weight is $mg\cos\theta$. Therefore



$$\sum F_\theta = ma_\theta :$$

$$mg\cos\theta = m\left(r\frac{d^2\theta}{dt^2} + 2\cancelto{0}{\frac{dr}{dt}}\frac{d\theta}{dt}\right). \quad \textbf{(1)}$$

Note that

$$\frac{d^2\theta}{dt^2} = \frac{dw}{dt} = \frac{dw}{d\theta}\frac{d\theta}{dt} = \frac{dw}{d\theta}w,$$

So (1) becomes

$$g\cos\theta = r\frac{dw}{d\theta}w.$$

Separating variables,

$$w\,dw = \frac{g}{r}\cos\theta\, d\theta. \quad \textbf{(2)}$$

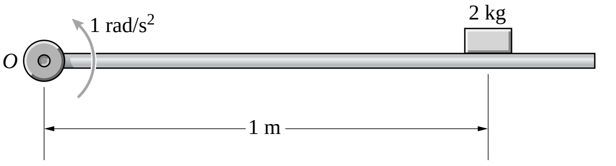
At A, $\theta = 45^\circ$ and $w = v_A/r = 17/6 = 2.83$ rad/s. Integrating (2),

$$\int_{2.83}^{w_B} w\,dw = \frac{g}{r}\int_{45^\circ}^{90^\circ}\cos\theta\, d\theta,$$

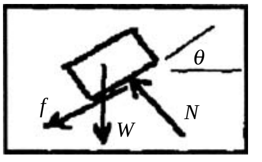
we obtain $w_B = 3.00$ rad/s. His velocity at B is $rw_B = 18.0$ m/s.

$\underline{v_B = 18.0 \text{ m/s.}}$

Problem 14.104* A 2-kg mass rests on a flat horizontal bar. The bar begins rotating *in the vertical plane* about O with a constant angular acceleration of 1 rad/s^2. The mass is observed to slip relative to the bar when the bar is 30° above the horizontal. What is the static coefficient of friction between the mass and the bar? Does the mass slip toward or away from O?

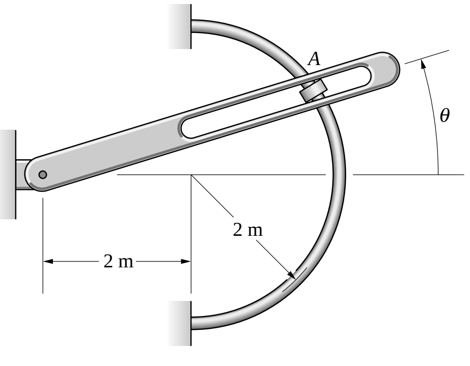


Solution: From Newton's second law for the radial component $-mg\sin\theta \pm \mu_s N = -mR\omega^2$, and for the normal component: $N - mg\cos\theta = mR\alpha$. Solve, and note that $\alpha = \dfrac{d\omega}{dt} = \omega\dfrac{d\omega}{d\theta} = 1 = const$, $\omega^2 = 2\theta$, since $\omega(0) = 0$, to obtain $-g\sin\theta \pm \mu_s(g\cos\theta + R\alpha) = -2R\theta$. For $\alpha = 1$, $R = 1$, this reduces to $\pm\mu_s(1 + g\cos\theta) = -2\theta + g\sin\theta$. Define the quantity $F_R = 2\theta - g\sin\theta$. If $F_R > 0$, the block will tend to slide away from O, the friction force will oppose the motion, and the negative sign is to be chosen. If $F_R < 0$, the block will tend to slide toward O, the friction force will oppose the motion, and the positive sign is to be chosen. The equilibrium condition is derived from the equations of motion: $\mathrm{sgn}(F_R)\mu_s(1 + g\cos\theta) = (2\theta - g\sin\theta)$, from which $\boxed{\mu_s = \mathrm{sgn}(F_R)\dfrac{2\theta - g\sin\theta}{1 + g\cos\theta} = 0.406}$.

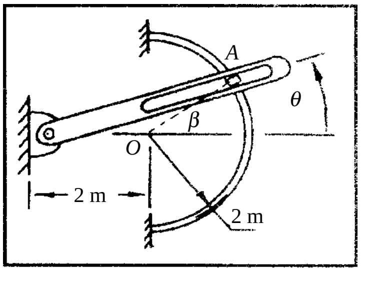


Since $F_r = -3.86 < 0$, the block will slide toward O.

Problem 14.105* The 0.25 N slider A is pushed along the circular bar by the slotted bar. The circular bar lies *in the horizontal plane*. The angular position of the slotted bar is $\theta = 10t^2$ rad. Determine the polar components of the total external force exerted on the slider at $t = 0.2$ s.



Solution: The interior angle β is between the radius from O to the slider A and the horizontal, as shown in the figure. The interior angle formed by the radius from C to the slider A and the line from O to the slider is $\beta - \theta$. The angle β is found by applying the law of sines: $\dfrac{2}{\sin\theta} = \dfrac{2}{\sin(\beta - \theta)}$ from which $\sin\theta = \sin(\beta - \theta)$ which is satisfied by $\beta = 2\theta$. The radial distance of the slider from the hinge point is also found from the sine law: $\dfrac{r}{\sin(180 - \beta)} = \dfrac{2}{\sin\theta}$, $r = \dfrac{2\sin 2\theta}{\sin\theta}$, from which $r = 4\cos\theta$. The radial component of the acceleration is $a_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$. The derivatives: $\dfrac{d\theta}{dt} = \dfrac{d}{dt}(10t^2) = 20t$. $\dfrac{d^2\theta}{dt^2} = 20$. $\dfrac{dr}{dt} = -4\sin\theta\left(\dfrac{d\theta}{dt}\right) = -(80\sin\theta)t$. $\dfrac{d^2r}{dt^2} = -80\sin\theta - (1600\cos\theta)t^2$. Substitute:



$$[a_r]_{t=0.2} = [a_r = -80\sin(10t^2) - (1600\cos(10t^2))t^2$$

$$- (4\cos(10t^2))(20t)^2]_{t=0.2}$$

$$= -149.0 \text{ m/s}^2.$$

From Newton's second law, the radial component of the external force is

$$\boxed{F_r = \left(\frac{W}{g}\right)a_r = -1.158 \text{ N.}}$$

The transverse acceleration is $a_\theta = r\left(\dfrac{d^2\theta}{dt^2}\right) + 2\left(\dfrac{dr}{dt}\right)\left(\dfrac{d\theta}{dt}\right)$. Substitute:

$$[a_\theta]_{t=0.2} = [a_\theta = (4\cos(10t^2))(20)$$

$$+ 2(-80\sin(10t^2))(t)(20)(t)]_{t=0.2} = 23.84 \text{ m/s}^2.$$

The transverse component of the external force is

$$\boxed{F_\theta = \left(\frac{W}{g}\right)a_\theta = 0.185 \text{ N}}$$

Problem 14.106* The 0.25 N slider A is pushed along the circular bar by the slotted bar. The circular bar lies *in the vertical plane*. The angular position of the slotted bar is $\theta = 10t^2$ rad. Determine the polar components of the total force exerted on the slider by the circular and slotted bars at $t = 0.25$ s.

Solution: Assume that the orientation in the vertical plane is such that the $\theta = 0$ line is horizontal. Use the solution to Problem 14.105. For positive values of θ the radial component of acceleration due to gravity acts toward the origin, which by definition is the same direction as the radial acceleration. The transverse component of the acceleration due to gravity acts in the same direction as the transverse acceleration. From which the components of the acceleration due to gravity in the radial and transverse directions are $g_r = g\sin\theta$ and $g_\theta = g\cos\theta$. These are to be added to the radial and transverse components of acceleration due to the motion. From Problem 14.105, $\theta = 10t^2$ rad

$$[a_r]_{t=0.25} = [-80\sin\theta - (1600\cos\theta)t^2$$

$$- (4\cos\theta)(20t)^2]_{t=0.25} = -209 \text{ m/s}^2.$$

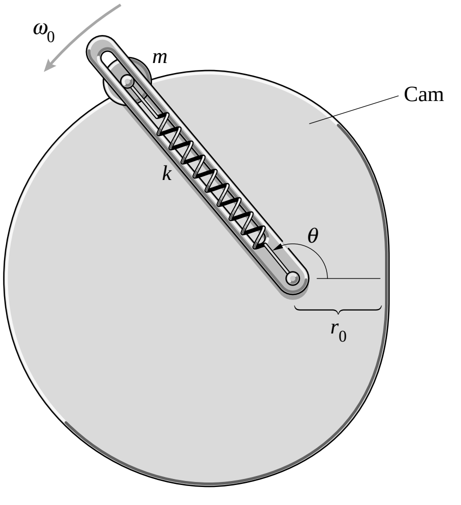
From Newton's second law for the radial component $F_r - mg\sin\theta = \left(\frac{W}{g}\right)a_r$, from which $F_r = -1.478$ N The transverse component of the acceleration is (from Problem 14.105)

$$[a_\theta]_{t=0.25} = [(4\cos\theta)(20)$$

$$+ 2(-80\sin\theta)(t)(20)(t)]_{t=0.25} = -52.14 \text{ m/s}^2.$$

From Newton's second law for transverse component $F_\theta - mg\cos\theta = \left(\frac{W}{g}\right)a_\theta$, from which $\boxed{F_\theta = -0.2025 \text{ N}}$

Problem 14.107* The slotted bar rotates *in the horizontal plane* with constant angular velocity ω_0. The mass m has a pin that fits into the slot of the bar. A spring holds the pin against the surface of the fixed cam. The surface of the cam is described by $r = r_0(2 - \cos\theta)$. Determine the polar components of the total external force exerted on the pin as functions of θ.



Solution: The angular velocity is constant, from which $\theta = \int \omega_0\,dt + C = \omega_0 t + C$. Assume that $\theta(t = 0) = 0$, from which $C = 0$. The radial acceleration is $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$. The derivatives: $\frac{d\theta}{dt} = \frac{d}{dt}(\omega_0 t) = \omega_0$, $\frac{d^2\theta}{dt^2} = 0$. $\frac{dr}{dt} = \frac{d}{dt}(r_0(2 - \cos\theta)) = r_0\sin\theta\left(\frac{d\theta}{dt}\right) = \omega_0 r_0\sin\theta$, $\frac{d^2r}{dt^2} = \frac{d}{dt}(\omega_0 r_0\sin\theta) = \omega_0^2 r_0\cos\theta$. Substitute: $a_r = \omega_0^2 r_0\cos\theta - r_0(2 - \cos\theta)(\omega_0^2) = 2r_0\omega_0^2(\cos\theta - 1)$. From Newton's second law the radial component of the external force is

$$\boxed{F_r = ma_r = 2mr_0\omega_0^2(\cos\theta - 1).}$$

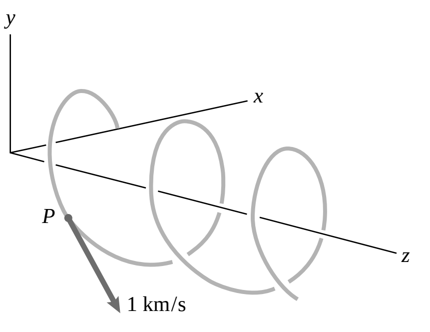
The transverse component of the acceleration is $a_\theta = r\frac{d^2\theta}{dt^2} + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)$. Substitute: $a_\theta = 2r_0\omega_0^2\sin\theta$. From Newton's second law, the transverse component of the external force is

$$\boxed{F_\theta = 2mr_0\omega_0^2\sin\theta}$$

Problem 14.108* In Problem 14.107, suppose that the unstretched length of the spring is r_0. Determine the smallest value of the spring constant k for which the pin will remain on the surface of the cam.

Solution: The spring force holding the pin on the surface of the cam is $F_r = k(r - r_0) = k(r_0(2 - \cos\theta) - r_0) = kr_0(1 - \cos\theta)$. This force acts toward the origin, which by definition is the same direction as the radial acceleration, from which Newton's second law for the pin is $\sum F = kr_0(1 - \cos\theta) = -ma_r$. From the solution to Problem 14.107, $kr_0(1 - \cos\theta) = -2mr\omega_0^2(\cos\theta - 1)$. Reduce and solve: $k = 2m\omega_0^2$. Since $\cos\theta \le 1$, $kr_0(1 - \cos\theta) \ge 0$, and $2mr_0\omega_0^2(\cos\theta - 1) \le 0$. If $k > 2m\omega_0^2$, Define $F_{\text{eq}} = kr_0(1 - \cos\theta) + 2mr\omega_0^2(\cos\theta - 1)$. If $F_{\text{eq}} > 0$ the spring force dominates over the range of θ, so that the pin remains on the cam surface. If $k < 2m\omega_0^2$, $F_{\text{eq}} < 0$ and the radial acceleration dominates over the range of θ, so that the pin will leave the cam surface at some value of θ. Thus $\boxed{k = 2m\omega_0^2}$ is the minimum value of the spring constant required to keep the pin in contact with the cam surface.

Problem 14.109 A charged particle P in a magnetic field moves along the spiral path described by $r = 1$ m, $\theta = 2z$ rad, where z is in meters. The particle moves along the path in the direction shown with a constant speed $|\mathbf{v}| = 1$ km/s. The mass of the particle is 1.67×10^{-27} kg. Determine the sum of the forces on the particle in terms of cylindrical coordinates.



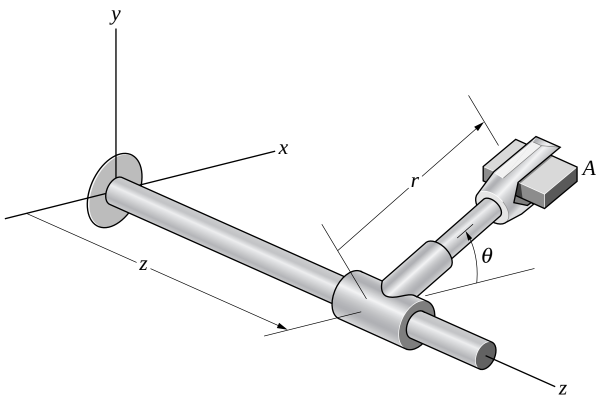
Solution: The force components in cylindrical coordinates are given by

$$\sum F_r = ma_r = m\left(\frac{d^2r}{dt^2} r\omega^2\right),$$

$$\sum F_\theta = ma_\theta = m\left(r\alpha + 2\left(\frac{dr}{dt}\right)\omega\right),$$

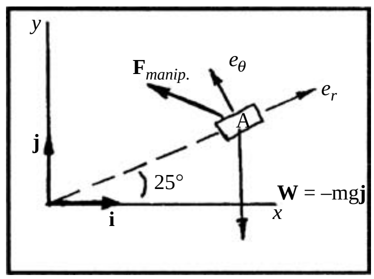
and $\sum F_z = ma_z = m\dfrac{d^2z}{dt^2}$. From the given information, $\dfrac{dr}{dt} = \dfrac{d^2r}{dt^2} = 0$. We also have that $\theta = 2z$. Taking derivatives of this, we see that $\dfrac{d\theta}{dt} = \omega = 2\dfrac{dz}{dt} = 2v_z$. Taking another derivative, we get $\alpha = 2a_z$. There is no radial velocity component so the constant magnitude of the velocity $|\mathbf{v}|^2 = v_\theta^2 + v_z^2 = r^2\omega^2 + v_z^2 = (1000\text{ m/s})^2$. Taking the derivative of this expression with respect to time, we get $r^2\left(2\omega\dfrac{d\omega}{dt}\right) + 2v_z\dfrac{dv_z}{dt} = 0$. Noting that $\dfrac{dv_z}{dt} = \dfrac{d^2z}{dt^2}$ and that $\alpha = \dfrac{d\omega}{dt}$, we can eliminate $\dfrac{dv_z}{dt}$ from the equation. We get $2r^2\omega\alpha + 2\left(\dfrac{\omega}{2}\right)\left(\dfrac{\alpha}{2}\right)$, giving $(2r^2 + 1/2)\omega\alpha = 0$. Since $\omega \ne 0$, $\alpha = 0$, and $a_z = 0$. Substituting these into the equations of motion, we get $\omega^2 = 4(1000)^2 5\text{ (rad/s)}^2$, and $\sum F_r = -mr\omega^2 = -1.34 \times 10^{-21}$ m/s^2, $\sum F_\theta = 0$ and $\sum F_z = 0$

Problem 14.110 At the instant shown, the cylindrical coordinates of the 4-kg part A held by the robotic manipulator are $r = 0.6$ m, $\theta = 25^\circ$, and $z = 0.8$ m. (The coordinate system is fixed with respect to the earth, and the y axis points upward). A's radial position is increasing at $\dfrac{dr}{dt} = 0.2$ m/s, and $\dfrac{d^2r}{dt^2} = -0.4$ m/s^2. The angle θ is increasing at $\dfrac{d\theta}{dt} = 1.2$ rad/s and $\dfrac{d^2\theta}{dt^2} = 2.8$ rad/s^2. The base of the manipulator arm is accelerating in the z direction at $\dfrac{d^2z}{dt^2} = 2.5$ m/s^2. Determine the force vector exerted on A by the manipulator in cylindrical coordinates.



Solution: The total force acting on part A in cylindrical coordinates is given by $\sum F_r = ma_r = m\left(\dfrac{d^2r}{dt^2} - r\omega^2\right)$, $\sum F_\theta = ma_\theta = m\left(r\alpha + 2\left(\dfrac{dr}{dt}\right)\omega\right)$, and $\sum F_z = ma_z = m\dfrac{d^2z}{dt^2}$. We are given the values of every term in the right hand side of these equations. (Recall the definitions of ω and α. Substituting in the known values, we get $\sum F_r = -5.06$ N, $\sum F_\theta = 8.64$ N, and $\sum F_z = 10.0$ N. These are the total forces acting on Part A, including the weight.

To find the forces exerted on the part by the manipulator, we need to draw a free body diagram of the part and resolve the weight into components along the various axes. We get $\sum \mathbf{F} = \sum \mathbf{F}_{\text{manip}} + \mathbf{W} = m\mathbf{a}$ where the components of $\sum \mathbf{F}$ have already been determined above. In cylindrical coordinates, the weight is given as $W = -mg\sin\theta\mathbf{e}_\text{r} - mg\cos\theta\mathbf{e}_\theta$. From the previous equation, $\sum \mathbf{F} = \sum \mathbf{F}_{\text{manip}} - mg\sin\theta\mathbf{e}_\text{r} - mg\cos\theta\mathbf{e}_\theta$. Substituting in terms of the components, we get $(\sum F_{\text{manip}})_z = 11.5$ (*newtons*), $(\sum F_{\text{manip}})_\theta = 44.2$ (*newtons*) and $(\sum F_{\text{manip}})_z = 10.0$ (*newtons*).



Problem 14.111 Suppose that the robotic manipulator in Problem 14.110 is used in a space station to investigate zero-g manufacturing techniques. During an interval of time, the manipulator is programmed so that the cylindrical coordinates of the 4-kg part A are $\theta = 0.15t^2$ rad, $r = 0.5(1 + \sin\theta)$ m, and $z = 0.8(1 + \theta)$ m Determine the force vector exerted on A by the manipulator at $t = 2$ s in terms of cylindrical coordinates.

Solution:

$$\theta = 0.15t^2 \text{ rad},$$

$$\frac{d\theta}{dt} = 0.3t \text{ rad/s},$$

$$\frac{d^2\theta}{dt^2} = 0.3 \text{ rad/s}^2.$$

$$r = 0.5(1 + \sin\theta) \text{ m},$$

$$\frac{dr}{dt} = 0.5\frac{d\theta}{dt}\cos\theta \text{ m/s},$$

$$\frac{d^2r}{dt^2} = 0.5\frac{d^2\theta}{dt^2}\cos\theta - 0.5\left(\frac{d\theta}{dt}\right)^2\sin\theta \text{ m/s}^2.$$

$$z = 0.8(1 + \theta) \text{ m},$$

$$\frac{dz}{dt} = 0.8\frac{d\theta}{dt} \text{ m/s},$$

$$\frac{d^2z}{dt^2} = 0.8\frac{d^2\theta}{dt^2} \text{ m/s}^2.$$

Evaluating these expressions at $t = 2$ s, the acceleration is

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta + \frac{d^2z}{dt^2}\mathbf{e}_z$$

$$= -0.259\mathbf{e}_r + 0.532\mathbf{e}_\theta + 0.24\mathbf{e}_z \text{ (m/s}^2\text{)}.$$

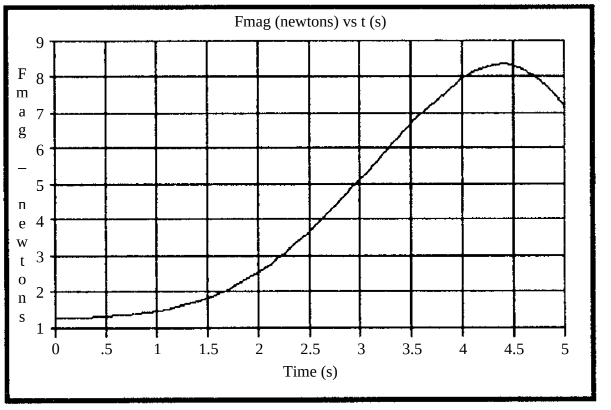
Therefore

$$\sum \mathbf{F} = m\mathbf{a}$$

$$= -1.04\mathbf{e}_r + 2.13\mathbf{e}_\theta + 0.96\mathbf{e}_z \text{ (N)}.$$

Problem 14.112* In Problem 14.111, draw a graph of the magnitude of the force exerted on part A by the manipulator as a function of time from $t = 0$ to $t = 5$ s and use it to estimate the maximum force during that interval of time.

Solution: Use a numerical solver to work problem 14.111 for a series of values of time during the required interval and plot the magnitude of the resulting force as a function of time. From the graph, the maximum force magnitude is approximately 8.4 N and it occurs at a time of about 4.4 seconds.



Problem 14.113 The International Space Station is in a circular orbit 225 miles above the earth's surface.

(a) What is the magnitude of the velocity of the space station?
(b) How long does it take to complete one revolution?

Solution: The radius of the orbit is

$$r_0 = R_E + 225 \text{ mi}$$

$$= 3960 + 225 \text{ mi}$$

$$= 2.21 \times 10^7 \text{ ft.}$$

(a) From Eq (14.24), the velocity is

$$v_0 = \sqrt{\frac{gR_E^2}{r_0}}$$

$$= \sqrt{\frac{(32.2)[(3960)(5280)]^2}{2.21 \times 10^7}}$$

$$= \underline{25200 \text{ ft/s}}(17200 \text{ mi/h}).$$

(b) Let T be the time required. Then $v_0 T = 2\pi r_0$,

so $T = \dfrac{2\pi r_0}{v_0} = \underline{5500 \text{ s}}$ (1.53 h).

Problem 14.114 The moon is approximately 383,000 km from the earth. Assume that the moon's orbit around the earth is circular with velocity given by Eq. (14.24).

(a) What is the magnitude of the moon's velocity?
(b) How long does it take to complete one revolution around the earth?

Solution:

$$R_E = 6370 \text{ km} = 6.37 \times 10^6 \text{ m}$$

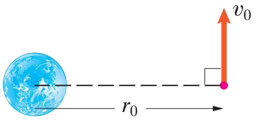
$$r_0 = 383000 \text{ km} = 3.83 \times 10^8 \text{ m}$$

$$\underline{v_0 = \sqrt{\frac{gR_E^2}{r_0}} = 1020 \text{ m/s}}$$

$$\text{Period} = 2\pi r_0/v_0$$

$$\underline{\text{Period} = 2.36 \times 10^6 \text{ s} = 27.3 \text{ days}}$$

Problem 14.115 Suppose that you place a satellite into an elliptic earth orbit with an initial radius $r_0 = 6700$ km and an initial velocity v_0 such that the maximum radius of the orbit is 13,400 km. (a) Determine v_0. (b) What is the magnitude of the satellite's velocity when it is at its maximum radius? (See Active Example 14.10).



Solution: We have

$$\varepsilon = \frac{r_0 v_0{}^2}{g R_E} - 1,\ r_{\max} = r_0\left(\frac{1+\varepsilon}{1-\varepsilon}\right),\ r_0 v_0 = r_{\max} v_{\max\,\text{radius}}.$$

Solving we find

$$\varepsilon = \frac{r_{\max} - r_0}{r_{\max} + r_0} = \frac{13{,}400 \text{ km} - 6700 \text{ km}}{13{,}400 \text{ km} + 6700 \text{ km}} = 0.333,$$

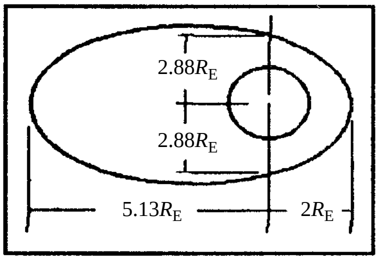
$$v_0 = \sqrt{g(1+\varepsilon)\frac{R_E{}^2}{r_0}} = \sqrt{(9.81 \text{ m/s}^2)(1.333)\frac{(6370 \text{ km})^2}{6700 \text{ km}}} = 8900 \text{ m/s},$$

$$v_{\max\ \text{radius}} = \frac{r_0}{r_{\max}} v_0 = \frac{6700 \text{ km}}{13400 \text{ km}}(8900 \text{ m/s}) = 4450 \text{ m/s}$$

$$\boxed{(a)\ v_0 = 8900 \text{ m/s},\quad (b) v_{\max\ \text{radius}} = 4450 \text{ m/s}.}$$

Problem 14.116 A satellite is given an initial velocity $v_0 = 6700$ m/s at a distance $r_0 = 2R_E$ from the center of the earth as shown in Fig. 14.18a. Draw a graph of the resulting orbit.

Solution: The graph is shown.



Problem 14.117 The time required for a satellite in a circular earth orbit to complete one revolution increases as the radius of the orbit increases. If you choose the radius properly, the satellite will complete one revolution in 24 hours. If a satellite is placed in such an orbit directly above the equator and moving from west to east, it will remain above the same point on the earth as the earth rotates beneath it. This type of orbit, conceived by Arthur C. Clarke, is called *geosynchronous*, and is used for communication and television broadcast satellites. Determine the radius of a geosynchronous orbit in km.

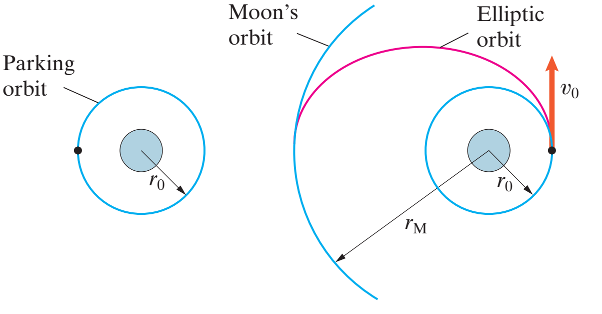
Solution: We have

$$v = \frac{2\pi r}{T}, \quad \frac{v^2}{r} = g\frac{R_E{}^2}{r^2}$$

$$r = \left(\frac{gR_E^2T^2}{4\pi^2}\right)^{1/3} = \left(\frac{[9.81\ \text{m/s}^2][6370\times10^3\ \text{m}]^2[24(60)(60)\ \text{s}]^2}{4\pi^2}\right)^{1/3}$$

$$\boxed{r = 42.2\times10^6\ \text{m} = 42{,}200\ \text{km}.}$$

Problem 14.118* You can send a spacecraft from the earth to the moon in the following way. First, launch the spacecraft into a circular "parking" orbit of radius r_0 around the earth (Fig. a). Then increase its velocity in the direction tangent to the circular orbit to a value v_0 such that it will follow an elliptic orbit whose maximum radius is equal to the radius r_M of the moon's orbit around the earth (Fig. b). The radius $r_M = 382942$ km. Let $r_0 = 6693$km. What velocity v_0 is necessary to send a spacecraft to the moon? (This description is simplified in that it disregards the effect of the moon's gravity.)



(a) (b)

Solution: Note that

$$R_E = 6370\ \text{km}, \quad r_0 = 6693\ \text{km}, \quad r_M = 382942\ \text{km}$$

First find the eccentricity:

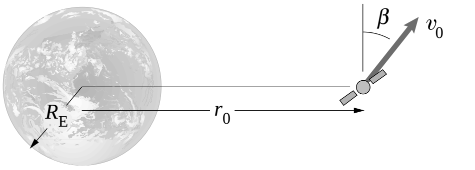
$$r_{max} = r_M = r_0\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \Rightarrow \varepsilon = \frac{r_M - r_0}{r_M + r_0}$$

Then use eq. 14.23

$$\varepsilon = \frac{r_0 v_0{}^2}{gR_E{}^2} - 1 \Rightarrow v_0 = R_E\sqrt{\frac{(1+\varepsilon)g}{r_0}} = R_E\sqrt{\frac{2gr_M}{r_0(r_0+r_M)}}$$

Putting in the numbers we have $\boxed{v_0 = 10820\ \text{m/s}}$

Problem 14.119* At $t = 0$, an earth satellite is a distance r_0 from the center of the earth and has an initial velocity v_0 in the direction shown. Show that the polar equation for the resulting orbit is



$$\frac{r}{r_0} = \frac{(\varepsilon + 1)\cos^2\beta}{[(\varepsilon + 1)\cos^2\beta - 1]\cos\theta - (\varepsilon + 1)\sin\beta\cos\beta\sin\theta + 1},$$

where $\varepsilon = \left(\dfrac{r_0 v_0^2}{g R_E^2}\right) - 1$.

Solution: We need to modify the solution in Section 14.5 to account for this new initial condition. At $\theta = 0$ (see Fig. 14.17)

$$v_r = \frac{dr}{dt} = v_0 \sin\beta$$

and

$$v_\theta = r\frac{d\theta}{dt} = v_0 \cos\beta.$$

Therefore Eq (14.15) becomes

$$r^2\frac{d\theta}{dt} = r v_\theta = r_0 v_0 \cos\beta \quad (1)$$

Following the same steps that led to Eq. (14.21) in terms of $u = 1/r$ yields

$$u = A\sin\theta + B\cos\theta + \frac{g R_E^2}{r_0^2 v_0^2 \cos^2\beta}. \quad (2)$$

At $\theta = 0$,

$$u = \frac{1}{r_0}. \quad (3)$$

Also, note that

$$v_r = \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -r_0 v_0 \cos\beta \frac{du}{d\theta},$$

where we used (1). Therefore, at $\theta = 0$

$$-r_0 v_0 \cos\beta \frac{du}{d\theta} = v_0 \sin\beta. \quad (4)$$

From (2),

$$\frac{du}{d\theta} = A\cos\theta - B\sin\theta. \quad (5)$$

From conditions (3) and (4) and Eq. (5),

$$B = \frac{1}{r_0} - \frac{g R_E^2}{r_0^2 v_0^2 \cos^2\beta}$$

and

$$A = -\frac{\sin\beta}{r_0\cos\beta}.$$

Substituting these expressions for A and B into Eq (2) yields the desired result.

Problem 14.120 The Acura NSX can brake from 120 km/h to a stop in a distance of 112 m. (a) If you assume that the vehicle's deceleration is constant, what are its deceleration and the magnitude of the horizontal force its tires exert on the road? (b) If the car's tires are at the limit of adhesion (i.e., slip is impending), and the normal force exerted on the car by the road equals the car's weight, what is the coefficient of friction μ_s? (This analysis neglects the effects of horizontal and vertical aerodynamic forces).

Solution:

(a) 120 km/h = 33.3 m/s.

$$a = \frac{dv}{dt} = \frac{dv}{ds}v.$$

Integrating,

$$\int_{33.3}^{0} v\,dv = \int_{0}^{112} a\,ds,$$

we obtain $a = -4.95$ m/s^2. The magnitude of the friction force is

$$f = m|a| = \left(\frac{13000}{9.81}\right)(4.95)$$

$$= 6560 \text{ N}.$$

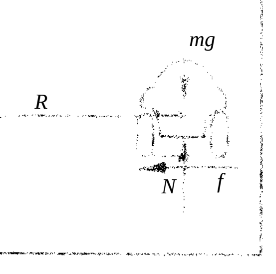
(b) The Normal force is the car's weight, so

$$\mu_s = \frac{f}{N} = \frac{6560}{13000}$$

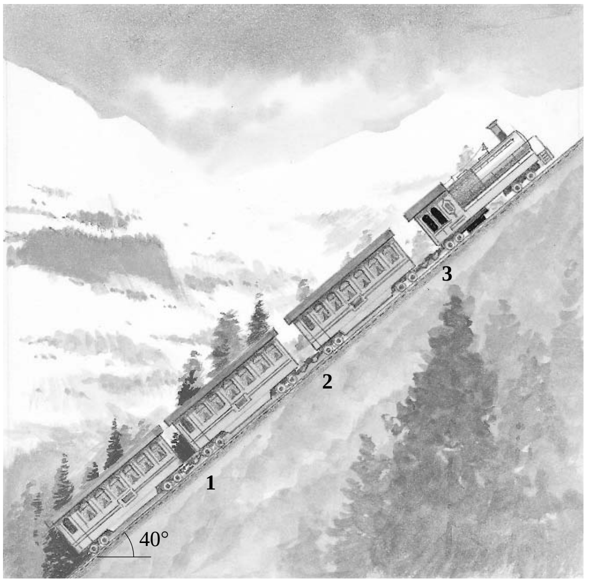
$$= 0.505.$$

Problem 14.121 Using the coefficient of friction obtained in Problem 14.120, determine the highest speed at which the NSX could drive on a flat, circular track of 600-m radius without skidding.

Solution: The free body diagram is at the right. The normal force is equal to the weight and the friction force has the same magnitude as in Problem 14.120 since $f = \mu_s N$. The equation of motion in the radial direction (from the center of curvature of the track to the car) is $\sum F_r = mv^2/R = f = \mu_s N = \mu_s mg$. Thus, we have that $mv^2/R = \mu_s mg$ or $v^2 = \mu_s Rg$. Inserting the numbers, we obtain $v = 36.7$ m/s $= 132$ km/h.



Problem 14.122 A cog engine hauls three cars of sightseers to a mountain top in Bavaria. The mass of each car, including its passengers, is 10,000 kg and the friction forces exerted by the wheels of the cars are negligible. Determine the forces in couplings 1, 2, and 3 if: (a) the engine is moving at constant velocity; (b) the engine is accelerating up the mountain at 1.2 m/s^2.



Solution: (a) The force in coupling 1 is

$$\boxed{F_1 = 10{,}000\ \text{g} \sin 40^\circ = 631\ \text{kN.}}$$

The force on coupling 2 is

$$\boxed{F_2 = 20{,}000\ \text{g} \sin(40) = 126.1\ \text{kN.}}$$

The force on coupling 3 is

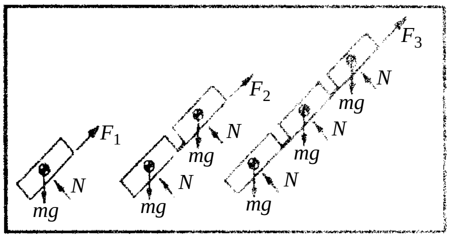
$$\boxed{F_3 = 30{,}000\ \text{g} \sin 40^\circ = 189.2\ \text{kN.}}$$

(b) From Newton's second law, $F_{1a} - mg \sin 40^\circ = ma$. Under constant acceleration up the mountain, the force on coupling 1 is

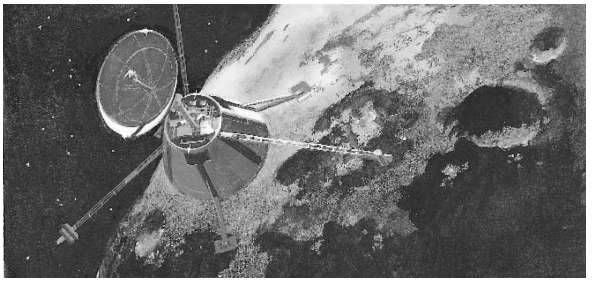
$$\boxed{F_{1a} = 10{,}000a + 10{,}000\ \text{g} \sin 40^\circ = 75.1\ \text{kN.}}$$

The force on coupling 2 is $\boxed{F_{2a} = 2F_{1a} = 150.1\ \text{kN.}}$

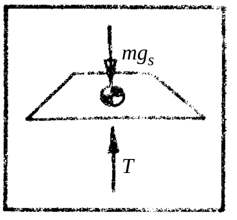
The force on coupling 3 is $\boxed{F_{2a} = 3\ F_{1a} = 225.2\ \text{kN}}$



Problem 14.123 In a future mission, a spacecraft approaches the surface of an asteroid passing near the earth. Just before it touches down, the spacecraft is moving downward at a constant velocity relative to the surface of the asteroid and its downward thrust is 0.01 N. The computer decreases the downward thrust to 0.005 N, and an onboard laser interferometer determines that the acceleration of the spacecraft relative to the surface becomes 5×10^{-6} m/s^2 downward. What is the gravitational acceleration of the asteroid near its surface?

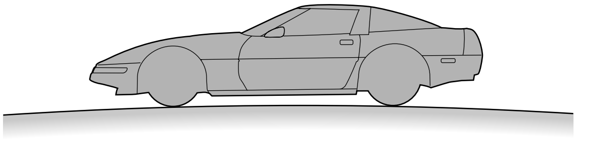


Solution: Assume that the mass of the spacecraft is negligible compared to mass of the asteroid. With constant downward velocity, the thrust balances the gravitational force: $0.01 - mg_s = 0$, where m is the mass of the space craft. With the change in thrust, this becomes $0.005 - mg_s = m(-5 \times 10^{-6})$ N/kg^2. Multiply the first equation by 0.005, the second by 0.01, and subtract: The result:

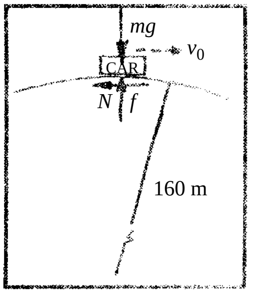


$$\boxed{g_s = \left(\frac{0.01(5 \times 10^{-6})}{(0.01 - 0.005)}\right) = 1 \times 10^{-5} \text{ N/kg}^2}$$

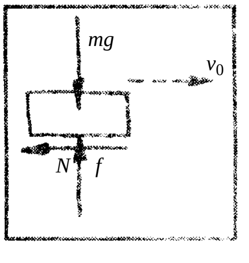
Problem 14.124 A car with a mass of 1470 kg, including its driver, is driven at 130 km/h over a slight rise in the road. At the top of the rise, the driver applies the brakes. The coefficient of static friction between the tires and the road is $\mu_s = 0.9$ and the radius of curvature of the rise is 160 m. Determine the car's deceleration at the instant the brakes are applied, and compare it with the deceleration on a level road.



Solution: First, note that 130 km/h = 36.11 m/s. We have a situation in which the car going over the rise reduces the normal force exerted on the car by the road and also reduces the braking force. The free body diagram of the car braking over the rise is shown at the right along with the free body diagram of the car braking on a level surface. For the car going over the rise, the equations of motion are $\sum F_t = -f = ma_t$, where f is the friction force. The normal equation is $\sum F_n = N - mg = mv^2/R$. The relation between friction and normal force is given as $f = \mu_s N$. Solving, we get $a_t = -1.49$ m/s^2.



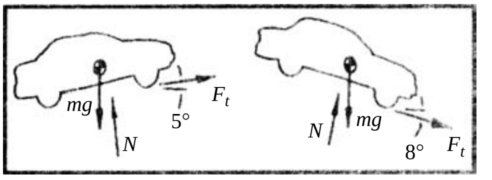
For the car braking on a level surface, the equations are $N - mg = 0$, $f = \mu_s N$, and $-f = ma_x$. Evaluating, we get $a_x = 8.83$ m/s^2. Note that the accelerations are VERY different. We conclude that at 130 km/h, a rise in the road with a radius of 160 m is not "slight". The car does not become airborne, but if the radius of curvature were smaller, the car would leave the road.



Problem 14.125 The car drives at constant velocity up the straight segment of road on the left. If the car's tires continue to exert the same tangential force on the road after the car has gone over the crest of the hill and is on the straight segment of road on the right, what will be the car's acceleration?

Solution: The tangential force on the left is, from Newton's second law, $F_t - mg\sin(5^\circ) = ma = 0$. On the right, from Newton's second law: $F_t + mg\sin(8^\circ) = ma$ from which the acceleration is

$$\boxed{a = g(\sin 5^\circ + \sin 8^\circ) = 0.2264\text{ g}}$$

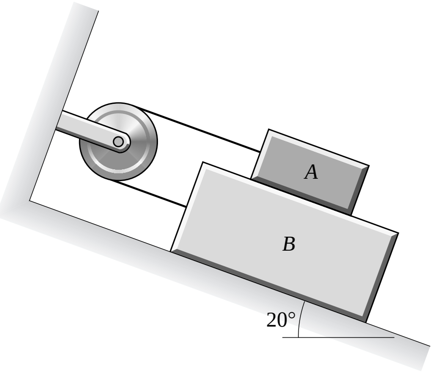


Problem 14.126 The aircraft carrier *Nimitz* weighs 91,000 tons. (A ton is 8896 N) Suppose that it is traveling at its top speed of approximately 15.4 m/s when its engines are shut down. If the water exerts a drag force of magnitude $88960v$ N, where v is the carrier's velocity in metre per second, what distance does the carrier move before coming to rest?

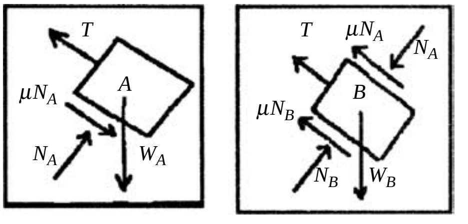
Solution: The force on the carrier is $F = -Kv$, where $K = 88960$. The acceleration is $a = \dfrac{F}{m} = -\dfrac{gK}{W}v$. Use the chain rule to write $v\dfrac{dv}{dx} = -\dfrac{gK}{W}v$. Separate variables and integrate: $dv = -\dfrac{gK}{W}dx$, $v(x) = -\dfrac{gK}{W}x + C$. The initial velocity: $v(0) = 15.4$ m/s, from which $C = v(0) = 50.63$, and $v(x) = v(0) - \dfrac{gK}{W}x$, from which, at rest,

$$\boxed{x = \frac{Wv(0)}{gK} = 4365\text{ m} = 4.36\text{ km}}$$

Problem 14.127 If $m_A = 10$ kg, $m_B = 40$ kg, and the coefficient of kinetic friction between all surfaces is $\mu_k = 0.11$, what is the acceleration of B down the inclined surface?



Solution: Choose a coordinate system with the origin at the wall and the x axis parallel to the plane surface. Denote $\theta = 20^\circ$. Assume that slip has begun. From Newton's second law for block A:



(1) $\sum F_x = -T + m_A g \sin\theta + \mu_k N_A = m_A a_A$,

(2) $\sum F_y = N_A - m_A g \cos\theta = 0$. From Newton's second law for block B:

(3) $\sum F_x = -T - \mu_k N_B - \mu_k N_A + m_B g \sin\theta = m_B a_B$,

(4) $\sum F_y = N_B - N_A - m_B g \cos\theta = 0$. Since the pulley is one-to-one, the sum of the displacements is $x_B + x_A = 0$. Differentiate twice:

(5) $a_B + a_A = 0$. Solving these five equations in five unknowns, $T = 49.63$ N, $N_A = 92.2$ N, $N_B = 460.9$ N, $a_A = -0.593$ m/s^2,

$\boxed{a_B = 0.593 \text{ m/s}^2}$

Problem 14.128 In Problem 14.127, if A weighs 89 N, B weighs 444.8 N, and the coefficient of kinetic friction between all surfaces is $\mu_k = 0.15$, what is the tension in the cord as B slides down the inclined surface?

Solution: From the solution to Problem 14.127,

(1) $\sum F_x = -T + m_A g \sin\theta + \mu_k N_A = m_A a_A$,

(2) $\sum F_y = N_A - m_A g \cos\theta = 0$. For block B:

(3) $\sum F_x = -T - \mu_k N_B - \mu_k N_A + m_B g \sin\theta = m_B a_B$,

(4) $\sum F_y = N_B - N_A - m_B g \cos\theta = 0$.

(5) $a_B + a_A = 0$ Solve by iteration:

$\boxed{T = 46.5 \text{ N}}$,

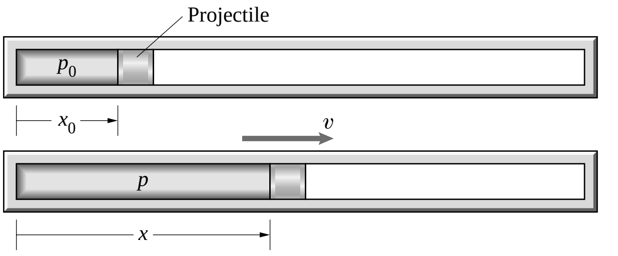
$N_A = 83.6$ N, $N_B - 501.7$ N, $a_A = -0.39$ m/s^2,

$a_B = 0.39$ m/s^2

Problem 14.129 A gas gun is used to accelerate projectiles to high velocities for research on material properties. The projectile is held in place while gas is pumped into the tube to a high pressure p_0 on the left and the tube is evacuated on the right. The projectile is then released and is accelerated by the expanding gas. Assume that the pressure p of the gas is related to the volume V it occupies by $pV^{\gamma} =$ constant, where γ is a constant. If friction can be neglected, show that the velocity of the projectile at the position x is

$$v = \sqrt{\frac{2p_0Ax_0^{\gamma}}{m(\gamma-1)}\left(\frac{1}{x_0^{\gamma-1}} - \frac{1}{x^{\gamma-1}}\right)},$$

where m is the mass of the projectile and A is the cross-sectional area of the tube.



Solution: The force acting on the projectile is $F = pA$ where p is the instantaneous pressure and A is the area. From $pV^{\gamma} = K$, where $K = p_0V_0^{\gamma}$ is a constant, and the volume $V = Ax$, it follows that $p = \frac{K}{(Ax)^{\gamma}}$, and the force is $F = KA^{1-\gamma}x^{-\gamma}$. The acceleration is

$$a = \frac{F}{m} = \frac{K}{m}A^{1-\gamma}x^{-\gamma}.$$

The equation to be integrated:

$v\frac{dv}{dx} = \frac{K}{m}A^{1-\gamma}x^{-\gamma}$, where the chain rule $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ has been used. Separate variables and integrate:

$$v^2 = 2\left(\frac{K}{m}\right)A^{1-\gamma}\int x^{-\gamma}\,dx + C = 2\left(\frac{K}{m}\right)A^{1-\gamma}\left(\frac{x^{1-\gamma}}{1-\gamma}\right) + C.$$

When $x = x_0$, $v_0 = 0$, therefore

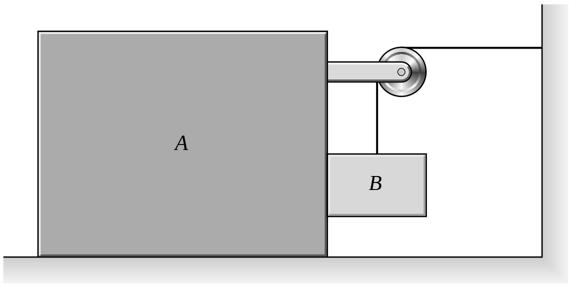
$$v^2 = 2\left(\frac{K}{m}\right)\left(\frac{A^{1-\gamma}}{1-\gamma}\right)(x^{1-\gamma} - x_0^{1-\gamma}).$$

Substitute $K = p_0V_0^{\gamma} = p_0A^{\gamma}x_0^{\gamma}$ and reduce:

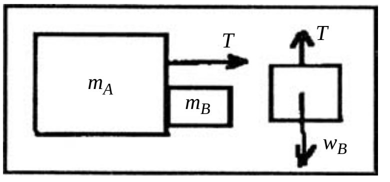
$v^2 = \frac{2p_0Ax_0^{\gamma}}{m(1-\gamma)}(x^{1-\gamma} - x_0^{1-\gamma})$. Rearranging:

$$\boxed{v = \sqrt{\frac{2p_0Ax_0^{\gamma}}{m(\gamma-1)}\left(\frac{1}{x_0^{\gamma-1}} - \frac{1}{x^{\gamma-1}}\right)}}$$

Problem 14.130 The weights of the blocks are $W_A = 120$ N, and $W_B = 20$ N and the surfaces are smooth. Determine the acceleration of block A and the tension in the cord.



Solution: Denote the tension in the cord near the wall by T_A. From Newton's second law for the two blocks:



$$\sum F_x = T_A = \left(\frac{W_A}{g} + \frac{W_B}{g}\right) a_A.$$

For block B: $\sum F_y = T_A - W_B = \frac{W_B}{g} a_B$. Since the pulley is one-to-one, as the displacement of B increases *downward* (negatively) the displacement of A increases *to the right* (positively), from which $x_A = -x_B$. Differentiate twice to obtain $a_A = -a_B$. Equate the expressions to obtain:

$a\left(\frac{W_A}{g} + \frac{W_B}{g}\right) = W_B + \frac{W_B}{g} a$, from which

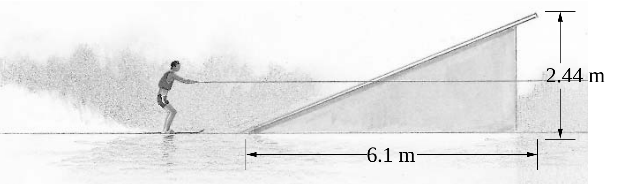
$$\boxed{a = g\left(\frac{W_B}{W_A + 2W_B}\right) = g\left(\frac{20}{160}\right) = \frac{9.81}{8} = 1.23 \text{ m/s}^2}$$

Problem 14.131 The 100-Mg space shuttle is in orbit when its engines are turned on, exerting a thrust force $\mathbf{T} = 10\mathbf{i} - 20\mathbf{j} + 10\mathbf{k}$ (kN) for 2 s. Neglect the resulting change in mass of the shuttle. At the end of the 2-s burn, fuel is still sloshing back and forth in the shuttle's tanks. What is the change in the velocity of the center of mass of the shuttle (including the fuel it contains) due to the 2-s burn?

Solution: At the completion of the burn, there are no external forces on the shuttle (it is in free fall) and the fuel sloshing is caused by internal forces that cancel, and the center of mass is unaffected. The change in velocity is

$$\boxed{\begin{aligned}\Delta\mathbf{v} &= \int_0^2 \frac{\mathbf{T}}{m}\,dt = \frac{2(10^4)}{10^5}\mathbf{i} - \frac{2(2\times10^4)}{10^5}\mathbf{j} + \frac{2(10^4)}{10^5}\mathbf{k} \\ &= 0.2\mathbf{i} - 0.4\mathbf{j} + 0.2\mathbf{k} \text{ (m/s)}\end{aligned}}$$

Problem 14.132 The water skier contacts the ramp with a velocity of 40.2 km/h parallel to the surface of the ramp. Neglecting friction and assuming that the tow rope exerts no force on him once he touches the ramp, estimate the horizontal length of the skier's jump from the end of the ramp.



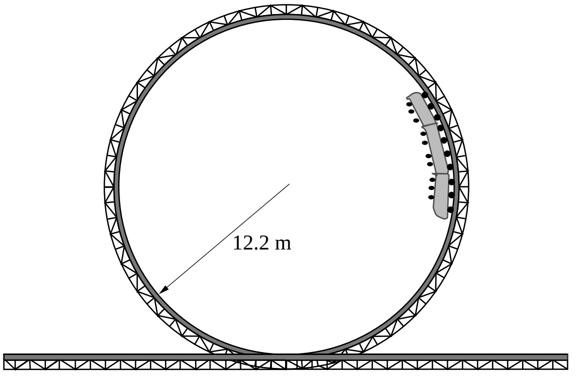
Solution: Break the path into two parts: (1) The path from the base to the top of the ramp, and (2) from the top of the ramp until impact with the water. Let u be the velocity parallel to the surface of the ramp, and let z be the distance along the surface of the ramp. From the chain rule, $u\dfrac{du}{dz} = -g\sin\theta$, where $\theta = \tan^{-1}\left(\dfrac{2.44}{6.1}\right) = 21.8°$. Separate variables and integrate: $u^2 = -(2g\sin\theta)z + C$. At the base of the ramp

$$u(0) = 40.2\ \text{km/h} = 11.17\ \text{m/s}$$

from which $C = (11.17^2) = 124.8$ and $u = \sqrt{C - (2g\sin\theta)z}$. At $z = \sqrt{2.44^2 + 6.1^2} = 6.56$ m $u = 8.78$ m/s. (2) In the second part of the path the skier is in free fall. The equations to be integrated are $\dfrac{dv_y}{dt} = -g$, $\dfrac{dy}{dt} = v_y$, with $v(0) = u\sin\theta = 8.78(0.3714) = 3.26$ m/s, $y(0) = 2.44$ m. $\dfrac{dv_x}{dt} = 0$, $\dfrac{dx}{dt} = v_x$, with $v_x(0) = u\cos\theta = 8.14$ m/s, $x(0) = 0$. Integrating: $v_y(t) = -gt + 3.26$ m/s. $y(t) = -\dfrac{g}{2}t^2 + 3.26t + 2.44$ m $v_x(t) = 8.14$ m/s, $x(t) = 8.14t$. When $y(t_{\text{impact}}) = 0$, the skier has hit the water. The impact time is $t_{\text{impact}}^2 + 2bt_{\text{impact}} + c = 0$ where $b = -\dfrac{3.26}{g}$, $c = -\dfrac{4.85}{g}$. The solution $t_{\text{impact}} = -b \pm \sqrt{b^2 - c} = 1.11$ s, $= -0.45$ s. The negative values has no meaning here. The horizontal distance is

$$\boxed{x(t_{\text{impact}}) = 8.14t_{\text{impact}} = 9.05\ \text{m}}$$

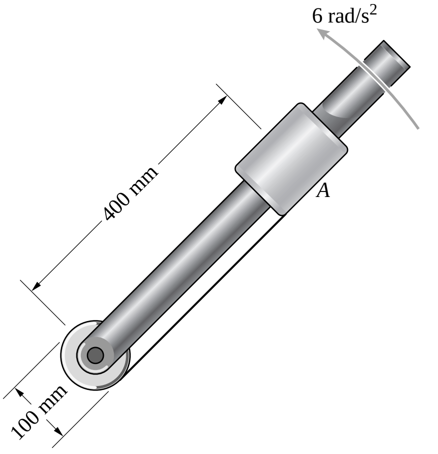
Problem 14.133 Suppose you are designing a roller-coaster track that will take the cars through a vertical loop of 12.2 m radius. If you decide that, for safety, the downward force exerted on a passenger by his seat at the top of the loop should be at least one-half the passenger's weight, what is the minimum safe velocity of the cars at the top of the loop?



Solution: Denote the normal force exerted on the passenger by the seat by N. From Newton's second law, at the top of the loop $-N - mg = -m\left(\dfrac{v^2}{R}\right)$, from which $-\dfrac{N}{m} = g - \dfrac{v^2}{R} = -\dfrac{g}{2}$. From which:

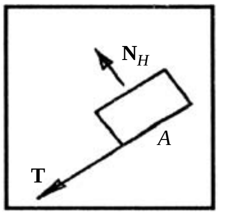
$$\boxed{v = \sqrt{\frac{3Rg}{2}} = 13.4\ \text{m/s}}$$

Problem 14.134 As the smooth bar rotates *in the horizontal plane*, the string winds up on the fixed cylinder and draws the 1-kg collar A inward. The bar starts from rest at $t = 0$ in the position shown and rotates with constant acceleration. What is the tension in the string at $t = 1$ s?



Solution: The angular velocity of the spool relative to the bar is $\alpha = 6$ rad/s^2. The acceleration of the collar relative to the bar is $\dfrac{d^2r}{dt^2} = -R\alpha = -0.05(6) = -0.3$ m/s^2. The take up velocity of the spool is

$$v_s = \int R\alpha\, dt = -0.05(6)t = -0.3t \text{ m/s.}$$



The velocity of the collar relative to the bar is $\dfrac{dr}{dt} = -0.3t$ m/s.
The velocity of the collar relative to the bar is $dr/dt = -0.3t$ m/s.
The position of the collar relative to the bar is $r = -0.15t^2 + 0.4$ m.
The angular acceleration of the collar is $\dfrac{d^2\theta}{dt^2} = 6$ rad/s^2. The angular velocity of the collar is $\dfrac{d\theta}{dt} = 6t$ rad/s. The radial acceleration is $a_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = -0.3 - (-0.15t^2 + 0.4)(6t)^2$. At $t = 1$ s the radial acceleration is $a_r = -9.3$ m/s^2, and the tension in the string is

$$\boxed{|T| = |ma_r| = 9.3 \text{ N}}$$

Problem 14.135 In Problem 14.134, suppose that the coefficient of kinetic friction between the collar and the bar is $\mu_k = 0.2$. What is the tension in the string at $t = 1$ s?

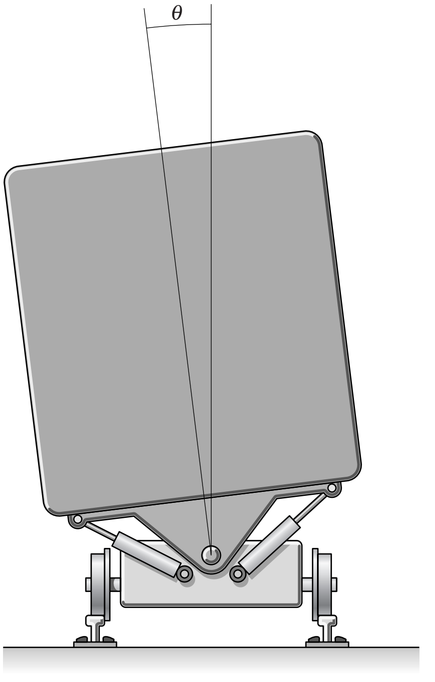
Solution: Use the results of the solution to Problem 14.134 At $t = 1$ s, the horizontal normal force is

$$N_H = |ma_\theta| = m\left|\left(r\frac{d^2\theta}{dt^2} + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)\right)\right| = 2.1 \text{ N,}$$

from which the total normal force is $N = \sqrt{N_H^2 + (mg)^2}$ From Newton's second law: $\left(-T + \mu_k\sqrt{N_H^2 + (mg^2)}\right)\mathbf{e}_r + N_H\mathbf{e}_\theta = ma_r\mathbf{e}_r + ma_\theta\mathbf{e}_\theta$, from which $-T + \mu_k\sqrt{N_H^2 + (mg)^2} = ma_r$. From the solution to Problem 14.152, $a_r = -9.3$ m/s^2. Solve: The tension is

$$\boxed{T = 11.306 \text{ N}}$$

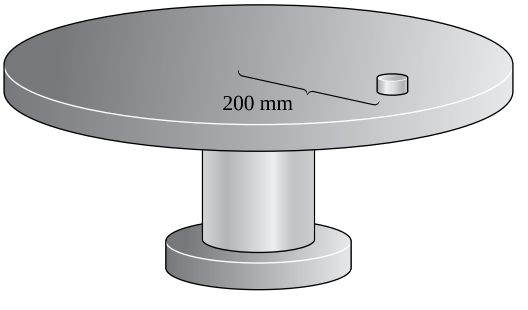
Problem 14.136 If you want to design the cars of a train to tilt as the train goes around curves in order to achieve maximum passenger comfort, what is the relationship between the desired tilt and θ, the velocity v of the train, and the instantaneous radius of curvature, ρ, of the track?



Solution: For comfort, the passenger should feel the total effects of acceleration down toward his feet, that is, apparent side (radial) accelerations should not be felt. This condition is achieved when the tilt θ is such that $mg\sin\theta - m(v^2/\rho)\cos\theta = 0$, from which

$$\boxed{\tan\theta = \frac{v^2}{\rho g}.}$$

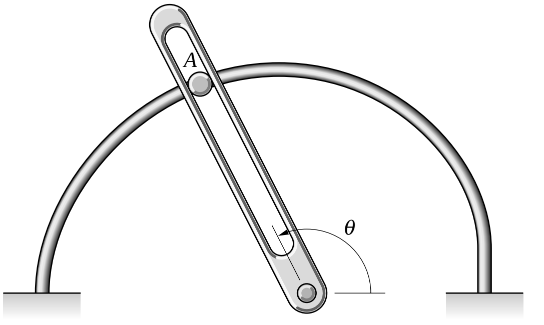
Problem 14.137 To determine the coefficient of static friction between two materials, an engineer at the U.S. National Institute of Standards and Technology places a small sample of one material on a horizontal disk whose surface is made of the other material and then rotates the disk from rest with a constant angular acceleration of 0.4 rad/s^2. If she determines that the small sample slips on the disk after 9.903 s, what is the coefficient of friction?



Solution: The angular velocity after $t = 9.903$ s is $\omega = 0.4t = 3.9612$ rad/s. The radial acceleration is $a_n = 0.2\omega^2 = 3.138$ m/s^2. The tangential acceleration is $a_t = (0.2)0.4 = 0.08$ m/s^2. At the instant before slip occurs, Newton's second law for the small sample is $\sum F = \mu_s N = \mu_s mg = m\sqrt{a_n^2 + a_t^2}$, from which

$$\boxed{\mu_s = \frac{\sqrt{a_n^2 + a_t^2}}{g} = 0.320}$$

Problem 14.138* The 1-kg slider A is pushed along the curved bar by the slotted bar. The curved bar lies *in the horizontal plane,* and its profile is described by $r = 2\left(\frac{\theta}{2\pi} + 1\right)$ m, where θ is in radians. The angular position of the slotted bar is $\theta = 2t$ rad. Determine the radial and transverse components of the total external force exerted on the slider when $\theta = 120^0$.



Solution: The radial position is $r = 2\left(\frac{t}{\pi} + 1\right)$. The radial velocity: $\frac{dr}{dt} = \frac{2}{\pi}$.

The radial acceleration is zero. The angular velocity: $\frac{d\theta}{dt} = 2$. The angular acceleration is zero. At $\theta = 120^0 = 2.09$ rad. From Newton's second law, the radial force is $F_r = ma_r$, from which

$$\boxed{\mathbf{F}_r = -\left[r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{e}_r = -10.67\mathbf{e}_r \text{ N}}$$

The transverse force is $F_\theta = ma_\theta$, from which

$$\boxed{\mathbf{F}_\theta = 2\left[\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)\right]\mathbf{e}_\theta = 2.55\mathbf{e}_\theta \text{ N}}$$

Problem 14.139* In Problem 14.138, suppose that the curved bar lies *in the vertical plane*. Determine the radial and transverse components of the total force exerted on A by the curved and slotted bars at $t = 0.5$ s.

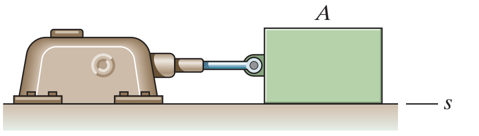
Solution: Assume that the curved bar is vertical such that the line $\theta = 0$ is horizontal. The weight has the components: $\mathbf{W} = (W \sin\theta)\mathbf{e}_r + (W \cos\theta)\mathbf{e}_\theta$. From Newton's second law: $F_r - W \sin\theta = ma_r$, and $F_\theta - W\cos\theta = ma_\theta$., from which $\mathbf{F}_r - g \sin 2t\mathbf{e}_r = -r(d\theta/dt)^2\mathbf{e}_r$, from which

$$F_r = \left(-2\left(\frac{t}{\pi} + 1\right)(2^2) + g \sin 2t\right),$$

at $t = 0.5$ s, $\boxed{F_r = -1.02 \text{ N}}$. The transverse component $F_\theta = 2\left(\frac{2}{\pi}\right)(2) + g\cos 2t = \left(\frac{8}{\pi} + g\cos 2t\right)$. At $t = 0.51$ s,

$$\boxed{F_\theta = 7.85 \text{ N}}$$

Problem 15.1 In Active Example 15.1, what is the velocity of the container when it has reached the position $s = 2$ m?



Solution: The 180-kg container A starts from rest at $s = 0$. The horizontal force (in newtons) is $F = 700 - 150s$. The coefficient of kinetic friction is $\mu_k = 0.26$.

$$U_{12} = T_2 - T_1$$

$$\int_0^2 (700 - 150s - 0.26[180(9.81)])ds = \frac{1}{2}(180 \text{ kg})v_2^2 - 0$$

$$700(2) - \frac{1}{2}(150)(2)^2 - (0.26[180(9.81)][2]) = 90v_2^2$$

$$\boxed{v_2 = 1.42 \text{ m/s.}}$$

Problem 15.2 The mass of the Sikorsky UH-60A helicopter is 9300 kg. It takes off vertically with its rotor exerting a constant upward thrust of 112 kN. Use the principle of work and energy to determine how far it has risen when its velocity is 6 m/s.

Strategy: Be sure to draw the free-body diagram of the helicopter.

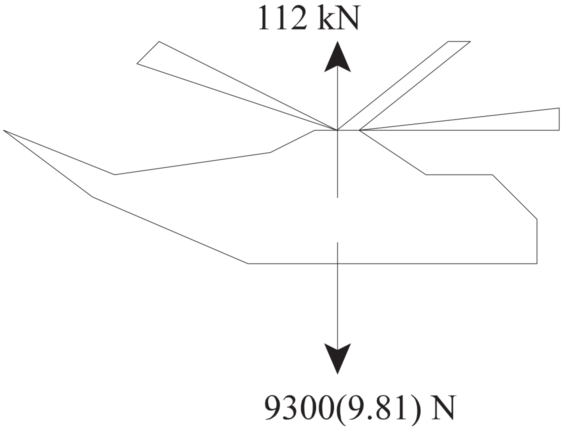
Solution:

$$U_{12} = T_2 - T_1$$

$$[(112000 - 9300[9.81])\text{N}]h$$

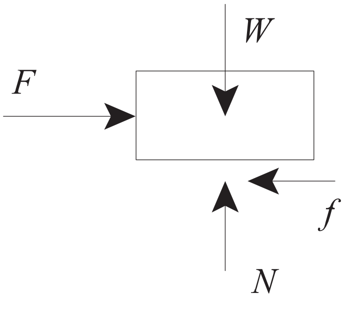
$$= \frac{1}{2}(9300 \text{ kg})(6 \text{ m/s})^2$$

$$\boxed{h = 8.06 \text{ m.}}$$



Problem 15.3 The 20-N box is at rest on the horizontal surface when the constant force $F = 5$ N is applied. The coefficient of kinetic friction between the box and the surface is $\mu_k = 0.2$. Determine how fast the box is moving when it has moved 2 m from its initial position (a) by applying Newton's second law; (b) by applying the principle of work and energy.

Solution:



(a) The equations of motion can be used to find the acceleration

$$\Sigma F_x : F - f = \frac{W}{g}a,\ \Sigma F_y : N - W = 0,$$

$$f = \mu_k N$$

Solving we have

$$a = g\left(\frac{F}{W} - \mu_k\right) = (9.81\,\text{m/s}^2)\left(\frac{5\text{ N}}{20\text{ N}} - 0.2\right) = 0.49\text{ m/s}^2$$

Now we integrate to find the velocity at the new position

$$a = v\frac{dv}{ds} \Rightarrow \int_0^v v\,dv = \int_0^{2\text{ m}} a\,ds \Rightarrow \frac{v^2}{2} = a(2\text{ m}) = (0.49\text{ m/s}^2)(2\text{ m})$$

$$\boxed{v = 1.4\text{ m/s}}$$

(b) Using the principle of work and energy we have (recognizing that $N = W$)

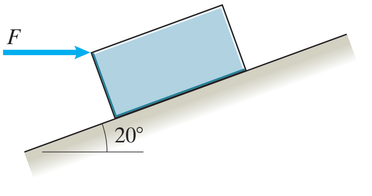
$$U_{12} = T_2 - T_1$$

$$(F - \mu_k N)d = \frac{1}{2}\left(\frac{W}{g}\right)v^2 - 0$$

$$v^2 = 2g\left(\frac{F}{W} - \mu_k\right)d = 2(9.81\text{ m/s}^2)\left(\frac{5\text{ N}}{20\text{ N}} - 0.2\right)(2\text{ m})$$

$$\boxed{v = 1.4\text{ m/s}}$$

Problem 15.4 At the instant shown, the 30-N box is moving up the smooth inclined surface at 2 m/s. The constant force $F = 15$ N. How fast will the box be moving when it has moved 1 m up the surface from its present position?



Solution:

$$U_{12} = T_2 - T_1$$

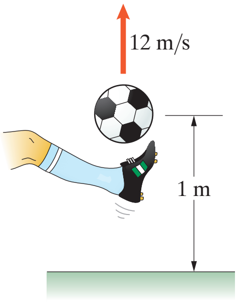
$$[(15\text{ N})\cos 20^\circ - (30\text{ N})\sin 20^\circ](1\text{ m})$$

$$= \frac{1}{2}\left(\frac{30\text{ N}}{9.81\text{ m/s}^2}\right)v^2 - \frac{1}{2}\left(\frac{30\text{ N}}{9.81\text{ m/s}^2}\right)(2\text{ m/s})^2$$

Solving we find

$$\boxed{v = 2.55\text{ m/s.}}$$

Problem 15.5 The 0.45-kg soccer ball is 1 m above the ground when it is kicked straight upward at 10 m/s. By using the principle of work and energy, determine: (a) how high above the ground the ball goes, (b) the magnitude of the ball's velocity when it falls back to a height of 1 m above the ground, (c) the magnitude of the ball's velocity immediately before it hits the ground.



Solution:

(a) Find the height above the ground

$$mg(1\text{ m} - h) = 0 - \frac{1}{2}mv_0^2,$$

$$h = \frac{v_0{}^2}{2g} + 1\text{ m} = \frac{(10\text{ m/s})^2}{2(9.81\text{ m/s}^2)} + 1\text{ m} = 6.10\text{ m}$$

(b) When the ball returns to the same level, the velocity must be equal to the initial velocity (but now it is moving downward) because the net work is zero

$$v = 10\text{ m/s} \downarrow$$

(c) The velocity just before it hits the ground

$$mg(1\text{ m}) = \frac{1}{2}mv^2 - \frac{1}{2}mv_0{}^2$$

$$v^2 = v_0{}^2 + 2g(1\text{ m}) = (10\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(1\text{ m})$$

$$v = 10.9\text{ m/s.} \downarrow$$

$$\boxed{(a)\ h = 6.10\text{ m},\quad (b)\ v = 10.0\text{ m/s},\quad (c)\ v = 10.9\text{ m/s.}}$$

Problem 15.6 Assume that the soccer ball in Problem 15.5 is stationary the instant before it is kicked upward at 12 m/s. The duration of the kick is 0.02 s. What average power is transferred to the ball during the kick?

Solution:

$$U_{12} = \frac{1}{2}(0.45\text{ kg})(12\text{ m/s})^2 - 0 = 32.4\text{ N-m}$$

$$\boxed{\text{Power} = \frac{U_{12}}{\Delta t} = \frac{32.4\text{ N-m}}{0.02\text{ s}} = 1.62\text{ kW}}$$

Problem 15.7 The 2000-N drag racer starts from rest and travels a quarter-kilometre course. It completes the cours in 4.524 seconds and crosses the finish line traveling at 325.77 km/h. (a) How much work is done on the car as it travels the course? (b) Assume that the horizontal force exerted on the car is constant and use the principle of work and energy to determine it.

Solution:

(a) The work is equal to the change in kinetic energy.

$$U = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{2000\text{ N}}{9.81\text{ m/s}^2}\right)\left[(325.77\text{ km/h})\left(\frac{1000}{3600}\right)\right]^2$$

$$\boxed{U = 8.35\times 10^5\text{ N-m}}$$

(b) The work is equal to the force times the distance

$$U = Fd \Rightarrow F = \frac{U}{d} = \frac{8.35\times 10^5\text{ N-m}}{\frac{1}{4}(1000\text{ m})} = 3339\text{ N}$$

$$\boxed{F = 3339\text{ N}}$$

Problem 15.8 The 2000-N drag racer starts from rest and travels a quarter-kilometre course. It completes the course in 4.524 seconds and crosses the finish line traveling at 325.77 km/h. Assume that the horizontal force exerted on the car is constant. Determine (a) the maximum power and (b) the average power transferred to the car as it travels the quarter-kilometre course.

Solution: From problem 15.7 we know that the force is

(a) The maximum power occurs when the car has reached its maximum velocity

$$P = Fv = (3339\text{ N})(325.77\text{ km/h})\left(\frac{1000}{3600}\right) = 3.02\times 10^5\text{ N-m/s.}$$

(b) The average power is equal to the change in kinetic energy divided by the time.

$$P_{\text{ave}} = \frac{\frac{1}{2}mv^2}{\Delta t} = \frac{\frac{1}{2}\left(\frac{2000\text{ N}}{9.81\text{ m/s}^2}\right)\left[(325.77\text{ km/h})\left(\frac{1000}{3600}\right)\right]^2}{4.524\text{ s}}$$

$$= 1.845\times 10^5\text{ N-m/s.}$$

$$\boxed{(a)\ 3.02\times 10^5\text{ N-m/s,}\quad (b)\ 1.845\times 10^5\text{ N-m/s.}}$$

Problem 15.9 As the 32,000-N airplane takes off, the tangential component of force exerted on it by its engines is $\Sigma F_t = 45{,}000$ N. Neglecting other forces on the airplane, use the principle of work and energy to determine how much runway is required for its velocity to reach 200 km/h.

Solution:

$$U_{12} = \frac{1}{2}mv^2 \Rightarrow Fd = \frac{1}{2}mv^2 \Rightarrow d = \frac{mv^2}{2F}$$

$$d = \frac{\left(\dfrac{32{,}000\text{ N}}{9.81\text{ m/s}^2}\right)\left[(200\text{ km/h})\left(\dfrac{1000}{3600}\right)\right]^2}{2(45{,}000\text{ N})} = 112\text{ m}.$$

$$\boxed{d = 112\text{ m}}$$

Problem 15.10 As the 32,000-N airplane takes off, the tangential component of force exerted on it by its engines is $\Sigma F_t = 45{,}000$ N. Neglecting other forces on the airplane, determine (a) the maximum power and (b) the average power transferred to the airplane as its velocity increases from zero to 200 km/h.

Solution:

(a) The maximum power occurs when the velocity is a maximum

$$P = Fv = (45{,}000\text{ N})\left[200\text{ km/h}\frac{1000}{3600}\right] = 2.5\times 10^6\text{ N-m/s}.$$

(b) To find the average power we need to know the time that it takes to reach full speed

$$a = \frac{F}{m} = \frac{45{,}000\text{ N}}{\left(\dfrac{32{,}000\text{ N}}{9.81\text{ m/s}^2}\right)} = 13.8\text{ m/s}^2$$

$$v = at \Rightarrow t = \frac{v}{a} = \frac{200\text{ km/h}\dfrac{1000}{3600}}{13.8\text{ m/s}^2} = 4.03\text{ s}.$$

Now, the average power is the change in kinetic energy divided by the time

$$P_{\text{ave}} = \frac{\frac{1}{2}mv^2}{t} = \frac{\frac{1}{2}\left(\dfrac{32{,}000\text{ N}}{9.81\text{ m/s}^2}\right)\left(200\text{ km/h}\dfrac{1000}{3600}\right)^2}{4.03\text{ s}} = 1.25\times 10^6\text{ N-m/s}.$$

$$\boxed{\text{(a) } 2.5\times 10^6\text{ N-m/s},\quad \text{(b) } 1.25\times 10^6\text{ N-m/s}.}$$

Problem 15.11 The 32,000-N airplane takes off from rest in the position $s = 0$. The total tangential force exerted on it by its engines and aerodynamic drag (in Newtons) is given as a function of its position s by $\Sigma F_t = 45{,}000 - 5.2s$. Use the principle of work and energy to determine how fast the airplane is traveling when its position is $s = 950$ m.

Solution:

$$U_{12} = \int_0^{950} (45{,}000 - 5.2s)\, ds$$

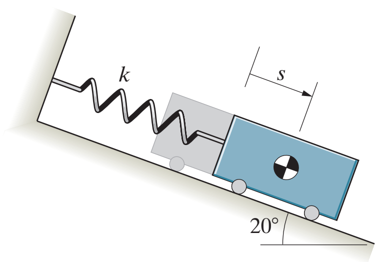
$$= (45{,}000)(950) - \frac{1}{2}(5.2)(950)^2 = 40.4 \times 10^6 \text{ N-m}$$

$$U_{12} = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{32{,}000 \text{ N}}{9.81 \text{ m/s}^2}\right)v^2$$

Solving, we find

$$\boxed{v = 157.4 \text{ m/s.}}$$

Problem 15.12 The spring ($k = 20$ N/m) is unstretched when $s = 0$. The 5-kg cart is moved to the position $s = -1$ m and released from rest. What is the magnitude of its velocity when it is in the position $s = 0$?



Solution: First we calculate the work done by the spring and by gravity

$$U_{12} = \int_{-1 \text{ m}}^{0} (-ks + mg\sin 20^\circ)\, ds$$

$$= \frac{1}{2}k(-1 \text{ m})^2 + mg\sin 20^\circ(1 \text{ m})$$

$$= \frac{1}{2}(20 \text{ N/m})(-1 \text{ m})^2 + (5 \text{ kg})(9.81 \text{ m/s}^2)\sin 20^\circ(1 \text{ m})$$

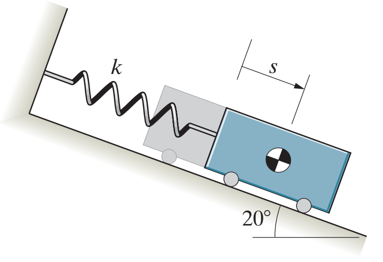
$$= 26.8 \text{ N-m.}$$

Now using work and energy

$$U_{12} = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2U_{12}}{m}} = \sqrt{\frac{2(26.8 \text{ N-m})}{5 \text{ kg}}} = 3.27 \text{ m/s.}$$

$$\boxed{v = 3.27 \text{ m/s.}}$$

Problem 15.13 The spring ($k = 20$ N/m) is unstretched when $s = 0$. The 5-kg cart is moved to the position $s = -1$ m and released from rest. What maximum distance down the sloped surface does the cart move relative to its initial position?



Solution: The cart starts from a position of rest, and when it reaches the maximum position, it is again at rest. Therefore, the total work must be zero.

$$U_{12} = \int_{-1\text{ m}}^{s} (-ks + mg\sin 20^\circ)\,ds$$

$$= -\frac{1}{2}k(s^2 - [-1\text{ m}]^2) + mg\sin 20^\circ(s - [-1\text{ m}])$$

$$= -\frac{1}{2}(20\text{ N/m})(s^2 - [-1\text{ m}]^2) + (5\text{ kg})(9.81\text{ m/s}^2)\sin 20^\circ(s - [-1\text{ m}]) = 0$$

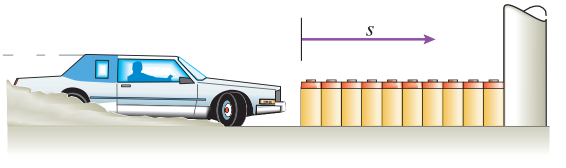
This is a quadratic equation that has the two solutions

$$s_1 = -1\text{ m},\ s_2 = 2.68\text{ m}.$$

The distance relative to the initial is $s = s_2 + 1$ m.

$$\boxed{s = 3.68\text{ m.}}$$

Problem 15.14 The force exerted on a car by a prototype crash barrier as the barrier crushes is $F = -(120s + 40s^3)$ N, where s is the distance in metre from the initial contact. The effective length of the barrier is 18 m. How fast can a 5000-N car be moving and be brought to rest within the effective length of the barrier?



Solution: The barrier can provide a maximum amount of work given by

$$U_{12} = \int_0^{18} -(120s + 40s^3)\,ds$$

$$= -\frac{1}{2}(120)(18)^2 - \frac{1}{4}(40)(18)^4 = -1.07 \times 10^6\text{ N-m}.$$

Using work and energy, we have

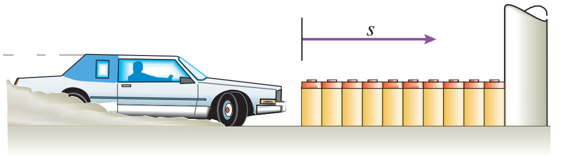
$$U_{12} = 0 - \frac{1}{2}\,mv^2$$

$$-1.07 \times 10^6\text{ N-m} = -\frac{1}{2}\left(\frac{5000\text{ N}}{9.81\text{ m/s}^2}\right)v^2$$

Solving for the velocity, we find

$$\boxed{v = 64.8\text{ m/s.}}$$

Problem 15.15 A 5000-N car hits the crash barrier at 80 km/h and is brought to rest in 0.11 seconds. What average power is transferred from the car during the impact?



Solution: The average power is equal to the change in kinetic energy divided by the time

$$P = \frac{\frac{1}{2}mv^2}{\Delta t} = \frac{\frac{1}{2}\left(\frac{5000\text{ N}}{9.81\text{ m/s}^2}\right)\left(80\text{ km/h}\frac{1000}{3600}\right)^2}{0.11\text{ s}} = 1.14\times 10^6\text{ N-m/s.}$$

$$\boxed{P = 1.14\times 10^6\text{ N-m/s.}}$$

Problem 15.16 A group of engineering students constructs a sun-powered car and tests it on a circular track with a 1000-m radius. The car, with a weight of 460 N including its occupant, starts from rest. The total tangential component of force on the car is

$$\Sigma F_t = 30 - 0.2s\text{ N},$$

where s is the distance (in ft) the car travels along the track from the position where it starts.

(a) Determine the work done on the car when it has gone a distance $s = 120$ m.

(b) Determine the magnitude of the *total* horizontal force exerted on the car's tires by the road when it is at the position $s = 120$ m.

Solution:

(a) $$\boxed{U = \int_0^{120\text{ m}} [30 - 0.2\ s]\text{ N }ds = 2160\text{ N-m}}$$

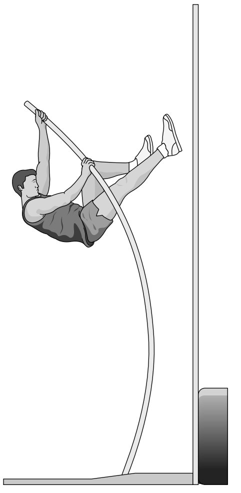
(b) $$2160\text{ N-m} = \frac{1}{2}\left(\frac{460\text{ N}}{9.81\text{ m/s}^2}\right)v^2 \Rightarrow v = 9.6\text{ m/s}$$

$$F_t = [30 - 0.2(120)] = 6\text{ N}$$

$$F_n = m\frac{v^2}{\rho} = \left(\frac{460\text{ N}}{9.81\text{ m/s}^2}\right)\frac{(9.6\text{ m/s})^2}{1000\text{ m}} = 4.32\text{ N}$$

$$\boxed{F = \sqrt{{F_t}^2 + {F_n}^2} = \sqrt{(6\text{ N})^2 + (4.32\text{ N})^2} = 7.39\text{ N}}$$

Problem 15.17 At the instant shown, the 160-N vaulter's center of mass is 8.5 m above the ground, and the vertical component of his velocity is 4 m/s. As his pole straightens, it exerts a vertical force on the vaulter of magnitude $180 + 2.8y^2$ N, where y is the vertical position of his center of mass *relative to its position at the instant shown*. This force is exerted on him from $y = 0$ to $y = 4$ m, when he releases the pole. What is the maximum height above the ground reached by the vaulter's center of mass?



Solution: The work done on him by the pole is

$$U_{\text{pole}} = \int_0^4 (180 + 2.8\ y^2)\,dy$$

$$= 180(4) + 2.8\frac{(4)^3}{3} = 780\ \text{N-m}.$$

Let $y_{\max}$ be his maximum height above the ground. The work done by his weight from the instant shown to the maximum height is

$$-160(y_{\max} - 8.5) = U_{\text{weight}},\ \text{or}\ U_{\text{weight}} + U_{\text{pole}}$$

$$= mv_2^2/2 - mv_1^2/2$$

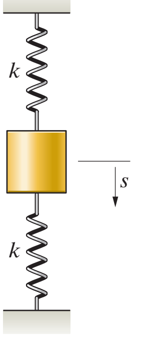
$$780 - 160(y_{\max} - 8.5) = 0 - \frac{1}{2}\left(\frac{160}{9.81}\right)(4)^2.$$

Solving, $y_{\max} = 14.2$ m

Problem 15.18 The springs ($k = 25$ N/cm) are unstretched when $s = 0$. The 50-N weight is released from rest in the position $s = 0$.

(a) When the weight has fallen 1 cm, how much work has been done on it by each spring?

(b) What is the magnitude of the velocity of the weight when it has fallen 1 cm?



Solution:

(a) The work done by each spring

$$U_{12} = \int_0^{1\,\text{cm}} -ks\,ds = -\frac{1}{2}(25\ \text{N/cm})(1\,\text{cm})^2 = -12.5\ \text{N-cm}.$$

(b) The velocity is found from the work-energy equation. The total work includes the work done by both springs and by gravity

$$U_{12} = (50\ \text{N})(1\ \text{cm}) - 2(12.5\ \text{N-cm}) = 25\ \text{N-cm}.$$

$$U_{12} = \frac{1}{2}\,mv^2 = \frac{1}{2}\left(\frac{50\ \text{N}}{9.81\ \text{m/s}^2}\right)v^2$$

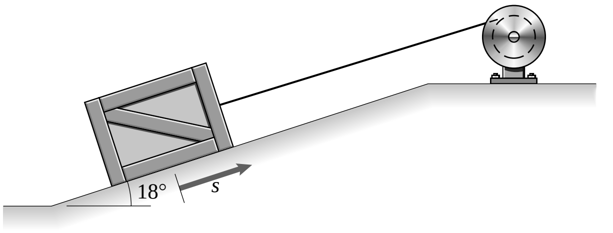
Solving for the velocity we find $v = 0.31$ m/s.

$\boxed{\text{(a)} -12.5\ \text{N-cm, (b) } 0.31\ \text{m/s.}}$

Problem 15.19 The coefficients of friction between the 160-kg crate and the ramp are $\mu_s = 0.3$ and $\mu_k = 0.28$.

(a) What tension T_0 must the winch exert to start the crate moving up the ramp?

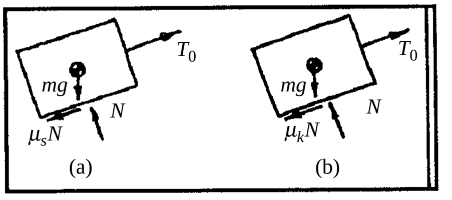
(b) If the tension remains at the value T_0 after the crate starts sliding, what total work is done on the crate as it slides a distance $s = 3$ m up the ramp, and what is the resulting velocity of the crate?



Solution:

(a) The tension is $T_0 = W\sin\theta + \mu_s$ N, from which

$$\boxed{T_0 = mg(\sin\theta + \mu_s\cos\theta) = 932.9 \text{ N.}}$$



(b) The work done on the crate by (non-friction) external forces is

$$U_{\text{weight}} = \int_0^3 T_0\,ds - \int_0^3 (mg\sin\theta)\,ds = 932.9(3) - 1455.1$$

$$= 1343.5 \text{ N-m.}$$

The work done on the crate by friction is

$$U_f = \int_0^3 (-\mu_k\ N)\,ds = -3\mu_k mg\cos\theta = -1253.9 \text{ N-m.}$$

From the principle of work and energy is

$$U_{\text{weight}} + U_f = \frac{1}{2}mv^2,$$

from which

$$v = \sqrt{\frac{6(T_0 - mg(\sin\theta + \mu_k\cos\theta))}{m}}$$

$$\boxed{v = 1.06 \text{ m/s}}$$

Problem 15.20 In Problem 15.19, if the winch exerts a tension $T = T_0(1 + 0.1s)$ after the crate starts sliding, what total work is done on the crate as it slides a distance $s = 3$ m up the ramp, and what is the resulting velocity of the crate?

Solution: The work done on the crate is

$$U = \int_0^3 T\,ds - \int_0^3 (mg\sin\theta)\,ds - \mu_k\int_0^3 (mg\cos\theta)\,ds,$$

from which

$$U = T_0\left[\left(s + 0.05s^2\right)\right]_0^3 - (mg\sin\theta)(3) - \mu_k(mg\cos\theta)(3).$$

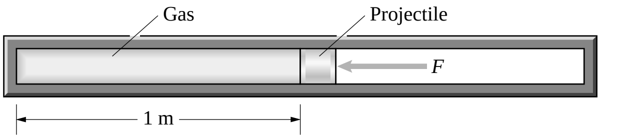
From the solution to Problem 15.19, $T_0 = 932.9$ N-m, from which the total work done is

$$\boxed{U = 3218.4 - 1455.1 - 1253.9 = 509.36 \text{ N-m.}}$$

From the principle of work and energy, $U = \frac{1}{2}mv^2$, from which

$$\boxed{v = \sqrt{\frac{2U}{m}} = 2.52 \text{ m/s}}$$

Problem 15.21 The 200-mm-diameter gas gun is evacuated on the right of the 8-kg projectile. On the left of the projectile, the tube contains gas with pressure $p_0 = 1 \times 10^5 Pa$ (N/m^2). The force F is slowly increased, moving the projectile 0.5 m to the left from the position shown. The force is then removed and the projectile accelerates to the right. If you neglect friction and assume that the pressure of the gas is related to its volume by $pV =$ constant, what is the velocity of the projectile when it has returned to its original position?



Solution: The constant is $K = pV = 1 \times 10^5(1)(0.1)^2\pi = 3141.6$ N-m. The force is $F = pA$. The volume is $V = As$, from which the pressure varies as the inverse distance: $p = \frac{K}{As}$, from which $F = \frac{K}{s}$.

The work done by the gas is

$$U = \int_{0.5}^{1} F\,ds = \int_{0.5}^{1} \frac{K}{x}\,ds = [K\ln(s)]_{0.5}^{1.0} = K\ln(2).$$

From the principle of work and energy, the work done by the gas is equal to the gain in kinetic energy:

$$K\ln(2) = \frac{1}{2}mv^2, \text{ and } v^2 = \frac{2K}{m}\ln(2),$$

$$\boxed{v = \sqrt{\frac{2K}{m}\ln(2)} = 23.33 \text{ m/s}}$$

Note: The argument of ln(2) is dimensionless, since it is ratio of two distances.

Problem 15.22 In Problem 15.21, if you assume that the pressure of the gas is related to its volume by $pV =$ constant while it is compressed (an isothermal process) and by $pV^{1.4} =$ constant while it is expanding (an isentropic process), what is the velocity of the projectile when it has returned to its original position?

Solution: The isothermal constant is $K = 3141.6$ N-m from the solution to Problem 15.21. The pressure at the leftmost position is

$$p = \frac{K}{A(0.5)} = 2 \times 10^5 \text{ N/m}^2.$$

The isentropic expansion constant is

$$K_e = pV^{1.4} = (2 \times 10^5)(A^{1.4})(0.5^{1.4}) = 596.5 \text{ N-m}$$

The pressure during expansion is

$$p = \frac{K_e}{(As)^{1.4}} = \frac{K_e}{A^{1.4}}s^{-1.4}.$$

The force is $F = pA = K_eA^{-0.4}s^{-1.4}$. The work done by the gas during expansion is

$$U = \int_{0.5}^{1.0} F\,ds = \int_{0.5}^{1.0} K_eA^{-0.4}s^{-1.4}\,ds = K_eA^{-0.4}\left[\frac{s^{-0.4}}{-0.4}\right]_{0.5}^{1.0}$$

$$= 1901.8 \text{ N-m}.$$

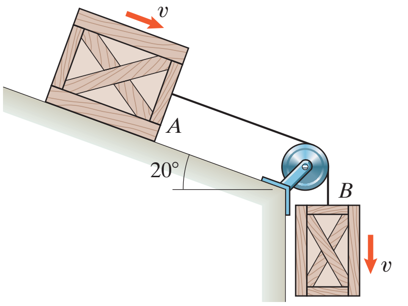
From the principle of work and energy, the work done is equal to the gain in kinetic energy,

$$\int_{0.5}^{1} F\,ds = \frac{1}{2}mv^2,$$

from which the velocity is

$$\boxed{v = \sqrt{\frac{2(1901.8)}{m}} = 21.8 \text{ m/s}}.$$

Problem 15.23 In Example 15.2, suppose that the angle between the inclined surface and the horizontal is increased from 20° to 30°. What is the magnitude of the velocity of the crates when they have moved 400 mm?



Solution: Doing work–energy for the system

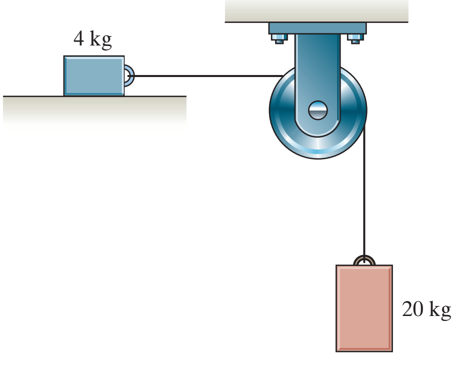
$$\int_0^{0.4} (m_A g \sin 30^\circ - \mu_k m_A g \cos 30^\circ + m_B g)\, ds = \frac{1}{2}(m_A + m_B)v_2^2$$

$$[40 \sin 30^\circ - (0.15)(40) \cos 30^\circ + 30](9.81)(0.4) = \frac{1}{2}(70)v_2^2$$

Solving for the velocity we find

$$\boxed{v_2 = 2.24 \text{ m/s.}}$$

Problem 15.24 The system is released from rest. The 4-kg mass slides on the smooth horizontal surface. By using the principle of work and energy, determine the magnitude of the velocity of the masses when the 20-kg mass has fallen 1 m.

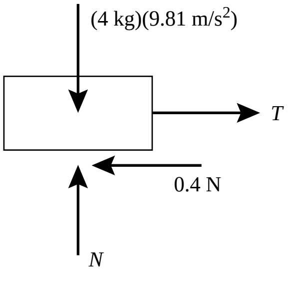


Solution: Write work-energy for system

$$U = (20 \text{ kg})(9.81 \text{ m/s}^2)(1 \text{ m}) = \frac{1}{2}(24 \text{ kg})v^2 \Rightarrow \boxed{v = 4.04 \text{ m/s}}$$

Problem 15.25 Solve Problem 15.24 if the coefficient of kinetic friction between the 4-kg mass and the horizontal surface is $\mu_k = 0.4$.

Solution:



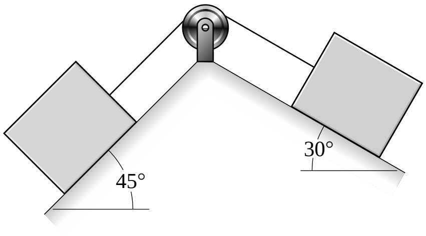
$$\sum F_y : N - (4 \text{ kg})(9.81 \text{ m/s}^2) = 0 \Rightarrow N = 39.24 \text{ N}$$

Write work-energy for system

$$U = [(20 \text{ kg})(9.81 \text{ m/s}^2) - 0.4(39.24 \text{ N})](1 \text{ m}) = 180.5 \text{ N-m}$$

$$180.5 \text{ N-m} = \frac{1}{2}(24 \text{ kg})v^2 \Rightarrow \boxed{v = 3.88 \text{ m/s}}$$

Problem 15.26 Each box weighs 50 N and the inclined surfaces are smooth. The system is released from rest. Determine the magnitude of the velocities of the boxes when they have moved 1 m.



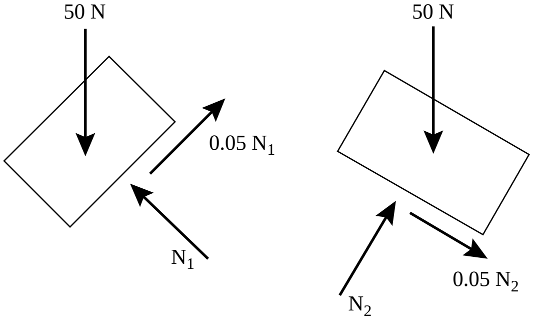
Solution: Write work-energy for the system

$$U = (50\ \text{N} \sin 45^\circ)(1\ \text{m}) - (50\ \text{N} \sin 30^\circ)(1\ \text{m}) = 10.36\ \text{N-m}$$

$$10.36\ \text{N-m} = \frac{1}{2}\left(\frac{100\ \text{N}}{9.81\ \text{m/s}^2}\right)v^2 \Rightarrow \boxed{v = 1.43\ \text{m/s}}$$

Problem 15.27 Solve Problem 15.26 if the coefficient of kinetic friction between the boxes and the inclined surfaces is $\mu_k = 0.05$.

Solution:



$$\sum F_{\nwarrow} : N_1 - (50\ \text{N}) \sin 45^\circ = 0$$

$$\sum F_{\nearrow} : N_2 - (50\ \text{N}) \cos 30^\circ = 0$$

$N_1 = 35.4$ N, $N_2 = 43.3$ N

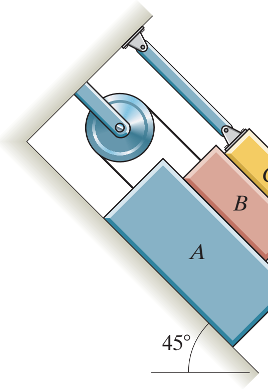
Work-energy for the system

$$U = (50\ \text{N} \sin 45^\circ)(1\ \text{m}) - (0.05)(35.4\ \text{N})(1\ \text{m})$$

$$- (50\ \text{N} \sin 30)(1\ \text{m}) - (0.05)(43.3\ \text{N})(1\ \text{m}) = 6.42\ \text{N-m}$$

$$6.42\ \text{N-m} = \frac{1}{2}\left(\frac{100\ \text{N}}{9.81\ \text{m/s}^2}\right)v^2 \Rightarrow \boxed{v = 1.12\ \text{m/s}}$$

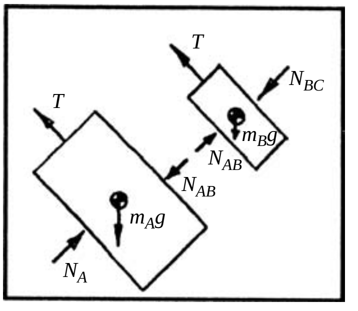
Problem 15.28 The masses of the three blocks are $m_A = 40$ kg, $m_B = 16$ kg, and $m_C = 12$ kg. Neglect the mass of the bar holding C in place. Friction is negligible. By applying the principle of work and energy to A and B individually, determine the magnitude of their velocity when they have moved 500 mm.



Solution: Denote $b = 0.5$ m. Since the pulley is one-to-one, denote $|v_A| = |v_B| = v$. The principle of work and energy for weight A is

$$\int_0^b (m_A g \sin\theta - T)\,ds = \frac{1}{2} m_A v^2,$$

and for weight B $\int_0^b (T - m_B g \sin\theta)\,ds = \frac{1}{2} m_B v^2.$



Add the two equations:

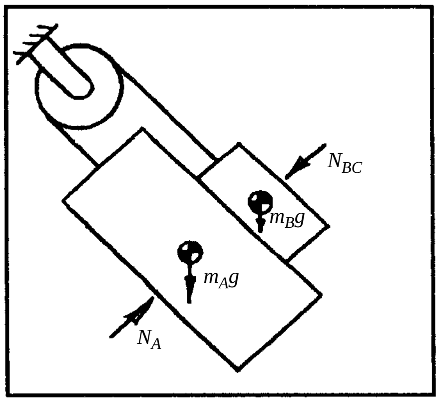
$$(m_A - m_B)gb\sin\theta = \tfrac{1}{2}(m_A + m_B)v^2.$$

Solve: $\boxed{|v_A| = |v_B| = \sqrt{\dfrac{2(m_A - m_B)gb\sin\theta}{(m_A + m_B)}} = 1.72 \text{ m/s}}$

Problem 15.29 Solve Problem 15.28 by applying the principle of work and energy to the system consisting of A, B, the cable connecting them, and the pulley.

Solution: Choose a coordinate system with the origin at the pulley axis and the positive x axis parallel to the inclined surface. Since the pulley is one-to-one, $x_A = -x_B$. Differentiate to obtain $v_A = -v_B$. Denote $b = 0.5$ m. From the principle of work and energy the work done by the external forces on the complete system is equal to the gain in kinetic energy,

$$\int_0^{x_A} m_A g \sin\theta\,ds + \int_0^{x_B} m_B g \sin\theta\,ds = \frac{1}{2} m_A v_A^2 + \frac{1}{2} m_B v_B^2,$$

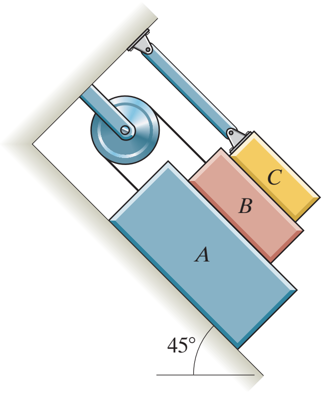


from which

$$(m_B - m_A)gb\sin\theta = \tfrac{1}{2}(m_A + m_B)v_A^2$$

and $\boxed{|v_A| = |v_B| = \sqrt{\dfrac{(m_A - m_B)}{(m_A + m_B)} 2gb\sin\theta} = 1.72 \text{ m/s.}}$

Problem 15.30 The masses of the three blocks are $m_A = 40$ kg, $m_B = 16$ kg, and $m_C = 12$ kg. The coefficient of kinetic friction between all surfaces is $\mu_k = 0.1$. Determine the magnitude of the velocity of blocks A and B when they have moved 500 mm. (See Example 15.3.)



Solution: We will apply the principles of work — energy to blocks A and B individually in order to properly account for the work done by internal friction forces.

$$\int_0^b (m_A g \sin\theta - T - \mu_k N_A - \mu_k N_{AB})\, ds = \frac{1}{2} m_A v^2,$$

$$\int_0^b (T - m_B g \sin\theta - \mu_k N_{BC} - \mu_k N_{AB})\, ds = \frac{1}{2} m_B v^2.$$

Adding the two equations, we get

$$([m_A - m_B] g \sin\theta - \mu_k [N_A + 2N_{AB} + N_{BC}]) b = \frac{1}{2}(m_A + m_B) v^2$$

The normal forces are

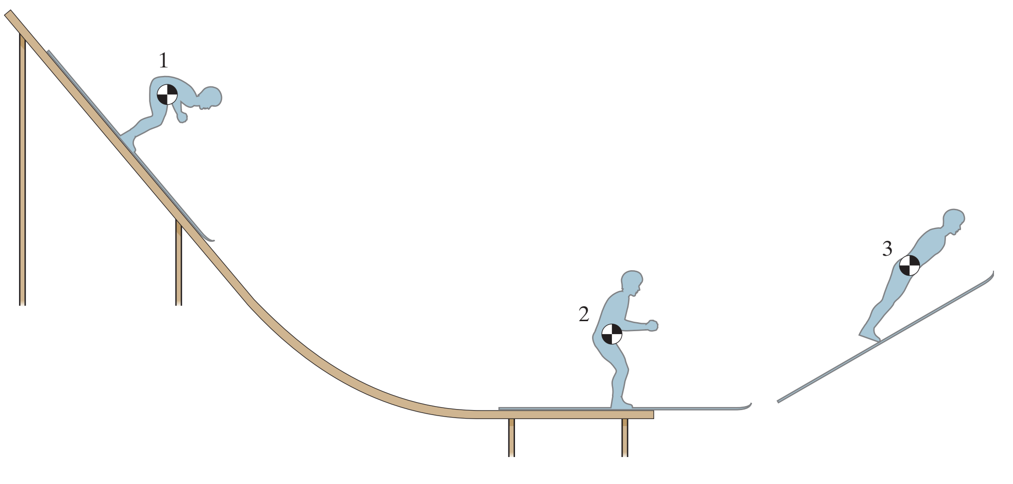
$$N_A = (m_A + m_B + m_C) g \cos\theta,$$

$$N_{AB} = (m_B + m_C) g \cos\theta,$$

$$N_{BC} = (m_C) g \cos\theta.$$

Substitute and solve $\Rightarrow$ $\boxed{v = 1.14 \text{ m/s.}}$

Problem 15.31 In Example 15.5, suppose that the skier is moving at 20 m/s when he is in position 1. Determine the horizontal component of his velocity when he reaches position 2, 20 m below position 1.



Solution:

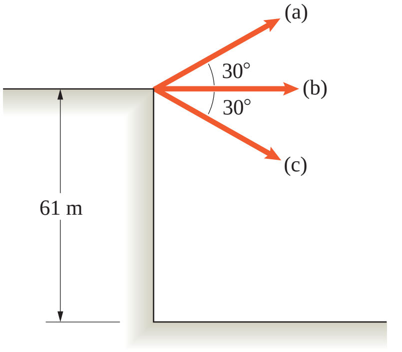
$$U_{12} = -mg(y_2 - y_1) = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$-m(9.81)(0 - 20) = \frac{1}{2} m v_2^2 - \frac{1}{2} m (20)^2$$

Solving for v_2 we find

$$\boxed{v_2 = 28.1 \text{ m/s.}}$$

Problem 15.32 Suppose that you stand at the edge of a 61 m cliff and throw rocks at 9.1 m/s in the three directions shown. Neglecting aerodynamic drag, use the principle of work and energy to determine the magnitude of the velocity of the rock just before it hits the ground in each case.



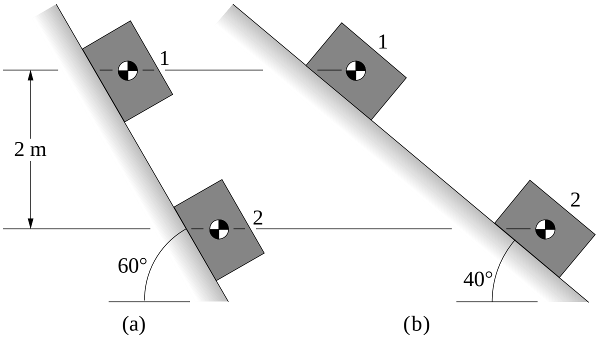
Solution:

$$U = m(9.81\text{ m/s}^2)(61\text{ m}) = \frac{1}{2}mv^2 - \frac{1}{2}m(9.1\text{ m/s})^2$$

$$\Rightarrow \boxed{v = 35.7\text{ m/s}}$$

Note that the answer does not depend on the initial angle.

Problem 15.33 The 30-kg box is sliding down the smooth surface at 1 m/s when it is in position 1. Determine the magnitude of the box's velocity at position 2 in each case.



Solution: The work done by the weight is the same in both cases.

$$U = -m(9.81\text{ m/s}^2)(0 - 2\text{ m}) = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m(1\text{ m/s})^2$$

$$\Rightarrow \boxed{v = 6.34\text{ m/s}}$$

Problem 15.34 Solve Problem 15.33 if the coefficient of kinetic friction between the box and the inclined surface is $\mu_k = 0.2$.

Solution: The work done by the weight is the same, however, the work done by friction is different.

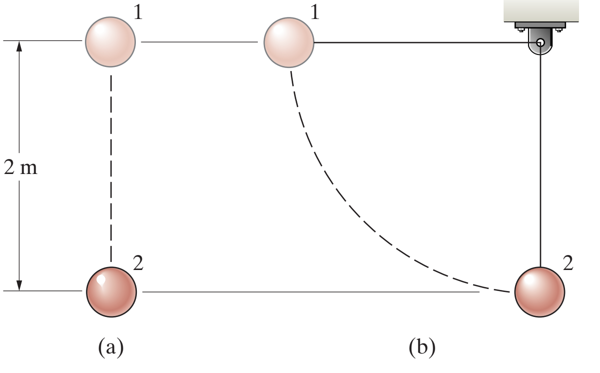
(a) $$U = -m(9.81\text{ m/s}^2)(0 - 2\text{ m}) - (0.2)[m(9.81\text{ m/s}^2)\cos 60°]\left[\frac{2\text{ m}}{\sin 60°}\right]$$

$$U = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m(1\text{ m/s})^2 \Rightarrow \boxed{v_2 = 5.98\text{ m/s}}$$

(b) $$U = -m(9.81\text{ m/s}^2)(0 - 2\text{ m}) - (0.2)[m(9.81\text{ m/s}^2)\cos 40°]\left[\frac{2\text{ m}}{\sin 40°}\right]$$

$$U = \frac{1}{2}m{v_2}^2 - \frac{1}{2}m(1\text{ m/s})^2 \Rightarrow \boxed{v_2 = 5.56\text{ m/s}}$$

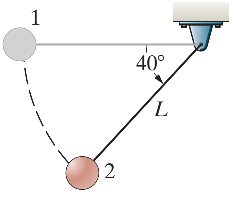
Problem 15.35 In case (a), a 5-N ball is released from rest at position 1 and falls to position 2. In case (b), the ball is released from rest at position 1 and swings to position 2. For each case, use the principle of work and energy to determine the magnitude of the ball's velocity at position 2. (In case (b), notice that the force exerted on the ball by the string is perpendicular to the ball's path.)



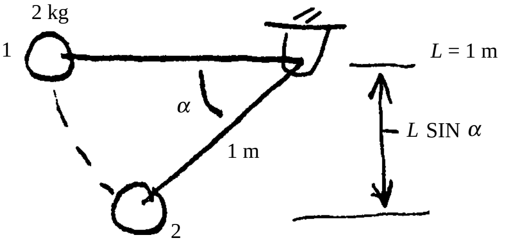
Solution: The work is independent of the path, so both cases are the same.

$$U = -m(9.81\text{m/s}^2)(0 - 2\text{ m}) = \frac{1}{2}m{v_2}^2 - 0 \Rightarrow \boxed{v_2 = 6.26\text{ m/s}}$$

Problem 15.36 The 2-kg ball is released from rest in position 1 with the string horizontal. The length of the string is $L = 1$ m. What is the magnitude of the ball's velocity when it is in position 2?



Solution:



$$U_{12} = -\int_0^{-L\sin\alpha} mg\mathbf{j}\cdot ds\mathbf{j}$$

$$U_{12} = -mg(-L\sin\alpha) = (2)(9.81)(1)\sin 40^\circ$$

$$U_{12} = 12.61 \text{ N-m}$$

$$U_{12} = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \qquad v_1 \equiv 0$$

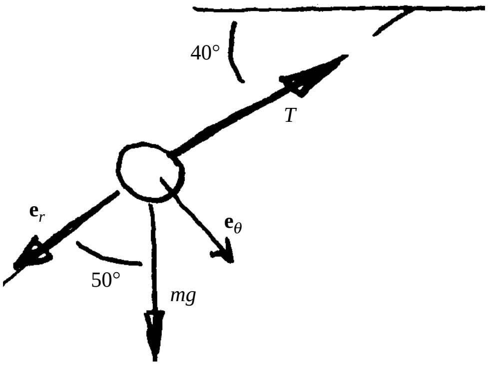
$$\tfrac{1}{2}(2)v_2^2 = 12.61$$

$$\underline{v_2 = 3.55 \text{ m/s}}$$

Problem 15.37 The 2-kg ball is released from rest in position 1 with the string horizontal. The length of the string is $L = 1$ m. What is the tension in the string when the ball is in position 2?

Strategy: Draw the free-body diagram of the ball when it is in position 2 and write Newton's second law in terms of normal and tangential components.

Solution: $m = 2$ kg



$$\sum F_r: \quad -T + mg\cos 50^\circ = -mv_2^2/L$$

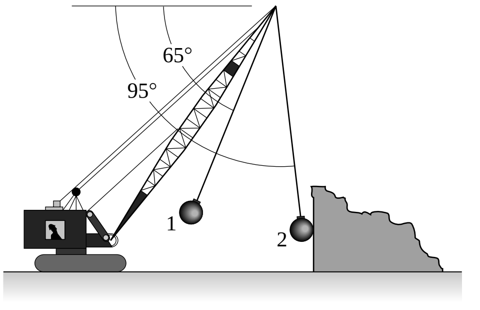
From Problem 15.36,

$$v_2 = 3.55 \text{ m/s}$$

$$T = mg\cos 50^\circ + mv^2/L$$

$$\underline{T = 37.8 \text{ N}}$$

Problem 15.38 The 400-N wrecker's ball swings at the end of a 25-m cable. If the magnitude of the ball's velocity at position 1 is 4 m/s, what is the magnitude of its velocity just before it hits the wall at position 2?



Solution:

$U = -(400 \text{ N})(-25 \text{ m} \sin 95^\circ - [-25 \text{ m} \sin 65^\circ])$

$U = \frac{1}{2}\left(\frac{400 \text{ N}}{9.81 \text{ m/s}^2}\right)({v_2}^2 - [4 \text{ m/s}]^2)$

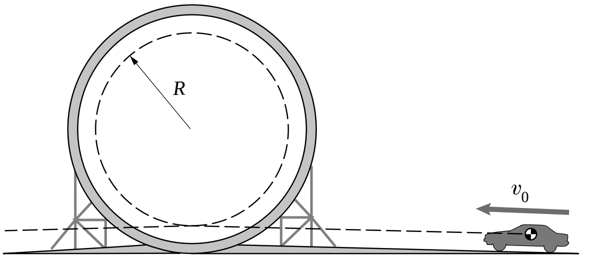
$\Rightarrow \boxed{v_2 = 7.75 \text{ m/s}}$

Problem 15.39 The 400-N wrecker's ball swings at the end of a 25-m cable. If the magnitude of the ball's velocity at position 1 is 4 m/s, what is the maximum tension in the cable as the ball swings from position 1 to position 2?

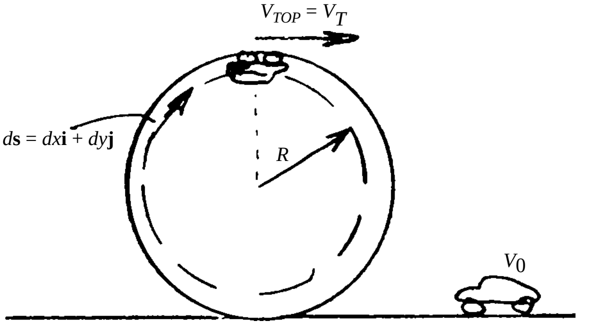
Solution: From the solution to Problem 15.37, the tension in the cable is $T = mg \sin\alpha + m\frac{v^2}{L}$. From the solution to Problem 15.38, $v^2 = 2gL[\sin\alpha - \sin 65^\circ](65^\circ \le \alpha \le 95^\circ)$, from which $T = 3\ mg \sin \alpha - 2\ mg \sin 65^\circ$. The maximum tension occurs when $\sin\alpha$ is a maximum in the interval $(65^\circ \le \alpha \le 95^\circ)$, from which

$\boxed{T = 3\ mg \sin 90^\circ - 2\ mg \sin 65^\circ = 474.9516 \text{ N.}}$

Problem 15.40 A stunt driver wants to drive a car through the circular loop of radius $R = 5$ m. Determine the minimum velocity v_0 at which the car can enter the loop and coast through without losing contact with the track. What is the car's velocity at the top of the loop?



Solution: First, let us find V_T



$\sum F_n$: $N + mg = mV_T^2/R$

For minimum velocity, $N \to 0$

$mg = mV_T^2/R$

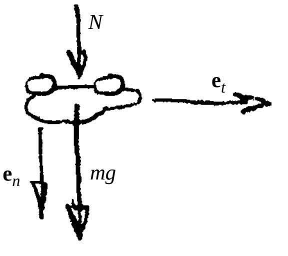
$\underline{V_T = \sqrt{Rg} = 7.00 \text{ m/s}}$

Now find V_0 using work-energy

$$U_{0T} = \int_0^{10} -mg\mathbf{j} \cdot (dx\mathbf{i} + dy\mathbf{j})$$

$$U_{0T} = \int_0^{10} -mg\,dy = -mgy$$

$U_{0T} = -98.1m$ (N-m)



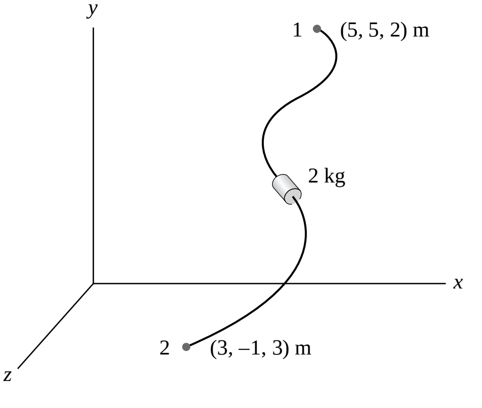
Also, $U_{0T} = \frac{1}{2}\cancel{m}V_T^2 - \frac{1}{2}\cancel{m}V_0^2 = -98.1\cancel{m}$ (N-m)

Solving for V_0 $(V_T = 7.00 \text{ m/s})$

$V_0^2 = V_T^2 + (98.1)(2)$

$\underline{V_0 = 15.68 \text{ m/s}}$

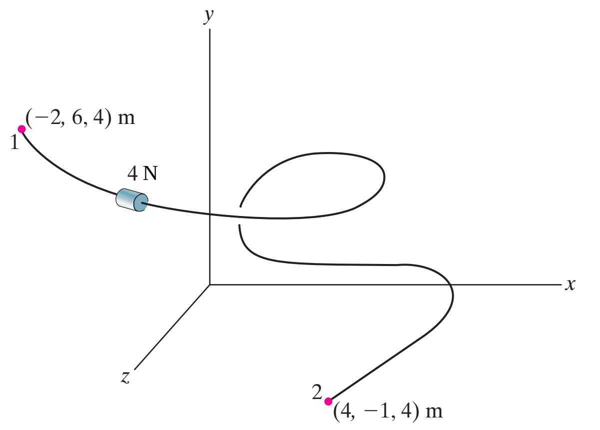
Problem 15.41 The 2-kg collar starts from rest at position 1 and slides down the smooth rigid wire. The y-axis points upward. What is the magnitude of the velocity of the collar when it reaches position 2?



Solution: The work done by the weight is $U_{\text{weight}} = mgh$, where $h = y_1 - y_2 = 5 - (-1) = 6$ m. From the principle of work and energy, $mgh = \frac{1}{2}mv^2$, from which

$\boxed{v = \sqrt{2\,gh} = 10.85 \text{ m/s}}$

Problem 15.42 The 4-N collar slides down the smooth rigid wire from position 1 to position 2. When it reaches position 2, the magnitude of its velocity is 24 m/s. What was the magnitude of its velocity at position 1?



Solution:

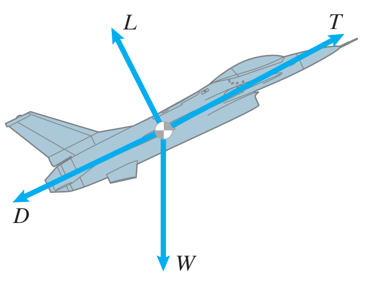
$$U = (4\text{ N})(6 - [-1])\text{ m} = \frac{1}{2}\left(\frac{4\text{ N}}{9.81\text{ m/s}^2}\right)([24\text{ m/s}]^2 - v_1^2)$$

$$\Rightarrow \boxed{v_1 = 20.9\text{ m/s}}$$

Problem 15.43 The forces acting on the 125 kN airplane are the thrust T and drag D, which are parallel to the airplane's path, the lift L, which is perpendicular to the path, and the weight W. The airplane climbs from an altitude of 914 m to an altitude of 3048 m. During the climb, the magnitude of its velocity decreases from 244 m/s to 183 m/s.

(a) What work is done on the airplane by its lift during the climb?

(b) What work is done by the thrust and drag combined?



Solution:

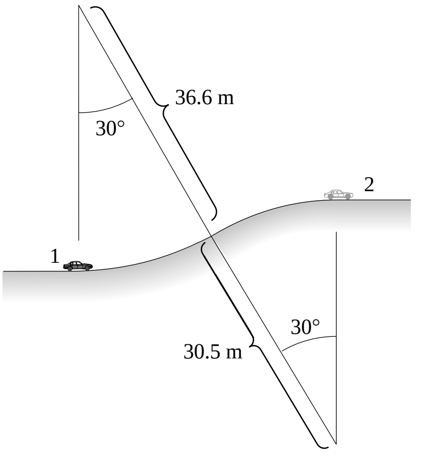
(a) The work due to the lift L is zero since it acts perpendicular to the motion.

(b) $U = U_{L+D} - (125000\text{ N})(3048 - 914)\text{ m}$

$$U = \frac{1}{2}\left(\frac{125000\text{ N}}{9.81\text{ m/s}^2}\right)([183\text{ m/s}]^2 - [244\text{ m/s}^2])$$

$$\Rightarrow \boxed{U_{L+D} = 10.07 \times 10^7\text{ N-m}}$$

Problem 15.44 The 10.7 kN car is traveling 64.4 km/h at position 1. If the combined effect of the aerodynamic drag on the car and the tangential force exerted on its wheels by the road is that they exert no net tangential force on the car, what is the magnitude of the car's velocity at position 2?



Solution: The initial velocity is

$$v_1 = 64.4 \text{ km/h} = 17.9 \text{ m/s}$$

The change in elevation of the car is

$$h = 36.6(1 - \cos 30^\circ) + 30.5(1 - \cos 30^\circ)$$

$$= 67.1(1 - \cos 30^\circ) = 8.98 \text{ m}$$

The initial kinetic energy is

$$\frac{1}{2}\left(\frac{W}{g}\right)v_1^2 = 173895 \text{ N}$$

The work done by gravity is

$$U_{\text{gravity}} = \int_0^h (-W)\,ds = -Wh = -10700(h) = -95903.9 \text{ N-m}.$$

From the principle of work and energy the work done is equal to the gain in kinetic energy:

$$U_{\text{gravity}} = \frac{1}{2}\left(\frac{W}{g}\right)v_2^2 - \frac{1}{2}\left(\frac{W}{g}\right)v_1^2,$$

from which

$$\boxed{v_2 = \sqrt{\frac{2g(-95903.9 + 173895)}{W}} = 11.96 \text{ m/s}}$$

Problem 15.45 The 10.7 kN car is traveling 64.4 km/h at position 1. If the combined effect of aerodynamic drag on the car and the tangential force exerted on its wheels by the road is that they exert a constant 1.78 kN tangential force on the car in the direction of its motion, what is the magnitude of the car's velocity at position 2?

Solution: From the solution to Problem 15.44, the work done by gravity is $U_{\text{gravity}} = -95903.9$ N-m due to the change in elevation of the car of $h = 8.98$ m, and $\frac{1}{2}\left(\frac{W}{g}\right)v_1^2 = 173895$ N-m.

The length of road between positions 1 and 2 is

$$s = 36.6(30^\circ)\left(\frac{\pi}{180^\circ}\right) + 30.5(30^\circ)\left(\frac{\pi}{180^\circ}\right) = 35.1 \text{ m}.$$

The work done by the tangential force is

$$U_{\text{tgt}} = \int_0^s 1780\,ds = 1780\,(35.1) = 62468.5 \text{ N-m}.$$

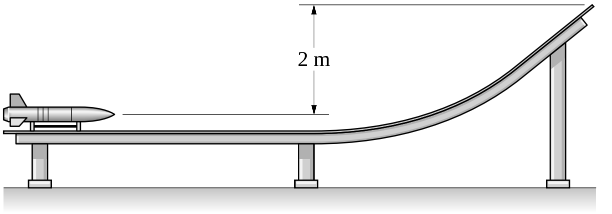
From the principle of work and energy

$$U_{\text{gravity}} + U_{\text{tgt}} = \frac{1}{2}\left(\frac{W}{g}\right)v_2^2 - \frac{1}{2}\left(\frac{W}{g}\right)v_1^2,$$

from which

$$\boxed{\begin{aligned} v &= \sqrt{\frac{2(9.81)(-95903.9 + 62468.5 + 173895)}{2400}} \\ &= 16.1 \text{ m/s} = 57.9 \text{ km/h} \end{aligned}}$$

Problem 15.46 The mass of the rocket is 250 kg. Its engine has a constant thrust of 45 kN. The length of the launching ramp is 10 m. If the magnitude of the rocket's velocity when it reaches the end of the ramp is 52 m/s, how much work is done on the rocket by friction and aerodynamic drag?



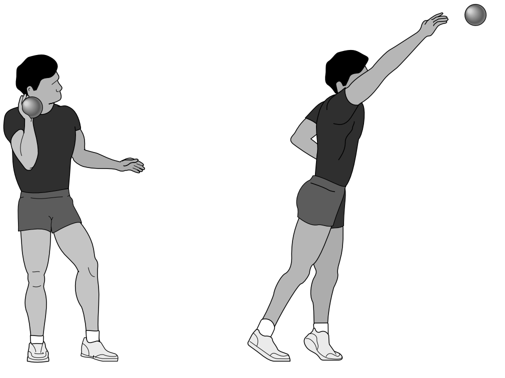
Solution:

$U = U_{Fr+Dr} + (45\ \text{kN})(10\ \text{m}) - (250\ \text{kg})(9.81\ \text{m/s}^2)(2\ \text{m})$

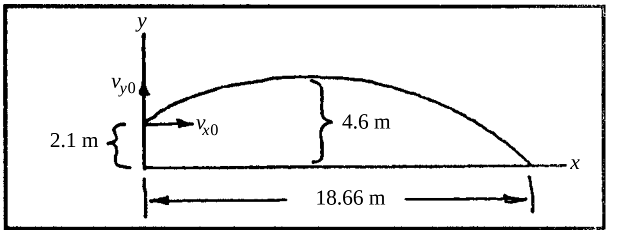
$U = \frac{1}{2}(250\ \text{kg})(52\ \text{m/s})^2$

$\boxed{U_{Fr+Dr} = -107\ \text{kN-m}}$

Problem 15.47 A bioengineer interested in energy requirements of sports determines from videotape that when the athlete begins his motion to throw the 7.25-kg shot (Fig. a), the shot is stationary and 1.50 m above the ground. At the instant the athlete releases it (Fig. b), the shot is 2.10 m above the ground. The shot reaches a maximum height of 4.60 m above the ground and travels a horizontal distance of 18.66 m from the point where it was released. How much work does the athlete do on the shot from the beginning of his motion to the instant he releases it?



(a) (b)



Solution: Let v_{x0} and v_{y0} be the velocity components at the instant of release. Using the chain rule,

$$a_y = \frac{dv_y}{dt} = \frac{dv_y}{dy}\frac{dy}{dt} = \frac{dv_y}{dy}v_y = -g,$$

and integrating,

$$\int_{v_{y0}}^{0} v_y\, dv_y = -g\int_{2.1}^{4.6} dy.$$

$-\frac{1}{2}v_{y0}^2 = -g(4.6 - 2.1)$, we find that $v_{y0} = 7.00$ m/s. The shot's x and y coordinates are given by $x = v_{x0}t$, $y = 2.1 + v_{y0}t - \frac{1}{2}gt^2$. Solving the first equation for t and substituting it into the second,

$$y = 2.1 + v_{y0}\left(\frac{x}{v_{x0}}\right) - \frac{1}{2}g\left(\frac{x}{v_{x0}}\right)^2$$

Setting $x = 18.66$ m, $y = 0$ in this equation and solving for v_{x0} gives $v_{x0} = 11.1$ m/s. The magnitude of the shot's velocity at release is $U_2 = \sqrt{v_{x0}^2 + v_{y0}^2} = 13.1$ m/s.

Let U_A be the work he does

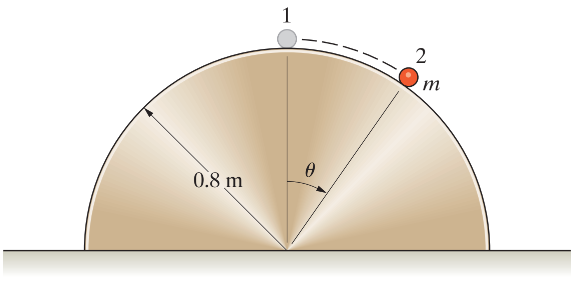
$$U_A - mg(y_2 - y_1) = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

$$U_A - (7.25\ \text{kg})(9.81\ \text{m/s}^2)(2.1 - 1.5) = \tfrac{1}{2}(7.25)(13.1)^2 - 0,$$

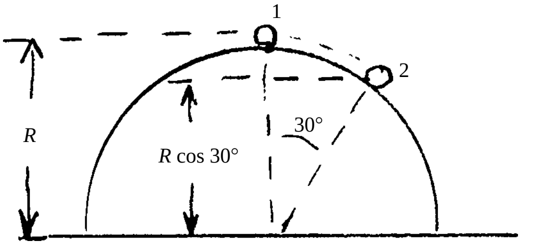
or $U_A = 666$ N-m

Problem 15.48 A small pellet of mass $m = 0.2$ kg starts from rest at position 1 and slides down the smooth surface of the cylinder to position 2, where $\theta = 30^\circ$.

(a) What work is done on the pellet as it slides from position 1 to position 2?
(b) What is the magnitude of the pellet's velocity at position 2?



Solution:



$v_1 = 0 \quad R = 0.8$ m

$$U_{12} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$U_{12} = (0.5)(0.2)v_2^2 = 0.1v_2^2$$

The work is

$$U_{12} = (0.2)(9.81)(0.8 - 0.8\cos 30^\circ)$$

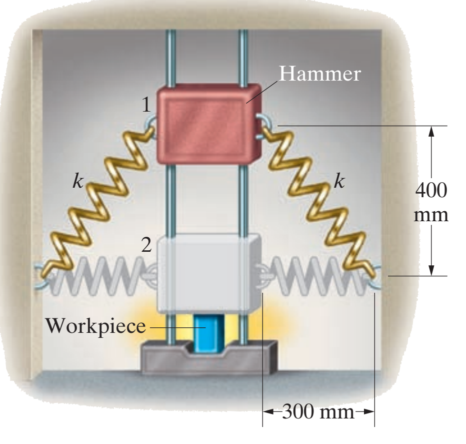
$$= 0.210 \text{ N-m.}$$

(a) $\underline{U_{12} = 0.210 \text{ (N-m)}}$

(b) $0.1v_2^2 = 0.210$ (N-m)

$$\underline{v_2 = 1.45 \text{ m/s}}$$

Problem 15.49 In Active Example 15.4, suppose that you want to increase the value of the spring constant k so that the velocity of the hammer just before it strikes the workpiece is 4 m/s. what is the required value of k?



Solution: The 40-kg hammer is released from rest in position 1. The springs are unstretched when in position 2. Neglect friction.

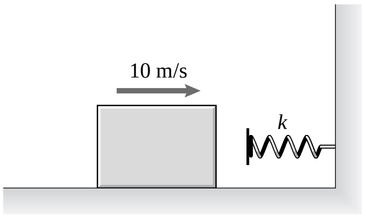
$$U_{12} = mgh + 2\frac{1}{2}kd^2 = \frac{1}{2}\,mv_2^2$$

$$k = \frac{m}{d^2}\left(\frac{v_2^2}{2} - gh\right)$$

$$k = \frac{40 \text{ kg}}{(0.2 \text{ m})^2}\left(\frac{[4 \text{ m}]^2}{2} - [9.81\text{m/s}^2][0.4\text{m}]\right)$$

$$\boxed{k = 4080 \text{ N/m.}}$$

Problem 15.50 Suppose that you want to design a bumper that will bring a 50-N. package moving at 10 m/s to rest 152.4 mm from the point of contact with bumper. If friction is negligible, what is the necessary spring constant k?



Solution: From the principle of work and energy, the work done on the spring must equal the change in kinetic energy of the package within the distance 152.4 m.

$$\frac{1}{2}kS^2 = \frac{1}{2}\left(\frac{W}{g}\right)v^2$$

from which

$$\boxed{k = \left(\frac{W}{g}\right)\left(\frac{v}{S}\right)^2 = \left(\frac{50}{9.81}\right)\left(\frac{10}{0.152}\right)^2 = 22060 \text{ N/m}}$$

Problem 15.51 In Problem 15.50, what spring constant is necessary if the coefficient of kinetic friction between the package and the floor is $\mu_k = 0.3$ and the package contacts the bumper moving at 10 m/s?

Solution: The work done on the spring over the stopping distance is

$$U_S = \int_0^S F\,ds = \int_0^S ks\,ds = \frac{1}{2}kS^2.$$

The work done by friction over the stopping distance is

$$U_f = \int_0^S F\,ds = \int_0^S \mu_k W\,ds = \mu_k WS.$$

From the principle of work and energy the work done must equal the kinetic energy of the package:

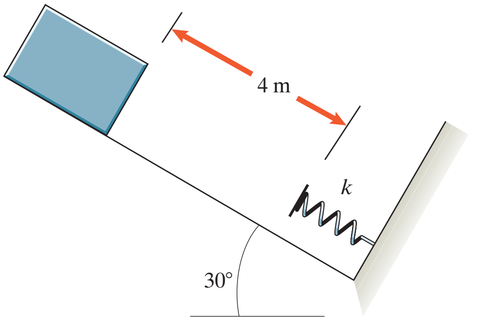
$$\frac{1}{2}kS^2 + \mu_k WS = \frac{1}{2}\left(\frac{W}{g}\right)v^2,$$

from which, for $S = 0.152$ m,

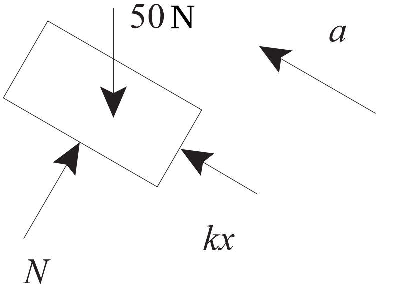
$$\boxed{k = \left(\frac{W}{g}\right)\frac{(v^2 - 2\,g\mu_k S)}{S^2} = 21863 \text{ N/m}}$$

Problem 15.52 The 50-N package starts from rest, slides down the smooth ramp, and is stopped by the spring.

(a) If you want the package to be brought to rest 0.5 m from the point of contact, what is the necessary spring constant k?
(b) What maximum deceleration is the package subjected to?



Solution:



(a) Find the spring constant

$$U_{12} = mgh - \frac{1}{2}kx^2 = 0$$

$$k = \frac{2mgh}{x^2} = \frac{2(50\text{ N})(4.5\text{ m})\sin 30^\circ}{(0.5\text{ m})^2}$$

$$\boxed{k = 900\text{ N/m}}$$

(b) The maximum deceleration occurs when the spring reaches the maximum compression (the force is then the largest).

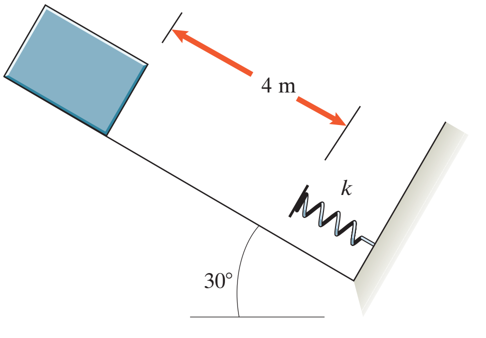
$$kx - mg\sin\theta = ma$$

$$a = \frac{k}{m}x - g\sin\theta$$

$$a = \frac{(900\text{ N/m})}{\left(\dfrac{50\text{ N}}{9.81\text{ m/s}^2}\right)}(0.5\text{ m}) - (9.81\text{m/s}^2)\sin 30^\circ$$

$$\boxed{a = 83.4\text{ m/s}^2}$$

Problem 15.53 The 50-N package starts from rest, slides down the smooth ramp, and is stopped by the spring. The coefficient of static friction between the package and the ramp is μ k = 0.12. If you want the package to be brought to rest 0.5 m from the point of contact, what is the necessary spring constant k?



Solution: Find the spring constant

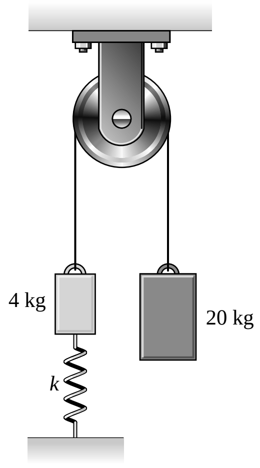
$$U_{12} = mgd\sin\theta - \mu_k mg\cos\theta\, d - \frac{1}{2}kx^2 = 0$$

$$k = \frac{2mgd}{x^2}(\sin\theta - \mu_k\cos\theta)$$

$$k = \frac{2(50\text{ N})(4.5\text{m})}{(0.5\text{ m})^2}(\sin 30^\circ - 0.12\cos 30^\circ)$$

$$\boxed{k = 713\text{ N/m.}}$$

Problem 15.54 The system is released from rest with the spring unstretched. The spring constant $k = 200$ N/m. Determine the magnitude of the velocity of the masses when the right mass has fallen 1 m.



Solution: When the larger mass falls 1 m, the smaller mass rises 1 m and the spring stretches 1 m. For the system of two masses, springs, and the cable,

$$U_{12} = \int_0^1 (-ks)\,ds + \int_0^1 (-m_1 g)\,ds + \int_0^1 m_2 g\,ds$$

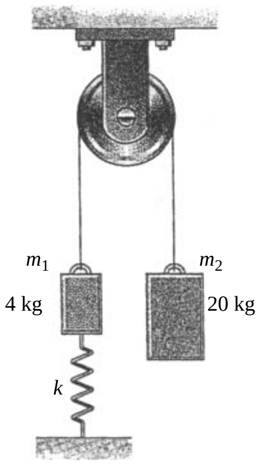
$$U_{12} = -\tfrac{1}{2}ks^2\Big|_0^1 - m_1 gs\Big|_0^1 + m_2 gs\Big|_0^1$$

$$U_{12} = -\tfrac{1}{2}k - 4(9.81) + (20)(9.81)$$

$$U_{12} = 56.96 \text{ N-m}$$

Also $U_{12} = \frac{1}{2}(m_1 + m_2)V_f^2 - 0$

Solving $\underline{V_f = 2.18 \text{ m/s}}$



Problem 15.55 The system is released from rest with the spring unstretched. The spring constant $k = 200$ N/m. What maximum downward velocity does the right mass attain as it falls?

Solution: From the solution to Problem 15.54,

$$U_{12} = -\frac{1}{2}Ks^2 + (m_2 - m_1)gs$$

and

$$U_{12} = \frac{1}{2}(m_1 + m_2)V^2$$

For all s. Setting these equal, we get

$$\frac{1}{2}(m_1 + m_2)V^2 = (m_2 - m_1)gs - \frac{1}{2}Ks^2 \quad \textbf{(1)}$$

Solve for $\dfrac{dv}{ds}$ and set $\dfrac{dv}{ds}$ to zero

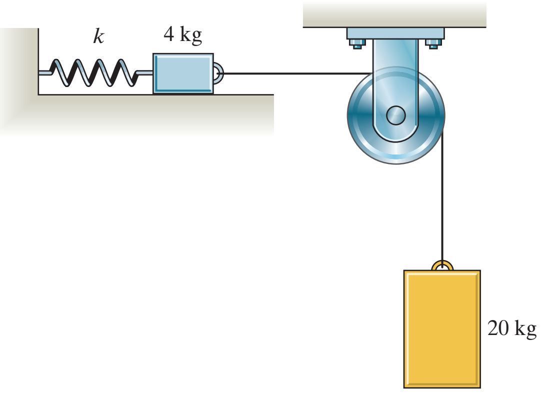
$$\frac{1}{2}(m_1 + m_2)2v\frac{dv}{ds} = (m_2 - m_1)g - Ks = 0$$

The extreme value for V occurs at

$$S = \frac{(m_2 - m_1)g}{K} = 0.785 \text{ m}$$

Substituting this back into (1) and solving, we get $\underline{V = 2.27 \text{ m/s}}$

Problem 15.56 The system is released from rest. The 4-kg mass slides on the smooth horizontal surface. The spring constant is $k = 100$ N/m, and the tension in the spring when the system is released is 50 N. By using the principle of work and energy, determine the magnitude of the velocity of the masses when the 20-kg mass has fallen 1 m.



Solution:

$50 \text{ N} = (100 \text{ N/m})x_1 \ \Rightarrow \ x_1 = 0.5 \text{ m}$

$$U = (20 \text{ kg})(9.81 \text{ m/s}^2)(1 \text{ m}) - \frac{1}{2}(100 \text{ N/m})([1.5 \text{ m}]^2 - [0.5 \text{ m}]^2)$$

$$U = \frac{1}{2}(24 \text{ kg})({v_2}^2 - 0)$$

$\boxed{v_2 = 2.83 \text{ m/s}}$

Problem 15.57 Solve Problem 15.56 if the coefficient of kinetic friction between the 4-kg mass and the horizontal surface is $\mu_k = 0.4$.

Solution:

$50 \text{ N} = (100 \text{ N/m})x_1 \ \Rightarrow \ x_1 = 0.5 \text{ m}$

$$U = (20 \text{ kg})(9.81 \text{ m/s}^2)(1 \text{ m}) - \frac{1}{2}(100 \text{ N/m})([1.5 \text{ m}]^2 - [0.5 \text{ m}]^2)$$

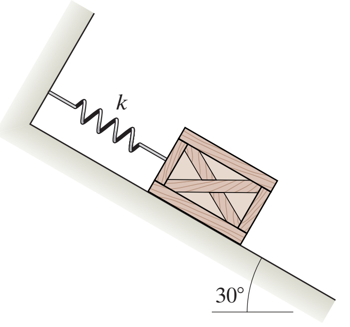
$$- (0.4)(4 \text{ kg})(9.81 \text{ m/s}^2)(1 \text{ m})$$

$$U = \frac{1}{2}(24 \text{ kg})({v_2}^2 - 0)$$

$\boxed{v_2 = 2.59 \text{ m/s}}$

Problem 15.58 The 40-N crate is released from rest on the smooth inclined surface with the spring unstretched. The spring constant is $k = 8$ N/m.

(a) How far down the inclined surface does the crate slide before it stops?
(b) What maximum velocity does the crate attain on its way down?



Solution: At an arbitrary distance s down the slope we have:

$$U = (40\text{ N})(s\sin 30^\circ) - \frac{1}{2}(8\text{ N/m})s^2 = \frac{1}{2}\left(\frac{40\text{ N}}{9.81\text{ m/s}^2}\right)v^2$$

(a) When it stops, we set $v = 0$ and solve for $\boxed{s = 5\text{ m}}$

(b) Solving for v^2, we have

$$v^2 = 0.491(20s - 4s^2) \Rightarrow \frac{dv^2}{ds} = 0.491(20 - 8s) = 0 \Rightarrow s = 2.5\text{ m}$$

Using $s = 2.5$ m $\Rightarrow$ $\boxed{v_{\max} = 3.5\text{ m/s}}$

Problem 15.59 Solve Problem 15.58 if the coefficient of kinetic friction between the 4-kg mass and the horizontal surface is $\mu_k = 0.2$.

Solution: At an arbitrary distance s down the slope we have:

$$U = (40\text{ N})(s\sin 30^\circ) - \frac{1}{2}(8\text{ N/m})s^2 - (0.2)(40\text{ N}\cos 30^\circ)s$$

$$= \frac{1}{2}\left(\frac{40\text{ N}}{9.81\text{ m/s}^2}\right)v^2$$

(a) When it stops, we set $v = 0$ and solve for $\boxed{s = 3.27\text{ m}}$

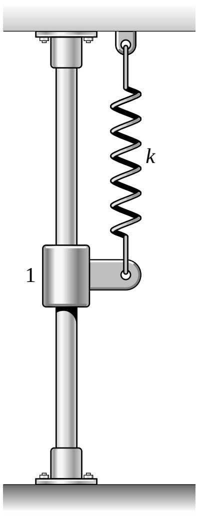
(b) Solving for v^2, we have

$$v^2 = 0.491(13.07s - 4s^2) \Rightarrow \frac{dv^2}{ds} = 0.491(13.07 - 8s) = 0$$

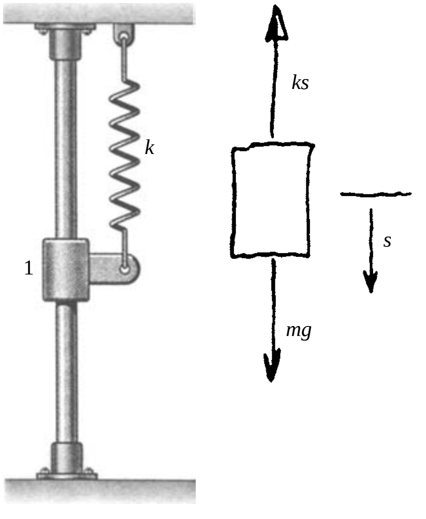
$$\Rightarrow s = 1.63\text{ m}$$

Using $s = 1.63$ m $\Rightarrow$ $\boxed{v_{\max} = 2.29\text{ m/s}}$

Problem 15.60 The 4-kg collar starts from rest in position 1 with the spring unstretched. The spring constant is $k = 100$ N/m. How far does the collar fall relative to position 1?



Solution:



$V_0 = V_f = 0$

Let position 2 be the location where the collar comes to rest

$$U_{12} = -\frac{Ks^2}{2} + mgs$$

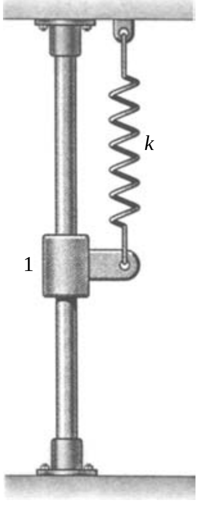
Also $U_{12} = \frac{1}{2}mV_f^2 - \frac{1}{2}mV_0^2 = 0$

Thus $0 = mgs - \dfrac{Ks^2}{2}$

$$s(2mg - Ks) = 0$$

Solving, $\underline{s = 0.785 \text{ m}}$.

Problem 15.61 In position 1 on the smooth bar, the 4-kg collar has a downward velocity of 1 m/s and the spring is unstretched. The spring constant is $k = 100$ N/m. What maximum downward velocity does the collar attain as it falls?



Solution: The work is

$$U_{12} = -\frac{Ks^2}{2} + mgs$$

Also,

$$U_{12} = \frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2$$

where $m = 4$ kg and $V_1 = 1$ m/s

Thus,

$$\frac{1}{2}mV_2^2 - \frac{1}{2}mV_1^2 = -\frac{Ks^2}{2} + \text{ mgs} \quad \textbf{(1)}$$

Finding $\frac{dV_2}{ds}$, and setting it to zero,

$$mV_2\frac{dV_2}{ds} = -Ks + mg = 0$$

$$s = mg/k = 0.392 \text{ m}$$

Solving (1) for V_2 we get $\underline{V_2 = 2.20 \text{ m/s}}$

Problem 15.62 The 4-kg collar starts from rest in position 1 on the smooth bar. The tension in the spring in position 1 is 20 N. The spring constant is $k = 100$ N/m. How far does the collar fall relative to position 1?

Solution: For this problem, we need a new reference for the spring. If the tension in the spring is 20 N

$T = k\delta_0$ and $K = 100$ N/m

$\delta_0 = 0.2$ m. (the initial stretch)

In this case, we have

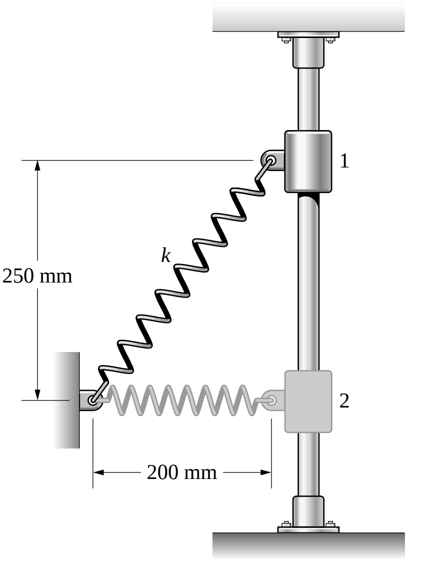
$$U_{12} = -\frac{Ks^2}{2}\bigg|_{0.2}^{s+0.2} + mgs\bigg|_{0.2}^{s+0.2}$$

Also $V_0 = V_f = 0 \qquad \therefore U_{12} = 0$

$$0 = -\frac{K(s+0.2)^2}{2} + \frac{K(0.2)^2}{2} + mg(s+0.2) - mg(0.2)$$

Solving, we get $\underline{s = 0.385 \text{ m}}$

Problem 15.63 The 4-kg collar is released from rest at position 1 on the smooth bar. If the spring constant is $k = 6$ kN/m and the spring is unstretched in position 2, what is the velocity of the collar when it has fallen to position 2?



Solution: Denote $d = 200$ mm, $h = 250$ mm. The stretch of the spring in position 1 is $S_1 = \sqrt{h^2 + d^2} - d = 0.120$ m and at 2 $S_2 = 0$. The work done by the spring on the collar is

$$U_{\text{spring}} = \int_{0.12}^{0} (-ks)\,ds = \left[-\frac{1}{2}ks^2\right]_{0.120}^{0} = 43.31 \text{ N-m}.$$

The work done by gravity is

$$U_{\text{gravity}} = \int_{0}^{-h} (-mg)\,ds = mgh = 9.81 \text{ N-m}.$$

From the principle of work and energy $U_{\text{spring}} + U_{\text{gravity}} = \frac{1}{2}mv^2$, from which

$$v = \sqrt{\left(\frac{2}{m}\right)(U_{\text{spring}} + U_{\text{gravity}})} = 5.15 \text{ m/s}$$

Problem 15.64 The 4-kg collar is released from rest in position 1 on the smooth bar. The spring constant is $k = 4$ kN/m. The tension in the spring in position 2 is 500 N. What is the velocity of the collar when it has fallen to position 2?

Solution: Denote $d = 200$ mm, $h = 250$ mm. The stretch of the spring at position 2 is

$$S_2 = \frac{T}{k} = \frac{500}{4000} = 0.125 \text{ m}.$$

The unstretched length of the spring is $L = d - S_2 = 0.2 - 0.125 = 0.075$ m. The stretch of the spring at position 1 is $S_1 = \sqrt{h^2 + d^2} - L = 0.245$ m. The work done by the spring is

$$U_{\text{spring}} = \int_{S_1}^{S_2} (-ks)\,ds = \frac{1}{2}k(S_1^2 - S_2^2) = 88.95 \text{ N-m}.$$

The work done by gravity is $U_{\text{gravity}} = mgh = 9.81$ N-m. From the principle of work and energy is $U_{\text{spring}} + U_{\text{gravity}} = \frac{1}{2}mv^2$, from which

$$v = \sqrt{\frac{2(U_{\text{spring}} + U_{\text{gravity}})}{m}} = 7.03 \text{ m/s}$$

Problem 15.65 The 4-kg collar starts from rest in position 1 on the smooth bar. Its velocity when it has fallen to position 2 is 4 m/s. The spring is unstretched when the collar is in position 2. What is the spring constant k?

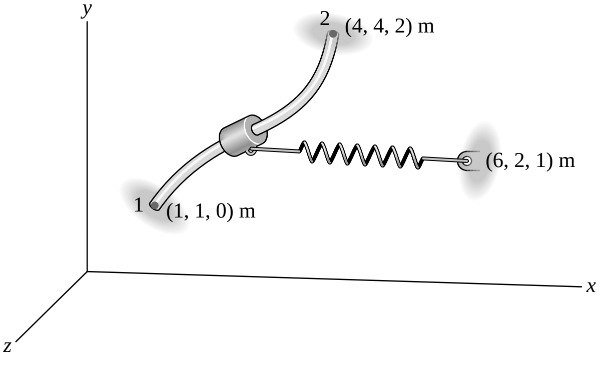
Solution: The kinetic energy at position 2 is $\frac{1}{2}mv^2 = 32$ N-m. From the solution to Problem 15.63, the stretch of the spring in position 1 is $S_1 = \sqrt{h^2 + d^2} - d = 0.120$ m. The potential of the spring is

$$U_{\text{spring}} = \int_{S_1}^{0} (-ks)\,ds = \frac{1}{2}kS_1^2.$$

The work done by gravity is $U_{\text{gravity}} = mgh = 9.81$ N-m. From the principle of work of work and energy, $U_{\text{spring}} + U_{\text{gravity}} = \frac{1}{2}mv^2$. Substitute and solve:

$$k = \frac{2\left(\frac{1}{2}mv^2 - U_{\text{gravity}}\right)}{S_1^2} = 3082 \text{ N/m}$$

Problem 15.66 The 10-kg collar starts from rest at position 1 and slides along the smooth bar. The y-axis points upward. The spring constant is $k = 100$ N/m and the unstretched length of the spring is 2 m. What is the velocity of the collar when it reaches position 2?



Solution: The stretch of the spring at position 1 is

$$S_1 = \sqrt{(6-1)^2 + (2-1)^2 + (1-0)^2} - 2 = 3.2 \text{ m}.$$

The stretch of the spring at position 2 is

$$S_2 = \sqrt{(6-4)^2 + (2-4)^2 + (1-2)^2} - 2 = 1 \text{ m}.$$

The work done by the spring is

$$U_{\text{spring}} = \int_{S_1}^{S_2} (-ks)\,ds = \frac{1}{2}k(S_1^2 - S_2^2) = 460.8 \text{ N-m}.$$

The work done by gravity is

$$U_{\text{gravity}} = \int_0^h (-mg)\,ds = -mgh = -(10)(9.81)(4-1)$$

$$= -294.3 \text{ N-m}.$$

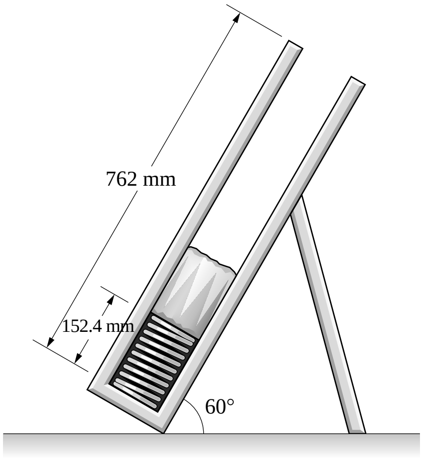
From the principle of work and energy:

$$U_{\text{spring}} + U_{\text{gravity}} = \frac{1}{2}mv^2,$$

from which

$$v = \sqrt{\frac{2(U_{\text{spring}} + U_{\text{gravity}})}{m}} = 5.77 \text{ m/s}$$

Problem 15.67 A spring-powered mortar is used to launch 44.5 N packages of fireworks into the air. The package starts from rest with the spring compressed to a length of 152.4 mm. The unstretched length of the spring is 762 mm. If the spring constant is $k = 1762$ N/m, what is the magnitude of the velocity of the package as it leaves the mortar?



Solution: Equating the work done to the change in the kinetic energy,

$$-\tfrac{1}{2}k(S_2^2 - S_1^2) - mg(y_2 - y_1) = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 :$$

$$-\tfrac{1}{2}(1762)[0 - (0.61\text{m})^2] - (44.5)(0.61 \sin 60^\circ \text{ft}) = \tfrac{1}{2}(44.5/9.81)v_2^2 - 0.$$

Solving, we obtain $v_2 = 39.3$ m/s.

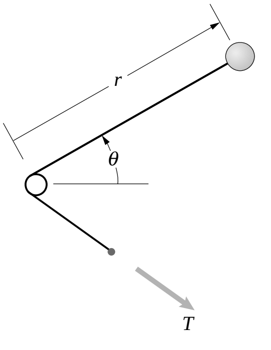
Problem 15.68 Suppose that you want to design the mortar in Problem 15.67 to throw the package to a height of 45.7 m above its initial position. Neglecting friction and drag, determine the necessary spring constant.

Solution: See the solution of Problem 15.67. Let v_2 be the velocity as the package leaves the barrel. To reach 45.7 m, $mg(45.7 - 0.61 \sin 60^\circ) = \frac{1}{2}m(v_2 \sin 60^\circ)^2$. Solving, we obtain $v_2 = 34.4$ m/s. Work and energy inside the barrel is

$$-\tfrac{1}{2}k[0 - (0.61\text{m})^2] - (44.5)(0.61 \sin 60^\circ \text{m}) = \tfrac{1}{2}(44.5/9.81)(34.4)^2 - 0,$$

which gives $\underline{k = 14549 \text{ N/m.}}$

Problem 15.69 Suppose an object has a string or cable with *constant* tension T attached as shown. The force exerted on the object can be expressed in terms of polar coordinates as $\mathbf{F} = -T\mathbf{e}_r$. Show that the work done on the object as it moves along an *arbitrary* plane path from a radial position r_1 to a radial position r_2 is $U_{12} = -T(r_1 - r_2)$.



Solution: The work done on the object is

$$U = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{s}.$$

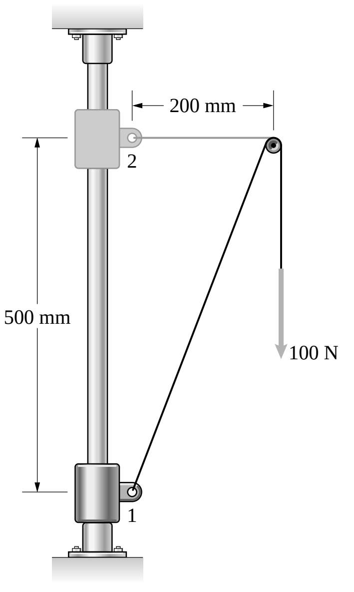
Suppose that the arbitrary path is defined by $d\mathbf{r} = (dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta)$, and the work done is

$$U = \int_{r_1}^{r_2} \mathbf{F} \cdot d\mathbf{r} = \int_{r_1}^{r_2} -T(\mathbf{e}_r \cdot \mathbf{e}_r)\,dr + \int_{r_1}^{r_2} T(r\mathbf{e}_r \cdot \mathbf{e}_\theta)rd\theta$$

$$= -\int_{r_1}^{r_2} T\,dr = -T(r_2 - r_1)$$

since $\mathbf{e}_r \cdot \mathbf{e}_s = 0$ by definition.

Problem 15.70 The 2-kg collar is initially at rest at position 1. A constant 100-N force is applied to the string, causing the collar to slide up the smooth vertical bar. What is the velocity of the collar when it reaches position 2? (See Problem 15.69.)



Solution: The constant force on the end of the string acts through a distance $s = \sqrt{0.5^2 + 0.2^2} - 0.2 = 0.3385$ m. The work done by the constant force is $U_F = Fs = 33.85$ N-m. The work done by gravity on the collar is

$$U_{\text{gravity}} = \int_0^h (-mg)\,ds = -mgh = -(2)(9.81)(0.5)$$

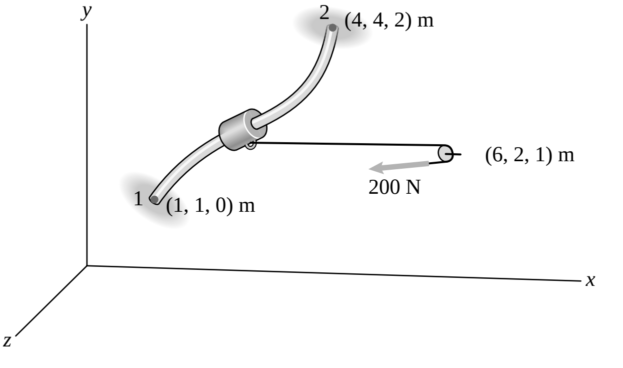
$$= -9.81 \text{ N-m.}$$

From the principle of work and energy:

$$U_F + U_{\text{gravity}} = \frac{1}{2}mv^2,$$

from which $v = \sqrt{\dfrac{2(U_F + U_{\text{gravity}})}{m}} = 4.90$ m/s

Problem 15.71 The 10-kg collar starts from rest at position 1. The tension in the string is 200 N, and the y axis points upward. If friction is negligible, what is the magnitude of the velocity of the collar when it reaches position 2? (See Problem 15.69.)



Solution: The constant force moves a distance

$$s = \sqrt{(6-1)^2 + (2-1)^2 + (1-0)^2}$$

$$- \sqrt{(6-4)^2 + (2-4)^2 + (1-2)^2} = 2.2 \text{ m.}$$

The work done by the constant force is

$$U_F = \int_0^s F\,ds = Fs = 439.2 \text{ N-m.}$$

The work done by gravity is

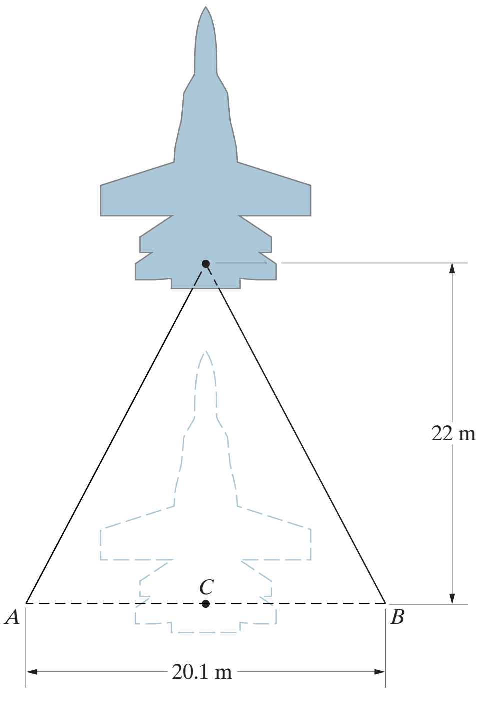
$$U_{\text{gravity}} = \int_0^h (-mg)\,ds = -mgh = -(10)(9.81)(3)$$

$$= -294.3 \text{ N-m.}$$

From the principle of work and energy $U_F + U_{\text{gravity}} = \frac{1}{2}mv^2$, from which

$$\boxed{v = \sqrt{\frac{2(U_F + U_{\text{gravity}})}{10}} = 5.38 \text{ m/s}}$$

Problem 15.72 As the F/A-18 lands at 64 m/s, the cable from A to B engages the airplane's arresting hook at C. The arresting mechanism maintains the tension in the cable at a constant value, bringing the 115.6 kN airplane to rest at a distance of 22 m. What is the tension in the cable? (See Problem 15.69.)



Solution:

$$U = -2T\left(\sqrt{(22\text{ m})^2 + (10.1\text{ m})^2} - 10.1\text{ m}\right)$$

$$= \frac{1}{2}\left(\frac{115600\text{ N}}{9.81\text{ m/s}^2}\right)(0 - [64\text{ m/s}]^2)$$

$$\boxed{T = 858.5\text{ kN}}$$

Problem 15.73 If the airplane in Problem 15.72 lands at 73.2 m/s, what distance does it roll before the arresting system brings it to rest?

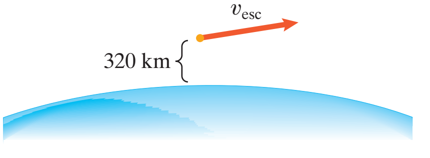
Solution:

$$U = -2(858500)\left(\sqrt{s^2 + (10.1\text{ m})^2} - 10.1\text{ m}\right)$$

$$= \frac{1}{2}\left(\frac{115600}{9.81\text{ m/s}^2}\right)(0 - [64\text{ m/s}]^2)$$

Solving we find $\boxed{s = 26.61\text{ m}}$

Problem 15.74 A spacecraft 320 km above the surface of the earth is moving at escape velocity $v_{esc} = 10{,}900$ m/s. What is its distance from the center of the earth when its velocity is 50 percent of its initial value? The radius of the earth is 6370 km. (See Example 15.6.)



Solution:

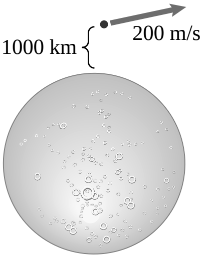
$$U_{12} = mgR_E^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$r_2 = \left[\frac{v_2^2 - v_1^2}{2gR_E^2} + \frac{1}{r_1}\right]^{-1}$$

$$r_2 = \left[\frac{(5450\text{ m/s})^2 - (10{,}900\text{ m/s})^2}{2(9.81\text{ m/s}^2)(6{,}370{,}000\text{ m})^2} + \frac{1}{6{,}690{,}000\text{ m}}\right]^{-1}$$

$$\boxed{r_2 = 26{,}600\text{ km.}}$$

Problem 15.75 A piece of ejecta is thrown up by the impact of a meteor on the moon. When it is 1000 km above the moon's surface, the magnitude of its velocity (relative to a nonrotating reference frame with its origin at the center of the moon) is 200 m/s. What is the magnitude of its velocity just before it strikes the moon's surface? The acceleration due to gravity at the surface of the moon is 1.62 m/s^2. The moon's radius is 1738 km.



Solution: The kinetic energy at $h = 1000$ km is

$$\left[\frac{m}{2}v^2\right]_{R_M+h} = 2 \times 10^4 \text{ N-m.}$$

The work done on the ejecta as it falls from 1000 km is

$$U_{\text{ejecta}} = \int_{R_{M+h}}^{R_M} (-W_{\text{ejecta}})\, ds = \int_{R_M+h}^{R_M} \left(-mg_M \frac{R_M^2}{s^2}\, ds\right)$$

$$= \left[mg_M \frac{R_M^2}{s}\right]_{R_M+h}^{R_M} = mg_M R_M \frac{h}{R_M + h},$$

$U_{\text{ejecta}} = 1.028 \text{ m} \times 10^6$ N-m. From the principle of work and energy, at the Moon's surface:

$$U_{\text{ejecta}} = \left[\frac{m}{2}v^2\right]_{\text{surface}} - \left[\frac{m}{2}v^2\right]_{R_M+h}$$

from which $\boxed{v_{\text{surface}} = \sqrt{2(1.028 \times 10^6 + 2 \times 10^4)} = 1448 \text{ m/s}}$

Problem 15.76 A satellite in a circular orbit of radius r around the earth has velocity $v = \sqrt{gR_E^2/r}$, where $R_E = 6370$ km is the radius of the earth. Suppose you are designing a rocket to transfer a 900-kg communication satellite from a circular parking orbit with 6700-km radius to a circular geosynchronous orbit with 42,222-km radius. How much work must the rocket do on the satellite?

Solution: Denote the work to be done by the rocket by U_{rocket}. Denote $R_{\text{park}} = 6700$ km, $R_{\text{geo}} = 42222$ km. The work done by the satellite's weight as it moves from the parking orbit to the geosynchronous orbit is

$$U_{\text{transfer}} = \int_{R_{\text{park}}}^{R_{\text{geo}}} F\,ds = \int_{R_{\text{park}}}^{R_{\text{geo}}} \left(-mg\frac{R_E^2}{s^2}\,ds\right)$$

$$= \left[mg\frac{R_E^2}{s}\right]_{R_{\text{park}}}^{R_{\text{geo}}} = mgR_E^2\left(\frac{1}{R_{\text{geo}}} - \frac{1}{R_{\text{park}}}\right),$$

$U_{\text{transfer}} = -4.5 \times 10^9$ N-m. From the principle of work and energy:

$$U_{\text{transfer}} + U_{\text{rocket}} = \left[\frac{1}{2}mv^2\right]_{\text{geo}} - \left[\frac{1}{2}mv^2\right]_{\text{park}}.$$

from which

$$U_{\text{rocket}} = \left[\frac{1}{2}mv^2\right]_{\text{geo}} - \left[\frac{1}{2}mv^2\right]_{\text{park}} - U_{\text{transfer}}.$$

Noting

$$\left[\frac{1}{2}mv^2\right]_{\text{geo}} = \frac{m}{2}\left(\frac{gR_E^2}{R_{\text{geo}}}\right) = 4.24 \times 10^9 \text{ N-m},$$

$$\left[\frac{1}{2}mv^2\right]_{\text{park}} = \frac{m}{2}\left(g\frac{R_E^2}{R_{\text{park}}}\right) = 2.67 \times 10^{10} \text{ N-m},$$

from which $\boxed{U_{\text{rocket}} = 2.25 \times 10^{10} \text{ N-m}}$

Problem 15.77 The force exerted on a charged particle by a magnetic field is $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where q and $\mathbf{v}$ are the charge and velocity of the particle and $\mathbf{B}$ is the magnetic field vector. Suppose that other forces on the particle are negligible. Use the principle of work and energy to show that the magnitude of the particle's velocity is constant.

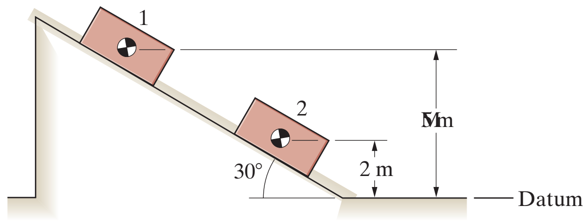
Solution: The force vector $\mathbf{F}$ is given by a cross product involving $\mathbf{v}$. This means that the force vector is ALWAYS perpendicular to the velocity vector. Hence, the force field does no work on the charged particle - it only changes the direction of its motion. Hence, if work is zero, the change in kinetic energy is also zero and the velocity of the charged particle is constant.

Problem 15.78 The 10-N box is released from rest at position 1 and slides down the smooth inclined surface to position 2.

(a) If the datum is placed at the level of the floor as shown, what is the sum of the kinetic and potential energies of the box when it is in position 1?

(b) What is the sum of the kinetic and potential energies of the box when it is in position 2?

(c) Use conservation of energy to determine the magnitude of the box's velocity when it is in position 2.



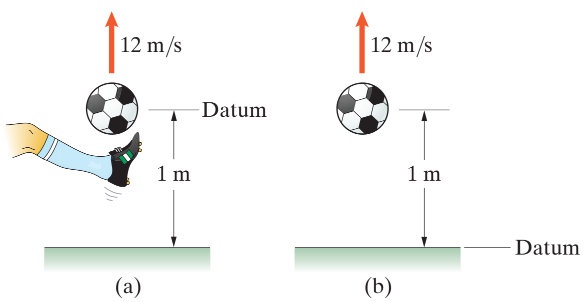
Solution:

(a) $\boxed{T_1 + V_1 = 0 + (10\text{ N})(5\text{ m}) = 50\text{ N-m}}$

(b) $\boxed{T_1 + V_1 = T_2 + V_2 = 50\text{ N-m}}$

(c) $50\text{ N-m} = \dfrac{1}{2}\left(\dfrac{10\text{ N}}{9.81\text{ m/s}^2}\right) v_2{}^2 + (10\text{ N})(2\text{ m}) \Rightarrow \boxed{v_2 = 7.67\text{ m/s}}$

Problem 15.79 The 0.45-kg soccer ball is 1 m above the ground when it is kicked upward at 12 m/s. Use conservation of energy to determine the magnitude of the ball's velocity when it is 4 m above the ground. Obtain the answer by placing the datum (a) at the level of the ball's initial position and (b) at ground level.



Solution:

(a) $T_1 = \dfrac{1}{2}(0.45\text{ kg})(12\text{ m/s})^2,\ \ V_1 = 0$

$T_2 = \dfrac{1}{2}(0.45\text{ kg})v_2{}^2,\ \ V_2 = (0.45\text{ kg})(9.81\text{ m/s}^2)(3\text{ m})$

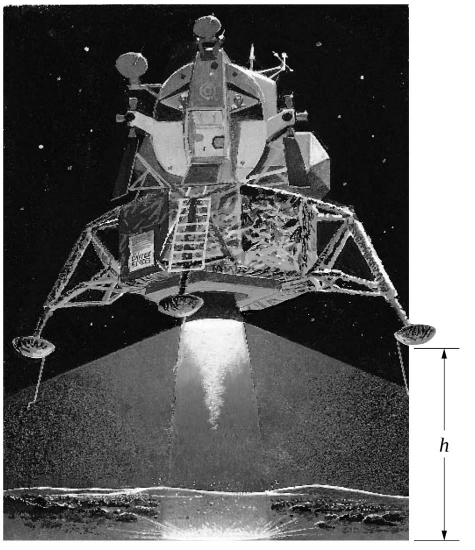
$T_1 + V_1 = T_2 + V_2 \ \Rightarrow\ \boxed{v_2 = 9.23\text{ m/s}}$

(b) $T_1 = \dfrac{1}{2}(0.45\text{ kg})(12\text{ m/s})^2,\ \ V_1 = (0.45\text{ kg})(9.81\text{ m/s}^2)(1\text{ m})$

$T_2 = \dfrac{1}{2}(0.45\text{ kg})v_2{}^2,\ V_2 = (0.45\text{ kg})(9.81\text{ m/s}^2)(4\text{ m})$

$T_1 + V_1 = T_2 + V_2 \ \Rightarrow\ \boxed{v_2 = 9.23\text{ m/s}}$

Problem 15.80 The Lunar Module (LM) used in the Apollo moon landings could make a safe landing if the magnitude of its vertical velocity at impact was no greater than 5 m/s. Use conservation of energy to determine the maximum height h at which the pilot could shut off the engine if the vertical velocity of the lander is (a) 2 m/s downward and (b) 2 m/s upward. The acceleration due to gravity at the moon's surface is 1.62 m/s^2.



Solution: Use conservation of energy Let state 1 be at the max height and state 2 at the surface. Datum is at the lunar surface.

(a), (b) $\frac{1}{2}mv_1^2 + mgh = \frac{1}{2}mv_2^2 + 0$

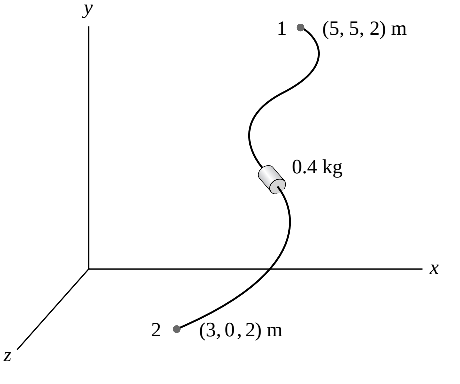
$$v_2 = 5 \text{ m/s} \quad g = 1.62 \text{ m/s}^2 \quad v_1 = \pm 2 \text{ m/s}$$

(m cancels from the equation.)

$$h = \frac{1}{2g}(v_2^2 - v_1^2)$$

(The sign of V_1 does not matter since v_1^2 is the only occurrence of v_1 in the relationship). Solving $\underline{h = 6.48 \text{ m}}$

Problem 15.81 The 0.4-kg collar starts from rest at position 1 and slides down the smooth rigid wire. The y axis points upward. Use conservation of energy to determine the magnitude of the velocity of the collar when it reaches point 2.



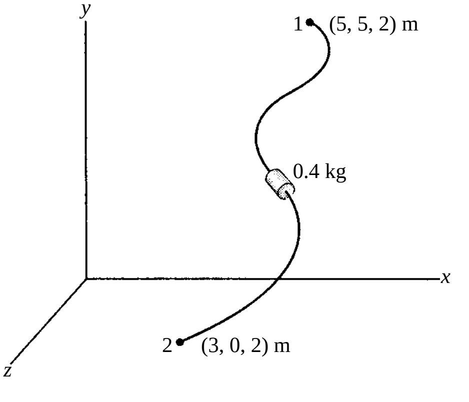
Solution: Assume gravity acts in the $-y$ direction and that $y = 0$ is the datum. By conservation of energy, $\frac{1}{2}mv^2 + V =$ constant where $V = mgy$. Thus,

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2$$

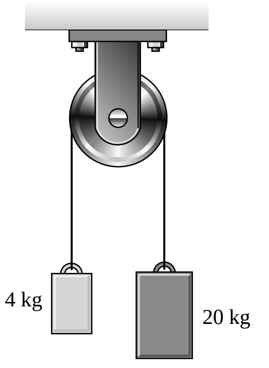
$m = 0.4$ kg, $g = 9.81$ m/s^2, $v_1 = 0$, $y_2 = 0$, and $y_1 = 5$ m. Thus

$$0 + (0.4)(9.81)(5) = \frac{1}{2}(0.4)v_2^2 + 0$$

$\underline{v_2 = 9.90 \text{ m/s}}$



Problem 15.82 At the instant shown, the 20-kg mass is moving downward at 1.6 m/s. Let d be the downward displacement of the mass relative to its present position. Use conservation of energy to determine the magnitude of the velocity of the 20-kg mass when $d = 1$ m.



Solution:

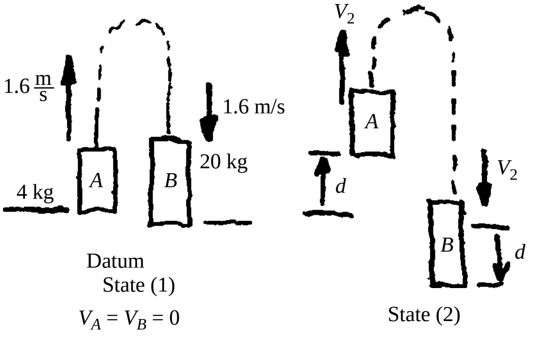
$m_1 = 4$ kg

$m_2 = 20$ kg

$v_1 = 1.6$ m/s

$g = 9.81$ m/s^2

$d = 1$ m



Energy for the system is conserved

$$\left(\tfrac{1}{2}m_1v_1^2 + 0\right) + \left(\tfrac{1}{2}m_2v_1^2 + 0\right) = \tfrac{1}{2}m_1v_2^2 + \tfrac{1}{2}m_2v_2^2 + m_1g(d) - m_2g(d)$$

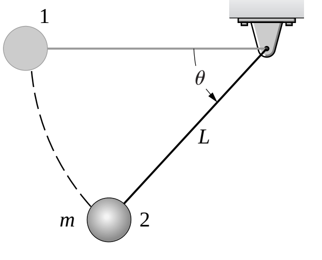
$$(m_1 + m_2)v_1^2 = (m_1 + m_2)v_2^2 + 2(m_1 - m_2)gd$$

Substituting known values and solving $\underline{v_2 = 3.95 \text{ m/s}}$

Problem 15.83 The mass of the ball is $m = 2$ kg and the string's length is $L = 1$ m. The ball is released from rest in position 1 and swings to position 2, where $\theta = 40^\circ$.

(a) Use conservation of energy to determine the magnitude of the ball's velocity at position 2.

(b) Draw graphs of the kinetic energy, the potential energy, and the total energy for values of θ from zero to 180°.



Solution:

$m = 2$ kg

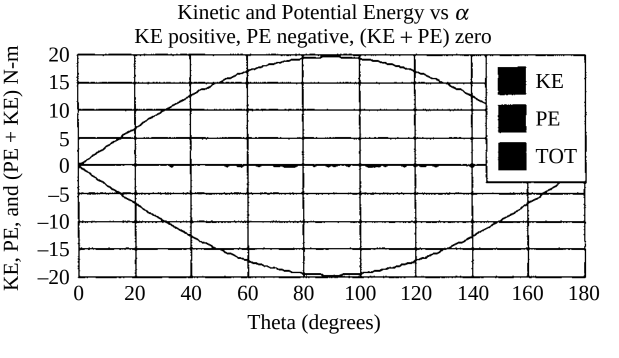
$L = 1$ m

Use conservation of energy State 1 $\theta = 0$; State 2, $0 < \theta < 180^\circ$
Datum: $\theta = 0$, $v_1 = 0$, $g = 9.81$ m/s^2

$$\tfrac{1}{2}mv_1^2 + mg(0) = \tfrac{1}{2}mv_2^2 + mg(-L\sin\theta)$$

$KE = \tfrac{1}{2}mv_2^2 \quad V = -mgL\sin\theta$ for all θ.
Total energy is always zero (datum value).

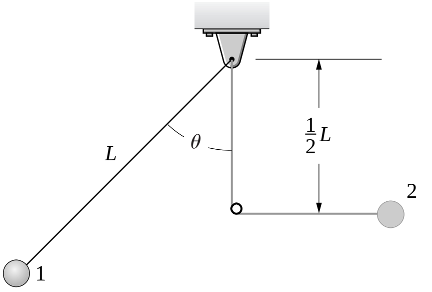
(a) Evaluating at $\theta = 40^\circ$, $\underline{v_2 = 3.55\ m/s}$



Problem 15.84 The mass of the ball is $m = 2$ kg and the string's length is $L = 1$ m. The ball is released from rest in position 1. When the string is vertical, it hits the fixed peg shown.

(a) Use conservation of energy to determine the minimum angle θ necessary for the ball to swing to position 2.

(b) If the ball is released at the minimum angle θ determined in part (a), what is the tension in the string just before and just after it hits the peg?



$m = 2$ kg

$L = 1$ m

Solution: Energy is conserved. $v_1 = v_2 = 0$ Use $\theta = 90^\circ$ as the datum.

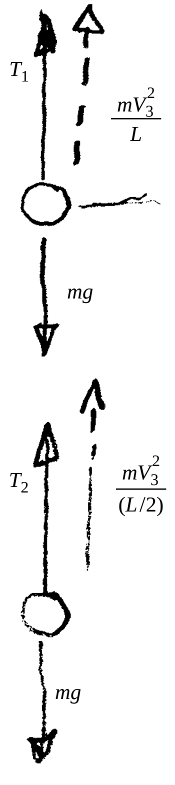
(a) $$\frac{1}{2}mv_1^2 + mg(-L\cos\theta_1) = \frac{1}{2}mv_2^2 - mg\frac{L}{2}$$

$$0 - mgL\cos\theta_1 = 0 - mg\frac{L}{2}$$

$$\cos\theta_1 = \frac{1}{2}$$

$$\theta = 60^\circ$$

(b) Use conservation of energy to determine velocity at the lowest point, (state 3) $(v_1 \equiv 0)$



$$\frac{1}{2}mv_1^2 - mgL\cos 60^\circ = \frac{1}{2}mv_3^2 - mgL$$

$$\frac{1}{2}mv_3^2 = mgL - mgL/2$$

$$v_3^2 = gL = 9.81\ \frac{\text{m}^2}{\text{s}^2}$$

$v_3 = 3.13$ m/s at $\theta = 0^\circ$.

Before striking the peg

$$T_1 - mg = mv_3^2/L$$

$$T_1 = (2)(9.81) + (2)(9.81)/(1)$$

$$\underline{T_1 = 39.2 \text{ N}}$$

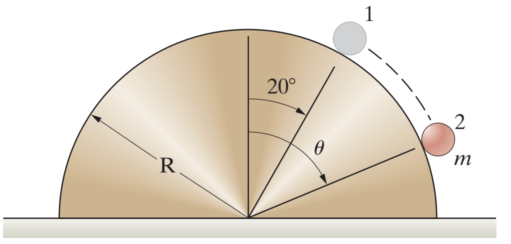
After striking the peg.

$$T - mg = mv_3^2/(L/2)$$

$$T = (2)(9.81) + 2[(2)(9.81)/1]$$

$$T = 58.9 \text{ N}$$

Problem 15.85 A small pellet of mass $m = 0.2$ kg starts from rest at position 1 and slides down the smooth surface of the cylinder to position 2. The radius $R = 0.8$ m. Use conservation of energy to determine the magnitude of the pellet's velocity at position 2 if $\theta = 45°$.



Solution: Use the ground as the datum

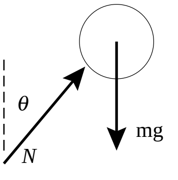
$$T_1 = 0,\ \ V_1 = (0.2\ \text{kg})(9.81\ \text{m/s}^2)(0.8\ \text{m})\cos 20^\circ$$

$$T_2 = \frac{1}{2}(0.2\ \text{kg})v_2{}^2,\ \ V_2 = (0.2\ \text{kg})(9.81\ \text{m/s}^2)(0.8\ \text{m})\cos 45^\circ$$

$$T_1 + V_1 = T_2 + V_2 \ \Rightarrow\ \boxed{v_2 = 1.91\ \text{m/s}}$$

Problem 15.86 In Problem 15.85, what is the value of the angle θ at which the pellet loses contact with the surface of the cylinder?

Solution: In position 2 we have



$$\sum F_{\nearrow} : N - mg\cos\theta = -m\frac{v_2{}^2}{R} \ \Rightarrow\ N = m\left(g\cos\theta - \frac{v_2{}^2}{R}\right)$$

When the pellet leaves the surface $N = 0 \ \Rightarrow\ v_2^2 = Rg\cos\theta$

Now do work-energy.

$$T_1 = 0,\ \ V_1 = mgR\cos 20^\circ$$

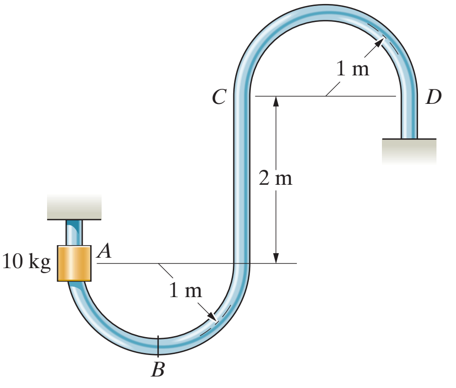
$$T_2 = \frac{1}{2}mv_2{}^2 = \frac{1}{2}mRg\cos\theta,\ \ V_2 = mgR\cos\theta$$

$$T_1 + V_1 = T_2 + V_2 \ \Rightarrow\ 0 + mgR\cos 20^\circ = \frac{3}{2}mgR\cos\theta$$

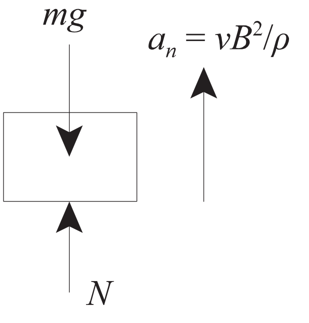
Solving we find $\boxed{\theta = \cos^{-1}\left(\frac{2}{3}\cos 20^\circ\right) = 51.2^\circ}$

Problem 15.87 The bar is smooth. The 10-kg slider at A is given a downward velocity of 6.5 m/s.

(a) Use conservation of energy to determine whether the slider will reach point C. If it does, what is the magnitude of its velocity at point C?
(b) What is the magnitude of the normal force the bar exerts on the slider as it passes point B?



Solution:



(a) Find the velocity at C.

$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_C^2 + mgh$$

$$v_C = \sqrt{v_A^2 - 2gh} = \sqrt{(6.5\text{ m/s})^2 - 2(9.81\text{ m/s}^2)(2\text{ m})}$$

This equation has a real solution, hence it is possible to reach point C.

$$\boxed{Yes, v_C = 1.73\text{ m/s.}}$$

(b) Find the velocity at point B

$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_B^2 - mgh,$$

$$v_B = \sqrt{v_A^2 + 2gh} = \sqrt{(6.5\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(1\text{ m})} = 7.87\text{ m/s.}$$

Now find the normal force

$$\Sigma F_y : N - mg = m\frac{v_B^2}{\rho}$$

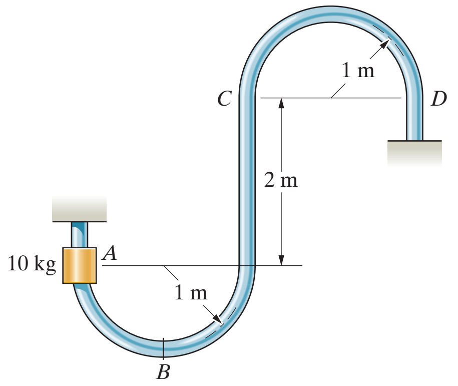
$$N = m\left(g + \frac{v_B^2}{\rho}\right) = (10\text{ kg})\left([9.81\text{ m/s}^2] + \frac{[7.87\text{ m/s}]^2}{1\text{ m}}\right)$$

$$\boxed{N = 717\text{ N.}}$$

Problem 15.88 The bar is smooth. The 10-kg slider at A is given a downward velocity of 7.5 m/s.

(a) Use conservation of energy to determine whether the slider will reach point D. If it does, what is the magnitude of its velocity at point D?

(b) What is the magnitude of the normal force the bar exerts on the slider as it passes point B?



Solution:

(a) We will first find the velocity at the highest point (half way between C and D).

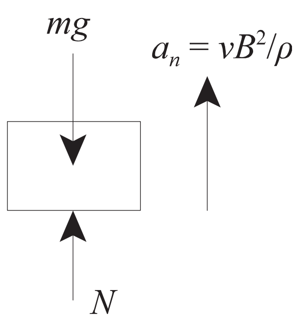
$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_D^2 + mgh$$

$$v_D = \sqrt{v_A^2 - 2gh} = \sqrt{(7.5 \text{ m/s})^2 - 2(9.81 \text{ m/s}^2)(3 \text{ m})}$$

$$v_D = \sqrt{-2.61} \text{ m/s}.$$

This equation does not have a solution in terms of real numbers which means that it cannot reach the highest point.

$\boxed{\text{No.}}$



(b) Find the velocity at point B

$$\frac{1}{2}mv_A^2 + 0 = \frac{1}{2}mv_B^2 - mgh,$$

$$v_B = \sqrt{v_A^2 + 2gh} = \sqrt{(7.5 \text{ m/s})^2 + 2(9.81 \text{ m/s}^2)(1 \text{ m})} = 8.71 \text{ m/s}.$$

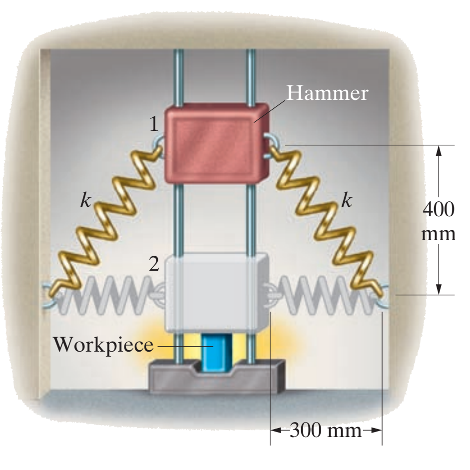
Now find the normal force

$$\Sigma F_y : N - mg = m\frac{v_B^2}{\rho}$$

$$N = m\left(g + \frac{v_B^2}{\rho}\right) = (10 \text{ kg})\left([9.81 \text{ m/s}^2] + \frac{[8.71 \text{ m/s}]^2}{1 \text{ m}}\right)$$

$$\boxed{N = 857 \text{ N.}}$$

Problem 15.89 In Active Example 15.7, suppose that you want to increase the value of the spring constant k so that the velocity of the hammer just before it strikes the workpiece is 4 m/s. Use conservation of energy to determine the required value of k.



Solution:

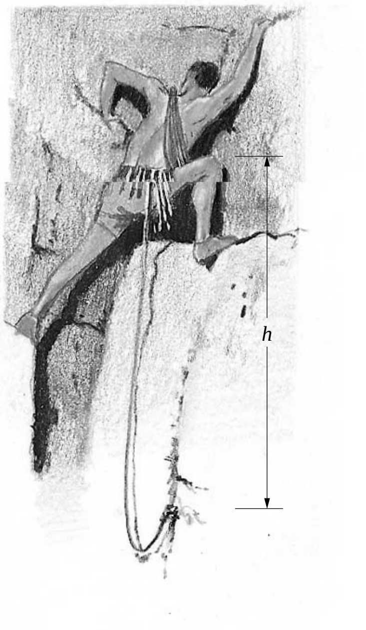
$$2\left(\frac{1}{2}ks^2\right) + mgh = \frac{1}{2}mv^2$$

$$k = \frac{m(v^2 - 2gh)}{2s^2}$$

$$k = \frac{(40 \text{ kg})([4 \text{ m/s}]^2 - 2[9.81 \text{ m/s}^2][0.4 \text{ m}])}{2([0.5 \text{ m}] - [0.3 \text{ m}])^2}$$

$$\boxed{k = 4080 \text{ N/m.}}$$

Problem 15.90 A rock climber of weight W has a rope attached a distance h below him for protection. Suppose that he falls, and assume that the rope behaves like a linear spring with unstretched length h and spring constant $k = C/h$, where C is a constant. Use conservation of energy to determine the maximum force exerted on the climber by the rope. (Notice that the maximum force is independent of h, which is a reassuring result for climbers: The maximum force resulting from a long fall is the same as that resulting from a short one.)



Solution: Choose the climber's center of mass before the fall as the datum. The energy of the climber before the fall is zero. As the climber falls, his energy remains the same:

$$0 = \frac{1}{2}mv^2 - Wy,$$

where y is positive downward. As the rope tightens, the potential energy stored in the rope becomes

$$V_{\text{rope}} = \frac{1}{2}k(y - 2h)^2.$$

At maximum extension the force on the climber is

$$F = -\frac{\partial V}{\partial y} = -k(y - 2h).$$

When the velocity of the falling climber is zero, $0 = -Wy + \frac{1}{2}k(y - 2h)^2$, from which: $y^2 + 2by + c = 0$, where

$$b = -\left(2h + \frac{W}{k}\right),$$

and $c = +4h^2$. The solution is

$$y = \left(2h + \frac{W}{k}\right) \pm \left(\frac{W}{k}\right)\sqrt{\frac{4\,kh}{W} + 1}.$$

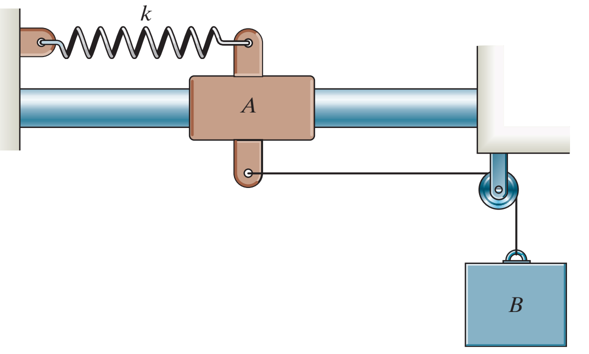
Substitute:

$$F = -W\left(1 \pm \sqrt{1 + \frac{4C}{W}}\right).$$

The positive sign applies, and the force is

$$\boxed{F = -W\left(1 + \sqrt{1 + \frac{4C}{W}}\right)} \text{ (directed upward).}$$

Problem 15.91 The collar A slides on the smooth horizontal bar. The spring constant $k = 40$ N/m. The weights are $W_A = 30$ N and $W_B = 60$ N. As the instant shown, the spring is unstretched and B is moving downward at 4 m/s. Use conservation of energy to determine the velocity of B when it has moved downward 2 m from its current position. (See Example 15.8.)



Solution: Notice that the collars have the same velocity

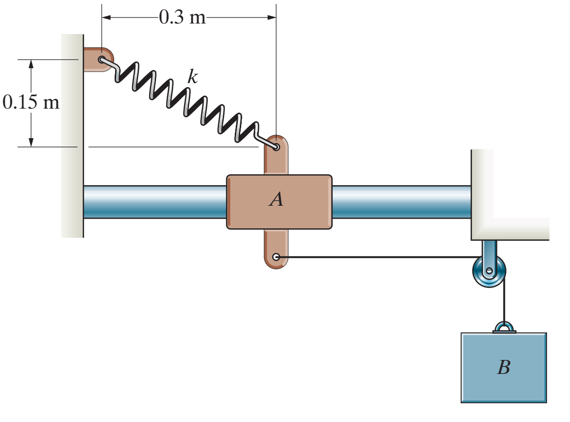
$$\frac{1}{2}\left(\frac{W_A + W_B}{g}\right)v_1^2 + W_B h = \frac{1}{2}\left(\frac{W_A + W_B}{g}\right)v_2^2 + \frac{1}{2}kh^2$$

$$v_2 = \sqrt{v_1^2 + \left(\frac{2\,W_B h - kh^2}{W_A + W_B}\right)g}$$

$$v_2 = \sqrt{(4\text{ m/s})^2 + \left(\frac{2[60\text{ N}][2\text{ m}] - [40\text{ N/m}][2\text{ m}]^2}{90\text{ N}}\right)(9.81\text{ m/s}^2)}$$

$$\boxed{v_2 = 4.97\text{ m/s.}}$$

Problem 15.92 The spring constant $k = 700$ N/m. The masses $m_A = 14$ kg and $m_B = 18$ kg. The horizontal bar is smooth. At the instant shown, the spring is unstretched and the mass B is moving downward at 1 m/s. How fast is B moving when it has moved downward 0.2 m from its present position?



Solution: The unstretched length of the spring is

$$\delta = \sqrt{(0.3\text{ m})^2 + (0.15\text{ m})^2}$$

$$= 0.335\text{ m.}$$

When B has moved 0.2 m, the length of the spring is

$$\Delta = \sqrt{(0.5\text{ m})^2 + (0.15\text{ m})^2} = 0.522\text{ m.}$$

Conservation of energy is

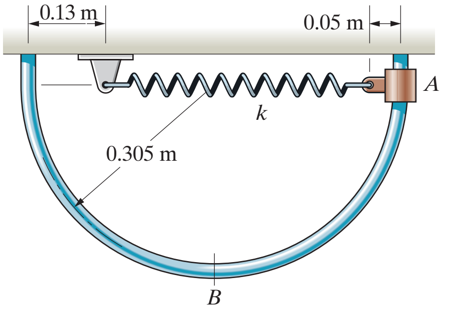
$$\frac{1}{2}(m_A + m_B)v_1^2 = \frac{1}{2}(m_A + m_B)v_2^2 - m_B gh + \frac{1}{2}k(\Delta - \delta)^2$$

$$v_2 = \sqrt{v_1^2 + \frac{2m_B gh - k(\Delta - \delta)^2}{m_A + m_B}}$$

$$v_2 = \sqrt{(1\text{ m/s})^2 + \frac{2(18\text{ kg})(9.81\text{ m/s}^2)(0.2\text{ m}) - (700\text{ N/m})(0.187\text{ m})^2}{32\text{ kg}}}$$

$$\boxed{v_2 = 1.56\text{ m/s.}}$$

Problem 15.93 The semicircular bar is smooth. The unstretched length of the spring is 0.254 m. The 5-N collar at A is given a downward velocity of 6 m/s, and when it reaches B the magnitude of its velocity is 15 m/s. Determine the spring constant k.



Solution: The stretch distances for the spring at A and B are

$\delta_A = 0.226$ m

$\delta_B = 0.098$ m

Conservation of energy gives

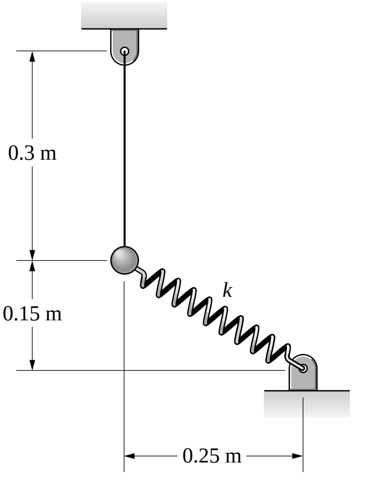
$$\frac{1}{2}mv_A^2 + \frac{1}{2}k\delta_A^2 + mgh = \frac{1}{2}mv_B^2 + \frac{1}{2}k\delta_B^2$$

$$k = \frac{m[v_B^2 - v_A^2 - 2gh]}{\delta_A^2 - \delta_B^2}$$

$$k = \left(\frac{5 \text{ N}}{9.81 \text{ m/s}^2}\right)\left[\frac{(15 \text{ m/s})^2 - (6 \text{ m/s})^2 - 2(9.81 \text{ m/s}^2)(0.305 \text{ m})}{(0.226 \text{ m})^2 - (0.098 \text{ m})^2}\right]$$

$\boxed{k = 2249 \text{ N/m.}}$

Problem 15.94 The mass $m = 1$ kg, the spring constant $k = 200$ N/m, and the unstretched length of the spring is 0.1 m. When the system is released from rest in the position shown, the spring contracts, pulling the mass to the right. Use conservation of energy to determine the magnitude of the velocity of the mass when the string and spring are parallel.



Solution: The stretch of the spring in position 1 is

$$S_1 = \sqrt{(0.15)^2 + (0.25)^2} - 0.1 = 0.192 \text{ m}.$$

The stretch in position 2 is

$$S_2 = \sqrt{(0.3 + 0.15)^2 + (0.25)^2} - 0.3 - 0.1 = 0.115 \text{ m}.$$

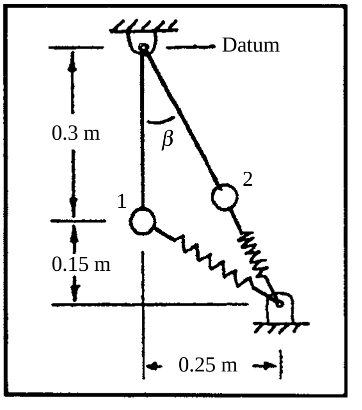
The angle $\beta = \arctan(0.25/0.45) = 29.1^\circ$. Applying conservation of energy,

$$\tfrac{1}{2}mv_1^2 + \tfrac{1}{2}kS_1^2 - mg(0.3) = \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}kS_2^2 - mg(0.3\cos\beta):$$

$$0 + \tfrac{1}{2}(200)(0.192)^2 - (1)(9.81)(0.3) = \tfrac{1}{2}(1)v_2^2 + \tfrac{1}{2}(200)(0.115)^2$$

$$- (1)(9.81)(0.3\cos 29.1^\circ).$$

Solving, $v_2 = 1.99$ m/s



Problem 15.95 In problem 15.94, what is the tension in the string when the string and spring are parallel?

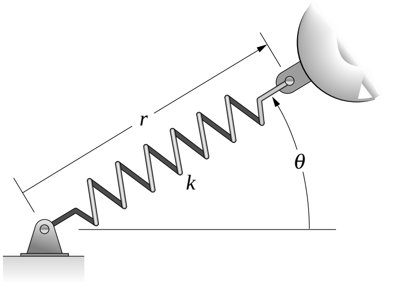
Solution: The free body diagram of the mass is: Newton's second law in the direction normal to the path is

$$T - kS_2 - mg\cos\beta = ma_n:$$

$$T - (200)(0.115) - (1)(9.81)\cos 29.1^\circ = (1)(v_2^2/0.3).$$

We obtain, $T = 44.7$ N.

Problem 15.96 The force exerted on an object by a *nonlinear* spring is $\mathbf{F} = -[k(r - r_0) + q(r - r_0)^3]\mathbf{e}_r$, where k and q are constants and r_0 is the unstretched length of the spring. Determine the potential energy of the spring in terms of its stretch $S = r - r_0$.



Solution: Note that $dS = dr$. The work done in stretching the spring is

$$V = -\int \boldsymbol{F} \cdot d\mathbf{r} + C = -\int \boldsymbol{F} \cdot (dr\boldsymbol{e}_r + r\,d\theta\boldsymbol{e}_\theta) + C$$

$$= \int [k(r - r_0) + q(r - r_0)^2]\,dr + C,$$

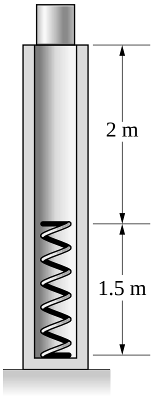
$$V = \int [kS + qS^3]\,dS + C.$$

Integrate:

$$\boxed{V = \frac{k}{2}S^2 + \frac{q}{4}S^4},$$

where $C = 0$, since $\boldsymbol{F} = 0$ at $S = 0$.

Problem 15.97 The 20-kg cylinder is released at the position shown and falls onto the linear spring ($k = 3000$ N/m). Use conservation of energy to determine how far down the cylinder moves after contacting the spring.



Solution: Choose the base of the cylinder as a datum. The potential energy of the piston at rest is $V_1 = mg(3.5) = 686.7$ N-m. The conservation of energy condition after the spring has compressed to the point that the piston velocity is zero is $mgh + \frac{1}{2}k(h - 1.5)^2 = mg(3.5)$, where h is the height above the datum. From which $h^2 + 2bh + c = 0$, where

$$b = -\left(\frac{3}{2} - \frac{mg}{k}\right)$$

and $\quad c = 2.25 - \dfrac{7\,mg}{k}.$

The solution is $h = -b \pm \sqrt{b^2 - c} = 1.95$ m, $n = 0.919$ m. The value $h = 1.95$ m has no physical meaning, since it is above the spring. The downward compression of the spring is

$$\boxed{S = 1.5 - 0.919 = 0.581 \text{ m}}$$

Problem 15.98 The 20-kg cylinder is released at the position shown and falls onto the *nonlinear* spring. In terms of the stretch S of the spring, its potential energy is $V = \frac{1}{2}kS^2 + \frac{1}{2}qS^4$, where $k = 3000$ N/m and $q = 4000$ N/m^3. What is the velocity of the cylinder when the spring has been compressed 0.5 m?

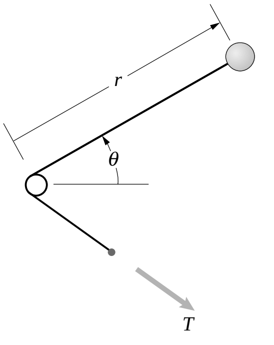
Solution: Note that $S = 1.5 - h$ where h is the height above the datum, from which $h = 1.5 - S$. Use the solution to Problem 15.97. The conservation of energy condition when the spring is being compressed is $\frac{1}{2}mv^2 + V_{\text{spring}} + mg(1.5 - S) = mg(3.5)$, from which

$$v = \sqrt{7.0g - 2g(1.5 - S) - 2\frac{V_{\text{spring}}}{m}}.$$

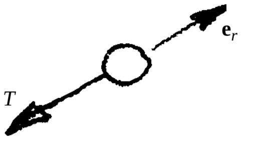
The potential energy in the spring is $V_{\text{spring}} = \frac{1}{2}(3000)(0.5^2) + \frac{1}{4}(4000)(0.5^4) = +437.5$ N-m.

Substitute numerical values to obtain $\boxed{v = 2.30 \text{ m/s}}$

Problem 15.99 The string exerts a force of constant magnitude T on the object. Use polar coordinates to show that the potential energy associated with this force is $V = Tr$.



Solution:



$dV = -\mathbf{F} \cdot d\mathbf{r}$

$V = -\int_{\text{DATUM}}^{r} - T\mathbf{e}_r \cdot dr\mathbf{e}_r$

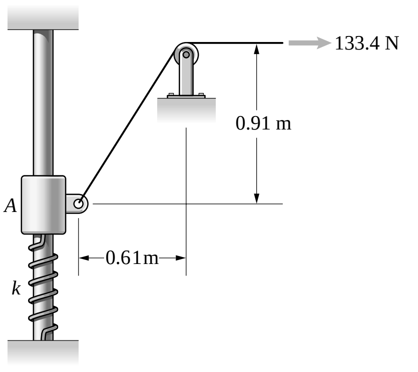
$V = Tr\Big|_{\text{DATUM}}^{r}$

$V = Tr - Tr_{\text{DATUM}}$

Let $r_{\text{DATUM}} = 0$

$\underline{V = Tr}$

Problem 15.100 The system is at rest in the position shown, with the 53.4 N collar A resting on the spring (292 N/m), when a constant 133.4 N force is applied to the cable. What is the velocity of the collar when it has risen 0.305 m? (See Problem 15.99.)



Solution: Choose the rest position as the datum. At rest, the compression of the spring is

$$S_1 = \frac{-W}{k} = -0.183 \text{ m}.$$

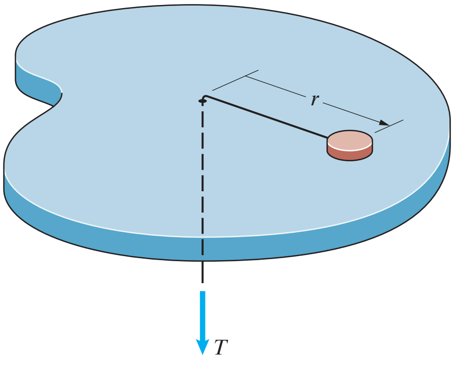
When the collar rises 0.31 m the stretch is $S_2 = S_1 + 0.3 = 0.127$ m When the collar rises 0.31 m the constant force on the cable has acted through a distance

$$s = \sqrt{0.91^2 + 0.61^2} - \sqrt{(0.91 - 0.31)^2 - 0.61^2} = 0.237 \text{ m}.$$

The work done on the system is $U_s = \frac{1}{2}k(S_1^2 - S_2^2) - mg(1) + Fs$. From the conservation of energy $U_s = \frac{1}{2}mv^2$ from which

$$\boxed{v = \sqrt{\frac{k}{m}(S_1^2 - S_2^2) - 2g + \frac{2Fs}{m}} = 2.57 \text{ m/s}.}$$

Problem 15.101 A 1-kg disk slides on a smooth horizontal table and is attached to a string that passes through a hole in the table. A constant force $T = 10$ N is exerted on the string. At the instant shown, $r = 1$ m and the velocity of the disk in terms of polar coordinates is $\mathbf{v} = 6\mathbf{e}_\theta$ (m/s). Use conservation of energy to determine the magnitude of the velocity of the disk when $r = 2$m. (See Problem 15.99.)



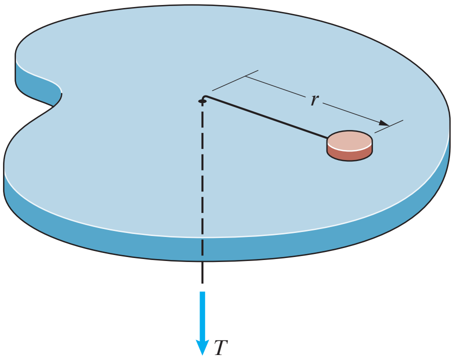
Solution:

$$\frac{1}{2}mv_1^2 + Tr_1 = \frac{1}{2}mv_2^2 + Tr_2$$

$$v_2 = \sqrt{v_1^2 + 2\frac{T}{m}(r_1 - r_2)} = \sqrt{(6 \text{ m/s})^2 + 2\left(\frac{10 \text{ N}}{1 \text{ kg}}\right)([1 \text{ m}] - [2 \text{ m}])}$$

$$\boxed{v_2 = 4 \text{ m/s}.}$$

Problem 15.102 A 1-kg disk slides on a smooth horizontal table and is attached to a string that passes through a hole in the table. A constant force $T = 10$ N is exerted on the string. At the instant shown, $r = 1$ m and the velocity of the disk in terms of polar coordinates is $\mathbf{v} = 8\mathbf{e}_\theta$ (m/s). Because this is central-force motion, the product of the radial position r and the transverse component of velocity v_θ is constant. Use this fact and conservation of energy to determine the velocity of the disk in terms of polar coordinates when $r = 2$m.



Solution: We have

$$\frac{1}{2}mv_1^2 + Tr_1 = \frac{1}{2}m(v_{2r}^2 + v_{2\theta}^2) + Tr_2, \quad r_1 v_1 = r_2 v_{2\theta}$$

Solving we find

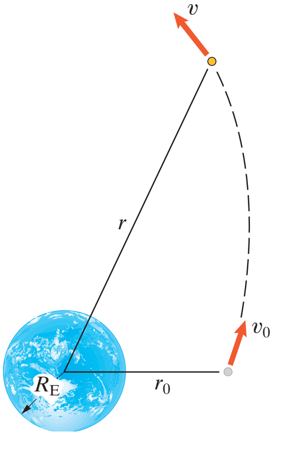
$$v_{2\theta} = \frac{r_1}{r_2}v_1 = \frac{1\text{ m}}{2\text{ m}}(8\text{ m/s}) = 4\text{ m/s}$$

$$v_{2r} = \sqrt{v_1^2 - v_{2\theta}^2 + 2\frac{T}{m}(r_1 - r_2)}$$

$$= \sqrt{(8\text{ m/s})^2 - (4\text{ m/s})^2 + 2\left(\frac{10\text{ N}}{1\text{ kg}}\right)([1\text{ m}] - [2\text{ m}])} = 5.29\text{ m/s}.$$

$$\boxed{\mathbf{v} = (5.29\mathbf{e}_r + 4\mathbf{e}_\theta)\text{ m/s.}}$$

Problem 15.103 A satellite initially is inserted into orbit at a distance $r_0 = 8800$ km from the center of the earth. When it is at a distance $r = 18{,}000$ km from the center of the earth, the magnitude of its velocity is $v = 7000$ m/s. Use conservation of energy to determine its initial velocity v_0. The radius of the earth is 6370 km. (See Example 15.9.)



Solution:

$$\frac{1}{2}mv_0^2 - \frac{mgR_E^2}{r_0} = \frac{1}{2}mv^2 - \frac{mgR_E^2}{r}$$

$$v_0 = \sqrt{v^2 + 2gR_E^2\left(\frac{1}{r_0} - \frac{1}{r}\right)}$$

$$v_0 = \sqrt{(7000\text{ m/s})^2 + 2(9.81\text{ m/s}^2)(6.37\times10^6\text{ m})^2\left(\frac{1}{8.8\times10^6\text{ m}} - \frac{1}{18\times10^6\text{ m}}\right)}$$

$$\boxed{v_0 = 9760\text{ m/s.}}$$

Problem 15.104 Astronomers detect an asteroid 100,000 km from the earth moving at 2 km/s relative to the center of the earth. Suppose the asteroid strikes the earth. Use conservation of energy to determine the magnitude of its velocity as it enters the atmosphere. (You can neglect the thickness of the atmosphere in comparison to the earth's 6370-km radius.)

Solution: Use the solution to Problem 15.103. The potential energy at a distance r is

$$V = -\frac{mgR_E^2}{r}.$$

The conservation of energy condition at

$$r_0 : \frac{1}{2}mv_0^2 - \frac{mgR_E^2}{r_0} = \frac{1}{2}mv^2 - \frac{mgR_E^2}{r}.$$

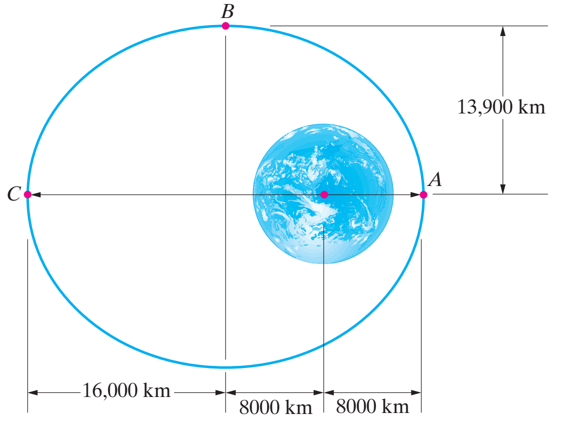
Solve:

$$v = \sqrt{v_0^2 + 2gR_E^2\left(\frac{1}{r} - \frac{1}{r_0}\right)}.$$

Substitute: $r_0 = 1 \times 10^8$ m, $v_0 = 2 \times 10^3$ m/s, and $r = 6.37 \times 10^6$ m. The velocity at the radius of the earth is

$$\boxed{v = 11 \text{ km/s}}$$

Problem 15.105 A satellite is in the elliptic earth orbit shown. Its velocity in terms of polar coordinates when it is at the perigee A is $\mathbf{v} = 8640\mathbf{e}_\theta$ (m/s). Determine the velocity of the satellite in terms of polar coordinates when it is at point B.



Solution: We have

$$r_A = 8000 \text{ km} = 8 \times 10^6 \text{ m}$$

$$r_B = \sqrt{13,900^2 + 8000^2} \text{ km} = 1.60 \times 10^7 \text{ m}.$$

Energy and angular momentum are conserved. Therefore

$$\frac{1}{2}mv_A^2 - \frac{mgR_E^2}{r_A} = \frac{1}{2}m(v_{Br}^2 + v_{B\theta}^2) - \frac{mgR_E^2}{r_B}, r_Av_A = r_Bv_{B\theta}$$

Solving we have

$$v_{B\theta} = \frac{r_A}{r_B}v_A = \frac{8 \times 10^6 \text{ m}}{1.60 \times 10^7 \text{ m}}(8640 \text{ m/s}) = 4310 \text{ m/s},$$

$$v_{Br} = \sqrt{v_A^2 - v_{B\theta}^2 + 2gR_E^2\left(\frac{1}{r_B} - \frac{1}{r_A}\right)} = 2480 \text{ m/s}.$$

$$\boxed{\mathbf{v} = (2480\mathbf{e}_r + 4310\mathbf{e}_\theta) \text{ m/s.}}$$

Problem 15.106 Use conservation of energy to determine the magnitude of the velocity of the satellite in Problem 15.105 at the apogee C. Using your result, confirm numerically that the velocities at perigee and apogee satisfy the relation $r_A v_A = r_C v_C$.

Solution: From Problem 15.105, $r_A = 8000$ km, $r_C = 24000$ km, $v_A = 8640$ m/s, $g = 9.81$ m/s^2, $R_E = 6370$ km.
From conservation of energy,

$$\frac{1}{2}mv_A^2 - \frac{mgR_E^2}{r_A} = \frac{1}{2}mv_c^2 - \frac{mgR_E^2}{r_c}$$

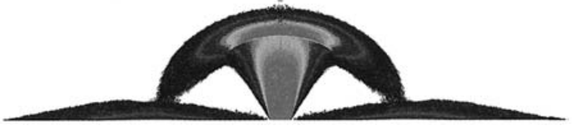
Factor m out of the equation, convert all distances to meters, and solve for v_c. Solving, $\underline{v_C = 2880 \text{ m/s}}$

Does $r_A v_A = r_C v_C$

Substituting the known values, we have

$$r_A v_A = r_C v_C = 6.91 \times 10^{10} \text{ m}^2/\text{s}$$

Problem 15.107 The *Voyager* and *Galileo* spacecraft have observed volcanic plumes, believed to consist of condensed sulfur or sulfur dioxide gas, above the surface of the Jovian satellite Io. The plume observed above a volcano named Prometheus was estimated to extend 50 km above the surface. The acceleration due to gravity at the surface is 1.80 m/s^2. Using conservation of energy and neglecting the variation of gravity with height, determine the velocity at which a solid particle would have to be ejected to reach 50 km above Io's surface.



Solution: Conservation of energy yields: $T_1 + V_1 = T_2 + V_2$: Using the forms for a constant gravity field, we get $\frac{1}{2}mv_1^2 + 0 = 0 + mgy_2$ Evaluating, we get $\frac{1}{2}v_1^2 = (1.8)(50{,}000)$, or $v_1 = 424$ m/s

Problem 15.108 Solve Problem 15.107 using conservation of energy and accounting for the variation of gravity with height. The radius of Io is 1815 km.

Solution: Conservation of energy yields: $T_1 + V_1 = T_2 + V_2$: Only the form of potential energy changes from that used in Problem 15.107. Here we get

$$\frac{1}{2}mv_1^2 - \frac{mgR_I^2}{R_I} = 0 - \frac{mgR_I^2}{r_I}.$$

Evaluating,

$$\frac{1}{2}v_1^2 - \frac{(1.8)(1{,}815{,}000)^2}{1{,}815{,}000} = -\frac{(1.8)(1{,}815{,}000)^2}{1{,}815{,}000 + 50{,}000}.$$

or $v_1 = 419$ m/s

Problem 15.109* What is the relationship between Eq. (15.21), which is the gravitational potential energy neglecting the variation of the gravitational force with height, and Eq. (15.23), which accounts for the variation? Express the distance from the center of the earth as $r = R_E + y$, where R_E is the earth's radius and y is the height above the surface, so that Eq. (15.23) can be written as

$$V = -\frac{mgR_E}{1 + \dfrac{y}{R_E}}.$$

By expanding this equation as a Taylor series in terms of y/R_E and assuming that $y/R_E \ll 1$, show that you obtain a potential energy equivalent to Eq. (15.21).

Solution: Define $y/R_E = \varepsilon$

$$V = -\frac{mgR_E{}^2}{r} = -\frac{mgR_E{}^2}{R_E + y} = -\frac{mgR_E}{1 + \dfrac{y}{R_E}} = -\frac{mgR_E}{1+\varepsilon}$$

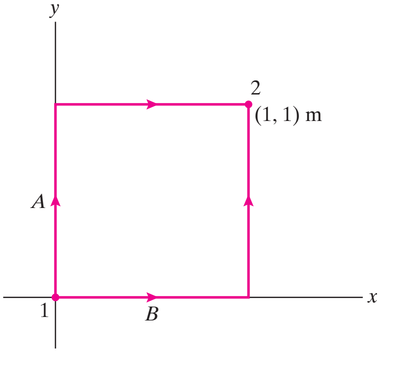
$$V = V|_{\varepsilon=0} + \left.\frac{dV}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + \cdots$$

$$V = -mgR_E + \left.\frac{mgR_E}{(1+\varepsilon)^2}\right|_{\varepsilon=0} \varepsilon = -mgR_E + mgR_E\frac{y}{R_E}$$

$$= -mgR_E + mgy \quad \text{QED}$$

Problem 15.110 The potential energy associated with a force **F** acting on an object is $V = x^2 + y^3$ N-m, where x and y are in meters.

(a) Determine **F**.
(b) Suppose that the object moves from position 1 to position 2 along path A, and then moves from position 1 to position 2 along path B. Determine the work done by **F** along each path.



Solution:

(a) $F_x = -\dfrac{dV}{dx} = -2x, \quad F_y = -\dfrac{dV}{dy} = -3y^2$

$\boxed{\mathbf{F} = -2x\mathbf{i} - 3y^2\mathbf{j} \text{ N.}}$

(b)

$$W_{12A} = \int_0^1 (-3y^2)\,dy + \int_0^1 (-2x)\,dx = -(1)^3 - (1)^2 = -2 \text{ N-m}$$

$$W_{12B} = \int_0^1 (-2x)\,dx + \int_0^1 (-3y^2)\,dy = -(1)^2 - (1)^3 = -2 \text{ N-m}$$

$\boxed{W_{12A} = W_{12B} = -2 \text{ N-m.}}$

Problem 15.111 An object is subjected to the force $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ (N), where x and y are in meters.

(a) Show that $\mathbf{F}$ is *not* conservative.
(b) Suppose the object moves from point 1 to point 2 along the paths A and B shown in Problem 15.110. Determine the work done by $\mathbf{F}$ along each path.

Solution:

(a) A necessary and sufficient condition that $\mathbf{F}$ be conservative is $\nabla \times \mathbf{F} = 0$.

$$\nabla \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{bmatrix} = \mathbf{i}0 - \mathbf{j}0 + (-1-1)\mathbf{k}$$

$$= -2\mathbf{k} \neq 0.$$

Therefore $\mathbf{F}$ is non conservative.

(b) The integral along path B is

$$\boxed{\begin{aligned} U_B &= \int_0^1 (y\mathbf{i} - x\mathbf{j})|_{y=0} \cdot \mathbf{i}\,dx + \int_0^1 (y\mathbf{i} - x\mathbf{j})|_{x=1} \cdot \mathbf{j}\,dy \\ &= 0 - 1 = -1 \text{ N-m} \end{aligned}}.$$

Along path A:

$$\boxed{\begin{aligned} U_A &= \int_0^1 (y\mathbf{i} - x\mathbf{j})|_{x=0} \cdot \mathbf{j}\,dy + \int_0^1 (y\mathbf{i} - x\mathbf{j})|_{y=1} \cdot \mathbf{i}\,dx \\ &= 0 + 1 = +1 \text{ N-m} \end{aligned}}$$

Problem 15.112 In terms of polar coordinates, the potential energy associated with the force $\mathbf{F}$ exerted on an object by a *nonlinear* spring is

$$V = \tfrac{1}{2}k(r - r_0)^2 + \tfrac{1}{4}q(r - r_0)^4,$$

where k and q are constants and r_0 is the unstretched length of the spring. Determine $\mathbf{F}$ in terms of polar coordinates. (See Active Example 15.10.)

Solution: The force is given by

$$\boxed{\begin{aligned} \mathbf{F} &= -\nabla V = -\left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta\right)\left(\frac{1}{2}k(r - r_0)^2 + \frac{1}{4}q(r - r_0)^4\right) \\ &= -[k(r - r_0) + q(r - r_0)^3]\mathbf{e}_r \end{aligned}}$$

Problem 15.113 In terms of polar coordinates, the force exerted on an object by a *nonlinear* spring is

$$\mathbf{F} = -(k(r - r_0) + q(r - r_0)^3)\mathbf{e}_r,$$

where k and q are constants and r_0 is the unstretched length of the spring. Use Eq. (15.36) to show that $\mathbf{F}$ is conservative. (See Active Example 15.10.)

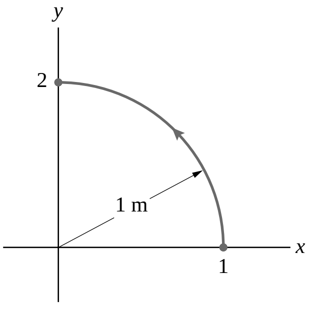
Solution: A necessary and sufficient condition that $\mathbf{F}$ be conservative is $\nabla \times \mathbf{F} = 0$.

$$\nabla \times \mathbf{F} = \frac{1}{r}\begin{bmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ -[k(r - r_0) + q(r - r_0)^3] & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{r}[0\mathbf{e}_r - 0r\mathbf{e}_\theta + 0\mathbf{e}_z] = 0.$$ $\mathbf{F}$ is conservative.

Problem 15.114 The potential energy associated with a force **F** acting on an object is $V = -r\sin\theta + r^2\cos^2\theta$ N-m, where r is in metre.

(a) Determine **F**.
(b) If the object moves from point 1 to point 2 along the circular path, how much work is done by **F**?



Solution: The force is

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta\right)(-r\sin\theta + r^2\cos^2\theta).$$

$$\boxed{\mathbf{F} = (\sin\theta - 2r\cos^2\theta)\mathbf{e}_r + (\cos\theta + 2r\sin\theta\cos\theta)\mathbf{e}_\theta}.$$

The work done is $U_{1,2} = \int_{1,2} \mathbf{F}\cdot d\mathbf{r}$,

where $d\mathbf{r} = \mathbf{e}_r dr + r\mathbf{e}_\theta d\theta$. Since the path is everywhere normal to $\mathbf{e}_r$, the radial term does not contribute to the work. The integral is

$$U_{1,2} = \int_0^{\frac{\pi}{2}} (\cos\theta + 2r\cos\theta\sin\theta)r\,d\theta = \left[r\sin\theta - r^2\cos^2\theta\right]_0^{\frac{\pi}{2}}$$

$$= 1 + 1 = 2 \text{ N-m}$$

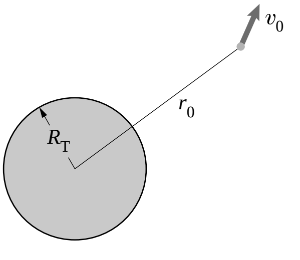
Check: Since the force is derivable from a potential, the system is conservative. In a conservative system the work done is $U_{1,2} = -(V_2 - V_1)$, where V_1, V_2 are the potentials at the beginning and end of the path. At $r = 1, \theta = 0, V_1 = 1$ N-m. At $r = 1$ m. $\theta = \frac{\pi}{2}$, $V_1 = -1$, from which $U_{1,2} = -(V_2 - V_1) = 2$ N-m. *check*.

Problem 15.115 In terms of polar coordinates, the force exerted on an object of mass m by the gravity of a hypothetical two-dimensional planet is

$$\mathbf{F} = -\left(\frac{mg_T R_T}{r}\right)\mathbf{e}_r,$$

where g_T is the acceleration due to gravity at the surface, R_T is the radius of the planet, and r is the distance from the center of the planet.

(a) Determine the potential energy associated with this gravitational force.
(b) If the object is given a velocity v_0 at a distance r_0, what is its velocity v as a function of r?



Solution:

(a) The potential is

$$V = -\int \mathbf{F}\cdot d\mathbf{r} + C = \int \left(\frac{mg_T R_T}{r}\right)\mathbf{e}_r\cdot(\mathbf{e}_r\,dr) + C$$

$$= mg_T R_T \ln(r) + C,$$

where C is the constant of integration. Choose $r = R_T$ as the datum, from which $C = -mg_T R_T \ln(R_T)$, and

$$\boxed{V = mg_T R_T \ln\left(\frac{r}{R_T}\right)}$$

(*Note*: Alternatively, the choice of $r = 1$ *length-unit* as the datum, from which $C = mg_M R_T \ln(1)$, yields $V = mg_T R_T \ln\left(\frac{r}{1}\right) = mg_T R_T \ln(r)$.)

(b) From conservation of energy,

$$\frac{1}{2}mv^2 + mg_T R_T \ln\left(\frac{r}{R_T}\right) = \frac{1}{2}mv_0^2 + mg_T R_T \ln\left(\frac{r_0}{R_T}\right).$$

Solve for the velocity

$$\boxed{v = \sqrt{v_0^2 + 2_{g_T}\ln\left(\frac{r_0}{r}\right)}}$$

Problem 15.116 By substituting Eqs. (15.27) into Eq. (15.30), confirm that $\nabla \times \mathbf{F} = 0$ if $\mathbf{F}$ is conservative.

Solution: Eq. 15.30 is

$$\boxed{\begin{aligned}\nabla \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{\partial V}{\partial x} & -\dfrac{\partial V}{\partial y} & -\dfrac{\partial V}{\partial z} \end{bmatrix} = \mathbf{i}\left(-\frac{\partial^2 V}{\partial y \partial z} + \frac{\partial^2 V}{\partial y \partial z}\right) \\ -\mathbf{j}\left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial x \partial z}\right) + \mathbf{k}\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial x \partial y}\right) = 0\end{aligned}}$$

Thus, $\mathbf{F}$ is conservative.

Problem 15.117 Determine which of the following are conservative.

(a) $\mathbf{F} = (3x^2 - 2xy)\mathbf{i} - x^2\mathbf{j}$;
(b) $\mathbf{F} = (x - xy^2)\mathbf{i} + x^2y\mathbf{j}$;
(c) $\mathbf{F} = (2xy^2 + y^3)\mathbf{i} + (2x^2y - 3xy^2)\mathbf{j}$.

Solution: Use Eq. (15.30)

(a) $$\nabla \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 - 2xy & -x^2 & 0 \end{bmatrix}$$

$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(-2x + 2x) = 0.$$

Force is conservative.

(b) $$\nabla \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x - xy^2 & x^2y & 0 \end{bmatrix}$$

$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(2xy + 2xy) = \mathbf{k}(4xy) \neq 0$$

Force is non-conservative.

(c) $$\nabla \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy^2 + y^3 & 2x^2y - 3xy^2 & 0 \end{bmatrix}$$

$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(4xy - 3y^2 - 4xy - 3y^2)$$

$$= \mathbf{k}(-6y^2) \neq 0.$$

Force is non-conservative.

Problem 15.118 The driver of a 12000 N car moving at 40 km/h applies an increasing force on the brake pedal. The magnitude of the resulting frictional force exerted on the car by the road is $f = 250 + 6s$ N, where s is the car's horizontal position (in feet) relative to its position when the brakes were applied. Assuming that the car's tires do not slip, determine the distance required for the car to stop

(a) by using Newton's second law and
(b) by using the principle of work and energy.

Solution:

(a) Newton's second law:

$$\left(\frac{W}{g}\right)\frac{dv}{dt} = -f,$$

where f is the force on the car in opposition to the motion. Use the chain rule:

$$\left(\frac{W}{g}\right)v\frac{dv}{ds} = -f = -(250 + 6s).$$

Integrate and rearrange:

$$v^2 = -\left(\frac{2g}{W}\right)(250s + 3s^2) + C.$$

At $s = 0$, $v(0) = 40 \times 1000/3600 = 11.1$ m/s,

from which $C = (11.1^2) = v_1^2$. The velocity is

$$v^2 = -\left(\frac{2g}{W}\right)(250s + 3s^2) + v_1^2 \text{ (m/s)}^2.$$

At $v = 0$, $s^2 + 2bs + c = 0$, where

$$b = \frac{125}{3} = 41.67, c = -\frac{Wv_1^2}{6g} = -25119.$$

The solution: $s = -b \pm \sqrt{b^2 - c} = 122.2$ m, from which $\boxed{s = 122.2 \text{ m}}$.

(b) Principle of work and energy: The energy of the car when the brakes are first applied is

$$\frac{1}{2}\left(\frac{W}{g}\right)v_1^2 = 6789 \text{ N-m}.$$

The work done is

$$U = \int_0^s f\,ds = -\int_0^s (250 + 6s)\,ds = -(250s + 3s^2).$$

From the principle of work and energy, after the brakes are applied,

$$U = \frac{1}{2}\left(\frac{W}{g}\right)v_2^2 - \frac{1}{2}\left(\frac{W}{g}\right)v_1^2.$$

Rearrange:

$$.\frac{1}{2}\left(\frac{W}{g}\right)v_2^2 = .\frac{1}{2}\left(\frac{W}{g}\right)v_1^2 - (250s + 3s^2).$$

When the car comes to a stop, $v_2 = 0$, from which

$$.0 = .\frac{1}{2}\left(\frac{W}{g}\right)v_1^2 - (250s + 3s^2).$$

Reduce: $s^2 + 2bs + c = 0$, where

$$b = \frac{125}{3} = 41.67, c = -\frac{Wv_1^2}{6g} = -25119.$$

The solution $s = -b \pm \sqrt{b^2 - c} = 122.2$ m, from which $\boxed{s = 122.2 \text{ m}}$.

Problem 15.119 Suppose that the car in Problem 15.118 is on wet pavement and the coefficients of friction between the tires and the road are $\mu_s = 0.4$ and $\mu_k = 0.35$. Determine the distance for the car to stop.

Solution: The initial velocity of the vehicle is $v_1 = 40$ km/h $=$ 11.1 m/s (a) Assume that the force $f = 250 + 6s$ lb applies until the tire slips. Slip occurs when $f = 250 + 6s = \mu_s W$, from which $s_{\text{slip}} =$ 758.3 m. The work done by the friction force is

$$U_f = \int_0^{s_{\text{slip}}} -f\,ds + \int_{s_{\text{slip}}}^{s_{\text{stop}}} -\mu_k W = -(250s_{\text{slip}} + 3s_{\text{slip}}{}^2)$$

$$-\mu_k W(s_{\text{stop}} - s_{\text{slip}}) = 1.27 \times 10^6 - 4200(s_{\text{stop}}).$$

From the principle of work and energy:

$$U_f = 0 - \left(\frac{1}{2}mv_1^2\right) = -6789 \text{ N-m},$$

from which $\boxed{s_{\text{stop}} = 300.8 \text{ m}}$.

Problem 15.120 An astronaut in a small rocket vehicle (combined mass = 450 kg) is hovering 100 m above the surface of the moon when he discovers that he is nearly out of fuel and can exert the thrust necessary to cause the vehicle to hover for only 5 more seconds. He quickly considers two strategies for getting to the surface:

(a) Fall 20 m, turn on the thrust for 5 s, and then fall the rest of the way;
(b) fall 40 m, turn on the thrust for 5 s, and then fall the rest of the way.

Which strategy gives him the best chance of surviving? How much work is done by the engine's thrust in each case? ($g_{\text{moon}} = 1.62$ m/s^2)

Solution: Assume $g = 1.62$ m/s^2 and that the fuel mass is negligible. Since the thruster causes the vehicle to hover, the thrust is $T = mg$. The potential energy at $h_1 = 100$ m is $V_1 = mgh$.

(a) Consider the first strategy: The energy condition at the end of a 20 m fall is $mgh = \frac{1}{2}mv_2^2 + mgh_2$, where $h_2 = h_1 - 20 = 80$ m, from which $\frac{1}{2}mv_2^2 = mg(h_1 - h_2)$, from which $v_2 = \sqrt{2g(h_1 - h_2)} = 8.05$ m/s. The work done by the thrust is

$$U_{\text{thrust}} = -\int_{h_2}^{h_3} F\,dh = -mg(h_3 - h_2),$$

where $F = mg$, acting upward, h_3 is the altitude at the end of the thrusting phase. The energy condition at the end of the thrusting phase is $mgh = \frac{1}{2}mv_3^2 + mgh_3 + U_{\text{thrust}}$, from which $mgh = \frac{1}{2}mv_3^2 + mgh_2$. It follows that the velocities $v_3 = v_2 = 8.05$ m/s, that is, the thruster does not reduce the velocity during the time of turn-on. The height at the end of the thruster phase is $h_3 = h_2 - v_3 t = 80 - (8.04)(5) = 39.75$ m. The energy condition at the beginning of the free fall after the thruster phase is $\frac{1}{2}mv_3^2 + mgh_3 = 43558.3$ N-m, which, by conservation of energy is also the energy at impact: is $\frac{1}{2}mv_4^2 = \frac{1}{2}mv_3^2 + mgh_3 = 43558.3$ N-m, from which

$$\boxed{v_4 = \sqrt{\frac{2(43558.3)}{m}} = 13.9 \text{ m/s at impact.}}$$

(b) Consider strategy (b): Use the solution above, with $h_2 = h_1 - 40 = 60$ m The velocity at the end of the free fall is $v_2 = \sqrt{2\text{ g}(h_1 - h_2)} = 11.38$ m/s. The velocity at the end of the thruster phase is $v_3 = v_2$. The height at the end of the thruster phase is $h_3 = h_2 - v_2 t = 3.08$ m. The energy condition at impact is: $\frac{1}{2}mv_4^2 = \frac{1}{2}mv_3^2 + mgh_3 = 31405$ N-m. The impact velocity is

$$\boxed{v_4 = \sqrt{\frac{2(31405)}{m}} = 11.8 \text{ m/s}}.$$

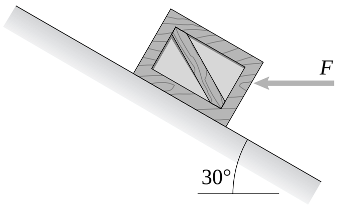
He should choose strategy (b) since the impact velocity is reduced by $\Delta v = 13.91 - 11.81 = 2.1$ m/s. The work done by the engine in strategy (a) is

$$U_{\text{thrust}} = \int_{h_3}^{h_3} F\,dh = mg(h_3 - h_2) = -29.3 \text{ kN-m}.$$

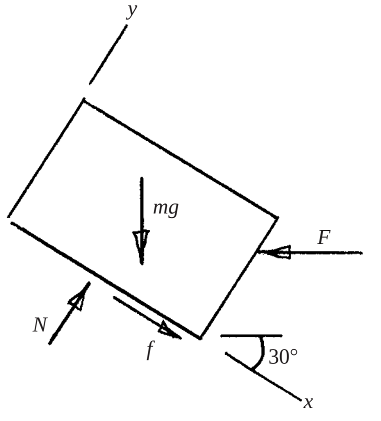
The work done by the engine in strategy (b) is

$$\boxed{U_{\text{thrust}} = \int_{h_2}^{h_3} F\,dh = mg(h_3 - h_2) = -41.5 \text{ kN-m}}$$

Problem 15.121 The coefficients of friction between the 20-kg crate and the inclined surface are $\mu_s = 0.24$ and $\mu_k = 0.22$. If the crate starts from rest and the horizontal force $F = 200$ N, what is the magnitude of the velocity of the crate when it has moved 2 m?



Solution:



$$\Sigma F_y = N - F\sin 30^\circ - mg\cos 30^\circ = 0,$$

so $N = F\sin 30^\circ + mg\cos 30^\circ = 270$ N.

The friction force necessary for equilibrium is

$$f = F\cos 30^\circ - mg\sin 30^\circ = 75.1 \text{ N}.$$

Since $\mu_s N = (0.24)(270) = 64.8$ N, the box will slip up the plane and $f = \mu_k N$. From work and energy,

$$(F\cos 30^\circ - mg\sin 30^\circ - \mu_k N)(2m) = \tfrac{1}{2}mv_2^2 - 0,$$

we obtain $v_2 = 1.77$ m/s.

Problem 15.122 The coefficients of friction between the 20-kg crate and the inclined surface are $\mu_s = 0.24$ and $\mu_k = 0.22$. If the crate starts from rest and the horizontal force $F = 40$ N. What is the magnitude of the velocity of the create when it has moved 2 m?

Solution: See the solution of Problem 15.121. The normal force is

$$N = F\sin 30^\circ + mg\cos 30^\circ = 190 \text{ N}.$$

The friction force necessary for equilibrium is

$$f = F\cos 30^\circ - mg\sin 30^\circ = -63.5 \text{ N}.$$

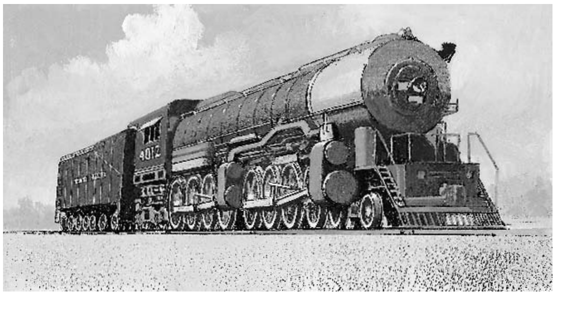
Since $\mu_s N = (0.24)(190) = 45.6$ N, the box will slip down the plane and the friction force is $\mu_k N$ up the plane.

From work and energy,

$$(mg\sin 30^\circ - F\cos 30^\circ - \mu_k N)(2m) = \tfrac{1}{2}mv_2^2 - 0,$$

we obtain $v_2 = 2.08$ m/s.

Problem 15.123 The Union Pacific Big Boy locomotive weighs 5.29 million lb, and the traction force (tangential force) of its drive wheels is 600 480 N. If you neglect other tangential forces, what distance is required for the train to accelerate from zero to 96.5 km/h?



Solution: The potential associated with the force is

$$V = -\int_0^s F\,ds = -Fs.$$

The energy at rest is zero. The energy at $v = 96.5$ km/h $= 26.8$ m/s is

$$0 = \frac{1}{2}\left(\frac{W}{g}\right)v^2 + V,$$

from which $\boxed{s = \frac{1}{2}\left(\frac{W}{gF}\right)v^2 = 323.4 \text{ m}}$

Problem 15.124 In Problem 15.123, suppose that the acceleration of the locomotive as it accelerates from zero to 96.5 km/h is $(F_0/m)\ (1 - v/88)$, where $F_0 =$ 600480 N, m is the mass of the locomotive, and v is its velocity in metre per second.

(a) How much work is done in accelerating the train to 96.5 km/h?
(b) Determine the locomotive's velocity as a function of time.

Solution: [Note: F is not a force, but an acceleration, with the dimensions of acceleration.]

(a) The work done by the force is equal to the energy acquired by the locomotive in attaining the final speed, in the absence of other tangential forces. Thus the work done by the traction force is

$$U = \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{W}{g}\right)(88^2) = 20.88 \times 10^8 \text{ N-m}$$

(b) From Newton's second law $m\dfrac{dv}{dt} = mF$, from which

$$\frac{dv}{dt} = \left(\frac{F_0}{m}\right)\left(1 - \frac{v}{88}\right).$$

Separate variables:

$$\frac{dv}{\left(1 - \dfrac{v}{88}\right)} = \left(\frac{F_0}{m}\right) dt.$$

Integrate:

$$\ln\left(1 - \frac{v}{88}\right) = -\left(\frac{F_0}{88m}\right) t + C_1.$$

Invert:

$$v(t) = 88\left(1 - Ce^{\frac{-F_0}{88m}t}\right).$$

At $t = 0$, $v(0) = 0$, from which $C = 1$. The result:

$$\boxed{v(t) = 88\left(1 - e^{\frac{-8F_0}{88W}t}\right)}$$

Check: To demonstrate that this is a correct expression, it is used to calculate the work done: Note that

$$U = \int_0^s mF\,ds = \int_0^T mF\left(\frac{ds}{dt}\right)dt = \int_0^T mFv\,dt.$$

For brevity write $K = \dfrac{gF_0}{88W}$.

Substitute the velocity into the force:

$$mF = F_0\left(1 - \frac{v}{88}\right) = F_0 e^{-Kt}.$$

The integral

$$U = \int_0^T mFv\,dt = \int_0^T 88F_0 e^{Kt}(1 - e^{-Kt})\,dt$$

$$U = 88F_0 \int_0^T (e^{-Kt} - e^{-2Kt})\,dt$$

$$= -\frac{88F_0}{K}\left[e^{-Kt} - \frac{1}{2}e^{-2Kt}\right]_0^T$$

$$= -\frac{88F_0}{K}\left[e^{-KT} - \frac{e^{-2KT}}{2} - \frac{1}{2}\right].$$

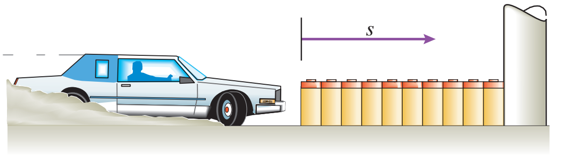
The expression for the velocity is asymptotic in time to the limiting value of 96.5 km/h: in strict terms the velocity never reaches 96.5 km/h; in practical terms the velocity approaches within a few tenths of percent of 96.5 km/h within the first few minutes. Take the limit of the above integral:

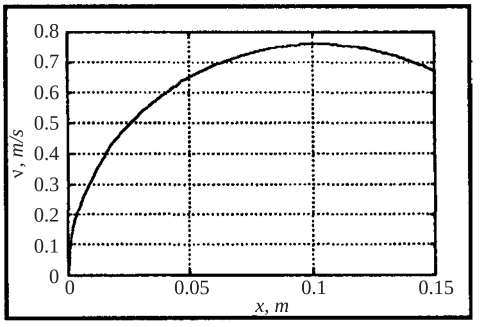
$$\lim_{T\to\infty}\int_0^T mFv\,dt = \lim_{T\to\infty} -\frac{88F_0}{K}\left[e^{-KT} - \frac{e^{-2KT}}{2} - \frac{1}{2}\right]$$

$$= \frac{88F_0}{2K} = \frac{1}{2}\frac{W}{g}(88^2) \equiv \text{kinetic energy},$$

which checks, and confirms the expression for the velocity. *check*.

Problem 15.125 A car traveling 104.6 km/h hits the crash barrier described in Problem 15.14. Determine the maximum deceleration to which the passengers are subjected if the car weighs (a) 11120 N and (b) 22240 N.





Solution: From Problem 15.14 we know that the force in the crash barrier is given by

$$F = -(120s + 40s^3)\ \text{N}.$$

The maximum deceleration occurs when the spring reaches its maximum deflection. Using work and energy we have

$$\frac{1}{2}mv^2 + \int_0^s F\,ds = 0$$

$$\frac{1}{2}mv^2 - \int_0^s (120s + 40s^3)\,ds = 0$$

$$\frac{1}{2}mv^2 = 60s^2 + 10s^4$$

This yields an equation that we can solve for the distance s at which the car stops.

(a) Using $m = \dfrac{11120\ \text{N}}{9.81\ \text{m/s}^2}$ and solving, we find that

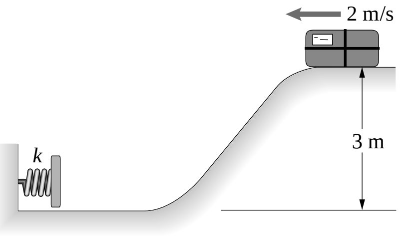
$$s = 14.68\ \text{m},\ a = \frac{F}{m} = \frac{120s + 40s^3}{m} = 113.2\ \text{m/s}^2$$

(b) Using $m = \dfrac{5000\ \text{N}}{9.81\ \text{m/s}^2}$ and solving, we find that

$$s = 12\ \text{m},\ a = \frac{F}{m} = \frac{120s + 40s^3}{m} = 138.4\ \text{m/s}^2$$

$$\boxed{\text{(a) } 113.2\ \text{m/s}^2,\ \text{(b) } 138.4\ \text{m/s}^2.}$$

Problem 15.126 In a preliminary design for a mail-sorting machine, parcels moving at 2 m/s slide down a smooth ramp and are brought to rest by a linear spring. What should the spring constant be if you don't want the 10-N parcel to be subjected to a maximum deceleration greater than $10g$'s?



Solution: From Newton's second law, the acceleration after contact with the spring is given by:

$$\frac{W}{g}\left(\frac{dv}{dt}\right) = -F = -kS,$$

where k is the spring constant and S is the stretch of the spring. Rearrange:

$$\left(\frac{dv}{dt}\right) = -\frac{gk}{W}S.$$

This expression has two unknowns, k and S. S is determined as follows: Choose the bottom of the ramp as the datum. The energy at the top of the ramp is

$$\frac{1}{2}\left(\frac{W}{g}\right)v^2 + V,$$

where V is the potential energy of the package due to gravity: $V = Wh$ where $h = 3$ m. The conservation of energy condition after contact with the spring is

$$\frac{1}{2}\left(\frac{W}{g}\right)v_0^2 + Wh = \frac{1}{2}\left(\frac{W}{g}\right)v_1^2 + \frac{1}{2}kS^2.$$

When the spring is fully compressed the velocity is zero, and

$$S = \sqrt{\frac{W}{gk}v_0^2 + 2\left(\frac{W}{k}\right)h}.$$

Substitute into the expression for the acceleration:

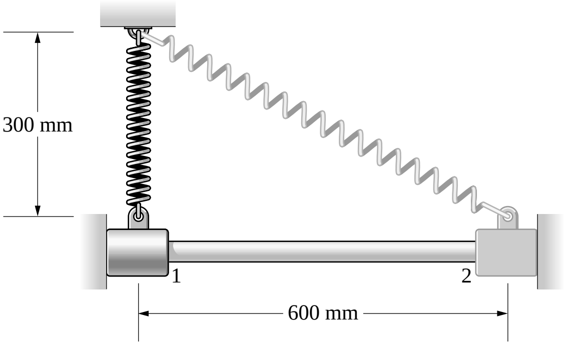
$$\left(\frac{dv}{dt}\right) = -\sqrt{k}\sqrt{\frac{gv_0^2}{W} + \frac{2g^2h}{W}}$$

(where the negative sign appears because $\frac{dv}{dt} = -10$ g), from which

$$k = \frac{\left(\frac{dv}{dt}\right)^2}{\left(\frac{gv_0^2}{W} + \frac{2g^2h}{W}\right)}.$$

Substitute numerical values: $v_0 = 2$ m/s, $W = 10$ N, $h = 3$ m, $\left(\frac{dv}{dt}\right) = -10$ g m/s^2, from which $\boxed{k = 156.1 \text{ N/m}}$

Problem 15.127 When the 1-kg collar is in position 1, the tension in the spring is 50 N, and the unstretched length of the spring is 260 mm. If the collar is pulled to position 2 and released from rest, what is its velocity when it returns to position 1?

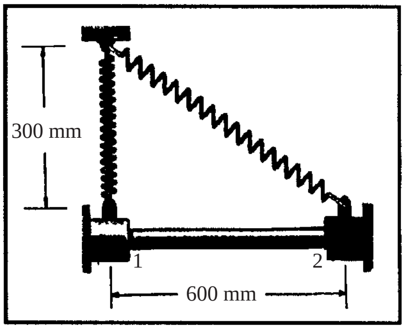


Solution: The stretched length of the spring in position 1 is $S_1 = 0.3 - 0.26 = 0.04$ m. The stretched length of the spring in position 2 is $S_2 = \sqrt{0.3^2 + 0.6^2} - 0.26 = 0.411$ m. The spring constant is

$$k = \frac{50}{S_1} = 1250 \text{ N/m}.$$

The potential energy of the spring in position 2 is $\frac{1}{2}kS_2^2$. The potential energy of the spring in position 1 is $\frac{1}{2}kS_1^2$. The energy in the collar at position 1 is $\frac{1}{2}kS_2^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}kS_1^2$, from which

$$v_1 = \sqrt{\frac{k}{m}(S_2^2 - S_1^2)} = 14.46 \text{ m/s}$$



Problem 15.128 When the 1-kg collar is in position 1, the tension in the spring is 100 N, and when the collar is in position 2, the tension in the spring is 400 N.

(a) What is the spring constant k?
(b) If the collar is given a velocity of 15 m/s at position 1, what is the magnitude of its velocity just before it reaches position 2?

Solution:

(a) Assume that the dimensions defining locations 1 and 2 remain the same, and that the unstretched length of the spring changes from that given in Problem 15.141. The stretched length of the spring in position 1 is $S_1 = 0.3 - S_0$, and in position 2 is $S_2 = \sqrt{0.3^2 + 0.6^2} - S_0$. The two conditions:

$$\sqrt{0.6^2 + 0.3^2} - S_0 = \frac{400}{k}, 0.3 - S_0 = \frac{100}{k}.$$

Subtract the second from the first, from which $k = 809$ N/m. Substitute and solve: $S_0 = 0.176$ m, and $S_1 = 0.124$ m, $S_2 = 0.494$ m.

(b) The energy at the onset of motion at position 1 is

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kS_1^2.$$

At position 2:

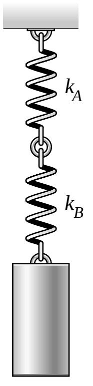
$$\frac{1}{2}mv_1^2 + \frac{1}{2}kS_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kS_2^2,$$

from which

$$\boxed{v_2 = \sqrt{v_1^2 + \frac{k}{m}(S_1^2 - S_2^2)} = 6.29 \text{ m/s}}$$

Problem 15.129 The 30-N weight is released from rest with the two springs ($k_A = 30$ N/m, $k_B = 15$ N/m) unstretched.

(a) How far does the weight fall before rebounding?
(b) What maximum velocity does it attain?



Solution: Choose the datum as the initial position.

(a) The work done as the weight falls is: for the springs

$$U_{\text{spring}} = \int_0^{-S_A} k_A s\, ds + \int_0^{-S_B} k_B s\, ds = -\frac{1}{2}k_A S_A^2 - \frac{1}{2}k_B S_B^2.$$

For the weight

$$U_{\text{weight}} = \int_0^{-(S_A+S_B)} - W\, ds = W(S_A + S_B).$$

From the principle of work and energy: $U_{\text{springs}} + U_{\text{weight}} = (mv^2/2)$. At the juncture of the two springs the sum of the forces is $k_A S_A - k_B S_B = 0$, from which $S_B = \dfrac{k_A}{k_B} S_A$, from which

$$-\left(\frac{1}{2}\right) k_A S_A^2 \left(1 + \frac{k_A}{k_B}\right) + W S_A \left(1 + \frac{k_A}{k_B}\right) = \left(\frac{1}{2}mv^2\right)$$

At the maximum extension the velocity is zero, from which

$$S_A = \frac{2W}{k_A} = 2 \text{ m}, \; S_B = \left(\frac{k_A}{k_B}\right) s_A = 4 \text{ m}.$$

The total fall of the weight is $S_A + S_B = 6$ m

(b) The maximum velocity occurs at

$$\frac{d}{dS_A}\left(\frac{1}{2}mv^2\right) = \frac{d}{dS_A}(U_{\text{spring}} + U_{\text{weight}})$$

$$= -k_A S_A \left(1 + \frac{k_A}{k_B}\right) + W\left(1 + \frac{k_A}{k_B}\right) = 0,$$

from which

$$[S_A]_{v\,\max} = \frac{W}{k_A} = 1 \text{ m}.$$

The maximum velocity is

$$|v_{\max}| = \left[\sqrt{\frac{2(U_{\text{spring}} + U_{\text{weight}})}{m}}\right]_{s_A=1} = 9.82 \text{ m/s}$$

Check: Replace the two springs with an equivalent spring of stretch $S = S_A + S_B$, with spring constant k_{eq}, from which

$$S = \frac{F}{k_A} + \frac{F}{k_B} = \frac{F}{k_{eq}}$$

from which

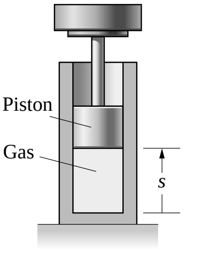
$$k_{eq} = \frac{F}{S} = \frac{F}{S_A + S_B} = \frac{F}{\dfrac{F}{k_A} + \dfrac{F}{k_A}} = \frac{k_A + k_B}{k_A k_B} = 10 \text{ N/m}.$$

From conservation of energy $0 = mv^2/2 + k_{eq}S^2/2 - WS$. Set $v = 0$ and solve: $S = 2W/k_{eq} = 6$ m is the maximum stretch. *check.* The velocity is a maximum when

$$\frac{d}{dS}\left(\frac{1}{2}mv^2\right) = W - k_{eq}S = 0,$$

from which $[S]_{v=mv_{\max}} = 3$ m, and the maximum velocity is $v = 9.82$ m/s. *check.*

Problem 15.130 The piston and the load it supports are accelerated upward by the gas in the cylinder. The total weight of the piston and load is 1000 N. The cylinder wall exerts a constant 50-N frictional force on the piston as it rises. The net force exerted on the piston by pressure is $(p_2 - p_{\text{atm}})A$, where p is the pressure of the gas, $p_{\text{atm}} = 2117\ \text{N/m}^2$ is the atmospheric pressure, and $A = 1\ \text{m}^2$ is the cross-sectional area of the piston. Assume that the product of p and the volume of the cylinder is constant. When $s = 1$ m, the piston is stationary and $p = 5000\ \text{N/m}^2$. What is the velocity of the piston when $s = 2$ m?



Solution: At the rest position, $p_0 A s = p_0 V = K$, where $V = 1\ \text{ft}^3$, from which $K = p_0$. Denote the datum: $s_0 = 1$ ft. The potential energy of the piston due to the gas pressure after motion begins is

$$V_{\text{gas}} = -\int_{s_0}^{s} F\,ds = -\int_{s_0}^{s} (p - p_{\text{atm}})A\,ds$$

$$= p_{\text{atm}} A(s - s_0) - \int_{s_0}^{s} pA\,ds.$$

From which

$$V_{\text{gas}} = p_{\text{atm}} A(s - s_0) - K\int_{s_0}^{s} \frac{ds}{s} = p_{\text{atm}} A(s - s_0) - K\ln\left(\frac{s}{s_0}\right).$$

The potential energy due to gravity is

$$V_{\text{gravity}} = -\int_{s_0}^{s} (-W)\,ds = W(s - s_0).$$

The work done by the friction is

$$U_{\text{friction}} = \int_{s_0}^{s} (-f)\,ds = -f(s - s_0), \text{ where } f = 50\ \text{N}.$$

From the principle of work and energy:

$$U_{\text{friction}} = \frac{1}{2}\left(\frac{W}{g}\right)v^2 + V_{\text{gas}} + V_{\text{gravity}}$$

Rearrange:

$$\frac{1}{2}\left(\frac{W}{g}\right)v^2 = U_{\text{friction}} - V_{\text{gas}} - V_{\text{gravity}}. \text{ At } s = 2\ \text{m},$$

$$\frac{1}{2}\left(\frac{W}{g}\right)v^2 = -(-1348.7) - (1000) - 50 = 298.7\ \text{N-m},$$

from which $v = \sqrt{\dfrac{2(298.7)g}{W}} = 2.42\ \text{m/s}$

Problem 15.131 When a 22,000-kg rocket's engine burns out at an altitude of 2 km, the velocity of the rocket is 3 km/s and it is traveling at an angle of 60° relative to the horizontal. Neglect the variation in the gravitational force with altitude.

(a) If you neglect aerodynamic forces, what is the magnitude of the velocity of the rocket when it reaches an altitude of 6 km?
(b) If the actual velocity of the rocket when it reaches an altitude of 6 km is 2.8 km/s, how much work is done by aerodynamic forces as the rocket moves from 2 km to 6 km altitude?

Solution: Choose the datum to be 2 km altitude.

(a) The energy is $\frac{1}{2}mv_0^2$ at the datum. The energy condition of the rocket when it reaches 6 km is $\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgh$, where $h = (6-2)\times 10^3 = 4\times 10^3$ m. Rearrange the energy expression: $v^2 = v_0^2 - 2gh$, from which the velocity at 6 km is $v = \sqrt{v_0^2 - 2gh} = 2.987$ km/s

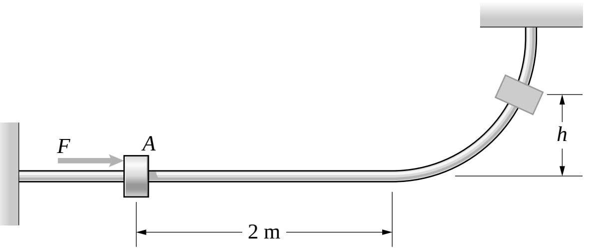
(b) Define U_{aero} to be the work done by the aerodynamic forces. The energy condition at 6 km is

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 + mgh - U_{\text{aero}}.$$

Rearrange:

$$U_{\text{aero}} = +\tfrac{1}{2}mv^2 + mgh - \tfrac{1}{2}mv_0^2 = -1.19\times 10^{10}\ \text{N-m}$$

Problem 15.132 The 12-kg collar A is at rest in the position shown at $t = 0$ and is subjected to the tangential force $F = 24 - 12t^2$ N for 1.5 s. Neglecting friction, what maximum height h does the collar reach?



Solution: Choose the datum at the initial point. The strategy is to determine the velocity at the end of the 1.5 s and then to use work and energy methods to find the height h. From Newton's second law:

$$m\frac{dv}{dt} = F = 24 - 12t^2.$$

Integrating:

$$v = \frac{1}{m}\int_0^{1.5}(24 - 12t^2)\,dt = \left(\frac{1}{m}\right)[24t - 4t^3]_0^{1.5} = 1.875 \text{ m/s}.$$

[*Note*: The displacement during this time must not exceed 2 m. Integrate the velocity:

$$s = \left(\frac{1}{m}\right)\int_0^{1.5}(24t - 4t^3)\,dt$$

$$= \left(\frac{1}{m}\right)[12t^2 - t^4]_0^{1.5} = 1.82 \text{ m} < 2 \text{ m},$$

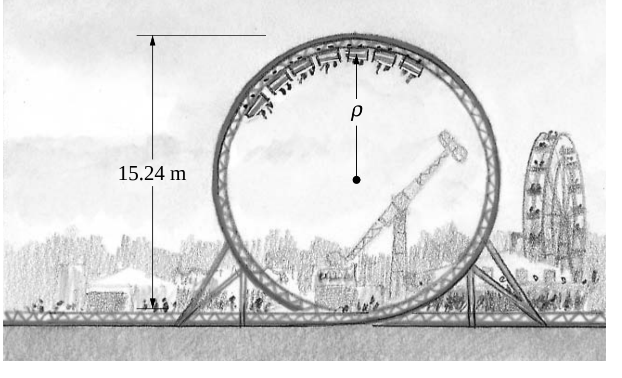
so the collar is still at the datum level at the end of 1.5 s.] The energy condition as the collar moves up the bar is

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 + mgh.$$

At the maximum height h, the velocity is zero, from which

$$h = \frac{v_0^2}{2g} = 0.179 \text{ m}$$

Problem 15.133 Suppose that, in designing a loop for a roller coaster's track, you establish as a safety criterion that at the top of the loop the normal force exerted on a passenger by the roller coaster should equal 10 percent of the passenger weight. (That is, the passenger's "effective weight" pressing him down into his seat is 10 percent of his actual weight.) The roller coaster is moving at 18.9 m/s when it enters the loop. What is the necessary instantaneous radius of curvature ρ of the track at the top of the loop?



Solution: The energy at the top of the loop is

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv_{\text{top}}^2 + mgh,$$

where $v_0 = 18.9$ m/s, $h = 15.24$ m, and $g = 9.81\,\text{m/s}^2$, from which $v_{\text{top}} = \sqrt{v_0^2 - 2gh} = 7.62$ m/s. From Newton's second law:

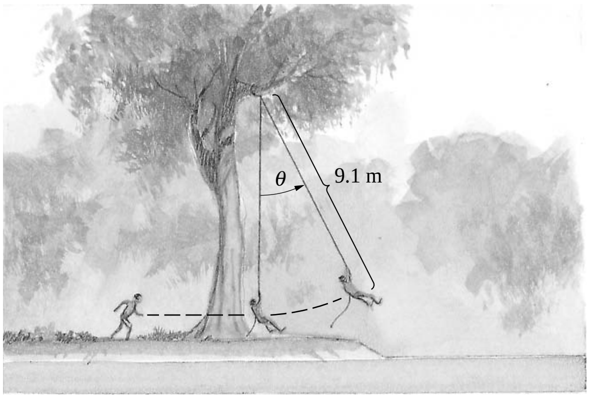
$$m\left(\frac{v_{\text{top}}^2}{\rho}\right) = (1.1)\text{ mg},$$

from which

$$\rho = \frac{v_{\text{top}}^2}{1.1\text{ g}} = 5.39 \text{ m}$$

Problem 15.134 A 800.6 N student runs at 4.57 m/s, grabs a rope, and swings out over a lake. He releases the rope when his velocity is zero.

(a) What is the angle θ when he releases the rope?
(b) What is the tension in the rope just before he release it?
(c) What is the maximum tension in the rope?



Solution:

(a) The energy condition after the seizure of the rope is

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}mv^2 + mgL(1 - \cos\theta),$$

where $v_0 = 4.57$ m/s, $L = 9.1$ m. When the velocity is zero, $v_0^2 = 2gL(1 - \cos\theta)$, from which

$$\cos\theta = 1 - \frac{v_0^2}{2gL} = 0.883, \theta = 27.9^\circ$$

(b) From the energy equation $v^2 = v_0^2 - 2gL(1 - \cos\theta)$. From Newton's second law, $(W/g)(v^2/L) = T - W\cos\theta$, from which

$$T = \left(\frac{W}{g}\right)\left(\frac{v^2}{L}\right) + W\cos\theta = 707.2 \text{ N}.$$

(c) The maximum tension occurs at the angle for which

$$\frac{dT}{d\theta} = 0 = -2W\sin\theta - W\sin\theta,$$

from which $\theta = 0$, from which

$$T_{\max} = W\left(\frac{v_0^2}{gL} + 1\right) = 987.5 \text{ N}$$

Problem 15.135 If the student in Problem 15.134 releases the rope when $\theta = 25^\circ$, what maximum height does he reach relative to his position when he grabs the rope?

Solution: Use the solution to Problem 15.134. [The height when he releases the rope is $h_1 = L(1 - \cos 25^\circ) = 0.856$ m.] Before he releases the rope, the total energy is

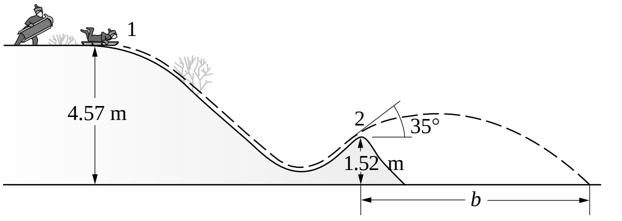
$$\frac{1}{2}\left(\frac{W}{g}\right)v_0^2 - WL = \frac{1}{2}\left(\frac{W}{g}\right)v^2 - WL\cos\theta.$$

Substitute $v_0 = 4.57$ m/s, $\theta = 25^\circ$ and solve: $v = 2.02$ m/s. The horizontal component of velocity is $v\cos\theta = 1.83$ m/s. From conservation of energy:

$$W(0.856) + \tfrac{1}{2}m(2.02^2) = Wh + \tfrac{1}{2}m(1.83^2)$$

from which $h = 0.893$ m.

Problem 15.136 A boy takes a running start and jumps on his sled at position 1. He leaves the ground at position 2 and lands in deep snow at a distance of $b = 7.62$ m. How fast was he going at 1?



Solution: The components of velocity at the point of leaving the ground are $v_y = v_2 \sin\theta$ and $v_x = v_2 \cos\theta$, where $\theta = 35^\circ$. The path is

$$y = -\frac{8}{2}t^2 + (v_2 \sin\theta)t + h,$$

where $h = 1.52$ m, and $x = (v_2 \cos\theta)t$. At impact $y = 0$, from which $t_{\text{impact}}^2 + 2bt_{\text{impact}} + c = 0$, where $b = \frac{v_2 \sin\theta}{g}$ [not to be confused with the b in the drawing], $c = -\frac{2h}{g}$. From which, since the time is positive, the time of impact is

$$(1)\ t_{\text{impact}} = \frac{v_2 \sin\theta}{g}\left(1 + \sqrt{1 + \frac{2gh}{v_2^2 \sin^2\theta}}\right).$$

The range is (2) $x(v_1) = b = (v_2 \cos\theta)t_{\text{impact}}$.

The velocity v_2 is found in terms of the initial velocity from the energy conditions: Choose the datum at the point where he leaves the ground. The energy after motion begins but before descent is under way is $\frac{1}{2}mv_1^2 + mgh_1$, where h_1 is the height above the point where he leaves the ground, $h_1 = 4.57 - 1.52 = 3.05$ m. The energy as he leaves the ground is $\frac{1}{2}mv_1^2 + mgh_1 = \frac{1}{2}mv_2^2$, from which (3) $v_2 = \sqrt{v_1^2 + 2gh_1}$. The function $x(v_1) = (v_2 \cos\theta)t_{\text{impact}} - b$, where $b = 7.62$ m, was graphed as a function of initial velocity, using the equations (2) and (3) above, to find the zero crossing. The value was refined by iteration to yield $v_1 = 1.44$ m/s. The other values were $v_2 = 7.86$ m/s, and the time in the air before impact was $t_{\text{impact}} = 1.182$ s. *Check*: An analytical solution is found as follows: Combine (1) and (2)

$$b = \frac{v_2^2 \sin\theta \cos\theta}{g}\left(1 + \sqrt{1 + \frac{2gh}{v_2^2 \sin^2\theta}}\right).$$

Invert this algebraically to obtain

$$v_2 = b\sqrt{\frac{g}{2\cos\theta(b \sin\theta + h \sin\theta)}} = 7.86 \text{ m/s}.$$

Use $v_1^2 = v_2^2 = v_2^2 - 2g(h_1 - h_2)$, from which $v_1 = 1.44$ m/s. *check.*

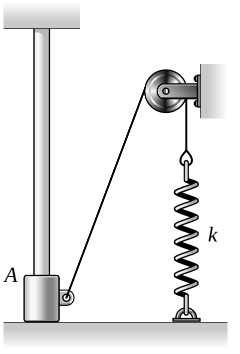
Problem 15.137 In Problem 15.136, if the boy starts at 1 going 4.57 m/s, what distance b does he travel through the air?

Solution: Use the solution to Problem 15.136. The distance $b = (v_2 \cos\theta)t_{\text{impact}}$, where

$$t_{\text{impact}} = \frac{v_2 \sin\theta}{g}\left(1 + \sqrt{1 + \frac{2gh}{v_2^2 \sin^2\theta}}\right),$$

and $v_2 = \sqrt{v_1^2 + 2gh_1}$. Numerical values are: $h = 1.52$ m, $\theta = 35^\circ$, $h_1 = 3.05$ m, $v_1 = 4.57$ m/s. Substituting, $b = 9.51$ m.

Problem 15.138 The 1-kg collar A is attached to the linear spring ($k = 500$ N/m) by a string. The collar starts from rest in the position shown, and the initial tension in the spring is 100 N. What distance does the collar slide up the smooth bar?



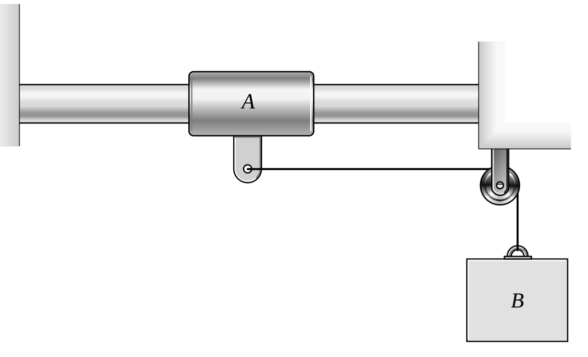
Solution: The deflection of the spring is

$$S = \frac{100}{k} = 0.2 \text{ m}.$$

The potential energy of the spring is $V_{\text{spring}} = \frac{1}{2}kS^2$. The energy condition after the collar starts sliding is $V_{\text{spring}} = \frac{1}{2}mv^2 + mgh$. At the maximum height, the velocity is zero, from which

$$h = \frac{V_{\text{spring}}}{mg} = \frac{k}{2mg}S^2 = 1.02 \text{ m}$$

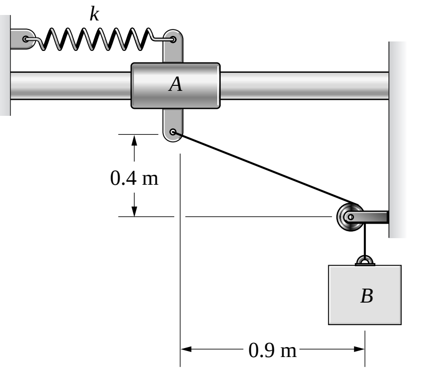
Problem 15.139 The masses $m_A = 40$ kg and $m_B = 60$ kg. The collar A slides on the smooth horizontal bar. The system is released from rest. Use conservation of energy to determine the velocity of the collar A when it has moved 0.5 m to the right.



Solution: Placing the datum for B at its initial position, conservation of energy gives $T_1 + V_1 = T_2 + V_2$: Evaluating, we get

$$0 + 0 = \frac{1}{2}(40)v^2 + \frac{1}{2}(60)v^2 - (60)(9.81)(0.5) \text{ or } v = 2.43 \text{ m/s}.$$

Problem 15.140 The spring constant is $k = 850$ N/m, $m_A = 40$ kg, and $m_B = 60$ kg. The collar A slides on the smooth horizontal bar. The system is released from rest in the position shown with the spring unstretched. Use conservation of energy to determine the velocity of the collar A when it has moved 0.5 m to the right.



Solution: Let v_A and v_B be the velocities of A and B when A has moved 0.5 m. The component of $A's$ velocity parallel to the cable equals $B's$ velocity: $v_A \cos 45^\circ = v_B$. B$'s$ downward displacement during $A's$ motion is

$$\sqrt{(0.4)^2 + (0.9)^2} - \sqrt{(0.4)^2 + (0.4)^2} = 0.419 \text{ m.}$$

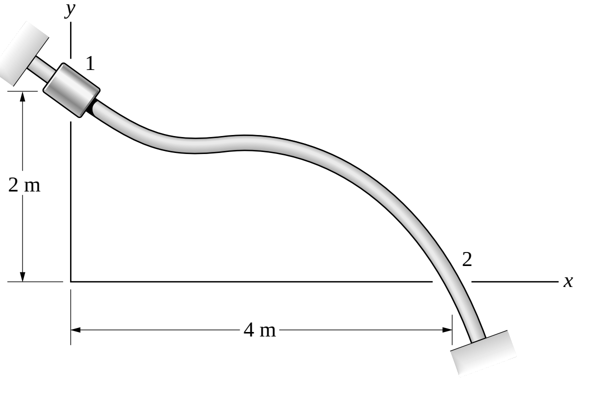
Conservation of energy is $T_1 + V_2 = T_2 + V_2$:

$$0 + 0 = \tfrac{1}{2}(40)v_A{}^2 + \tfrac{1}{2}(60)(v_A \cos 45^\circ)^2$$

$$+ \tfrac{1}{2}(850)(0.5)^2 - (60)(9.81)(0.419).$$

Solving, $v_A = 2.00$ m/s.

Problem 15.141 The y axis is vertical and the curved bar is smooth. If the magnitude of the velocity of the 4-N slider is 6 m/s at position 1, what is the magnitude of its velocity when it reaches position 2?



Solution: Choose the datum at position 2. At position 2, the energy condition is

$$\frac{1}{2}\left(\frac{W}{g}\right)v_1^2 + Wh = \frac{1}{2}\left(\frac{W}{g}\right)v_2^2,$$

where $h = 2$, from which

$$v_2 = \sqrt{v_1^2 + 2\,gh} = \sqrt{6^2 + 2\,g(2)} = 8.67 \text{ m/s}$$

Problem 15.142 In Problem 15.141, determine the magnitude of the velocity of the slider when it reaches position 2 if it is subjected to the additional force $\mathbf{F} = 3x\mathbf{i} - 2\mathbf{j}$ (N) during its motion.

Solution:

$$U = \int \mathbf{F} \cdot d\mathbf{r} = \int_2^0 (-2)\,dy + \int_0^4 3x\,dx$$

$$= [-2y]_2^0 + \left[\frac{3}{2}x^2\right]_0^4 = 4 + 24 = 28 \text{ N-m}.$$

From the solution to Problem 15.141, the energy condition at position 2 is

$$\frac{1}{2}\left(\frac{W}{g}\right)v_1^2 + Wh + U = \frac{1}{2}\left(\frac{W}{g}\right)v_2^2,$$

from which

$$v_2 = \sqrt{v_1^2 + 2gh + \frac{2g(28)}{W}} = \sqrt{6^2 + 2g(2) + \frac{2g(28)}{4}}$$

$$= 14.24 \text{ m/s}$$

Problem 15.143 Suppose that an object of mass m is beneath the surface of the earth. In terms of a polar coordinate system with its origin at the earth's center, the gravitational force on the object is $-(mgr/R_E)\mathbf{e}_r$, where R_E is the radius of the earth. Show that the potential energy associated with the gravitational force is $V = mgr^2/2R_E$.

Solution: By definition, the potential associated with a force $\mathbf{F}$ is

$$V = -\int \mathbf{F} \cdot d\mathbf{r}.$$

If $d\mathbf{r} = \mathbf{e}_r\,dr + r\mathbf{e}_\theta\,d\theta$, then

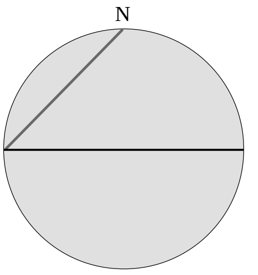
$$V = -\int \left(-\frac{\mathrm{mgr}}{\mathrm{R_E}}\right)\mathbf{e}_r \cdot \mathbf{e}_r\,dr - \int \left(-\frac{\mathrm{mgr}}{\mathrm{R_E}}\right)\mathbf{e}_r \cdot \mathbf{e}_\theta r\,d\theta$$

$$= -\int \left(-\frac{\mathrm{mgr}}{\mathrm{R_E}}\right)dr = \left(\frac{mgr^2}{2R_E}\right)$$

Problem 15.144 It has been pointed out that if tunnels could be drilled straight through the earth between points on the surface, trains could travel between these points using gravitational force for acceleration and deceleration. (The effects of friction and aerodynamic drag could be minimized by evacuating the tunnels and using magnetically levitated trains.) Suppose that such a train travels from the North Pole to a point on the equator. Determine the magnitude of the velocity of the train

(a) when it arrives at the equator and
(b) when it is halfway from the North Pole to the equator. The radius of the earth is $R_E = 6372$ km.

(See Problem 15.143.)



Solution: The potential associated with gravity is

$$V_{\text{gravity}} = \frac{mgr^2}{2R_E}.$$

With an initial velocity at the North Pole of zero, from conservation of energy, at any point in the path

$$\left(\frac{mgr^2}{2R_E}\right)_{NP} = \frac{1}{2}mv^2 + \left(\frac{mgr^2}{2R_E}\right).$$

(a) At the equator, the conservation of energy condition reduces to

$$\left(\frac{mgR_E}{2}\right) = \frac{1}{2}mv_{EQ}^2 + \left(\frac{mgR_E}{2}\right),$$

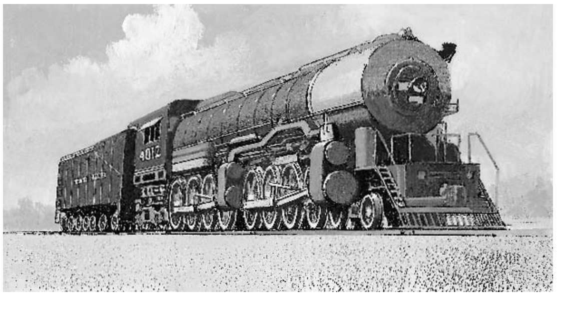
from which $v_{EQ} = 0$

(b) At the midway point, $r = R_E \sin 45^\circ = \dfrac{R_E}{\sqrt{2}}$, and from conservation of energy

$$\left(\frac{mgR_E}{2}\right) = \frac{1}{2}mv_M^2 + \left(\frac{mgR_E}{4}\right),$$

from which $v_M = \sqrt{\dfrac{gR_E}{2}} = 5590.6$ m/s $= 20126$ km/h.

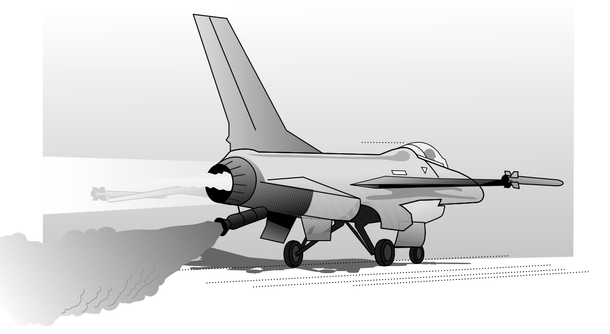
Problem 15.145 In Problem 15.123, what is the maximum power transferred to the locomotive during its acceleration?



Solution: From Problem 15.123, the drive wheel traction force $F = 135{,}000$ lb is a constant, and the final velocity is $v = 60$ mi/h $=$ 88 ft/s. The power transferred is $P = Fv$, and since the force is a constant, *by inspection* the maximum power transfer occurs at the maximum velocity, from which $P = Fv = (135000)(88) = 11.88 \times 10^6$ ft-lb/sec $= 21{,}600$ hp.

Problem 15.146 Just before it lifts off, the 10,500-kg airplane is traveling at 60 m/s. The total horizontal force exerted by the plane's engines is 189 kN, and the plane is accelerating at 15 m/s^2.

(a) How much power is being transferred to the plane by its engines?
(b) What is the total power being transferred to the plane?



Solution:

(a) The power being transferred by its engines is

$$P = Fv = (189 \times 10^3)(60) = 1.134 \times 10^7 \text{ Joule/s} = 11.3 \text{ MW}.$$

(b) Part of the thrust of the engines is accelerating the airplane: From Newton's second law,

$$m\frac{dv}{dt} = T = (10.5 \times 10^3)(15) = 157.5 \text{ kN}.$$

The difference $(189 - 157.5) = 31.5$ kN is being exerted to overcome friction and aerodynamic losses.

(b) The total power being transferred to the plane is

$$P_t = (157.5 \times 10^3)(60) = 9.45 \text{ MW}$$

Problem 15.147 The "Paris Gun" used by Germany in World War I had a range of 120 km, a 37.5-m barrel, a muzzle velocity of 1550 m/s and fired a 120-kg shell.

(a) If you assume the shell's acceleration to be constant, what maximum power was transferred to the shell as it traveled along the barrel?
(b) What average power was transferred to the shell?

Solution: From Newton's second law, $m\dfrac{dv}{dt} = F$, from which, for a constant acceleration,

$$v = \left(\frac{F}{m}\right)t + C.$$

At $t = 0$, $v = 0$, from which $C = 0$. The position is

$$s = \frac{F}{2m}t^2 + C.$$

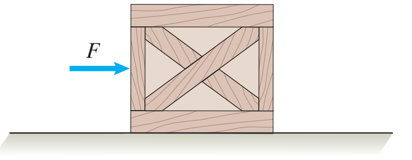
At $t = 0$, $s = 0$, from which $C = 0$. At $s = 37.5$ m, $v = 1550$ m/s, from which $F = 3.844 \times 10^6$ N and $t = 4.84 \times 10^{-2}$ s is the time spent in the barrel.

The power is $P = Fv$, and since F is a constant and v varies monotonically with time, the maximum power transfer occurs just before the muzzle exit: $P = F(1550) = 5.96 \times 10^9$ joule/s = 5.96 GW. (b) From Eq. (15.18) the average power transfer is

$$P_{\text{ave}} = \frac{\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}{t} = 2.98 \times 10^9 \text{ W} = 2.98 \text{ GW}$$

Problem 16.1 The 20-kg crate is stationary at time $t = 0$. It is subjected to a horizontal force given as a function of time (in newtons) by $F = 10 + 2t^2$.

(a) Determine the magnitude of the linear impulse exerted on the crate from $t = 0$ to $t = 4$ s.
(b) Use the principle of impulse and momentum to determine how fast the crate is moving at $t = 4$ s.



Solution:

(a) The impulse

$$I = \int_{t_0}^{t_1} F\,dt = \int_0^4 (10 + 2t^2)\,dt = 10(4) + \frac{2}{3}(4)^3 = 82.7 \text{ N-s}$$

$\boxed{I = 82.7 \text{ N-s.}}$

(b) Use the principle of impulse and momentum

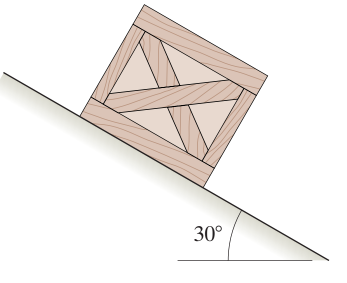
$mv_0 + I = mv_1$

$$0 + 82.7 \text{ N-s} = (20 \text{ kg})v \Rightarrow v = \frac{82.7 \text{ N-s}}{20 \text{ kg}}$$

$\boxed{v = 4.13 \text{ m/s.}}$

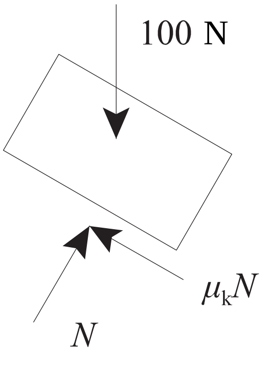
Problem 16.2 The 100-N crate is released from rest on the inclined surface at time $t = 0$. The coefficient of kinetic friction between the crate and the surface is $\mu_k = 0.18$.

(a) Determine the magnitude of the linear impulse due to the forces acting on the crate from $t = 0$ to $t = 2$ s.
(b) Use the principle of impulse and momentum to determine how fast the crate is moving at $t = 2$ s.



Solution: We have

$$\Sigma F \nearrow: N - (100 \text{ N})\cos 30^\circ = 0 \Rightarrow N = 86.6 \text{ N}$$



(a) Then, along the slope the impulse is

$I = (W \sin 30^\circ - \mu_k N)t$

$I = ([100 \text{ N}]\sin 30^\circ - [0.18][86.6\text{N}])(2 \text{ s})$

$\boxed{I = 68.8 \text{ N-s.}}$

(b) Using the principle of impulse-momentum,

$mv_1 + I = mv_2$

$$0 + 68.8 \text{ N-s} = \left(\frac{100 \text{ N}}{9.81 \text{ m/s}^2}\right) v_2$$

Solving we find

$\boxed{v_2 = 6.75 \text{ m/s.}}$

Problem 16.3 The mass of the helicopter is 9300 kg. It takes off vertically at time $t = 0$. The pilot advances the throttle so that the upward thrust of its engine (in kN) is given as a function of time in seconds by $T = 100 + 2t^2$.

(a) Determine the magnitude of the linear impulse due to the forces acting on the helicopter from $t = 0$ to $t = 3$ s.

(b) Use the principle of impulse and momentum to determine how fast the helicopter is moving at $t = 3$ s.

Solution:

(a) The impulse - using the total force (T and the weight).

$$I = \int_{t_1}^{t_2} F\,dt = \int_{t_1}^{t_2} (T - mg)\,dt$$

$$= \int_0^3 (100 + 2t^2 - 9.3[9.81])\,dt = (8.77)(3) + \frac{2}{3}(3)^3 = 44.3 \text{ kN-s.}$$

$\boxed{I = 44.3 \text{ kN-s.}}$

(b) Using the principle of impulse - momentum,

$$mv_1 + I = mv_2$$

$$0 + 44.3 \text{ kN-s} = (9300 \text{ kg})v_2$$

$\boxed{v_2 = 4.76 \text{ m/s.}}$

Problem 16.4 A 150 million-kg cargo ship starts from rest. The total force exerted on it by its engines and hydrodynamic drag (in newtons) can be approximated as a function of time in seconds by $\Sigma F_t = 937{,}500 - 0.65t^2$. Use the principle of impulse and momentum to determine how fast the ship is moving in 16 minutes.

Solution: The impulse is

$$I = \int_{t_1}^{t_2} F\,dt = \int_0^{16(60)} (937{,}500 - 0.65t^2)\,dt$$

$$= (937{,}500)(960) - \frac{1}{3}(0.65)(960)^3 = 7.08 \times 10^8 \text{ N-s.}$$

Using the principle of impulse and momentum, we have

$$mv_1 + I = mv_2$$

$$0 + 7.08 \times 10^8 \text{ N-s} = (150 \times 10^6 \text{ kg})v_2$$

Solving, we find

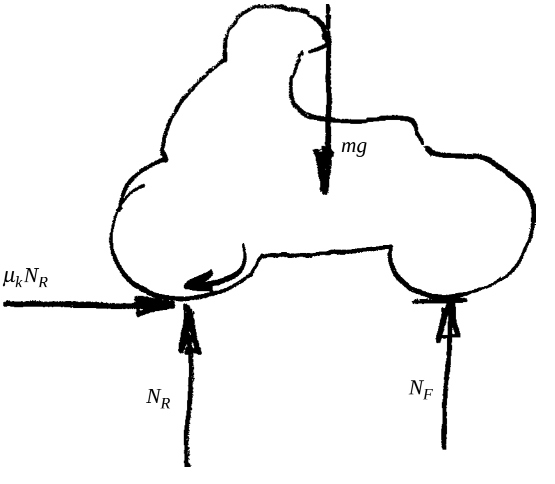
$\boxed{v_2 = 4.72 \text{ m/s (9.18 knots).}}$

Problem 16.5 The combined mass of the motorcycle and rider is 136 kg. The coefficient of kinetic friction between the motorcycle's tires and the road is $\mu_k = 0.6$. The rider starts from rest and spins the rear (drive) wheel. The normal force between the rear wheel and road is 790 N.

(a) What impulse does the friction force on the rear wheel exert in 2 s?

(b) If you neglect other horizontal forces, what velocity is attained by the motorcycle in 2 s?

Solution:



$m = 136$ kg

$g = 9.81$ m/s^2

$N_R = 790$ N

$$\sum F_x = \mu_k N_R = m\frac{dv_x}{dt}$$

$$\text{Imp} = \int_0^2 \mu_k N_R \, dt = \mu_k N_R t \Big|_0^2$$

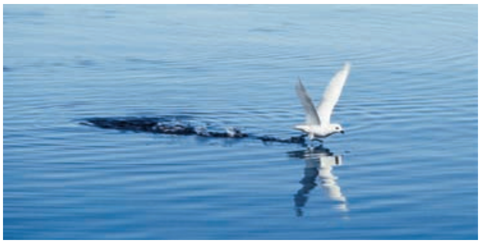
(a) $\underline{\text{Imp} = (0.6 \times 790 \times 2) = 948 \text{ N-s}}$

(b) $$\int_0^2 \mu_k N_R \, dt = m\int_0^v dv = \text{Imp}$$

948 N-s $= mv$

$\underline{v = 948 \text{ N-s}/136 \text{ kg} = 6.97 \text{ m/s}}$

Problem 16.6 A bioengineer models the force generated by the wings of the 0.2-kg snow petrel by an equation of the form $F = F_0(1 + \sin \omega t)$, where F_0 and ω are constants. From video measurements of a bird taking off, he estimates that $\omega = 18$ and determines that the bird requires 1.42 s to take off and is moving at 6.1 m/s when it does. Use the principle of impulse and momentum to determine the constant F_0.



Solution:

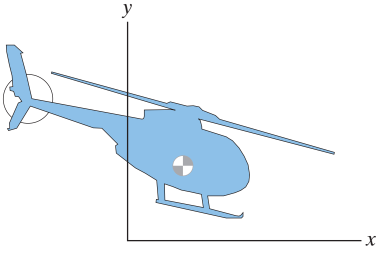
$$\int_0^t F_0(1 + \sin \omega t)\, dt = mv$$

$$F_0\left(t - \frac{1}{\omega}\cos \omega t\right)_0^t = F_0\left(t + \frac{1}{\omega}[1 - \cos \omega t]\right) = mv$$

$$F_0 = \frac{mv}{t + \dfrac{1}{\omega}(1 - \cos \omega t)} = \frac{(0.2 \text{ kg})\,(6.1 \text{ m/s})}{(1.42 \text{ s}) + \dfrac{1}{18 \text{ rad/s}}(1 - \cos[(18)(1.42)])}$$

$$\boxed{F_0 = 0.856 \text{ N.}}$$

Problem 16.7 In Active Example 16.1, what is the average total force acting on the helicopter from $t = 0$ to $t = 10$ s?



Solution: From Active Example 16.1 we know the total impulse that occurs during the time. Then

$$\mathbf{F}\Delta t = \mathbf{I} \Rightarrow \mathbf{F} = \frac{\mathbf{I}}{\Delta t}$$

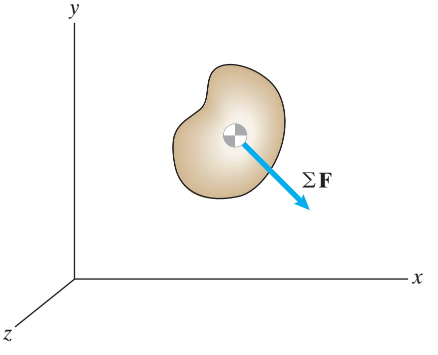
$$\mathbf{F} = \frac{(36{,}000\mathbf{i} + 3600\mathbf{j}) \text{ N-s}}{10 \text{ s}}$$

$$\boxed{\mathbf{F} = (3600\mathbf{i} + 360\mathbf{j}) \text{ N.}}$$

Problem 16.8 At time $t = 0$, the velocity of the 15-kg object is $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ (m/s). The total force acting on it from $t = 0$ to $t = 4$ s is

$$\Sigma\mathbf{F} = (2t^2 - 3t + 7)\mathbf{i} + 5t\mathbf{j} + (3t + 7)\mathbf{k} \text{ (N).}$$

Use the principle of impulse and momentum to determine its velocity at $t = 4$ s.



Solution: Working in components we have

$$(15)(2) + \int_0^4 (2t^2 - 3t + 7)\, dt = (15)v_{2x}$$

$$(15)(3) + \int_0^4 5t\, dt = (15)v_{2y}$$

$$(15)(-5) + \int_0^4 (3t + 7)\, dt = (15)v_{2z}$$

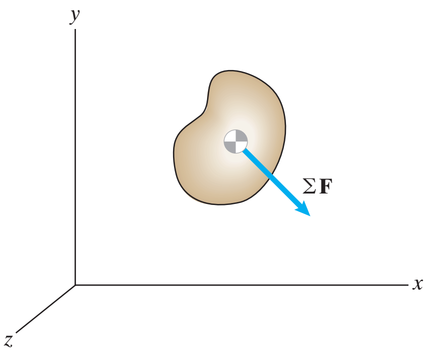
Solving we find $v_{2x} = 5.11$ m/s, $v_{2y} = 5.67$ m/s, $v_{2z} = -1.53$ m/s.

$$\boxed{\mathbf{v}_2 = (5.11\mathbf{i} + 5.67\mathbf{j} - 1.53\mathbf{k}) \text{ m/s.}}$$

Problem 16.9 At time $t = 0$, the velocity of the 15-kg object is $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ (m/s). The total force acting on it from $t = 0$ to $t = 4$ s is

$$\Sigma\mathbf{F} = (2t^2 - 3t + 7)\mathbf{i} + 5t\mathbf{j} + (3t + 7)\mathbf{k} \text{ (N)}.$$

What is the average total force on the object during the interval of time from $t = 0$ to $t = 4$ s?



Solution: The components of the impulse are

$$I_x = \int_0^4 (2t^2 - 3t + 7)\,dt = \frac{2}{3}(4)^3 - \frac{3}{2}(4)^2 + 7(4) = 46.7 \text{ N},$$

$$I_y = \int_0^4 5t\,dt = \frac{5}{2}(4)^2 = 40 \text{ N},$$

$$I_z = \int_0^4 (3t + 7)\,dt = \frac{3}{2}(4)^2 + 7(4) = 52 \text{ N}.$$

The average force is given by

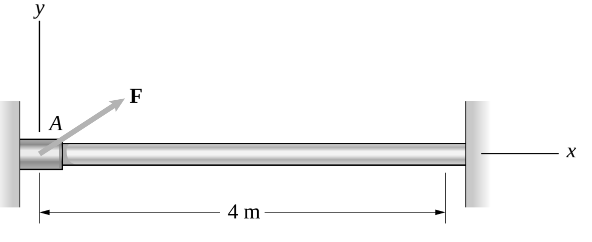
$$\mathbf{F}_{\text{ave}} = \frac{\mathbf{I}}{\Delta t} = \frac{(46.7\mathbf{i} + 40\mathbf{j} + 52\mathbf{k}) \text{ N}}{4 \text{ s}}$$

$$\boxed{\mathbf{F}_{\text{ave}} = (11.7\mathbf{i} + 10\mathbf{j} + 13\mathbf{k})\text{N}.}$$

Problem 16.10 The 1-N collar A is initially at rest in the position shown on the smooth horizontal bar. At $t = 0$, a force

$$\mathbf{F} = \left(\tfrac{1}{20}\right)t^2\mathbf{i} + \left(\tfrac{1}{10}\right)t\mathbf{j} - \left(\tfrac{1}{30}\right)t^3\mathbf{k} \text{ (N)}$$

is applied to the collar, causing it to slide along the bar. What is the velocity of the collar at $t = 2$ s?



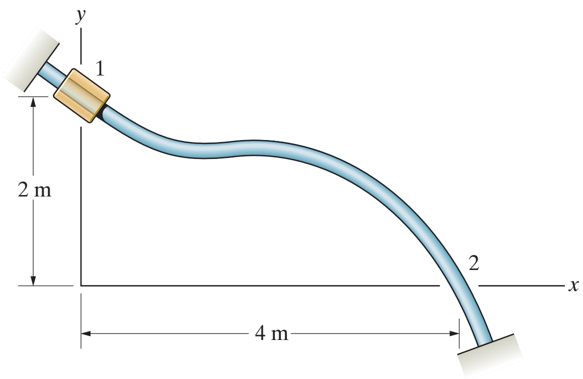
Solution: The impulse applied to the collar is $\int_{t_1}^{t_2} \sum \mathbf{F}\,dt = mv_{x2} - mv_{x1}$: Evaluating, we get

$$\int_0^2 \frac{1}{20}t^2\,dt = (1/9.81)v_{x2},$$

or $\left[\frac{1}{60}t^3\right]_0^2 = (1/9.81)v_{x2}.$

Hence, $v_{x2} = 1.31$ m/s.

Problem 16.11 The y axis is vertical and the curved bar is smooth. The 4-N slider is released from rest in position 1 and requires 1.2 s to slide to position 2. What is the magnitude of the average tangential force acting on the slider as it moves from position 1 to position 2?



Solution: We will use the principle of work and energy to find the velocity at position 2.

$$T_1 + W_{1\to 2} = T_2$$

$$0 + (4\text{ N})(2\text{ m}) = \frac{1}{2}\left(\frac{4\text{ N}}{9.81\text{ m/s}^2}\right)v_2{}^2 \Rightarrow v_2 = 6.26\text{ m/s}.$$

Now, using the principle of impulse - momentum we can find the average tangential force

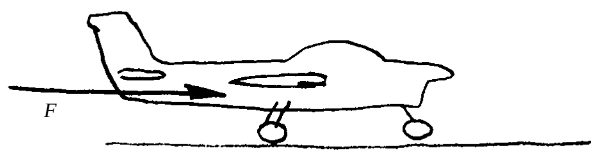
$$mv_1 + F_{\text{ave}}\Delta t = mv_2$$

$$F_{\text{ave}} = m\frac{(v_2 - v_1)}{\Delta t} = \left(\frac{4\text{ N}}{9.81\text{ m/s}^2}\right)\frac{(6.26\text{ m/s} - 0)}{1.2\text{ s}}$$

$$\boxed{F_{\text{ave}} = 2.13\text{ N.}}$$

Problem 16.12 During the first 5 s of the 14,200-kg airplane's takeoff roll, the pilot increases the engine's thrust at a constant rate from 22 kN to its full thrust of 112 kN.

(a) What impulse does the thrust exert on the airplane during the 5 s?
(b) If you neglect other forces, what total time is required for the airplane to reach its takeoff speed of 46 m/s?



Solution:

$$m = 14200\text{ kg}$$

$$F = (22000 + 18000\,t)\text{ N}$$

$$\text{Imp} = \int_0^5 (22000 + 18000\,t)\,dt(\text{N-s})$$

$$\text{Imp} = 22000\,t + 9000\,t^2\Big|_0^5$$

(a) $\underline{\text{Imp} = 335000\text{ N-s} = 335\text{ kN-s}}$

$$\int_0^{t_f} F\,dt = mv_f - m\cancel{v_0}^0$$

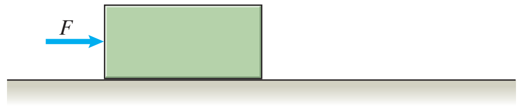
$$\int_0^5 (22000 + 18000\text{ t})\,dt + \int_5^t (112000)\,dt = mv_f$$

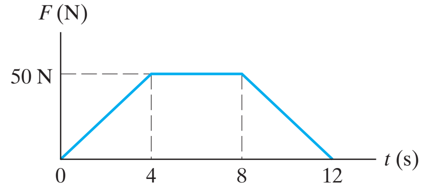
$$335000 + 112000\,t\Big|_5^t = (14200)(46)$$

$$112000(t - 5) + 335000 = (14200)(46)$$

(b) $\underline{t = 7.84\text{ s}}$

Problem 16.13 The 10-kg box starts from rest on the smooth surface and is subjected to the horizontal force described in the graph. Use the principle of impulse and momentum to determine how fast the box is moving at $t = 12$ s.





Solution: The impulse is equal to the area under the curve in the graph

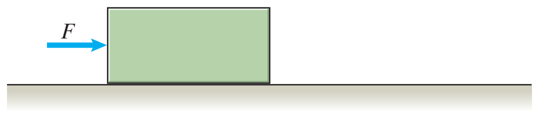
$$I = \frac{1}{2}(50 \text{ N})(4 \text{ s}) + (50 \text{ N})(4 \text{ s}) + \frac{1}{2}(50 \text{ N})(4 \text{ s}) = 400 \text{ N-s}.$$

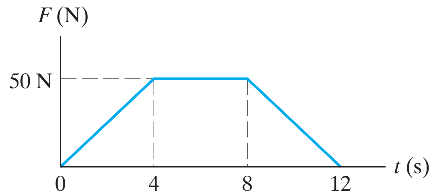
Using the principle of impulse and momentum we have

$$mv_1 + I = mv_2 \Rightarrow 0 + (400 \text{ N-s}) = (10 \text{ kg})v_2$$

Solving we find $\boxed{v_2 = 40 \text{ m/s.}}$

Problem 16.14 The 10-kg box starts from rest and is subjected to the horizontal force described in the graph. The coefficients of friction between the box and the surface are $\mu_s = \mu_k = 0.2$. Determine how fast the box is moving at $t = 12$ s.





Solution: The box will not move until the force F is able to overcome friction. We will first find this critical time.

$$N = W = (10 \text{ kg})\,(9.81 \text{ m/s}^2) = 98.1 \text{ N}$$

$$f = \mu N = (0.2)(98.1 \text{ N}) = 19.6 \text{ N}$$

$$F = \frac{50 \text{ N}}{4 \text{ s}} t = 19.6 \text{ N} \Rightarrow t_{cr} = 1.57 \text{ s}.$$

The impulse from t_{cr} to 12 s is

$$I = \frac{(50 \text{ N} + 19.6 \text{ N})}{2}(4 \text{ s} - 1.57 \text{ s}) + (50 \text{ N})\,(4 \text{ s}) + \frac{(50 \text{ N})}{2}(4 \text{ s})$$

$$-(19.6 \text{ N})(12 \text{ s} - 1.57 \text{ s}) = 180 \text{ N-s}.$$

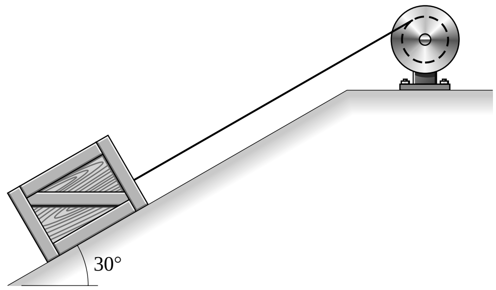
The principle of impulse and momentum gives

$$mv_1 + I = mv_2 \Rightarrow 0 + 180 \text{ N-s} = (10 \text{ kg})v_2.$$

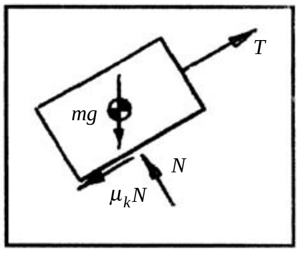
Solving we find $\boxed{v_2 = 18.0 \text{ m/s.}}$

Problem 16.15 The crate has a mass of 120 kg, and the coefficients of friction between it and the sloping dock are $\mu_s = 0.6$ and $\mu_k = 0.5$. The crate starts from rest, and the winch exerts a tension $T = 1220$ N.

(a) What impulse is applied to the crate during the first second of motion?
(b) What is the crate's velocity after 1 s?



Solution: The motion starts only if $T - mg\sin 30^\circ > \mu_s mg \cos 30^\circ$, from which $631.4 > 611.7$. The motion indeed starts.



(a) The impulse in the first second is

$$\boxed{\begin{aligned}\int_{t_1}^{t_2} F\,dt &= \int_0^1 (T - mg\sin 30^\circ - mg\mu_k \cos 30^\circ)\,dt \\ &= 121.7\text{ t} = 121.7\text{ N-s}\end{aligned}}$$

(b) The velocity is $\boxed{v = \dfrac{121.7}{120} = 1.01\text{ m/s}}$

Problem 16.16 Solve Problem 16.15 if the crate starts from rest at $t = 0$ and the winch exerts a tension $T = 1220 + 200t$ N.

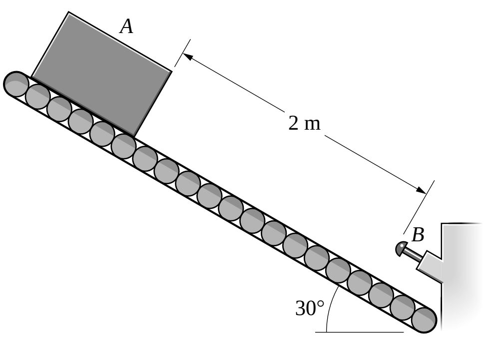
Solution: From the solution to Problem 16.15, motion will start.

(a) The impulse at the end of 1 second is

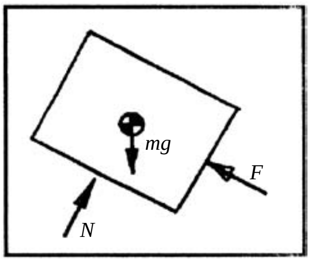
$$\boxed{\begin{aligned}\int_{t_1}^{t_2} F\,dt &= \int_0^1 (1220 + 200t - mg\sin 30^0 - \mu_k mg\cos 30^0)\,dt \\ &= [1220t + 100t^2 - 1098.3t]_0^1 = 221.7\text{ N-s}\end{aligned}}$$

(b) The velocity is $\boxed{v = \dfrac{221.7}{m} = \dfrac{221.7}{120} = 1.85\text{ m/s}}$

Problem 16.17 In an assembly-line process, the 20-kg package A starts from rest and slides down the smooth ramp. Suppose that you want to design the hydraulic device B to exert a constant force of magnitude F on the package and bring it to a stop in 0.15 s. What is the required force F?



Solution: Use conservation of energy to obtain the velocity of the crate at point of contact with device B. $mgh = \frac{1}{2}mv_B^2$, where $h = 2\sin 30^\circ = 1$ m, from which $v_B = \sqrt{2g} = 4.43$ m/s. The impulse to be exerted by B is $\int_{t_1}^{t_2} F\,dt = mv_B = 88.6$ N-s. The constant force to be applied by device B is $F - mg\sin 30^\circ = \dfrac{88.6}{0.15} = 590.67$ N, from which $\boxed{F = 688.8\text{ N}}$



Problem 16.18 The 20-kg package A starts from rest and slides down the smooth ramp. If the hydraulic device B exerts a force of magnitude $F = 540(1 + 0.4t^2)$ N on the package, where t is in seconds measured from the time of first contact, what time is required to bring the package to rest?

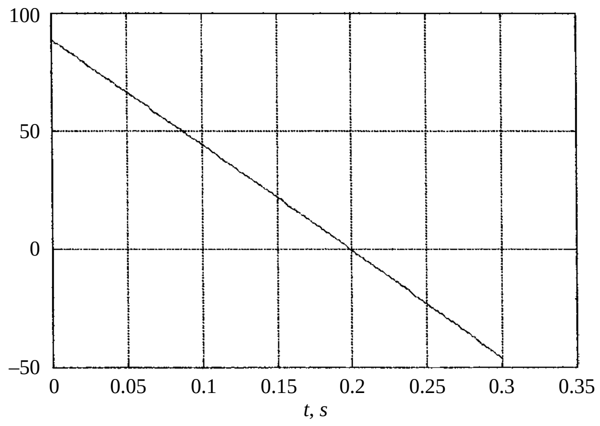
Solution: See the solution of Problem 16.17. The velocity at first contact is 4.43 m/s. Impulse and momentum is

$$\int_0^t [mg \sin 30^\circ - 540(1 + 0.4t^2)]\, dt = 0 - 4.43 \text{ m}.$$

Integrating yields

$$mgt \sin 30^\circ - 540\left(t + \frac{0.4t^3}{3}\right) + 4.43 \text{ m} = 0.$$

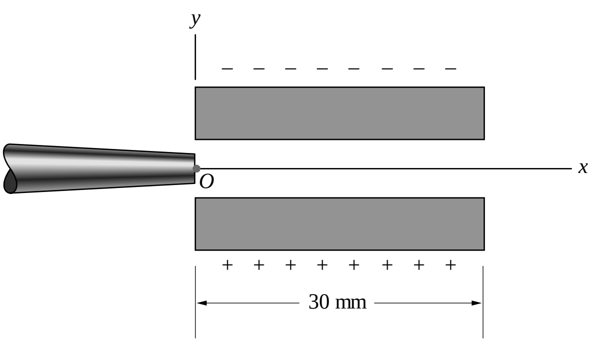
The graph of the left side of this equation as a function of t is shown. By examining calculated results, we estimate the solution to be $\underline{t = 0.199 \text{ s}}$.



Problem 16.19 In a cathode-ray tube, an electron (mass $= 9.11 \times 10^{-31}$ kg) is projected at O with velocity $\mathbf{v} = (2.2 \times 10^7)\mathbf{i}$ (m/s). While the electron is between the charged plates, the electric field generated by the plates subjects it to a force $\mathbf{F} = -eE\mathbf{j}$. The charge of the electron is $e = 1.6 \times 10^{-19}$ C (coulombs), and the electric field strength is $E = 15 \sin(\omega t)$ kN/C, where the circular frequency $\omega = 2 \times 10^9$ s^{-1}.

(a) What impulse does the electric field exert on the electron while it is between the plates?

(b) What is the velocity of the electron as it leaves the region between the plates?



Solution: The x component of the velocity is unchanged. The time spent between the plates is $t = \dfrac{0.03}{2.2 \times 10^7} = 1.36 \times 10^{-9}$ s.

(a) The impulse is

$$\int_{t_1}^{t_2} F\, dt = \int_0^t (-eE)\, dt = \int_0^t -e(15 \times 10^3)(\sin \omega t)\, dt$$

$$= \left[\frac{(15 \times 10^3)e}{\omega} \cos \omega t\right]_0^{1.36 \times 10^{-9}}$$

$$\boxed{\int_{t_1}^{t_2} F\, dt = -2.3 \times 10^{-24} \text{ N-s}}$$

The y component of the velocity is

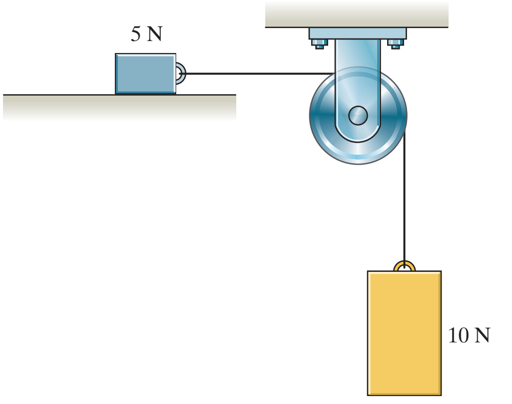
$$v_y = \frac{-2.3 \times 10^{-24}}{9.11 \times 10^{-31}} = -2.52 \times 10^6 \text{ m/s}.$$

(b) The velocity on emerging from the plates is

$$\boxed{\mathbf{v} = 22 \times 10^6 \mathbf{i} - 2.5 \times 10^6 \mathbf{j} \text{ m/s.}}$$

Problem 16.20 The two weights are released from rest at time $t = 0$. The coefficient of kinetic friction between the horizontal surface and the 5-N weight is $\mu_k = 0.4$. Use the principle of impulse and momentum to determine the magnitude of the velocity of the 10-N weight at $t = 1$ s.

Strategy: Apply the principle to each weight individually.

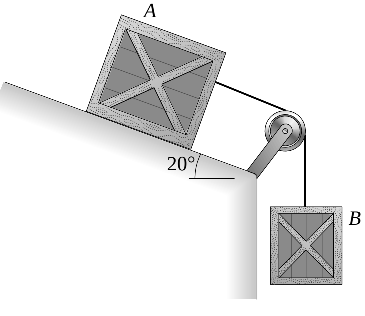


Solution:

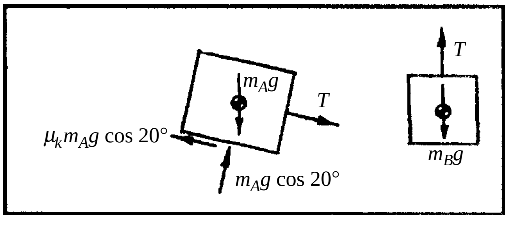
Impulse $= (10\ \text{N})(1\ \text{s}) - 0.4(5)(1\ \text{s}) = 8\ \text{N-s}$

$$8\ \text{lb-s} = \left(\frac{15\ \text{N}}{9.81\ \text{m/s}^2}\right) v \Rightarrow \boxed{v = 5.23\ \text{m/s}}$$

Problem 16.21 The two crates are released from rest. Their masses are $m_A = 40$ kg and $m_B = 30$ kg, and the coefficient of kinetic friction between crate A and the inclined surface is $\mu_k = 0.15$. What is the magnitude of the velocity of the crates after 1 s?



Solution: The force acting to move crate A is



$$F_A = T + m_A g(\sin 20° - \mu_k \cos 20°)$$

$$= T + 78.9\ \text{N},$$

where T is the tension in the cable.

The impulse, since the force is constant, is

$$(T + 78.9)t = m_A v.$$

For crate B,

$$F_B = -T + m_B g = -T + 294.3.$$

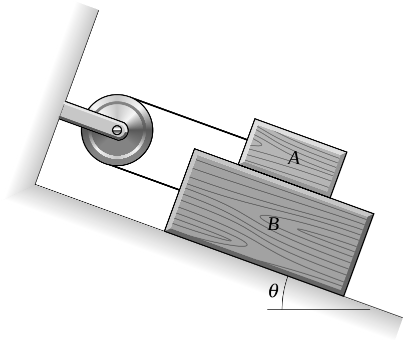
The impulse, since the force is constant, is

$$(-T + 294.3)t = m_B v.$$

For $t = 1$ s, add and solve: $78.9 + 294.3 = (40 + 30)v$, from which

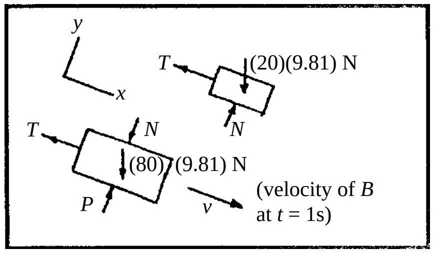
$$\boxed{v = 5.33\ \text{m/s}}$$

Problem 16.22 The two crates are released from rest. Their masses are $m_A = 20$ kg and $m_B = 80$ kg, and the surfaces are smooth. The angle $\theta = 20^\circ$. What is the magnitude of the velocity after 1 s?



Strategy: Apply the principle of impulse and momentum to each crate individually.

Solution: The free body diagrams are as shown:



Crate B: $\int_{t_1}^{t_2} \sum F_x \, dt = mv_{x2} - mv_{x1}$:

$$\int_0^1 [(80)(9.81)\sin 20^\circ - T]\, dt = (80)(v - 0).$$

Crate A: $\int_{t_1}^{t_2} \sum F_x \, dt = mv_{x2} - mv_{x1}$:

$$\int_0^1 [(20)(9.81)\sin 20^\circ - T]\, dt = (20)[(-v) - 0]$$

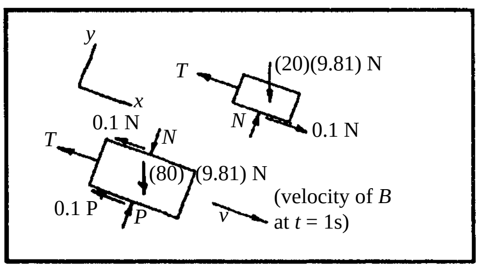
Subtracting the second equation from the first one,

$$\int_0^1 (80 - 20)(9.81)\sin 20^\circ \, dt = (80 + 20)v.$$

Solving, we get $v = 2.01$ m/s.

Problem 16.23 The two crates are released from rest. Their masses are $m_A = 20$ kg and $m_B = 80$ kg. The coefficient of kinetic friction between the contacting surfaces is $\mu_k = 0.1$. The angle $\theta = 20^\circ$. What is the magnitude of the velocity of crate A after 1 s?

Solution: The free body diagrams are as shown:



The sums of the forces in the y direction equal zero:

$$\sum F_y = N - (20)(9.81)\cos 20^\circ = 0 \quad N = 184 \text{ N},$$

$$\sum F_y = P - N - (80)(9.81)\cos 20^\circ = 0 \quad P = 922 \text{ N}$$

Crate B: $\int_{t_1}^{t_2} \sum F_x \, dt = mv_{x2} - mv_{x1}$:

$$\int_0^1 [(80)(9.81)\sin 20^\circ - 0.1\ P - 0.1\ \text{N-T}]\, dt = (80)(v - 0). \quad (1)$$

Crate A: $\int_{t_1}^{t_2} \sum F_x \, dt = mv_{x2} - mv_{x1}$:

$$\int_0^1 [(20)(9.81)\sin 20^\circ + 0.1\ \text{N-T}]\, dt = (20)[(-v) - 0]. \quad (2)$$

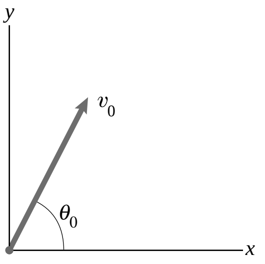
Subtracting Equation (2) from Equation (1),

$$\int_0^1 [(80 - 20)(9.81)\sin 20^\circ - 0.1\ \text{P} - 0.2\ \text{N}]\, dt = (80 + 20)v.$$

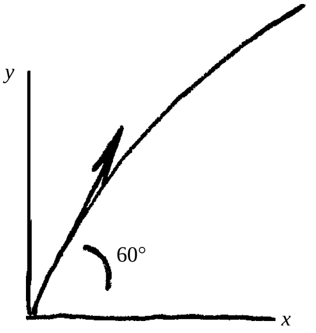
Solving, $v = 0.723$ m/s.

Problem 16.24 At $t = 0$, a 20-kg projectile is given an initial velocity $v_0 = 20$ m/s at $\theta_0 = 60^\circ$ above the horizontal.

(a) By using Newton's second law to determine the acceleration of the projectile, determine its velocity at $t = 3$ s.
(b) What impulse is applied to the projectile by its weight from $t = 0$ to $t = 3$ s?
(c) Use the principle of impulse and momentum to determine the projectile's velocity at $t = 3$ s.



Solution:



$\mathbf{a} = -g\mathbf{j}$

$a_x = 0$

$a_y = -g$

$v_{x_0} = v_0 \cos 60^\circ = 10$ m/s

$v_{y_0} = v_0 \sin 60^\circ = 17.32$ m/s

$x_0 = y_0 = 0$

$v_0 = 20$ m/s

$a_x = 0 \qquad a_y = -g$

$v_x = v_0 \cos 60^\circ \quad v_y = v_0 \sin 60^\circ - gt$

$\mathbf{v} = (v_0 \cos 60^\circ)\mathbf{i} + (v_0 \sin 60^\circ - gt)\mathbf{j}$ (m/s)

At $t = 3\ s$,

(a) $\underline{\mathbf{v} = 10\,\mathbf{i} - 12.1\,\mathbf{j} \text{ (m/s)}}$

$\mathbf{I}_G =$ Impulse due to gravity

$\mathbf{F}_G = -mg\mathbf{j}$

$$\mathbf{I}_G = -\int_0^3 mg\,\mathbf{j}\,dt$$

$\mathbf{I}_G = -mg\,t|_0^3\,\mathbf{j} = -3\,mg\,\mathbf{j}$ (N-s)

(b) $\underline{\mathbf{I}_G = -589\,\mathbf{j} \text{ (N-s)}}$

$I_G = m\mathbf{v}(3) - m\mathbf{v}_0$

$-589\mathbf{j} = mv_x\,\mathbf{i} + mv_y\,\mathbf{j} - mv_{xo}\,\mathbf{i} - mv_{yo}\,\mathbf{j}$

$x: \quad 0 = mv_x - mv_{xo}$

$v_x = v_{xo} = 10$ m/s

$y: \quad -589 = mv_y - mv_{yo}$

$20\,v_y = (20)(17.32) - 589$

$v_y = -12.1$ m/s

(c) $\underline{\mathbf{v} = 10\,\mathbf{i} - 12.1\,\mathbf{j} \text{ (m/s) at } t = 3 \text{ s}}$

Problem 16.25 A soccer player kicks the stationary 0.45-kg ball to a teammate. The ball reaches a maximum height above the ground of 2 m at a horizontal distance of 5 m form the point where it was kicked. The duration of the kick was 0.04 seconds. Neglecting the effect of aerodynamic drag, determine the magnitude of the average force the player everted on the ball.

Solution: We will need to find the initial velocity of the ball. In the y direction we have

$$v_y = \sqrt{2gh} = \sqrt{2(9.81\ \text{m/s}^2)(2\ \text{m})}$$

$$= 6.26\ \text{m/s}.$$

The time of flight is given by

$$t = \frac{v_y}{g} = \frac{(6.26\ \text{m/s})}{(9.81\ \text{m/s}^2)} = 0.639\ \text{s}.$$

In the x direction we have

$$v_x = \frac{d}{t} = \frac{(5\ \text{m})}{(0.639\ \text{s})} = 7.83\ \text{m/s}.$$

The total velocity is then

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(6.26\ \text{m/s})^2 + (7.83\ \text{m/s})^2} = 10.0\ \text{m/s}.$$

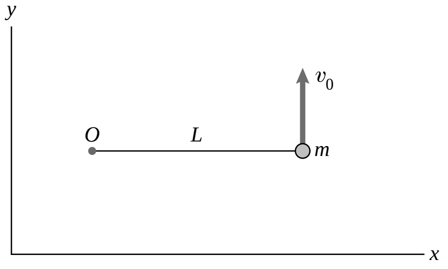
The principle of impulse and momentum then gives

$$F\Delta t = mv \Rightarrow F = \frac{mv}{\Delta t} = \frac{(0.45\ \text{kg})(10.0\ \text{m/s})}{0.04\ \text{s}}.$$

$$\boxed{F = 113\ \text{N}.}$$

Problem 16.26 An object of mass $m = 2$ kg slides with constant velocity $v_0 = 4$ m/s on a horizontal table (seen from above in the figure). The object is connected by a string of length $L = 1$ m to the fixed point O and is in the position shown, with the string parallel to the x axis, at $t = 0$.

(a) Determine the x and y components of the force exerted on the mass by the string as functions of time.
(b) Use your results from part (a) and the principle of impulse and momentum to determine the velocity vector of the mass at $t = 1$ s.



Strategy: To do part (a), write Newton's second law for the mass in terms of polar coordinates.

Solution:

$$\mathbf{T} = mv^2/L\,\mathbf{e}_N$$

$$-\mathbf{e}_N = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$

$$v_0 = rw:$$

$$4 = (1)w \quad w = 4 \text{ rad/s}$$

$$\frac{d\theta}{dt} = w = 4 \text{ rad/s}$$

$$\theta = 4t \text{ rad}$$

$$\mathbf{T} = -(mv^2/L)\cos(4t)\mathbf{i} - (mv^2/L)\sin(4t)\mathbf{j}$$

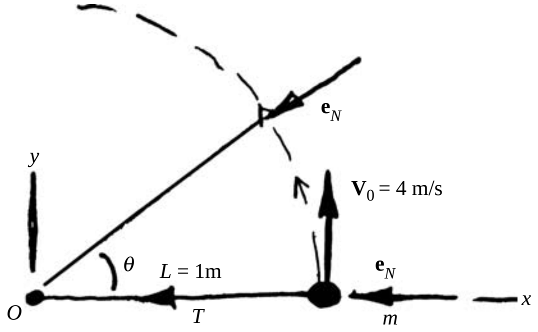
$$\underline{\mathbf{T} = -32\cos 4t\,\mathbf{i} - 32\sin 4t\,\mathbf{j} \text{ (N)}}$$

$$T_x = -32\cos 4t \text{ N}$$

$$\underline{T_y = -32\sin 4t \text{ N}}$$

$$\int_0^1 \mathbf{F}\,dt = m\mathbf{v}_f - m\mathbf{v}_0$$

$$\int_0^1 \mathbf{T}\,dt = mv_x\mathbf{i} + mv_y\mathbf{j} - m(4)\mathbf{j}$$



$$\begin{cases} x\colon \displaystyle\int_0^1 (-32\cos 4t)\,dt = mv_x \\ y\colon \displaystyle\int_0^1 (-32\sin 4t)\,dt = mv_y - 4\text{ m} \end{cases}$$

$$m = 2 \text{ kg}$$

$$x\colon \left.\frac{-32\sin 4t}{4}\right|_0^1 = 2v_x \quad + \quad \left.\frac{32\cos 4t}{4}\right|_0^1 = 2v_y - 8$$

Solving $\underline{\mathbf{v} = 3.03\mathbf{i} - 2.61\mathbf{j} \text{ (m/s)}}$

Problem 16.27 A rail gun, which uses an electromagnetic field to accelerate an object, accelerates a 30-g projectile from zero to 5 km/s in 0.0004 s. What is the magnitude of the average force exerted on the projectile?

Solution:

$$\boxed{F_{\text{ave}} = \frac{(0.03 \text{ kg})(5000 \text{ m/s} - 0)}{0.0004 \text{ s}} = 375 \text{ kN}}$$

Problem 16.28 The mass of the boat and its passenger is 420 kg. At time $t = 0$, the boat is moving at 14 m/s and its motor is turned off. The magnitude of the hydrodynamic drag force on the boat (in newtons) is given as a function of time by $830(1 - 0.08t)$. Determine how long it takes for the boat's velocity to decrease to 5 m/s.

Solution: The principle of impulse and momentum gives

$$mv_1 + \int_{t_1}^{t_2} F\,dt = mv_2$$

$$(420\text{ kg})\,(14\text{ m/s}) - \int_0^t 830(1 - 0.08t)\,dt = (420\text{ kg})\,(5\text{ m/s})$$

$$-830(t - 0.04t^2) = -3780$$

$$t^2 - 25t + 114 = 0$$

$$t = \frac{25 - \sqrt{25^2 - 4(1)(114)}}{2} = 5.99\text{ s.}$$

$$\boxed{t = 5.99\text{ s.}}$$

Problem 16.29 The motorcycle starts from rest at $t = 0$ and travels along a circular track with 300-m radius. From $t = 0$ to $t = 10$ s, the component of the total force on the motorcycle tangential to its path is $\Sigma F_t = 600$ N. The combined mass of the motorcycle and rider is 150 kg. Use the principle of impulse and momentum to determine the magnitude of the motorcycle's velocity at $t = 10$ s. (See Active Example 16.2.)

Solution:

$$(600\text{ N})(10\text{ s}) = (150\text{ kg})v \Rightarrow \boxed{v = 40\text{ m/s}}$$

Problem 16.30 Suppose that from $t = 0$ to $t = 10$ s, the component of the total tangential force on the motorcycle in Problem 16.29 is given as a function of time by $\Sigma F_t = 460 + 3t^2$ N. The combined mass of the motorcycle and rider is 150 kg. Use the principle of impulse and momentum to determine the magnitude of the motorcycle's velocity at $t = 10$ s. (See Active Example 16.2.)

Solution:

$$\int_0^{10\text{ s}} (460 + 3t^2)\text{ N}\,dt = (150\text{ kg})v \Rightarrow \boxed{v = 37.3\text{ m/s}}$$

Problem 16.31 The titanium rotor of a Beckman Coulter ultracentrifuge used in biomedical research contains 2-gram samples at a distance of 41.9 mm from the axis of rotation. The rotor reaches its maximum speed of 130,000 rpm in 12 minutes.

(a) Determine the average tangential force exerted on a sample during the 12 minutes the rotor is acceleration.
(b) When the rotor is at its maximum speed, what normal acceleration are samples subjected to?

Solution:

(a) Using the principle of impulse and momentum we have

$$0 + F_{\text{ave}}\Delta t = mv$$

$$F_{\text{ave}} = \frac{mv}{\Delta t}$$

$$= \frac{(0.002 \text{ kg})\left(130,000\dfrac{\text{rev}}{\text{min}}\right)\left(\dfrac{2\pi \text{ rad}}{\text{rev}}\right)\left(\dfrac{1 \text{ min}}{60 \text{ s}}\right)(0.0419 \text{ m})}{(12 \text{ min})\left(\dfrac{60 \text{ s}}{\text{min}}\right)}$$

$$\boxed{F_{\text{ave}} = 0.00158 \text{ N}}$$

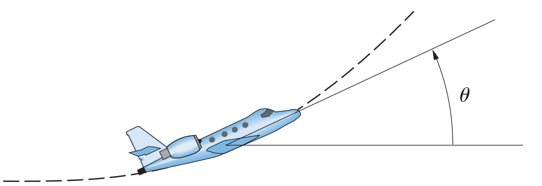
(b)

$$a_n = \frac{v^2}{\rho} = \frac{(r\omega)^2}{r} = r\omega^2$$

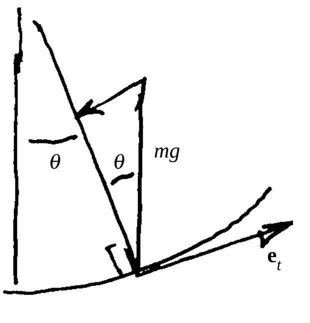
$$= (0.0419 \text{ m})\left(\left[130{,}000\frac{\text{rev}}{\text{min}}\right]\left[\frac{2\pi \text{ rad}}{\text{rev}}\right]\left[\frac{1 \text{ min}}{60 \text{ s}}\right]\right)^2$$

$$\boxed{a_n = 7.77 \times 10^6 \text{ m/s}^2.}$$

Problem 16.32 The angle θ between the horizontal and the airplane's path varies from $\theta = 0$ to $\theta = 30^\circ$ at a constant rate of 5 degrees per second. During this maneuver, the airplane's thrust and aerodynamic drag are again balanced, so that the only force exerted on the airplane in the direction tangent to its path is due to its weight. The magnitude of the airplane's velocity when $\theta = 0$ is 120 m/s. Use the principle of impulse and momentum to determine the magnitude of the velocity when $\theta = 30^\circ$.



Solution:



$$w = \frac{d\theta}{dt} = 5^\circ/\text{s}, \quad \text{constant} = 0.0873 \ \frac{\text{rad}}{\text{s}}$$

It takes 6 seconds to go from $\theta = 0^\circ$ to $\theta = 30^\circ$. The resisting force is

$$\mathbf{F}_t = -mg\sin\theta\mathbf{e}_t$$

$$\mathbf{F}_t = m\frac{dv}{dt}\mathbf{e}_t - \cancel{m}g\sin\theta\mathbf{e}_t$$

$$= \cancel{m}\frac{dv}{dt}\mathbf{e}_t - g\int_0^6 \sin\theta\, dt = \int_{120}^{v_f} dv$$

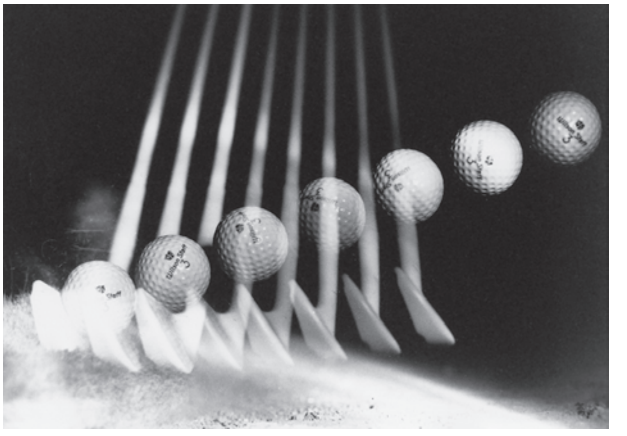
$$-g\int_0^{30^\circ} \sin\theta\frac{dt}{d\theta}d\theta = V_f - 120 \text{ m/s}$$

$$-\frac{g}{w}(-\cos\theta)\Big|_0^{30^\circ} = V_f - 120 \text{ m/s}$$

$$V_f = 120 + \frac{g}{w}(\cos 30^\circ - 1)$$

$\underline{V_f = 105 \text{ m/s}}$

Problem 16.33 In Example 16.3, suppose that the mass of the golf ball is 0.046 kg and its diameter is 43 mm. The club is in contact with the ball for 0.0006 s, and the distance the ball travels in one 0.001-s interval is 50 mm. What is the magnitude of the average impulsive force exerted by the club?



Solution: Using the principle of impulse and momentum

$$0 + F_{\text{ave}}\Delta t = mv$$

$$F_{\text{ave}} = \frac{mv}{\Delta t} = \frac{(0.046 \text{ kg})\left(\frac{0.05 \text{ m}}{0.001 \text{ s}}\right)}{0.0006 \text{ s}}$$

$$\boxed{F_{\text{ave}} = 3830 \text{ N.}}$$

Problem 16.34 In a test of an energy-absorbing bumper, a 12700 N car is driven into a barrier at 8 km/h. The duration of the impact is 0.4 seconds. When the car rebounds from the barrier, the magnitude of its velocity is 1.6 km/h.

(a) What is the magnitude of the average horizontal force exerted on the car during the impact?
(b) What is the average deceleration of the car during the impact?

Solution: The velocities are

$$v_1 = 8 \text{ km/h} = 2.22 \text{ m/s}, \quad v_2 = 1.6 \text{ km/h} = 0.44 \text{ m/s}.$$

(a) Using the principle of impulse and momentum we have

$$-mv_1 + F_{\text{ave}}\Delta t = mv_2$$

$$F_{\text{ave}} = m\frac{(v_1 + v_2)}{\Delta t} = \frac{12700 \text{ N}}{9.81 \text{ m/s}^2}\left(\frac{2.22 \text{ m/s} + 0.44 \text{ m/s}}{0.4 \text{ s}}\right)$$

$$\boxed{F_{\text{ave}} = 8609 \text{ N}}$$

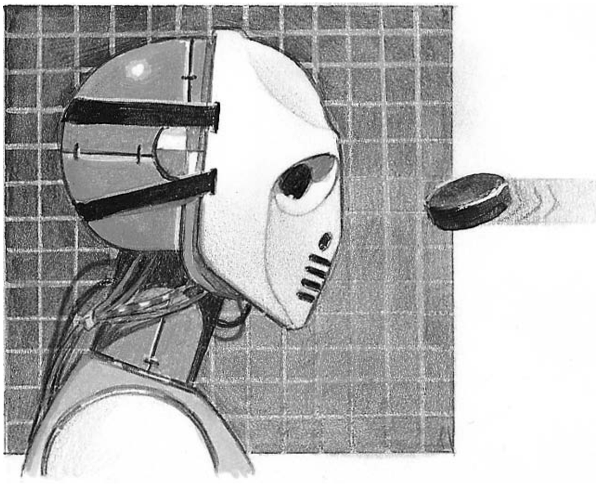
(b) The average deceleration of the car during impact is

$$a = \frac{v_2 - (-v_1)}{\Delta t} = \frac{(2.22 \text{ m/s} + 0.44 \text{ m/s})}{0.4 \text{ s}}$$

$$\boxed{a = 6.65 \text{ m/s}^2}$$

Problem 16.35 A bioengineer, using an instrumented dummy to test a protective mask for a hockey goalie, launches the 170-g puck so that it strikes the mask moving horizontally at 40 m/s. From photographs of the impact, she estimates the duration to be 0.02 s and observes that the puck rebounds at 5 m/s.

(a) What linear impulse does the puck exert?
(b) What is the average value of the impulsive force exerted on the mask by the puck?



Solution:

(a) The linear impulse is

$$\int_{t_1}^{t_2} F\,dt = F_{\text{ave}}(t_2 - t_1) = mv_2 - mv_1.$$

The velocities are $v_2 = -5$ m/s, and $v_1 = 40$ m/s, from which

$$\int_{t_1}^{t_2} F\,dt = F_{\text{ave}}(t_2 - t_1) = (0.17)(-5 - 40) = -7.65 \text{ N-s},$$

where the negative sign means that the force is directed parallel to the negative x axis.

(b) The average value of the force is

$$\boxed{F_{\text{ave}} = \frac{-7.65}{0.02} = -382.5 \text{ N}}$$

Problem 16.36 A fragile object dropped onto a hard surface breaks because it is subjected to a large impulsive force. If you drop a 0.56 N watch from 1.22 m above the floor, the duration of the impact is 0.001 s, and the watch bounces 51 mm. above the floor, what is the average value of the impulsive force?

Solution: The impulse is

$$\int_{t_1}^{t_2} F\,dt = F_{\text{ave}}(t_2 - t_1) = \left(\frac{W}{g}\right)(v_2 - v_1).$$

The weight of the watch is

$$W = 0.56 \text{ N},$$

and its mass is

$$\left(\frac{W}{g}\right) = 0.0571 \text{ kg}$$

The velocities are obtained from energy considerations (the conservation of energy in free fall):

$$v_1 = \sqrt{2gh} = \sqrt{2(9.81)(1.22)} = 4.88 \text{ m/s}.$$

$$v_2 = -\sqrt{2gh} = -\sqrt{2(9.81)(0.051)} = -1.0 \text{ m/s}.$$

The average value of the impulsive force is

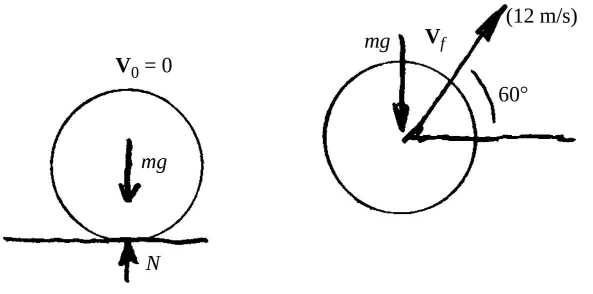
$$\boxed{F_{\text{ave}} = \frac{(0.0571 \times 10^{-3})(-1 - 4.88)}{1 \times 10^{-3}} = -334 \text{ N}}$$

Problem 16.37 The 0.45-kg soccer ball is given a kick with a 0.12-s duration that accelerates it from rest to a velocity of 12 m/s at 60° above the horizontal.

(a) What is the magnitude of the average total force exerted on the ball during the kick?
(b) What is the magnitude of the average force exerted on the ball by the player's foot during the kick?

Strategy: Use Eq. (16.2) to determine the average total force on the ball. To determine the average force exerted by the player's foot, you must subtract the ball's weight from the average total force.

Solution:



$$\int \mathbf{F}\,dt = \mathbf{F}_{\text{avg}}\Delta t = m\mathbf{V}_f - m\mathbf{V}_0$$

$$\mathbf{F}_{AV}(0.12) = 0.45(12\cos 60^\circ\mathbf{i} + 12\sin 60^\circ\mathbf{j})$$

$$\mathbf{F}_{AV} = 22.5\mathbf{i} + 39.0\mathbf{j}\text{ N}$$

(a) $\underline{|\mathbf{F}_{AV}| = 45.0\text{ N}}$ $\quad mg = (0.45)(9.81) = 4.41\text{ N}$

$$\mathbf{F}_{AV} = \mathbf{F}_{\text{FOOT}} + \mathbf{F}_G$$

$$\mathbf{F}_{AV} = \mathbf{F}_{\text{FOOT}} - mg\mathbf{j}$$

$$\mathbf{F}_{\text{FOOT}} = \mathbf{F}_{AV} + mg\mathbf{j}$$

$$\mathbf{F}_{\text{FOOT}} = 22.5\mathbf{i} + 39.0\mathbf{j} + 4.41\mathbf{j}$$

(b) $|\mathbf{F}_{\text{FOOT}}| = 48.9\text{ N}$

Problem 16.38 An entomologist measures the motion of a 3-g locust during its jump and determines that the insect accelerates from rest to 3.4 m/s in 25 ms (milliseconds). The angle of takeoff is 55° above the horizontal. What are the horizontal and vertical components of the average impulsive force exerted by the locust's hind legs during the jump?

Solution: The impulse is

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = \mathbf{F}_{\text{ave}}(t_2 - t_1) = m(\mathbf{v}_2) = m(3.4\cos 55^\circ\mathbf{i} + 3.4\sin 55^\circ\mathbf{j}),$$

from which

$$\mathbf{F}_{\text{ave}}(2.5\times 10^{-2}) = (5.85\times 10^{-3})\mathbf{i} + (8.36\times 10^{-3})\mathbf{j}\text{ N-s.}$$

The average total force is

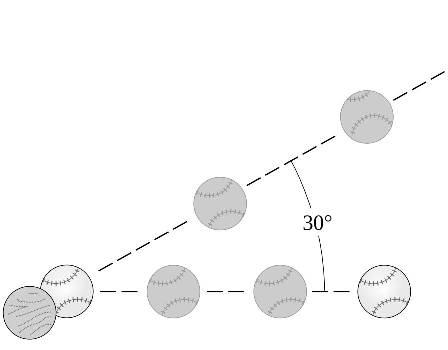
$$\mathbf{F}_{\text{ave}} = \frac{1}{2.5\times 10^{-2}}((5.85\times 10^{-3})\mathbf{i} + (8.36\times 10^{-3})\mathbf{j})$$

$$= 0.234\mathbf{i} + 0.334\mathbf{j}\text{ N.}$$

The impulsive force is

$$\boxed{\mathbf{F}_{\text{imp}} = \mathbf{F}_{\text{ave}} - (-mg\mathbf{j}) = 0.234\mathbf{i} + 0.364\mathbf{j}\text{ N}}$$

Problem 16.39 A 1.4 N baseball is 0.91 m above the ground when it is struck by a bat. The horizontal distance to the point where the ball strikes the ground is 54.9 m. Photographs studies indicate that the ball was moving approximately horizontally at 30.5 m/s before it was struck, the duration of the impact was 0.015 s, and the ball was traveling at 30° above the horizontal after it was struck. What was the magnitude of the average impulsive force exerted on the ball by the bat?



Solution: The impulse is

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = \mathbf{F}_{\text{ave}}(t_2 - t_1) = \left(\frac{W}{g}\right)(\mathbf{v}_2 - \mathbf{v}_1).$$

The velocity v_2 is determined from the trajectory. The path is

$$y = -\frac{gt^2}{2} + (v_2 \sin 30^\circ)t + y_0,$$

$$x = (v_2 \cos 30^\circ)t$$

where v_2 is the magnitude of the velocity at the point of leaving the bat, and $y_0 = 0.91$ m. At $x = 54.9$ m, $t = 54.9/(v_2 \cos 30^\circ)$, and $y = 0$. Substitute and reduce to obtain

$$v_2 = 54.9\sqrt{\frac{g}{2\cos^2 30^\circ(54.9\tan 30^\circ + y_0)}} = 24.6 \text{ m/s}.$$

From which:

$$\mathbf{F}_{\text{ave}} = \left(\frac{1}{0.015}\right)\left(\frac{W}{g}\right)((v_2\cos 30^\circ)\mathbf{i} + (v_2 \sin 30^\circ)\mathbf{j} - (-30.5\mathbf{i}))$$

$$= 489.3\mathbf{i} + 116.1\mathbf{j}.$$

Subtract the weight of the baseball: $\mathbf{F}_{\text{imp}} = \mathbf{F}_{\text{ave}} - (-W\mathbf{j}) = 489.3\mathbf{i} + 117.5\mathbf{j}$, from which

$$\boxed{|\mathbf{F}_{\text{imp}}| = 503.2 \text{ N}}$$

Problem 16.40 Paraphrasing the official rules of racquetball, a standard racquetball is 56 mm in diameter, weighs 0.4 N, and bounces between 68 and 72 centimetres from a 100-cm drop at a temperature between 70 and 74 degrees Fahrenheit Suppose that a ball bounces 71 cm when it is dropped from a 100-cm height. If the duration of the impact is 0.08 s, what average force is exerted on the ball by the floor?

Solution: The velocities before and after the impact are

$$v_1 = \sqrt{2(9.81 \text{ m/s}^2)(1 \text{ m})} = 4.43 \text{ m/s},$$

$$v_2 = \sqrt{2(9.81 \text{ m/s}^2)(0.71)} = 3.73 \text{ m/s},$$

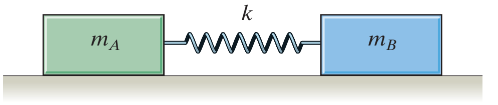
Using the principle of impulse and momentum we have

$$-mv_1 + F_{\text{ave}}\Delta t = mv_2$$

$$F_{\text{ave}} = m\frac{v_1 + v_2}{\Delta t} = (0.4 \text{ N})\left(\frac{1}{9.81 \text{ m/s}^2}\right)\frac{(4.43 \text{ m/s} + 3.73 \text{ m/s})}{0.08 \text{ s}}$$

$$\boxed{F_{\text{ave}} = 4.16 \text{ N.}}$$

Problem 16.41 The masses $m_A = m_B$. The surface is smooth. At $t = 0$, A is stationary, the spring is unstretched, and B is given a velocity v_0 toward the right.



(a) In the subsequent motion, what is the velocity of the common center of mass of A and B?
(b) What are the velocities of A and B when the spring is unstretched?

Strategy: To do part (b), think about the motions of the masses relative to their common center of mass.

Solution:

(a) The velocity of the center of mass does not change because there are no external forces on the system

$$v_c = \frac{m_A v_0 + m_B 0}{(m_A + m_B)} = \frac{v_0}{2} \quad \boxed{v_C = \frac{v_0}{2}}.$$

(b) Looking at the system from a reference frame moving with the center of mass, we have an oscillatory system with either the masses moving towards the center or away from the center. Translating back to the ground reference system, this means

$$\boxed{\begin{array}{ll} \text{Either} & v_A = v_0(\text{to the right}),\ v_B = 0 \\ \text{or} & v_A = 0,\ v_B = v_0(\text{to the right}). \end{array}}$$

Problem 16.42 In Problem 16.41, $m_A = 40$ kg, $m_B = 30$ kg, and $k = 400$ N/m. The two masses are released from rest on the smooth surface with the spring stretched 1 m. What are the magnitudes of the velocities of the masses when the spring is unstretched?

Solution: From the solution of Problem 16.41, (1) $m_A v_A + m_B v_B = 0$: or, evaluating, $40v_A + 30v_B = 0$. Energy is conserved, Thus, (2) $\frac{1}{2}kS^2 = \frac{1}{2}m_A v_A{}^2 + \frac{1}{2}m_B v_B{}^2$. Evaluating, we get

$$\frac{1}{2}(400)(1)^2 = \frac{1}{2}(40)v_A^2 + \frac{1}{2}(30)v_B^2$$

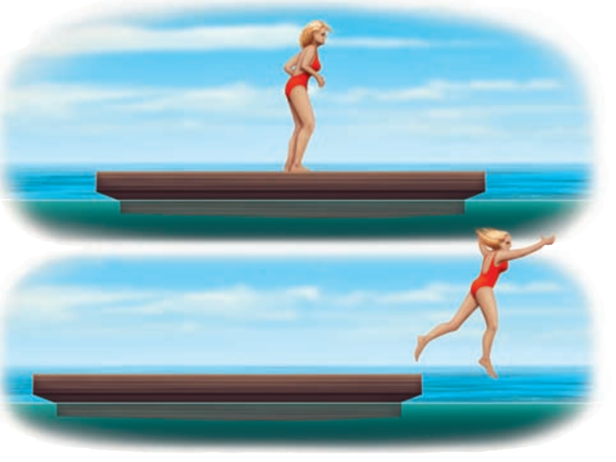
Solving Equations (1) and (2),

$|\mathbf{v}_A| = 2.07$ m/s,

$|\mathbf{v}_B| = 2.76$ m/s.

Problem 16.43 A girl weighing 356 N stands at rest on a 1446 N floating platform. She starts running at 3.05 m/s *relative to the platform* and runs off the end. Neglect the horizontal force exerted on the platform by the water.

(a) After she starts running, what is her velocity relative to the water?
(b) While she is running, what is the velocity of the common center of mass of the girl and the platform relative to the water? (See Active Example 16.4.)



Solution:

(a) Momentum is conserved.

$$0 = m_g v_g + m_b v_p,\ v_{g/p} = v_g - v_p$$

$$0 = (356\ \text{N})v_g + (1446\ \text{N})v_p,\ (3.05\ \text{m/s}) = v_g - v_p$$

Solving we find

$$v_g = \frac{(3.05\ \text{m/s})(1446\ \text{N})}{1802\ \text{N}} = 2.45\ \text{m/s}.$$

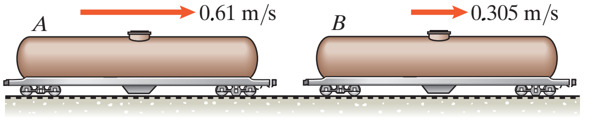
$\boxed{v_g = 2.45\ \text{m/s to the right.}}$

(b) Since momentum is conserved, the velocity of the center of mass is always zero.

$\boxed{v_{cm} = 0.}$

Problem 16.44 Two railroad cars with weights $W_A = 533.8$ kN and $W_B = 311.4$ kN collide and become coupled together. Car A is full, and car B is half full, of carbolic acid. When the cars collide, the acid in B sloshes back and forth violently.

(a) Immediately after the impact, what is the velocity of the common center of mass of the two cars?
(b) When the sloshing in B has subsided, what is the velocity of the two cars?

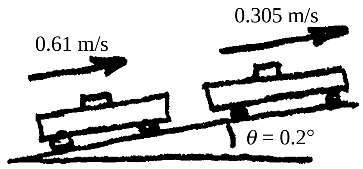


Solution:

(a) $$\boxed{\begin{aligned} v_{\text{center of mass}} &= \frac{(533800\ \text{N})(0.61\ \text{m/s}) + (311400\ \text{N})(0.305\ \text{m/s})}{(533800\ \text{N} + 311400\ \text{N})} \\ &= 0.497\ \text{m/s} \end{aligned}}$$

(b) After the sloshing stops $\boxed{v_{\text{cars}} = v_{\text{center of mass}} = 0.497\ \text{m/s}}$

Problem 16.45 Suppose that the railroad track in Problem 16.44 has a constant slope of 0.2 degrees upward toward the right. If the cars are 1.83 m apart at the instant shown, what is the velocity of their common center of mass immediately after they become coupled together?



Solution: Time to couple (both accelerate at the same rate) is $t = \dfrac{1.83\ \text{m}}{(0.61\ \text{m/s} - 0.305\ \text{m/s})} = 6\ \text{s}$.

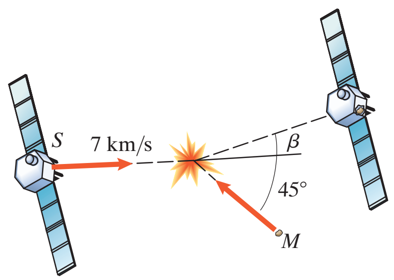
Impulse—momentum is now

$$\left(\frac{533800\ \text{N}}{9.81\ \text{m/s}^2}\right)(0.61\ \text{m/s}) + \left(\frac{311400\ \text{N}}{9.81\ \text{m/s}^2}\right)(0.305\ \text{m/s})$$

$$-(845200\ \text{N})\sin 0.2^\circ(6\ \text{s}) = \left(\frac{845200\ \text{N}}{9.81\ \text{m/s}^2}\right) v_{\text{center of mass}}$$

$$\boxed{v_{\text{center of mass}} = 0.292\ \text{m/s}}$$

Problem 16.46 The 400-kg satellite S traveling at 7 km/s is hit by a 1-kg meteor M traveling at 12 km/s. The meteor is embedded in the satellite by the impact. Determine the magnitude of the velocity of their common center of mass after the impact and the angle β between the path of the center of mass and the original path of the satellite.



Solution:

(a, b) $m_A v_A + m_B v_B = (m_A + m_B) v_f$

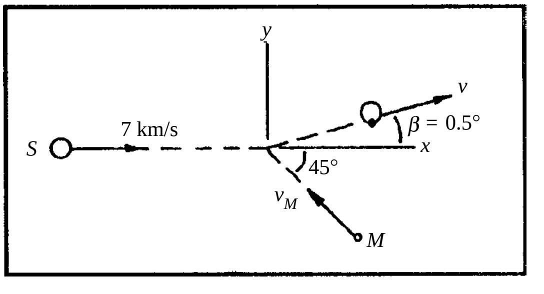
$(9600)(2) + (5400)(1) = (15000) v_f$

$v_f = \dfrac{246}{150}$ m/s

$\underline{v_f = 1.64 \text{ m/s to the right}}$

Problem 16.47 The 400-kg satellite S traveling at 7 km/s is hit by a 1-kg meteor M. The meteor is embedded in the satellite by the impact. What would the magnitude of the velocity of the meteor need to be to cause the angle β between the original path of the satellite and the path of the center of mass of the combined satellite and meteor after the impact to be 0.5°? What is the magnitude of the velocity of the center of mass after the impact?

Solution: Conservation of linear momentum yields $(400)(7\mathbf{i}) + (1)(-v_m \sin 45^\circ \mathbf{i} + v_m \cos 45^\circ \mathbf{j}) = (400+1)(v \cos 0.5^\circ i + v \sin 0.5^\circ \mathbf{j})$. Equating $\mathbf{i}$ and $\mathbf{j}$ components, we get $(400)(7) - v_m \cos 45^\circ = 401\ v \cos 0.5^\circ$; $v_m \sin 45^\circ = 401\ v \sin 0.5^\circ$ and solving, we obtain $v_m = 34.26$ km/s : $v = 6.92$ km/s.

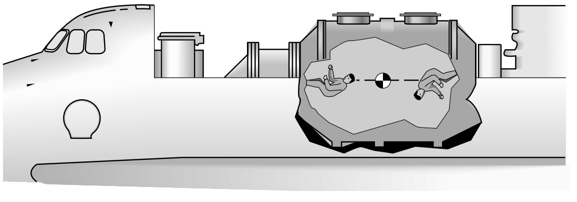


Problem 16.48 A 68-kg astronaut is initially stationary at the left side of an experiment module within an orbiting space shuttle. The 105,000-kg shuttle's center of mass is 4 m to the astronaut's right. He launches himself toward the center of mass at 1 m/s *relative to the shuttle*. He travels 8 m relative to the shuttle before bringing himself to rest at the opposite wall of the experiment module.

(a) What is the change in the magnitude of the shuttle's velocity relative to its original velocity while the astronaut is in motion?

(b) What is the change in the magnitude of the shuttle's velocity relative to its original velocity after his "flight"?

(c) Where is the shuttle's center of mass relative to the astronaut after his "flight"?



Solution: Consider the motion of the astronaut (A) and shuttle (S) relative to a reference frame that is stationary with respect to their common center of mass. During the astronaut's motion,

$$m_A v_A + m_S v_S = 0$$

and $v_A - v_S = 1$.

Solving these two equations, we obtain

(a) $v_S = -0.000647$ m/s.

(b) After his flight $v_A = v_S$, so $v_S = 0$.

(c) It is 4 m to his left.

Problem 16.49 An 356 N boy sitting in a stationary 89 N wagon wants to simulate rocket propulsion by throwing bricks out of the wagon. Neglect horizontal forces on the wagon's wheels. If the boy has three bricks weighing 44.5 N each and throws them with a horizontal velocity of 3.05 m/s relative to the wagon, determine the velocity he attains (a) if he throws the bricks one at a time and (b) if he throws them all at once.

Solution:

(a) The boy (B) in the wagon (w) throws one brick (b) at a time.

First brick:

$$0 = (m_B + m_w + 2m_b)v_{w1} + m_b v_{b1},$$

$$v_{b1} - v_{w1} = -3.05.$$

Solving, $v_{w1} = 0.234$ m/s.

Second brick:

$$(m_B + m_w + 2m_b)v_{w1} = (m_B + m_w + m_b)v_{w2} + m_b v_{b2},$$

$$v_{b2} - v_{w2} = -3.05.$$

Solving, $v_{w2} = 0.489$ m/s.

Third brick :

$$(m_B + m_w + m_b)v_{w2} = (m_B + m_w)v_{w3} + m_b v_{b3},$$

$$v_{b3} - v_{w3} = -3.05.$$

Solving, $\underline{v_{w3} = 0.765 \text{ m/s}}$.

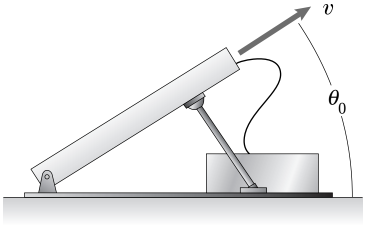
(b) All the bricks are thrown at once.

$$0 = (m_B + m_w)v_w + 3m_b v_b,$$

$$v_b - v_w = -3.05 \text{ m/s}.$$

Solving, $\underline{v_w = 0.704 \text{ m/s}}$.

Problem 16.50 A catapult designed to throw a line to ships in distress throws a 2-kg projectile. The mass of the catapult is 36 kg, and it rests on a smooth surface. It the velocity of the projectile *relative to the earth* as it leaves the tube is 50 m/s at $\theta_0 = 30^\circ$ relative to the horizontal, what is the resulting velocity of the catapult toward the left?



Solution:

$$0 = m_c v_c + m_p(50\cos 30^\circ).$$

Solving,

$$v_c = \frac{-(2)(50\cos 30^\circ)}{36} = -2.41 \text{ m/s}.$$

Problem 16.51 The catapult, which has a mass of 36 kg and throws a 2-kg projectile, rests on a smooth surface. The velocity of the projectile *relative to the catapult* as it leaves the tube is 50 m/s at $\theta_0 = 30^\circ$ relative to the horizontal. What is the resulting velocity of the catapult toward the left?

Solution:

$0 = m_c v_c + m_p v_{px},$

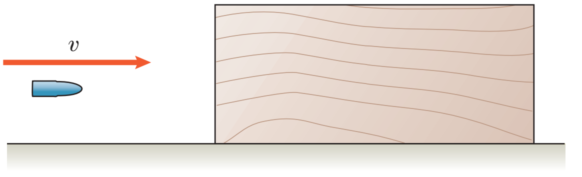
where

$v_{px} - v_c = 50 \cos 30^\circ.$

Solving,

$v_c = -2.28$ m/s.

Problem 16.52 A bullet with a mass of 3.6 grams is moving horizontally with velocity v and strikes a 5-kg block of wood, becoming embedded in it. After the impact, the bullet and block slide 24 mm across the floor. The coefficient of kinetic friction between the block and the floor is $\mu_k = 0.4$. Determine the velocity v.



Solution: Momentum is conserved through the collision and then work energy is used to finish the problem

$$m_b v = (M + m_b)v_2, \quad \frac{1}{2}(M + m_b){v_2}^2 - \mu_k(M + m_b)gd = 0$$

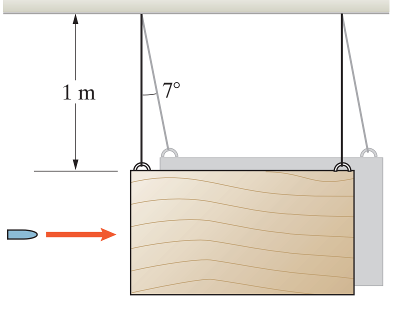
Solving we find

$v_2 = \sqrt{2\mu_k g d},$

$$v = \left(\frac{M + m_b}{m_b}\right)\sqrt{2\mu_k g d} = \left(\frac{5.0036 \text{ kg}}{0.0036 \text{ kg}}\right)\sqrt{2(0.4)(9.81 \text{ m/s}^2)(0.024 \text{ m})}$$

$\boxed{v = 603 \text{ m/s.}}$

Problem 16.53 A 28-g bullet hits a suspended 45-kg block of wood and becomes embedded in it. The angle through which the wires supporting the block rotate as a result of the impact is measured and determined to be 7°. What was the bullet's velocity?



Solution: Momentum is conserved through the collision and then work-energy is used to finish the problem.

$$m_b v_b = (M + m_b)v_2, \quad \frac{1}{2}(M + m_b){v_2}^2 = (M + m_b)gL(1 - \cos\theta)$$

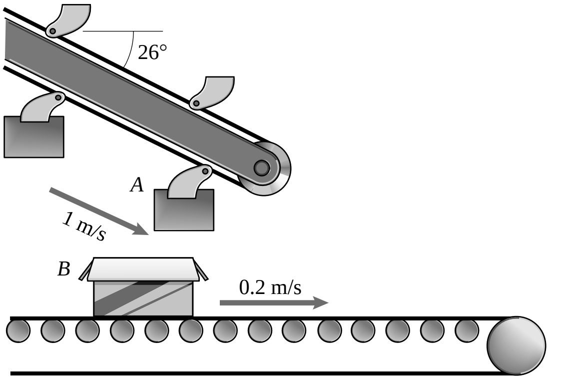
Solving we have

$$v_2 = \sqrt{2gL(1 - \cos\theta)} = \sqrt{2(9.81 \text{ m/s}^2)(1 \text{ m})(1 - \cos 7^\circ)} = 0.382 \text{ m/s},$$

$$v_b = \left(\frac{M + m_b}{m_b}\right)v_2 = \left(\frac{45 + 0.028}{0.028}\right)(0.382 \text{ m/s}) = 615 \text{ m/s}.$$

$\boxed{v_b = 615 \text{ m/s.}}$

Problem 16.54 The overhead conveyor drops the 12-kg package A into the 1.6-kg carton B. The package is tacky and sticks to the bottom of the carton. If the coefficient of friction between the carton and the horizontal conveyor is $\mu_k = 0.2$, what distance does the carton slide after impact?



Solution: The horizontal velocity of the package (A) *relative to the carton* (B) is

$$v_A = (1)\cos 26^\circ - 0.2 = 0.699 \text{ m/s}.$$

Let v be the velocity of the combined object relative to the belt.

$$m_A v_A = (m_A + m_B)v.$$

Solving, $$v = \frac{m_A v_A}{m_A + m_B} = \frac{(12)(0.699)}{12 + 1.6}$$

$$= 0.617 \text{ m/s}.$$

Use work and energy to determine the sliding distance d:

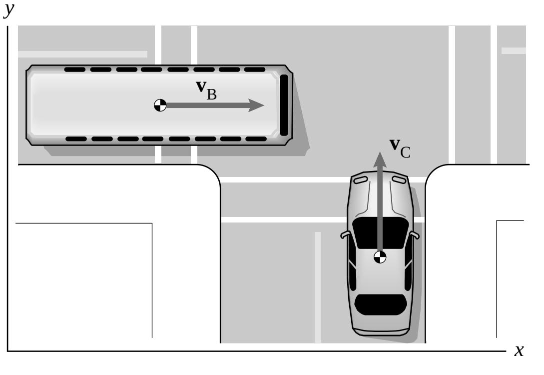
$$-\mu_k(m_A + m_B)gd = 0 - \tfrac{1}{2}(m_A + m_B)v^2,$$

$$d = \frac{v^2}{2\mu_k g} = \frac{(0.617)^2}{2(0.2)(9.81)}$$

$$d = 0.0969 \text{ m}.$$

Problem 16.55 A 53376 N bus collides with a 12454 N car. The velocity of the bus before the collision is $\mathbf{v}_B = 5.5\mathbf{i}$ (m/s) and the velocity of the car is $\mathbf{v}_C = 10\mathbf{j}$ (m/s). The two vehicles become entangled and remain together after the collision. The coefficient of kinetic friction between the vehicles' tires and the road is $\mu_k = 0.6$.

(a) What is the velocity of the common center of mass of the two vehicles immediately after the collision?
(b) Determine the approximate final position of the common center of mass of the vehicles relative to its position when the collision occurred. (Assume that the tires skid, not roll, on the road.)



Solution:

(a) The collision (impulse - momentum).

$$\left(\frac{53376 \text{ N}}{9.81 \text{ m/s}^2}\right)(5.5\mathbf{i} \text{ m/s}) + \left(\frac{12454 \text{ N}}{9.81 \text{ m/s}^2}\right)(10\mathbf{j} \text{ m/s}) = \left(\frac{65830 \text{ N}}{9.81 \text{ m/s}^2}\right)\mathbf{v}$$

$$\boxed{\mathbf{v} = (4.45\mathbf{i} + 1.90\mathbf{j}) \text{ m/s}, \quad v = 4.84 \text{ m/s}, \quad \theta = 23.2^\circ}$$

(b) The skid after the accident (work–energy).

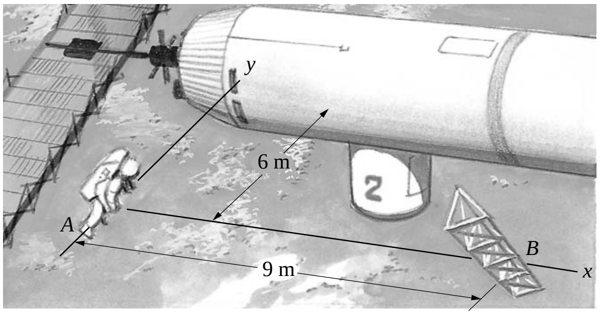
$$\frac{1}{2}\left(\frac{65830 \text{ N}}{9.81 \text{ m/s}^2}\right)(4.84 \text{ m/s})^2 - (0.6)(65830 \text{ N})s \Rightarrow s = 1.99 \text{ m}$$

The final position is

$$\boxed{\mathbf{r} = (1.99 \text{ m})(\cos 23.2^\circ\mathbf{i} + \sin 23.2^\circ\mathbf{j}) = (1.83\mathbf{i} + 0.78\mathbf{j}) \text{ m}}$$

Problem 16.56 The velocity of the 200-kg astronaut A relative to the space station is $40\mathbf{i} + 30\mathbf{j}$ (mm/s). The velocity of the 300-kg structural member B relative to the station is $-20\mathbf{i} + 30\mathbf{j}$ (mm/s). When they approach each other, the astronaut grasps and clings to the structural member.

(a) What is the velocity of their common center of mass when they arrive at the station?
(b) Determine the approximate position at which they contact the station.



Solution:

(a) The velocity of the center of mass after the collision

$$(200\ \text{kg})(0.04\mathbf{i} + 0.03\mathbf{j})\ \text{m/s} + (300\ \text{kg})(-0.02\mathbf{i} + 0.03\mathbf{j})\ \text{m/s}$$

$$= (500\ \text{kg})\ \mathbf{v}$$

$$\boxed{\mathbf{v} = (0.004\mathbf{i} + 0.03\mathbf{j})\ \text{m/s}}$$

(b) The time to arrive at the station is $t = \dfrac{6\ \text{m}}{0.03\ \text{m/s}} = 200$ s.

The center of mass of the two bodies starts at

$$\mathbf{r}_0 = \frac{(200\ \text{kg})(0) + (300\ \text{kg})(9\mathbf{i})\ \text{m}}{500\ \text{kg}} = 5.4\mathbf{i}\ \text{m}$$

The position upon arrival is

$$\boxed{\mathbf{r} = (5.4\mathbf{i})\ \text{m} + [(0.004\mathbf{i} + 0.03\mathbf{j})\ \text{m/s}](200\ \text{s}) = (6.2\mathbf{i} + 6\mathbf{j})\ \text{m}}$$

Problem 16.57 The weights of the two objects are $W_A = 5$ N and $W_B = 8$ N. Object A is moving at $v_A = 2$ m/s and undergoes a perfectly elastic impact with the stationary object B. Determine the velocities of the objects after the impact.

Solution: Momentum is conserved and the coefficient of restitution is also used.

$$m_A v_A = m_A v'_A + m_B v'_B, \quad e v_A = v'_B - v'_A$$

$$(5\ \text{N})\ (2\ \text{m/s}) = (5\ \text{N})v'_A + (8\ \text{N})v'_B,\ (2\ \text{m/s}) = v'_B - v'_A$$

Solving, we find $v'_A = -0.462$ m/s, $v'_B = 1.54$ m/s.

Therefore $\boxed{v'_A = 0.462\ \text{m/s to the left},\quad v'_B = 1.54\ \text{m/s to the right.}}$

Problem 16.58 The weights of the two objects are $W_A = 5$ N and $W_B = 8$ N. Object A is moving at $v_A = 2$ m/s and undergoes a direct central impact with the stationary object B. The coefficient of restitution is $e = 0.8$. Determine the velocities of the objects after the impact.

Solution: Momentum is conserved and the coefficient of restitution is also used.

$$m_A v_A = m_A v_A' + m_B v_B', \quad e v_A = v_B' - v_A'$$

$$(5\text{ N})(2\text{ m/s}) = (5\text{ N})v_A' + (8\text{ N})v_B', (0.8)(2\text{ m/s}) = v_B' - v_A'$$

Solving, we find $v_A' = -0.215$ m/s, $v_B' = 1.38$ m/s.

Therefore $\boxed{v_A' = 0.462\text{ m/s to the left}, \quad v_B' = 1.54\text{ m/s to the right.}}$

Problem 16.59 The objects A and B with velocities $v_A = 20$ m/s and $v_B = 4$ m/s undergo a direct central impact. Their masses are $m_A = 8$ kg and $m_B = 12$ kg. After the impact, the object B is moving to the right at 16 m/s. What is the coefficient of restitution?

Solution: Momentum is conserved, the coefficient of restitution is used.

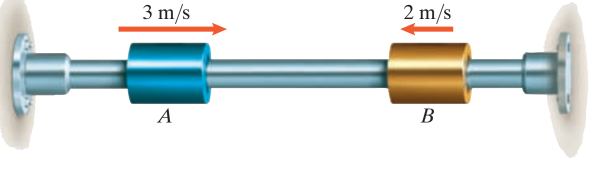
$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B', \quad e(v_A - v_B) = v_B' - v_A'$$

$$(8\text{ kg})\,(20\text{ m/s}) + (12\text{ kg})\,(4\text{ m/s}) = (8\text{ kg})v_A' + (12\text{ kg})\,(16\text{ m/s})$$

$$e(20\text{ m/s} - 4\text{ m/s}) = (16\text{ m/s}) - v_A'$$

Solving we find $v_A' = 2.0$ m/s, $\boxed{e = 0.875.}$

Problem 16.60 The 8-kg mass A and the 12-kg mass B slide on the smooth horizontal bar with the velocities shown. The coefficient of restitution is $e = 0.2$. Determine the velocities of the masses after they collide. (See Active Example 16.5).



Solution: Momentum is conserved, and the coefficient of restitution is used.

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B', \quad e(v_A - v_B) = v_B' - v_A'$$

Putting in the numbers we have

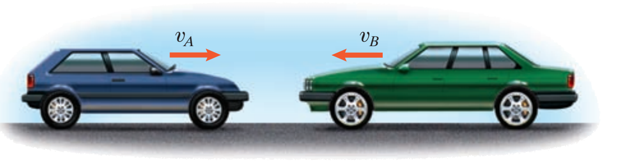
$$(8\text{ kg})\,(3\text{ m/s}) + (12\text{ kg})(-3\text{ m/s}) = (8\text{ kg})v_A' + (12\text{ kg})v_B'$$

$$(0.2)([3\text{ m/s}] - [-2\text{ m/s}]) = v_B' - v_A'$$

Solving we find $v_A' = -0.6$ m/s, $v_B' = 0.4$ m/s.

Thus $\boxed{v_A' = 0.6\text{ m/s to the left}, \quad v_B' = 0.4\text{ m/s to the right.}}$

Problem 16.61 In a study of the effects of an accident on simulated occupants, the 1900 N car with velocity $v_A = 30$ km/h collides with the 2800 N car with velocity $v_B = 20$ km/h. The coefficient of restitution of the impact is $e = 0.15$. What are the velocities of the cars immediately after the collision?



Solution: Momentum is conserved, and the coefficient of restitution is used.

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B', \quad e(v_A - v_B) = v_B' - v_A'$$

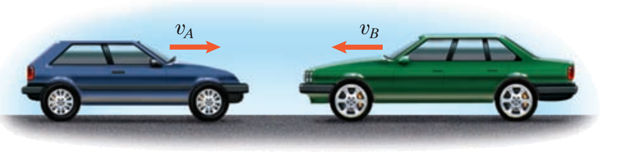
$$(1900 \text{ N})\,(30 \text{ km/h}) + (2800 \text{ N})(-20 \text{ km/h}) = (1900 \text{ N})\,v_A' + (2800 \text{ N})v_B'$$

$$(0.15)([30 \text{ km/h}] - [-20 \text{ km/h}]) = v_B' - v_A'$$

Solving we find $v_A' = -4.26$ km/h, $v_B' = 3.24$ km/h.

Converting into ft/s we find $\boxed{v_A' = 1.18 \text{ m/s to the left}, \;\; v_B' = 0.9 \text{ m/s to the right.}}$

Problem 16.62 In a study of the effects of an accident on simulated occupants, the 1900 N car with velocity $v_A = 30$ km/h collides with the 2800 N car with velocity $v_B = 20$ km/h. The coefficient of restitution of the impact is $e = 0.15$. The duration of the collision is 0.22 s. Determine the magnitude of the average acceleration to which the occupants of each car are subjected.



Solution: The velocities before the collision are converted into ft/s. The velocities after the collision were calculated in the preceding problem. The velocities are

$$v_A = 8.33 \text{ m/s}, \;\; v_B = 5.55 \text{ m/s}, \;\; v_A' = -1.18 \text{ m/s}, \;\; v_B' = 0.9 \text{ m/s}.$$

The average accelerations are

$$a_A = \frac{\Delta v}{\Delta t} = \left| \frac{(-1.18 \text{ m/s}) - (8.33 \text{ m/s})}{0.22 \text{ s}} \right| = -43.23 \text{ m/s}^2$$

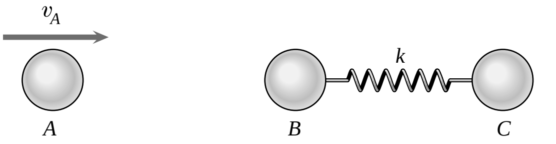
$$a_B = \frac{\Delta v}{\Delta t} = \left| \frac{(0.9 \text{ m/s}) - (-5.55 \text{ m/s})}{0.22 \text{ s}} \right| = 29.32 \text{ m/s}^2$$

$$\boxed{a_A = -43.23 \text{ m/s}^2, \;\; a_B = 29.32 \text{ m/s}^2.}$$

Problem 16.63 The balls are of equal mass m. Balls B and C are connected by an unstretched spring and are stationary. Ball A moves toward ball B with velocity v_A. The impact of A with B is perfectly elastic ($e = 1$).

(a) What is the velocity of the common center of mass of B and C immediately after the impact?

(b) What is the velocity of the common center of mass of B and C at time t after the impact?

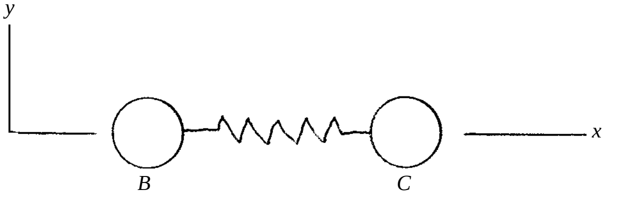


Solution: Consider the impact of balls A and B. From the equations

$$mv_A = mv_A' + mv_B',$$

$$e = 1 = \frac{v_B' - v_A'}{v_A},$$

we obtain $v_A' = 0$, $v_B' = v_A$.



(a) The position of the center of mass is

$$\mathbf{x} = \frac{mx_B + mx_C}{m + m} = \frac{x_B + x_C}{2},$$

so

$$\frac{d\mathbf{x}}{dt} = \frac{1}{2}\left(\frac{dx_B}{dt} + \frac{dx_C}{dt}\right).$$

Immediately after the impact $dx_B/dt = v_A$ and $dx_C/dt = 0$, so

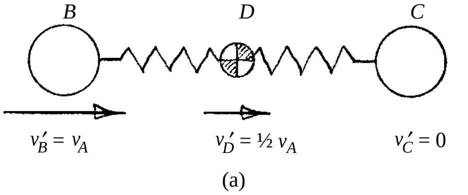
$$\frac{d\mathbf{x}}{dt} = \frac{1}{2}v_A.$$

(b) With no external forces,

$$\frac{d\mathbf{x}}{dt} = \text{const.} = \frac{1}{2}v_A.$$

Problem 16.64 In Problem 16.63, what is the maximum compressive force in the spring as a result of the impact?

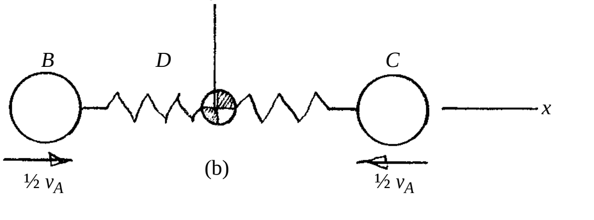
Solution: See the solution of Problem 16.63. Just after the collision of A and B, B is moving to the right with velocity v_A, C is stationary, and the center of mass D of B and C is moving to the right with velocity $\frac{1}{2}v_A$ (Fig. a). Consider the motion in terms of a reference frame that is moving with D (Fig. b). Relative to this reference frame, B is moving to the right with velocity $\frac{1}{2}v_A$ and C is moving to the left with velocity $\frac{1}{2}v_A$. There total kinetic energy is



(a)

$$\frac{1}{2}m\left(\frac{1}{2}v_A\right)^2 + \frac{1}{2}m\left(\frac{1}{2}v_A\right)^2 = \frac{1}{4}mv_A^2.$$

When the spring has brought B and C to rest relative to D, their kinetic energy has been transformed into potential energy of the spring. This is when the compressive force in the spring is greatest. Setting $\frac{1}{4}mv_A^2 = \frac{1}{2}kS^2$, we find that the compression of the spring is



(b)

$$S = -v_A\sqrt{\frac{m}{2k}}.$$

Therefore the maximum compressive force is

$$k|S| = v_A\sqrt{\frac{mk}{2}}.$$

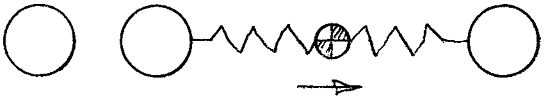
Problem 16.65* The balls are of equal mass m. Balls B and C are connected by an unstretched spring and are stationary. Ball A moves toward ball B with velocity v_A. The impact of A with B is perfectly elastic ($e = 1$). Suppose that you interpret this as an impact between ball A and an "object" D consisting of the connected balls B and C.

(a) What is the coefficient of restitution of the impact between A and D?

(b) If you consider the total energy after the impact to be the sum of the kinetic energies, $\frac{1}{2}m(v'_A)^2 + \frac{1}{2}(2\,m)(v'_D)^2$, where v'_D is the velocity of the center of mass of D after the impact, how much energy is "lost" as a result of the impact?

(c) How much energy is actually lost as a result of the impact? (This problem is an interesting model for one of the mechanisms for energy loss in impacts between objects. The energy "loss" calculated in part (b) is transformed into "internal energy"—the vibrational motions of B and C relative to their common center of mass.)

Solution: See the solution of Problem 16.135. Just after the impact of A and B, A is stationary and the center of mass D of B and C is moving with velocity $\frac{1}{2}v_A$.



(a) The coefficient of restitution is

$$e = \frac{v'_D - v'_A}{v_A} = \frac{\frac{1}{2}v_A - 0}{v_A} = \frac{1}{2}.$$

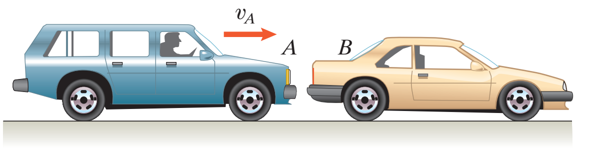
(b) The energy before the impact is $\frac{1}{2}mv_A^2$. The energy after is

$$\frac{1}{2}m(v'_A)^2 + \frac{1}{2}(2m)(v'_D)^2 = \frac{1}{4}mv_A^2.$$

The energy "lost" is $\frac{1}{4}mv_A^2$.

(c) No energy is actually lost. The total kinetic energy of A, B, and C just after the impact is $\frac{1}{2}mv_A^2$.

Problem 16.66 Suppose that you investigate an accident in which a 1300 kg car A struck a parked 1200 kg car B. All four of car B's wheels were locked, and skid marks indicate that it slid 8 m after the impact. If you estimate the coefficient of kinetic friction between B's tires and the road to be $\mu_k = 0.8$ and the coefficient of restitution of the impact to be $e = 0.4$, what was A's velocity v_A just before the impact? (Assume that only one impact occurred.)



Solution: We can use work-energy to find the velocity of car B just after the impact. Then we use conservation of momentum and the coefficient of restitution to solve for the velocity of A. In general terms we have

$$\frac{1}{2}m_B v_B'^2 - \mu_k m_B g d = 0 \Rightarrow v_B' = \sqrt{2\mu_k g d}$$

$$m_A v_A = m_A v_A' + m_B v_B', \quad e v_A = v_B' - v_A'$$

Putting in the numbers we have

$$v_B' = \sqrt{2(0.8)(9.81 \text{ m/s}^2)(8 \text{ m})} = 11.2 \text{ m/s},$$

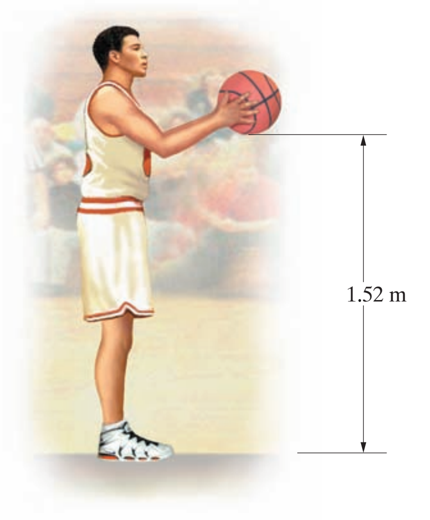
$$(1300 \text{ kg})\, v_A = (1300 \text{ kg})\, v_A' + (1200 \text{ kg})\,(11.2 \text{ m/s})$$

$$(0.2)v_A = (9.81 \text{ m/s}^2) - v_A'$$

Solving the last two equations simultaneously we have

$$v_A' = 6.45 \text{ m/s}, \quad \boxed{v_A = 16.8 \text{ m/s}}$$

Problem 16.67 When the player releases the ball from rest at a height of 1.52 m above the floor, it bounces to a height of 1.07 m. If he throws the ball downward, releasing it at 0.91 m above the floor, how fast would he need to throw it so that it would bounce to a height of 3.66 m?



Solution: When dropped from 1.52 m, the ball hits the floor with a speed

$$v_{\text{before}} = \sqrt{2(9.81 \text{ m/s}^2)(1.52 \text{ m})} = 5.47 \text{ m/s}$$

In order to rebound to 1.07 m, it must leave the floor with a speed

$$v_{\text{after}} = \sqrt{2(9.81 \text{ m/s}^2)(1.07 \text{ m})} = 4.58 \text{ m/s}$$

The coefficient of restitution is therefore $e = \dfrac{4.58 \text{ m/s}}{5.47 \text{ m/s}} = 0.837$

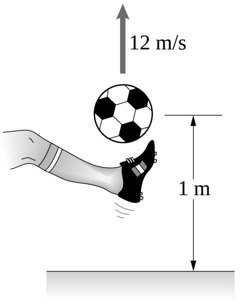
To bounce to a height of 3.66 m we need a rebound velocity of

$$v_{\text{rebound}} = \sqrt{2(9.81 \text{ m/s}^2)(3.66 \text{ m})} = 8.47 \text{ m/s}$$

Therefore, the ball must have a downward velocity of $\dfrac{8.47 \text{ m/s}}{0.837} =$ 10.13 m/s before it hits the floor. To find the original velocity when it leaves his hands,

$$\frac{1}{2}mv^2 + m(9.81 \text{ m/s}^2)(0.91\text{m}) = \frac{1}{2}\, m\,(10.13 \text{ m/s})^2 \Rightarrow \boxed{v = 9.2 \text{ m/s}}$$

Problem 16.68 The 0.45-kg soccer ball is 1 m above the ground when it is kicked upward at 12 m/s. If the coefficient of restitution between the ball and the ground is $e = 0.6$, what maximum height above the ground does the ball reach on its first bounce?



Solution: We must first find the velocity with which the ball strikes the ground. Then we analyze the impact. Finally, we analyze the post impact bounce.

Kick-to-Bounce Phase:- Use Cons. of Energy Datum is the ground level.

$\frac{1}{2}mv_0^2 + mgh_0 = \frac{1}{2}mv_1^2 + mg(0)$

Impact occurs at v_1

$v_0 = 12$ m/s, $h_0 = 1$ m, $m = 0.45$ kg,

$g = 9.81$ m/s^2

Solving, $\underline{v_1 = -12.79 \text{ m/s}}$ (downward)

Impact:

$$e = \frac{-v_1'}{v_1} = 0.6$$

$v_1' = 7.67$ m/s upward after impact

Post Impact:

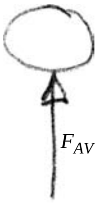
$\frac{1}{2}m(v_1')^2 + 0 = 0 + mgh_2$

$$h_2 = \frac{(v_1')^2}{2g}$$

$\underline{h_2 = 3.00 \text{ m}}$

Problem 16.69 The 0.45-kg soccer ball is stationary just before it is kicked upward at 12 m/s. If the impact lasts 0.02 s, what average force is exerted on the ball by the player's foot?

Solution: (Neglect gravity during the impact) Details of kick (all in y direction)



$v_0 = 0$

$v_1 = 12$ m/s

$m = 0.45$ kg

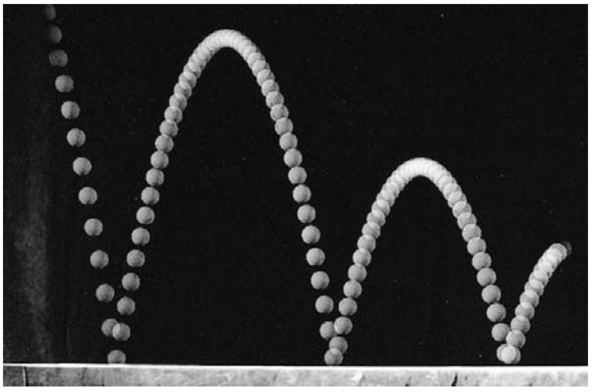
$\mathbf{j}$: $\int F_{AV}\,dt = F_{AV}\Delta t = mv_1 - mv_0$

$F_{AV}\Delta t = mv_1 = (0.45)(12) = 5.40$ N-s

$\Delta t = 0.02$ s,

Solving $\underline{F_{AV} = 270 \text{ N}}$

Problem 16.70 By making measurements directly from the photograph of the bouncing golf ball, estimate the coefficient of restitution.



Solution: For impact on a stationary surface, the coefficient of restitution is defined to be $e = -v'_A/v_A$. (Since the impact and rebound velocities have opposite signs, e is positive.) (See Eq. (16.19)). From the conservation of energy, $\frac{1}{2}m_A(v'_A)^2 = m_A gh$, the velocity is proportional to the square root of the rebound height, so that if $h_1, h_2, \ldots h_N, \ldots$ are successive rebound heights, then an estimate of e is $e = \sqrt{h_{i+1}/h_i}$. Measurements are $h_1 = 5.1$ cm, $h_2 = 3.1$ cm, from which

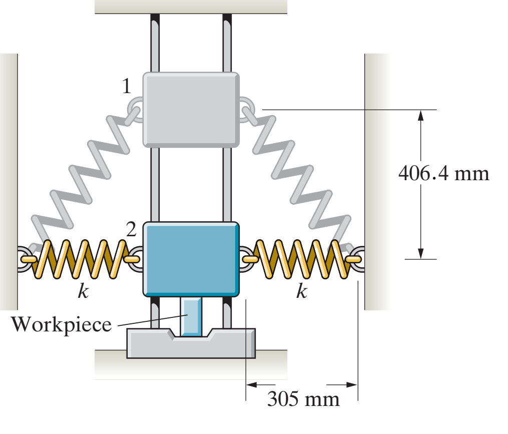
$$\boxed{e = \sqrt{3.1/5.1} = 0.78}$$

Problem 16.71 If you throw the golf ball in Problem 16.70 horizontally at 0.61m/s and release it 1.22 m above the surface, what is the distance between the first two bounces?

Solution: The normal velocity at impact is $v_{An} = -\sqrt{2g(1.22)} = -4.89$ m/s (downward). The rebound normal velocity is (from Eq (16.19)) $v'_{An} = -ev_{An} = -(0.78)(-4.89) = 3.81$ m/s (upward). From the conservation of energy for free fall the first rebound height is $h = (v'_{An})^2/2g = 0.74$ m/s. From the solution of Newton's second law for free fall, the time spent between rebounds is twice the time to fall from the maximum height: $t = 2\sqrt{2h/g} = 0.778$ s from which the distance between bounces is:

$$\boxed{d = v_0 t = 2t = 0.48 \text{ m}}$$

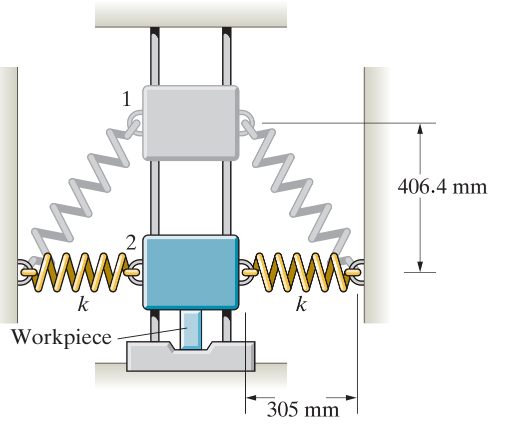
Problem 16.72 In a forging operation, the 100-N weight is lifted into position 1 and released from rest. It falls and strikes a workpiece in position 2. If the weight is moving at 15 m/s immediately before the impact and the coefficient of restitution is $e = 0.3$, what is the velocity of the weight immediately after impact?



Solution: The strategy is to treat the system as an in-line impact on a rigid, immovable surface. From Eq. (16.16) with B's velocity equal to zero: $v'_A = -ev_A$, from which

$$\boxed{v'_A = -0.3(-15) = 4.5 \text{ m/s}}$$

Problem 16.73 The 100-N weight is released from rest in position 1. The spring constant is $k = 1751$ N/m, and the springs are unstretched in position 2. If the coefficient of restitution of the impact of the weight with the workpiece in position 2 is $e = 0.6$, what is the magnitude of the velocity of the weight immediately after the impact?



Solution: Work-energy will be used to find the velocity just before the collision. Then the coefficient of restitution will give the velocity after impact.

$$2\left(\frac{1}{2}kx^2\right) + mgh = \frac{1}{2}\,mv^2$$

$$v = \sqrt{2\frac{k}{m}x^2 + 2gh}$$

$$= \sqrt{2\left(\frac{1751\ \text{N/m}}{\left[\dfrac{100\ \text{N}}{9.81\ \text{m/s}^2}\right]}\right)(0.51\ \text{m} - 0.305\ \text{m})^2 + 2(9.81\ \text{m/s}^2)(0.406\ \text{m})} = 4.73\ \text{m/s}$$

$v' = ev = (0.6)(4.73\ \text{m/s}) = 2.84\ \text{m/s}.$

$\boxed{v' = 2.84\ \text{m/s}.}$

Problem 16.74* A bioengineer studying helmet design uses an experimental apparatus that launches a 2.4-kg helmet containing a 2-kg model of the human head against a rigid surface at 6 m/s. The head, suspended within the helmet, is not immediately affected by the impact of the helmet with the surface and continues to move to the right at 6 m/s, so the head then undergoes an impact with the helmet. If the coefficient of restitution of the helmet's impact with the surface is 0.85 and the coefficient of restitution of the subsequent impact of the head with the helmet is 0.15, what is the velocity of the head after its initial impact with the helmet?

Solution: The helmet's rebound velocity is

$$v_{\text{helmet}} = (0.85)(6\ \text{m/s}) = 5.1\ \text{m/s}$$

The collision of the helmet and head

$$(2\ \text{kg})(6\ \text{m/s}) + (2.4\ \text{kg})(-5.1\ \text{m/s}) = (2\ \text{kg})v_{\text{head}}{}' + (2.4\ \text{kg})v_{\text{helmet}}{}'$$

$$0.15(6 - [-5.1])\ \text{m/s} = v_{\text{helmet}}{}' - v_{\text{head}}{}'$$

Solving we find $\boxed{v_{\text{head}}{}' = -0.963\ \text{m/s}}$

Problem 16.75*

(a) If the duration of the impact of the head with the helmet in Problem 16.74 is 0.004 s, what is the magnitude of the average force exerted on the head by the impact?

(b) Suppose that the simulated head alone strikes the rigid surface at 6 m/s, the coefficient of restitution is 0.5, and the duration of the impact is 0.0002 s. What is the magnitude of the average force exerted on the head by the impact?

Solution: See the solution to Problem 16.74

(a) $(2 \text{ kg})(6 \text{ m/s}) - F_{\text{ave}}(0.004 \text{ s}) = (2 \text{ kg})(-0.963 \text{ m/s})$

$$\Rightarrow \boxed{F_{\text{ave}} = 3.48 \text{ kN}}$$

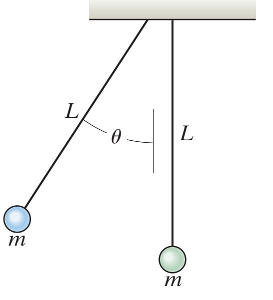
(b) The velocity of the head after the collision is

$$v = 0.5(6 \text{ m/s}) = 3 \text{ m/s}$$

$$(2 \text{ kg})(6 \text{ m/s}) - F_{\text{ave}}(0.0002 \text{ s}) = (2 \text{ kg})(-3 \text{ m/s})$$

$$\Rightarrow \boxed{F_{\text{ave}} = 90 \text{ kN}}$$

Problem 16.76 Two small balls, each of 1-N weight, hang from strings of length $L = 3$ m. The left ball is released from rest with $\theta = 35°$. The coefficient of restitution of the impact is $e = 0.9$. Through what maximum angle does the right ball swing?



Solution: Using work-energy and conservation of momentum we have

$$mgL(1 - \cos\theta) = \frac{1}{2}\, m{v_A}^2 \Rightarrow v_A = \sqrt{2gL(1 - \cos\theta)}$$

$$\left.\begin{aligned} mv_A &= mv_A' + mv_B' \\ ev_A &= v_B' - v_A' \end{aligned}\right\} \Rightarrow v_B' = \frac{1+e}{2} v_A$$

$$\frac{1}{2}\, m{v_B'}^2 = mgL(1 - \cos\phi) \Rightarrow \phi = \cos^{-1}\left(1 - \frac{{v_B'}^2}{2gL}\right)$$

Putting in the numbers we find

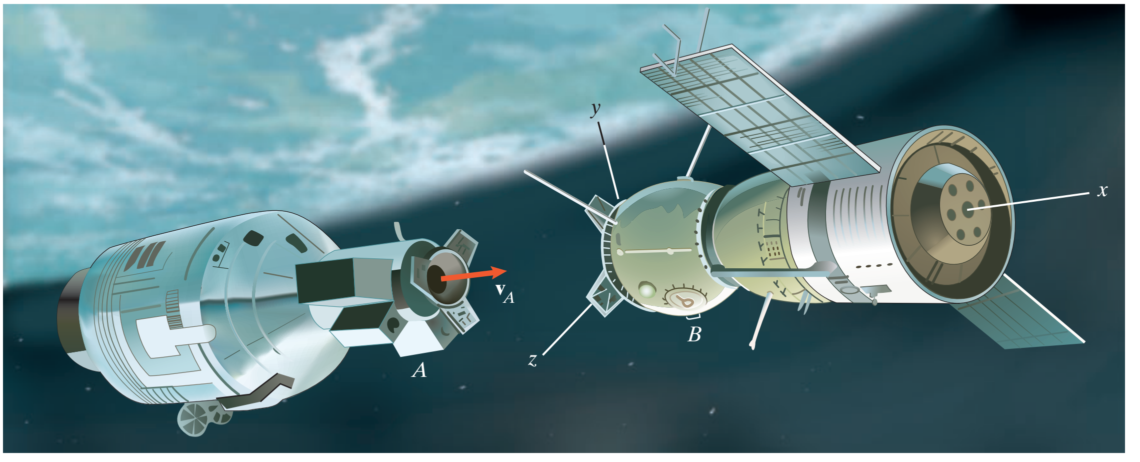
$$v_A = \sqrt{2(9.81 \text{ m/s}^2)(3 \text{ m})(1 - \cos 35°)} = 3.26 \text{ m/s},$$

$$v_B' = \frac{1.9}{2}(3.26 \text{ m/s}) = 3.1 \text{ m/s},$$

$$\phi = \cos^{-1}\left(1 - \frac{[3.1 \text{ m/s}]^2}{2[9.81 \text{ m/s}^2][3 \text{ m}]}\right) = 33.2°.$$

$$\boxed{\phi = 33.2°.}$$

Problem 16.77 In Example 16.6, if the *Apollo* command-service module approaches the *Soyuz* spacecraft with velocity $0.25\mathbf{i} + 0.04\mathbf{j} + 0.01\mathbf{k}$ (m/s) and the docking is successful, what is the velocity of the center of mass of the combined vehicles afterward?



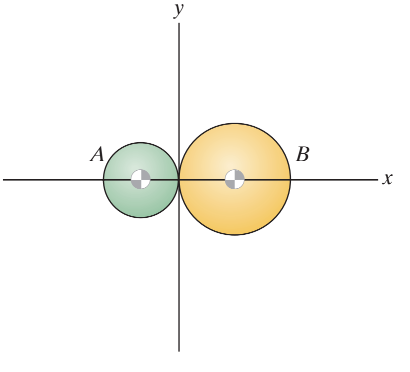
Solution: Momentum is conserved so

$$m_A\mathbf{v}_A = (m_A + m_S)\mathbf{v}_{\text{comb}} \Rightarrow \mathbf{v}_{\text{comb}} = \frac{m_A}{m_A + m_S}\mathbf{v}_A$$

$$\mathbf{v}_{\text{comb}} = \frac{18}{18 + 6.6}(0.25\mathbf{i} + 0.04\mathbf{j} + 0.01\mathbf{k})\ \text{m/s}.$$

$$\boxed{\mathbf{v}_{\text{comb}} = (0.183\mathbf{i} + 0.0293\mathbf{j} + 0.00732\mathbf{k})\ \text{m/s}.}$$

Problem 16.78 The 3-kg object A and 8-kg object B undergo an oblique central impact. The coefficient of restitution is $e = 0.8$. Before the impact, $\mathbf{v}_A = 10\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$ (m/s) and $\mathbf{v}_B = -2\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$ (m/s). What are the velocities of A and B after the impact?



Solution: Tangent to the impact plane the velocities do not change. In the impact plane we have

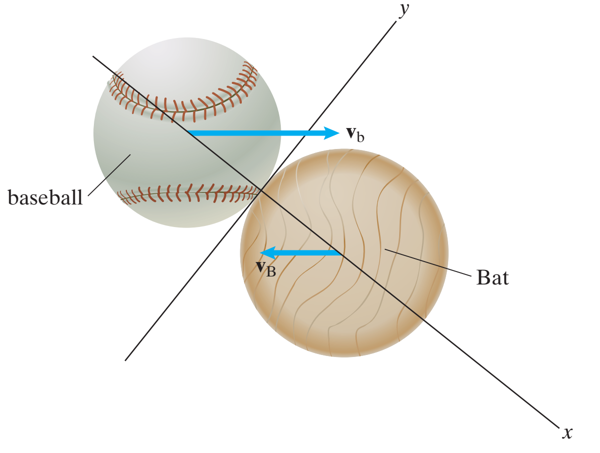
$$(3\ \text{kg})(10\ \text{m/s}) + (8\ \text{kg})(-2\ \text{m/s}) = (3\ \text{kg})v_{Ax}{}' + (8\ \text{kg})v_{Bx}{}'$$

$$0.8(10 - [-2])\ \text{m/s} = v_{Bx}{}' - v_{Ax}{}'$$

Solving we find $v_{Ax}{}' = -5.71$ m/s, $v_{Bx}{}' = 3.89$ m/s

Thus $\boxed{\mathbf{v}_A{}' = (-5.71\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})\ \text{m/s},\ \ \mathbf{v}_B{}' = (3.89\mathbf{i} - 6\mathbf{j} + 5\mathbf{k})\ \text{m/s}}$

Problem 16.79 A baseball bat (shown with the bat's axis perpendicular to the page) strikes a thrown baseball. Before their impact, the velocity of the baseball is $\mathbf{v}_b = 132(\cos 45^\circ \mathbf{i} + \cos 45^\circ \mathbf{j})$ (m/s) and the velocity of the bat is $\mathbf{v}_B = 60(-\cos 45^\circ \mathbf{i} - \cos 45^\circ \mathbf{j})$(m/s). Neglect the change in the velocity of the bat due to the direct central impact. The coefficient of restitution is $e = 0.2$. What is the ball's velocity after the impact? Assume that the baseball and the bat are moving horizontally. Does the batter achieve a potential hit or a foul ball?



Solution: Tangent to the impact plane, the velocity does not change. Since we are neglecting the change in velocity of the bat, then

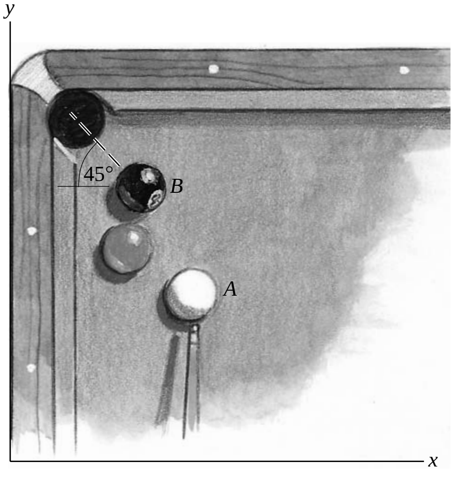
$$0.2([132\cos 45^\circ] - [-60\cos 45^\circ]) \text{ m/s} = (-60\cos 45^\circ) \text{ m/s} - v_{\text{ball}x}'$$

Solving we find $v_{\text{ball}x}' = -69.6$ m/s

Thus $\boxed{\mathbf{v}_{\text{ball}}' = (-69.6\mathbf{i} + 132\cos 45^\circ \mathbf{j}) \text{ m/s} = (-69.6\mathbf{i} + 93.3\mathbf{j}) \text{ m/s}}$

The ball is foul.

Problem 16.80 The cue gives the cue ball A a velocity parallel to the y axis. The cue ball hits the eight ball B and knocks it straight into the corner pocket. If the magnitude of the velocity of the cue ball just before the impact is 2 m/s and the coefficient of restitution is $e = 1$, what are the velocity vectors of the two balls just after the impact? (The balls are of equal mass.)



Solution: Denote the line from the 8-ball to the corner pocket by BP. This is an oblique central impact about BP. Resolve the cue ball velocity into components parallel and normal to BP. For a 45° angle, the unit vector parallel to BP is $\mathbf{e}_{BP} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$, and the unit vector normal to BP is $\mathbf{e}_{BPn} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Resolve the cue ball velocity before impact into components: $\mathbf{v}_A = v_{AP}\mathbf{e}_{BP} + v_{APn}\mathbf{e}_{BPn}$. The magnitudes v_{AP} and v_{APn} are determined from

$$\sqrt{|v_{AP}\mathbf{e}_{BP}|^2 + |v_{APn}\mathbf{e}_{BPn}|^2} = |\mathbf{v}_A| = 2 \text{ m/s}$$

and the condition of equality imposed by the 45° angle, from which $v_{AP} = v_{APn} = \sqrt{2}$ m/s. The cue ball velocity after impact is $\mathbf{v}_A' = v_{AP}'\mathbf{e}_{BP} + v_{APn}\mathbf{e}_{BPn}$, (since the component of $\mathbf{v}_A$ that is at right angles to BP will be unchanged by the impact). The velocity of the 8-ball after impact is $\mathbf{v}_{BP}' = v_{BP}'\mathbf{e}_{BP}$. The unknowns are the magnitudes v_{BP}' and v_{AP}'. These are determined from the conservation of linear momentum along BP and the coefficient of restitution.

$$m_A v_{AP} = m_A v_{AP}' + m_B v_{BP}',$$

and

$$1 = \frac{v_{BP}' - v_{AP}'}{v_{AP}}.$$

For $m_A = m_B$, these have the solution $v_{AP}' = 0$, $v_{BP}' = v_{AP}$, from which

$$\boxed{\mathbf{v}_A' = v_{APn}\mathbf{e}_{BPn} = \mathbf{i} + \mathbf{j} \text{ (m/s)}}$$

and

$$\boxed{\mathbf{v}_B' = v_{AP}\mathbf{e}_{BP} = -\mathbf{i} + \mathbf{j} \text{ (m/s)}.}$$

Problem 16.81 In Problem 16.80, what are the velocity vectors of the two balls just after impact if the coefficient of restitution is $e = 0.9$?

Solution: Use the results of the solution to Problem 16.80, where the problem is solved as an oblique central impact about the line from the 8-ball to the corner pocket. Denote the line from the 8-ball to the corner pocket by BP. The unit vector parallel to BP is $\mathbf{e}_{BP} = \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j})$, and the unit vector normal to BP is $\mathbf{e}_{BPn} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Resolve the cue ball velocity before impact into components:

$$\mathbf{v}_A = v_{AP}\mathbf{e}_{BP} + v_{APn}\mathbf{e}_{BPn},$$

where, from Problem 16.80, $v_{AP} = v_{APn} = \sqrt{2}$ m/s. The velocity of the 8-ball after impact is $\mathbf{v}'_{BP} = v'_{BP}\mathbf{e}_{BP}$. The unknowns are the magnitudes v'_{BP} and v'_{AP}. These are determined from the conservation of linear momentum along BP and the coefficient of restitution.

$$m_A v_{AP} = m_A v'_{AP} + m_B v'_{BP},$$

and

$$e = \frac{v'_{BP} - v'_{AP}}{v_{AP}}.$$

For $m_A = m_B$, these have the solution

$$v'_{AP} = \left(\frac{1}{2}\right)(1 - e)v_{AP} = 0.05 v_{AP},$$

and

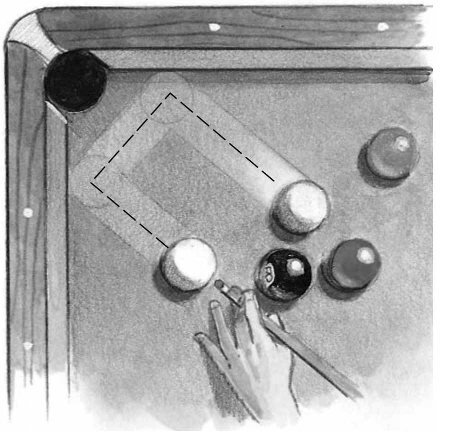
$$v'_{BP} = \left(\frac{1}{2}\right)(1 + e)v_{AP} = 0.95 v_{AP}.$$

The result:

$$\boxed{\begin{aligned}\mathbf{v}'_A &= v_{AP}\mathbf{e}_{BP} + v_{APn}\mathbf{e}_{BPn} = (-0.05\mathbf{i} + 0.05\mathbf{j} + \mathbf{i} + \mathbf{j})\\ &= 0.95\mathbf{i} + 1.05\mathbf{j} \text{ m/s}\end{aligned}}$$

$$\boxed{v'_B = 0.95(-\mathbf{i} + \mathbf{j}) \text{ (m/s)}}$$

Problem 16.82 If the coefficient of restitution is the same for both impacts, show that the cue ball's path after two banks is parallel to its original path.



Solution: The strategy is to treat the two banks as two successive oblique central impacts. Denote the path from the cue ball to the first bank impact as CP1, the path from the first impact to the second as CP2, and the final path after the second bank as CP3. The cue ball velocity along CP1 is

$$\mathbf{v}_{A1} = v_{A1x}\mathbf{i} + v_{A1y}\mathbf{j},$$

and the angle is

$$\alpha = \tan^{-1}\left(\frac{v_{Ax1}}{y_{Ay1}}\right).$$

The component $v_{A1y}\mathbf{j}$ is unchanged by the impact. The x component after the first impact is $v_{A2x} = -ev_{A1x}$, from which the velocity of the cue ball along path CP2 is

$$\mathbf{v}_{A2} = -ev_{A1x}\mathbf{i} + v_{A1y}\mathbf{j}.$$

The angle is

$$\beta = \tan^{-1}\left(\frac{-ev_{A1x}}{v_{A1y}}\right).$$

The x component of the velocity along path CP2 is unchanged after the second impact, and the y component after the second impact is $v_{A3y} = -ev_{A1y}$. The velocity along the path CP3 is

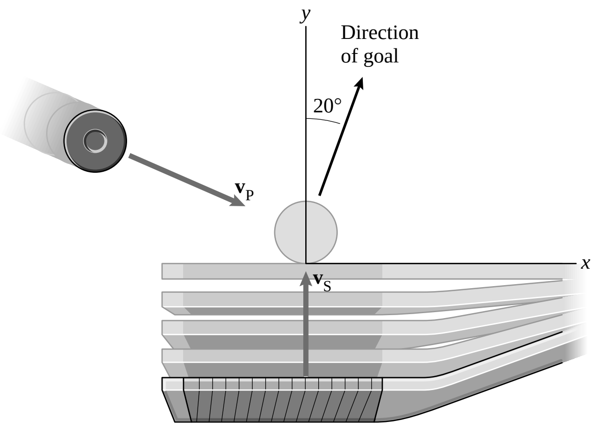
$$\mathbf{v}_{A3} = -ev_{A1x}\mathbf{i} - ev_{A1y}\mathbf{j},$$

and the angle is

$$\gamma = \tan^{-1}\left(\frac{-ev_{A1x}}{-ev_{A1y}}\right) = \alpha.$$

The sides of the table at the two banks are at right angles; the angles $\boxed{\alpha = \gamma}$ show that the paths CP1 and CP3 are parallel.

Problem 16.83 The velocity of the 170-g hockey puck is $\mathbf{v}_\mathrm{P} = 10\mathbf{i} - 4\mathbf{j}$ (m/s). If you neglect the change in the velocity $\mathbf{v}_\mathrm{S} = v_\mathrm{S}\mathbf{j}$ of the stick resulting from the impact, and if the coefficient of restitution is $e = 0.6$, what should v_S be to send the puck toward the goal?



Solution: The strategy is to treat the collision as an *oblique central impact* with a moving object of infinite mass. The horizontal component of the puck velocity is unchanged by the impact. The vertical component of the velocity after impact must satisfy the condition $\tan^{-1}(v'_{Px}/v'_{Py}) = \tan^{-1}(10/v'_{Py}) = 20^\circ$, from which the velocity of the puck after impact must be $v'_{Py} = 27.47$ m/s. Assume for the moment that the hockey stick has a finite mass, and consider only the y component of the puck velocity. The conservation of linear momentum and the definition of the coefficient of restitution are

$$m_P v_{Py} + m_S v_S = m_P v'_{Py} + m_S v'_S,$$

and

$$e = \frac{v'_S - v'_{Py}}{v_{Py} - v_S}.$$

These two simultaneous equations have the solution

$$v'_{Py} = (1/(m_P + m_S))(m_S(1+e)v_S + (m_P - em_S)v_{Py}).$$

Divide numerator and denominator on the right by m_S and take the limit as

$$m_S \to \infty,\ v'_{Py} = \lim_{m_S\to\infty}\left(1/\left(\frac{m_P}{m_S}+1\right)\right)((1+e)v_S$$

$$+\left(\frac{m_P}{m_S} - e\right)v_{Py}\Big) = (1+e)v_S - ev_{Py}.$$

Substitute the values: $v'_{Py} = 27.47$ m/s, $e = 0.6$, and $v_{Py} = -4$ m/s and solve:

$$\boxed{v_S = \frac{v'_{Py} + ev_{Py}}{(1+e)} = 15.67 \text{ m/s}}$$

Problem 16.84 In Problem 16.83, if the stick responds to the impact the way an object with the same mass as the puck would, and if the coefficient of restitution is $e = 0.6$, what should v_S be to send the puck toward the goal?

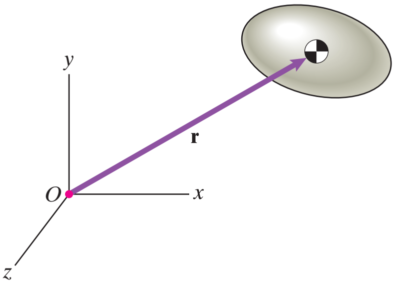
Solution: Use the solution to Problem 16.83, where m_S has a finite mass,

$$v'_{Py} = \left(\frac{1}{m_P + m_S}\right)(m_S(1+e)v_S + (m_P - em_S)v_{Py}).$$

Substitute $m_P = m_S$, $e = 0.6$, $v'_{Py} = 27.47$ m/s, and $v_{Py} = -4$ m/s, and solve:

$$\boxed{v_S = \frac{2v'_{Py} - (1-e)v_{Py}}{(1+e)} = 35.3 \text{ m/s}}$$

Problem 16.85 At the instant shown ($t_1 = 0$), the position of the 2-kg object's center of mass is $\mathbf{r} = 6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ (m) and its velocity is $\mathbf{v} = -16\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}$ (m/s). No external forces act on the object. What is the object's angular momentum about the origin O at $t_2 = 1$ s?



Solution:

$$\mathbf{H}_O = (6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) \text{ m} \times (2 \text{ kg})(-16\mathbf{i} + 8\mathbf{j} - 12\mathbf{k}) \text{ m/s}$$

$$\boxed{\mathbf{H}_O = (-128\mathbf{i} + 80\mathbf{j} + 224\mathbf{k}) \text{ kg-m}^2/s}$$

Problem 16.86 Suppose that the total external force on the 2-kg object shown in Problem 16.85 is given as a function of time by $\Sigma\mathbf{F} = 2t\mathbf{i} + 4\mathbf{j}$(N). At time $t_1 = 0$, the object's position and velocity are $\mathbf{r} = \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$.

(a) Use Newton's second law to determine the object's velocity $\mathbf{v}$ and position $\mathbf{r}$ as functions of time.

(b) By integrating $\mathbf{r} \times \Sigma\mathbf{F}$ with respect to time from $t_1 = 0$ to $t_2 = 6$ s, determine the angular impulse about O exerted on the object during this interval of time.

(c) Use the results of part (a) to determine the change in the object's angular momentum from $t_1 = 0$ to $t_2 = 6$ s.

Solution:

(a) $\mathbf{F} = (2t\mathbf{i} + 4\mathbf{j})\text{N} = (2 \text{ kg})\mathbf{a}, \quad \mathbf{a} = (t\mathbf{i} + 2\mathbf{j}) \text{ m/s}^2$

$$\boxed{\mathbf{v} = \left(\frac{t^2}{2}\mathbf{i} + 2t\mathbf{j}\right) \text{ m/s}, \quad \mathbf{r} = \left(\frac{t^3}{6}\mathbf{i} + t^2\mathbf{j}\right) \text{ m}}$$

(b) Angular Impulse $= \int_0^{6 \text{ s}} \mathbf{M}_O dt$

$$= \int_0^{6 \text{ s}} \mathbf{r} \times \mathbf{F} dt = \int_0^{6 \text{ s}} \left[\left(\frac{t^3}{6}\mathbf{i} + t^2\mathbf{j}\right) \text{ m}\right] \times [(2t\mathbf{i} + 4\mathbf{j})\text{N}]dt$$

$$= \int_0^{6 \text{ s}} \left(-\frac{4t^3}{3}\mathbf{k}\right) \text{N-m}dt$$

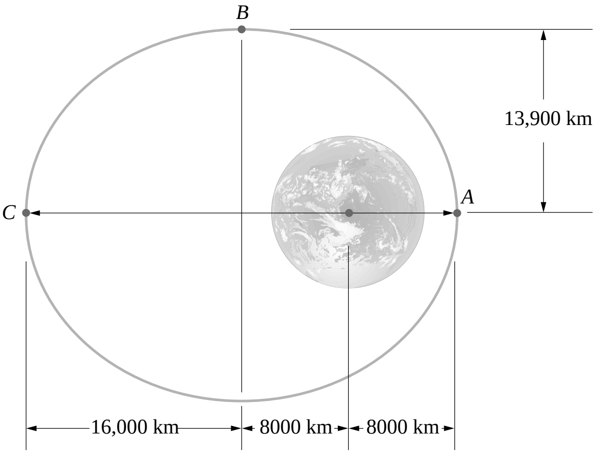
$$\boxed{\text{Angular Impulse} = (-432\mathbf{k}) \text{ kg-m}^2/s}$$

(c) $$\boxed{\Delta\mathbf{H}_O = \int_0^{6 \text{ s}} \mathbf{M}_O dt = (-432\mathbf{k}) \text{ kg-m}^2/s}$$

Problem 16.87 A satellite is in the elliptic earth orbit shown. Its velocity at perigee A is 8640 m/s. The radius of the earth is 6370 km.

(a) Use conservation of angular momentum to determine the magnitude of the satellite's velocity at apogee C.
(b) Use conservation of energy to determine the magnitude of the velocity at C.

(See Example 16.8.)



Solution:

(a) $r_A v_A = r_C v_C = |\mathbf{H}_0|$

$(8000)(8640) = (24000)v_C$

$\underline{v_C = 2880 \text{ m/s}}$

(b) $\dfrac{1}{2}mv_A^2 - \dfrac{mgR_E^2}{r_A} = \dfrac{1}{2}mv_C^2 - \dfrac{mgR_E^2}{r_C}$

$\dfrac{v_A^2}{2} - \dfrac{gR_E^2}{r_A} = \dfrac{v_C^2}{2} - \dfrac{gR_E^2}{r_C}$

where $v_A = 8640$ m/s, $g = 9.81$ m/s, $R_E = 6370000$ m, $r_A = 8{,}000{,}000$ m, $r_C = 24{,}000{,}000$ m.

Solving for v_C, $\underline{v_C = 2880 \text{ m/s}}$

Problem 16.88 For the satellite in Problem 16.87, determine the magnitudes of the radial velocity v_r and transverse velocity v_θ at B. (See Example 16.8.)

Solution: Use conservation of energy to find the velocity magnitude at B. Then use conservation of angular momentum to determine the components.

$$\frac{1}{2}mV_A^2 - \frac{mgR_E^2}{r_A} = \frac{1}{2}mV_B^2 - \frac{mgR_E^2}{r_B}$$

where $r_A = 8 \times 10^6$ m, $v_A = 8640$ m/s

$$r_B = \sqrt{(8 \times 10^6)^2 + (13.9 \times 10^6)^2}$$

$$r_B = 16 \times 10^6 \text{ m}, \quad R_E = 6.370 \times 10^6 \text{ m}$$

Solving, we get $v_B = 4990$ m/s

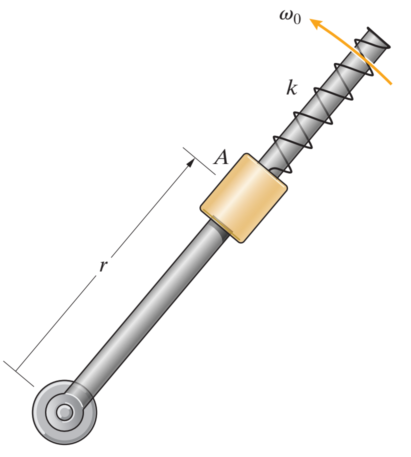
From conservation of angular momentum

$$r_A v_A = r_B v_\theta$$

Solving, $v_\theta = 4320$ m/s

Finally $v_r = \sqrt{v_B^2 - v_\theta^2} = 2500$ m/s

Problem 16.89 The bar rotates *in the horizontal plane* about a smooth pin at the origin. The 2-kg sleeve A slides on the smooth bar, and the mass of the bar is negligible in comparison to the mass of the sleeve. The spring constant $k = 40$ N/m, and the spring is unstretched when $r = 0$. At $t = 0$, the radial position of the sleeve is $r = 0.2$ m and the angular velocity of the bar is $w_0 = 6$ rad/s. What is the angular velocity of the bar when $r = 0.25$ m?



Solution: Since the spring force is radial, it does not affect the angular momentum which is constant.

$$(0.2\text{ m})(2\text{ kg})[(0.2\text{ m})(6\text{ rad/s})] = (0.25\text{ m})(2\text{ kg})[(0.25\text{ m})\omega]$$

$$\boxed{\omega = 3.84\text{ rad/s}}$$

Problem 16.90 At $t = 0$, the radial position of the sleeve A in Problem 16.89 is $r = 0.2$ m, the radial velocity of the sleeve is $v_r = 0$ and the angular velocity of the bar is $w_0 = 6$ rad/s. What are the angular velocity of the bar and the radial velocity of the sleeve when $r = 0.25$ m?

Solution: From problem 16.89 we have $\boxed{\omega = 3.84\text{ rad/s}}$

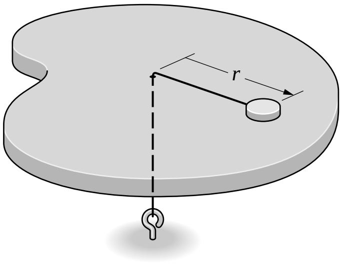
Use work - energy to find the radial velocity

$$\frac{1}{2}(2\text{ kg})(0^2 + [(0.2\text{ m})(6\text{ rad/s})^2]) + \frac{1}{2}(40\text{ N/m})(0.2\text{ m})^2 =$$

$$\frac{1}{2}(2\text{ kg})(v_r{}^2 + [(0.25\text{ m})(3.84\text{ rad/s})^2]) + \frac{1}{2}(40\text{ N/m})(0.25\text{ m})^2$$

$$\boxed{v_r = \pm 0.262\text{ m/s}}$$

Problem 16.91 A 2-kg disk slides on a smooth horizontal table and is connected to an elastic cord whose tension is $T = 6r$ N, where r is the radial position of the disk in meters. If the disk is at $r = 1$ m and is given an initial velocity of 4 m/s in the transverse direction, what are the magnitudes of the radial and transverse components of its velocity when $r = 2$ m? (See Active Example 16.7.)



Solution: The strategy is to (a) use the principle of conservation of angular momentum to find the transverse velocity and (b) use the conservation of energy to find the radial velocity. The angular momentum the instant after $t = 0$ is $(\mathbf{r} \times m\mathbf{v})_o = H_0\mathbf{e}_z = (mrv_\theta)_o\mathbf{e}_z$, from which $H_0 = 8$ kg-m^2/s. In the absence of external transverse forces, the angular momentum impulse vanishes:

$$\int_{t_1}^{t_2} \left(\mathbf{r} \times \sum \mathbf{F}\right) dt = 0 = \mathbf{H}_2 - \mathbf{H}_1,$$

so that $\mathbf{H}_1 = \mathbf{H}_2$, that is, the angular momentum is constant. At

$$r = 2, \boxed{v_\theta = \frac{H_0}{mr} = \frac{8}{4} = 2 \text{ m/s.}}$$

From conservation of energy:

$$\tfrac{1}{2}mv_{r_0}^2 + \tfrac{1}{2}mv_{\theta_o}^2 + \tfrac{1}{2}kS_o^2 = \tfrac{1}{2}mv_r^2 + \tfrac{1}{2}mv_\theta^2 + \tfrac{1}{2}kS^2.$$

Solve:

$$v_r = \sqrt{v_{r_0}^2 + v_{\theta_o}^2 - v_\theta^2 + \left(\frac{k}{m}\right)(S_o^2 - S^2)}.$$

Substitute numerical values: Noting $m = 2$ kg, $v_{r_0} = 0$, $v_{\theta_o} = 4$ m/s, $k = 6$ N/m, $r = 2$ m, $v_\theta = 2$ m/s, $S_o = 1$ m, $S = 2$ m from which $\boxed{v_r = \sqrt{3} \text{ m/s}}$. The velocity is $\boxed{\mathbf{v} = 1.732\mathbf{e}_r + 2\mathbf{e}_\theta}$ (m/s).

Problem 16.92 In Problem 16.91, determine the maximum value of r reached by the disk.

Solution: The maximum value is the stretch of the cord when $v_r = 0$. From the solution to Problem 16.91,

$$v_r^2 = v_{r_0}^2 + v_{\theta_o}^2 - v_\theta^2 + \left(\frac{k}{m}\right)(S_o^2 - S^2) = 0,$$

where $v_\theta = \dfrac{H_0}{mr}$ m/s, $v_{r_0}^2 = 0$, $v_{\theta_o} = \dfrac{H_0}{m(1)}$ m/s, $S_o = r_0 = 1$ m, $S = rm$, and $H_0 = 8$ kg-m^2/s. Substitute and reduce:

$$v_r^2 = 0 = \left(\frac{H_0^2}{m^2}\right)\left(1 - \frac{1}{r^2}\right) + \left(\frac{k}{m}\right)(1 - r^2).$$

Denote $x = r^2$ and reduce to a quadratic canonical form $x^2 + 2bx + c = 0$, where

$$b = -\left(\frac{1}{2}\right)\left(\frac{H_0^2}{km} + 1\right) = -3.167, \; c = \frac{H_0^2}{km} = 5.333.$$

Solve $r_{1,2}^2 = -b \pm \sqrt{b^2 - c} = 5.333, = 1$, from which the greatest positive root is $\boxed{r_{\max} = 2.31 \text{ m}}$

[*Check*: This value is confirmed by a graph of the value of

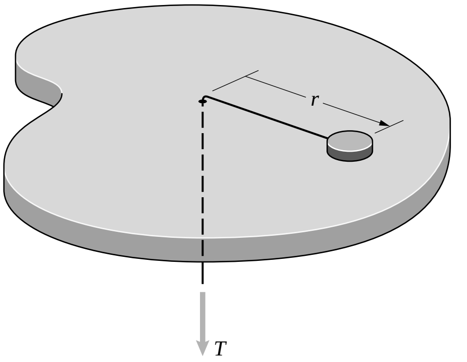
$$f(r) = \left(\frac{H_0^2}{m^2}\right)\left(1 - \frac{1}{r^2}\right) + \left(\frac{k}{m}\right)(1 - r^2)$$

to find the zero crossing. *check*.]

Problem 16.93 A 1-kg disk slides on a smooth horizontal table and is attached to a string that passes through a hole in the table.

(a) If the mass moves in a circular path of constant radius $r = 1$ m with a velocity of 2 m/s, what is the tension T?

(b) Starting from the initial condition described in part (a), the tension T is increased in such a way that the string is pulled through the hole at a constant rate until $r = 0.5$ m. Determine the value of T as a function of r while this is taking place.

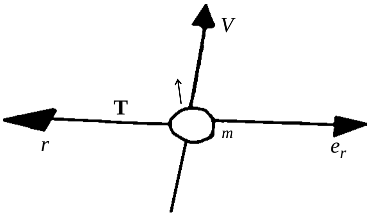


Solution:

(a) Circular motion

$$\mathbf{T} = -mv^2/r\mathbf{e}_r$$

$$\underline{|\mathbf{T}| = (1)(2)^2/1 = 4 \text{ N}}$$



(b) By conservation of angular momentum,

$$mr_0v_0 = mrv_T, \quad \therefore v_T = \frac{r_0v_0}{r}$$

and from Newton's second law

$$|\mathbf{T}| = mv_T^2/r = m\left(\frac{r_0v_0}{r}\right)^2 \Big/ r$$

$$\underline{T = (1)\left(\frac{(1)(2)}{r}\right)^2 \Bigg/ r = 4/r^3 \text{ N}}$$

Problem 16.94 In Problem 16.93, how much work is done on the mass in pulling the string through the hole as described in part (b)?

Solution: The work done is

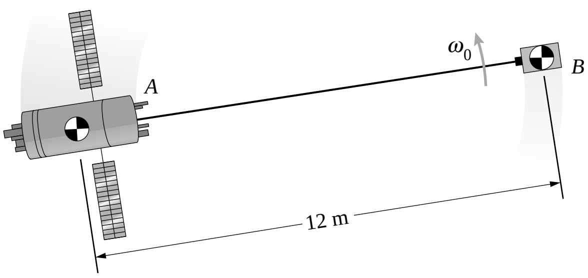
$$U_{12} = \int_1^{0.5}\left(-\frac{mr_0^2r_0^2}{r^3}\right)\mathbf{e}_r \cdot dr\mathbf{e}_r$$

$$= -mr_0^2v_0^2\int_1^{0.5}\frac{dr}{r^3} = -(1)(1)^2(2)^2\left[\frac{1}{-2r^2}\right]_1^{0.5}$$

$$= -4\left[-\frac{1}{2(0.5)^2} + \frac{1}{2}\right]$$

$$\underline{U_{12} = -4[-1.5] = 6 \text{ N-m}}$$

Problem 16.95 Two gravity research satellites ($m_A = 250$ kg, $m_B = 50$ kg) are tethered by a cable. The satellites and cable rotate with angular velocity $\omega_0 = 0.25$ revolution per minute. Ground controllers order satellite A to slowly unreel 6 m of additional cable. What is the angular velocity afterward?



Solution: The satellite may be rotating in (a) a vertical plane, or (b) in the horizontal plane, or (c) in some intermediate plane. The strategy is to determine the angular velocity for the three possibilities.

Case (a) Assume that the system rotates in the x-y plane, with y positive upward. Choose the origin of the coordinates at the center of mass of the system. The distance along the cable from the center of mass to A is

$$\frac{12m_B}{m_A + m_B} = 2 \text{ m, from which the distance to B is 10 m.}$$

Assume that both satellites lie on the x axis at $t = 0$. The radius position of satellite A is $\mathbf{r}_A = -2(\mathbf{i}\cos\omega_0 t + \mathbf{j}\sin\omega_0 t)$, and the radius position of satellite B is $\mathbf{r}_B = 10(\mathbf{i}\cos\omega_0 t + \mathbf{j}\sin\omega_0 t)$. The acceleration due to gravity is

$$\mathbf{W} = -\frac{mgR_E^2}{r^2}\mathbf{j} = -mg'\mathbf{j},$$

from which $\mathbf{W}_A = -m_A g'\mathbf{j}$ and $\mathbf{W}_B = -m_B g'\mathbf{j}$ (or, alternatively). The angular momentum impulse is

$$\int_{t_1}^{t_2} (\mathbf{r} \times \sum \mathbf{F})\, dt = \int_{t_1}^{t_2} (\mathbf{r}_A \times \mathbf{W}_A + \mathbf{r}_B \times W_B)\, dt.$$

Carry out the indicated operations:

$$\mathbf{r}_A \times \mathbf{W}_A = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2(\cos\omega_0 t) & -2(\sin\omega_0 t) & 0 \\ 0 & -m_A g' & 0 \end{bmatrix}$$

$$= 2\, m_A g' \cos\omega_0 t\mathbf{k}.$$

$$\mathbf{r}_B \times \mathbf{W}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 10(\cos\omega_0 t) & 10(\sin\omega_0 t) & \\ 0 & -m_B g' & 0 \end{bmatrix}$$

$$= -10 m_B g' \cos\omega_0 t\mathbf{k}.$$

Substitute into the angular momentum impulse:

$$\int_{t_1}^{t_2} (\mathbf{r} \times \sum \mathbf{F})\, dt = 0 = \mathbf{H}_2 - \mathbf{H}_1,$$

from which $\mathbf{H}_1 = \mathbf{H}_2$; that is, the angular momentum is conserved. From Newton's second law, the center of mass remains unchanged as the cable is slowly reeled out.

A repeat of the argument above for *any* additional length of cable leads to the same result, namely, the angular momentum is constant, from which the angular momentum is conserved as the cable is reeled out. The angular momentum of the original system is

$$\mathbf{r}_A \times m_A\mathbf{v} + \mathbf{r}_B \times m_B\mathbf{v} = 4m_A\omega_0\mathbf{k} + 100m_B\omega_0\mathbf{k}$$

$$= 157.1 \text{ kg-m}^2/\text{s},$$

in magnitude, where $\omega_0 = 0.026$ rad/s. After 6 meters is reeled out, the distance along the cable from the center of mass to A is

$$\frac{m_B(6 + 12)}{m_A + m_B} = 3 \text{ m},$$

from which the distance to B is 15 m. The new angular velocity when the 6 m is reeled out is

$$\boxed{\omega = \frac{H}{3^2 m_A + 15^2 m_B} = 0.0116 \text{ rad/s} = 0.1111 \text{ rpm}}$$

Case (b): Assume that the system rotates in the x-z plane, with y positive upward. As above, choose the origin of the coordinates at the center of mass of the system. Assume that both satellites lie on the x axis at $t = 0$. The radius position of satellite A is $\mathbf{r}_A = -2(\mathbf{i}\cos\omega_0 t + \mathbf{k}\sin\omega_0 t)$, and the radius position of satellite B is $\mathbf{r}_B = 10(\mathbf{i}\cos\omega_0 t + \mathbf{k}\sin\omega_0 t)$. The force due to gravity is $\mathbf{W}_A = -m_A g'\mathbf{j}$ and $\mathbf{W}_B = -m_B g'\mathbf{j}$. The angular momentum impulse is

$$\int_{t_1}^{t_2} (\mathbf{r} \times \sum \mathbf{F})\, dt = \int_{t_1}^{t_2} (\mathbf{r}_A \times \mathbf{W}_A + \mathbf{r}_B \times W_B)\, dt.$$

Carry out the indicated operations:

$$\mathbf{r}_A \times \mathbf{W}_A = 2m_A g' \cos\omega_0 t\mathbf{i},$$

$$\mathbf{r}_B \times \mathbf{W}_B = -10m_B g' \cos\omega_0 t\mathbf{i}.$$

Substitute into the angular momentum impulse:

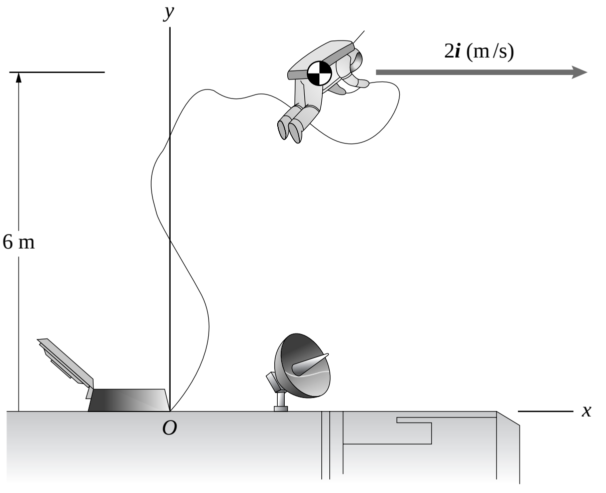
$$\int_{t_1}^{t_2} (\mathbf{r} \times \sum \mathbf{F})\, dt = 0 = \mathbf{H}_2 - \mathbf{H}_1,$$

from which $\mathbf{H}_1 = \mathbf{H}_2$; that is, the angular momentum is conserved. By a repeat of the argument given in Case (a), the new angular velocity is $\omega = 0.0116$ rad/s $= 0.111$ rpm.

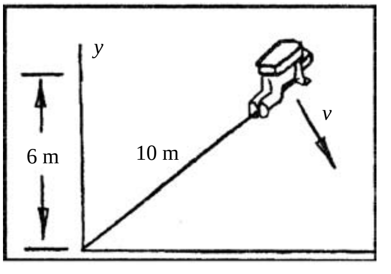
Case (c): Since the angular momentum is conserved, a repeat of the above for *any orientation of the system relative to the gravity vector* leads to the same result.

Problem 16.96 The astronaut moves in the x–y plane at the end of a 10-m tether attached to a large space station at O. The total mass of the astronaut and his equipment is 120 kg.

(a) What is the astronaut's angular momentum about O before the tether becomes taut?
(b) What is the magnitude of the component of his velocity perpendicular to the tether immediately after the tether becomes taut?



Solution:



(a) The angular momentum by definition is

$$(\mathbf{r} \times m\mathbf{v}) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 6 & 0 \\ (120)2 & 0 & 0 \end{bmatrix} = -6(2)(120)\mathbf{k}$$

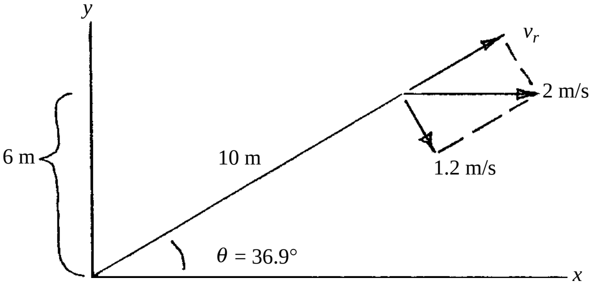
$$= -1440\mathbf{k} \text{ (kg-m}^2\text{/s)}.$$

(b) From conservation of angular momentum,
(b) $\mathbf{H} = -1440\mathbf{k} = -(10)(120)v\mathbf{k}$, from which

$$\boxed{v = \frac{1440}{(10)(120)} = 1.2 \text{ m/s}}$$

Problem 16.97 The astronaut moves in the $x-y$ plane at the end of a 10-m tether attached to a large space station at O. The total mass of the astronaut and his equipment is 120 kg. The coefficient of restitution of the "impact" that occurs when he comes to the end of the tether is $e = 0.8$. What are the x and y components of his velocity immediately after the tether becomes taut?

Solution: From the solution of Problem 16.96, his velocity perpendicular to the tether is 1.2 m/s.



Before the tether becomes taught, his component of velocity parallel to the tether is

$v_r = 2\cos 36.9^\circ = 1.6$ m/s.

After it becomes taught,

$v_r' = -ev_r = -(0.8)(1.6) = -1.28$ m/s.

The x and y components of his velocity are

$v_x' = v_r' \cos 36.9^\circ + (1.2)\sin 36.9^\circ$

$= -0.304$ m/s

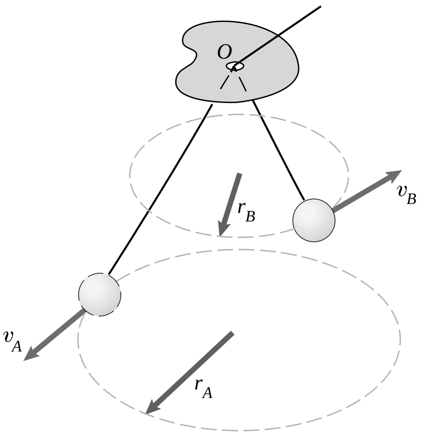
and

$v_y' = v_r' \sin 36.9^\circ - (1.2)\cos 36.9^\circ$

$= -1.728$ m/s.

Problem 16.98 A ball suspended from a string that goes through a hole in the ceiling at O moves with velocity v_A in a horizontal circular path of radius r_A. The string is then drawn through the hole until the ball moves with velocity v_B in a horizontal circular path of radius r_B. Use the principle of angular impulse and momentum to show that $r_A v_A = r_B v_B$.

Strategy: Let $\mathbf{e}$ be a unit vector that is perpendicular to the ceiling. Although this is not a central-force problem — the ball's weight does not point toward O — you can show that $\mathbf{e} \cdot (\mathbf{r} \times \sum \mathbf{F}) = 0$, so that $\mathbf{e} \cdot \mathbf{H}_O$ is conserved.



Solution: Assume that the motion is in the x-y plane, and that the ball lies on the positive x axis at $t = 0$. The radius vector

$$\mathbf{r}_A = r_A(\mathbf{i} \cos \omega_A t + \mathbf{j} \sin \omega_A t),$$

where ω_A is the angular velocity of the ball in the path. The velocity is

$$\mathbf{v}_A = -\mathbf{j} r_A \omega_A \sin \omega_A t + \mathbf{j} r_A \omega_A \cos \omega_A t.$$

The angular momentum per unit mass about the axis normal to the ceiling is

$$\left(\frac{\mathbf{r} \times m\mathbf{v}}{m}\right) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_A \cos \omega_A t & r_A \sin \omega_A t & 0 \\ -r_A \omega_A \sin \omega_A t & r_A \omega_A \cos \omega_A t & 0 \end{bmatrix} = \mathbf{k}(r_A^2 \omega_A).$$

Define the unit vector parallel to this angular momentum vector, $\mathbf{e} = \mathbf{k}$. From the principle of angular impulse and momentum, the external forces do not act to change this angular momentum. This is shown as follows:

The external force is the weight, $\mathbf{W} = -mg\mathbf{k}$. The momentum impulse is

$$\int_{t_1}^{t_2} (\mathbf{r} \times \sum \mathbf{F})\, dt = \int_{t_1}^{t_2} (\mathbf{r}_A \times \mathbf{W})\, dt.$$

Carry out the operation

$$\mathbf{r}_A \times \mathbf{W} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_A \cos \omega_A t & r_A \sin \omega_A t & 0 \\ 0 & 0 & -mg \end{bmatrix} = r_A mg \cos \omega_A t \mathbf{j},$$

from which

$$\int_{t_1}^{t_2} \mathbf{j} r_A mg \cos \omega t\, dt = -\mathbf{j}(r_A \omega_A mg)(\sin \omega_A t_2 - \sin \omega_A t_1)$$

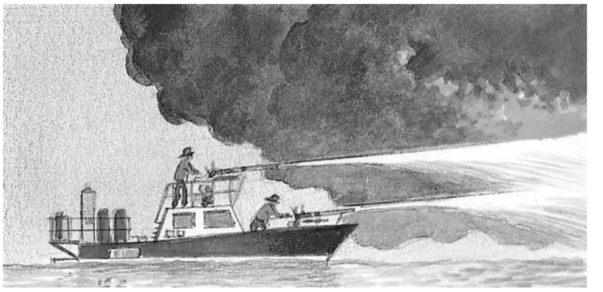
$$= \mathbf{H}_2 - H_1.$$

Since this has no component parallel to the unit vector $\mathbf{e} = \mathbf{k}$, the angular momentum along the axis normal to the ceiling is unaffected by the weight, that is, the projection of the angular momentum impulse due to the external forces on the unit vector normal to the ceiling is zero $\mathbf{e} \cdot \mathbf{H}_2 = \mathbf{e} \cdot \mathbf{H}_1 = 0$, hence the *angular momentum normal to the ceiling is conserved*. This result holds true for any length of string, hence

$$(\mathbf{r} \times \mathbf{v})_A = \mathbf{k} r_A^2 \omega_A = (\mathbf{r} \times \mathbf{v})_B = \mathbf{k} r_B^2 \omega_B,$$

from which $r_A^2 \omega_A = r_B^2 \omega_B$. Since $v_A = r_A \omega_A$, $v_B = r_B \omega_B$, the result can be expressed $\boxed{v_A r_A = v_B r_B}$

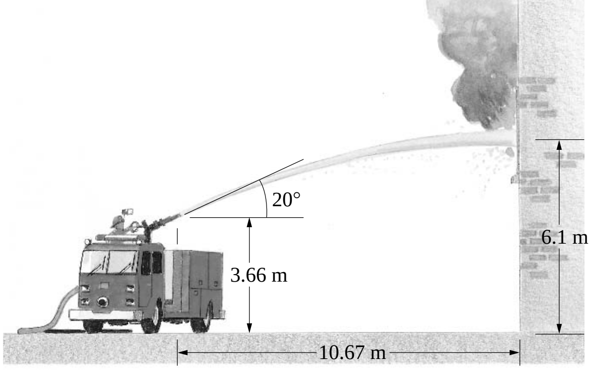
Problem 16.99 The Cheverton fire-fighting and rescue boat can pump 3.8 kg/s of water from each of its two pumps at a velocity of 44 m/s. If both pumps point in the same direction, what total force do they exert on the boat.



Solution: The magnitude of the total force is

$2(3.8 \text{ kg/s})(44 \text{ m/s}) = 334$ N.

Problem 16.100 The mass flow rate of water through the nozzle is 23.3 kg/s. Determine the magnitude of the horizontal force exerted on the truck by the flow of the water.



Solution: We must determine the velocity with which the water exits the nozzle. Relative to the end of the nozzle, the x-coordinate of a particle of water is $x = v_0 \cos 20^\circ t$ and the y coordinate is

$y = v_0 \sin 20^\circ t - \frac{1}{2}(9.81)t^2$.

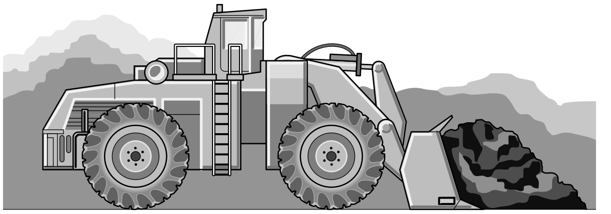
Setting $x = 10.67$ m and $y = 2.44$ m and eliminating t we obtain

$$2.44 = v_0 \sin 20^\circ \left(\frac{10.67}{v_0 \cos 20^\circ}\right) - \frac{1}{2}(9.81)\left(\frac{10.67}{v_0 \cos 20^\circ}\right)^2$$

From this equation, $v_0 = 20.94$ m/s. The horizontal force exerted by the flow of water is

$$\frac{dm_f}{dt} v_f \cos 20^\circ = (23.3 \text{ kg/s})(20.94)\cos 20^\circ = 458.1 \text{ N}.$$

Problem 16.101 The front-end loader moves at a constant speed of 3.2 km/h scooping up iron ore. The constant horizontal force exerted on the loader by the road is 1780 N. What weight of iron ore is scooped up in 3 s?



Solution:

$$1780\text{ N} = \left(\frac{dm_f}{dt}\right)\left(\frac{3.2\times 1000}{3600}\right) \Rightarrow \frac{dm_f}{dt} = 2002.5\text{ N-s/m}$$

In 3 s $m_f = (2002.5\text{ N-s/m})(3\text{ s}) = 6007.5\text{ N-s}^2\text{/m}$

$$\boxed{W = m_f g = (6007.5)(9.81\text{ m/s}^2) = 58933.6\text{ N}}$$

Problem 16.102 The snowblower moves at 1 m/s and scoops up 750 kg/s of snow. Determine the force exerted by the entering flow of snow.

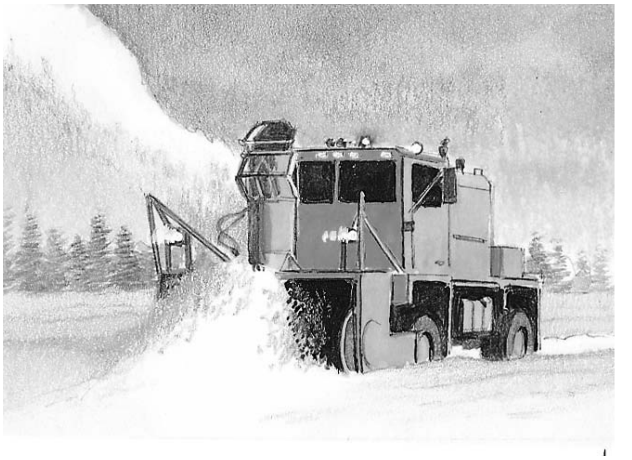
Solution: The mass flow rate is

$$\left(\frac{dm_f}{dt}\right) = 750\text{ kg/s.}$$

The velocity is $v_f = 1$ m/s. The force exerted by the entering flow of snow is

$$\boxed{F = \left(\frac{dm_f}{dt}\right)v_f = 750(1) = 750\text{ N}}$$

Problem 16.103 The snowblower scoops up 750 kg/s of snow. It blows the snow out the side at 45° above the horizontal from a port 2 m above the ground and the snow lands 20 m away. What horizontal force is exerted on the blower by the departing flow of snow?



Solution: The strategy is to use the solution of Newton's second law to determine the exit velocity.

From Newton's second law (ignoring drag) $m_f \dfrac{dv_y}{dt} = -m_f g$, from which $v_y = -gt + v_f \sin 45^\circ$ (m/s),

$v_x = v_f \cos 45^\circ$ (m/s),

and

$$y = -\frac{g}{2}t^2 + \left(v_f \sin 45^\circ\right) t + 2 \text{ m},$$

$$x = \left(v_f \cos 45^\circ\right) t.$$

At $y = 0$, the time of impact is $t_{\text{imp}}^2 + 2bt_{\text{imp}} + c = 0$, where $b = -\dfrac{v_f \sin 45^\circ}{g}$, $c = -\dfrac{4}{g}$.

The solution:

$$t_{\text{imp}} = -b \pm \sqrt{b^2 - c}$$

$$= \frac{v_f}{\sqrt{2}g}\left(1 \pm \sqrt{1 + \frac{8g}{v_f^2}}\right).$$

Substitute:

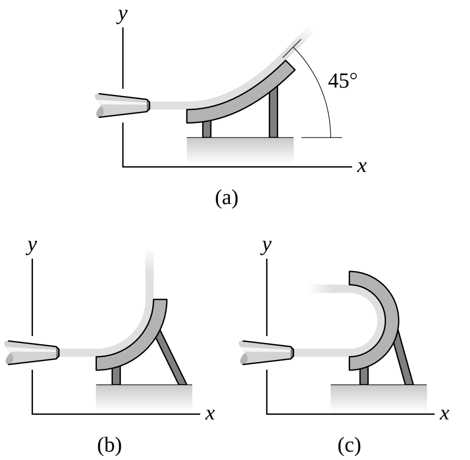
$$x = 20 = \left(\frac{v_f}{\sqrt{2}}\right)\left(\frac{v_f}{\sqrt{2}g}\right)\left(1 \pm \sqrt{1 + \frac{8g}{v_f^2}}\right).$$

This equation is solved by iteration using **TK Solver Plus** to yield $v_f = 13.36$ m/s.

The horizontal force exerted on the blower is

$$\boxed{F = \left(\frac{dm_f}{dt}\right) v_f = 750(13.36)\cos 45^\circ = 7082.7 \text{ kN}}$$

Problem 16.104 A nozzle ejects a stream of water horizontally at 40 m/s with a mass flow rate of 30 kg/s, and the stream is deflected in the horizontal plane by a plate. Determine the force exerted on the plate by the stream in cases (a), (b), and (c). (See Example 16.11.)



Solution: Apply the strategy used in Example 16.7. The exit velocity is $\mathbf{v}_{fe} = v_0(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$ where θ is the total angle of deflection of the stream, and v_0 is the magnitude of the stream velocity. The inlet velocity is $\mathbf{v}_{fi} = v_0\mathbf{i}$. The force on the plate exerted by the exit stream is in a direction opposite the stream flow, whereas the force exerted on the plate by the inlet stream is in the direction of stream flow. The sum of the forces *exerted on the plate* (see Eq. (16.25)) is

$$\sum \mathbf{F} = (dm_f/dt)(\mathbf{v}_{fi} - \mathbf{v}_{fe}),$$

from which

$$\sum \mathbf{F} = (dm_f/dt)v_0(\mathbf{i}(1 - \cos\theta) - \mathbf{j}\sin\theta).$$

The mass flow is $(dm_f/dt) = 30$ kg/s and the stream velocity is $v_0 = 40$ m/s. For Case(a), $\theta = 45^\circ$,

$$\boxed{\sum \mathbf{F} = (30)(40)(0.2929\mathbf{i} - 0.7071\mathbf{j}) = 351.5\mathbf{i} - 848.5\mathbf{j} \text{ (N)}.}$$

The force is downward to the right. For Case (b) $\theta = 90^\circ$,

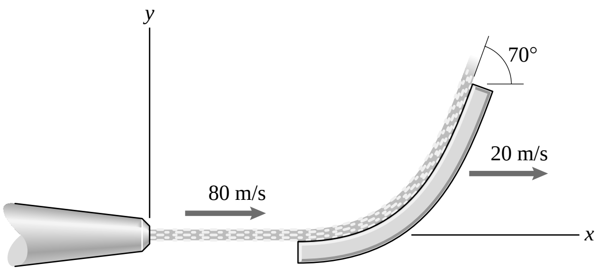
$$\boxed{\sum \mathbf{F} = (30)(40)(\mathbf{i} - \mathbf{j}) = 1200\mathbf{i} - 1200\mathbf{j} \text{ (N)}}$$

For Case (c) $\theta = 180^\circ$,

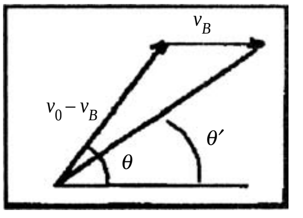
$$\boxed{\sum \mathbf{F} = (30)(40)(2\mathbf{i}) = 2400\mathbf{i} \text{ (N)}}$$

Problem 16.105* A stream of water with velocity $80\mathbf{i}$ (m/s) and a mass flow of 6 kg/s strikes a turbine blade moving with constant velocity $20\mathbf{i}$ (m/s).

(a) What force is exerted on the blade by the water?
(b) What is the magnitude of the velocity of the water as it leaves the blade?



Solution: Denote the fixed reference frame as the *nozzle frame*, and the moving blade frame as the *blade rest frame*. Assume that the discharge angle (70°) is referenced to the blade rest frame, so that the magnitude of the stream velocity and the effective angle of discharge in the blade rest frame is not modified by the velocity of the blade. Denote the velocity of the blade by v_B. The inlet velocity of the water relative to the blade is $\mathbf{v}_{fi} = (v_0 - v_B)\mathbf{i}$. The magnitude $(v_0 - v_B)$ is the magnitude of the stream velocity as it flows along the contour of the blade. At exit, the magnitude of the discharge velocity in the blade rest frame is the inlet velocity *in the blade rest frame* $|\mathbf{v}_{fe}| = (v_0 - v_B) =$ 60 m/s, and the vector velocity is $\mathbf{v}_{fe} = v_{fe}(\mathbf{i}\cos 70^\circ + \mathbf{j}\sin 70^\circ)$ in the blade rest frame. From Eq. (16.25) the sum of the forces on the blade is



$$\sum \mathbf{F} = (dm_f/dt)((v_0 - v_B - v_{fe}\cos 70^\circ)\mathbf{i} - \mathbf{j}(v_{fe}\sin 70^\circ)).$$

(a) Substitute numerical values:

$$(dm_f/dt) = 6 \text{ kg/s},$$

$$v_0 = 80 \text{ m/s},$$

$$v_B = 20 \text{ m/s},$$

from which

$$\boxed{\sum \mathbf{F} = 236.9\mathbf{i} - 338.3\mathbf{j}}$$

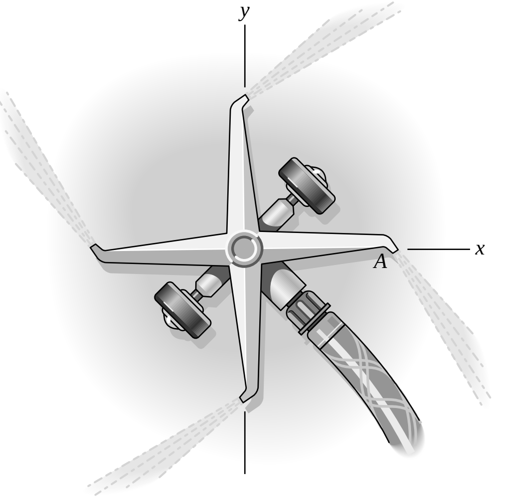
(b) Assume that the magnitude of the velocity of the water as it leaves the blade is required *in the nozzle frame*. The velocity of the water leaving the blade in the nozzle frame (see vector diagram) is $\mathbf{v}_{\text{ref}} = 20\mathbf{i} + 60\cos 70^\circ\mathbf{i} + 60\sin 70^\circ\mathbf{j} = 40.52\mathbf{i} + 56.38\mathbf{j}$ (m/s). The magnitude of the velocity is

$$\boxed{|\mathbf{v}| = 69.4 \text{ m/s.}}$$

Problem 16.106 At the instant shown, the nozzle A of the lawn sprinkler is located at $(0.1, 0, 0)$ m. Water exits each nozzle at 8 m/s relative to the nozzle with a mass flow rate of 0.22 kg/s. At the instant shown, the flow relative to the nozzle at A is in the direction of the unit vector

$$\mathbf{e} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}.$$

Determine the total moment about the z axis exerted on the sprinkler by the flows from all four nozzles.



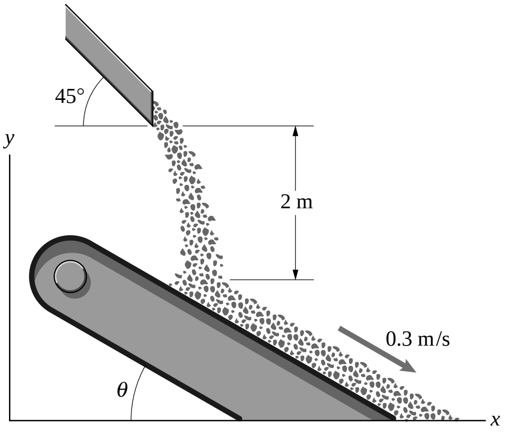
Solution:

$$\mathbf{e} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad \mathbf{r} = 0.1\mathbf{i}, \quad \mathbf{v} = (8 \text{ m/s})\mathbf{e}, \quad \frac{dm_f}{dt} = 0.22 \text{ kg/s}$$

$$\mathbf{M} = 4\left(-\frac{dm_f}{dt}\right)(\mathbf{r}\times\mathbf{v}) = (0.406 \text{ N-m})(\mathbf{j}+\mathbf{k})$$

The moment about the z-axis is $\boxed{M_z = \mathbf{M}\bullet\mathbf{k} = 0.406 \text{ N-m}}$

Problem 16.107 A 45-kg/s flow of gravel exits the chute at 2 m/s and falls onto a conveyor moving at 0.3 m/s. Determine the components of the force exerted on the conveyor by the flow of gravel if $\theta = 0$.



Solution: The horizontal component of the velocity of the gravel flow is $v_x = 2\cos 45^\circ = \sqrt{2}$ m/s. From Newton's second law, (using the chain rule) the vertical component of the velocity is

$$v_y = -\sqrt{(v\sin 45^\circ)^2 + 2gh} = -\sqrt{2 + 2(9.81)(2)} = -6.422 \text{ m/s}.$$

The mass flow rate is $\left(\frac{dm_f}{dt}\right) = 45$ kg/s. The force exerted on the belt is

$$\boxed{\sum\mathbf{F} = \left(\frac{dm_f}{dt}\right)((v_x - 0.3)\mathbf{i} + v_y\mathbf{j}) = 50.1\mathbf{i} - 289\mathbf{j} \text{ (N)}}$$

Problem 16.108 Solve Problem 16.107 if $\theta = 30^\circ$.

Solution: Use the solution to Problem 16.107 as appropriate. From Problem 16.107:

$$v_x = 2\cos 45^\circ = \sqrt{2} \text{ m/s}, \quad v_y = -\sqrt{(v \sin 45^\circ)^2 + 2gh}$$

$$= -\sqrt{2 + 2(9.81)(2)} = -6.422 \text{ m/s}.$$

The velocity of the conveyor belt is

$$\mathbf{v}_B = v_{Bx}\mathbf{i} + v_{By}\mathbf{j}$$

$$= 0.3(\mathbf{i}\cos 30^\circ - \mathbf{j}\sin 30^\circ) = 0.2598\mathbf{i} - 0.15\mathbf{j} \text{ (m/s)}.$$

The magnitude of the velocity of the gravel:

$$v_{\text{mag}} = \sqrt{(v_x - v_{Bx})^2 + (v_y - v_{by})^2}$$

$$= 6.377 \text{ (m/s)}.$$

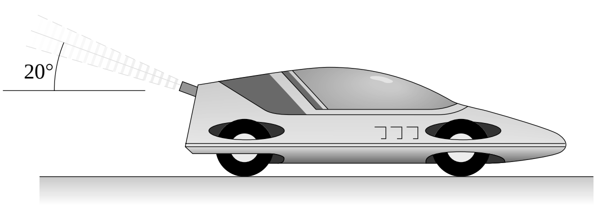
The angle of impact:

$$\beta = \tan^{-1}\left(\frac{v_y - v_{bx}}{v_x - v_{bx}}\right) = -79.6^\circ.$$

The force on the belt is

$$\boxed{\sum \mathbf{F} = \left(\frac{dm_f}{dt}\right)(v_{\text{mag}})(\mathbf{j}\cos\beta + \mathbf{j}\sin\beta) = 51.9\mathbf{i} - 282.2\mathbf{j} \text{ (N)}}$$

Problem 16.109 Suppose that you are designing a toy car that will be propelled by water that squirts from an internal tank at 3.05 m/s relative to the car. The total weight of the car and its water "fuel" is to be 8.9 N. If you want the car to achieve a maximum speed of 3.66 m/s, what part of the total weight must be water?

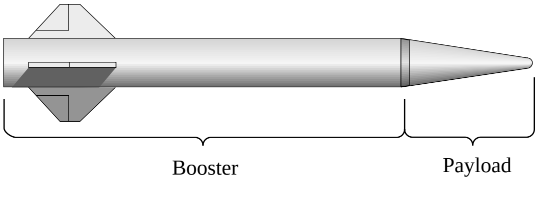


Solution: See Example 16.10

$$3.66 \text{ m/s} - 0 = (3.05 \text{ m/s})\cos 20^\circ \ln\left(\frac{8.9\text{ N}}{W}\right) \Rightarrow W = 2.48 \text{ N}$$

The water must be $\boxed{W_{\text{water}} = (8.9 \text{ N}) - (2.48 \text{ N}) = 6.42 \text{ N}}$

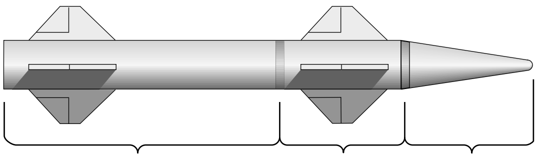
Problem 16.110 The rocket consists of a 1000-kg payload and a 9000-kg booster. Eighty percent of the booster's mass is fuel, and its exhaust velocity is 1200 m/s. If the rocket starts from rest and external forces are neglected, what velocity will it attain? (See Example 16.10.)



Solution:

$$\boxed{v = (1200 \text{ m/s})\ln\left(\frac{10{,}000 \text{ kg}}{1{,}000 \text{ kg} + 0.2[9{,}000 \text{ kg}]}\right) = 1530 \text{ m/s}}$$

Problem 16.111* The rocket consists of a 1000-kg payload and a booster. The booster has two stages whose total mass is 9000 kg. Eighty percent of the mass of each stage is fuel, and the exhaust velocity of each stage is 1200 m/s. When the fuel of stage 1 is expended, it is discarded and the motor of stage 2 is ignited. Assume that the rocket starts from rest and neglect external forces. Determine the velocity attained by the rocket if the masses of the stages are $m_1 = 6000$ kg and $m_2 = 3000$ kg. Compare your result to the answer to Problem 16.110.



Solution:

First burn, $\begin{cases} \text{Full mass} = 10{,}000 \text{ kg} \\ \text{Empty mass} = 4{,}000 \text{ kg} + 0.2(6{,}000 \text{ kg}) = 5{,}200 \text{ kg} \end{cases}$

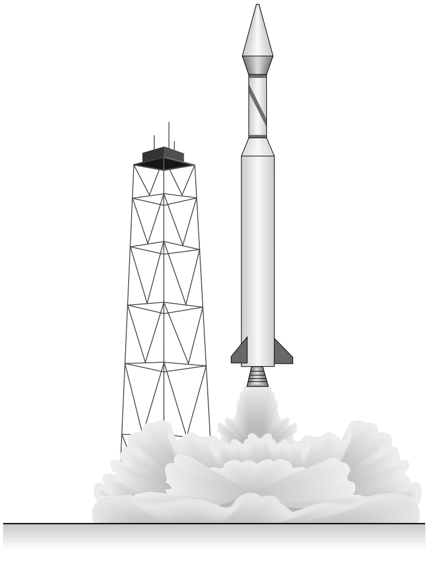
Second burn $\begin{cases} \text{Full mass} = 4{,}000 \text{ kg} \\ \text{Empty mass} = 1{,}000 \text{ kg} + 0.2(300 \text{ kg}) = 1{,}600 \text{ kg} \end{cases}$

$$\boxed{\begin{aligned} v &= (12{,}000 \text{ m/s}) \ln\left(\frac{10{,}000 \text{ kg}}{5200 \text{ kg}}\right) + (12{,}000 \text{ m/s}) \ln\left(\frac{4{,}000 \text{ kg}}{1600 \text{ kg}}\right) \\ &= 1880 \text{ m/s} \end{aligned}}$$

Much faster using stages.

Problem 16.112 A rocket of initial mass m_0 takes off straight up. Its exhaust velocity v_f and the mass flow rate of its engine $m_f = dm_f/dt$ are constant. Show that, during the initial part of the flight, when aerodynamic drag is negligible, the rocket's upward velocity as a function of time is

$$v = v_f \ln\left(\frac{m_0}{m_0 - \dot{m}_f t}\right) - gt.$$



Solution: Start by adding a gravity term to the second equation in Example 16.10. (Newton's second law). We get

$$\sum F_x = \frac{dm_f}{dt} \, v_f - mg = m\frac{dv_x}{dt}.$$

Substitute

$$\frac{dm}{dt} = -\frac{dm_f}{dt}$$

as in the example and divide through by m to get

$$-v_f \left(\frac{1}{m}\right) dmdt - g = \frac{dv_x}{dt}.$$

Integrate with respect to time gives the relation

$$-v_f \int_0^t \left(\frac{1}{m}\right) \frac{dm}{dt} dt$$

$$-\int_0^t g dt = \int_0^t \frac{dv_x}{dt} dt.$$

Simplifying the integrals and setting appropriate limits when we change the variable of integration, we get

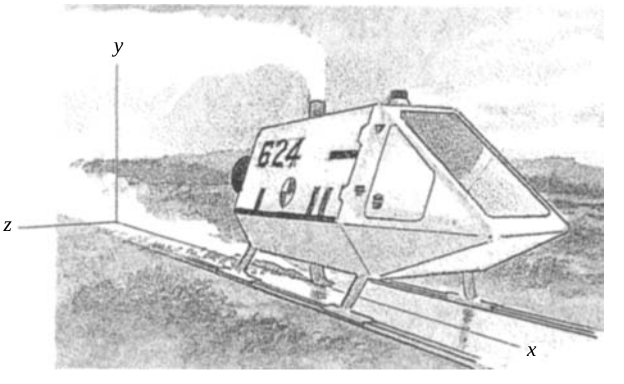
$$-v_f \int_{m_0}^{m} \frac{dm}{m} - g\int_0^t dt = \int_0^v dv_x.$$

Integrating and evaluating at the limits of integration, we get

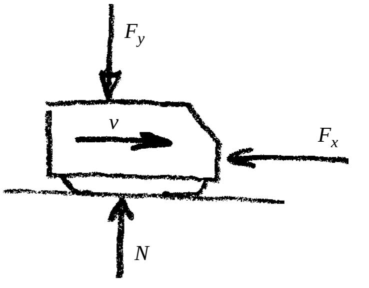
$$v = v_f \ln(m_0/m) - gt.$$

Recalling that $m = m_0 - m_f t$ and substituting this in, we get the desired result.

Problem 16.113 The mass of the rocket sled in Active Example 16.9 is 440 kg. Assuming that the only significant force acting on the sled in the direction of its motion is the force exerted by the flow of water entering it, what distance is required for the sled to decelerate from 300 m/s to 100 m/s?



Solution: From Example 16.9, the force in the x direction (the only force that affects the speed) is



$$F_x = -\rho A v^2$$

From Newton's Second law

$$ma_x = -\rho A v^2$$

$$a_x = v\frac{dv_x}{ds} = -\left(\frac{\rho A}{m}\right)v^2$$

$$\int_{300}^{100} \frac{dv}{v} = -\left(\frac{\rho A}{m}\right)\int_0^{s_f} ds$$

$$\ln(v)\Big|_{300}^{100} = -\frac{\rho A}{m}s_f$$

where

$$\rho = 1000 \text{ kg/m}^3,$$

$$A = 0.01 \text{ m}^2,$$

$$m = 440 \text{ kg}.$$

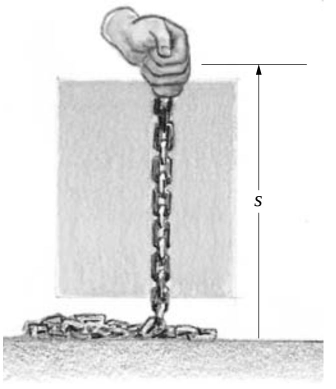
Solving

$$\underline{s_f = 48.3 \text{ m}}$$

Problem 16.114* Suppose that you grasp the end of a chain that weighs 43.8 N/m and lift it straight up off the floor at a constant speed of 0.61 m/s.

(a) Determine the upward force F you must exert as a function of the height s.
(b) How much work do you do in lifting the top of the chain to $s = 1.22$ m?

Strategy: Treat the part of the chain you have lifted as an object that is gaining mass.



Solution: The force is the sum of the "mass flow" reaction and the weight of suspended part of the chain,

$$F = mg + v\left(\frac{dm}{dt}\right).$$

(a) Substitute:

$$\left(\frac{dm}{dt}\right) = \left(\frac{43.8}{9.81}\right) v \text{ kg/s}.$$

The velocity is 0.61 m/s.
The suspended portion of the chain weighs $43.8s$ N. The force required to lift the chain is

$$43.8s + \left(\frac{43.8}{9.81}v\right) v = 43.8s + \frac{43.8(0.61)^2}{9.81}$$

(b) The work done is

$$\boxed{\int_0^{1.22} F\,ds = \left[\left(\frac{16.3}{g}\right)s + \left(\frac{43.8}{2}\right)s^2\right]_0^{1.22} = 34.6 \text{ N-m}}$$

Problem 16.115* Solve Problem 16.114, assuming that you lift the end of the chain straight up off the floor with a constant acceleration of 0.61 m/s^2.

Solution: Assume that the velocity is zero at $s = 0$. The mass of the chain currently suspended is $m = \left(\frac{3}{g}\right)s$. Use the solution to Problem 16.114. From Newton's second law,

$$m\left(\frac{dv}{dt}\right) = F - v\left(\frac{dm}{dt}\right) - mg.$$

The velocity is expressed in terms of s as follows: The acceleration is constant: $dv/dt = 0.61$ m/s^2. Use the chain rule $v(dv/ds) = 0.61$. Integrate: $v^2 = 0.37$ s, where it is assumed that the velocity is zero at $s = 0$, from which

$$F = m\left(\frac{dv}{dt}\right) + 2\sqrt{s}\left(\frac{dm}{dt}\right) + 43.8s$$

Substitute:

$$m = \left(\frac{43.8}{g}\right)s,\quad \frac{dm}{dt} = \left(\frac{43.8}{g}\right)v = \left(\frac{26.6}{g}\right)\sqrt{s},$$

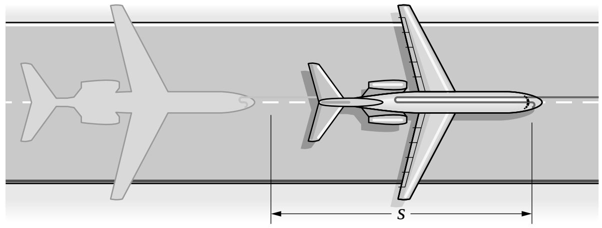
from which

$$\boxed{F = \left(\frac{80}{g} + 43.8\right)s = 11.15 \text{ s}}$$

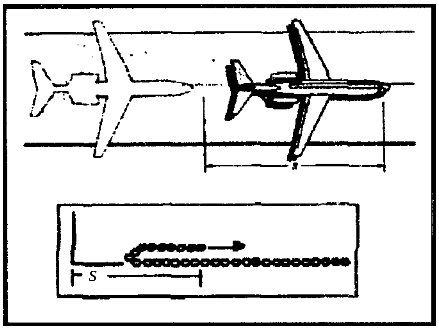
(b) The work done is

$$\boxed{\int_0^4 F\,ds = \left[\left(\frac{80}{g} + 43.8\right)\frac{s^2}{2}\right]_0^{1.22} = 38.66}$$

Problem 16.116* It has been suggested that a heavy chain could be used to gradually stop an airplane that rolls past the end of the runway. A hook attached to the end of the chain engages the plane's nose wheel, and the plane drags an increasing length of the chain as it rolls. Let m be the airplane's mass and v_0 its initial velocity, and let ρ_L be the mass per unit length of the chain. Neglecting friction and aerodynamic drag, what is the airplane's velocity as a function of s?



Solution: Assume that the chain is laid out lengthwise along the runway, such that the aircraft hook seizes the nearest end as the aircraft proceeds down the runway. As the distance s increases, the length of chain being dragged is $\frac{s}{2}$ (see figure). The mass of the chain being dragged is $\frac{\rho L^s}{2}$. The mass "flow" of the chain is



$$\frac{d\left(\frac{\rho L^s}{2}\right)}{dt} = \frac{\rho L^v}{2}.$$

From Newton's second law,

$$\left(\frac{\rho L^s}{2} + m\right)\frac{dv}{dt} = -\frac{\rho L^{v^2}}{2}.$$

Use the chain rule and integrate:

$$\left(\frac{\rho L^s}{2} + m\right)\frac{dv}{ds} = -\frac{\rho L^v}{2}, \quad \frac{dv}{v} = -\frac{\rho L}{2\left(\frac{\rho L^s}{2} + m\right)}\,ds,$$

$$\ln(v) = -\ln\left(m + \frac{\rho L^s}{2}\right) + C.$$

For $v = v_0$ at $s = 0$, $C = \ln(mv_0)$, and

$$\boxed{v = \frac{mv_0}{m + \frac{\rho L^s}{2}}.}$$

Problem 16.117* In Problem 16.116, the frictional force exerted on the chain by the ground would actually dominate other forces as the distance s increases. If the coefficient of kinetic friction between the chain and the ground is μ_k and you neglect all other forces except the frictional force, what is the airplane's velocity as a function of s?

Solution: Assume that the chain layout is configured as shown in Problem 16.116. From Problem 16.116, the weight of the chain being dragged at distance s is $\frac{\rho L^{gs}}{2}$. From Newton's second law

$$\left(m + \frac{\rho L^s}{2}\right)\frac{dv}{dt} = -\frac{\mu_k \rho L^{gs}}{2},$$

where only the friction force is considered. Use the chain rule and integrate:

$$\left(m + \frac{\rho L^s}{2}\right)v\frac{dv}{ds} = -\frac{\mu_k \rho L^{gs}}{2}, \; v\,dv = -\mu_k g \frac{\rho L^s}{2\left(m + \frac{\rho L^s}{2}\right)}\,ds,$$

from which

$$\frac{v^2}{2} = \frac{-2\mu_k g}{\rho L}\left(m + \frac{\rho L^s}{2} - m\ln\left(m + \frac{\rho_L s}{m}\right)\right) + C.$$

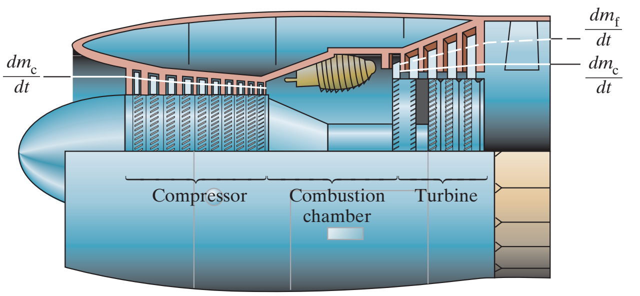
For $v = v_0$ at $s = 0$,

$$C = \frac{v_0^2}{2} + \frac{2\mu_k g m}{\rho L}(1 - \ln(m)),$$

from which, after reduction

$$\boxed{v^2 = v_0^2 - 2g\mu_k\left(s - \frac{2m}{\rho L}\ln\left(1 + \frac{\rho_L s}{2m}\right)\right).}$$

Problem 16.118 A turbojet engine is being operated on a test stand. The mass flow rate of air entering the compressor is 13.5 kg/s, and the mass flow rate of fuel is 0.13 kg/s. The effective velocity of air entering the compressor is zero, and the exhaust velocity is 500 m/s. What is the thrust of the engine? (See Example 16.12.)



Solution: The sum of the mass flows is

$$\left(\frac{dm}{dt}\right) = \left(\frac{dm_c}{dt}\right) + \left(\frac{dm_f}{dt}\right) = 13.5 + 0.13 = 13.63 \text{ kg/s}.$$

The inlet velocity is zero, and the exit velocity is 500 m/s. The thrust is

$$\boxed{T = 13.63(500) = 6820 \text{ N}}$$

Problem 16.119 A turbojet engine is in an airplane flying at 400 km/h. The mass flow rate of air entering the compressor is 13.5 kg/s and the mass flow rate of fuel is 0.13 kg/s. The effective velocity of the air entering the inlet is equal to the airplane's velocity, and the exhaust velocity (relative to the airplane) is 500 m/s. What is the thrust of the engine? (See Example 16.12.)

Solution: Use the "rest frame" of the engine to determine the thrust. Use the solution to Problem 16.118. The inlet velocity is

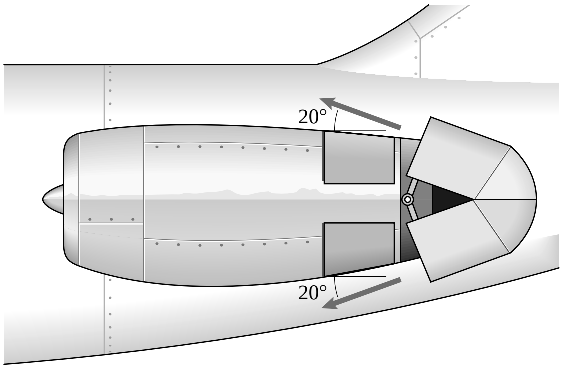
$$v_i = 400\left(\frac{10^3}{3600}\right) = 111.11 \text{ m/s}.$$

The thrust is

$$T = \left(\frac{dm_c}{dt} + \frac{dm_f}{dt}\right)500 - \left(\frac{dm_c}{dt}\right)111.11,$$

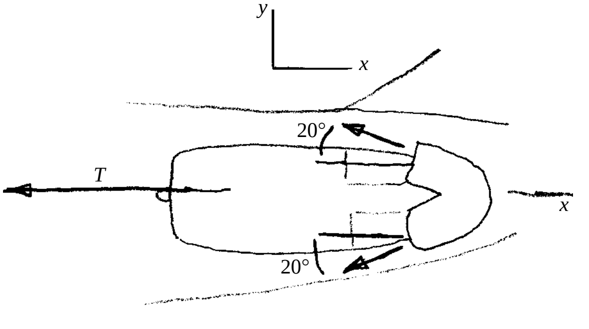
$$\boxed{T = 13.63(500) - (111.11)13.5 = 5315 \text{ N}}$$

Problem 16.120 A turbojet engine's thrust reverser causes the exhaust to exit the engine at 20° from the engine centerline. The mass flow rate of air entering the compressor is 44 kg/s, and the air enters at 60 m/s. The mass flow rate of fuel is 1.5 kg/s, and the exhaust velocity is 370 m/s. What braking force does the engine exert on the airplane? (See Example 16.12.)



Solution:

$$\mathbf{T} = \left[\left(\frac{dm_c dt}{+}\frac{dm_f}{dt}\right)v_e - \frac{dm_c}{dt}v_i\right]_x(-\mathbf{i})$$



where

$$\frac{dm_c}{dt} = 44 \text{ kg/s}$$

$$\frac{dm_f}{dt} = 1.5 \text{ kg/s}$$

$$v_{e_x} = -370\cos(20^\circ) \text{ m/s}$$

$$v_i = +60 \text{ m/s}$$

Solving,

$$T = -18500 \text{ N} = -18.5 \text{ KN (to the Right)}$$

$$|T| = 18.5 \text{ KN}$$

Problem 16.121 The total external force on a 10-kg object is constant and equal to $90\mathbf{i} - 60\mathbf{j} + 20\mathbf{k}$ (N). At $t = 2$ s, the object's velocity is $-8\mathbf{i} + 6\mathbf{j}$ (m/s).

(a) What impulse is applied to the object from $t = 2$ s to $t = 4$ s?
(b) What is the object's velocity at $t = 4$ s?

Solution:

(a) The impulse is

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = 90(4-2)\mathbf{i} - 60(4-2)\mathbf{j} + 20(4-2)\mathbf{k}$$

$$\boxed{= 180\mathbf{i} - 120\mathbf{j} + 40\mathbf{k}\ \text{(N-s)}.}$$

(b) The velocity is

$$m\mathbf{v}_2 - m\mathbf{v}_1 = \int_{t_1}^{t_2} \mathbf{F}\,dt,$$

from which

$$\mathbf{v}_2 = \mathbf{v}_1 + \left(\frac{1}{m}\right)\int_{t_1}^{t_2} \mathbf{F}\,dt = \left(-8 + \frac{180}{10}\right)\mathbf{i} + \left(6 - \frac{120}{10}\right)\mathbf{j}$$

$$+ \left(\frac{40}{10}\right)\mathbf{k}\ \boxed{= 10\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}\ \text{(m/s)}}$$

Problem 16.122 The total external force on an object is $\mathbf{F} = 10t\mathbf{i} + 60\mathbf{j}$ (N). At $t = 0$, the object's velocity is $\mathbf{v} = 20\mathbf{j}$ (m/s). At $t = 12$ s, the x component of its velocity is 48 m/s.

(a) What impulse is applied to the object from $t = 0$ to $t = 6$ s?
(b) What is the object's velocity at $t = 6$ s?

Solution:

(a) The impulse is

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = [5t^2]_0^6\mathbf{i} + [60t]_0^6\mathbf{j} = 180\mathbf{i} + 360\mathbf{j}\ \text{(N-s)}$$

(b) The mass of the object is found from the x component of the velocity at 12 s.

$$m = \frac{\int_{t_1}^{t_2} F\,dt}{48 - 0} = \frac{5(12^2)}{48} = 15\ \text{kg}.$$

The velocity at 6 s is

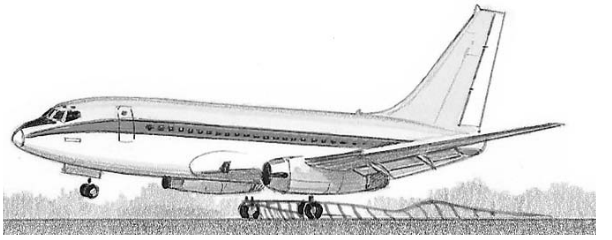
$$m\mathbf{v}_2 - m\mathbf{v}_1 = \int_{t_1}^{t_2} \mathbf{F}\,dt,$$

from which

$$\boxed{\mathbf{v}_2 = 20\mathbf{j} + \frac{180}{15}\mathbf{i} + \frac{360}{15}\mathbf{j} = 12\mathbf{i} + 44\mathbf{j}\ \text{(m/s)}.}$$

Problem 16.123 An aircraft arresting system is used to stop airplanes whose braking systems fail. The system stops a 47.5-Mg airplane moving at 80 m/s in 9.15 s.

(a) What impulse is applied to the airplane during the 9.15 s?
(b) What is the average deceleration to which the passengers are subjected?



Solution:

(a) The impulse is

$$\boxed{\int_{t_1}^{t_2} F\,dt = mv_2 - mv_1 = 47500(80) = 3.8 \times 10^6 \text{ N-s.}}$$

(b) The average force is

$$F_{\text{ave}} = \frac{\int_{t_1}^{t_2} F\,dt}{t_2 - t_1} = \frac{3.8 \times 10^6}{9.15} = 4.153 \times 10^5 \text{ N.}$$

From Newton's second law

$$\boxed{\left(\frac{dv}{dt}\right)_{\text{ave}} = \frac{F_{\text{ave}}}{47500} = 8.743 \text{ m/s}^2}$$

Problem 16.124 The 1895 Austrain 150-mm howitzer had a 1.94-m-long barrel, possessed a muzzle velocity of 300 m/s, and fired a 38-kg shell. If the shell took 0.013 s to travel the length of the barrel, what average force was exerted on the shell?

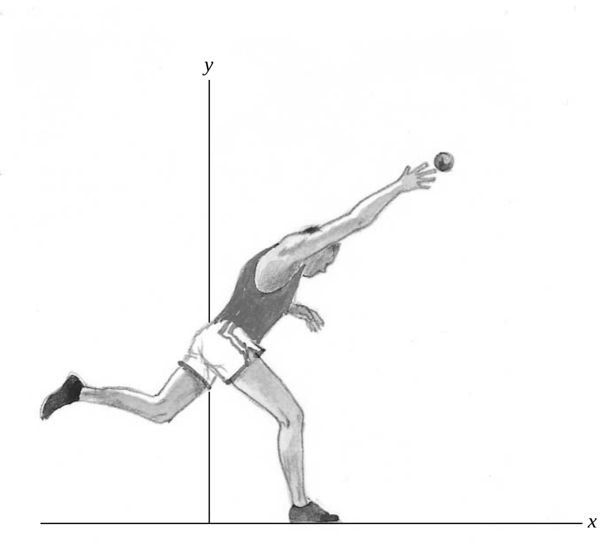
Solution: The average force is

$$\boxed{F_{\text{ave}} = \frac{\int_{t_1}^{t_2} F\,dt}{t_2 - t_1} = \frac{(38)(300)}{0.013} = 877{,}000 \text{ N}}$$

Problem 16.125 An athlete throws a shot put weighing 71.2 N. When he releases it, the shot is 2.13 m above the ground and its components of velocity are $v_x = 9.45$ m/s and $v_y = 7.92$ m/s.

(a) Suppose the athlete accelerates the shot from rest in 0.8 s, and assume as a first approximation that the force **F** he exerts on the shot is constant. Use the principle of impulse and momentum to determine the x and y components of **F**.

(b) What is the horizontal distance from the point where he releases the shot to the point where it strikes the ground?



Solution:

(a) Let F_x, F_y be the components of the force exerted by the athlete. The impulse is

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = \mathbf{i}(F_x)(t_2 - t_1) + \mathbf{j}(F_y - W)(t_2 - t_1)$$

$$= \mathbf{i}\left(\frac{W}{g}\right)(9.45) + \mathbf{j}\left(\frac{W}{g}\right)(7.92)$$

from which

$$\boxed{F_x = \left(\frac{W}{g}\right)\left(\frac{9.45}{0.8}\right) = 85.8 \text{ N.}}$$

$$\boxed{F_y = \left(\frac{W}{g}\right)\left(\frac{7.92}{0.8}\right) + W = 143.2 \text{ N.}}$$

(b) From the conservation of energy for the vertical component of the motion, the maximum height reached is

$$h = \left(\frac{1}{2g}\right)v_y^2 + 2.13 = 5.33 \text{ m.}$$

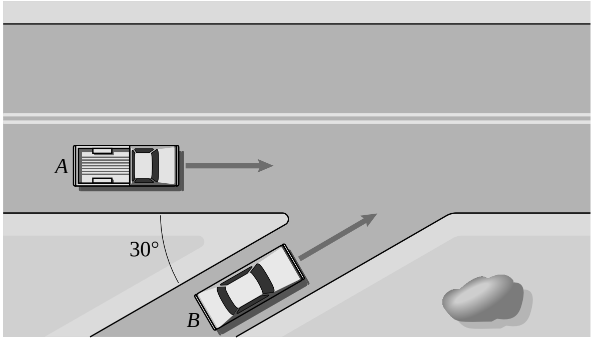
From the solution of Newton's second law for free fall, the time of flight is

$$t_f = \sqrt{\frac{2(h - 2.13)}{g}} + \sqrt{\frac{2h}{g}} = 1.85 \text{ s}$$

and the horizontal distance is $\boxed{D = 9.45 t_f = 17.5 \text{ m}}$

Problem 16.126 The 26688 N pickup truck A moving at 12.2 m/s collides with the 17792 N car B moving at 9.1m/s.

(a) What is the magnitude of the velocity of their common center of mass after the impact?
(b) Treat the collision as a perfectly plastic impact. How much kinetic energy is lost?



Solution:

(a) From the conservation of linear momentum,

$$\left(\frac{W_A}{g}\right)v_A + \left(\frac{W_B}{g}\right)v_B\cos\theta = \left(\frac{W_A+W_B}{g}\right)v_x,$$

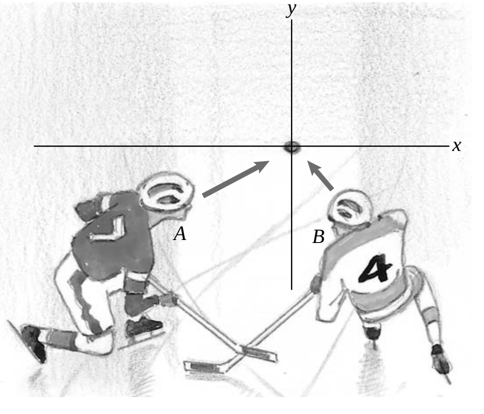
$$\left(\frac{W_B}{g}\right)v_B\sin\theta = \left(\frac{W_A+W_B}{g}\right)v_y.$$

Substitute numerical values, with $\theta = 30^\circ$ to obtain $v_x =$ 10.48 m/s, v_y= 1.83 m/s, from which $v = \sqrt{v_x^2+v_y^2} = 10.64$ m/s

(b) The kinetic energy before the collision minus the kinetic energy after the collision is the loss in kinetic energy:

$$\boxed{\frac{1}{2}\left(\frac{W_A}{g}\right)v_A^2 + \frac{1}{2}\left(\frac{W_B}{g}\right)v_B^2 - \frac{1}{2}\left(\frac{W_A+W_B}{g}\right)v^2 = 21{,}2853 \text{ N-m}}$$

Problem 16.127 Two hockey players ($m_A = 80$ kg, $m_B = 90$ kg) converging on the puck at $x = 0$, $y = 0$ become entangled and fall. Before the collision, $\mathbf{v}_A = 9\mathbf{i} + 4\mathbf{j}$ (m/s) and $\mathbf{v}_B = -3\mathbf{i} + 6\mathbf{j}$ (m/s). If the coefficient of kinetic friction between the players and the ice is $\mu_k = 0.1$, what is their approximate position when they stop sliding?



Solution: The strategy is to determine the velocity of their combined center of mass immediately after the collision using the conservation of linear momentum, and determine the distance using the conservation of energy. The conservation of linear momentum is $9m_A - 3m_B = (m_A + m_B)v_x$, and $4m_A + 6m_B = (m_A + m_B)v_y$, from which $v_x = 2.65$ m/s, and $v_y = 5.06$ m/s, from which $v = \sqrt{v_x^2 + v_y^2} = 5.71$ m/s, and the angle of the path is

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = 62.4^\circ.$$

The work done by friction is

$$\int_0^s -\mu_k g(m_A+m_B)\,ds = -\mu_k g(m_A+m_B)s,$$

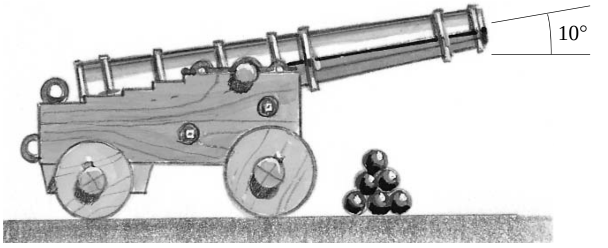
where s is the distance the players slide after collision. From the conservation of work and energy $\frac{1}{2}(m_A+m_B)v^2 - \mu_k g(m_A+m_B)s = 0$, from which

$$s = \frac{v^2}{2\mu_k g} = 16.6 \text{ m}.$$

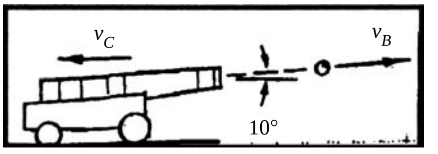
The position of the players after they stop sliding is

$$\boxed{\mathbf{r} = s(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) = 7.7\mathbf{i} + 14.7\mathbf{j} \text{ (m)}}$$

Problem 16.128 The cannon weighed 1780 N, fired a cannonball weighing 44.5 N, and had a muzzle velocity of 61 m/s. For the 10° elevation angle shown, determine (a) the velocity of the cannon after it was fired and (b) the distance the cannonball traveled. (Neglect drag.)



Solution: Assume that the height of the cannon mouth above the ground is negligible. (a) Relative to a reference frame moving with the cannon, the cannonball's velocity is 61 m/s at. The conservation of linear momentum condition is $-m_C v_C + m_B(v_B \cos 10^\circ - v_C) = 0$, from which



$$v_C = \frac{m_B v_B \cos 10^\circ}{m_B + m_C} = 1.464 \text{ m/s}.$$

The velocity of the cannon ball relative to the ground is

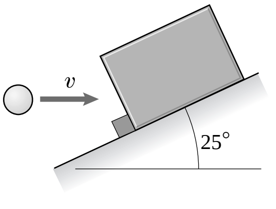
$$v_B = (61 \cos 10^\circ - v_C)\mathbf{i} + 61 \sin 10^\circ \mathbf{j} = 58.57\mathbf{i} + 10.59\mathbf{j} \text{ m/s},$$

from which $v_{0x} = 58.57$ m/s, $v_{0y} = 10.59$ m/s. The maximum height is (from the conservation of energy for free fall) $h_{\max} = v_{0y}^2/(2g) = 5.72$ m/s, where the height of the cannon mouth above the ground is negligible. The time of flight (from the solution of Newton's second law for free fall) is twice the time required to fall from the maximum height,

$$t_{\text{flight}} = 2\sqrt{\frac{2h_{\max}}{g}} = 2.16 \text{ s}.$$

From which the range is $\boxed{x_{\text{impact}} = v_{0x} t_{\text{flight}} = 126.5 \text{ m}}$ since v_{0x} is constant during the flight.

Problem 16.129 A 1-kg ball moving horizontally at 12 m/s strikes a 10-kg block. The coefficient of restitution of the impact is $e = 0.6$, and the coefficient of kinetic friction between the block and the inclined surface is $\mu_k = 0.4$. What distance does the block slide before stopping?



Solution: First we analyze the impact between the ball (b) and the block (B). The component of the ball's velocity parallel to the inclined surface is $v_b = (12) \cos 25^\circ = 10.9$ m/s. Solving the equations

$$m_b v_b = m_b v_b' + m_B v_B',$$

$$e = \frac{v_B' - v_b'}{v_b},$$

we obtain $v_B' = 1.58$ m/s. We use work and energy to determine the distance d the block slides:

$$(-m_B g \sin 25^\circ - \mu_k m_B g \cos 25^\circ)d = 0 - \frac{1}{2} m_B (v_B')^2.$$

Solving yields $\underline{d = 0.162 \text{ m}}$.

Problem 16.130 A Peace Corps volunteer designs the simple device shown for drilling water wells in remote areas. A 70-kg "hammer," such as a section of log or a steel drum partially filled with concrete, is hoisted to $h = 1$ m and allowed to drop onto a protective cap on the section of pipe being pushed into the ground. The combined mass of the cap and section of pipe is 20 kg. Assume that the coefficient of restitution is nearly zero.

(a) What is the velocity of the cap and pipe immediately after the impact?

(b) If the pipe moves 30 mm downward when the hammer is dropped, what resistive force was exerted on the pipe by the ground? (Assume that the resistive force is constant during the motion of the pipe.)

Solution: The conservation of momentum principle for the hammer, pipe and cap is $m_H v_H + m_p v_p = (m_H + m_p)v$, where v_H, v_p are the velocity of the hammer and pipe, respectively, before impact, and v is the velocity of their combined center of mass immediately after impact. The velocity of the hammer before impact (from the conservation of energy for a free fall from height h) is $v_H = \sqrt{2gh} = \sqrt{2(9.81)(1)} = 4.43$ m/s. Since $v_P = 0$,

$$\boxed{v = \frac{m_H}{m_H + m_P} v_H = \frac{70}{90}(4.43) = 3.45 \text{ m/s.}}$$

(b) The work done by the resistive force exerted on the pipe by the cap is $\int_0^s -F\,ds = -Fs$ N-m, where $s = 0.03$ m. The work and energy principle for the hammer, pipe and cap is $-Fs + (m_H + m_P)gs = \frac{1}{2}(m_H + m_P)v^2$, from which

$$\boxed{\begin{aligned} F &= \frac{(m_H + m_P)(v^2 + 2gs)}{2s} = \frac{(90)(3.45^2 + 2(9.81)(0.03))}{2(0.03)} \\ &= 18{,}700 \text{ N} \end{aligned}}$$

Problem 16.131 A tugboat (mass $= 40$ Mg) and a barge (mass $= 160$ Mg) are stationary with a slack hawser connecting them. The tugboat accelerates to 2 knots (1 knot $= 1852$ m/h) before the hawser becomes taut. Determine the velocities of the tugboat and the barge just after the hawser becomes taut (a) if the "impact" is perfectly plastic ($e = 0$) and (b) if the "impact" is perfectly elastic ($e = 1$). Neglect the forces exerted by the water and the tugboat's engines.

Solution: The tugboat's initial velocity is $v_T = 2\left(1852\ \frac{\text{m}}{\text{h}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 1.03$ m/s. The equations governing the "impact" are

$$m_T v_T = m_T v'_T + m_B v'_B,$$

$$e = \frac{v'_B - v'_T}{v_T}.$$

(a) With $e = 0$, we obtain

$$v'_T = v'_B = 0.206 \text{ m/s (0.4 knots).}$$

(b) With $e = 1$, we obtain

$$v'_T = -0.617 \text{ m/s } (-1.2 \text{ knots}),$$

$$v'_B = 0.412 \text{ m/s (0.8 knots).}$$

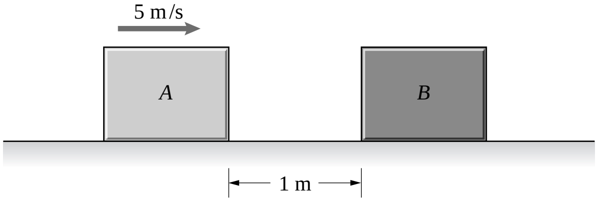
Problem 16.132 In Problem 16.131, determine the magnitude of the impulsive force exerted on the tugboat in the two cases if the duration of the "impact" is 4 s. Neglect the forces exerted by the water and the tugboat's engines during this period.

Solution: Since the tugboat is stationary at $t = t_1$, the linear impulse is

$$\int_{t_1}^{t_2} F\,dt = F_{ave}(t_2 - t_1) = m_B v'_B,$$

from which $F_{ave} = m_B v'_B/4 = (4 \times 10^4)v'_B$. For Case (a) $v'_B = 0.206$ m/s, from which $\boxed{F_{ave} = 8230 \text{ N}}$. Case (b). $v'_B = 0.412$ m/s, $\boxed{F_{ave} = 16{,}500 \text{ N}}$

Problem 16.133 The 10-kg mass A is moving at 5 m/s when it is 1 m from the stationary 10-kg mass B. The coefficient of kinetic friction between the floor and the two masses is $\mu_k = 0.6$, and the coefficient of restitution of the impact is $e = 0.5$. Determine how far B moves from its initial position as a result of the impact.



Solution: Use work and energy to determine A's velocity just before impact:

$$-\mu_k mg(1) = \tfrac{1}{2}mv_A^2 - \tfrac{1}{2}m(5)^2.$$

Solving, $v_A = 3.64$ m/s. Now analyze the impact:

$$mv_A = mv'_A + mv'_B,$$

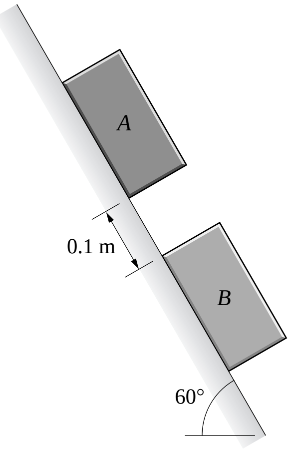
$$e = \frac{v'_B - v'_A}{v_A}.$$

Solving these two equations, we obtain $v'_B = 2.73$ m/s. We use work and energy to determine how far B slides:

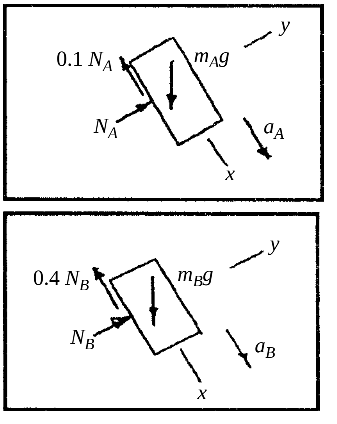
$$-\mu_k mgd = 0 - \tfrac{1}{2}m(v'_B)^2.$$

Solving, $d = 0.632$ m.

Problem 16.134 The kinetic coefficients of friction between the 5-kg crates A and B and the inclined surface are 0.1 and 0.4, respectively. The coefficient of restitution between the crates is $e = 0.8$. If the crates are released from rest in the positions shown, what are the magnitudes of their velocities immediately after they collide?



Solution: The free body diagrams of A and B are shown. From the diagram of A, we have $N_A = m_A g \cos 60^\circ = (5)(9.81)\cos 60^\circ = 24.5$ N



$$\sum F_x = m_A g \sin 60^\circ - 0.1\ N_A = m_A a_A.$$

Solving, $(5)(9.81)\sin 60^\circ - 0.1(24.5) = (5)a_A$, $a_A = 8.01$ m/s^2. The velocity of A is $v_A = a_A t$ and its position is $x_A = \frac{1}{2}a_A t^2$. From the free body diagram of B, we have $N_B = N_A = 24.5$ N and $\sum F_x = m_A g \sin 60^\circ - 0.4 N_B = m_B a_B$. Solving we have $(5)(9.81)\sin 60^\circ - 0.4(24.5) = (5)a_B$, or $a_B = 6.53$ m/s^2. The velocity of B is $v_B = a_B t$ and its position is $x_B = a_B t^2/2$. To find the time of impact, set

$$x_A = x_B + 0.1: \quad \frac{1}{2}a_A t^2 = \frac{1}{2}a_B t^2 + 0.1$$

Solving for t

$$t = \sqrt{\frac{2(0.1)}{a_A - a_B}} = \sqrt{\frac{2(0.1)}{8.01 - 6.53}} = 0.369 \text{ s}.$$

The velocities at impact are

$$v_A = a_A t = (8.01)(0.369) = 2.95 \text{ m/s} \quad v_B = a_B t = (6.53)(0.369)$$

$$= 2.41 \text{ m/s}$$

Conservation of linear momentum yields

$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B:$$

$$(5)(2.95) + (5)(2.41) = (5)v'_A + (5)v'_B \quad \textbf{(1)}$$

The coefficient of restitution is

$$e = \frac{v'_B - v'_A}{v_A - v_B}:$$

Evaluating, we have

$$0.8 = \frac{v'_B - v'_A}{2.95 - 2.41} \quad \textbf{(2)}.$$

Solving equations (1) and (2), $v'_A = 2.46$ m/s, $v'_B = 2.90$ m/s.

Problem 16.135 Solve Problem 16.134 if crate A has a velocity of 0.2 m/s down the inclined surface and crate B is at rest when the crates are in the positions shown.

Solution: From the solution of Problem 16.134, A's acceleration is $a_A = 8.01$ m/s^2 so A's velocity is

$$\int_{0.2}^{v} dv = \int_0^t a_A \, dt v_A = 0.2 + a_A t$$

and its position is $x_A = 0.2t + \frac{1}{2}a_A t^2$. From the solution of Problem 16.136, the acceleration, velocity, and position of B are $a_B = 6.53$ m/s^2 $v_B = a_B t$ $x_B = \frac{1}{2}a_B t^2$, At impact, $x_A = x_B + 0.1$; $0.2t + \frac{1}{2}a_A t^2 = \frac{1}{2}a_B t^2 + 0.1$. Solving for t, we obtain $t = 0.257$ s so, $v_A = 0.2 + (8.01)(0.257) = 2.26$ m/s; $v_B = (6.53)(0.257) = 1.68$ m/s

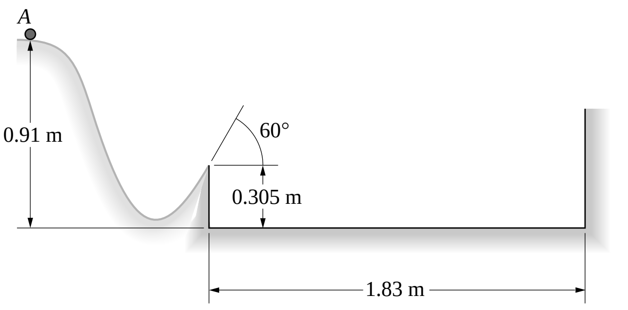
$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B :$$

$$(5)(2.26) + (5)(1.68) = (5)v'_A + (5)v'_B \quad \textbf{(1)};$$

$$e = \frac{v'_B - v'_A}{v_A - v_B} : 0.8 = \frac{v'_B - v'_A}{2.26 - 1.68} \qquad \textbf{(2)}.$$

Solving Equations (1) and (2), $v'_A = 1.74$ m/s $v'_B = 2.20$ m/s.

Problem 16.136 A small object starts from rest at A and slides down the smooth ramp. The coefficient of restitution of the impact of the object with the floor is $e = 0.8$. At what height above the floor does the object hit the wall?



Solution: The impact with the floor is an oblique central impact in which the horizontal component of the velocity is unchanged. The strategy is

(a) determine the time between leaving the ramp and impact with the wall,

(b) determine the time between the first bounce and impact with the wall

(c) from the velocity and height after the first bounce, determine the height at the time of impact with the wall. The steps in this process are:

(1) *The velocities on leaving the ramp*. The velocity at the bottom edge of the ramp is (from the conservation of energy) $v = \sqrt{2gh} = \sqrt{2(9.81)(0.91 - 0.305)} = 3.46$ m/s. The horizontal component of the velocity is $v_x = v\cos 60^\circ = 1.73$ m/s. The vertical component of the velocity at the bottom edge of the ramp is $v_y = v\sin 60^\circ = 3$ m/s.

(2) *The maximum height after leaving the ramp*: From the conservation of energy, the maximum height reached is

$$h_{\max} = \frac{v_y^2}{2g} + 0.305 = 0.762 \text{ m}.$$

(3) *The velocity and maximum height after the first bounce*: The velocity of the first impact (from the conservation of energy for a free fall) is $v_{\text{yimpact}} = -\sqrt{2gh_{\max}} = -3.86$ m/s. The vertical velocity after the first bounce (see equation following Eq. (16.17)) is $v'_y = -ev_{\text{yimpact}} = 0.8(3.86) = 3.09$ m/s. The maximum height after the first bounce is

$$h'_y = \frac{(v'_y)^2}{2g} = 0.49 \text{ m}.$$

(4) *The time of impact with the wall, the time at the first bounce, and the time between the first bounce and impact with the wall*. The time required to reach the wall is

$$t_w = \frac{1.83}{v_x} = 1.058 \text{ s}.$$

The time at the first bounce (from a solution to Newton's second law for a free fall) is

$$t_b = \sqrt{\frac{2(h_{\max} - 0.305)}{g}} + \sqrt{\frac{(2)h_{\max}}{g}} = 0.7 \text{ s}.$$

The time between the first bounce and wall impact is $t_{w-b} = t_w - t_b = 0.3582$ s.

(5) *Is the ball on an upward or downward part of its path when it strikes the wall*? The time required to reach maximum height after the first bounce is

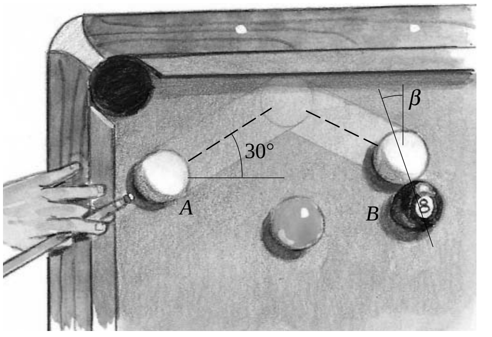
$$t'_{bh} = \sqrt{\frac{2h'_y}{g}} = 0.3154 \text{ s}.$$

Since $t_{w-b} > t'_{bh}$, the ball is on a downward part of its trajectory when it impacts the wall.

(6) *The height at impact with the wall*: The height of impact is (from a solution to Newton's second law for free fall)

$$\boxed{h_w = h'_y - \frac{g}{2}(t_{w-b} - t'_{bh})^2 = 0.48 \text{ m/s}}$$

Problem 16.137 The cue gives the cue ball A a velocity of magnitude 3 m/s. The angle $\beta = 0$ and the coefficient of restitution of the impact of the cue ball and the eight ball B is $e = 1$. If the magnitude of the eight ball's velocity after the impact is 0.9 m/s, what was the coefficient of restitution of the cue ball's impact with the cushion? (The balls are of equal mass.)



Solution: The strategy is to treat the two impacts as oblique central impacts. Denote the paths of the cue ball as $P1$ before the bank impact, $P2$ after the bank impact, and $P3$ after the impact with the 8-ball. The velocity of the cue ball is

$$\mathbf{v}_{AP1} = 3(\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ) = 2.6\mathbf{i} + 1.5\mathbf{j} \text{ (m/s)}.$$

The x component is unchanged by the bank impact. The y component after impact is $v_{AP2y} = -ev_{AP1y} = -1.5e$, from which the velocity of the cue ball after the bank impact is $\mathbf{v}_{AP2} = 2.6\mathbf{i} - 1.5e\mathbf{j}$. At impact with the 8-ball, the x component is unchanged. The y component after impact is obtained from the conservation of linear momentum and the coefficient of restitution. The two equations are

$$m_A v_{AP2y} = m_A v_{AP3y} + m_B v_{BP3y}$$

$$\text{and} \quad 1 = \frac{v_{BP3y} - v_{AP3y}}{v_{AP2y}}.$$

For $m_A = m_B$, these equations have the solution $v_{AP3y} = 0$ and $v_{BP3y} = v_{AP2y}$, from which the velocities of the cue ball and the 8-ball after the second impact are $\mathbf{v}_{AP3} = 2.6\mathbf{i}$ (m/s), and $\mathbf{v}_{BP3} = -1.5e\mathbf{j}$ (m/s). The magnitude of the 8-ball velocity is $v_{BP3} = 0.9$ m/s, from which

$$\boxed{e = \frac{0.9}{1.5} = 0.6}$$

Problem 16.138 What is the solution to Problem 16.137 if the angle $\beta = 10^\circ$?

Solution: Use the results of the solution to Problem 16.137. The strategy is to treat the second collision as an oblique central impact about the line P when $\beta = 10^\circ$. The unit vector parallel to the line P is

$$\mathbf{e}_P = (\mathbf{i}\sin 10^\circ - \mathbf{j}\cos 10^\circ) = 0.1736\mathbf{i} - 0.9848\mathbf{j}.$$

The vector normal to the line P is $\mathbf{e}_{Pn} = 0.9848\mathbf{i} + 0.1736\mathbf{j}$. The projection of the velocity $\mathbf{v}_{AP2} = v_{AP3}\mathbf{e}_P + v_{AP3n}\mathbf{e}_{Pn}$. From the solution to Problem 16.13, $\mathbf{v}_{AP2} = 2.6\mathbf{i} - 1.5e\mathbf{j}$, from which the two simultaneous equations for the new components: $2.6 = 0.173\, v_{AP3} + 0.9848 v_{AP3n}$, and $-1.5e = -0.9848 v_{AP3} + 0.1736 v_{AP3n}$. Solve: $v_{AP3} = 0.4512 + 1.477e$, $v_{AP3n} = 2.561 - 0.2605e$. The component of the velocity normal to the line P is unchanged by impact. The change in the component parallel to P is found from the conservation of linear momentum and the coefficient of restitution:

$$m_A v_{AP3} = m_A v'_{AP3} + m_B v'_{BP3},$$

$$\text{and} \quad 1 = \frac{v'_{BP3} - v'_{AP3}}{v_{AP3}}.$$

For $m_A = m_B$, these equations have the solution $v'_{AP3} = 0$ and $v'_{BP3} = v_{AP3}$. From the value of $v'_{BP3} = 0.9$ m/s, $0.9 = 0.4512 + 1.477e$, from which

$$\boxed{e = \frac{0.9 - 0.4512}{1.477} = 0.304}$$

Problem 16.139 What is the solution to Problem 16.137 if the angle $\beta = 15^\circ$ and the coefficient of restitution of the impact between the two balls is $e = 0.9$?

Solution: Use the solution to Problem 16.137. The strategy is to treat the second collision as an oblique central impact about P when $\beta = 15^\circ$. The unit vector parallel to the line P is

$$\mathbf{e}_P = (\mathbf{i}\sin 15^\circ - \mathbf{j}\cos 15^\circ) = 0.2588\mathbf{i} - 0.9659\mathbf{j}.$$

The vector normal to the line P is

$$\mathbf{e}_{Pn} = 0.9659\mathbf{i} + 0.2588\mathbf{j}.$$

The projection of the velocity $\mathbf{v}_{AP2} = v_{AP3}\mathbf{e}_P + v_{AP3n}\mathbf{e}_{Pn}$. From the solution to Problem 16.139, $\mathbf{v}_{AP2} = 2.6\mathbf{i} - 1.5e\mathbf{j}$, from which the two simultaneous equations for the new components: $2.6 = 0.2588v_{AP3} + 0.9659v_{AP3n}$, and $-1.5e = -0.9659v_{AP3} + 0.2588v_{AP3n}$.

Solve: $v_{AP3} = 0.6724 + 1.449e$, $v_{AP3n} = 2.51 - 0.3882e$. The component of the velocity normal to the line P is unchanged by impact. The velocity of the 8-ball after impact is found from the conservation of linear momentum and the coefficient of restitution:

$$m_A v_{AP3} = m_A v'_{AP3} + m_B v'_{BP3},$$

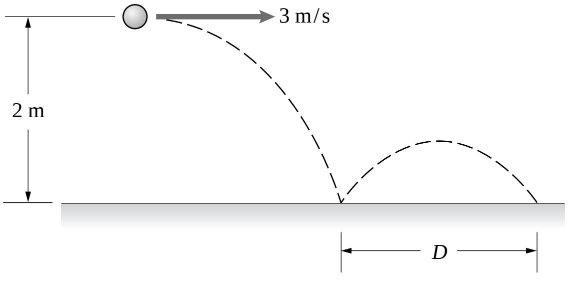
and $$e_B = \frac{v'_{BP3} - v'_{AP3}}{v_{AP3}},$$

where $e_B = 0.9$. For $m_A = m_B$, these equations have the solution

$$v'_{BP3} = \left(\frac{1}{2}\right)(1 + e_B)v_{AP3} = 0.95v_{AP3}.$$

From the value of $v'_{BP3} = 0.9$ m/s, $0.9 = 0.95(0.6724 + 1.449e)$, from which $\boxed{e = 0.189}$

Problem 16.140 A ball is given a horizontal velocity of 3 m/s at 2 m above the smooth floor. Determine the distance D between the ball's first and second bounces if the coefficient of restitution is $e = 0.6$.



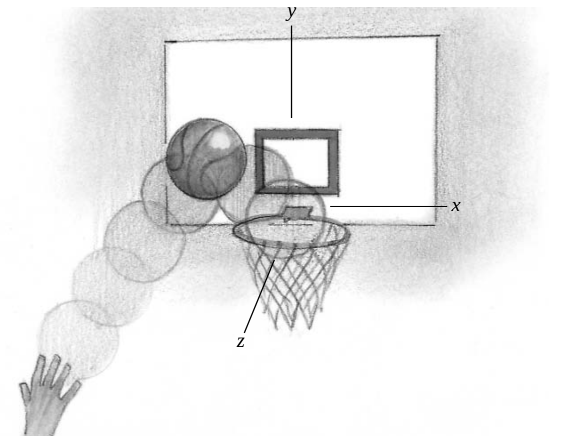
Solution: The strategy is to treat the impact as an oblique central impact with a rigid surface. The horizontal component of the velocity is unchanged by the impact. The vertical velocity after impact is $v'_{Ay} = -ev_{Ay}$. The vertical velocity before impact is $v_{Ay} = -\sqrt{2\,gh} = -\sqrt{2(9.81)2} = -6.264$ m/s, from which $v'_{Ay} = -0.6(-6.264) = 3.759$ m/s. From the conservation of energy for a free fall, the height of the second bounce is

$$h' = \frac{(v'_{Ay})^2}{2g} = 0.720 \text{ m}.$$

From the solution of Newton's second law for free fall, the time between the impacts is twice the time required to fall from height h, $t = 2\sqrt{\frac{2h'}{g}} = 0.7663$ s, and the distance D is

$$\boxed{D = v_0 t = (3)(0.7663) = 2.30 \text{ m}}$$

Problem 16.141* A basketball dropped from a height of 1.22 m rebounds to a height of 0.91 m. In the layup shot shown, the magnitude of the ball's velocity is 1.52 m/s, and the angles between its velocity vector and the positive coordinate axes are $\theta_x = 42^\circ$, $\theta_y = 68^\circ$, and $\theta_z = 124^\circ$ just before it hits the backboard. What are the magnitude of its velocity and the angles between its velocity vector and the positive coordinate axes just after the ball hits the backboard?



Solution: Using work and energy to determine the dropped ball's velocity just before and just after it hits the floor,

$$mg(1.22 \text{ m}) = \frac{1}{2}mv^2,$$

$$mg(0.91 \text{ m}) = \frac{1}{2}m(v')^2,$$

we obtain $v = 4.9$ m/s and $v' = -4.24$ m/s. The coefficient of restitution is

$$e = -\frac{v'}{v} = 0.866.$$

The ball's velocity just before it hits the backboard is

$$\mathbf{v} = 1.52(\cos 42^\circ \mathbf{i} + \cos 68^\circ \mathbf{j} + \cos 124^\circ \mathbf{k})$$

$$= 1.13\mathbf{i} + 0.57\mathbf{j} - 0.85\mathbf{k} \text{ (m/s)}.$$

The z component of velocity just after the impact is

$$v_z' = -ev_z = -(0.866)(-0.85)$$

$$= 0.74 \text{ m/s}.$$

The ball's velocity after impact is $\mathbf{v}' = 1.13\mathbf{i} + 0.57\mathbf{j} + 0.74\mathbf{k}$ (m/s). Its magnitude is $|\mathbf{v}'| = 1.47$ m/s, and

$$\theta_x = \arccos\left(\frac{1.13}{1.47}\right) = 39.5^\circ,$$

$$\theta_y = \arccos\left(\frac{0.57}{1.47}\right) = 67.1^\circ,$$

$$\theta_z = \arccos\left(\frac{0.74}{1.47}\right) = 59.8^\circ.$$

Problem 16.142* In Problem 16.141, the basketball's diameter is 241.3 mm., the coordinates of the center of basket rim are $x = 0$, $y = 0$, and $z = 305$ mm., and the backboard lies in the x-y plane. Determine the x and y coordinates of the point where the ball must hit the backboard so that the center of the ball passes through the center of the basket rim.

Solution: See the solution of Problem 16.141. The ball's velocity after impact in m/s is

$$\mathbf{v}' = 1.13\mathbf{i} + 0.57\mathbf{j} + 0.74\mathbf{k} \text{ m/s}$$

From impact to the center of the rim, the distance the center of the ball moves in the z direction is $305 - \frac{1}{2}(241.3) = 184.4$ mm. The time required is

$$t = \frac{0.1844 \text{ m}}{0.74 \text{ m/s}} = 0.250 \text{ s}.$$

Setting

$$x = 0 = x_0 + v_{xo}t$$

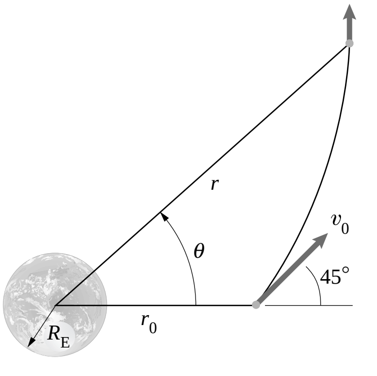
$$= x_0 + (1.13)(0.250)$$

and $y = 0 = y_0 + v_{y0}t - \frac{1}{2}gt^2$

$$= y_0 + (0.57)(0.250) - \frac{1}{2}(9.81)(0.305)(0.250)^2,$$

we obtain $x_0 = -282.7$ mm., $y_0 = 163.1$ mm.

Problem 16.143 A satellite at $r_0 = 16090$ km from the center of the earth is given an initial velocity $v_0 = 6096$ m/s in the direction shown. Determine the magnitude of the transverse component of the satellite's velocity when $r = 32180$ km. (The radius of the earth is 6372 km.)



Solution: By definition,

$$H_0 = |\mathbf{r} \times m\mathbf{v}| = mr_0v_0 \sin 45^\circ$$

$$= m(16090)(1000)(6096) \sin 45^\circ$$

$$H_0 = 6.93 \text{ m} \times 10^{10} \text{ kg-m}^2/\text{s}.$$

The gravitational force

$$\mathbf{F} = -\frac{mgR_E^2}{r^2}\mathbf{e}_r,$$

where $\mathbf{e}_r$ is a unit vector parallel to the radius vector $\mathbf{r}$. Since $(\mathbf{r} \times \mathbf{e}_r) = 0$, it follows that the angular momentum impulse is

$$\int_{t_1}^{t_2} \left(\mathbf{r} \times \sum \mathbf{F}\right) dt = 0 = \mathbf{H}_1 - \mathbf{H}_0,$$

from which $\mathbf{H}_0 = \mathbf{H}_1$, and the angular momentum is conserved. Thus the angular momentum at the distance 32180 km is $H_0 = mr_1v_{\theta 1} \sin 90^\circ$. From which

$$\boxed{v_{\theta 1} = \frac{6.93 \times 10^{10}}{3.21 \times 10^7} = 2155 \text{ m/s}}$$

is the magnitude of the transverse velocity.

Problem 16.144 In Problem 16.143, determine the magnitudes of the radial and transverse components of the satellite's velocity when $r = 24135$ km.

Solution: From the solution to Problem 16.143, the constant angular momentum of the satellite is $H_0 = 6.93 \text{ m} \times 10^{10}$ kg-m^2/s. The transverse velocity at $r = 24135$ km is

$$\boxed{v_\theta = \frac{H}{m(24135)(1000)} = 2569.5 \text{ m/s}}$$

From conservation of energy:

$$\left(\frac{1}{2}mv_0^2\right)_{r=16090} - \frac{mgR_E^2}{r_0} = \left(\frac{1}{2}mv_1^2\right)_{r=24135} - \frac{mgR_E^2}{r_1},$$

from which

$$v_1^2 = v_0^2 + 2gR_E^2\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

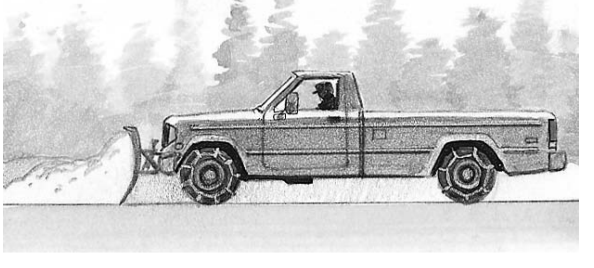
$$= (6096)^2 + 2(9.81)(6372000)^2\left(\frac{1}{24135000} - \frac{1}{16090000}\right)$$

$$= 0.207 \times 10^8 \,(\text{m/s})^2.$$

The radial velocity is

$$\boxed{v_r = \sqrt{v_1^2 - v_\theta^2} = 3520.4 \text{ m/s}}$$

Problem 16.145 The snow is 0.61 m deep and weighs 3142 N/m^3, the snowplow is 2.44 m wide, and the truck travels at 8 km/h. What force does the snow exert on the truck?



Solution: The velocity of the truck is

$v = 8$ km/h $= 2.22$ m/s.

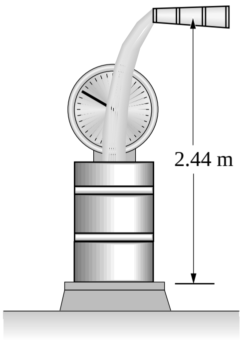
The mass flow is the product of the depth of the snow, the width of the plow, the mass density of the snow, and the velocity of the truck, from which

$$\left(\frac{dm_f}{dt}\right) = (0.61)\left(\frac{3142}{g}\right)(2.44)(2.22) = 1058.3 \text{ kg/s}.$$

The force is

$$\boxed{F = \left(\frac{dm_f}{dt}\right) v = 2375.2 \text{ N}}$$

Problem 16.146 An empty 244.6 N drum, 0.91 m in diameter, stands on a set of scales. Water begins pouring into the drum at 5338 N/min from 2.44 m above the bottom of the drum. The weight density of water is approximately 9802 N/m^3. What do the scales read 40 s after the water starts pouring?



Solution: The mass flow rate is

$$\left(\frac{dm_f}{dt}\right) = \left(\frac{5338}{g}\right)\left(\frac{1}{60}\right) = 9.07 \text{ kg/s}$$

After 40 seconds, the volume of water in the drum is

$$\text{volume} = \frac{5338}{9802}\left(\frac{40}{60}\right) = 0.363 \text{ m}^3.$$

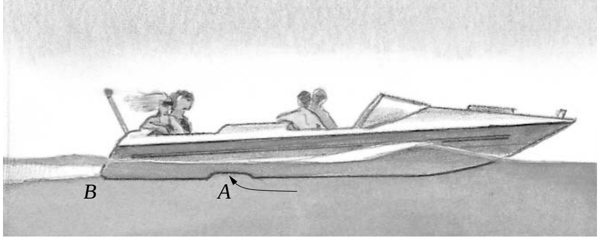
The height of water in the drum is

$$h_{\text{water}} = \frac{\text{volume}}{\pi(0.455^2)} = 0.553 \text{ m}.$$

The velocity of the water stream at point of impact is (from the conservation of energy for free fall) $v = \sqrt{2g(2.44 - h_{\text{water}})} = 6.081$ m/s. The scale reading is

$$\boxed{\begin{aligned} W &= (\text{volume})(9802) + 244.6 + \left(\frac{dm_f}{dt}\right) \\ v &= 0.363(9802) + 244.6 + 9.07(6.081) = 3856.4 \text{ N} \end{aligned}}$$

Problem 16.147 The ski boat's jet propulsive system draws water in at A and expels it at B at 24.4 m/s relative to the boat. Assume that the water drawn in enters with no horizontal velocity relative to the surrounding water. The maximum mass flow rate of water through the engine is 36.5 kg/s. Hydrodynamic drag exerts a force on the boat of magnitude $1.5v$ N, where v is the boat's velocity in feet per second. Neglecting aerodynamic drag, what is the ski boat's maximum velocity?



Solution: Use the boat's "rest" frame of reference. The force exerted by the inlet mass flow on the boat is

$$F_{\text{inlet}} = -\left(\frac{dm_f}{dt}\right) v = -36.5 \text{ v}.$$

The force exerted by the exiting mass flow on the boat is

$$F_{\text{exit}} = \left(\frac{dm_f}{dt}\right)(24.4) = 36.5(24.4)$$

The hydrodynamic drag is $F_{\text{drag}} = -1.5$ v. At top speed the sum of the forces vanishes:

$$\sum F = F_{\text{inlet}} + F_{\text{exit}} + F_{\text{drag}} = 0$$

$$= -36.5v + 36.5(24.4) - 1.5v = -38v + 36.5(24.4) = 0,$$

from which

$$\boxed{v = \frac{36.5(24.4)}{38} = 23.4 \text{ m/s}}$$

Problem 16.148 The ski boat of Problem 16.147 weighs 12454 N. The mass flow rate of water through the engine is 36 kg/s, and the craft starts from rest at $t = 0$. Determine the boat's velocity (a) at $t = 20$ s and (b) $t = 60$ s.

Solution: Use the solution to Problem 16.147: the sum of the forces on the boat is

$$\sum F = -38v + 36.5(24.4) \text{ N}.$$

From Newton's second law

$$\frac{W}{g}\left(\frac{dv}{dt}\right) = -38v + 890.6.$$

Separate variables and integrate:

$$\frac{dv}{23.4 - v} = \frac{38g}{W}\,dt,$$

from which

$$\ln(23.4 - v) = -\frac{38g}{W}t + C.$$

At $t = 0$, $v = 0$, from which $C = \ln(23.4)$, from which

$$\ln\left(1 - \frac{v}{23.4}\right) = -\frac{38g}{W}t.$$

Invert:

$$v(t) = 23.4\left(1 - e^{-\frac{38g}{W}t}\right).$$

Substitute numerical values:

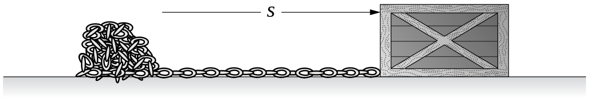
$W = 12454$ N,

$g = 9.81$ m/s^2.

(a) At $t = 20$ s, $\boxed{v = 10.54 \text{ m/s}}$,

(b) At $t = 60$ s, $\boxed{v = 19.52 \text{ m/s}}$

Problem 16.149* A crate of mass m slides across a smooth floor pulling a chain from a stationary pile. The mass per unit length of the chain is ρ_L. If the velocity of the crate is v_0 when $s = 0$, what is its velocity as a function of s?



Solution: The "mass flow" of the chain is

$$\left(\frac{dm_f}{dt}\right) = \rho_L v.$$

The force exerted by the "mass flow" is $F = \rho_L v^2$. From Newton's second law

$$(\rho_L s + m)\left(\frac{dv}{dt}\right) = -\rho_L v^2.$$

Use the chain rule:

$$(\rho_L s + m)v\frac{dv}{ds} = -\rho_L v^2.$$

Separate variables and integrate:

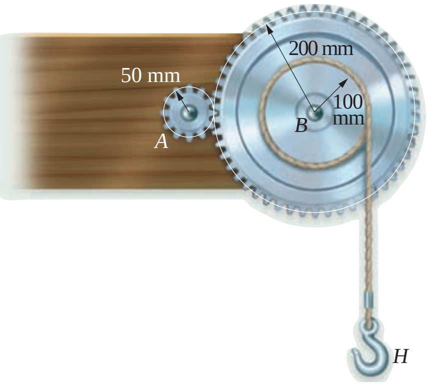
$$\ln(v) = -\ln\left(s + \frac{m}{\rho_L}\right) + C,$$

from which

$$C = \ln\left(\frac{mv_0}{\rho_L}\right).$$

Reduce and solve: $\boxed{v = \frac{mv_0}{(\rho_L s + m)}}$

Problem 17.1 In Active Example 17.1, suppose that at a given instant the hook H is moving downward at 2 m/s. What is the angular velocity of gear A at that instant?



Solution: The angular velocity of gear B is

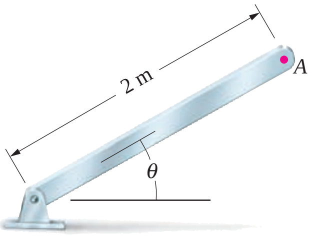
$$\omega_B = \frac{v_H}{r_H} = \frac{2 \text{ m/s}}{0.1 \text{ m}} = 20 \text{ rad/s}.$$

The gears are connected through the common velocity of the contact points

$$r_B\omega_B = r_A\omega_A \Rightarrow \omega_A = \frac{r_B}{r_A}\omega_B = \frac{0.2 \text{ m}}{0.05 \text{ m}}(20 \text{ rad/s}) = 80 \text{ rad/s}.$$

$\boxed{\omega_A = 80 \text{ rad/s counterclockwise.}}$

Problem 17.2 The angle θ (in radians) is given as a function of time by $\theta = 0.2\pi t^2$. At $t = 4$ s, determine the magnitudes of (a) the velocity of point A and (b) the tangential and normal components of acceleration of point A.



Solution: We have

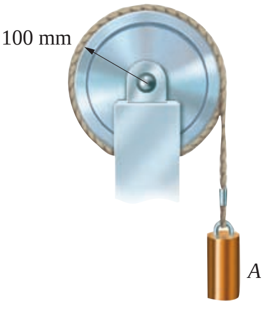
$$\theta = 0.2\pi t^2, \quad \omega = \frac{d\theta}{dt} = 0.4\pi t, \quad \alpha = \frac{d\omega}{dt} = 0.4\pi.$$

Then

(a) $v = r\omega = (2)(0.4\pi)(4) = 10.1 \text{ m/s}.$ $\boxed{v = 10.1 \text{ m/s.}}$

(b) $a_n = r\omega^2 = (2)[(0.4\pi)(4)]^2 = 50.5 \text{ m/s}^2,$
$a_t = r\alpha = (2)(0.4\pi) = 2.51 \text{ m/s}^2.$ $\boxed{\begin{aligned} a_n &= 50.5 \text{ m/s}^2, \\ a_t &= 2.51 \text{ m/s}^2. \end{aligned}}$

Problem 17.3 The mass A starts from rest at $t = 0$ and falls with a constant acceleration of 8 m/s^2. When the mass has fallen one meter, determine the magnitudes of (a) the angular velocity of the pulley and (b) the tangential and normal components of acceleration of a point at the outer edge of the pulley.



Solution: We have

$$a = 8 \text{ m/s}^2, v = \sqrt{2as} = \sqrt{2(8 \text{ m/s}^2)(1 \text{ m})} = 4 \text{ m/s},$$

$$\omega = \frac{v}{r} = \frac{4 \text{ m/s}}{0.1 \text{ m}} = 40 \text{ rad/s},$$

$$\alpha = \frac{a}{r} = \frac{8 \text{ m/s}^2}{0.1 \text{ m}} = 80 \text{ rad/s}^2.$$

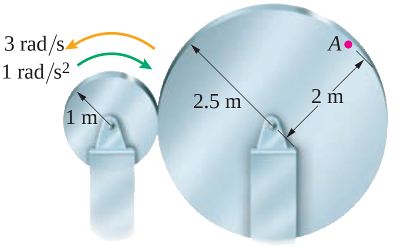
(a) $\boxed{\omega = 40 \text{ rad/s.}}$

(b) $a_t = r\alpha = (0.1 \text{ m})(80 \text{ rad/s}^2) = 8 \text{ m/s}^2,$
$a_n = r\omega^2 = (0.1 \text{ m})(40 \text{ rad/s})^2 = 160 \text{ m/s}^2.$ $\boxed{\begin{aligned} a_t &= 8 \text{ m/s}^2, \\ a_n &= 160 \text{ m/s}^2. \end{aligned}}$

Problem 17.4 At the instant shown, the left disk has an angular velocity of 3 rad/s counterclockwise and an angular acceleration of 1 rad/s^2 clockwise.

(a) What are the angular velocity and angular acceleration of the right disk? (Assume that there is no relative motion between the disks at their point of contact.)

(b) What are the magnitudes of the velocity and acceleration of point A?



Solution:

(a)
$$\boxed{\begin{aligned} r_L\omega_L &= r_R\omega_R \Rightarrow \omega_R = \frac{r_R}{r_L}\omega_L = \frac{1\text{ m}}{2.5\text{ m}}(3\text{ rad/s}) = 1.2\text{ rad/s} \\ r_L\alpha_L &= r_R\alpha_R \Rightarrow \alpha_R = \frac{r_R}{r_L}\alpha_L = \frac{1\text{ m}}{2.5\text{ m}}(1\text{ rad/s}^2) = 0.4\text{ rad/s}^2 \end{aligned}}$$

(b) $v_A = (2\text{ m})(1.2\text{ rad/s}) = 2.4\text{ m/s}$

$$a_{At} = (2\text{ m})(0.4\text{ rad/s}^2) = 0.8\text{ m/s}^2$$

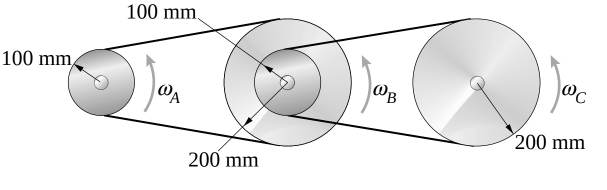
$$a_{An} = (2\text{ m})(1.2\text{ rad/s})^2 = 2.88\text{ m/s}^2$$

$$\Rightarrow \boxed{\begin{aligned} v_A &= 2.4\text{ m/s} \\ a_A &= \sqrt{(0.8)^2 + (2.88)^2}\text{ m/s}^2 = 2.99\text{ m/s}^2 \end{aligned}}$$

Problem 17.5 The angular velocity of the left disk is given as a function of time by $\omega_A = 4 + 0.2t$ rad/s.

(a) What are the angular velocities ω_B and ω_C at $t = 5$ s?

(b) Through what angle does the right disk turn from $t = 0$ to $t = 5$ s?



Solution:

$$\omega_A = (4 + 0.2t)\text{ rad/s}$$

$$r_A\omega_A = r_B\omega_B \;\Rightarrow\; \omega_B = \frac{0.1\text{ m}}{0.2\text{ m}}\omega_A = 0.5\,\omega_A$$

$$r_B\omega_B = r_C\omega_C \;\Rightarrow\; \omega_C = \frac{0.1\text{ m}}{0.2\text{ m}}\omega_B = 0.25\,\omega_A$$

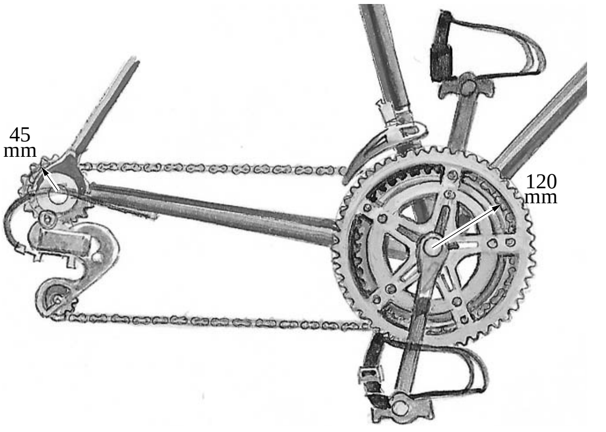
(a) At $t = 5$ s $\quad \omega_A = (4 + 0.2[5])\text{ rad/s} = 5\text{ rad/s}$

$$\boxed{\begin{aligned} \omega_B &= 0.5(5\text{ rad/s}) = 2.5\text{ rad/s} \\ \omega_C &= 0.25(5\text{ rad/s}) = 1.25\text{ rad/s} \end{aligned}}$$

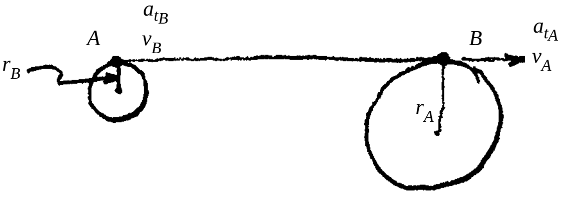
(b) $\omega_C = 0.25\,\omega_A = 0.25(4 + 0.2t)\text{ rad/s} = (1 + 0.05t)\text{ rad/s}$

$$\boxed{\theta_C = \int_0^{5\text{ s}} \omega_C\,dt = [t + 0.025t^2]_0^{5\text{ s}} = 5.625\text{ rad}}$$

Problem 17.6 (a) If the bicycle's 120-mm sprocket wheel rotates through one revolution, through how many revolutions does the 45-mm gear turn? (b) If the angular velocity of the sprocket wheel is 1 rad/s, what is the angular velocity of the gear?



Solution: The key is that the tangential accelerations and tangential velocities along the chain are of constant magnitude



(a) $\theta_B = 2.67$ rev

(b) $v_B = r\omega_B \qquad v_A = r_A\omega_A$

$v_A = v_B$

$v_B = (0.045)\omega_B \qquad v_A = (0.120)(1)$

$$\omega_B = \left(\frac{120}{45}\right) \text{ rad/s} = 2.67 \text{ rad/s}$$

$$r_B\frac{d\theta_B}{dt} = r_A\frac{d\theta_A}{dt}$$

Integrating, we get

$r_B\theta_B = r_A\theta_A \qquad r_A = 0.120$ m

$r_B = 0.045$ m

$$\theta_B = \left(\frac{120}{45}\right)(1) \text{ rev} \qquad \theta_A = 1 \text{ rev.}$$

Problem 17.7 The rear wheel of the bicycle in Problem 17.6 has a 330-mm radius and is rigidly attached to the 45-mm gear. It the rider turns the pedals, which are rigidly attached to the 120-mm sprocket wheel, at one revolution per second, what is the bicycle's velocity?

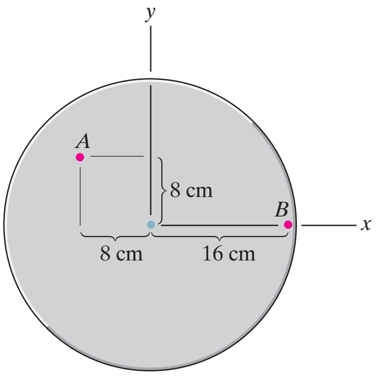
Solution: The angular velocity of the 120 mm sprocket wheel is $\omega = 1$ rev/s $= 2\pi$ rad/s. Use the solution to Problem 17.6. The angular velocity of the 45 mm gear is

$$\omega_{45} = 2\pi\left(\frac{120}{45}\right) = 16.76 \text{ rad/s.}$$

This is also the angular velocity of the rear wheel, from which the velocity of the bicycle is

$v = \omega_{45}(330) = 5.53$ m/s.

Problem 17.8 The disk is rotating about the origin with a constant clockwise angular velocity of 100 rpm. Determine the x and y components of velocity of points A and B (in cm/s).



Solution:

$$\omega = 100 \text{ rpm}\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 10.5 \text{ rad/s}.$$

Working with point A

$$r_A = \sqrt{(8\text{ cm})^2 + (8\text{ cm})^2} = 11.3 \text{ cm}, \ \ \theta_A = \tan^{-1}\left(\frac{8\text{ cm}}{8\text{ cm}}\right) = 45^\circ$$

$$v_A = r_A\omega = (11.3 \text{ cm})(10.5 \text{ rad/s}) = 118 \text{ cm/s}$$

$$\mathbf{v}_A = v_A(\cos\theta_A\mathbf{i} + \sin\theta_A\mathbf{j}) = (118 \text{ cm/s})(\cos 45^\circ\mathbf{i} + \sin 45^\circ\mathbf{j})$$

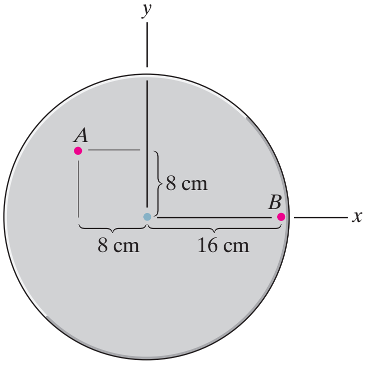
$$\boxed{\mathbf{v}_A = (83.8\mathbf{i} + 83.8\mathbf{j}) \text{ cm/s.}}$$

Working with point B

$$r_B = 16 \text{ in}, \ \ v_B = r_B\omega = (16 \text{ cm})(10.5 \text{ rad/s}) = 168 \text{ cm/s}$$

$$\boxed{\mathbf{v}_\text{B} = -(168\mathbf{j}) \text{ cm/s.}}$$

Problem 17.9 The disk is rotating about the origin with a constant clockwise angular velocity of 100 rpm. Determine the x and y components of acceleration of points A and B (in cm/s^2).



Solution:

$$\omega = 100 \text{ rpm}\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 10.5 \text{ rad/s}.$$

Working with point A

$$r_A = \sqrt{(8\text{ cm})^2 + (8\text{ cm})^2} = 11.3 \text{ cm}, \ \ \theta_A = \tan^{-1}\left(\frac{8\text{ cm}}{8\text{ cm}}\right) = 45^\circ$$

$$a_A = r_A\omega^2 = (11.3 \text{ cm})(10.5 \text{ rad/})^2 = 1240 \text{ cm/s}^2$$

$$\mathbf{a}_A = a_A(\cos\theta_A\mathbf{i} - \sin\theta_A\mathbf{j}) = (1240 \text{ cm/s}^2)(\cos 45^\circ\mathbf{i} - \sin 45^\circ\mathbf{j})$$

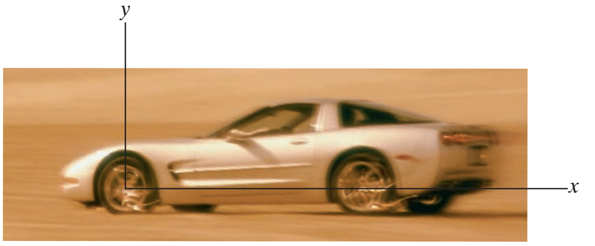
$$\boxed{\mathbf{a}_A = (877\mathbf{i} - 877\mathbf{j}) \text{ cm/s}^2.}$$

Working with point B

$$r_B = 16 \text{ in}, a_B = r_B\omega^2 = (16 \text{ cm})(10.5 \text{ rad/s})^2 = 1750 \text{ cm/s}^2$$

$$\boxed{\mathbf{a}_\text{B} = -(1750\mathbf{i}) \text{ cm/s}^2.}$$

Problem 17.10 The radius of the Corvette's tires is 30 cm. It is traveling at 80 km/h when the driver applies the brakes, subjecting the car to a deceleration of 25 m/s^2. Assume that the tires continue to roll, not skid, on the road surface. At that instant, what are the magnitudes of the tangential and normal components of acceleration (in m/s^2) of a point at the outer edge of a tire relative to a nonrotating coordinate system with its origin at the center of the tire?



Solution: We have

$$\omega = \frac{v}{r} = \frac{\left(\dfrac{80 \times 1000}{3600}\right)}{(0.3 \text{ m})} = 74.1 \text{ rad/s},$$

$$\alpha = \frac{a}{r} = \frac{25 \text{ m/s}^2}{(0.3 \text{ m})} = 83.33 \text{ rad/s}^2.$$

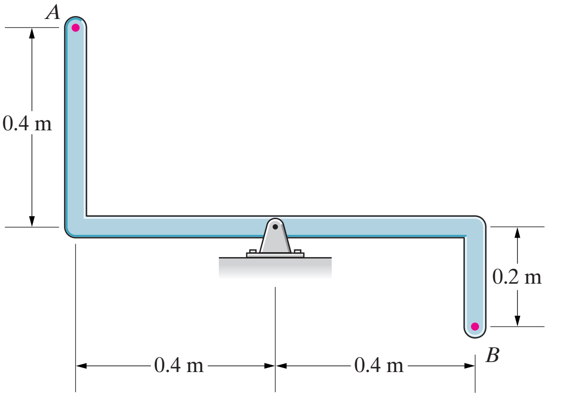
Relative to the given coordinate system (which is not an inertial coordinate system)

$$a_t = r\alpha = (0.3 \text{ m})(83.33 \text{ rad/s}^2) = 25 \text{ m/s}^2$$

$$a_n = r\omega^2 = (0.3 \text{ m})(74.1 \text{ rad/s})^2 = 1647.2 \text{ m/s}^2.$$

$$\boxed{\begin{aligned} a_t &= 25 \text{ m/s}^2, \\ a_n &= 1647.2 \text{ m/s}^2. \end{aligned}}$$

Problem 17.11 If the bar has a counterclockwise angular velocity of 8 rad/s and a clockwise angular acceleration of 40 rad/s^2, what are the magnitudes of the accelerations of points A and B?



Solution:

$$\mathbf{a}_A = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$

$$= (-40 \text{ rad/s}^2\mathbf{k}) \times (-0.4\mathbf{i} + 0.4\mathbf{j}) \text{ m} - (8 \text{ rad/s})^2(-0.4\mathbf{i} + 0.4\mathbf{j}) \text{ m}$$

$$= (41.6\mathbf{i} - 9.6\mathbf{j}) \text{ m/s}^2$$

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/O} - \omega^2 \mathbf{r}_{B/O}$$

$$= (-40 \text{ rad/s}^2\mathbf{k}) \times (0.4\mathbf{i} - 0.2\mathbf{j}) \text{ m} - (8 \text{ rad/s})^2(0.4\mathbf{i} - 0.2\mathbf{j}) \text{ m}$$

$$= (-33.6\mathbf{i} - 3.2\mathbf{j}) \text{ m/s}^2$$

$$\boxed{\begin{aligned} a_A &= \sqrt{(41.6)^2 + (-9.6)^2} \text{ m/s}^2 = 42.7 \text{ m/s}^2 \\ a_B &= \sqrt{(-33.6)^2 + (-3.2)^2} \text{ m/s}^2 = 33.8 \text{ m/s}^2 \end{aligned}}$$

Problem 17.12 Consider the bar shown in Problem 17.11. If $|\mathbf{v}_A| = 3$ m/s and $|\mathbf{a}_A| = 28$ m/s^2, what are $|\mathbf{v}_B|$ and $|\mathbf{a}_B|$?

Solution:

$$v_A = \omega r \Rightarrow 3 \text{ m/s} = \omega\sqrt{(0.4)^2 + (0.4)^2} \text{ m} \Rightarrow \omega = 5.30 \text{ rad/s}$$

$$a_{An} = \omega^2 r = (5.3 \text{ rad/s})^2\sqrt{(0.4)^2 + (0.4)^2} \text{ m} = 15.9 \text{ m/s}^2$$

$$a_{At} = \sqrt{a_A{}^2 - a_{An}{}^2} = \sqrt{(28)^2 - (15.9)^2} \text{ m/s}^2 = 23.0 \text{ m/s}^2$$

$$a_{At} = \alpha r \Rightarrow 23.0 \text{ m/s}^2 = \alpha\sqrt{(0.4)^2 + (0.4)^2} \text{ m} \Rightarrow \alpha = 40.7 \text{ rad/s}$$

$$\boxed{v_B = \omega\sqrt{(0.4)^2 + (-0.2)^2} \text{ m} = 2.37 \text{ m/s}}$$

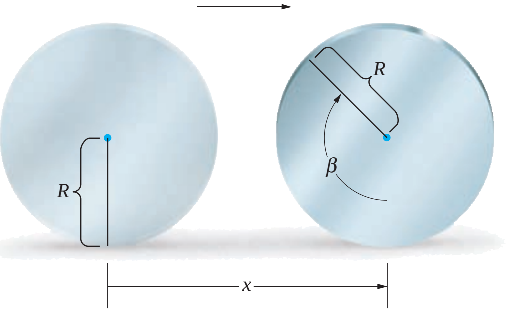
$$a_{Bt} = \alpha\sqrt{(0.4)^2 + (-0.2)^2} = 18.2 \text{ m/s}^2$$

$$a_{Bn} = \omega^2\sqrt{(0.4)^2 + (-0.2)^2} = 12.6 \text{ m/s}^2$$

$$\boxed{a_B = \sqrt{(18.2)^2 + (12.6)^2} \text{ m/s}^2 = 22.1 \text{ m/s}^2}$$

Problem 17.13 A disk of radius $R = 0.5$ m rolls on a horizontal surface. The relationship between the horizontal distance x the center of the disk moves and the angle β through which the disk rotates is $x = R\beta$. Suppose that the center of the disk is moving to the right with a constant velocity of 2 m/s.

(a) What is the disk's angular velocity?
(b) Relative to a nonrotating reference frame with its origin at the center of the disk, what are the magnitudes of the velocity and acceleration of a point on the edge of the disk?

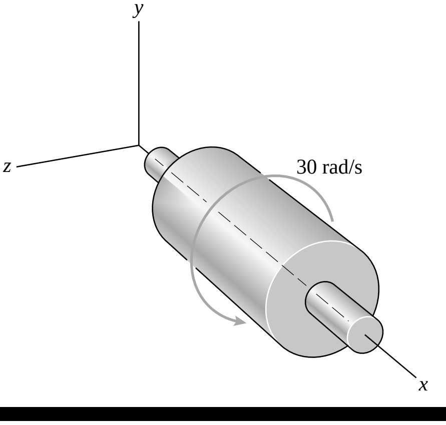


Solution:

(a) $x = R\beta \Rightarrow \dot{x} = R\dot{\beta} \Rightarrow v = R\omega \Rightarrow \boxed{\omega = \dfrac{v}{R} = \dfrac{2 \text{ m/s}}{0.5 \text{ m}} = 4 \text{ rad/s}}$

(b) $$\boxed{\begin{aligned} v &= R\omega = (0.5 \text{ m})(4 \text{ rad/s}) = 2 \text{ m/s} \\ a &= a_n = R\omega^2 = (0.5 \text{ m})(4 \text{ rad/s})^2 = 8 \text{ m/s}^2 \end{aligned}}$$

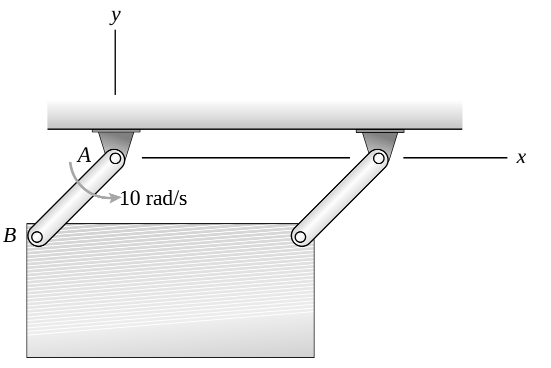
Problem 17.14 The turbine rotates relative to the coordinate system at 30 rad/s about a fixed axis coincident with the x axis. What is its angular velocity vector?



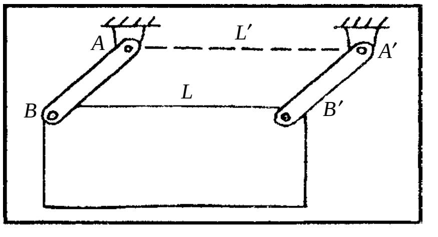
Solution: The angular velocity vector is parallel to the x axis, with magnitude 30 rad/s. By the right hand rule, the positive direction coincides with the positive direction of the x axis.

$\boxed{\boldsymbol{\omega} = 30\mathbf{i}}$ (rad/s).

Problem 17.15 The rectangular plate swings in the $x-y$ plane from arms of equal length. What is the angular velocity of (a) the rectangular plate and (b) the bar AB?



Solution: Denote the upper corners of the plate by B and B', and denote the distance between these points (the length of the plate) by L. Denote the suspension points by A and A', the distance separating them by L'. By inspection, since the arms are of equal length, and since $L = L'$, the figure $AA'B'B$ is a parallelogram. By definition, the opposite sides of a parallelogram remain parallel, and since the fixed side AA' does not rotate, then BB' cannot rotate, so that the plate does not rotate and



$\boxed{\boldsymbol{\omega}_{BB'} = 0}$.

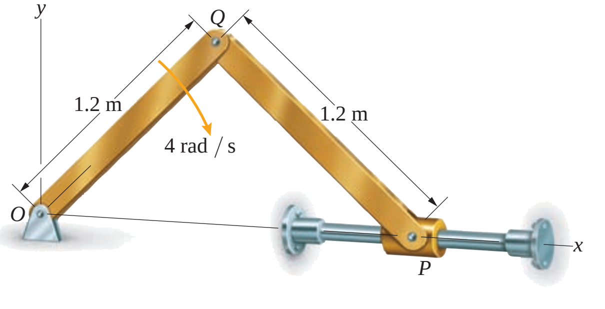
Similarly, by inspection the angular velocity of the bar AB is

$\boxed{\boldsymbol{\omega}_{AB} = 10\mathbf{k}}$ (rad/s),

where by the right hand rule the direction is along the positive z axis (out of the paper).

Problem 17.16 Bar OQ is rotating in the clockwise direction at 4 rad/s. What are the angular velocity vectors of the bars OQ and PQ?

Strategy: Notice that if you know the angular velocity of bar OQ, you also know the angular velocity of bar PQ.

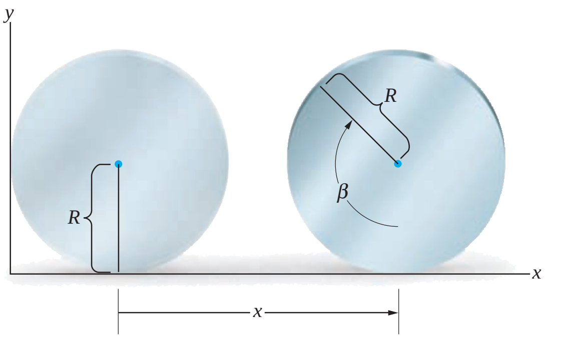


Solution: The magnitudes of the angular velocities are the same. The directions are opposite

$$\boxed{\boldsymbol{\omega}_{OQ} = -(4 \text{ rad/s})\mathbf{k}, \quad \boldsymbol{\omega}_{PQ} = (4 \text{ rad/s})\mathbf{k},}$$

Problem 17.17 A disk of radius $R = 0.5$ m rolls on a horizontal surface. The relationship between the horizontal distance x the center of the disk moves and the angle β through which the disk rotates is $x = R\beta$. Suppose that the center of the disk is moving to the right with a constant velocity of 2 m/s.

(a) What is the disk's angular velocity?
(b) What is the disk's angular velocity vector?



Solution:

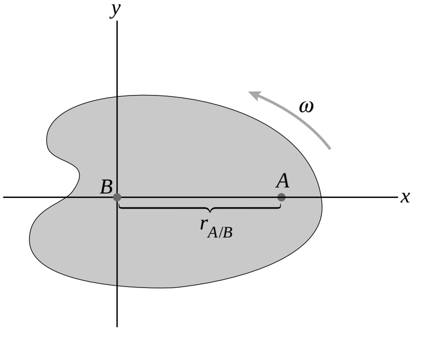
(a) $x = R\beta \Rightarrow \dot{x} = R\dot{\beta} \Rightarrow v = R\omega$

$$\Rightarrow \boxed{\omega = \frac{v}{R} = \frac{2\text{ m/s}}{0.5\text{ m}} = 4\text{ rad/s CW}}$$

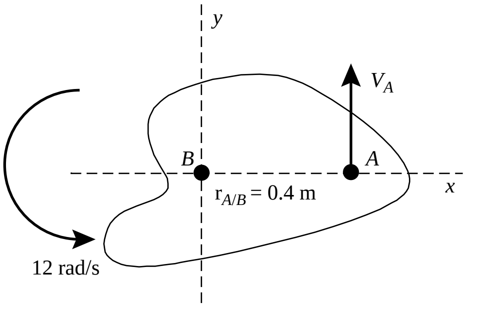
(b) $\boxed{\boldsymbol{\omega} = -(4\text{ rad/s})\mathbf{k}}$

Problem 17.18 The rigid body rotates with angular velocity $\omega = 12$ rad/s. The distance $r_{A/B} = 0.4$ m.

(a) Determine the x and y components of the velocity of A relative to B by representing the velocity as shown in Fig. 17.10b.
(b) What is the angular velocity vector of the rigid body?
(c) Use Eq. (17.5) to determine the velocity of A relative to B.



Solution:



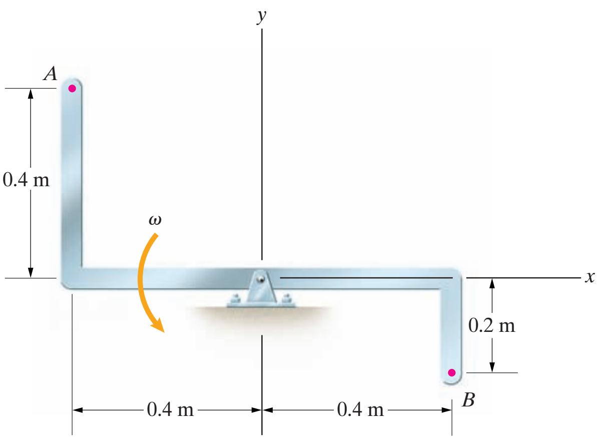
(a) $\boxed{\begin{aligned} v_{Ax} &= 0 \\ v_{Ay} &= (12\text{ rad/s})(0.4\text{ m}) = 4.8\text{ m/s} \end{aligned}}$

(b) $\boxed{\boldsymbol{\omega} = (12\text{ rad/s})\mathbf{k}}$

(c) $\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{A/B} = (12\text{ rad/s})\mathbf{k} \times (0.4\text{ m})\mathbf{i}$

$$\boxed{\mathbf{v}_A = (4.8\text{ m/s})\mathbf{j}}$$

Problem 17.19 The bar is rotating in the counterclockwise direction with angular velocity ω. The magnitude of the velocity of point A is 6 m/s. Determine the velocity of point B.



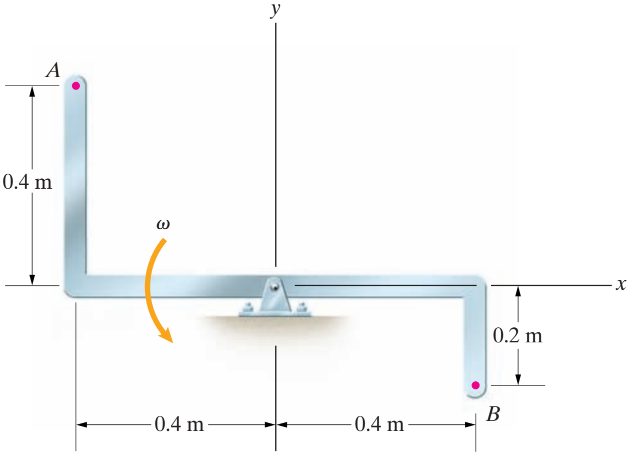
Solution:

$$\omega = \frac{v}{r} = \frac{6\ \text{m/s}}{\sqrt{2}(0.4\ \text{m})} = 10.6\ \text{rad/s}.$$

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_B = (10.6\ \text{rad/s})\mathbf{k} \times (0.4\mathbf{i} - 0.2\mathbf{j})\ \text{m}$$

$$\boxed{\mathbf{v}_B = (2.12\mathbf{i} + 4.24\mathbf{j})\ \text{m/s.}}$$

Problem 17.20 The bar is rotating in the counterclockwise direction with angular velocity ω. The magnitude of the velocity of point A relative to point B is 6 m/s. Determine the velocity of point B.



Solution:

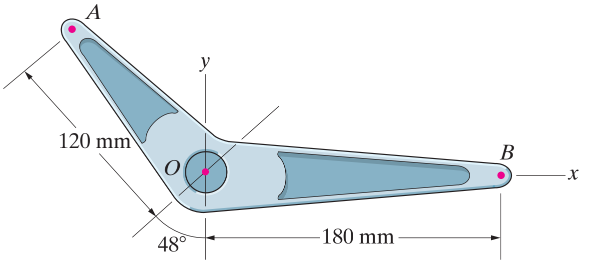
$$r_{A/B} = \sqrt{(0.8\ \text{m})^2 + (0.6\ \text{m})^2} = 1\ \text{m}$$

$$\omega = \frac{v}{r_{A/B}} = \frac{6\ \text{m/s}}{1\ \text{m}} = 6\ \text{rad/s}.$$

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_B = (6\ \text{rad/s})\mathbf{k} \times (0.4\mathbf{i} - 0.2\mathbf{j})\ \text{m}$$

$$\boxed{\mathbf{v}_B = (1.2\mathbf{i} + 2.4\mathbf{j})\ \text{m/s.}}$$

Problem 17.21 The bracket is rotating about point O with counterclockwise angular velocity ω. The magnitude of the velocity of point A relative to point B is 4 m/s. Determine ω.



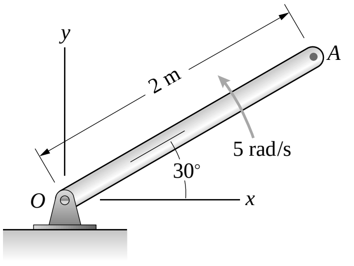
Solution:

$$r_{B/A} = \sqrt{(0.18 + 0.12\cos 48^\circ)^2 + (0.12\sin 48^\circ)^2} = 0.275\ \text{m}$$

$$\omega = \frac{v_{B/A}}{r_{B/A}} = \frac{4\ \text{m/s}}{0.275\ \text{m}} = 14.5\ \text{rad/s}.$$

$$\boxed{\omega = 14.5\ \text{rad/s.}}$$

Problem 17.22 Determine the x and y components of the velocity of point A.



Solution: The velocity of point A is given by:

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O}.$$

Hence, $$\mathbf{v}_A = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ 2\cos 30^\circ & 2\sin 30^\circ & 0 \end{vmatrix}$$

$$= -10\sin 30^\circ \mathbf{i} + 10\cos 30^\circ \mathbf{j},$$

or $\mathbf{v}_A = -5\mathbf{i} + 8.66\mathbf{j}$ (m/s).

Problem 17.23 If the angular velocity of the bar in Problem 17.22 is constant, what are the x and y components of the velocity of Point A 0.1 s after the instant shown?

Solution: The angular velocity is given by

$$\omega = \frac{d\theta}{dt} = 5 \text{ rad/s},$$

$$\int_0^\theta d\theta = \int_0^t 5\,dt,$$

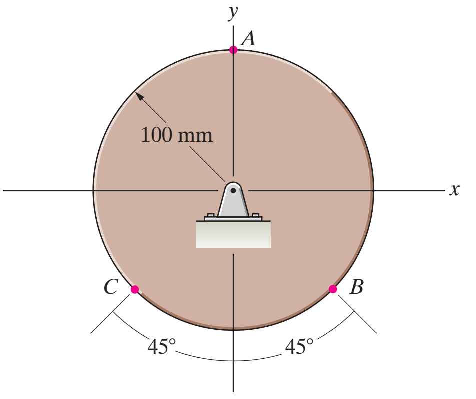
and $\theta = 5t$ rad.

At $t = 0.1$ s, $\theta = 0.5$ rad $= 28.6^\circ$.

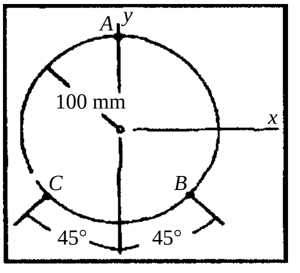
$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ 2\cos 28.6^\circ & 2\sin 28.6^\circ & 0 \end{vmatrix}.$$

Hence, $\mathbf{v}_A = -10\sin 28.6^\circ \mathbf{i} + 10\cos 28.6^\circ \mathbf{j} = -4.78\mathbf{i} + 8.78\mathbf{j}$.

Problem 17.24 The disk is rotating about the z axis at 50 rad/s in the clockwise direction. Determine the x and y components of the velocities of points A, B, and C.



Solution: The velocity of A is given by $\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O}$, or $\mathbf{v}_A = 0 + (-50\mathbf{k}) \times (0.1\mathbf{j}) = 5\mathbf{i}$ (m/s).



For B, we have

$$\mathbf{v}_B = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{B/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -50 \\ 0.1\cos 45^\circ & -0.1\sin 45^\circ & 0 \end{vmatrix}$$

$$= -3.54\mathbf{i} - 3.54\mathbf{j} \text{ (m/s)},$$

For C, we have

$$\mathbf{v}_c = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{C/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -50 \\ -0.1\cos 45^\circ & -0.1\sin 45^\circ & 0 \end{vmatrix}$$

$$= -3.54\mathbf{i} + 3.54\mathbf{j} \text{ (m/s)}.$$

Problem 17.25 Consider the rotating disk shown in Problem 17.24. If the magnitude of the velocity of point A relative to point B is 4 m/s, what is the magnitude of the disk's angular velocity?

Solution:

$$\mathbf{v}_0 = 0 \quad \boldsymbol{\omega} = \omega\mathbf{k} \quad r = 0.1 \text{ m}$$

$$\mathbf{v}_B = \mathbf{v}_0 + \omega\mathbf{k} \times \mathbf{r}_{OB}$$

$$= \omega\mathbf{k} \times (r\cos 45^\circ\mathbf{i} - r\sin 45^\circ\mathbf{j})$$

$$= (r\omega\cos 45^\circ)\mathbf{j} + (r\omega\sin 45^\circ)\mathbf{i}.$$

$$\mathbf{v}_A = \mathbf{v}_0 + \omega\mathbf{k} \times \mathbf{r}_{OA} = \omega\mathbf{k} \times r\mathbf{j}$$

$$= -r\omega\mathbf{i}.$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B},$$

$$\mathbf{v}_{A/B} = \mathbf{v}_A - \mathbf{v}_B,$$

$$\mathbf{v}_{A/B} = (-r\omega - r\omega\sin 45^\circ)\mathbf{i} - r\omega\cos 45^\circ\mathbf{j}$$

$$= r\omega(-1 - \sin 45^\circ)\mathbf{i} - r\omega\cos 45^\circ\mathbf{j}.$$

We know

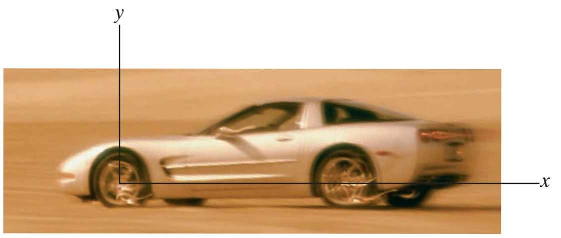
$$|\mathbf{v}_{A/B}| = 4 \text{ m/s}, \quad r = 0.1 \text{ m}$$

$$|\mathbf{v}_{A/B}| = \sqrt{[r\omega(-1 - \sin 45^\circ)]^2 + [-r\omega\cos 45^\circ]^2}$$

Solving for ω, $\omega = 21.6$ rad/s (direction undetermined).

Problem 17.26 The radius of the Corvette's tires is 30 cm. It is traveling at 100 km/h. Assume that the tires roll, not skid, on the road surface.

(a) What is the angular velocity of its wheels?
(b) In terms of the earth-fixed coordinate system shown, determine the velocity (in m/s) of the point of the tire with coordinates (−30 cm, 0,0).



Solution:

(a) $\omega = \dfrac{v}{r} = \dfrac{\left(\dfrac{100 \times 1000}{3600}\right)}{(0.3)} = 92.6 \text{ rad/s.}$ $\boxed{\omega = 92.6 \text{ rad/s.}}$

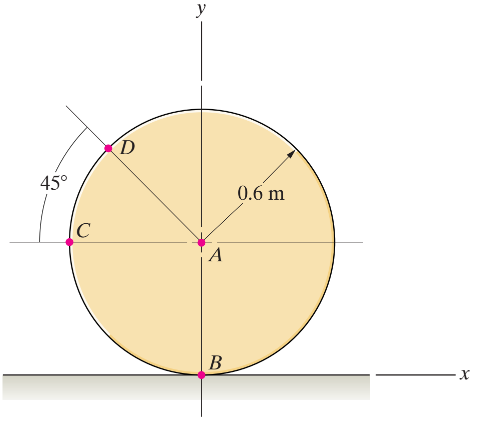
(b)

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{v}_P = -\left(\frac{100 \times 1000}{3600} \text{ m/s}\right)\mathbf{i} + (92.6 \text{ rad/s})\mathbf{k} \times \left(-[0.3 \text{ m}]\,\mathbf{i}\right)$$

$$\boxed{\mathbf{v}_P = (-27.8\mathbf{i} - 27.8\mathbf{j}) \text{ m/s.}}$$

Problem 17.27 Point A of the rolling disk is moving toward the right. The magnitude of the velocity of point C is 5 m/s. determine the velocities of points B and D.



Solution: Point B is the center of rotation (zero velocity).

$r_{C/B} = \sqrt{2}(0.6 \text{ m}) = 0.849 \text{ m},$

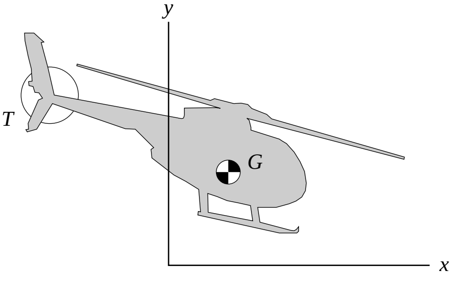
$$\omega = \frac{v_C}{r_{C/B}} = \frac{5 \text{ m/s}}{0.849 \text{ m}} = 5.89 \text{ rad/s}$$

Therefore

$$\mathbf{v}_D = \boldsymbol{\omega} \times \mathbf{r}_{D/B} = -(5.89 \text{ rad/s})\mathbf{k} \times [-0.6\cos 45^\circ \mathbf{i} + (0.6 + 0.6\sin 45^\circ)\mathbf{j}]$$

$$\boxed{\mathbf{v}_D = (6.04\mathbf{i} + 2.5\mathbf{j}) \text{ m/s}, \mathbf{v}_B = 0.}$$

Problem 17.28 The helicopter is in planar motion in the x–y plane. At the instant shown, the position of its center of mass, G, is $x = 2$ m, $y = 2.5$ m, and its velocity is $\mathbf{v}_G = 12\mathbf{i} + 4\mathbf{j}$ (m/s). The position of point T, where the tail rotor is mounted, is $x = -3.5$ m, $y = 4.5$ m. The helicopter's angular velocity is 0.2 (rad/s) clockwise. What is the velocity of point T?



Solution: The position of T relative to G is

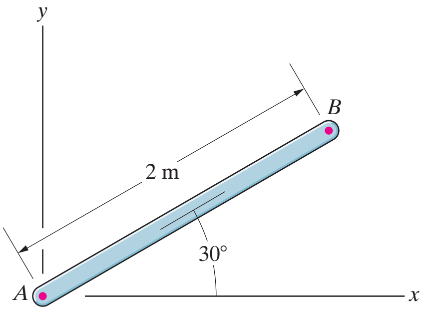
$$\mathbf{r}_{T/G} = (-3.5 - 2)\mathbf{i} + (4.5 - 2.5)\mathbf{j} = -5.5\mathbf{i} + 2\mathbf{j} \text{ (m)}.$$

The velocity of T is

$$\mathbf{v}_T = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{T/G} = 12\mathbf{i} + 4\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -0.2 \\ -5.5 & 2 & 0 \end{vmatrix}$$

$$= 12.4\mathbf{i} + 5.1\mathbf{j} \text{ (m/s)}$$

Problem 17.29 The bar is moving in the $x-y$ plane and is rotating in the counterclockwise direction. The velocity of point A relative to the reference frame is $\mathbf{v}_A = 12\mathbf{i} - 2\mathbf{j}$ (m/s). The magnitude of the velocity of point A relative to point B is 8 m/s. what is the velocity of point B relative to the reference frame?



Solution:

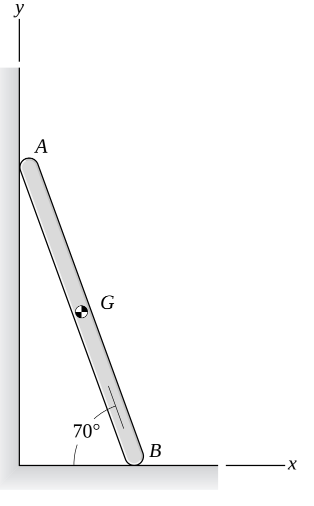
$$\omega = \frac{v}{r} = \frac{8 \text{ m/s}}{2 \text{ m}} = 4 \text{ rad/s},$$

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} = (12\mathbf{i} - 2\mathbf{j}) \text{ m/s} + (4 \text{ rad/s})\mathbf{k} \times (2 \text{ m})(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$$

$$\boxed{\mathbf{v}_B = (8\mathbf{i} + 4.93\mathbf{j}) \text{ m/s.}}$$

Problem 17.30 Points A and B of the 2-m bar slide on the plane surfaces. Point B is moving to the right at 3 m/s. What is the velocity of the midpoint G of the bar?

Strategy: First apply Eq. (17.6) to points A and B to determine the bar's angular velocity. Then apply Eq. (17.6) to points B and G.



Solution: Take advantage of the constraints (B stays on the floor, A stays on the wall)

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$v_A\mathbf{j} = (3 \text{ m/s})\mathbf{i} + \omega\mathbf{k} \times (2 \text{ m})(-\cos 70^\circ \mathbf{i} + \sin 70^\circ \mathbf{j})$$

$$= (3 - 1.88\ \omega)\mathbf{i} + (-0.684\ \omega)\mathbf{j}$$

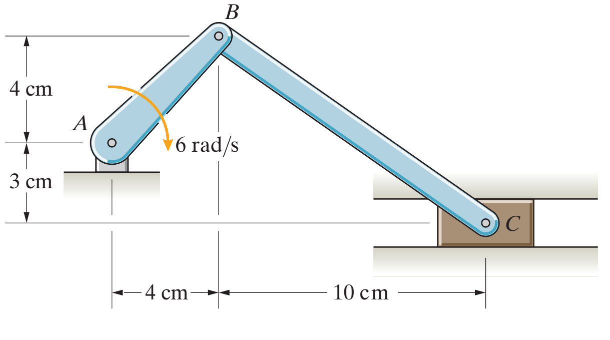
Equating $\mathbf{i}$ components we find $3 - 1.88\ \omega = 0 \Rightarrow \omega = 1.60$ rad/s
Now find the velocity of point G

$$\mathbf{v}_G = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{G/B}$$

$$= (3 \text{ m/s})\mathbf{i} + (1.60 \text{ rad/s})\mathbf{k} \times (1 \text{ m})(-\cos 70^\circ \mathbf{i} + \sin 70^\circ \mathbf{j})$$

$$\boxed{\mathbf{v}_G = (1.5\mathbf{i} - 0.546\mathbf{j}) \text{ m/s}}$$

Problem 17.31 Bar AB rotates at 6 rad/s in the clockwise direction. Determine the velocity (in cm/s) of the slider C.



Solution:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$\mathbf{v}_B = 0 - (6 \text{ rad/s})\mathbf{k} \times (4\mathbf{i} + 4\mathbf{j}) \text{ cm}$$

$$\mathbf{v}_B = (24\mathbf{i} - 24\mathbf{j}) \text{ cm/s.}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$\mathbf{v}_C = (24\mathbf{i} - 24\mathbf{j}) + \omega_{BC}\mathbf{k} \times (10\mathbf{i} - 7\mathbf{j})$$

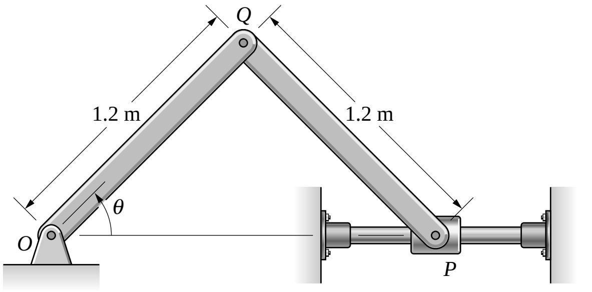
$$\mathbf{v}_C = (24 + 7\ \omega_{BC})\mathbf{i} + (-24 + 10\ \omega_{BC})\mathbf{j}$$

Slider C cannot move in the $\mathbf{j}$ direction, therefore

$$\omega_{BC} = \frac{24}{10} = 2.4 \text{ rad/s.}$$

$$\mathbf{v}_C = (24 + 7(2.4))\mathbf{i} = (43.8 \text{ cm/s})\mathbf{i} \quad \boxed{v_C = 43.8 \text{ cm/s to the right.}}$$

Problem 17.32 If $\theta = 45^\circ$ and the sleeve P is moving to the right at 2 m/s, what are the angular velocities of the bars OQ and PQ?



Solution: From the figure, $\mathbf{v}_0 = 0$, $\mathbf{v}_P = v_P\mathbf{i} = 2\ \mathbf{i}$ (m/s)

$$\mathbf{v}_Q = \mathbf{v}_0 + \omega_{OQ}\mathbf{k} \times (L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j})$$

$$\begin{cases} \mathbf{i}: & v_{Q_x} = -\omega_{OQ}L\sin\theta \qquad \mathbf{(1)} \\ \mathbf{j}: & v_{Q_y} = \omega_{OQ}L\cos\theta \qquad \mathbf{(2)} \end{cases}$$

$$\mathbf{v}_P = \mathbf{v}_Q + \omega_{QP}\mathbf{k} \times (L\cos\theta\mathbf{i} - L\sin\theta\mathbf{j})$$

$\mathbf{i}$: $2 = v_{Q_x} + \omega_{QP}L\sin\theta$ **(3)**

$\mathbf{j}$: $0 = v_{Q_y} + \omega_{QP}L\cos\theta$ **(4)**

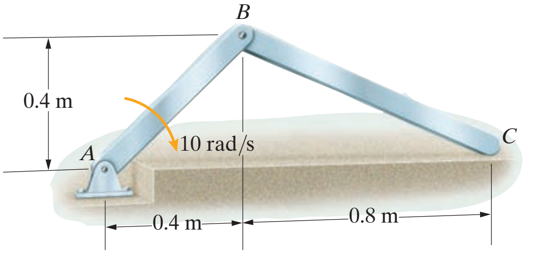
Eqns (1)–(4) are 4 eqns in the 4 unknowns ω_{OQ}, ω_{QP}, v_{Q_x}, v_{Q_y}
Solving, we get

$v_{Q_x} = 1$ m/s, $v_{Q_y} = -1$ m/s

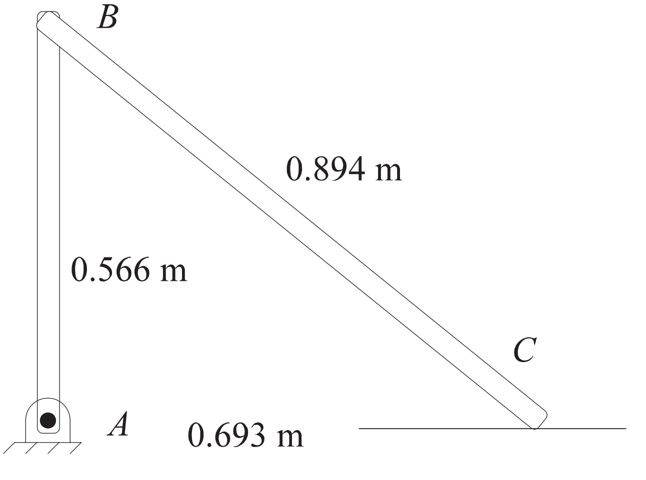
$\underline{\boldsymbol{\omega}_{OQ} = -1.18\mathbf{k}}$ (rad/s),

$\boldsymbol{\omega}_{QP} = 1.18\mathbf{k}$ (rad/s).

Problem 17.33 In Active Example 17.2, consider the instant when bar AB is vertical and rotating in the clockwise direction at 10 rad/s. Draw a sketch showing the positions of the two bars at that instant. Determine the angular velocity of bar BC and the velocity of point C.



Solution: The sketch is shown. We have



$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$= 0 - (10\text{ rad/s})\mathbf{k} \times (0.566\text{ m})\mathbf{j}$$

$$= (5.66\text{ m/s})\mathbf{i}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{BC}$$

$$= (5.66\text{ m/s})\mathbf{i}$$

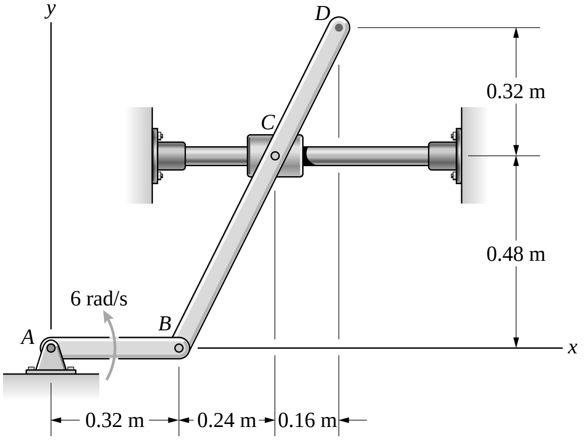
$$+ \omega_{BC}\mathbf{k} \times (0.693\mathbf{i} - 0.566\mathbf{j})\text{ m}$$

$$= (5.66 + 0.566\ \omega_{BC})\mathbf{i} + (0.693\ \omega_{BC})\mathbf{j}$$

Point C cannot move in the $\mathbf{j}$ direction, therefore

$$\boxed{\omega_{BC} = 0, \mathbf{v}_C = (5.66\text{ m/s})\mathbf{i}}$$

Problem 17.34 Bar AB rotates in the counterclockwise direction at 6 rad/s. Determine the angular velocity of bar BD and the velocity of point D.



Solution:

$\mathbf{v}_A = 0 \quad \boldsymbol{\omega}_{AB} = 6\,\mathbf{k}$

$\mathbf{v}_B = \mathbf{v}_A + \omega_{AB}\mathbf{k} \times \mathbf{r}_{B/A} = 6\mathbf{k} \times 0.32\mathbf{i} = 1.92\mathbf{j}$ m/s.

$\mathbf{v}_C = v_C\mathbf{i} = \mathbf{v}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{C/B}$:

$v_c\mathbf{i} = 1.92\mathbf{j} + \omega_{BD}\mathbf{k} \times (0.24\mathbf{i} + 0.48\mathbf{j})$.

$$\begin{cases} \mathbf{i}: & v_C = -0.48\omega_{BD} \\ \mathbf{j}: & 0 = 1.92 + 0.24\omega_{BD} \end{cases}$$

Solving, $\omega_{BD} = -8$,

$$\underline{\boldsymbol{\omega}_{BD} = -8\mathbf{k}\left(\frac{\text{rad}}{\text{s}}\right)},$$

$$\underline{\mathbf{v}_C = 3.84\mathbf{i}\ (\text{m/s})}.$$

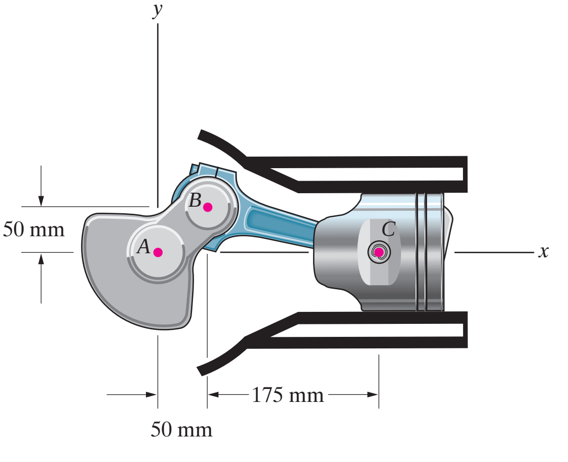
Now for the velocity of point D

$\mathbf{v}_D = \mathbf{v}_B + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{D/B}$

$= 1.92\mathbf{j} + (-8\mathbf{k}) \times (0.4\mathbf{i} + 0.8\mathbf{j})$

$\underline{\mathbf{v}_D = 6.40\mathbf{i} - 1.28\mathbf{j}\ (\text{m/s})}$.

Problem 17.35 At the instant shown, the piston's velocity is $\mathbf{v}_C = -14\mathbf{i}$(m/s). What is the angular velocity of the crank AB?



Solution:

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$

$= 0 + \omega_{AB}\mathbf{k} \times (0.05\mathbf{i} + 0.05\mathbf{j})$ m

$= -0.05\omega_{AB}\mathbf{i} + 0.05\omega_{AB}\mathbf{j}$

$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}$

$= (-0.05\omega_{AB}\mathbf{i} + 0.05\omega_{AB}\mathbf{j})$

$+ \omega_{BC}\mathbf{k} \times (0.175\mathbf{i} - 0.05\mathbf{j})$

$= (-0.05\omega_{AB} + 0.05\omega_{BC})\mathbf{i} + (0.05\omega_{AB} + 0.175\omega_{BC})\mathbf{j}$

We can now separate components and produce two equations in two unknowns

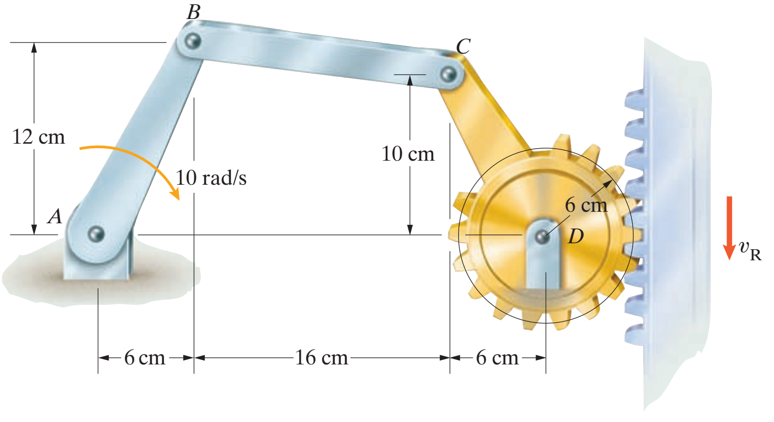
$-14 = -0.05\omega_{AB} + 0.05\omega_{BC},\ 0 = 0.05\omega_{AB} + 0.175\omega_{BC}$

Solving we find

$\omega_{BC} = -62.2$ rad/s, $\omega_{AB} = 218$ rad/s.

Thus $\boxed{\omega_{AB} = 218 \text{ rad/s counterclockwise.}}$

Problem 17.36 In Example 17.3, determine the angular velocity fo the bar AB that would be necessary so that the downward velocity of the rack $V_R = 120$ cm/s at the instant shown.



Solution: We have

$$\omega_{CD} = \frac{v_R}{r}$$

$$= \frac{120 \text{ cm/s}}{6 \text{ cm}} = 20 \text{ rad/s}.$$

Then

$$\mathbf{V}_B = \mathbf{V}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$= 0 + \omega_{AB}\mathbf{k} \times (6\mathbf{i} + 12\mathbf{j}) = -12\omega_{AB}\mathbf{i} + 6\omega_{AB}\mathbf{j}$$

$$\mathbf{V}_C = \mathbf{V}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= (-12\omega_{AB}\mathbf{i} + 6\omega_{AB}\mathbf{j}) + \omega_{BC}\mathbf{k} \times (16\mathbf{i} - 2\mathbf{j})$$

$$= (-12\omega_{AB} + 2\omega_{BC})\mathbf{i} + (6\omega_{AB} + 16\omega_{BC})\mathbf{j}$$

$$\mathbf{V}_D = \mathbf{V}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C}$$

$$= [(-12\omega_{AB} + 2\omega_{BC})\mathbf{i} + (6\omega_{AB} + 16\omega_{BC})\mathbf{j}] - (20)\mathbf{k} \times (6\mathbf{i} - 10\mathbf{j})$$

$$= (-12\omega_{AB} + 2\omega_{BC} - 200)\mathbf{i} + (6\omega_{AB} + 16\omega_{BC} - 120)\mathbf{j}$$

Point D cannot move, therefore

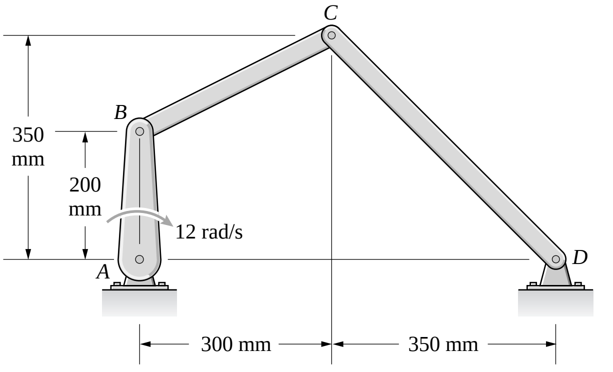
$$-12\omega_{AB} + 2\omega_{BC} - 200 = 0, \quad 6\omega_{AB} + 16\omega_{BC} - 120 = 0.$$

Solving, we find

$$\omega_{AB} = -14.5 \text{ rad/s}, \quad \omega_{BC} = 12.9 \text{ rad/s}.$$

$$\boxed{\omega_{AB} = 14.5 \text{ rad/s counterclockwise.}}$$

Problem 17.37 Bar AB rotates at 12 rad/s in the clockwise direction. Determine the angular velocities of bars BC and CD.



Solution: The strategy is analogous to that used in Problem 17.36. The radius vector AB is $\mathbf{r}_{B/A} = 200\mathbf{j}$ (mm). The angular velocity of AB is $\boldsymbol{\omega} = -12\mathbf{k}$ (rad/s). The velocity of point B is

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -12 \\ 0 & 200 & 0 \end{bmatrix} = 2400\mathbf{i} \text{ (mm/s)}.$$

The radius vector BC is $\mathbf{r}_{C/B} = 300\mathbf{i} + (350 - 200)\mathbf{j} = 300\mathbf{i} + 150\mathbf{j}$ (mm). The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 300 & 150 & 0 \end{bmatrix}$$

$$= (2400 - 150\omega_{BC})\mathbf{i} + \omega_{BC}300\mathbf{j} \text{ (mm/s)}.$$

The radius vector DC is $\mathbf{r}_{C/D} = -350\mathbf{i} + 350\mathbf{j}$ (mm). The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{C/D} = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ -350 & 350 & 0 \end{bmatrix}$$

$$= -350\omega_{CD}(\mathbf{i} + \mathbf{j}).$$

Equate the two expressions for $\mathbf{v}_C$, and separate components:

$$(2400 - 150\omega_{BC} + 350\omega_{CD})\mathbf{i} = 0,$$

and $$(300\omega_{BC} + 350\omega_{CD})\mathbf{j} = 0.$$

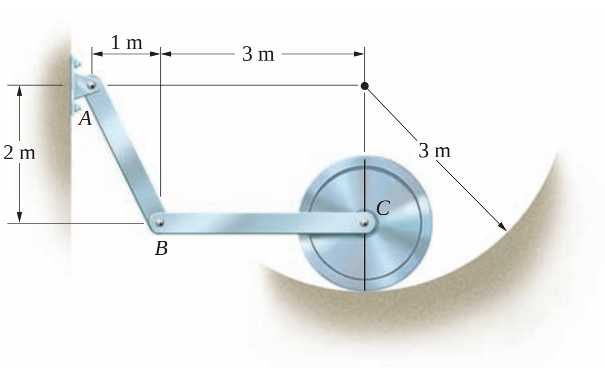
Solve: $\omega_{BC} = 5.33$ rad/s,

$$\boxed{\boldsymbol{\omega}_{BC} = 5.33\mathbf{k} \text{ (rad/s)}}.$$

$\omega_{CD} = -4.57$ rad/s,

$$\boxed{\boldsymbol{\omega}_{CD} = -4.57\mathbf{k} \text{ (rad/s)}}.$$

Problem 17.38 Bar AB is rotating at 10 rad/s in the counterclockwise direction. The disk rolls on the circular surface. Determine the angular velocities of bar BC and the disk at the instant shown.



Solution: The point "D" at the bottom of the wheel has zero velocity.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$= 0 + (10)\mathbf{k} \times (1\mathbf{i} - 2\mathbf{j}) = (20\mathbf{i} + 10\mathbf{j}) \text{ m/s}.$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= (20\mathbf{i} + 10\mathbf{j}) + \omega_{BC}\mathbf{k} \times (3\mathbf{i}) = (20)\mathbf{i} + (10 + 3\omega_{BC})\mathbf{j}$$

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C}$$

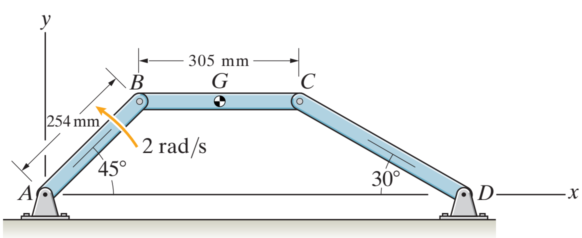
$$= (20)\mathbf{i} + (10 + 3\omega_{BC})\mathbf{j} + \omega_{CD}\mathbf{k} \times (-1\mathbf{j}) = (20 + \omega_{CD})\mathbf{i} + (10 + 3\omega_{BC})\mathbf{j}$$

Since the velocity of D is zero, we can set the components of velocity equal to zero and solve to find

$$\omega_{\text{disk}} = \omega_{CD} = -20 \text{ rad/s}, \quad \omega_{BC} = 3.33 \text{ rad/s}.$$

$$\boxed{\omega_{\text{disk}} = 20 \text{ rad/s clockwise}, \quad \omega_{BC} = 3.33 \text{ rad/s clockwise.}}$$

Problem 17.39 Bar AB rotates at 2 rad/s in the counterclockwise direction. Determine the velocity of the midpoint G of bar BC.



Solution: We first need to find the angular velocities of BC and CD

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = (2 \text{ rad/s})\mathbf{k} \times (0.254 \text{ m})(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j})$$

$$= (-0.354\mathbf{i} + 0.354\mathbf{j}) \text{ m/s}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = (-0.354\mathbf{i} + 0.354\mathbf{j}) \text{ m/s} + \omega_{BC}\mathbf{k} \times (0.305 \text{ m})\mathbf{i}$$

$$= [(-0.354 \text{ m/s})\mathbf{i} + (0.354 \text{ m/s} + \{0.305 \text{ m}\}\omega_{BC})\mathbf{j}]$$

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C}$$

$$= [(-0.354 \text{ m/s})\mathbf{i} + (0.354 \text{ m/s} + \{0.305 \text{ m}\}\omega_{BC})\mathbf{j}]$$

$$+ \omega_{CD}\mathbf{k} \times (0.254 \text{ m})[\sin 45^\circ \cot 30^\circ \mathbf{i} - \sin 45^\circ \mathbf{j}]$$

$$= [(-0.354 \text{ m/s} + \{0.18 \text{ m}\}\omega_{CD})\mathbf{i} + (0.354 \text{ m/s}$$

$$+ \{0.305 \text{ m}\}\, \omega_{BC} + \{0.31 \text{ m}\}\omega_{CD})\mathbf{j}]$$

Since D is fixed, we set both components to zero and solve for the angular velocities

$$\begin{aligned} -0.354 \text{ m/s} + \{0.18 \text{ m}\}\omega_{CD} &= 0 \\ 0.354 \text{ m/s} + \{0.305 \text{ m}\}\,\omega_{BC} + \{0.31 \text{ m}\}\omega_{CD} &= 0 \end{aligned} \Rightarrow \begin{aligned} \omega_{BC} &= -3.22 \text{ rad/s} \\ \omega_{CD} &= 2.00 \text{ rad/s} \end{aligned}$$

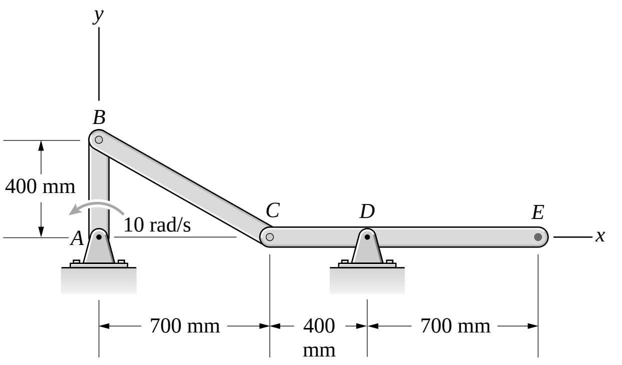
Now we can find the velocity of point G.

$$\mathbf{v}_G = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{G/B}$$

$$= (-0.354\mathbf{i} + 0.354\mathbf{j}) \text{ m/s} + (-3.22 \text{ rad/s})\mathbf{k} \times (0.152 \text{ mm})\mathbf{i}$$

$$\boxed{\mathbf{v}_G = (-0.354\,\mathbf{i} - 0.132\,\mathbf{j}) \text{ m/s}}$$

Problem 17.40 Bar AB rotates at 10 rad/s in the counterclockwise direction. Determine the velocity of point E.



Solution: The radius vector AB is $\mathbf{r}_{B/A} = 400\mathbf{j}$ (mm). The angular velocity of bar AB is $\boldsymbol{\omega}_{AB} = 10\mathbf{k}$ (rad/s). The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{AB} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 0 & 400 & 0 \end{bmatrix} = -4000\mathbf{i} \text{ (mm/s)}.$$

The radius vector BC is $\mathbf{r}_{C/B} = 700\mathbf{i} - 400\mathbf{j}$ (mm). The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = -4000\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 700 & -400 & 0 \end{bmatrix}$$

$$= (-4000 + 400\omega_{BC})\mathbf{i} + 700\omega_{BC}\mathbf{j}.$$

The radius vector CD is $\mathbf{r}_{C/D} = -400\mathbf{i}$ (mm). The point D is fixed (cannot translate). The velocity at point C is

$$\mathbf{v}_C = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{C/D} = (\omega_{CD}(\mathbf{k}) \times (-400\mathbf{i})) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ -400 & 0 & 0 \end{bmatrix}$$

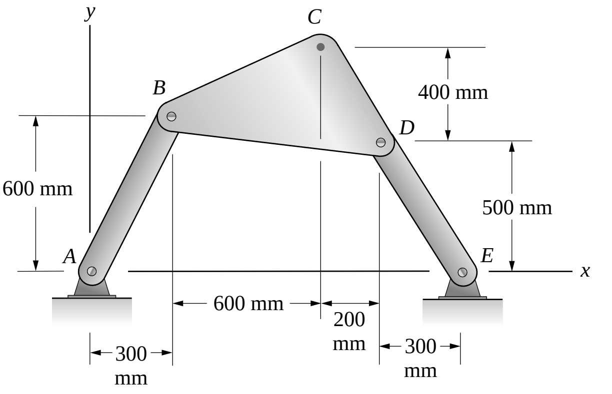
$$= -400\omega_{CD}\mathbf{j}.$$

Equate the two expressions for the velocity at point C, and separate components: $0 = (-4000 + 400\omega_{BC})\mathbf{i}$, $0 = (700\omega_{BC} + 400\omega_{CD})\mathbf{j}$. Solve: $\omega_{BC} = 10$ rad/s, $\omega_{CD} = -17.5$ rad/s. The radius vector DE is $\mathbf{r}_{D/E} = 700\mathbf{i}$ (mm). The velocity of point E is

$$\mathbf{v}_E = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/E} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -17.5 \\ 700 & 0 & 0 \end{bmatrix}$$

$\boxed{\mathbf{v}_E = -12250\mathbf{j} \text{ (mm/s)}}$.

Problem 17.41 Bar AB rotates at 4 rad/s in the counterclockwise direction. Determine the velocity of point C.



Solution: The angular velocity of bar AB is $\boldsymbol{\omega} = 4\mathbf{k}$ (rad/s). The radius vector AB is $\mathbf{r}_{B/A} = 300\mathbf{i} + 600\mathbf{j}$ (mm). The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 300 & 600 & 0 \end{bmatrix},$$

from which $\mathbf{v}_B = -2400\mathbf{i} + 1200\mathbf{j}$ (mm/s). The vector radius from B to C is $\mathbf{r}_{C/B} = 600\mathbf{i} + (900 - 600)\mathbf{j} = 600\mathbf{i} + 300\mathbf{j}$ (mm). The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 600 & 300 & 0 \end{bmatrix}$$

$$= (-2400 - 300\omega_{BC})\mathbf{i} + (1200 + 600\omega_{BC})\mathbf{j} \text{ (mm/s)}.$$

The radius vector from C to D is $\mathbf{r}_{D/C} = 200\mathbf{i} - 400\mathbf{j}$ (mm). The velocity of point D is

$$\mathbf{v}_D = \mathbf{v}_C + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 200 & -400 & 0 \end{bmatrix}$$

$$= \mathbf{v}_C + 400\omega_{BC}\mathbf{i} + 200\omega_{BC}\mathbf{j} \text{ (mm/s)}.$$

The radius vector from E to D is $\mathbf{r}_{D/E} = -300\mathbf{i} + 500\mathbf{j}$ (mm). The velocity of point D is

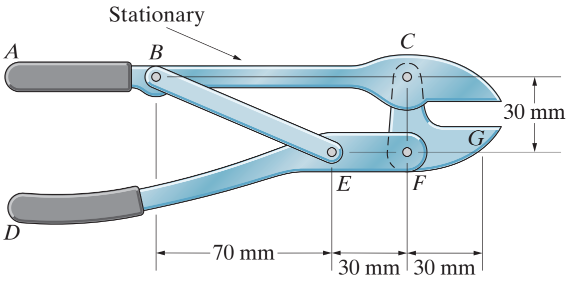
$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{DE} \\ -300 & 500 & 0 \end{bmatrix}$$

$$= -500\omega_{DE}\mathbf{i} - 300\omega_{DE}\mathbf{j} \text{ (mm/s)}.$$

Equate the expressions for the velocity of point D; solve for $\mathbf{v}_C$, to obtain one of two expressions for the velocity of point C. Equate the two expressions for $\mathbf{v}_C$, and separate components: $0 = (-500\omega_{DE} - 100\omega_{BC} + 2400)\mathbf{i}$, $0 = (1200 + 300\omega_{DE} + 800\omega_{BC})\mathbf{j}$. Solve $\omega_{DE} = 5.51$ rad/s, $\omega_{BC} = -3.57$ rad/s. Substitute into the expression for the velocity of point C to obtain

$$\boxed{\mathbf{v}_C = -1330\mathbf{i} - 941\mathbf{j} \text{ (mm/s)}.}$$

Problem 17.42 The upper grip and jaw of the pliers ABC is stationary. The lower grip DEF is rotating at 0.2 rad/s in the clockwise direction. At the instant shown, what is the angular velocity of the lower jaw CFG?



Solution:

$$\mathbf{v}_E = \mathbf{v}_B + \boldsymbol{\omega}_{BE} \times \mathbf{r}_{E/B} = 0 + \omega_{BE}\mathbf{k} \times (0.07\mathbf{i} - 0.03\mathbf{j}) \text{ m}$$

$$= \omega_{BE}(0.03\mathbf{i} + 0.07\mathbf{j}) \text{ m}$$

$$\mathbf{v}_F = \mathbf{v}_E + \boldsymbol{\omega}_{DEF} \times \mathbf{r}_{F/E} = \omega_{BE}(0.03\mathbf{i} + 0.07\mathbf{j}) \text{ m}$$

$$+ (-0.2 \text{ rad/s})\mathbf{k} \times (0.03 \text{ m})\mathbf{i}$$

$$= [(0.03 \text{ m})\omega_{BE}\mathbf{i} + (-0.006 \text{ m/s} + \{0.07 \text{ m}\}\omega_{BE})\mathbf{j}]$$

$$\mathbf{v}_C = \mathbf{v}_F + \boldsymbol{\omega}_{CFG} \times \mathbf{r}_{C/G}$$

$$= [(0.03 \text{ m})\omega_{BE}\mathbf{i} + (-0.006 \text{ m/s} + \{0.07 \text{ m}\}\omega_{BE})\mathbf{j}]$$

$$+ \omega_{CFG}\mathbf{k} \times (0.03 \text{ m})\mathbf{j}$$

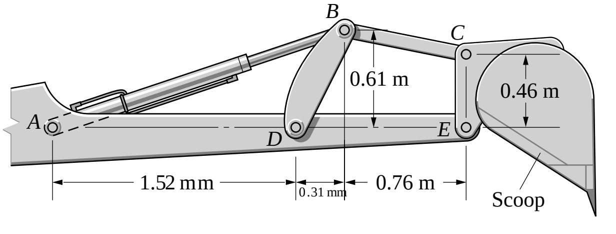
$$= [(0.03 \text{ m})(\omega_{BE} - \omega_{CFG})\mathbf{i} + (-0.006 \text{ m/s} + \{0.07 \text{ m}\}\omega_{BE})\mathbf{j}]$$

Since C is fixed we have

$$\left.\begin{array}{r}(0.03 \text{ m})(\omega_{BE} - \omega_{CFG}) = 0 \\ -0.006 \text{ m/s} + \{0.07 \text{ m}\}\omega_{BE} = 0\end{array}\right. \Rightarrow \begin{array}{l}\omega_{BE} = 0.0857 \text{ rad/s} \\ \omega_{CFG} = 0.0857 \text{ rad/s}\end{array}$$

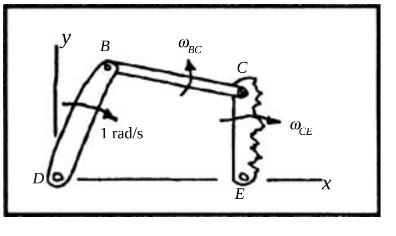
So we have $\boxed{\omega_{CFG} = 0.0857 \text{ rad/s CCW}}$

Problem 17.43 The horizontal member ADE supporting the scoop is stationary. If the link BD is rotating in the clockwise direction at 1 rad/s, what is the angular velocity of the scoop?



Solution: The velocity of B is $\mathbf{v}_B = \mathbf{v}_D + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{B/D}$. Expanding, we get

$$\mathbf{v}_B = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -0.31 \\ 0.31 & 0.61 & 0 \end{vmatrix} = 0.61\mathbf{i} - 0.31\mathbf{j} \text{ (m/s)}.$$



The velocity of C is

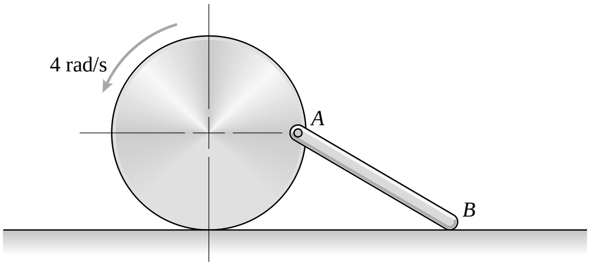
$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = 0.61\mathbf{i} - 0.31\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & +\omega_{BC} \\ 0.76 & -0.15 & 0 \end{vmatrix} \quad \textbf{(1)}.$$

We can also express the velocity of C as $\mathbf{v}_C = \mathbf{v}_E + \boldsymbol{\omega}_{CE} \times \mathbf{r}_{C/E}$ or

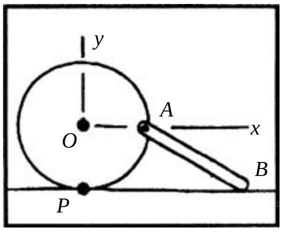
$$\mathbf{v}_C = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & +\omega_{CE} \\ 0 & 0.46 & 0 \end{vmatrix} \quad \textbf{(2)}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components in Equations (1) and (2) and solving, we obtain $\omega_{BC} = 0.4$ rad/s and $\omega_{CE} = -1.47$ rad/s.

Problem 17.44 The diameter of the disk is 1 m, and the length of the bar AB is 1 m. The disk is rolling, and point B slides on the plane surface. Determine the angular velocity of the bar AB and the velocity of point B.



Solution: Choose a coordinate system with the origin at O, the center of the disk, with x axis parallel to the horizontal surface. The point P of contact with the surface is stationary, from which



$$\mathbf{v}_P = 0 = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times -\mathbf{R} = \mathbf{v}_0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_0 \\ 0 & -0.5 & 0 \end{bmatrix} = \mathbf{v}_0 + 2\mathbf{i},$$

from which $\mathbf{v}_0 = -2\mathbf{i}$ (m/s). The velocity at A is $\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_{A/O}$.

$$\mathbf{v}_A = -2\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_0 \\ 0.5 & 0 & 0 \end{bmatrix} = -2\mathbf{i} + 2\mathbf{j} \text{ (m/s)}.$$

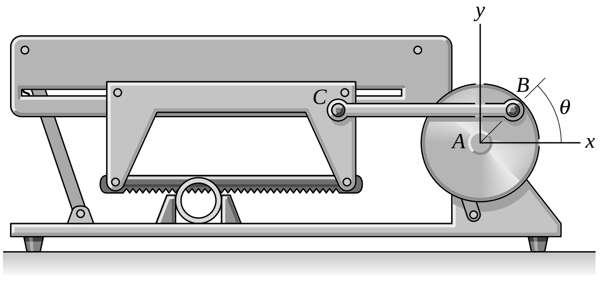
The vector from B to A is $\mathbf{r}_{A/B} = -\mathbf{i}\cos\theta + \mathbf{j}\sin\theta$ (m), where $\theta = \sin^{-1} 0.5 = 30^\circ$. The motion at point B is parallel to the x axis. The velocity at A is

$$\mathbf{v}_A = v_B\mathbf{i} + \boldsymbol{\omega} \times \mathbf{r}_{A/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ -0.866 & 0.5 & 0 \end{bmatrix}$$

$$= (v_B - 0.5\omega_{AB})\mathbf{i} - 0.866\omega_{AB}\mathbf{j} \text{ (m/s)}.$$

Equate and solve: $(-2 - 0.866\omega_{AB})\mathbf{j} = 0$, $(v_B - 0.5\omega_{AB} + 2)\mathbf{i} = 0$, from which $\boldsymbol{\omega}_{AB} = -2.31\mathbf{k}$ (rad/s), $\mathbf{v}_B = -3.15\mathbf{i}$ (m/s).

Problem 17.45 A motor rotates the circular disk mounted at A, moving the saw back and forth. (The saw is supported by a horizontal slot so that point C moves horizontally). The radius AB is 101.6 mm, and the link BC 355.6 mm long. In the position shown, $\theta = 45^\circ$ and the link BC is horizontal. If the angular velocity of the disk is one revolution per second counterclockwise, what is the velocity of the saw?



Solution: The radius vector from A to B is

$$\mathbf{r}_{B/A} = 0.1016\,(\mathbf{i}\cos 45^\circ + \mathbf{j}\sin 45^\circ) = 0.051\sqrt{2}(\mathbf{i}+\mathbf{j})\ (\text{m}).$$

The angular velocity of B is

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A},$$

$$\mathbf{v}_B = 0 + 2\pi(0.051\sqrt{2}) \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 0.102\pi\sqrt{2}(-\mathbf{i}+\mathbf{j})\ (\text{m/s}).$$

The radius vector from B to C is $\mathbf{r}_{C/B} = (0.1016\ \cos 45^\circ - 0.356)\mathbf{i}. = -\ 0.284\,\mathbf{i}$
The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -\,0.284 - 14 & 0 & 0 \end{bmatrix}$$

$$= -\,0.453\,\mathbf{i} + 0.453\,\mathbf{j} + \mathbf{j}\,(-\,0.284\,\omega_{BC})$$

$$= -\,0.453\,\mathbf{i} + (0.453 - 0.284\,\omega_{BC})\,\mathbf{j}$$

The saw is constrained to move parallel to the x axis, hence $0.453 - 0.284\ \omega_{BC} = 0$, and the saw velocity is

$$\boxed{\mathbf{v}_S = -\ 0.453\,\mathbf{i}\ (\text{m/s})}.$$

Problem 17.46 In Problem 17.45, if the angular velocity of the disk is one revolution per second counterclockwise and $\theta = 270^\circ$, what is the velocity of the saw?

Solution: The radius vector from A to B is $\mathbf{r}_{B/A} = -4\mathbf{j}$ (m).
The velocity of B is

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = 2\pi(-0.1016) \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = 0.203\pi\mathbf{i}\ (\text{m/s}).$$

The coordinates of point C are

$$(-0.356\ \cos\beta, +0.1016\sin 45^\circ) = (-0.31,\ 0.072)\ \text{m},$$

where $\quad \beta = \sin^{-1}\left(\dfrac{0.1016\,(1+\sin 45^\circ)}{0.356}\right) = 29.19^\circ.$

The coordinates of point B are $(0, -0.1016)$ in. The vector from C to B is

$$\mathbf{r}_{C/B} = (-0.31 - 0)\,\mathbf{i} + (0.072 - (-0.1016))\mathbf{j} = -\,0.31\mathbf{i} + \ 0.173\mathbf{j}\ (\text{m})$$

The velocity at point C is

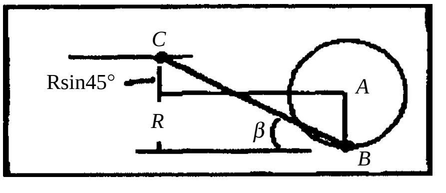
$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -0.31 & 0.173 & 0 \end{bmatrix}$$

$$= (0.203\pi - 0.173\omega_{BC})\mathbf{i} - 0.31\omega_{BC}\mathbf{j}.$$

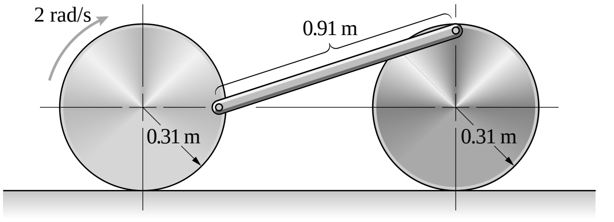
Since the saw is constrained to move parallel to the x axis, $-0.31\omega_{BC}\mathbf{j} = 0$, from which $\omega_{BC} = 0$, and the velocity of the saw is

$$\boxed{\mathbf{v}_C = 0.203\pi\mathbf{i} = 0.638\,\mathbf{i}\ (\text{m/s})}$$

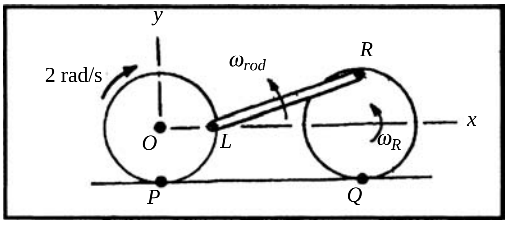
[*Note*: Since the vertical velocity at B reverses direction at $\theta = 270^\circ$, the angular velocity $\omega_{BC} = 0$ can be determined on physical grounds by inspection, simplifying the solution.]



Problem 17.47 The disks roll on a plane surface. The angular velocity of the left disk is 2 rad/s in the clockwise direction. What is the angular velocity of the right disk?



Solution: The velocity at the point of contact P of the left disk is zero. The vector from this point of contact to the center of the left disk is $\mathbf{r}_{O/P} = 0.31\mathbf{j}$ (m). The velocity of the center of the left disk is



$$\mathbf{v}_O = \boldsymbol{\omega} \times \mathbf{r}_{O/P} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -2 \\ 0 & 0.31 & 0 \end{bmatrix} = 0.61\mathbf{i} \text{ (m/s)}.$$

The vector from the center of the left disk to the point of attachment of the rod is $\mathbf{r}_{L/O} = 0.31\mathbf{i}$ (m). The velocity of the point of attachment of the rod to the left disk is

$$\mathbf{v}_L = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{L/O} = 0.61\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -2 \\ 0.31 & 0 & 0 \end{bmatrix}$$

$$= 0.61\mathbf{i} - 0.61\mathbf{j} \text{ (m/s)},$$

The vector from the point of attachment of the left disk to the point of attachment of the right disk is

$$\mathbf{r}_{R/L} = 0.91\,(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) \text{ (m)},$$

where $\theta = \sin^{-1}\left(\dfrac{0.31}{0.91}\right) = 19.47^\circ$.

The velocity of the point on attachment on the right disk is

$$\mathbf{v}_R = \mathbf{v}_L + \boldsymbol{\omega}_{\text{rod}} \times \mathbf{r}_{R/L} = \mathbf{v}_L + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{\text{rod}} \\ 0.863 & 1 & 0 \end{bmatrix}$$

$$= (0.61 - \omega_{\text{rod}})\mathbf{i} + (-0.61 + 0.863\omega_{\text{rod}})\mathbf{j} \text{ (m/s)}.$$

The velocity of point R is also expressed in terms of the contact point Q,

$$\mathbf{v}_R = \boldsymbol{\omega}_{RO} \times \mathbf{r}_{R/O} = \omega_{RO}(2) \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 0.31 & 0 \end{bmatrix}$$

$$= -0.61\omega_{RO}\mathbf{i} \text{ (m/s)}.$$

Equate the two expressions for the velocity $\mathbf{v}_R$ and separate components:

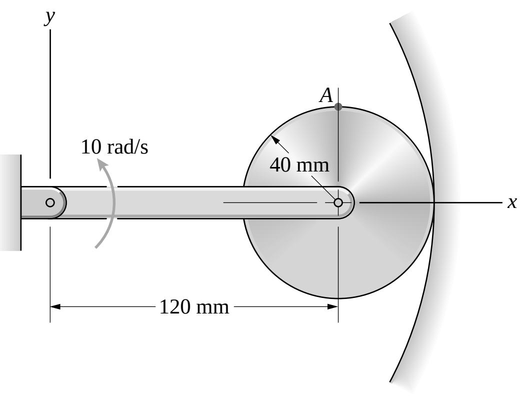
$$(0.61 - \omega_{\text{rod}} + 2\omega_{RO})\mathbf{i} = 0,$$

$$(-0.61 + 0.863\omega_{\text{rod}})\mathbf{j} = 0,$$

from which $\boxed{\boldsymbol{\omega}_{RO} = -0.65\mathbf{k} \text{ (rad/s)}}$

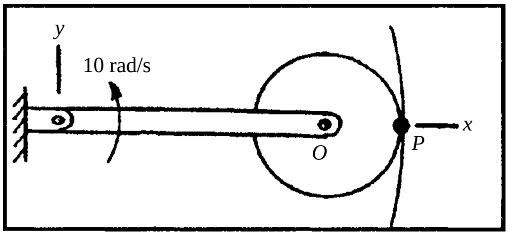
and $\omega_{\text{rod}} = 0.707$ rad/s.

Problem 17.48 The disk rolls on the curved surface. The bar rotates at 10 rad/s in the counterclockwise direction. Determine the velocity of point A.



Solution: The radius vector from the left point of attachment of the bar to the center of the disk is $\mathbf{r}_{\text{bar}} = 120\mathbf{i}$ (mm). The velocity of the center of the disk is

$$\mathbf{v}_O = \boldsymbol{\omega}_{\text{bar}} \times \mathbf{r}_{\text{bar}} = 10(120)\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 1200\mathbf{j} \text{ (mm/s)}.$$



The radius vector from the point of contact with the disk and the curved surface to the center of the disk is $\mathbf{r}_{O/P} = -40\mathbf{i}$ (m). The velocity of the point of contact of the disk with the curved surface is zero, from which

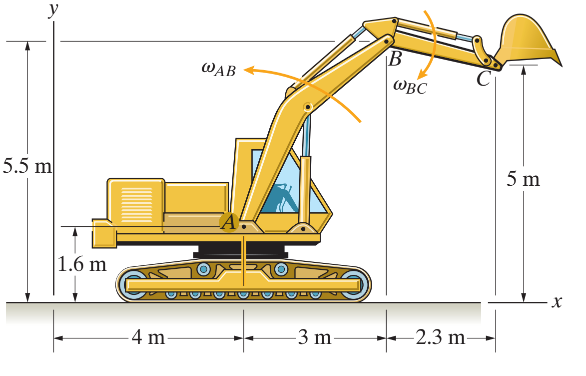
$$\mathbf{v}_O = \boldsymbol{\omega}_O \times \mathbf{r}_{O/P} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_O \\ -40 & 0 & 0 \end{bmatrix} = -40\omega_O\mathbf{j}.$$

Equate the two expressions for the velocity of the center of the disk and solve: $\omega_O = -30$ rad/s. The radius vector from the center of the disk to point A is $\mathbf{r}_{A/O} = 40\mathbf{j}$ (mm). The velocity of point A is

$$\mathbf{v}_A = \mathbf{v}_O + \boldsymbol{\omega}_O \times \mathbf{r}_{A/O} = 1200\mathbf{j} - (30)(40)\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boxed{= 1200\mathbf{i} + 1200\mathbf{j} \text{ (mm/s)}}$$

Problem 17.49 If $\omega_{AB} = 2$ rad/s and $\omega_{BC} = 4$ rad/s, what is the velocity of point C, where the excavator's bucket is attached?



Solution: The radius vector AB is

$$\mathbf{r}_{B/A} = 3\mathbf{i} + (5.5 - 1.6)\mathbf{j} = 3\mathbf{i} + 3.9\mathbf{j} \text{ (m)}.$$

The velocity of point B is

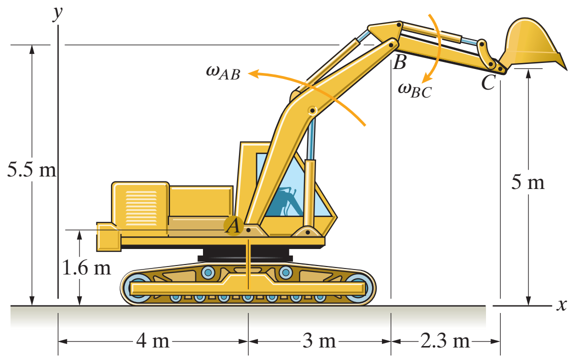
$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 3 & 3.9 & 0 \end{bmatrix} = -7.8\mathbf{i} + 6\mathbf{j} \text{ (m/s)}.$$

The radius vector BC is $\mathbf{r}_{C/B} = 2.3\mathbf{i} + (5 - 5.5)\mathbf{j} = 2.3\mathbf{i} - 0.5\mathbf{j}$ (m). The velocity at point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = -7.8\mathbf{i} + 6\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -4 \\ 2.3 & -0.5 & 0 \end{bmatrix}$$

$$\boxed{= -9.8\mathbf{i} - 3.2\mathbf{j} \text{ (m/s)}}$$

Problem 17.50 In Problem 17.49, if $\omega_{AB} = 2$ rad/s, what clockwise angular velocity ω_{BC} will cause the vertical component of the velocity of point C to be zero? What is the resulting velocity of point C?



Solution: Use the solution to Problem 17.49. The velocity of point B is

$$\mathbf{v}_B = -7.8\mathbf{i} + 6\mathbf{j} \text{ (m/s)}.$$

The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= -7.8\mathbf{i} + 6\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega_{BC} \\ 2.3 & -0.5 & 0 \end{bmatrix},$$

$$\mathbf{v}_C = (-7.8 - 0.5\omega_{BC})\mathbf{i} + (6 - 2.3\omega_{BC})\mathbf{j} \text{ (m/s)}.$$

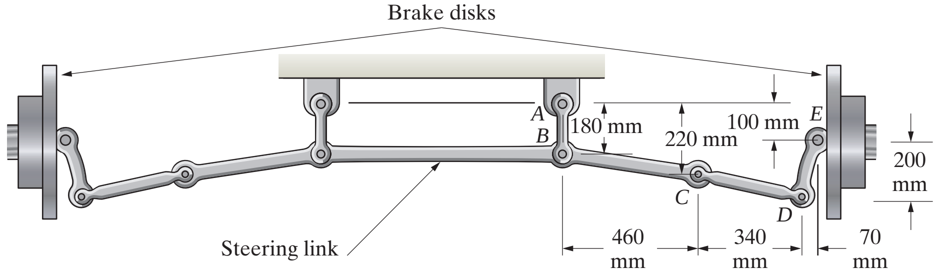
For the vertical component to be zero,

$$\boxed{\omega_{BC} = \frac{6}{2.3} = 2.61 \text{ rad/s clockwise.}}$$

The velocity of point C is

$$\boxed{\mathbf{v}_C = -9.1\mathbf{i} \text{ (m/s)}}$$

Problem 17.51 The steering linkage of a car is shown. Member DE rotates about fixed pin E. The right brake disk is rigidly attached to member DE. The tie rod CD is pinned at C and D. At the instant shown, the Pitman arm AB has a counterclockwise angular velocity of 1 rad/s. What is the angular velocity of the right brake disk?



Solution: Note that the steering link moves in translation only. Thus $\mathbf{v}_B = \mathbf{v}_C$.

$$\mathbf{v}_C = \mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + (1\mathbf{k}) \times (-0.18\mathbf{j}) = 0.18\mathbf{i}$$

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C} = 0.18\mathbf{i} + \omega_{CD}\mathbf{k} \times (0.34\mathbf{i} - 0.08\mathbf{j})$$

$$= (0.18 + 0.08\omega_{CD})\mathbf{i} + (0.34\omega_{CD})\mathbf{j}$$

$$\mathbf{v}_E = \mathbf{v}_D + \boldsymbol{\omega}_{DE} \times \mathbf{r}_{E/D}$$

$$= (0.18 + 0.08\omega_{CD})\mathbf{i} + (0.34\omega_{CD})\mathbf{j} + \omega_{DE}\mathbf{k} \times (0.07\mathbf{i} + 0.2\mathbf{j})$$

$$= (0.18 + 0.08\omega_{CD} - 0.2\omega_{DE})\mathbf{i} + (0.34\omega_{CD} + 0.07\omega_{DE})\mathbf{j}$$

Point E is not moving. Equating the components of the velocity of point E to zero, we have

$$0.18 + 0.08\omega_{CD} - 0.2\omega_{DE} = 0, \; 0.34\omega_{CD} + 0.07\omega_{DE} = 0.$$

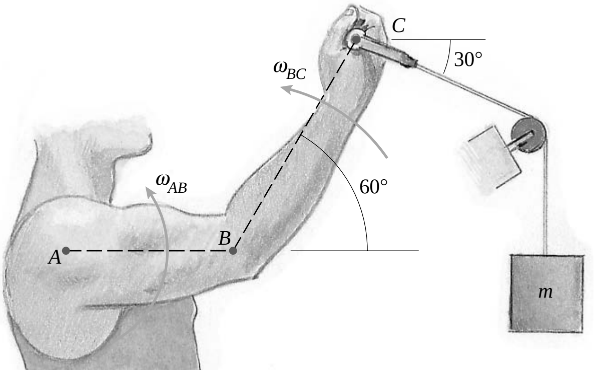
Solving these equations simultaneously, we find that

$$\omega_{CD} = -0.171 \text{ rad/s}, \quad \omega_{DE} = 0.832 \text{ rad/s}.$$

The angular velocity of the right brake disk is then

$$\boxed{\omega_{\text{disk}} = \omega_{DE} = 0.832 \text{ rad/s counterclockwise.}}$$

Problem 17.52 An athlete exercises his arm by raising the mass m. The shoulder joint A is stationary. The distance AB is 300 mm, and the distance BC is 400 mm. At the instant shown, $\omega_{AB} = 1$ rad/s and $\omega_{BC} = 2$ rad/s. How fast is the mass m rising?



Solution: The magnitude of the velocity of the point C parallel to the cable at C is also the magnitude of the velocity of the mass m. The radius vector AB is $\mathbf{r}_{B/A} = 300\mathbf{i}$ (mm). The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 300 & 0 & 0 \end{bmatrix} = 300\mathbf{j} \text{ (mm/s)}.$$

The radius vector BC is $\mathbf{r}_{C/B} = 400(\mathbf{i}\cos 60^\circ + \mathbf{j}\sin 60^\circ) = 200\mathbf{i} + 346.4\mathbf{j}$ (mm). The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = 300\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 200 & 346.4 & 0 \end{bmatrix}$$

$$= -692.8\mathbf{i} + 700\mathbf{j} \text{ (mm/s)}.$$

The unit vector parallel to the cable at C is $\mathbf{e}_C = -\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ = -0.866\mathbf{i} + 0.5\mathbf{j}$. The component of the velocity parallel to the cable at C is

$$\boxed{\mathbf{v}_C \cdot \mathbf{e}_C = 950 \text{ mm/s}},$$

which is the velocity of the mass m.

Problem 17.53 The distance AB is 305 mm, the distance BC is 406.4 mm, $\omega_{AB} = 0.6$ rad/s, and the mass m is rising at 610 mm/s. What is the angular velocity ω_{BC}?

Solution: The radius vector AB is $\mathbf{r}_{B/A} = 0.305\mathbf{i}$ (m). The velocity at point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.6 \\ 0.305 & 0 & 0 \end{bmatrix} = 2.19\,\mathbf{j} \text{ (m/s)}.$$

The radius vector BC is

$$\mathbf{r}_{C/B} = 0.4064\,(\mathbf{i}\cos 60 + \mathbf{j}\sin 60) = 0.2032\,\mathbf{i} + 0.3531\mathbf{j} \text{ (m)}$$

The velocity at C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = 7.2\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 0.2032 & 0.3531 & 0 \end{bmatrix}$$

$$= -\,0.3531\omega_{BC}\mathbf{i} + (2.19 + 0.2032\,\omega_{BC})\mathbf{j}.$$

The unit vector parallel to the cable at C is

$$\mathbf{e}_C = -\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 60^\circ = -0.866\mathbf{i} + 0.5\mathbf{j}.$$

The component of the velocity at C parallel to the cable is

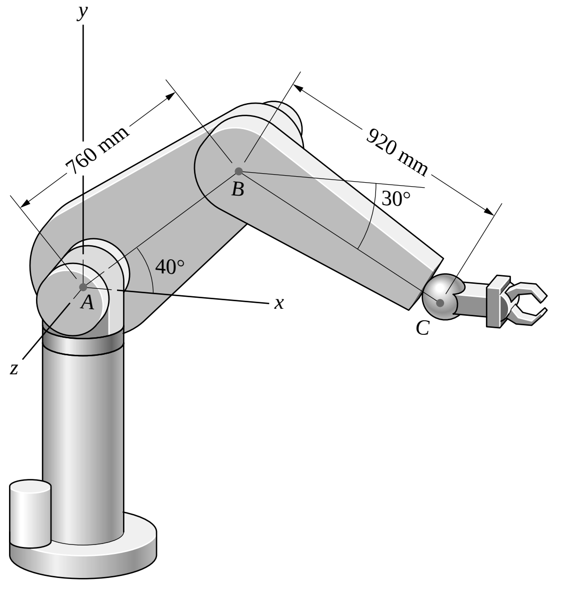
$$|\mathbf{v}_{CP}| = \mathbf{v}_C \cdot \mathbf{e}_C = +\,3.67\omega_{BC} + 1.22\,\omega_{BC} + 1.1 \text{ (m/s)}.$$

This is also the velocity of the rising mass, from which

$$4.89\omega_{BC} + 1.1 = \,0.61,$$

$$\boxed{\omega_{BC} = 1.28 \text{ rad/s}}$$

Problem 17.54 Points B and C are in the $x-y$ plane. The angular velocity vectors of the arms AB and BC are $\omega_{AB} = -0.2\mathbf{k}$ (rad/s), and $\omega_{BC} = 0.4\mathbf{k}$ (rad/s). What is the velocity of point C.



Solution: Locations of Points:

A: $(0, 0, 0)$ m

B: $(0.76\cos 40^\circ, 0.76\sin 40^\circ, 0)$ m

C: $(x_B + 0.92\cos 30^\circ, y_B - 0.92\sin 30^\circ, 0)$ m

or B: $(0.582, 0.489, 0)$,

C: $(1.379, 0.0285, 0)$ m

$\mathbf{r}_{B/A} = 0.582\mathbf{i} + 0.489\mathbf{j}$ (m)

$\mathbf{r}_{C/B} = 0.797\mathbf{i} - 0.460\mathbf{j}$ (m)

$$\mathbf{v}_A = 0, \omega_{AB} = -0.2\mathbf{k}\left(\frac{\text{rad}}{\text{s}}\right), \quad \omega_{BC} = 0.4\mathbf{k}\left(\frac{\text{rad}}{\text{s}}\right)$$

$\mathbf{v}_B = \mathbf{v}_A + \omega_{AB} \times \mathbf{r}_{B/A}$

$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC} \times \mathbf{r}_{C/B}$

$\mathbf{v}_B = (-0.2\mathbf{k}) \times (0.582\mathbf{i} + 0.489\mathbf{j})$

$\mathbf{v}_B = 0.0977\mathbf{i} - 0.116\mathbf{j}$ (m/s).

$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$

$\mathbf{v}_C = \mathbf{v}_B + 0.184\mathbf{i} + 0.319\mathbf{j}$ (m/s)

$\underline{\mathbf{v}_C = 0.282\mathbf{i} + 0.202\mathbf{j} \text{ (m/s)}}$.

Problem 17.55 If the velocity at point C of the robotic arm shown in Problem 17.54 is $\mathbf{v}_C = 0.15\mathbf{i} + 0.42\mathbf{j}$ (m/s), what are the angular velocities of the arms AB and BC?

Solution: From the solution to Problem 17.54,

$\mathbf{r}_{B/A} = 0.582\mathbf{i} + 0.489\mathbf{j}$ (m)

$\mathbf{r}_{C/B} = 0.797\mathbf{i} - 0.460\mathbf{j}$ (m)

$$\mathbf{v}_B = \omega_{AB}\mathbf{k} \times \mathbf{r}_{B/A} \quad (\mathbf{v}_A = 0)$$

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC}\mathbf{k} \times \mathbf{r}_{C/B}$$

We are given

$\mathbf{v}_C = -0.15\mathbf{i} + 0.42\mathbf{j} + 0\mathbf{k}$ (m/s).

Thus, we know everything in the $\mathbf{v}_C$ equation except ω_{AB} and ω_{BC}.

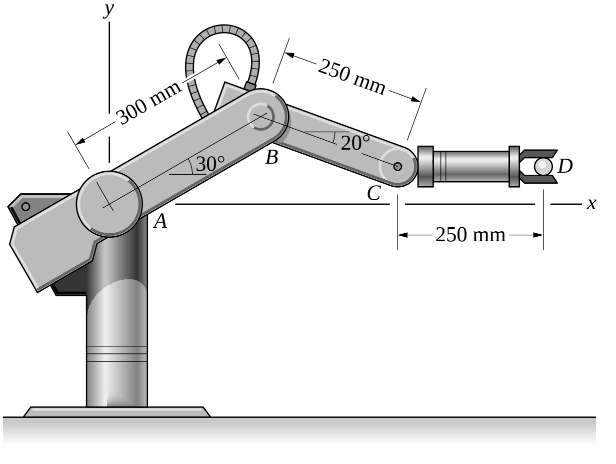
$\mathbf{v}_C = \omega_{AB}\mathbf{k} \times \mathbf{r}_{B/A} + \omega_{BC}\mathbf{k} \times \mathbf{r}_{C/B}$

This yields two scalar equations in two unknowns **i** and **j** components. Solving, we get

$\underline{\boldsymbol{\omega}_{AB} = 0.476\mathbf{k}}$ (rad/s),

$\underline{\boldsymbol{\omega}_{BC} = 0.179\mathbf{k}}$ (rad/s).

Problem 17.56 The link AB of the robot's arm is rotating at 2 rad/s in the counterclockwise direction, the link BC is rotating at 3 rad/s in the clockwise direction, and the link CD is rotating at 4 rad/s in the counterclockwise direction. What is the velocity of point D?



Solution: The velocity of B is

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A},$$

or $$\mathbf{v}_B = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0.3\cos 30^\circ & 0.3\sin 30^\circ & 0 \end{vmatrix}$$

$$= -0.3\mathbf{i} + 0.520\mathbf{j} \text{ (m/s)}.$$

The velocity of C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

or $$\mathbf{v}_C = -0.3\mathbf{i} + 0.520\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -3 \\ 0.25\cos 20^\circ & -0.25\sin 20^\circ & 0 \end{vmatrix}$$

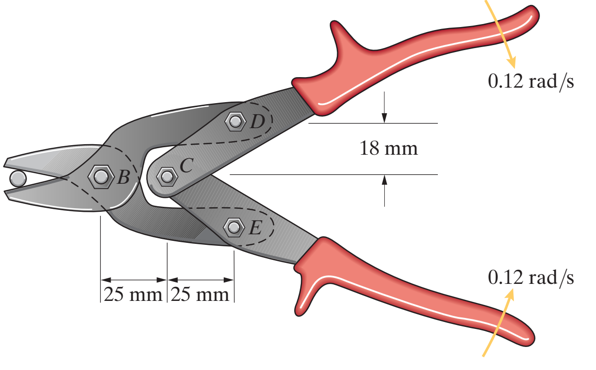
$$= -0.557\mathbf{i} - 0.185\mathbf{j} \text{ (m/s)}.$$

The velocity of D is

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C} = -0.557\mathbf{i} - 0.185\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 0.25 & 0 & 0 \end{vmatrix},$$

or $\mathbf{v}_D = -0.557\mathbf{i} + 0.815\mathbf{j}$ (m/s).

Problem 17.57 The person squeezes the grips of the shears, causing the angular velocities shown. What is the resulting angular velocity of the jaw *BD*?



Solution:

$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C} = 0 - (0.12 \text{ rad/s})\mathbf{k} \times (0.025\mathbf{i} + 0.018\mathbf{j}) \text{ m}$

$= (0.00216\mathbf{i} - 0.003\mathbf{j}) \text{ m/s}$

$\mathbf{v}_{\mathrm{B}} = \mathbf{v}_D + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{B/D}$

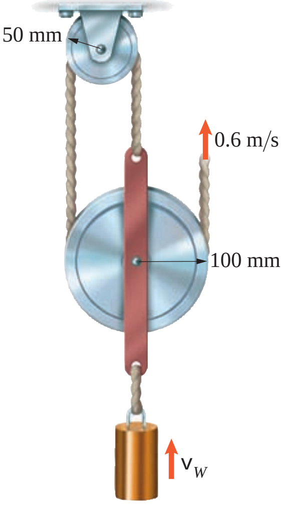
$= (0.00216\mathbf{i} - 0.003\mathbf{j}) \text{ m/s} + \omega_{BD}\mathbf{k} \times (-0.05\mathbf{i} - 0.018\mathbf{j}) \text{ m}$

$= (0.00216 \text{ m/s} + \{0.018 \text{ m}\}\omega_{BD})\mathbf{i} + (-0.003 \text{ m/s} - \{0.05 \text{ m}\}\omega_{BD})\mathbf{j}$

From symmetry we know that C and B do not move vertically

$-0.003 \text{ m/s} - \{0.05 \text{ m}\}\omega_{BD} = 0 \Rightarrow \boxed{\begin{array}{l}\omega_{BD} = -0.06 \text{ rad/s} \\ \omega_{BD} = 0.06 \text{ rad/s CW}\end{array}}$

Problem 17.58 Determine the velocity v_W and the angular velocity of the small pulley.



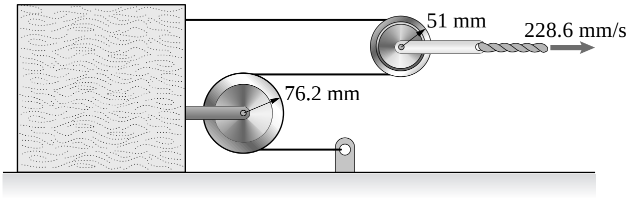
Solution: Since the radius of the bottom pulley is not given, we cannot use Eq (17.6) (or the equivalent). The strategy is to use the fact (derived from elementary principles) that the velocity of the center of a pulley is the mean of the velocities of the extreme edges, where the edges lie on a line normal to the motion, *taking into account the directions of the velocities at the extreme edges*. The center rope from the bottom pulley to the upper pulley moves upward at a velocity of v_W. Since the small pulley is fixed, the velocity of the center is zero, and the rope to the left moves downward at a velocity v_W, from which the left edge of the bottom pulley is moving at a velocity v_W downward. The right edge of the bottom pulley moves upward at a velocity of 0.6 m/s. The velocity of the center of the bottom pulley is the mean of the velocities at the extreme edges, from which $v_W = \dfrac{0.6 - v_W}{2}$.

Solve: $\boxed{v_W = \dfrac{0.6}{3} = 0.2 \text{ m/s}}$.

The angular velocity of the small pulley is

$$\boxed{\omega = \frac{v_W}{r} = \frac{0.2}{0.05} = 4 \text{ rad/s}}$$

Problem 17.59 Determine the velocity of the block and the angular velocity of the small pulley.



Solution: Denote the velocity of the block by v_B. The strategy is to determine the velocities of the extreme edges of a pulley by determining the velocity of the element of rope in contact with the pulley. The upper rope is fixed to the block, so that it moves to the right at the velocity of the block, from which the upper edge of the small pulley moves to the right at the velocity of the block. The fixed end of the rope at the bottom is stationary, so that the bottom edge of the large pulley is stationary. The center of the large pulley moves at the velocity of the block, from which the upper edge of the bottom pulley moves at twice the velocity of the block (since the velocity of the center is equal to the mean of the velocities of the extreme edges, one of which is stationary) from which the bottom edge of the small pulley moves at twice the velocity of the block. The center of the small pulley moves to the right at 9 in/s. The velocity of the center of the small pulley is the mean of the velocities at the extreme edges, from which

$$0.2286 = \frac{0.051 v_B + 0.025 v_B}{0.051} = \frac{0.076}{0.051} v_B,$$

from which

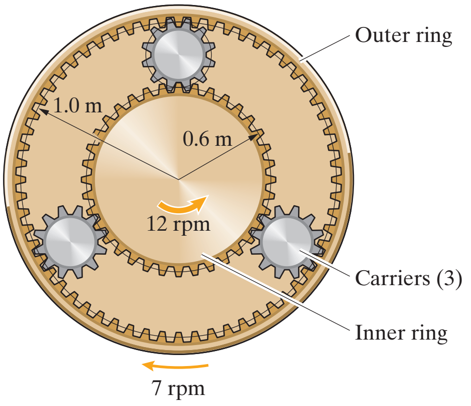
$$\boxed{v_B = \frac{0.051}{0.076}\, 0.2286 = 0.152 \text{ m/s}}.$$

The angular velocity of small pulley is given by

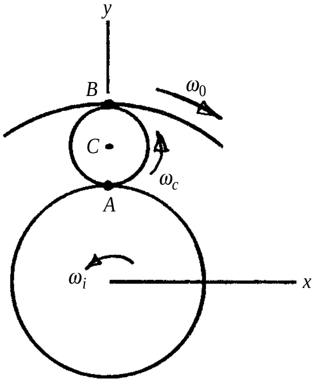
$$0.2286\,\mathbf{i} = 0.051\mathbf{v}_B\mathbf{i} + \boldsymbol{\omega} \times 0.051\mathbf{j} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ 0 & 0.051 & 0 \end{bmatrix} = 0.051\, v_B\mathbf{i} - 0.051\,\omega\mathbf{i},$$

from which $\boxed{\omega = \dfrac{0.305 - 0.2286}{0.051} = 1.5 \text{ rad/s}}$

Problem 17.60 The device shown is used in the semiconductor industry to polish silicon wafers. The wafers are placed on the faces of the carriers. The outer and inner rings are then rotated, causing the wafers to move and rotate against an abrasive surface. If the outer ring rotates in the clockwise direction at 7 rpm and the inner ring rotates in the counterclockwise direction at 12 rpm, what is the angular velocity of the carriers?



Solution: The velocity of pt. B is $\mathbf{v}_B = (1\text{ m})\omega_0\mathbf{i} = \omega_0\mathbf{i}$. The velocity of pt. A is $\mathbf{v}_A = -(0.6\text{ m})\omega_i\mathbf{i}$. Then



$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_C \times \mathbf{r}_{B/A} : \omega_0\mathbf{i} = -0.6\omega_i\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_C \\ 0 & 0.4 & 0 \end{vmatrix}.$$

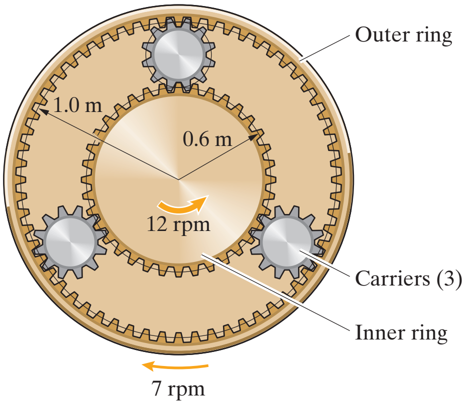
The $\mathbf{i}$ component of this equation is $\omega_0 = -0.6\omega_i - 0.4\omega_C$,

$$\text{so} \quad \omega_C = \frac{-0.6\omega_i - \omega_0}{0.4}$$

$$= \frac{-0.6(12\text{ rpm}) - 7\text{ rpm}}{0.4}$$

$$= -35.5\text{ rpm}.$$

Problem 17.61 In Problem 17.60, suppose that the outer ring rotates in the clockwise direction at 5 rpm and you want the centerpoints of the carriers to remain stationary during the polishing process. What is the necessary angular velocity of the inner ring?



Solution: See the solution of Problem 17.60. The velocity of pt. B is $\mathbf{v}_B = \omega_0\mathbf{i}$ and the angular velocity of the carrier is

$$\omega_C = \frac{-0.6\omega_i - \omega_0}{0.4}.$$

We want the velocity of pt. C to be zero:

$$\mathbf{v}_C = 0 = \mathbf{v}_B + \boldsymbol{\omega}_C \times \mathbf{r}_{C/B} = \omega_0\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_C \\ 0 & -0.2 & 0 \end{vmatrix}.$$

From this equation we see that $\omega_C = -5\omega_0$. Therefore the velocity of pt. A is

$$\mathbf{v}_A = \mathbf{v}_C + \boldsymbol{\omega}_C \times \mathbf{r}_{A/C}$$

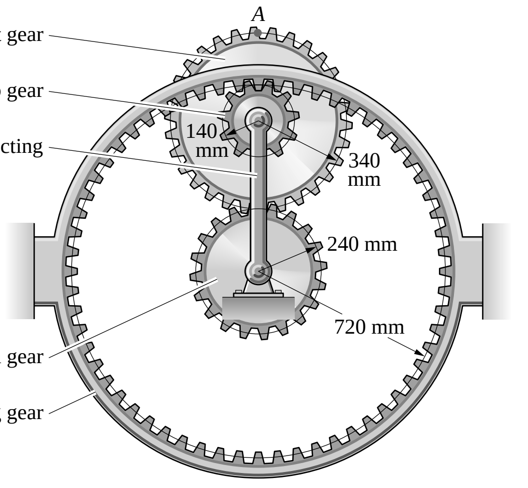
$$= 0 + (-5\omega_0\mathbf{k}) \times (-0.2\mathbf{j})$$

$$= -\omega_0\mathbf{i}$$

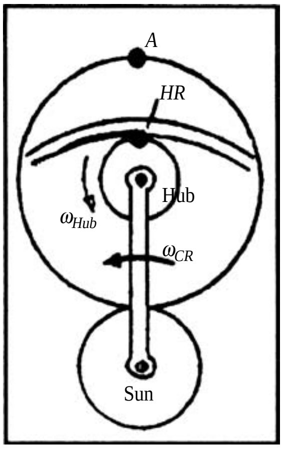
We also know that $\mathbf{v}_A = -(0.6\text{ m})\omega_i\mathbf{i}$,

$$\text{so} \quad \omega_i = \frac{\omega_0}{0.6} = \frac{5\text{ rpm}}{0.6} = 8.33\text{ rpm}.$$

Problem 17.62 The ring gear is fixed and the hub and planet gears are bonded together. The connecting rod rotates in the counterclockwise direction at 60 rpm. Determine the angular velocity of the sun gear and the magnitude of the velocity of point A.



Solution: Denote the centers of the sun, hub and planet gears by the subscripts Sun, Hub, and Planet, respectively. Denote the contact points between the sun gear and the planet gear by the subscript SP and the point of contact between the hub gear and the ring gear by the subscript HR. The angular velocity of the connecting rod is $\omega_{CR} = 6.28$ rad/s. The vector distance from the center of the sun gear to the center of the hub gear is $\mathbf{r}_{\text{Hub/Sun}} = (720 - 140)\mathbf{j} = 580\mathbf{j}$ (mm). The velocity of the center of the hub gear is



$$\mathbf{v}_{\text{Hub}} = \boldsymbol{\omega}_{CR} \times \mathbf{r}_{\text{Hub/Sun}} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2\pi \\ 0 & 580 & 0 \end{bmatrix} = -3644\mathbf{i} \text{ (mm/s)}$$

The angular velocity of the hub gear is found from

$$\mathbf{v}_{HR} = 0 = \mathbf{v}_{\text{Hub}} + \boldsymbol{\omega}_{\text{Hub}} \times 140\mathbf{j} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{\text{Hub}} \\ 0 & 140 & 0 \end{bmatrix} + \mathbf{v}_{\text{Hub}}$$

$$= -3644\mathbf{i} - 140\omega_{\text{Hub}}\mathbf{i},$$

from which

$$\omega_{\text{Hub}} = -\frac{3644}{140} = -26.03 \text{ rad/s}.$$

This is also the angular velocity of the planet gear. The linear velocity of point A is

$$\mathbf{v}_A = \boldsymbol{\omega}_{\text{Hub}} \times (340 - 140)\mathbf{j} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -26.03 \\ 0 & 200 & 0 \end{bmatrix}$$

$$\boxed{= 5206\mathbf{i} \text{ (mm/s)}}$$

The velocity of the point of contact with the sun gear is

$$\mathbf{v}_{PS} = \boldsymbol{\omega}_{\text{Hub}} \times (-480\mathbf{j}) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -26.03 \\ 0 & -480 & 0 \end{bmatrix}$$

$$= -12494.6\mathbf{i} \text{ (mm/s)}.$$

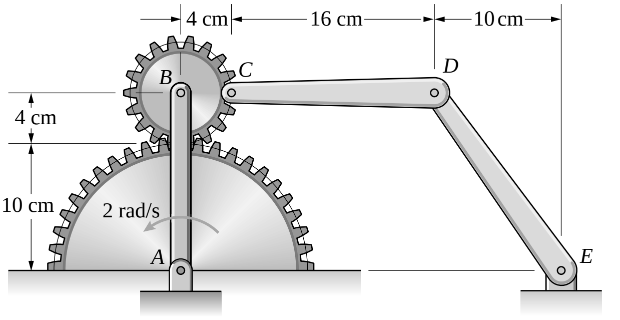
The angular velocity of the sun gear is found from

$$\mathbf{v}_{PS} = -12494.6\mathbf{i} = \boldsymbol{\omega}_{\text{Sun}} \times (240\mathbf{j}) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{\text{Sun}} \\ 0 & 240 & 0 \end{bmatrix}$$

$$= -240\omega_{\text{Sun}}\mathbf{i},$$

from which $\boxed{\omega_{\text{Sun}} = \frac{12494.6}{240} = 52.06 \text{ rad/s}}$

Problem 17.63 The large gear is fixed. Bar AB has a counterclockwise angular velocity of 2 rad/s. What are the angular velocities of bars CD and DE?



Solution: The strategy is to express vector velocity of point D in terms of the unknown angular velocities of CD and DE, and then to solve the resulting vector equations for the unknowns. The vector distance AB is $\mathbf{r}_{B/A} = 14\mathbf{j}$ (cm) The linear velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & 14 & 0 \end{bmatrix} = -28\mathbf{i} \text{ (cm/s)}.$$

The lower edge of gear B is stationary. The radius vector from the lower edge to B is $\mathbf{r}_B = 4\mathbf{j}$ (cm), The angular velocity of B is

$$\mathbf{v}_B = \boldsymbol{\omega}_B \times \mathbf{r}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_B \\ 0 & 4 & 0 \end{bmatrix} = -4\omega_B\mathbf{i} \text{ (cm/s)},$$

from which $\omega_B = -\dfrac{v_B}{4} = 7$ rad/s. The vector distance from B to C is $\mathbf{r}_{C/B} = 4\mathbf{i}$ (cm). The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_B \times \mathbf{r}_{C/B} = -28\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 7 \\ 4 & 0 & 0 \end{bmatrix}$$

$$= -28\mathbf{i} + 28\mathbf{j} \text{ (cm/s)}.$$

The vector distance from C to D is $\mathbf{r}_{D/C} = 16\mathbf{i}$ (cm), and from E to D is $\mathbf{r}_{D/E} = -10\mathbf{i} + 14\mathbf{j}$ (cm). The linear velocity of point D is

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C} = -28\mathbf{i} + 28\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ 16 & 0 & 0 \end{bmatrix}$$

$$= -28\mathbf{i} + (16\omega_{CD} + 28)\mathbf{j} \text{ (cm/s)}.$$

The velocity of point D is also given by

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{D/E} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{DE} \\ -10 & 14 & 0 \end{bmatrix}$$

$$= -14\omega_{DE}\mathbf{i} - 10\omega_{DE}\mathbf{j} \text{ (cm/s)}.$$

Equate components:

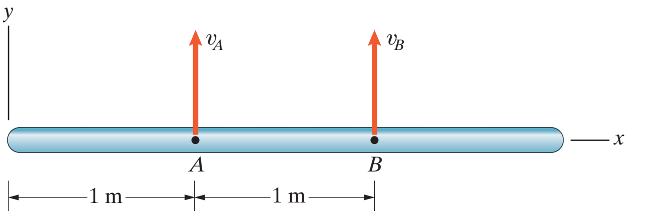
$$(-28 + 14\omega_{DE})\mathbf{i} = 0,$$

$$(16\omega_{CD} + 28 + 10\omega_{DE})\mathbf{j} = 0.$$

Solve: $\boxed{\omega_{DE} = 2 \text{ rad/s}}, \boxed{\omega_{CD} = -3 \text{ rad/s}}.$

The negative sign means a clockwise rotation.

Problem 17.64 If the bar has a clockwise angular velocity of 10 rad/s and $v_A = 20$ m/s, what are the coordinates of its instantaneous center of the bar, and what is the value of v_B?



Solution: Assume that the coordinates of the instantaneous center are (x_C, y_C), $\boldsymbol{\omega} = -\omega\mathbf{k} = -10\mathbf{k}$. The distance to point A is $\mathbf{r}_{A/C} = (1 - x_C)\mathbf{i} + y_C\mathbf{j}$. The velocity at A is

$$\mathbf{v}_A = 20\mathbf{j} = \boldsymbol{\omega} \times \mathbf{r}_{A/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega \\ 1 - x_C & y_C & 0 \end{bmatrix}$$

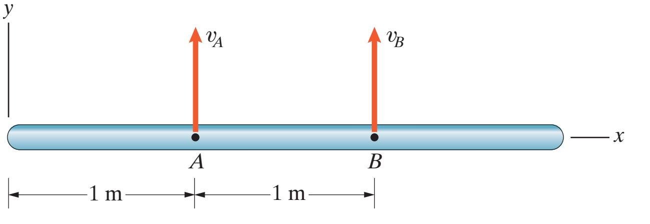
$$= y_C\omega\mathbf{i} - \omega(1 - x_C)\mathbf{j},$$

from which $y_C\omega\mathbf{i} = 0$, and $(20 + \omega(1 - x_C))\mathbf{j} = 0$.

Substitute $\omega = 10$ rad/s to obtain $y_C = 0$ and $x_C = 3$ m. The coordinates of the instantaneous center are $\boxed{(3, 0)\ (\text{m})}$. The vector distance from C to B is $\mathbf{r}_{B/C} = (2 - 3)\mathbf{i} = -\mathbf{i}$ (m). The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -10 \\ -1 & 0 & 0 \end{bmatrix} = -10(-\mathbf{j}) \boxed{= 10\mathbf{j}\ (\text{m/s})}$$

Problem 17.65 In Problem 17.64, if $v_A = 24$ m/s and $v_B = 36$ m/s, what are the coordinates of the instantaneous center of the bar, and what is its angular velocity?



Solution: Let (x_C, y_C) be the coordinates of the instantaneous center. The vectors from the instantaneous center and the points A and B are $\mathbf{r}_{A/C} = (1 - x_C)\mathbf{i} + y_C\mathbf{j}$ (m) and $\mathbf{r}_{B/C} = (2 - x_C)\mathbf{i} + y_C\mathbf{j}$. The velocity of A is given by

$$\mathbf{v}_A = 24\mathbf{j} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 1 - x_C & y_C & 0 \end{bmatrix}$$

$$= -\omega_{AB}y_C\mathbf{i} + \omega_{AB}(1 - x_C)\mathbf{j}\ (\text{m/s})$$

The velocity of B is

$$\mathbf{v}_B = 36\mathbf{j} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 2 - x_C & y_C & 0 \end{bmatrix}$$

$$= -y_C\omega_{AB}\mathbf{i} + \omega_{AB}(2 - x_C)\mathbf{j}\ (\text{m/s}).$$

Separate components:

$$24 - \omega_{AB}(1 - x_C) = 0,$$

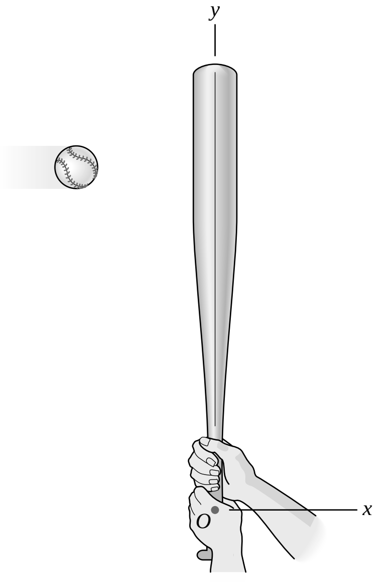
$$36 - \omega_{AB}(2 - x_C) = 0,$$

$$\omega_{AB}y_C = 0.$$

Solve: $\boxed{x_C = -1,}$ $\boxed{y_C = 0,}$

and $\boxed{\omega_{AB} = 12\ \text{rad/s}}$ *counter clockwise*.

Problem 17.66 The velocity of point O of the bat is $\mathbf{v}_O = -1.83\mathbf{i} - 4.27\mathbf{j}$ (m/s), and the bat rotates about the z axis with a counterclockwise angular velocity of 4 rad/s. What are the x and y coordinates of the bat's instantaneous center?



Solution: Let (x_C, y_C) be the coordinates of the instantaneous center. The vector from the instantaneous center to point O is $\mathbf{r}_{O/C} = -x_C\mathbf{i} - y_C\mathbf{j}$ (m). The velocity of point O is

$$\mathbf{v}_0 = -1.83\mathbf{i} - 4.27\mathbf{j} = \boldsymbol{\omega} \times \mathbf{r}_{O/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ -x_C & -y_C & 0 \end{bmatrix}$$

$$= y_C\omega\mathbf{i} - x_C\omega\mathbf{j} \text{ (m/s)}.$$

Equate terms and solve:

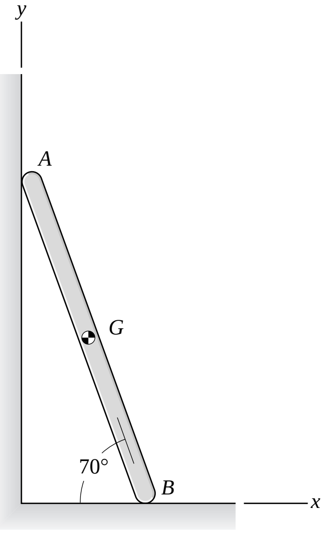
$$y_C = -\frac{1.83}{\omega} = -\frac{1.83}{4} = -0.46 \text{ m},$$

$$x_C = \frac{4.27}{\omega} = \frac{4.27}{4} = 1.07 \text{ m},$$

from which the coordinates are $\boxed{(1.07, -0.46) \text{ m}}$

Problem 17.67 Points A and B of the 1-m bar slide on the plane surfaces. The velocity of B is $\mathbf{v}_B = 2\mathbf{i}$ (m/s).

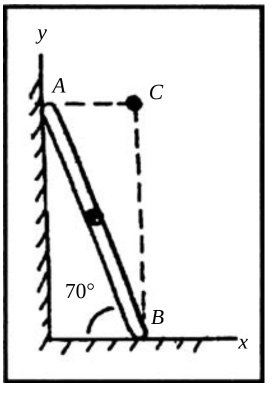
(a) What are the coordinates of the instantaneous center of the bar?
(b) Use the instantaneous center to determine the velocity at A.



Solution:

(a) A is constrained to move parallel to the y axis, and B is constrained to move parallel to the x axis. Draw perpendiculars to the velocity vectors at A and B. From geometry, the perpendiculars intersect at

$$\boxed{(\cos 70^\circ, \sin 70^\circ) = (0.3420, 0.9397)\text{ m}}.$$



(b) The vector from the instantaneous center to point B is

$$\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = 0.3420\mathbf{i} - (0.3420\mathbf{i} + 0.9397\mathbf{j}) = -0.9397\mathbf{j}$$

The angular velocity of bar AB is obtained from

$$\mathbf{v}_B = 2\mathbf{i} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 0 & -0.9397 & 0 \end{bmatrix}$$

$$= \omega_{AB}(0.9397)\mathbf{i},$$

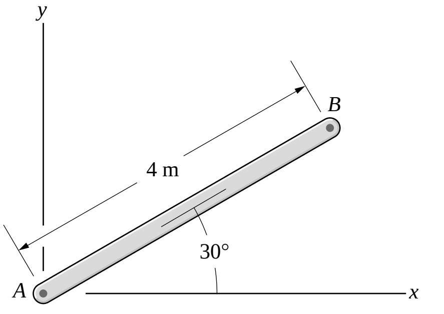
from which $\omega_{AB} = \dfrac{2}{0.9397} = 2.13$ rad/s.

The vector from the instantaneous center to point A is $\mathbf{r}_{A/C} = \mathbf{r}_A - \mathbf{r}_C = -0.3420\mathbf{i}$ (m). The velocity at A is

$$\mathbf{v}_A = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2.13 \\ -0.3420 & 0 & 0 \end{bmatrix}$$

$$\boxed{= -0.7279\mathbf{j}\text{ (m/s)}.}$$

Problem 17.68 The bar is in two-dimensional motion in the $x-y$ plane. The velocity of point A is $\mathbf{v}_A = 8\mathbf{i}$ (m/s), and B is moving in the direction parallel to the bar. Determine the velocity of B (a) by using Eq. (17.6) and (b) by using the instantaneous center of the bar.



Solution:

(a) The unit vector parallel to the bar is

$$\mathbf{e}_{AB} = (\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ) = 0.866\mathbf{i} + 0.5\mathbf{j}.$$

The vector from A to B is $\mathbf{r}_{B/A} = 4\mathbf{e}_{AB} = 3.46\mathbf{i} + 2\mathbf{j}$ (m). The velocity of point B is

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 8\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 3.46 & 2 & 0 \end{bmatrix}$$

$$\mathbf{v}_B = (8 - 2\omega_{AB})\mathbf{i} + 3.46\omega_{AB}\mathbf{j}.$$

But $\mathbf{v}_B$ is also moving parallel to the bar,

$$\mathbf{v}_B = v_B\mathbf{e}_{AB} = v_B(0.866\mathbf{i} + 0.5\mathbf{j}).$$

Equate, and separate components:

$$(8 - 2\omega_{AB} - 0.866v_B)\mathbf{i} = 0,$$

$$(3.46\omega_{AB} - 0.5v_B)\mathbf{j} = 0.$$

Solve: $\omega_{AB} = 1$ rad/s, $v_B = 6.93$ m/s, from which

$$\boxed{\mathbf{v}_B = v_B\mathbf{e}_{AB} = 6\mathbf{i} + 3.46\mathbf{j} \text{ (m/s)}}$$

(b) Let (x_C, y_C) be the coordinates of the instantaneous center. The vector from the center to A is

$$\mathbf{r}_{A/C} = \mathbf{r}_A - \mathbf{r}_C = -\mathbf{r}_C = -x_C\mathbf{i} - y_C\mathbf{j} \text{ (m)}.$$

The vector from the instantaneous center to B is

$$\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = (3.46 - x_C)\mathbf{i} + (2 - y_C)\mathbf{j}.$$

The velocity of point A is

$$\mathbf{v}_A = 8\mathbf{i} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ -x_C & -y_C & 0 \end{bmatrix}$$

$$= \omega_{AB}y_C\mathbf{i} - \omega_{AB}x_C\mathbf{j} \text{ (m/s)}.$$

From which $\boxed{x_C = 0}$, and $\omega_{AB}y_C = 8$. The velocity of point B is

$$\mathbf{v}_B = v_B\mathbf{e}_{AB} = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 3.46 - x_C & 2 - y_C & 0 \end{bmatrix}$$

$$= -\omega_{AB}(2 - y_C)\mathbf{i} + \omega_{AB}(3.46 - x_C)\mathbf{j}.$$

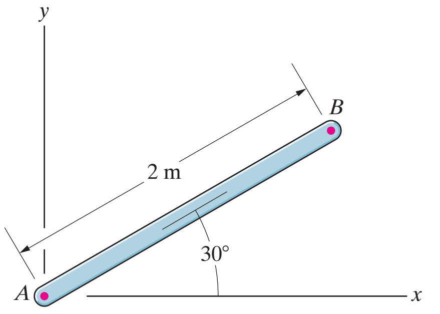
Equate terms and substitute

$\omega_{AB}y_C = 8$, and $x_C = 0$, to obtain: $(0.866v_B + 2\omega_{AB} - 8)\mathbf{i} = 0$, and $(0.5v_C - 3.46\omega_{AB})\mathbf{j} = 0$. These equations are algebraically identical with those obtained in Part (a) above (as can be shown by multiplying all terms by -1). Thus $\omega_{AB} = 1$ rad/s, $v_B = 6.93$ (m/s), and the velocity of B is that obtained in Part (a)

$$\boxed{\mathbf{v}_B = v_B\mathbf{e}_{AB} = 6\mathbf{i} + 3.46\mathbf{j} \text{ (m/s)}}.$$

Problem 17.69 Point A of the bar is moving at 8 m/s in the direction of the unit vector $0.966\mathbf{i} - 0.259\mathbf{j}$, and point B is moving in the direction of the unit vector $0.766\mathbf{i} + 0.643\mathbf{j}$.

(a) What are the coordinates of the bar's instantaneous center?
(b) What is the bar's angular velocity?



Solution: Assume the instantaneous center Q is located at (x, y).

Then

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_{A/Q}, \quad \mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_{B/Q}$$

$$(8\text{ m/s})(0.966\mathbf{i} - 0.259\mathbf{j}) = \omega\mathbf{k} \times (-x\mathbf{i} - y\mathbf{j})$$

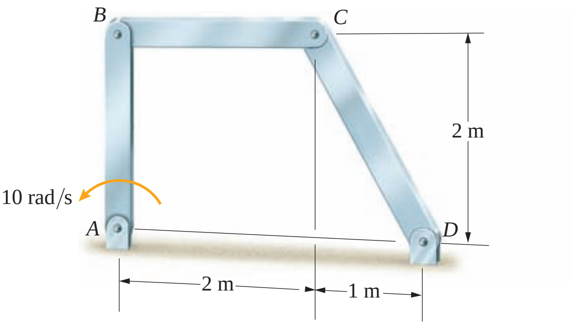
$$v_B(0.766\mathbf{i} + 0.643\mathbf{j}) = \omega\mathbf{k} \times ([\{2\text{ m}\}\cos 30^\circ - x]\mathbf{i}$$

$$+ [\{2\text{ m/s}\}\sin 30^\circ - y]\mathbf{j})$$

Expanding we have the four equations

$$\left.\begin{array}{l} 7.73\text{ m/s} = \omega y \\ -2.07\text{ m/s} = -\omega x \\ 0.766 v_B = \omega(y - 1\text{ m}) \\ 0.643 v_B = \omega(1.73\text{ m} - x) \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} x = 0.623\text{ m} \\ y = 2.32\text{ m} \end{array}}$$

Problem 17.70 Bar AB rotates with a counterclockwise angular velocity of 10 rad/s. At the instant shown, what are the angular velocities of bars BC and CD? (See Active Example 17.4.)



Solution: The location of the instantaneous center for BC is shown, along with the relevant distances. Using the concept of the instantaneous centers we have

$$v_B = (10)(2) = \omega_{BC}(4)$$

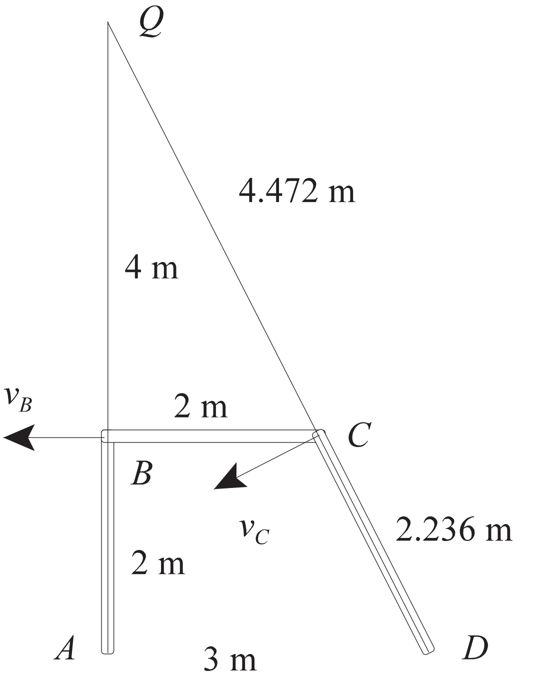
$$\omega_{BC} = 5 \text{ rad/s}$$

$$v_C = (4.472)(5) = 2.236(\omega_{CD})$$

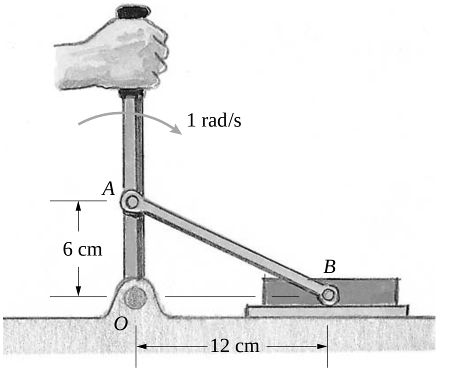
$$\omega_{CD} = 10 \text{ rad/s}$$

We determine the directions by inspection

$$\boxed{\begin{aligned} \omega_{BC} &= 5 \text{ rad/s clockwise.} \\ \omega_{CD} &= 10 \text{ rad/s counterclockwise.} \end{aligned}}$$



Problem 17.71 Use instantaneous centers to determine the horizontal velocity of B.



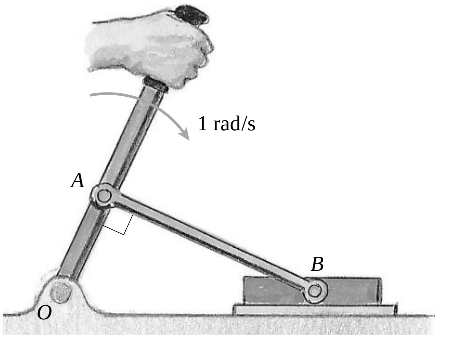
Solution: The instantaneous center of OA lies at O, by definition, since O is the point of zero velocity, and the velocity at point A is parallel to the x-axis:

$$\mathbf{v}_A = \boldsymbol{\omega}_{OA} \times \mathbf{r}_{A/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega_{OA} \\ 0 & 6 & 0 \end{bmatrix} = 6\mathbf{i} \text{ (cm/s)}.$$

A line perpendicular to this motion is parallel to the y axis. The point B is constrained to move on the x axis, and a line perpendicular to this motion is also parallel to the y axis. These two lines will not intersect at any finite distance from the origin, hence *at the instant shown the instantaneous center of bar AB is at infinity and the angular velocity of bar AB is zero*. At the instant shown, the bar AB translates only, from which the horizontal velocity of B is the horizontal velocity at A:

$$\boxed{\mathbf{v}_B = \mathbf{v}_A = 6\mathbf{i} \text{ (cm/s)}}.$$

Problem 17.72 When the mechanism in Problem 17.71 is in the position shown here, use instantaneous centers to determine the horizontal velocity of B.



Solution: The strategy is to determine the intersection of lines perpendicular to the motions at A and B. The velocity of A is parallel to the bar AB. A line perpendicular to the motion at A will be parallel to the bar OA. From the dimensions given in Problem 17.71, the length of bar AB is $r_{AB} = \sqrt{6^2 + 12^2} = 13.42$ cm. Consider the triangle OAB. The interior angle at B is

$$\beta = \tan^{-1}\left(\frac{6}{r_{AB}}\right) = 24.1^\circ,$$

and the interior angle at O is $\theta = 90^\circ - \beta = 65.9^\circ$. The unit vector parallel to the handle OA is $\mathbf{e}_{OA} = \mathbf{i}\cos\theta + \mathbf{j}\sin\theta$, and a point on the line is $\mathbf{L}_{OA} = L_{OA}\mathbf{e}_{OA}$, where L_{OA} is the magnitude of the distance of the point from the origin. A line perpendicular to the motion at B is parallel to the y axis. At the intersection of the two lines

$$L_{OA}\cos\theta = \frac{r_{AB}}{\cos\beta},$$

from which $L_{OA} = 36$ cm. The coordinates of the instantaneous center are (14.7, 32.9) (in.).

Check: From geometry, the triangle *OAB* and the triangle formed by the intersecting lines and the base are similar, and thus the interior angles are known for the larger triangle. From the law of sines

$$\frac{L_{OA}}{\sin 90^\circ} = \frac{r_{OB}}{\sin\beta} = \frac{r_{AB}}{\sin\beta\cos\beta} = 36 \text{ cm},$$

and the coordinates follow immediately from $\mathbf{L}_{OA} = L_{OA}\mathbf{e}_{OA}$. *check.* The vector distance from O to A is $\mathbf{r}_{A/O} = 6(\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) = 2.450\mathbf{i} + 5.478\mathbf{j}$ (cm). The angular velocity of the bar AB is determined from the known linear velocity at A.

$$\mathbf{v}_A = \boldsymbol{\omega}_{OA} \times \mathbf{r}_{A/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ 2.450 & 5.477 & 0 \end{bmatrix}$$

$$= 5.48\mathbf{i} - 2.45\mathbf{j} \text{ (cm/s)}.$$

The vector from the instantaneous center to point A is

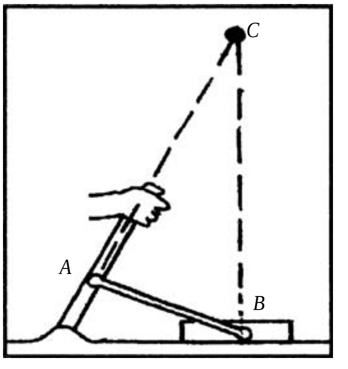
$$\mathbf{r}_{A/C} = \mathbf{r}_{OA} - \mathbf{r}_C = 6\mathbf{e}_{OA} - (14.7\mathbf{i} + 32.86\mathbf{j})$$

$$= -12.25\mathbf{i} - 27.39\mathbf{j} \text{ (cm)}$$

The velocity at point A is

$$\mathbf{v}_A = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ -12.25 & -27.39 & 0 \end{bmatrix}$$

$$= \omega_{AB}(27.39\mathbf{i} - 12.25\mathbf{j}) \text{ (cm/s)}.$$



Equate the two expressions for the velocity at point A and separate components, $5.48\mathbf{i} = 27.39\omega_{AB}$, $-2.45\mathbf{j} = -12.25\omega_{AB}\mathbf{j}$ (one of these conditions is superfluous) and solve to obtain $\omega_{AB} = 0.2$ rad/s, counterclockwise.

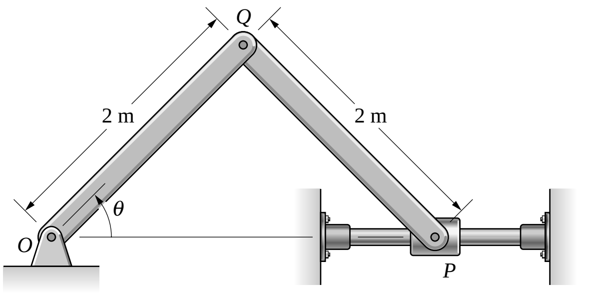
[*Check*: The distance *OA* is 6 cm. The magnitude of the velocity at A is $\omega_{OA}(6) = (1)(6) = 6$ cm/s. The distance to the instantaneous center from O is $\sqrt{14.7^2 + 32.9^2} = 36$ cm, and from C to A is $(36 - 6) = 30$ cm from which $30\,\omega_{AB} = 6$ cm/s, from which $\omega_{AB} = 0.2$ rad/s. *check.*]. The vector from the instantaneous center to point B is

$$\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = 14.7\mathbf{i} - (14.7\mathbf{i} + 32.86\mathbf{j} = -32.86\mathbf{j}) \text{ (cm)}$$

The velocity at point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.2 \\ 0 & -32.86 & 0 \end{bmatrix} \boxed{= 6.57\mathbf{i} \text{ (cm/s)}}$$

Problem 17.73 The angle $\theta = 45^\circ$, and the bar OQ is rotating in the counterclockwise direction at 0.2 rad/s. Use instantaneous centers to determine the velocity of the sleeve P.



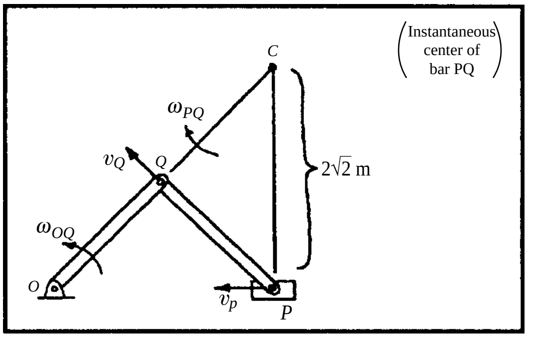
Solution: The velocity of Q is

$v_Q = 2\omega_{0Q} = 2(0.2) = 0.4$ m/s.

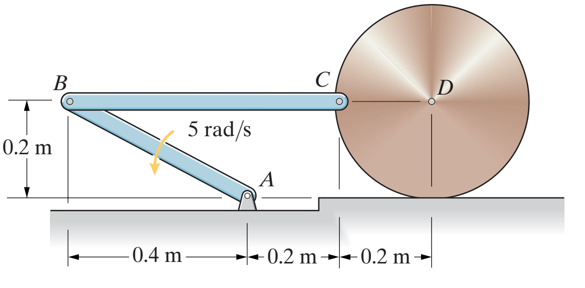
Therefore

$$|\overline{\omega}_{PQ}| = \frac{v_Q}{2\text{ m}} = \frac{0.4}{2} = 0.2 \text{ rad/s}$$

(clockwise) and $|\mathbf{v}_P| = 2\sqrt{2}\omega_{PQ} = 0.566$ m/s ($\mathbf{v}_P$ is to the left).



Problem 17.74 Bar AB is rotating in the counterclockwise direction at 5 rad/s. The disk rolls on the horizontal surface. Determine the angular velocity of bar BC.



Solution: First locate the instantaneous center

From the geometry we have

$$\frac{BC}{AE} = \frac{QB}{QA}$$

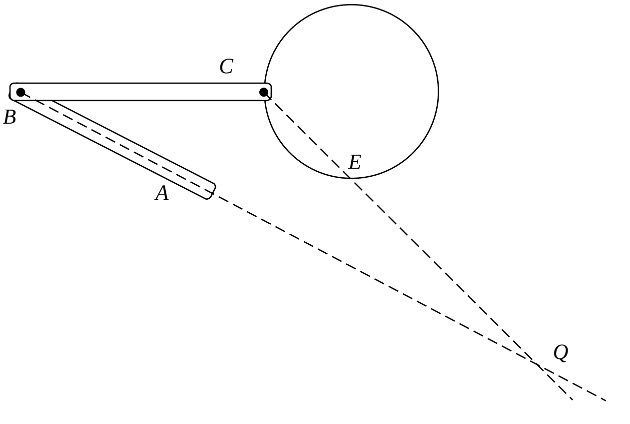
$$\frac{0.6\text{ m}}{0.4\text{ m}} = \frac{QB}{QB - \sqrt{0.2^2 + 0.4^2}\text{ m}}$$

Solving we find $BQ = 1.342$ m

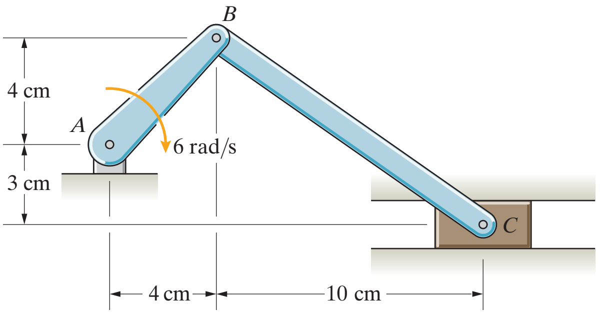
Now

$v_B = \omega_{AB}(AB) = \omega_{BC}(QB)$

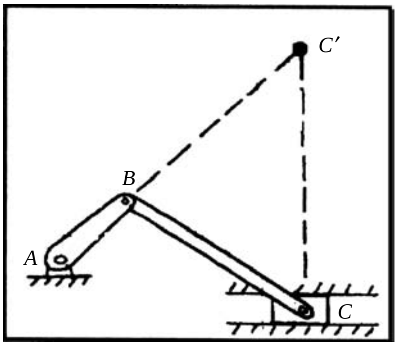
$$\boxed{\omega_{BC} = \frac{AB}{QB}(5\text{ rad/s}) = \frac{\sqrt{0.2^2+0.4^2}\text{ m}}{1.342\text{ m}}(5\text{ rad/s}) = 1.67\text{ rad/s CCW}}$$



Problem 17.75 Bar AB rotates at 6 rad/s in the clockwise direction. Use instantaneous centers to determine the angular velocity of bar BC.



Solution: Choose a coordinate system with origin at A and y axis vertical. Let C' denote the instantaneous center. The instantaneous center for bar AB is the point A, by definition, since A is the point of zero velocity. The vector AB is $\mathbf{r}_{B/A} = 4\mathbf{i} + 4\mathbf{j}$ (cm). The velocity at B is



$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -6 \\ 4 & 4 & 0 \end{bmatrix} = 24\mathbf{i} - 24\mathbf{j} \text{ (cm/s)}.$$

The unit vector parallel to AB is also the unit vector perpendicular to the velocity at B,

$$\mathbf{e}_{AB} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

The vector location of a point on a line perpendicular to the velocity at B is $\mathbf{L}_{AB} = L_{AB}\mathbf{e}_{AB}$, where L_{AB} is the magnitude of the distance from point A to the point on the line. The vector location of a point on a perpendicular to the velocity at C is $\mathbf{L}_C = (14\mathbf{i} + y\mathbf{j})$ where y is the y-coordinate of the point referenced to an origin at A. When the two lines intersect,

$$\frac{L_{AB}}{\sqrt{2}}\mathbf{i} = 14\mathbf{i},$$

and $y = \dfrac{L_{AB}}{\sqrt{2}} = 14$

from which $L_{AB} = 19.8$ cm, and the coordinates of the instantaneous center are (14, 14) (cm).

[*Check*: The line AC' is the hypotenuse of a right triangle with a base of 14 cm and interior angles of 45°, from which the coordinates of C' are (14, 14) cm *check*.]. The angular velocity of bar BC is determined from the known velocity at B. The vector from the instantaneous center to point B is

$$\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = 4\mathbf{i} + 4\mathbf{j} - 14\mathbf{i} - 14\mathbf{j} = -10\mathbf{i} - 10\mathbf{j}.$$

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -10 & -10 & 0 \end{bmatrix}$$

$$= \omega_{BC}(10\mathbf{i} - 10\mathbf{j}) \text{ (cm/s)}.$$

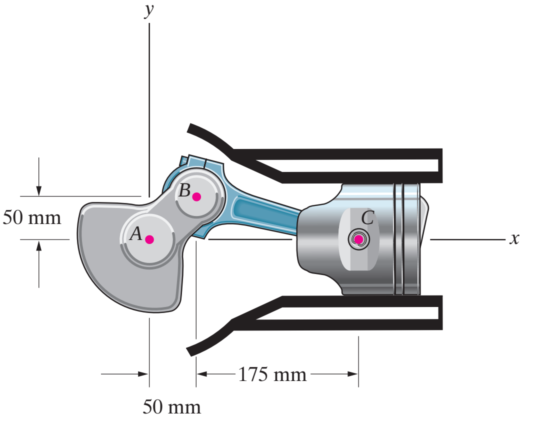
Equate the two expressions for the velocity: $24 = 10\omega_{BC}$, from which

$$\boxed{\omega_{BC} = 2.4 \text{ rad/s}},$$

Problem 17.76 The crank AB is rotating in the clockwise direction at 2000 rpm (revolutions per minute).

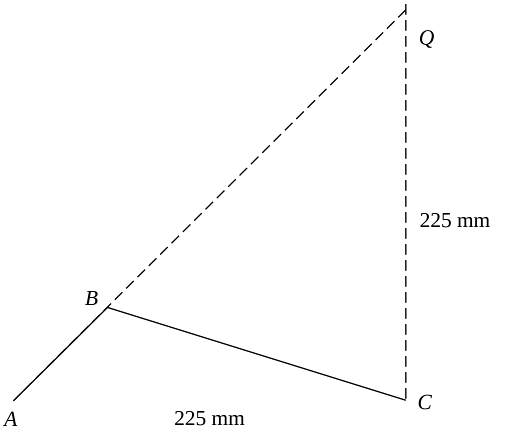
(a) At the instant shown, what are the coordinates of the instantaneous center of the connecting rod BC?

(b) Use instantaneous centers to determine the angular velocity of the connecting rod BC at the instant shown.



Solution:

$$\omega_{AB} = 2000 \text{ rpm}\left(\frac{2\pi \text{ rad}}{\text{rev}}\right)\left(\frac{\text{min}}{60 \text{ s}}\right) = 209 \text{ rad/s}$$



(a) The instantaneous center of BC is located at point Q

$$\boxed{Q = (225, 225) \text{ mm}}$$

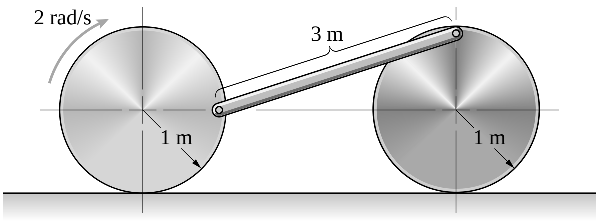
(b) $v_B = \omega_{AB}(AB) = \omega_{BC}(QB)$

$$\omega_{BC} = \frac{AB}{QB}\omega_{AB} = \frac{50\sqrt{2} \text{ mm}}{(225-50)\sqrt{2} \text{ mm}}(209 \text{ rad/s}) = 59.8 \text{ rad/s}$$

$$v_C = \omega_{BC}(QC) = (59.8 \text{ rad/s})(0.225 \text{ m}) = 13.5 \text{ m/s}$$

$$\boxed{\mathbf{v}_C = (13.5 \text{ m/s})\mathbf{i}}$$

Problem 17.77 The disks roll on the plane surface. The left disk rotates at 2 rad/s in the clockwise direction. Use the instantaneous centers to determine the angular velocities of the bar and the right disk.



Solution: Choose a coordinate system with the origin at the point of contact of the left disk with the surface, and the x axis parallel to the plane surface. Denote the point of attachment of the bar to the left disk by A, and the point of attachment to the right disk by B. The instantaneous center of the left disk is the point of contact with the surface. The vector distance from the point of contact to the point A is $\mathbf{r}_{A/P} = \mathbf{i} + \mathbf{j}$ (m). The velocity of point A is

$$\mathbf{v}_A = \omega_{LD} \times \mathbf{r}_{A/P} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix} = 2\mathbf{i} - 2\mathbf{j} \text{ (m/s)}.$$

The point on a line perpendicular to the velocity at A is $\mathbf{L}_A = L_A(\mathbf{i} + \mathbf{j})$, where L_A is the distance of the point from the origin. The point B is at the top of the right disk, and the velocity is constrained to be parallel to the x axis. A point on a line perpendicular to the velocity at B is $\mathbf{L}_B = (1 + 3\cos\theta)\mathbf{i} + y\mathbf{j}$ (m), where

$$\theta = \sin^{-1}\left(\frac{1}{3}\right) = 19.5^\circ.$$

At the intersection of these two lines $L_A = 1 + 3\cos\theta = 3.83$ ft, and the coordinates of the instantaneous center of the bar are (3.83, 3.83) (m). The angular velocity of the bar is determined from the known velocity of point A. The vector from the instantaneous center to point A is

$$\mathbf{r}_{A/C} = \mathbf{r}_A - \mathbf{r}_C = \mathbf{i} + \mathbf{j} - 3.83\mathbf{i} - 3.83\mathbf{j} = -2.83\mathbf{i} - 2.83\mathbf{j} \text{ (m)}.$$

The velocity of point A is

$$\mathbf{v}_A = \omega_{AB} \times \mathbf{r}_{A/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ -2.83 & -2.83 & 0 \end{bmatrix}$$

$$= \omega_{AB}(2.83\mathbf{i} - 2.83\mathbf{j}) \text{ (m/s)}.$$

Equate the two expressions and solve:

$$\boxed{\omega_{AB} = \frac{2}{2.83} = 0.7071 \text{ (rad/s)}} \text{ counterclockwise.}$$

The vector from the instantaneous center to point B is

$$\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = (1 + 3\cos\theta)\mathbf{i} + 2\mathbf{j} - 3.83\mathbf{i} - 3.83\mathbf{j} = -1.83\mathbf{j}.$$

The velocity of point B is

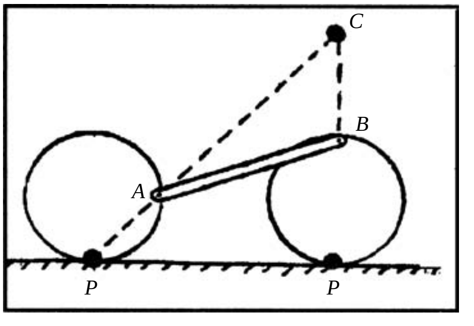
$$\mathbf{v}_B = \omega_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.7071 \\ 0 & -1.83 & 0 \end{bmatrix} = 1.294\mathbf{i} \text{ (m/s)}.$$

Using the fixed center at point of contact:

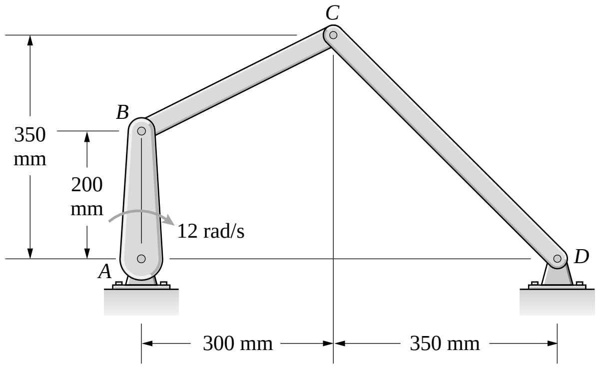
$$\mathbf{v}_B = \omega_{RD} \times \mathbf{r}_{B/P} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{RD} \\ 0 & 2 & 0 \end{bmatrix} = -2\omega_{RD}\mathbf{i} \text{ (m/s)}.$$

Equate the two expressions for $\mathbf{v}_B$ and solve:

$$\boxed{\omega_{RD} = -0.647 \text{ rad/s, clockwise.}}$$



Problem 17.78 Bar AB rotates at 12 rad/s in the clockwise direction. Use instantaneous centers to determine the angular velocities of bars BC and CD.



Solution: Choose a coordinate system with the origin at A and the x axis parallel to AD. The instantaneous center of bar AB is point A, by definition. The velocity of point B is normal to the bar AB. Using the instantaneous center A and the known angular velocity of bar AB the velocity of B is

$$\mathbf{v}_B = \omega \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -12 \\ 0 & 200 & 0 \end{bmatrix} = 2400\mathbf{i} \text{ (mm/s)}.$$

The unit vector perpendicular to the velocity of B is $\mathbf{e}_{AB} = \mathbf{j}$, and a point on a line perpendicular to the velocity at B is $\mathbf{L}_{AB} = L_{AB}\mathbf{j}$ (mm). The instantaneous center of bar CD is point D, by definition. The velocity of point C is constrained to be normal to bar CD. The interior angle at D is 45°, by inspection. The unit vector parallel to DC (and perpendicular to the velocity at C) is

$$\mathbf{e}_{DC} = -\mathbf{i}\cos 45^\circ + \mathbf{j}\sin 45^\circ = \left(\frac{1}{\sqrt{2}}\right)(-\mathbf{i} + \mathbf{j}).$$

The point on a line parallel to DC is

$$\mathbf{L}_{DC} = \left(650 - \frac{L_{DC}}{\sqrt{2}}\right)\mathbf{i} + \frac{L_{DC}}{\sqrt{2}}\mathbf{j} \text{ (mm)}.$$

At the intersection of these lines $\mathbf{L}_{AB} = \mathbf{L}_{DC}$, from which

$$\left(650 - \frac{L_{DC}}{\sqrt{2}}\right) = 0$$

and $L_{AB} = \dfrac{L_{DC}}{\sqrt{2}}$,

from which $L_{DC} = 919.2$ mm, and $L_{AB} = 650$ mm. The coordinates of the instantaneous center of bar BC are (0, 650) (mm). Denote this center by C'. The vector from C' to point B is

$$\mathbf{r}_{B/C'} = \mathbf{r}_B - \mathbf{r}_{C'} = 200\mathbf{j} - 650\mathbf{j} = -450\mathbf{j}.$$

The vector from C' to point C is

$$\mathbf{r}_{C/C'} = 300\mathbf{i} + 350\mathbf{j} - 650\mathbf{j} = 300\mathbf{i} - 300\mathbf{j} \text{ (mm)}.$$

The velocity of point B is

$$\mathbf{v}_B = \omega_{BC} \times \mathbf{r}_{B/C'} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 0 & -450 & 0 \end{bmatrix} = 450\omega_{BC}\mathbf{i} \text{ (mm/s)}.$$

Equate and solve: $2400 = 450\omega_{BC}$, from which

$$\boxed{\omega_{BC} = \frac{2400}{450} = 5.33 \text{ (rad/s)}}.$$

The angular velocity of bar CD is determined from the known velocity at point C. The velocity at C is

$$\mathbf{v}_C = \omega_{BC} \times \mathbf{r}_{C/C'} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5.33 \\ 300 & -300 & 0 \end{bmatrix}$$

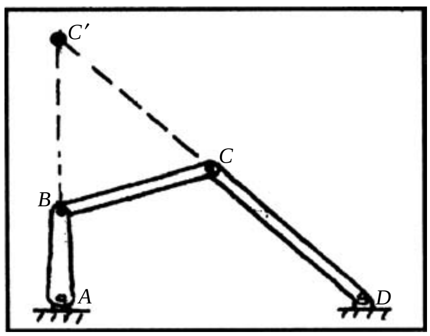
$$= 1600\mathbf{i} + 1600\mathbf{j} \text{ (mm/s)}.$$

The vector from D to point C is $\mathbf{r}_{C/D} = -350\mathbf{i} + 350\mathbf{j}$ (mm). The velocity at C is

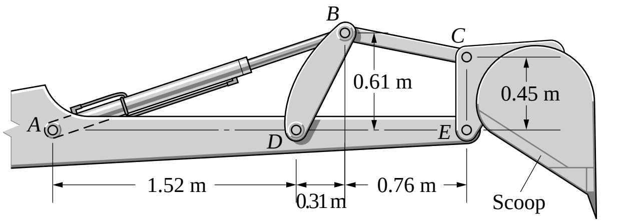
$$\mathbf{v}_C = \omega_{CD} \times \mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ -350 & 350 & 0 \end{bmatrix}$$

$$= -350\omega_{CD}\mathbf{i} - 350\omega_{CD}\mathbf{j} \text{ (mm/s)}.$$

Equate and solve: $\boxed{\omega_{CD} = -4.57 \text{ rad/s}}$ clockwise.



Problem 17.79 The horizontal member *ADE* supporting the scoop is stationary. The link *BD* is rotating in the clockwise direction at 1 rad/s. Use instantaneous centers to determine the angular velocity of the scoop.



Solution: The distance from D to B is $r_{BD} = \sqrt{0.31^2 + 0.61^2} = 0.68$ m. The distance from B to H is

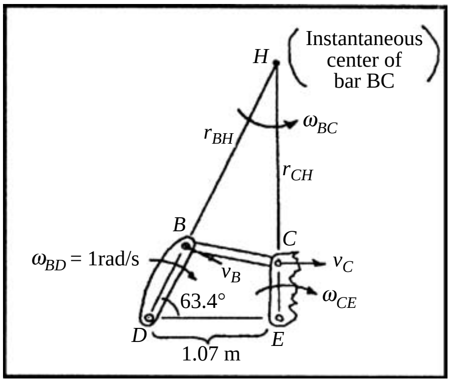
$$r_{BH} = \frac{1.07}{\cos 63.4^\circ} - r_{BD} = 1.7 \text{ m},$$

and the distance from C to H is $r_{CH} = 1.07 \tan 63.4^\circ - r_{CE} = 1.68$ m. The velocity of B is $v_B = r_{BD}\omega_{BD} = (0.68)(1) = 0.68$ m/s. Therefore

$$\omega_{BC} = \frac{v_B}{r_{BH}} = \frac{0.68}{1.7} = 0.4 \text{ rad/s}.$$

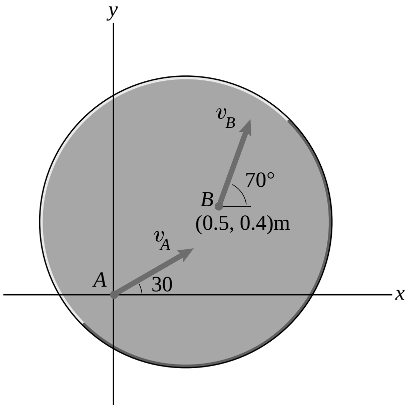
The velocity of C is $v_c = r_{CH}\omega_{BC} = (1.68)(0.4) = 0.67$, so the angular velocity of the scoop is

$$\omega_{CE} = \frac{v_C}{r_{CE}} = \frac{0.67}{0.46} = 1.47 \text{ rad/s}$$

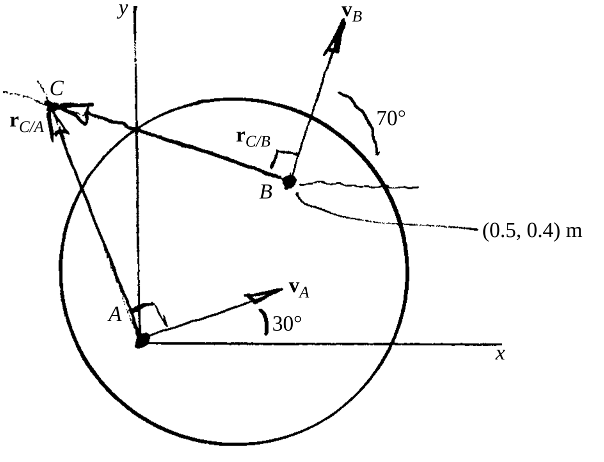


Problem 17.80 The disk is in planar motion. The directions of the velocities of points A and B are shown. The velocity of point A is $v_A = 2$ m/s.

(a) What are the coordinates of the disk's instantaneous center?
(b) Determine the velocity v_B and the disk's angular velocity.



Solution:



$\boldsymbol{\omega} = \omega \mathbf{k}$

$r_{c/A} = x_c \mathbf{i} + y_c \mathbf{j}$

$\mathbf{r}_{c/B} = (x_c - x_B)\mathbf{i} + (y_c - y_B)\mathbf{j}$

The velocity of C, the instantaneous center, is zero.

$\mathbf{v}_c = 0 = \mathbf{v}_A + \omega \mathbf{k} \times \mathbf{r}_{c/A}$

$$\begin{cases} 0 = v_{A_x} - \omega y_c \quad \mathbf{(1)} \\ 0 = v_{A_y} + \omega x_c \quad \mathbf{(2)} \end{cases}$$

where $v_{A_x} = v_A \cos 30^\circ = 2 \cos 30^\circ$ m/s

$v_{A_y} = v_A \sin 30^\circ = 1$ m/s

$v_{B_x} = v_B \cos 70^\circ$

$v_{B_y} = v_B \sin 70^\circ$

Also $\mathbf{v}_c = 0 = \mathbf{v}_B + \omega \mathbf{k} \times \mathbf{r}_{c/B}$

$0 = v_B \cos 70^\circ - \omega(y_c - y_B) \quad \mathbf{(3)}$

$0 = v_B \sin 70^\circ + \omega(x_c - x_B) \quad \mathbf{(4)}$

Eqns (1) $\rightarrow$ (4) are 4 eqns in the four unknowns ω, v_B, x_c, and y_c.

Solving,

$\omega = 2.351$ rad/s,

$\underline{\boldsymbol{\omega} = 2.351\mathbf{k} \text{ rad/s}}$,

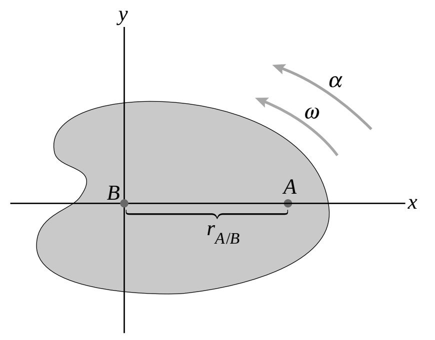
$\underline{v_B = 2.31 \text{ m/s}}$,

$\underline{x_c = -0.425 \text{ m}}$,

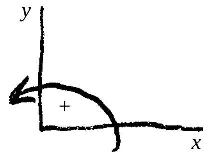
$\underline{y_c = 0.737 \text{ m}}$.

Problem 17.81 The rigid body rotates about the z axis with counterclockwise angular velocity $\omega = 4$ rad/s and counterclockwise angular acceleration $\alpha = 2$ rad/s^2. The distance $r_{A/B} = 0.6$ m.

(a) What are the rigid body's angular velocity and angular acceleration vectors?
(b) Determine the acceleration of point A relative to point B, first by using Eq. (17.9) and then by using Eq. (17.10).



Solution:



(a) By definition,

$$\boldsymbol{\omega} = 4\mathbf{k},$$

$$\boldsymbol{\alpha} = 6\mathbf{k}.$$

(b) $\mathbf{a}_{A/B} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) + \boldsymbol{\alpha} \times \mathbf{r}_{A/B}$

$$\mathbf{a}_{A/B} = 4\mathbf{k} \times (4\mathbf{k} \times 0.6\mathbf{i}) + 2\mathbf{k} \times 0.6\mathbf{i}$$

$$\underline{\mathbf{a}_{A/B} = -9.6\mathbf{i} + 1.2\mathbf{j} \text{ (m/s}^2\text{)}}.$$

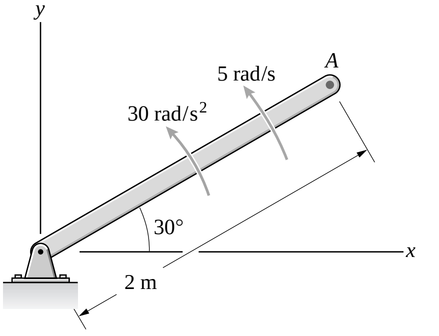
Using Eq. (17.10),

$$\mathbf{a}_{A/B} = \boldsymbol{\alpha} \times \mathbf{r}_{A/B} - \omega^2 r_{A/B}$$

$$= 2\mathbf{k} \times 0.6\mathbf{i} - 16(0.6)\mathbf{i}$$

$$\underline{\mathbf{a}_{A/B} = -9.6\mathbf{i} + 1.2\mathbf{j} \text{ (m/s}^2\text{)}}.$$

Problem 17.82 The bar rotates with a counterclockwise angular velocity of 5 rad/s and a counterclockwise angular acceleration of 30 rad/s^2. Determine the acceleration of A (a) by using Eq. (17.9) and (b) by using Eq. (17.10).



Solution:

(a) Eq. (17.9): $\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})$.

Substitute values:

$$\mathbf{a}_B = 0. \qquad \boldsymbol{\alpha} = 30\mathbf{k} \text{ (rad/s}^2\text{)},$$

$$\mathbf{r}_{A/B} = 2(\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ) = 1.732\mathbf{i} + \mathbf{j} \text{ (m)}.$$

$$\boldsymbol{\omega} = 5\mathbf{k} \text{ (rad/s)}.$$

Expand the cross products:

$$\boldsymbol{\alpha} \times \mathbf{r}_{A/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 30 \\ 1.732 & 1 & 0 \end{bmatrix} = -30\mathbf{i} + 52\mathbf{j} \text{ (m/s}^2\text{)}.$$

$$\boldsymbol{\omega} \times \mathbf{r}_{A/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ 1.732 & 1 & 0 \end{bmatrix} = -5\mathbf{i} + 8.66\mathbf{j} \text{ (m/s)}.$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ -5 & 8.66 & 0 \end{bmatrix} = -43.3\mathbf{i} - 25\mathbf{j} \text{ (m/s}^2\text{)}.$$

Collect terms: $\boxed{\mathbf{a}_A = -73.3\mathbf{i} + 27\mathbf{j} \text{ (m/s}^2\text{)}}$.

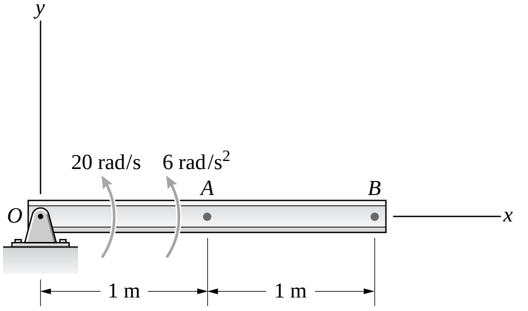
(b) Eq. (17.10): $\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} - \omega^2 \mathbf{r}_{A/B}$.

Substitute values, and expand the cross product as in Part (b) to obtain

$$\boxed{\mathbf{a}_A = -30\mathbf{i} + 52\mathbf{j} - (5^2)(1.732\mathbf{i} + \mathbf{j}) = -73.3\mathbf{i} + 27\mathbf{j} \text{ (m/s}^2\text{)}}$$

Problem 17.83 The bar rotates with a counterclockwise angular velocity of 20 rad/s and a counterclockwise angular acceleration of 6 rad/s^2.

(a) By applying Eq. (17.10) to point A and the fixed point O, determine the acceleration of A.
(b) By using the result of part (a) and Eq. (17.10), to points A and B, determine the acceleration point B.



Solution:

(a) $\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$

where

$\boldsymbol{\omega} = 20\mathbf{k}$ rad/s

$\boldsymbol{\alpha} = 6\mathbf{k}$ rad/s^2

$\mathbf{r}_{A/O} = 1\mathbf{i}$, and $\mathbf{a}_A = 0$

$\mathbf{a}_A = O + 6\mathbf{k} \times 1\mathbf{i} - 400(1\mathbf{i})$

$\underline{\mathbf{a}_A = -400\mathbf{i} + 6\mathbf{j}\ (\text{m/s}^2)}$.

(b) $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$

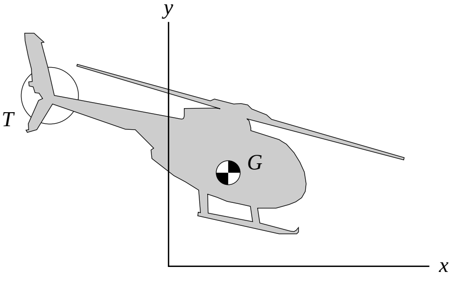
where

$\mathbf{r}_{B/A} = 1\mathbf{i}$

$\mathbf{a}_B = -400\mathbf{i} + 6\mathbf{j} + 6\mathbf{k} \times 1\mathbf{i} - 400(1\mathbf{i})$

$\underline{\mathbf{a}_B = -800\mathbf{i} + 12\mathbf{j}\ (\text{m/s}^2)}$.

Problem 17.84 The helicopter is in planar motion in the x–y plane. At the instant shown, the position of its center of mass G is $x = 2$ m, $y = 2.5$ m, its velocity is $\mathbf{v}_G = 12\mathbf{i} + 4\mathbf{j}$ (m/s), and its acceleration is $a_G = 2\mathbf{i} + 3\mathbf{j}$ (m/s^2). The position of point T where the tail rotor is mounted is $x = -3.5$ m, $y = 4.5$ m. The helicopter's angular velocity is 0.2 rad/s clockwise, and its angular acceleration is 0.1 rad/s^2 counterclockwise. What is the acceleration of point T?



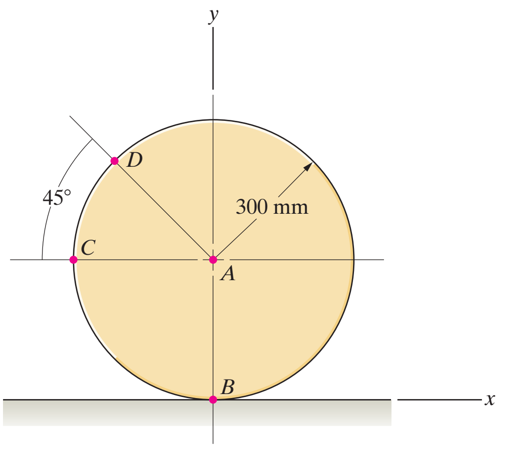
Solution: The acceleration of T is

$\mathbf{a}_T = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{T/G} - \omega^2 \mathbf{r}_{T/G}$;

$$\mathbf{a}_T = 2\mathbf{i} + 3\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.1 \\ -5.5 & 2 & 0 \end{vmatrix} - (0.2)^2(-5.5\mathbf{i} + 2\mathbf{j})$$

$= 2.02\mathbf{i} + 2.37\mathbf{j}\ (\text{m/s}^2)$.

Problem 17.85 Point A of the rolling disk is moving toward the right and accelerating toward the right. The magnitude of the velocity of point C is 2 m/s, and the magnitude of the acceleration of point C is 14 m/s^2. Determine the acceleration of points B and D. (See Active Example 17.5.)



Solution: First the velocity analysis

$$\mathbf{v}_C = \mathbf{v}_A + \omega \times \mathbf{r}_{C/A}$$

$$= r\omega\mathbf{i} - \omega\mathbf{k} \times (-r\mathbf{i}) = r\omega\mathbf{i} + r\omega\mathbf{j}$$

$$v_c = \sqrt{(r\omega)^2 + (r\omega)^2} = \sqrt{2}r\omega \Rightarrow \omega = \frac{v}{\sqrt{2}r} = \frac{2 \text{ m/s}}{\sqrt{2}(0.3 \text{ m})} = 4.71 \text{ rad/s}.$$

Now the acceleration analysis

$$\mathbf{a}_C = \mathbf{a}_A + \alpha \times \mathbf{r}_{C/A} - \omega^2\mathbf{r}_{C/A}$$

$$= r\alpha\mathbf{i} - \alpha\mathbf{k} \times (-r\mathbf{i}) - \omega^2(-r\mathbf{i}) = (r\alpha + \omega^2 r)\mathbf{i} + (\alpha r)\mathbf{j}$$

$$a_C = \sqrt{(\alpha r + \omega^2 r)^2 + (\alpha r)^2} = r\sqrt{(\alpha + \omega^2)^2 + \alpha^2}$$

$$(14 \text{ m/s}^2)^2 = (0.3 \text{ m})^2[(\alpha + [4.71]^2)^2 + \alpha^2]$$

Solving this quadratic equation for α we find $\alpha = 20.0$ rad/s^2.

Now

$$\mathbf{a}_B = r\omega^2\mathbf{j} = (0.3 \text{ m})(4.71 \text{ rad/s})^2\mathbf{j} \quad \boxed{\mathbf{a}_B = 6.67\mathbf{j} \text{ (m/s}^2)}$$

$$\mathbf{a}_D = \mathbf{a}_A + \alpha \times \mathbf{r}_{D/A} - \omega^2\mathbf{r}_{D/A}$$

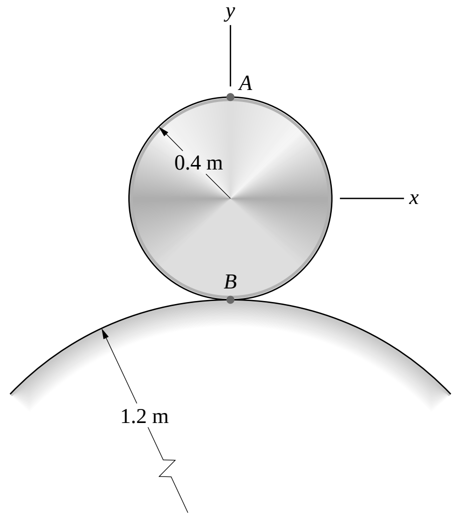
$$= r\alpha\mathbf{i} - \alpha\mathbf{k} \times (-r\cos 45^\circ\mathbf{i} + 4\sin 45^\circ\mathbf{j}) - \omega^2(-r\cos 45^\circ\mathbf{i} + 4\sin 45^\circ\mathbf{j})$$

$$= (r[1 + \sin 45^\circ]\alpha + r\omega^2\cos 45^\circ)\mathbf{i} + (r\alpha\cos 45^\circ - r\omega^2\sin 45^\circ)\mathbf{j}$$

Putting in the numbers we find $\boxed{\mathbf{a}_D = 14.9\mathbf{i} - 0.480\mathbf{j} \text{ (m/s}^2).}$

Problem 17.86 The disk rolls on the circular surface with a constant clockwise angular velocity of 1 rad/s. What are the accelerations of points A and B?

Strategy: Begin by determining the acceleration of the center of the disk. Notice that the center moves in a circular path and the magnitude of its velocity is constant.



Solution:

$\mathbf{v}_B = 0$

$\mathbf{v}_0 = \mathbf{v}_B + \omega\mathbf{k} \times \mathbf{r}_{O/B} = (-1\mathbf{k}) \times (0.4\mathbf{j})$

$\mathbf{v}_0 = 0.4\ \mathbf{i}$ m/s

Point O moves in a circle at constant speed. The acceleration of O is

$\mathbf{a}_0 = -v_0^2/(R+r)\mathbf{j} = (-0.16)/(1.2+0.4)\mathbf{j}$

$\underline{\mathbf{a}_0 = -0.1\mathbf{j}\ (\text{m/s}^2)}$.

$\mathbf{a}_B = \mathbf{a}_0 - \omega^2\mathbf{r}_{B/O} = -0.1\mathbf{j} - (1)^2(-0.4)\mathbf{j}$

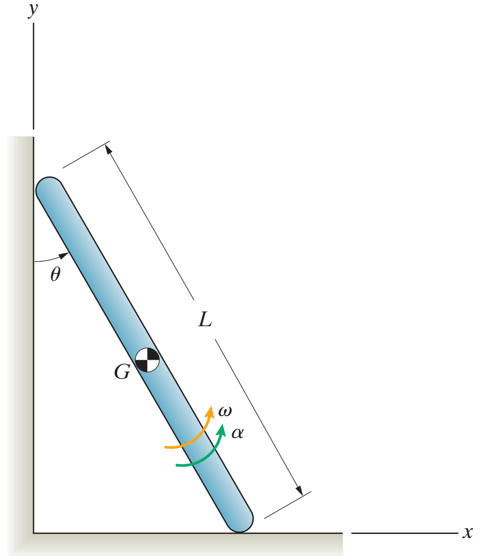
$\underline{\mathbf{a}_B = 0.3\mathbf{j}\ (\text{m/s}^2)}$.

$\mathbf{a}_A = \mathbf{a}_0 - \omega^2\mathbf{r}_{A/O} = -0.1\mathbf{j} - (1)^2(0.4)\mathbf{j}$

$\underline{\mathbf{a}_A = -0.5\mathbf{j}\ (\text{m/s}^2)}$.

Problem 17.87 The length of the bar is $L = 4$ m and the angle $\theta = 30^\circ$. The bar's angular velocity is $\omega = 1.8$ rad/s and its angular acceleration is $\alpha = 6$ rad/s^2. The endpoints of the bar slide on the plane surfaces. Determine the acceleration of the midpoint G.

Strategy: Begin by applying Eq. (17.10) to the endpoints of the bar to determine their accelerations.



Solution: Call the top point D and the bottom point B.

$$\mathbf{a}_D = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{D/B} - \omega^2\mathbf{r}_{D/B}$$

Put in the known constraints

$a_D\mathbf{j} = a_B\mathbf{i} + \alpha\mathbf{k} \times L(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) - \omega^2 L(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$

$a_D\mathbf{j} = (a_B - \alpha L\cos\theta + \omega^2 L\sin\theta)\mathbf{i} + (-\alpha L\sin\theta - \omega^2 L\cos\theta)\mathbf{j}$

Equating components we have

$a_B = \alpha L\cos\theta - \omega^2 L\sin\theta = (6)(4)\cos 30^\circ - (1.8)^2(4)\sin 30^\circ$

$= 14.3\ \text{m/s}^2$

$a_D = \alpha L\sin\theta - \omega^2 L\cos\theta = (6)(4)\sin 30^\circ - (1.8)^2(4)\cos 30^\circ$

$= -23.2\ \text{m/s}^2$

Now we can use either point as a base point to find the acceleration of point G. We will use B as the base point.

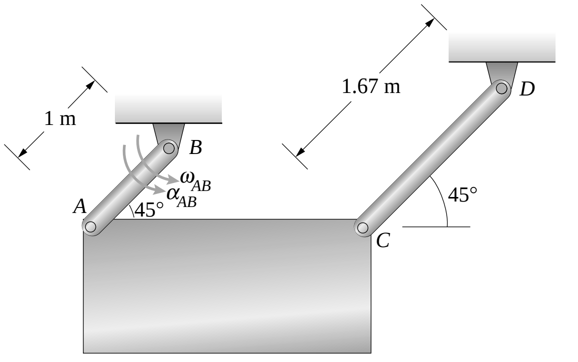
$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{G/B} - \omega^2\mathbf{r}_{GB}$

$= 14.3\mathbf{i} + 6\mathbf{k} \times (2)(-\sin 30^\circ\mathbf{i} + \cos 30^\circ\mathbf{j})$

$- (1.8)^2(2)(-\sin 30^\circ\mathbf{i} + \cos 30^\circ\mathbf{j})$

$\boxed{\mathbf{a}_G = 7.15\mathbf{i} - 11.6\mathbf{j}(\text{m/s}^2.)}$

Problem 17.88 The angular velocity and angular acceleration of bar AB are $\omega_{AB} = 2$ rad/s and $\alpha_{AB} = 10$ rad/s^2. The dimensions of the rectangular plate are 1 m $\times$ 2 m. What are the angular velocity and angular acceleration of the rectangular plate?



Solution: The instantaneous center for bar AB is point B, by definition. The instantaneous center for bar CD is point D, by definition. The velocities at points A and C are normal to the bars AB and CD, respectively. However, by inspection these bars are parallel at the instant shown, so that lines perpendicular to the velocities at A and C will never intersect—the instantaneous center of the plate AC is at infinity, hence *the plate only translates at the instant shown*, and $\boxed{\omega_{AC} = 0}$. If the plate is not rotating, the velocity at every point on the plate must be the same, and in particular, the vector velocity at A and C must be identical. The vector A/B is

$$\mathbf{r}_{A/B} = -\mathbf{i}\cos 45^\circ - \mathbf{j}\sin 45^\circ = \left(\frac{-1}{\sqrt{2}}\right)(\mathbf{i}+\mathbf{j}) \text{ (m)}.$$

The velocity at point A is

$$\mathbf{v}_A = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B} = \frac{-\omega_{AB}}{\sqrt{2}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \sqrt{2}(\mathbf{i}-\mathbf{j}) \text{ (m/s)}.$$

The vector C/D is

$$\mathbf{r}_{C/D} = (1.67)\,(-\mathbf{i}\cos 45^\circ - \mathbf{j}\sin 45^\circ) = -1.179(\mathbf{i}+\mathbf{j}) \text{ (m)}.$$

The velocity at point C is

$$\mathbf{v}_C = -1.179\omega_{CD} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 1.179\omega_{CD}(\mathbf{i}-\mathbf{j}) \text{ (m/s)}.$$

Equate the velocities $\mathbf{v}_C = \mathbf{v}_A$, separate components and solve: $\omega_{CD} = 1.2$ rad/s. Use Eq. (17.10) to determine the accelerations. The acceleration of point A is

$$\mathbf{a}_A = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} - \omega_{AB}^2 \mathbf{r}_{A/B} = -\frac{10}{\sqrt{2}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + \left(\frac{2^2}{\sqrt{2}}\right)(\mathbf{i}+\mathbf{j})$$

$$= 9.9\mathbf{i} - 4.24\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration of point C relative to point A is

$$\mathbf{a}_C = \mathbf{a}_A + \boldsymbol{\alpha}_{AC} \times \mathbf{r}_{C/A} = \mathbf{a}_A + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AC} \\ 2 & 0 & 0 \end{bmatrix}$$

$$= 9.9\mathbf{i} + (2\alpha_{AC} - 4.24)\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration of point C relative to point D is $\mathbf{a}_C = \mathbf{a}_D + \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{C/D} - \omega_{CD}^2 \mathbf{r}_{C/D}$. Noting $\mathbf{a}_D = 0$,

$$\mathbf{a}_C = -1.179\alpha_{CD} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} + 1.179\omega_{CD}^2(\mathbf{i}+\mathbf{j})$$

$$= (1.179\alpha_{CD} + 1.697)\mathbf{i} + (-1.179\alpha_{CD} + 1.697)\mathbf{j} \text{ (m/s}^2\text{)}.$$

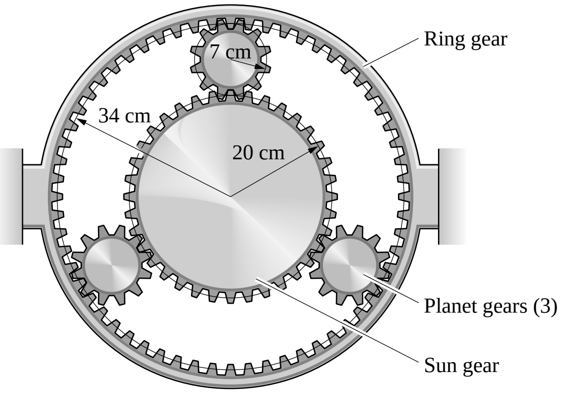
Equate the two expressions for the acceleration at point C and separate components:

$$(-9.9 + 1.179\alpha_{CD} + 1.697)\mathbf{i} = 0,$$

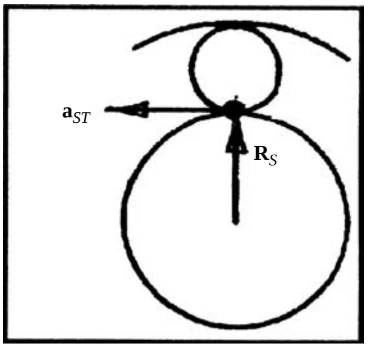
$$(2\alpha_{AC} - 4.24 + 1.179\alpha_{CD} - 1.697)\mathbf{j} = 0.$$

Solve: $\boxed{\alpha_{AC} = -1.13 \text{ (rad/s}^2\text{)}}$ (clockwise), $\alpha_{CD} = 6.96$ (rad/s^2) (counterclockwise).

Problem 17.89 The ring gear is stationary, and the sun gear has an angular acceleration of 10 rad/s^2 in the counterclockwise direction. Determine the angular acceleration of the planet gears.



Solution: The strategy is to use the tangential acceleration at the point of contact of the sun and planet gears, together with the constraint that the point of contact of the planet gear and ring gear is stationary, to determine the angular acceleration of the planet gear. The tangential acceleration of the sun gear at the point of contact with the top planet gear is



$$\mathbf{a}_{ST} = \boldsymbol{\alpha} \times \mathbf{r}_S = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 0 & 20 & 0 \end{bmatrix} = -200\mathbf{i} \text{ (cm/s}^2\text{)}.$$

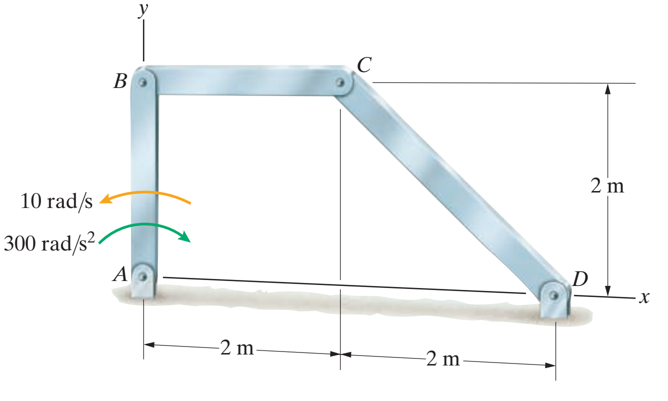
This is also the tangential acceleration of the planet gear at the point of contact. At the contact with the ring gear, the planet gears are stationary, hence the angular acceleration of the planet gear satisfies

$$\boldsymbol{\alpha}_P \times (-2\mathbf{r}_P) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_P \\ 0 & -14 & 0 \end{bmatrix} = -200\mathbf{i}$$

from which

$$\boxed{\alpha_P = -\frac{200}{14} = -14.29 \text{ (rad/s}^2\text{)}} \text{ (clockwise).}$$

Problem 17.90 In Example 17.6, what is the acceleration of the midpoint of bar BC?



Solution: From Example 17.6 we know that

$$\omega_{BC} = -10 \text{ rad/s},$$

$$\alpha_{BC} = 100 \text{ rad/s}^2,$$

$$\omega_{CD} = 10 \text{ rad/s},$$

$$\alpha_{CD} = -100 \text{ rad/s}^2.$$

To find the acceleration of the midpoint "G" of bar BC, we have

$$\mathbf{a}_B = \mathbf{a} + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}{}^2\mathbf{r}_{B/A}$$

$$= 0 - (300 \text{ rad/s}^2)\ \mathbf{k} \times (2 \text{ m})\mathbf{j} - (10 \text{ rad/s})^2(2 \text{ m})\mathbf{j}$$

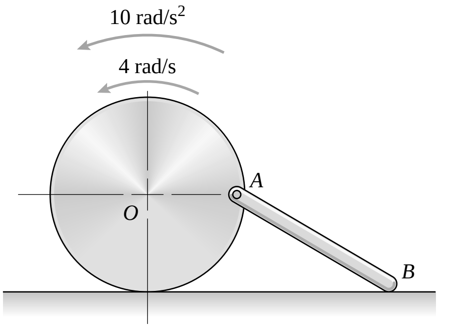
$$= (600\mathbf{i} - 200\mathbf{j}) \text{ m/s}^2$$

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G/B} - \omega_{BC}{}^2\mathbf{r}_{G/B}$$

$$= (600\mathbf{i} - 200\mathbf{j}) + (100\mathbf{k}) \times (1\mathbf{i}) - (-10)^2(1\mathbf{i})$$

$$\boxed{\mathbf{a}_G = (500\mathbf{i} - 100\mathbf{j}) \text{ m/s}^2.}$$

Problem 17.91 The 1-m-diameter disk rolls, and point B of the 1-m-long bar slides, on the plane surface. Determine the angular acceleration of the bar and the acceleration of point B.



Solution: Choose a coordinate system with the origin at O, the center of the disk, with x axis parallel to the horizontal surface. The point P of contact with the surface is stationary, from which

$$\mathbf{v}_P = 0 = \mathbf{v}_O + \boldsymbol{\omega}_O \times -\mathbf{R} = \mathbf{v}_O + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_O \\ 0 & -0.5 & 0 \end{bmatrix} = \mathbf{v}_O + 2\mathbf{i},$$

from which $\mathbf{v}_O = -2\mathbf{i}$ (m/s). The velocity at A is

$$\mathbf{v}_A = \mathbf{v}_O + \boldsymbol{\omega}_O \times \mathbf{r}_{A/O} = -2\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_O \\ 0.5 & 0 & 0 \end{bmatrix} = -2\mathbf{i} + 2\mathbf{j} \text{ (m/s)}$$

The motion at point B is constrained to be parallel to the x axis. The line perpendicular to the velocity of B is parallel to the y axis. The line perpendicular to the velocity at A forms an angle at 45° with the x axis. From geometry, the line from A to the fixed center is the hypotenuse of a right triangle with base $\cos 30^\circ = 0.866$ and interior angles 45°. The coordinates of the fixed center are $(0.5 + 0.866, 0.866) = (1.366, 0.866)$ in. The vector from the instantaneous center to the point A is $\mathbf{r}_{A/C} = \mathbf{r}_A - \mathbf{r}_C = -0.866\mathbf{i} - 0.866\mathbf{j}$ (m). The angular velocity of the bar AB is obtained from

$$\mathbf{v}_A = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ -0.866 & -0.866 & 0 \end{bmatrix}$$

$$= 0.866\omega_{AB}\mathbf{i} - 0.866\omega_{AB}\mathbf{j} \text{ (m/s)},$$

from which

$$\omega_{AB} = -\frac{2}{0.866} = -2.31 \text{ (rad/s)}.$$

The acceleration of the center of the rolling disk is $\mathbf{a}_O = -\alpha R\mathbf{i} = -10(0.5)\mathbf{i} = -5\mathbf{i}$ (m/s^2). The acceleration of point A is

$$\mathbf{a}_A = \mathbf{a}_O + \boldsymbol{\alpha}_O \times \mathbf{r}_{A/O} - \omega_O^2 \mathbf{r}_{A/O}$$

$$= -5\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ 0.5 & 0 & 0 \end{bmatrix} - 16(0.5)\mathbf{i}$$

$$= -13\mathbf{i} + 5\mathbf{j} \text{ (m/s}^2\text{)}.$$

The vector *B/A* is

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = (0.5 + \cos\theta)\mathbf{i} - 0.5\mathbf{j} - 0.5\mathbf{i}$$

$$= 0.866\mathbf{i} - 0.5\mathbf{j} \text{ (m)}.$$

The acceleration of point B is

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} - \omega_{AB}^2 \mathbf{r}_{A/B}$$

$$= -13\mathbf{i} + 5\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ -\cos\theta & \sin\theta & 0 \end{bmatrix}$$

$$- \omega_{AB}^2(-\mathbf{i}\cos\theta + \mathbf{j}\sin\theta).$$

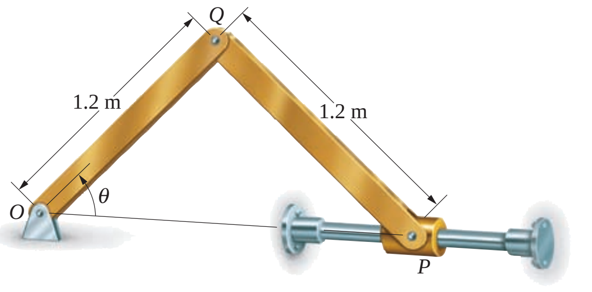
The constraint on B insures that the acceleration of B will be parallel to the x axis. Separate components:

$$a_B = -13 + 0.5\alpha_{AB} - \omega_{AB}^2(0.866),$$

$$0 = 5 + 0.866\alpha_{AB} + 0.5\omega_{AB}^2.$$

Solve: $\boxed{\alpha_{AB} = -8.85 \text{ (rad/s}^2\text{)}}$, where the negative sign means a clockwise rotation. $\boxed{\mathbf{a}_B = -22.04\mathbf{i} \text{ (m/s}^2\text{)}}$

Problem 17.92 If $\theta = 45^\circ$ and sleeve P is moving to the right with a constant velocity of 2 m/s, what are the angular accelerations of the bars OQ and PQ?



Solution:

$\mathbf{v}_0 = \mathbf{a}_0 = 0$, $\mathbf{v}_p = 2\mathbf{i}$, $\mathbf{a}_p = 0$

$\mathbf{r}_{Q/o} = 1.2\cos 45^\circ \mathbf{i} + 1.2 \sin 45^\circ \mathbf{j}$ m

$\mathbf{r}_{p/Q} = 1.2\cos 45^\circ \mathbf{i} - 1.2 \sin 45^\circ \mathbf{j}$ m

$\mathbf{v}_Q = \omega_{oQ}\mathbf{k} \times \mathbf{r}_{Q/o} = \omega_{oQ}\mathbf{k} \times (0.848\mathbf{i} + 0.848\mathbf{j})$

$$\begin{cases} v_{Qx} = -0.848\omega_{oQ} & \textbf{(1)} \\ v_{Qy} = 0.848\omega_{oQ} & \textbf{(2)} \end{cases}$$

$\mathbf{v}_p = \mathbf{v}_Q + \omega_{pQ}\mathbf{k} \times (0.848\mathbf{i} - 0.848\mathbf{j})$

$$\begin{cases} 2 = v_{Qx} + 0.848\omega_{pQ} & \textbf{(3)} \\ O = v_{Qy} + 0.848\omega_{pQ} & \textbf{(4)} \end{cases}$$

Solving eqns. (1)–(4),

$\omega_{oQ} = -1.179$ rad/s, $\omega_{pQ} = 1.179$ rad/s

$v_{Qx} = 1$ m/s $v_{Qy} = -1$ m/s

$\mathbf{a}_Q = \boldsymbol{\alpha}_{oQ} \times \mathbf{r}_{Q/o} - \omega_{oQ}^2 \mathbf{r}_{Q/o}$

$$\begin{cases} a_{Qx} = -0.848\alpha_{oQ} - 0.848\omega_{oQ}^2 & \textbf{(5)} \\ a_{Qy} = 0.848\alpha_{oQ} - 0.848\omega_{oQ}^2 & \textbf{(6)} \end{cases}$$

Also,

$\mathbf{a}_p = 0 = \mathbf{a}_Q + \alpha_{pQ}\mathbf{k} \times \mathbf{r}_{p/Q} - \omega_{pQ}^2 \mathbf{r}_{p/Q}$

$$\begin{cases} 0 = a_{Qx} + 0.848\alpha_{pQ} - 0.848\omega_{pQ}^2 & \textbf{(7)} \\ 0 = a_{Qy} + 0.848\alpha_{pQ} + 0.848\omega_{pQ}^2 & \textbf{(8)} \end{cases}$$

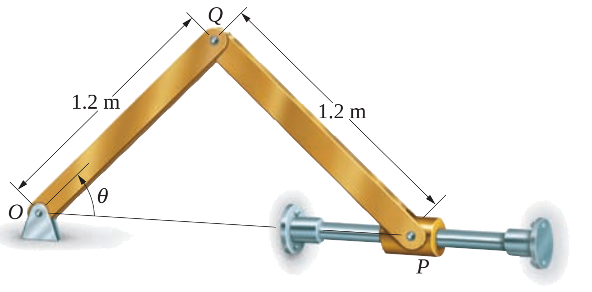
Solving eqns. (5)–(8), we get

$a_{Qx} = 0$, $a_{Qy} = 0$

$\underline{\alpha_{oQ} = 1.39 \text{ rad/s}^2}$ (clockwise)

$\underline{\alpha_{pQ} = 1.39 \text{ rad/s}^2}$ (counterclockwise)

Problem 17.93 Consider the system shown in Problem 17.92. If $\theta = 50^\circ$ and bar OQ has a constant clockwise angular velocity of 1 rad/s, what is the acceleration of sleeve P?



Solution:

$\boldsymbol{\omega}_{oQ} = -1\mathbf{k}$ rad/s, $\boldsymbol{\alpha}_{oQ} = 0$, $\mathbf{a}_0 = 0$

$$\mathbf{a}_Q = \mathbf{a}_0 + \boldsymbol{\alpha}_{oQ} \times \mathbf{r}_{Q/o} - \omega^2 \mathbf{r}_{Q/o}$$

$$\mathbf{a}_Q = 0 + 0 - (1)^2(1.2\cos 50^\circ \mathbf{i} + 1.2 \sin 50^\circ \mathbf{j})$$

$$\mathbf{a}_Q = -0.771\mathbf{i} - 0.919\mathbf{j} \text{ m/s}^2$$

$$\mathbf{a}_p = \mathbf{a}_Q + \alpha_{Qp}\mathbf{k} \times \mathbf{r}_{p/Q} - \omega_{Qp}^2 \mathbf{r}_{p/Q}$$

where $\mathbf{a}_p = a_p \mathbf{i}$

$$\mathbf{r}_{p/Q} = 1.2\cos 50^\circ \mathbf{i} - 1.2 \sin 50^\circ \mathbf{j}$$

$$\mathbf{i}: \ a_p = -0.771 + 1.2\alpha_{Qp} \sin 50^\circ - \omega_{Qp}^2 (1.2)\cos 50^\circ \quad \textbf{(1)}$$

$$\mathbf{j}: \ 0 = -0.919 + 1.2\alpha_{Qp}\cos 50^\circ + \omega_{Qp}^2(1.2)\sin 50^\circ \quad \textbf{(2)}$$

We have two eqns in three unknowns a_p, α_{Qp}, ω_{Qp}.

We need another eqn. To get it, we use the velocity relationships and determine ω_{QP}. Note $\mathbf{v}_p = v_p \mathbf{i}$.

$$\mathbf{v}_Q = \mathbf{v}_0 + \omega_{oQ} \times r_{Q/o} \quad \mathbf{v}_0 = 0$$

$$= (-1\mathbf{k}) \times [(1.2\cos 50^\circ)\mathbf{i} + (1.2 \sin 50)\mathbf{j}]$$

$$= .919\mathbf{i} - 0.771\mathbf{j} \text{ (m/s)}.$$

$$\mathbf{v}_P = \mathbf{v}_Q + \boldsymbol{\omega}_{QP} \times \mathbf{r}_{P/Q}$$

$$= \mathbf{v}_Q + \omega_{QP}\mathbf{k} \times (1.2\cos 50^\circ \mathbf{i} - 1.2\sin 50^\circ \mathbf{j}).$$

$$\mathbf{i}: \ v_P = 0.919 + 1.2\omega_{QP}\sin 50^\circ$$

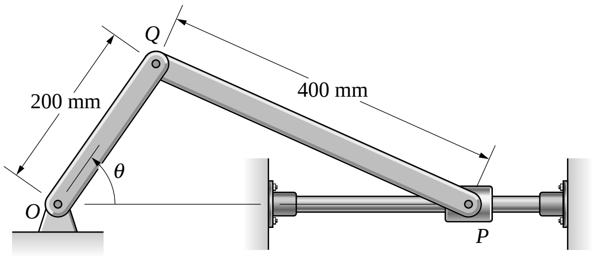
$$\mathbf{j}: \ O = -0.771 + 1.2\omega_{QP}\cos 50^\circ$$

Solving, $v_P = 1.839$ m/s, $\omega_{QP} = 1$ rad/s. Now going back to eqns. (1) and (2), we solve to get

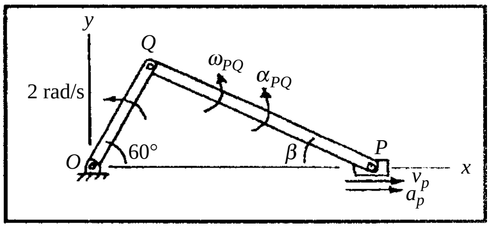
$$a_P = -1.54 \text{ m/s}^2$$

$$\alpha_{QP} = 0, \quad \text{(to the left)}$$

Problem 17.94 The angle $\theta = 60^\circ$, and bar OQ has a constant counterclockwise angular velocity of 2 rad/s. What is the angular acceleration of the bar PQ?



Solution: By applying the law of sines, the angle $\beta = 25.7^\circ$
The velocity of Q is $\mathbf{v}_Q = \mathbf{v}_0 + \boldsymbol{\omega}_{0Q} \times \mathbf{r}_{Q/O}$



$$\mathbf{v}_Q = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0.2\cos 60^\circ & 0.2\sin 60^\circ & 0 \end{vmatrix}$$

$$= -0.4\sin 60^\circ \mathbf{i} + 0.4\cos 60^\circ \mathbf{j}.$$

The velocity of P is

$$v_P\mathbf{i} = \mathbf{v}_Q + \boldsymbol{\omega}_{PQ} \times \mathbf{r}_{P/Q}$$

$$= -0.4\sin 60^\circ \mathbf{i} + 0.4\cos 60^\circ \mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{PQ} \\ 0.4\cos\beta & -0.4\sin\beta & 0 \end{vmatrix}.$$

Equating $\mathbf{j}$ components, we get $0 = 0.4\cos 60^\circ + 0.4\omega_{PQ}\cos\beta$, and obtain $\omega_{PQ} = -0.555$ rad/s. The acceleration of Q is

$$\mathbf{a}_Q = \mathbf{a}_0 + \alpha_{0Q} \times \mathbf{r}_{Q/0} - \omega_{0Q}^2 \mathbf{r}_{Q/0},$$

$$\text{or } \mathbf{a}_Q = 0 + 0 - (2)^2(0.2\cos 60^\circ \mathbf{i} + 0.2\sin 60^\circ \mathbf{j})$$

$$= -0.8\cos 60^\circ \mathbf{i} - 0.8\sin 60^\circ \mathbf{j}.$$

The acceleration of P is

$$a_P\mathbf{i} = \mathbf{a}_Q + \boldsymbol{\alpha}_{PQ} \times \mathbf{r}_{P/Q} - \omega_{PQ}^2 \mathbf{r}_{P/Q}$$

$$= -0.8\cos 60^\circ \mathbf{i} - 0.8\sin 60^\circ \mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{PQ} \\ 0.4\cos\beta & -0.4\sin\beta & 0 \end{vmatrix}$$

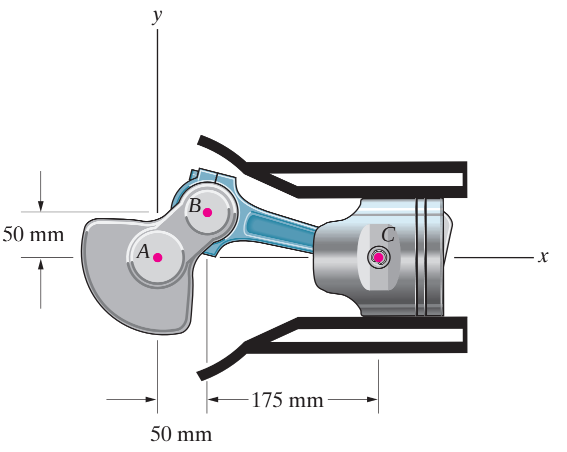
$$- (-0.555)^2(0.4\cos\beta \mathbf{i} - 0.4\sin\beta \mathbf{j}).$$

Equating $\mathbf{j}$ components

$$0 = -0.8\sin 60^\circ + 0.4\alpha_{PQ}\cos\beta + (0.555)^2 0.4\sin\beta.$$

Solving, we obtain $\alpha_{PQ} = 1.77$ rad/s^2.

Problem 17.95 At the instant shown, the piston's velocity and acceleration are $\mathbf{v}_C = -14\mathbf{i}$ (m/s) and $\mathbf{a}_C = -2200\mathbf{i}$ (m/s^2). What is the angular acceleration of the crank AB?



Solution: The velocity analysis:

$$\mathbf{v}_B = \mathbf{V}_A + \omega_{AB} \times \mathbf{r}_{B/A}$$

$$= 0 + \omega_{AB}\mathbf{k} \times (0.05\mathbf{i} + 0.05\mathbf{j})$$

$$= (-0.05\omega_{AB}\mathbf{i} + 0.05\omega_{AB}\mathbf{j})$$

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC} \times \mathbf{r}_{C/B}$$

$$= (-0.05\omega_{AB}\mathbf{i} + 0.05\omega_{AB}\mathbf{j}) + \omega_{BC}\mathbf{k} \times (0.175\mathbf{i} - 0.05\mathbf{j})$$

$$= (-0.05\omega_{AB} + 0.05\omega_{AB})\mathbf{i} + (0.05\omega_{AB} + 0.175\omega_{BC})\mathbf{j} = -14\mathbf{i}$$

Equating components and solving we find that $\omega_{AB} = 218$ rad/s, $\omega_{BC} = -62.2$ rad/s. The acceleration analysis:

$$\mathbf{a}_B = \mathbf{a}_A + \alpha_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}{}^2\mathbf{r}_{B/A}$$

$$= 0 + \alpha_{AB}\mathbf{k} \times (0.05\mathbf{i} + 0.05\mathbf{j}) - (218)^2(0.05\mathbf{i} + 0.05\mathbf{j})$$

$$= (-0.05\alpha_{AB} - 2370)\mathbf{i} + (0.05\alpha_{AB} - 2370)\mathbf{j}$$

$$\mathbf{a}_C = \mathbf{a}_B + \alpha_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}{}^2\mathbf{r}_{C/B}$$

$$= (-0.05\alpha_{AB} - 2370)\mathbf{i} + (0.05\alpha_{AB} - 2370)\mathbf{j} + \alpha_{BC}\mathbf{k} \times (0.175\mathbf{i} - 0.05\mathbf{j})$$

$$- (-62.2)^2(0.175\mathbf{i} - 0.05\mathbf{j})$$

$$= (-0.05\alpha_{AB} - 2370 + 0.05\alpha_{BC} - 678)\mathbf{i}$$

$$+ (0.05\alpha_{AB} - 2370 + 0.175\alpha_{BC} + 194)\mathbf{j}$$

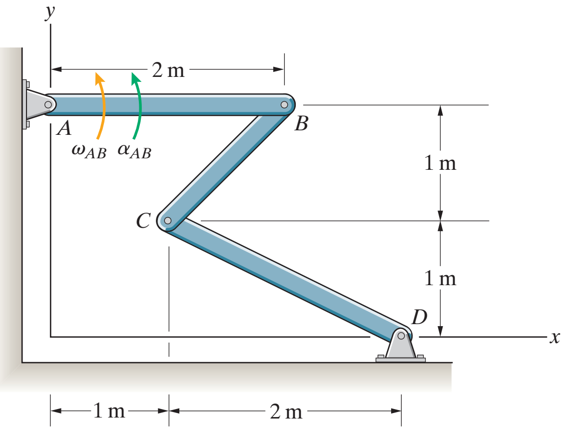
$$= -2200\mathbf{i}$$

Equating components and solving, we find

$$\alpha_{AB} = -3530 \text{ rad/s}^2, \quad \alpha_{BC} = 13{,}500 \text{ rad/s}^2$$

Thus $\boxed{\alpha_{AB} = 3530 \text{ rad/s}^2 \text{ clockwise.}}$

Problem 17.96 The angular velocity and angular acceleration of bar AB are $\omega_{AB} = 4$ rad/s and $\alpha_{AB} = -6$ rad/s^2. Determine the angular accelerations of bars BC and CD.



Solution: From 17.38 we know $\omega_{BC} = 2.67$ rad/s CCW, $\omega_{CD} = 2.67$ rad/s CW

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}{}^2\mathbf{r}_{B/A}$$

$$= 0 + (-6 \text{ rad/s}^2)\mathbf{k} \times (2 \text{ m})\mathbf{i} - (4 \text{ rad/s})^2(2 \text{ m})\mathbf{i} = (-32\mathbf{i} - 12\mathbf{j}) \text{ m/s}^2$$

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}{}^2\mathbf{r}_{C/B}$$

$$= (-32\mathbf{i} - 12\mathbf{j}) \text{ m/s}^2 + \alpha\mathbf{k} \times (-\mathbf{i} - \mathbf{j}) \text{ m} - (2.67 \text{ rad/s})^2(-\mathbf{i} - \mathbf{j}) \text{ m}$$

$$= [-24.9 \text{ m/s}^2 + \{1 \text{ m}\}\alpha_{BC}]\mathbf{i} + [-4.89 \text{ m/s}^2 - \{1 \text{ m}\}\alpha_{BC}]\mathbf{j}$$

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{D/C} - \omega_{CD}{}^2\mathbf{r}_{D/C}$$

$$= [-24.9 \text{ m/s}^2 + \{1 \text{ m}\}\alpha_{BC}]\mathbf{i} + [-4.89 \text{ m/s}^2 - \{1 \text{ m}\}\alpha_{BC}]\mathbf{j}$$

$$+ \alpha_{CD}\mathbf{k} \times (2\mathbf{i} - \mathbf{j}) \text{ m} - (2.67 \text{ rad/s})^2(2\mathbf{i} - \mathbf{j}) \text{ m}$$

$$= [-39.1 \text{ m/s}^2 + \{1 \text{ m}\}(\alpha_{BC} + \alpha_{CD})]\mathbf{i}$$

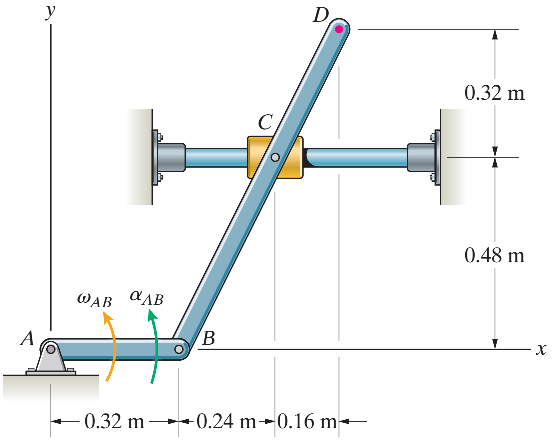
$$+ [2.22 \text{ m/s}^2 - \{1 \text{ m}\}\alpha_{BC} + \{2 \text{ m}\}\alpha_{CD}]\mathbf{j}$$

Since point D is fixed we have

$$\left.\begin{array}{l} 2.22 \text{ m/s}^2 - \{1 \text{ m}\}\alpha_{BC} + \{2 \text{ m}\}\alpha_{CD} = 0 \\ -39.1 \text{ m/s}^2 + \{1 \text{ m}\}(\alpha_{BC} + \alpha_{CD}) = 0 \end{array}\right\}$$

$$\Rightarrow \boxed{\begin{array}{l} \alpha_{BC} = 26.8 \text{ rad/s}^2 \text{ CCW} \\ \alpha_{CD} = 12.30 \text{ rad/s}^2 \text{ CCW} \end{array}}$$

Problem 17.97 The angular velocity and angular acceleration of bar AB are $\omega_{AB} = 2$ rad/s and $\alpha_{AB} = 8$ rad/s^2. What is the acceleration of point D?



Solution: First we must do a velocity analysis to find the angular velocity of BCD

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + (2\text{ rad/s})\mathbf{k} \times (0.32\text{ m})\mathbf{i} = (0.64\text{ m/s})\mathbf{j}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BCD} \times \mathbf{r}_{C/B} = (0.64\text{ m/s})\mathbf{j} + \omega_{BCD}\mathbf{k} \times (0.24\mathbf{i} + 0.48\mathbf{j})\text{ m}$$

$$= (-0.48\text{ m})\omega_{BCD}\mathbf{i} + (0.64\text{ m/s} + \{0.24\text{ m}\}\omega_{BCD})\mathbf{j}$$

Since C cannot move in the $\mathbf{j}$ direction we know

$$0.64\text{ m/s} + \{0.24\text{ m}\}\omega_{BCD} = 0 \Rightarrow \omega_{BCD} = -2.67\text{ rad/s}$$

Now we do the acceleration analysis

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}{}^2\mathbf{r}_{B/A}$$

$$= 0 + (8\text{ rad/s}^2)\mathbf{k} \times (0.32\text{ m})\mathbf{i} - (2\text{ rad/s})^2(0.32\text{ m})\mathbf{i}$$

$$= (-1.28\mathbf{i} + 2.56\mathbf{j})\text{ m/s}^2$$

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}{}^2\mathbf{r}_{C/B}$$

$$= (-1.28\mathbf{i} + 2.56\mathbf{j})\text{ m/s}^2 + \alpha_{BCD}\mathbf{k} \times (0.24\mathbf{i} + 0.48\mathbf{j})\text{ m}$$

$$- (-2.67\text{ rad/s})^2(0.24\mathbf{i} + 0.48\mathbf{j})\text{ m}$$

$$= (-2.99\text{ m/s}^2 - \{0.48\text{ m}\}\alpha_{BCD})\mathbf{i}$$

$$+ (-0.853\text{ m/s}^2 + \{0.24\text{ m}\}\alpha_{BCD})\mathbf{j}$$

Since C cannot move in the $\mathbf{j}$ direction we know

$$-0.853\text{ m/s}^2 + \{0.24\text{ m}\}\alpha_{BCD} = 0 \Rightarrow \alpha_{BCD} = 3.56\text{ rad/s}^2$$

Now we can find the acceleration of point D

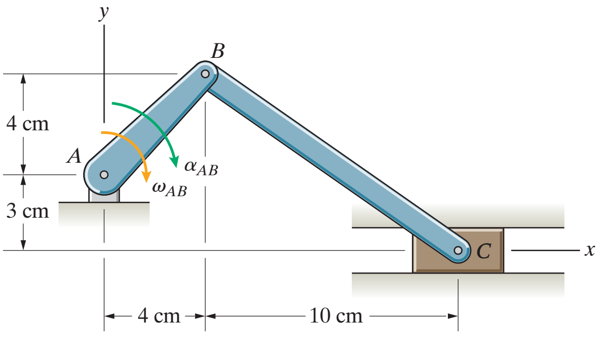
$$\mathbf{a}_D = \mathbf{a}_B + \boldsymbol{\alpha}_{BCD} \times \mathbf{r}_{D/B} - \omega_{BCD}{}^2\mathbf{r}_{D/B}$$

$$= (-1.28\mathbf{i} + 2.56\mathbf{j})\text{ m/s}^2 + (3.56\text{ rad/s}^2)\mathbf{k} \times (0.4\mathbf{i} + 0.8\mathbf{j})\text{ m}$$

$$- (-2.67\text{ rad/s})^2(0.4\mathbf{i} + 0.8\mathbf{j})\text{ m}$$

$$\boxed{\mathbf{a}_D = (-.697\mathbf{i} - 1.71\mathbf{j})\text{ m/s}^2}$$

Problem 17.98 The angular velocity $\omega_{AB} = 6$ rad/s. If the acceleration of the slider C is zero at the instant shown, what is the angular acceleration α_{AB}?



Solution: The velocity analysis:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$= 0 + (-6)\mathbf{k} \times (4\mathbf{i} + 4\mathbf{j})$$

$$= (24\mathbf{i} - 24\mathbf{j})$$

$$\mathbf{v}_c = \mathbf{v}_B + \boldsymbol{\omega}_{BC} + \mathbf{r}_{C/B}$$

$$= (24\mathbf{i} - 24\mathbf{j}) + \omega_{BC}\mathbf{k} \times (10\mathbf{i} - 7\mathbf{j}) = (24 + 7\omega_{BC})\mathbf{i} + (24 + 10\omega_{BC})\mathbf{j}$$

Since C cannot move in the $\mathbf{j}$ direction, we set the $\mathbf{j}$ component to zero and find that

$$\omega_{BC} = -2.4 \text{ rad/s.}$$

The acceleration analysis:

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - {\omega_{AB}}^2\mathbf{r}_{B/A}$$

$$= 0 - \alpha_{AB}\mathbf{k} \times (4\mathbf{i} + 4\mathbf{j}) - (6)^2(4\mathbf{i} + 4\mathbf{j}) = (4\alpha_{AB} - 144)\mathbf{i} + (-4\alpha_{AB} - 144)\mathbf{j}$$

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - {\omega_{BC}}^2\mathbf{r}_{C/B}$$

$$= (4\alpha_{AB} - 144)\mathbf{i} + (-4\alpha_{AB} - 144)\mathbf{j} + \alpha_{BC}\mathbf{k} \times (10\mathbf{i} - 7\mathbf{j}) - (-2.4)^2(10\mathbf{i} - 7\mathbf{j})$$

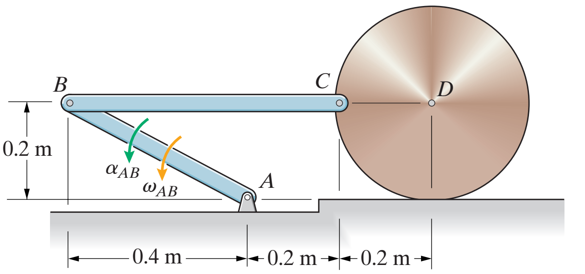
$$= (4\alpha_{AB} - 144 + 7\alpha_{BC} - 57.6)\mathbf{i} + (-4\alpha_{AB} - 144 + 10\alpha_{BC} + 40.3)\mathbf{j}$$

The acceleration of C is zero. Equating both components to zero and solving, we find that

$$\alpha_{AB} = 19.0 \text{ rad/s}^2, \alpha_{BC} = 18.0 \text{ rad/s}^2$$

Thus $\boxed{\alpha_{AB} = 19.0 \text{ rad/s}^2 \text{ clockwise.}}$

Problem 17.99 The angular velocity and angular acceleration of bar AB are $\omega_{AB} = 5$ rad/s and $\alpha_{AB} = 10$ rad/s^2. Determine the angular acceleration of bar BC.



Solution: Do a velocity analysis first to find all of the angular velocities. Let point E be the point on the wheel that is in contact with the ground.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + (5\text{ rad/s})\mathbf{k} \times (-0.4\mathbf{i} + 0.2\mathbf{j})\text{ m}$$

$$= (-\mathbf{i} - 2\mathbf{j})\text{ m/s}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = (-\mathbf{i} - 2\mathbf{j})\text{ m/s} + \omega_{BC}\mathbf{k} \times (0.6\text{ m})\mathbf{i}$$

$$= (-1\text{ m/s})\mathbf{i} + (-2\text{ m/s} + \{0.6\text{ m}\}\omega_{BC})\mathbf{j}$$

$$\mathbf{v}_E = \mathbf{v}_C + \boldsymbol{\omega}_{\text{wheel}} \times \mathbf{r}_{E/C} = (-1\text{ m/s})\mathbf{i} + (-2\text{ m/s} + \{0.6\text{ m}\}\omega_{BC})\mathbf{j}$$

$$+ \omega_{\text{wheel}}\mathbf{k} \times (0.2\mathbf{i} - 0.2\mathbf{j})\text{ m}$$

$$= (-1\text{ m/s} + \{0.2\text{ m}\}\omega_{\text{wheel}})\mathbf{i}$$

$$+ (-2\text{ m/s} + \{0.6\text{ m}\}\omega_{BC} + \{0.2\text{ m}\}\omega_{\text{wheel}})\mathbf{j}$$

Point E is the instantaneous center of the wheel. Therefore

$$\left.\begin{array}{l} -1\text{ m/s} + \{0.2\text{ m}\}\omega_{\text{wheel}} = 0 \\ -2\text{ m/s} + \{0.6\text{ m}\}\omega_{BC} + \{0.2\text{ m}\}\omega_{\text{wheel}} = 0 \end{array}\right\}$$

$$\Rightarrow \begin{array}{l} \omega_{BC} = 1.67\text{ rad/s} \\ \omega_{\text{wheel}} = 5\text{ rad/s} \end{array}$$

Now we do the acceleration analysis

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}{}^2\mathbf{r}_{B/A}$$

$$= 0 + (10\text{ rad/s}^2)\mathbf{k} \times (-0.4\mathbf{i} + 0.2\mathbf{j})\text{ m}$$

$$- (5\text{ rad/s})^2(-0.4\mathbf{i} + 0.2\mathbf{j})\text{ m} = (8\mathbf{i} - 9\mathbf{j})\text{ m/s}^2$$

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}{}^2\mathbf{r}_{C/B}$$

$$= (8\mathbf{i} - 9\mathbf{j})\text{ m/s}^2 + \alpha_{BC}\mathbf{k} \times (0.6\text{ m})\mathbf{i} - (1.67\text{ rad/s})^2(0.6\text{ m})\mathbf{i}$$

$$= (6.33\text{ m/s}^2)\mathbf{i} + (-9\text{ m/s}^2 + \{0.6\text{ m}\}\alpha_{BC})\mathbf{j}$$

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{\text{wheel}} \times \mathbf{r}_{D/C} - \omega_{\text{wheel}}{}^2\mathbf{r}_{D/C}$$

$$= (6.33\text{ m/s}^2)\mathbf{i} + (-9\text{ m/s}^2 + \{0.6\text{ m}\}\alpha_{BC})\mathbf{j} + \alpha_{\text{wheel}}\mathbf{k} \times (0.2\text{ m})\mathbf{i}$$

$$- (5\text{ rad/s})^2(0.2\text{ m})\mathbf{i}$$

$$= (1.33\text{ m/s}^2)\mathbf{i} + (-9\text{ m/s}^2 + \{0.6\text{ m}\}\alpha_{BC} + \{0.2\text{ m}\}\alpha_{\text{wheel}})\mathbf{j}$$

Finally we work down to point E

$$\mathbf{a}_E = \mathbf{a}_D + \boldsymbol{\alpha}_{\text{wheel}} \times \mathbf{r}_{E/D} - \omega_{\text{wheel}}{}^2\mathbf{r}_{E/D}$$

$$= (1.33\text{ m/s}^2)\mathbf{i} + (-9\text{ m/s}^2 + \{0.6\text{ m}\}\alpha_{BC} + \{0.2\text{ m}\}\alpha_{\text{wheel}})\mathbf{j}$$

$$+ \alpha_{\text{wheel}}\mathbf{k} \times (-0.2\text{ m})\mathbf{j} - (5\text{ rad/s})^2(-0.2\text{ m})\mathbf{j}$$

$$= (1.33\text{ m/s}^2 + \{0.2\text{ m}\}\alpha_{\text{wheel}})\mathbf{i}$$

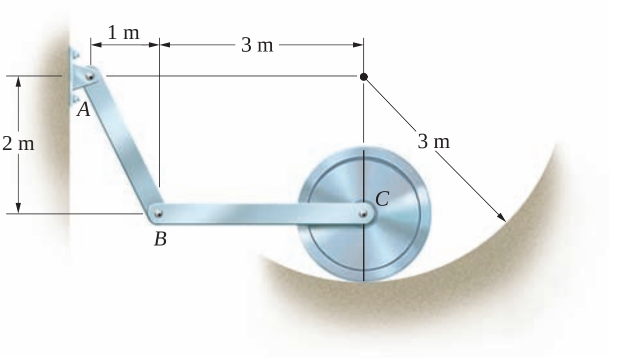
$$+ (-4\text{ m/s}^2 + \{0.6\text{ m}\}\alpha_{BC} + \{0.2\text{ m}\}\alpha_{\text{wheel}})\mathbf{j}$$

D moves horizontally therefore $\mathbf{a}_D \cdot \mathbf{j} = 0$
The wheel does not slip therefore $\mathbf{a}_E \cdot \mathbf{i} = 0$
We have

$$\left.\begin{array}{l} -9\text{ m/s}^2 + \{0.6\text{ m}\}\alpha_{BC} + \{0.2\text{ m}\}\alpha_{\text{wheel}} = 0 \\ 1.33\text{ m/s}^2 + \{0.2\text{ m}\}\alpha_{\text{wheel}} = 0 \end{array}\right\}$$

$$\Rightarrow \begin{array}{l} \alpha_{\text{wheel}} = -6.67\text{ rad/s}^2 \\ \alpha_{BC} = 17.2\text{ rad/s}^2 \end{array} \quad \boxed{\alpha_{BC} = 17.2\text{ rad/s}^2\text{ CCW}}$$

Problem 17.100 At the instant shown, bar AB is rotating at 10 rad/s in the counterclockwise direction and has a counterclockwise angular acceleration of 20 rad/s^2. The disk rolls on the circular surface. Determine the angular accelerations of bar BC and the disk.



Solution: The velocity analysis:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + (10\mathbf{k}) \times (1\mathbf{i} - 2\mathbf{j}) = 20\mathbf{i} + 10\mathbf{j}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = (20\mathbf{i} + 10\mathbf{j}) + \omega_{BC}\mathbf{k} \times (3\mathbf{i}) = (20)\mathbf{i} + (10 + 3\omega_{BC})\mathbf{j}$$

$$\mathbf{v}_C = \boldsymbol{\omega}_{\text{disk}} \times \mathbf{r} = \omega_{\text{disk}}\mathbf{k} \times (1\mathbf{j}) = -\omega_{\text{disk}}\ (1)\mathbf{i}$$

Equating the components of these two expressions for $\mathbf{v}_C$ and solving, we find

$$\omega_{BC} = -3.33 \text{ rad/s}, \quad \omega_{\text{disk}} = -20 \text{ rad/s}, \quad v_C = 20 \text{ m/s}.$$

The acceleration analysis (note that point C is moving in a circle of radius 2 m):

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - {\omega_{AB}}^2\mathbf{r}$$

$$= 0 + (20\mathbf{k}) \times (1\mathbf{i} - 2\mathbf{j}) - (10)^2(1\mathbf{i} - 2\mathbf{j}) = (-60\mathbf{i} + 220\mathbf{j})$$

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - {\omega_{BC}}^2\mathbf{r}_{C/B}$$

$$= (-60\mathbf{i} + 220\mathbf{j}) + \alpha_{BC}\mathbf{k} \times (3\mathbf{i}) - (-3.33)^2(3\mathbf{i}) = (-93.3)\mathbf{i} + (220 + 3\alpha_{BC})\mathbf{j}$$

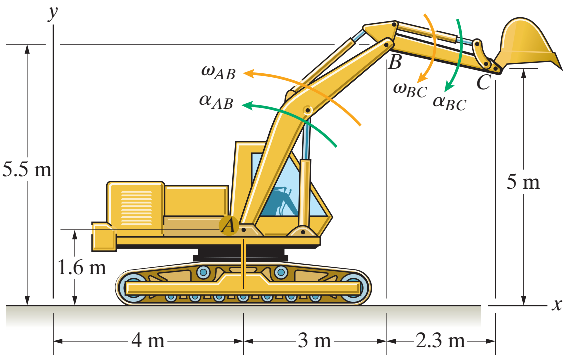
$$\mathbf{a}_C = \boldsymbol{\alpha}_{\text{disk}} \times \mathbf{r} + \frac{v_C^2}{2\text{ m}}\mathbf{j} = \alpha_{\text{disk}}\mathbf{k} \times (1\mathbf{j}) + \frac{(20)^2}{2}\mathbf{j} = -\alpha_{\text{disk}}(1)\mathbf{i} + 200\mathbf{j}$$

Equating the components of these two expressions for $\mathbf{a}_C$ and solving, we find

$\alpha_{BC} = -6.67$ rad/s^2, $\alpha_{\text{disk}} = 93.3$ rad/s^2.

$\boxed{\alpha_{BC} = 6.67 \text{ rad/s}^2 \text{ clockwise}, \ \alpha_{\text{disk}} = 93.3 \text{ rad/s}^2 \text{ counterclockwise.}}$

Problem 17.101 If $\omega_{AB} = 2$ rad/s, $\alpha_{AB} = 2$ rad/s^2, $\omega_{BC} = -1$ rad/s, and $\alpha_{BC} = -4$ rad/s^2, what is the acceleration of point C where the scoop of the excavator is attached?



Solution: The vector locations of points A, B, C are

$$\mathbf{r}_A = 4\mathbf{i} + 1.6\mathbf{j} \text{ (m)},$$

$$\mathbf{r}_B = 7\mathbf{i} + 5.5\mathbf{j} \text{ (m)}.$$

$$\mathbf{r}_C = 9.3\mathbf{i} + 5\mathbf{j} \text{ (m)}.$$

The vectors

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = 3\mathbf{i} + 3.9\mathbf{j} \text{ (m)},$$

$$\mathbf{r}_{C/B} = \mathbf{r}_C - \mathbf{r}_B = 2.3\mathbf{i} - 0.5\mathbf{j} \text{ (m)}.$$

The acceleration of point B is

$$\mathbf{a}_B = \alpha_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}.$$

$$\mathbf{a}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 3 & 3.9 & 0 \end{bmatrix} - (2^2)(3.0\mathbf{i} + 3.9\mathbf{j}),$$

$$\mathbf{a}_B = +2(-3.9\mathbf{i} + 3\mathbf{j}) - (4)(3\mathbf{i} + 3.9\mathbf{j})$$

$$= -19.8\mathbf{i} - 9.6\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration of point C in terms of the acceleration at point B is

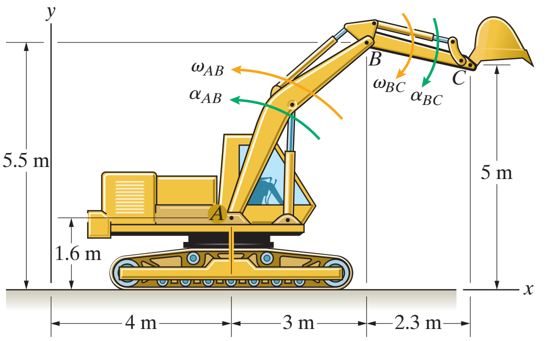
$$\mathbf{a}_C = \mathbf{a}_B + \alpha_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 (\mathbf{r}_{C/B})$$

$$= -19.8\mathbf{i} - 9.6\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -4 \\ 2.3 & -0.5 & 0 \end{bmatrix} - 1^2(2.3\mathbf{i} - 0.5\mathbf{j}),$$

$$\mathbf{a}_C = -19.8\mathbf{i} - 9.6\mathbf{j} - 2\mathbf{i} - 9.2\mathbf{j} - 2.3\mathbf{i} + 0.5\mathbf{j}$$

$$\boxed{= -24.1\mathbf{i} - 18.3\mathbf{j} \text{ (m/s}^2\text{)}}$$

Problem 17.102 If the velocity of point C of the excavator in Problem 17.101 is $\mathbf{v}_C = 4\mathbf{i}$ (m/s) and is constant, what are ω_{AB}, α_{AB}, ω_{BC}, α_{BC}?



Solution: The strategy is to determine the angular velocities ω_{AB}, ω_{BC} from the known velocity at point C, and the angular velocities α_{AB}, α_{BC} from the data that the linear acceleration at point C is constant.

The angular velocities: The vector locations of points A, B, C are

$$\mathbf{r}_A = 4\mathbf{i} + 1.6\mathbf{j} \text{ (m)},$$

$$\mathbf{r}_B = 7\mathbf{i} + 5.5\mathbf{j} \text{ (m)},$$

$$\mathbf{r}_C = 9.3\mathbf{i} + 5\mathbf{j} \text{ (m)}.$$

The vectors

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = 3\mathbf{i} + 3.9\mathbf{j} \text{ (m)},$$

$$\mathbf{r}_{C/B} = \mathbf{r}_C - \mathbf{r}_B = 2.3\mathbf{i} - 0.5\mathbf{j} \text{ (m)}.$$

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 3 & 3.9 & 0 \end{bmatrix} = -3.9\omega_{AB}\mathbf{i} + 3\omega_{AB}\mathbf{j}.$$

The velocity of C in terms of the velocity of B

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= -3.9\omega_{AB}\mathbf{i} + 3\omega_{AB}\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega_{BC} \\ 2.3 & -0.5 & 0 \end{bmatrix},$$

$$\mathbf{v}_C = -3.9\omega_{AB}\mathbf{i} + 3\omega_{AB}\mathbf{j} - 0.5\omega_{BC}\mathbf{i} - 2.3\omega_{BC}\mathbf{j} \text{ (m/s)}.$$

Substitute $\mathbf{v}_C = 4\mathbf{i}$ (m/s), and separate components:

$$4 = -3.9\omega_{AB} - 0.5\omega_{BC},$$

$$0 = 3\omega_{AB} - 2.3\omega_{BC}.$$

Solve: $\boxed{\omega_{AB} = -0.8787 \text{ rad/s}}$, $\boxed{\omega_{BC} = -1.146 \text{ rad/s}}$.

The angular accelerations: The acceleration of point B is

$$\mathbf{a}_B = \mathbf{a}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 3 & 3.9 & 0 \end{bmatrix} - \left(\omega_{AB}^2\right)(3\mathbf{i} + 3.9\mathbf{j}),$$

$$\mathbf{a}_B = -3.9\alpha_{AB}\mathbf{i} + 3\alpha_{AB}\mathbf{j} - 3\omega_{AB}^2\mathbf{i} - 3.9\omega_{AB}^2\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration of C in terms of the acceleration of B is

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$

$$= \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\alpha_{BC} \\ 2.3 & -0.5 & 0 \end{bmatrix} - \omega_{BC}^2(2.3\mathbf{i} - 0.5\mathbf{j})$$

$$\mathbf{a}_C = \left(-3.9\alpha_{AB} - 3\omega_{AB}^2\right)\mathbf{i} + \left(3\alpha_{AB} - 3.9\omega_{AB}^2\right)\mathbf{j}$$

$$+ (-0.5\alpha_{BC} - 2.3\omega_{BC}^2)\mathbf{i} + \left(-2.3\alpha_{BC} + 0.5\omega_{BC}^2\right)\mathbf{j}.$$

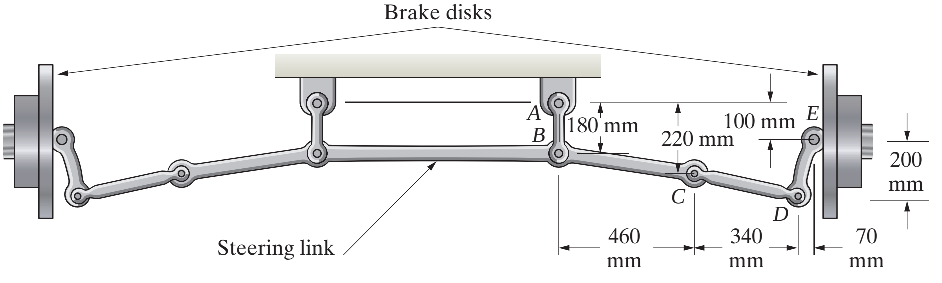
Substitute $\mathbf{a}_C = 0$ from the conditions of the problem, and separate components:

$$0 = -3.9\alpha_{AB} - 0.5\alpha_{BC} - 3\omega_{AB}^2 - 2.3\omega_{BC}^2,$$

$$0 = 3\alpha_{AB} - 2.3\alpha_{BC} - 3.9\omega_{AB}^2 + 0.5\omega_{BC}^2.$$

Solve: $\boxed{\alpha_{BC} = -2.406 \text{ rad/s}^2}$, $\boxed{\alpha_{AB} = -1.06 \text{ rad/s}^2}$.

Problem 17.103 The steering linkage of a car is shown. Member DE rotates about the fixed pin E. The right brake disk is rigidly attached to member DE. The tie rod CD is pinned at C and D. At the instant shown, the Pitman arm AB has a counterclockwise angular velocity of 1 rad/s and a clockwise angular acceleration of 2 rad/s^2. What is the angular acceleration of the right brake disk?



Solution: Note that the steering link translates, but does not rotate. The velocity analysis:

$$\mathbf{v}_C = \mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + (1\mathbf{k}) \times (-0.18\mathbf{j}) = 0.18\mathbf{i}$$

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C} = 0.18\mathbf{i} + \omega_{CD}\mathbf{k} \times (0.34\mathbf{i} - 0.08\mathbf{j})$$

$$= (0.18 + 0.08\omega_{CD})\mathbf{i} + (0.34\omega_{CD})\mathbf{j}$$

$$\mathbf{v}_E = \mathbf{v}_D + \boldsymbol{\omega}_{DE} \times \mathbf{r}_{E/D}$$

$$= (0.18 + 0.08\omega_{CD})\mathbf{i} + (0.34\omega_{CD})\mathbf{j} + \omega_{DE}\mathbf{k} \times (0.07\mathbf{i} + 0.2\mathbf{j})$$

$$= (0.18 + 0.08\omega_{CD} - 0.2\omega_{DE})\mathbf{i} + (0.34\omega_{CD} + 0.07\omega_{DE})\mathbf{j}$$

Since point E is fixed, we can set both components to zero and solve. We find

$$\omega_{CD} = -0.171 \text{ rad/s}, \quad \omega_{DE} = 0.832 \text{ rad/s}.$$

The acceleration analysis:

$$\mathbf{a}_C = \mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{C/B} - {\omega_{AB}}^2\mathbf{r}_{B/A}$$

$$= 0 + (-2\mathbf{k}) \times (-0.18\mathbf{j}) - (1)^2(-0.18\mathbf{j}) = (-0.36\mathbf{i} + 0.18\mathbf{j})$$

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{D/C} - {\omega_{CD}}^2\mathbf{r}_{D/C}$$

$$= (-0.36\mathbf{i} + 0.18\mathbf{j}) + \alpha_{CD}\mathbf{k} \times (0.34\mathbf{i} - 0.08\mathbf{j}) - (-0.171)^2(0.34\mathbf{i} - 0.08\mathbf{j})$$

$$= (-0.370 + 0.08\alpha_{CD})\mathbf{i} + (0.182 + 0.34\alpha_{CD})\mathbf{j}$$

$$\mathbf{a}_E = \mathbf{a}_D + \boldsymbol{\alpha}_{DE} \times \mathbf{r}_{E/D} - {\omega_{DE}}^2\mathbf{r}_{E/D}$$

$$= (-0.370 + 0.08\alpha_{CD})\mathbf{i} + (0.182 + 0.34\alpha_{CD})\mathbf{j}$$

$$+ \alpha_{DE}\mathbf{k} \times (0.07\mathbf{i} + 0.2\mathbf{j}) - (0.832)^2(0.07\mathbf{i} + 0.2\mathbf{j})$$

$$= (-.418 + 0.08\alpha_{CD} - 0.2\alpha_{DE})\mathbf{i} + (0.0436 + 0.34\alpha_{CD} + 0.07\alpha_{DE})\mathbf{j}$$

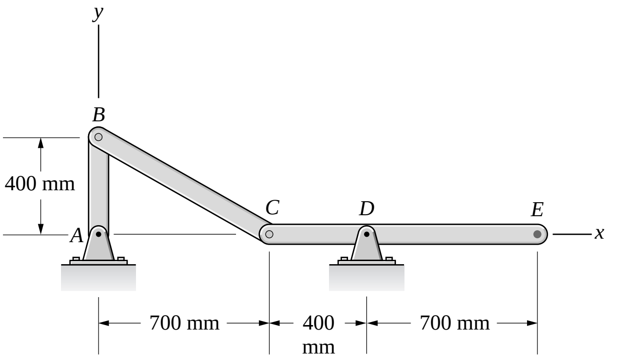
Since point E cannot move, we set both components of $\mathbf{a}_E$ to zero and solve. We find:

$$\alpha_{CD} = 0.278 \text{ rad/s}^2, \quad \alpha_{DE} = -1.98 \text{ rad/s}^2.$$

The angular acceleration of the right brake is the same as the angular acceleration of DE.

$$\boxed{\alpha_{\text{right brake}} = 1.98 \text{ rad/s}^2 \text{ clockwise.}}$$

Problem 17.104 At the instant shown, bar AB has no angular velocity but has a counterclockwise angular acceleration of 10 rad/s^2. Determine the acceleration of point E.



Solution: The vector locations of A, B, C and D are: $\mathbf{r}_A = 0$, $\mathbf{r}_B = 400\mathbf{j}$ (mm), $\mathbf{r}_C = 700\mathbf{i}$ (mm), $\mathbf{r}_D = 1100\mathbf{i}$ (mm). $\mathbf{r}_E = 1800\mathbf{i}$ (mm) The vectors

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = 400\mathbf{j} \text{ (mm)}.$$

$$\mathbf{r}_{C/B} = \mathbf{r}_C - \mathbf{r}_B = 700\mathbf{i} - 400\mathbf{j} \text{ (mm)},$$

$$\mathbf{r}_{C/D} = \mathbf{r}_C - \mathbf{r}_D = -400\mathbf{i} \text{ (mm)}$$

(a) *Get the angular velocities* ω_{BC}, ω_{CD}. The velocity of point B is zero. The velocity of C in terms of the velocity of B is

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC} \times \mathbf{r}_{C/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 700 & -400 & 0 \end{bmatrix}$$

$$= +400\omega_{BC}\mathbf{i} + 700\omega_{BC}\mathbf{j} \text{ (mm/s)}.$$

The velocity of C in terms of the velocity of point D

$$\mathbf{v}_C = \omega_{CD} \times \mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ -400 & 0 & 0 \end{bmatrix}$$

$$= -400\omega_{CD}\mathbf{j} \text{ (mm/s)}.$$

Equate the expressions for $\mathbf{v}_C$ and separate components: $400\omega_{BC} = 0$, $700\omega_{BC} = -400\omega_{CD}$. Solve: $\omega_{BC} = 0$ rad/s, $\omega_{CD} = 0$ rad/s.

(b) *Get the angular accelerations*. The acceleration of point B is

$$\mathbf{a}_B = \alpha_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 0 & 400 & 0 \end{bmatrix}$$

$$= -4000\mathbf{i} \text{ (mm/s}^2\text{)}.$$

The acceleration of point C in terms of the acceleration of point B:

$$\mathbf{a}_C = \mathbf{a}_B + \alpha_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$

$$= -4000\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 700 & -400 & 0 \end{bmatrix}.$$

$$\mathbf{a}_C = -4000\mathbf{i} + 400\alpha_{BC}\mathbf{i} + 700\alpha_{BC}\mathbf{j} \text{ (mm/s}^2\text{)}.$$

The acceleration of point C in terms of the acceleration of point D:

$$\mathbf{a}_C = \alpha_{CD} \times \mathbf{r}_{C/D} - \omega_{CD}^2\mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ -400 & 0 & 0 \end{bmatrix}$$

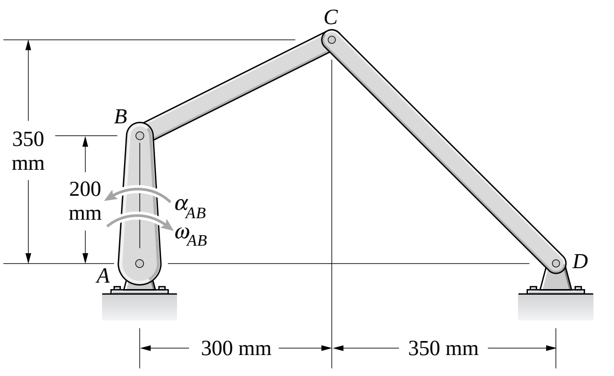
$$= -400\alpha_{CD}\mathbf{j} \text{ (mm/s}^2\text{)}.$$

Equate the expressions and separate components: $-4000 + 400\alpha_{BC} = 0$, $700\alpha_{BC} = -400\alpha_{CD}$.
Solve: $\alpha_{BC} = 10$ rad/s^2, $\alpha_{CD} = -17.5$ rad/s^2, The acceleration of point E in terms of the acceleration of point D is

$$\mathbf{a}_E = \alpha_{CD} \times \mathbf{r}_{E/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -17.5 \\ 700 & 0 & 0 \end{bmatrix}$$

$$\boxed{= -12250\mathbf{j} \text{ (mm/s}^2\text{)}} \text{ (clockwise)}$$

Problem 17.105 If $\omega_{AB} = 12$ rad/s and $\alpha_{AB} = 100$ rad/s^2, what are the angular accelerations of bars BC and CD?



Solution: The vector locations of A, B, C and D are: $\mathbf{r}_A = 0$, $\mathbf{r}_B = 200\mathbf{j}$ (mm), $\mathbf{r}_C = 300\mathbf{i} + 350\mathbf{j}$ (mm), $\mathbf{r}_D = 650\mathbf{i}$ (mm). The vectors

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = 200\mathbf{j} \text{ (mm)}.$$

$$\mathbf{r}_{C/B} = \mathbf{r}_C - \mathbf{r}_B = 300\mathbf{i} + 150\mathbf{j} \text{ (mm)},$$

$$\mathbf{r}_{C/D} = \mathbf{r}_C - \mathbf{r}_D = -350\mathbf{i} + 350\mathbf{j} \text{ (mm)}$$

(a) *Get the angular velocities ω_{BC}, ω_{CD}*. The velocity of point B is

$$\mathbf{v}_B = \omega_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -12 \\ 0 & 200 & 0 \end{bmatrix} = 2400\mathbf{i} \text{ (mm/s)}.$$

The velocity of C in terms of the velocity of B is

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 300 & 150 & 0 \end{bmatrix}$$

$$= 2400\mathbf{i} - 150\omega_{BC}\mathbf{i} + 300\omega_{BC}\mathbf{j} \text{ (mm/s)}.$$

The velocity of C in terms of the velocity of point D

$$\mathbf{v}_C = \omega_{CD} \times \mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ -350 & 350 & 0 \end{bmatrix}$$

$$= -350\omega_{CD}\mathbf{i} - 350\omega_{CD}\mathbf{j} \text{ (mm/s)}.$$

Equate the expressions for $\mathbf{v}_C$ and separate components: $2400 - 150\omega_{BC} = -350\omega_{CD}$, $300\omega_{BC} = -350\omega_{CD}$. Solve: $\omega_{BC} = 5.33$ rad/s, $\omega_{CD} = -4.57$ rad/s.

(b) *Get the angular accelerations*. The acceleration of point B is

$$\mathbf{a}_B = \alpha_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 100 \\ 0 & 200 & 0 \end{bmatrix}$$

$$- \omega_{AB}^2(200\mathbf{j})$$

$$= -20{,}000\mathbf{i} - 28{,}800\mathbf{j} \text{ (mm/s}^2\text{)}.$$

The acceleration of point C in terms of the acceleration of point B:

$$\mathbf{a}_C = \mathbf{a}_B + \alpha_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B}$$

$$= \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 300 & 150 & 0 \end{bmatrix} - \omega_{BC}^2(300\mathbf{i} + 150\mathbf{j}).$$

$$\mathbf{a}_C = (-20{,}000 - 150\alpha_{BC} - 300\omega_{BC}^2)\mathbf{i}$$

$$+ (-28{,}800 + 300\alpha_{BC} - 150\omega_{BC}^2)\mathbf{j} \text{ (mm/s}^2\text{)}.$$

The acceleration of point C in terms of the acceleration of point D:

$$\mathbf{a}_C = \alpha_{CD} \times \mathbf{r}_{C/D} - \omega_{CD}^2 \mathbf{r}_{C/D}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ -350 & 350 & 0 \end{bmatrix} - \omega_{CD}^2(-350\mathbf{i} + 350\mathbf{j})$$

$$\mathbf{a}_C = -350\alpha_{CD}\mathbf{i} - 350\alpha_{CD}\mathbf{j} + 350\omega_{CD}^2\mathbf{i} - 350\omega_{CD}^2\mathbf{j} \text{ (mm/s}^2\text{)}.$$

Equate the expressions and separate components:

$$-20{,}000 - 150\alpha_{BC} - 300\omega_{BC}^2 = -350\alpha_{CD} + 350\omega_{CD}^2,$$

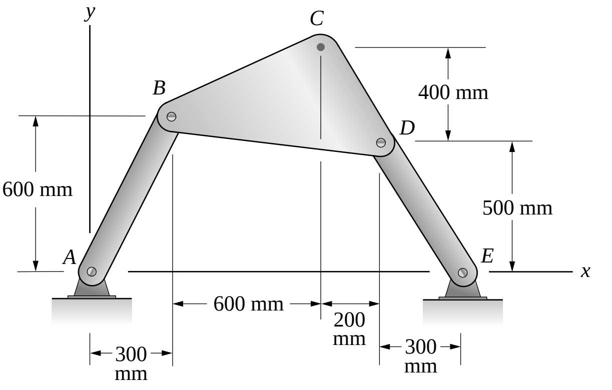
$$-28{,}800 + 300\alpha_{BC} - 150\omega_{BC}^2 = -350\alpha_{CD} - 350\omega_{CD}^2.$$

Solve: $\boxed{\alpha_{BC} = -22.43 \text{ rad/s}^2}$,

$\boxed{\alpha_{CD} = 92.8 \text{ rad/s}^2}$,

where the negative sign means a clockwise acceleration.

Problem 17.106 If $\omega_{AB} = 4$ rad/s counterclockwise and $\alpha_{AB} = 12$ rad/s^2 counterclockwise, what is the acceleration of point C?



Solution: The velocity of B is

$$\mathbf{v}_B = \mathbf{v}_A + \omega_{AB} \times \mathbf{r}_{B/A}$$

$$= \mathbf{O} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 0.3 & 0.6 & 0 \end{vmatrix}$$

$$= -0.6\omega_{AB}\mathbf{i} + 0.3\omega_{AB}\mathbf{j}.$$

The velocity of D is

$$\mathbf{v}_D = \mathbf{v}_B + \omega_{BD} \times \mathbf{r}_{D/B}$$

$$= -0.6\omega_{AB}\mathbf{i} + 0.3\omega_{AB}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BD} \\ 0.8 & -0.1 & 0 \end{vmatrix}. \quad \textbf{(1)}$$

We can also express the velocity of D as

$$\mathbf{v}_D = \mathbf{v}_E + \omega_{DE} \times \mathbf{r}_{D/E} = \mathbf{O} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{DE} \\ -0.3 & 0.5 & 0 \end{vmatrix}. \quad \textbf{(2)}$$

Equating **i** and **j** components in Eqns. (1) and (2), we obtain

$$-0.6\omega_{AB} + 0.1\omega_{BD} = -0.5\omega_{DE}, \quad \textbf{(3)}$$

$$0.3\omega_{AB} + 0.8\omega_{BD} = -0.3\omega_{DE}. \quad \textbf{(4)}$$

Solving these two eqns with $\omega_{AB} = 4$ rad/s, we obtain

$$\omega_{BD} = -3.57 \text{ rad/s}, \quad \omega_{DE} = 5.51 \text{ rad/s}.$$

The acceleration of B is

$$\mathbf{a}_B = \mathbf{a}_A + \alpha_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A}$$

$$= \mathbf{O} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 0.3 & 0.6 & 0 \end{vmatrix} - \omega_{AB}^2(0.3\mathbf{i} + 0.6\mathbf{j})$$

$$= (-0.6\alpha_{AB} - 0.3\omega_{AB}^2)\mathbf{i} + (0.3\alpha_{AB} - 0.6\omega_{AB}^2)\mathbf{j}.$$

The acceleration of D is

$$\mathbf{a}_D = \mathbf{a}_B + \alpha_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B}$$

$$= (-0.6\alpha_{AB} - 0.3\omega_{AB}^2)\mathbf{i} + (0.3\alpha_{AB} - 0.6\omega_{AB}^2)\mathbf{j}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BD} \\ 0.8 & -0.1 & 0 \end{vmatrix} - \omega_{BD}^2(0.8\mathbf{i} - 0.1\mathbf{j}). \quad \textbf{(5)}$$

We can also express the acceleration of D as

$$\mathbf{a}_D = \mathbf{a}_E + \alpha_{DE} \times \mathbf{r}_{D/E} - \omega_{DE}^2\mathbf{r}_{D/E}$$

$$= \mathbf{O} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{DE} \\ -0.3 & 0.5 & 0 \end{vmatrix} - \omega_{DE}^2(-0.3\mathbf{i} + 0.5\mathbf{j}). \quad \textbf{(6)}$$

Equating **i** and **j** components in Eqns. (5) and (6), we obtain

$$-0.6\alpha_{AB} - 0.3\omega_{AB}^2 + 0.1\alpha_{BD} - 0.8\omega_{BD}^2$$

$$= -0.5\alpha_{DE} + 0.3\omega_{DE}^2, \quad \textbf{(7)}$$

$$0.3\alpha_{AB} - 0.6\omega_{AB}^2 + 0.8\alpha_{BD} + 0.1\omega_{BD}^2$$

$$= -0.3\alpha_{DE} - 0.5\omega_{DE}^2. \quad \textbf{(8)}$$

Solving these two eqns with $\alpha_{AB} = 12$ rad/s^2, we obtain

$$\alpha_{BD} = -39.5 \text{ rad/s}^2.$$

The acceleration of C is

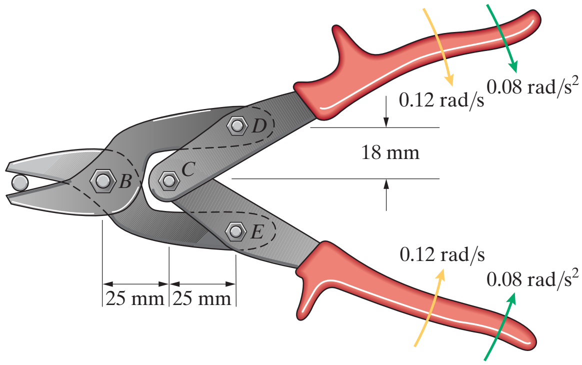
$$\mathbf{a}_C = \mathbf{a}_B + \alpha_{BD} \times \mathbf{r}_{C/B} - \omega_{BD}^2\mathbf{r}_{C/B}$$

$$= (-0.6\alpha_{AB} - 0.3\omega_{AB}^2)\mathbf{i} + (0.3\alpha_{AB} - 0.6\omega_{AB}^2)\mathbf{j}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BD} \\ 0.6 & 0.3 & 0 \end{vmatrix} - \omega_{BD}^2(0.6\mathbf{i} + 0.3\mathbf{j}). \quad \textbf{(9)}$$

$$\mathbf{a}_C = -7.78\mathbf{i} - 33.5\mathbf{j} \text{ (m/s}^2\text{)}.$$

Problem 17.107 The angular velocities and angular accelerations of the grips of the shears are shown. What is the resulting angular acceleration of the jaw BD?



Solution: From 17.57 we know that $\omega_{BD} = -0.06$ rad/s

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{D/C} - {\omega_{CD}}^2 \mathbf{r}_{D/C}$$

$$= 0 - (0.08 \text{ rad/s}^2)\mathbf{k} \times (0.025\mathbf{i} + 0.018\mathbf{j}) \text{ m}$$

$$- (0.12 \text{ rad/s})^2 (0.025\mathbf{i} + 0.018\mathbf{j}) \text{ m}$$

$$= (0.00108\mathbf{i} - 0.00226\mathbf{j}) \text{ m/s}^2$$

$$\mathbf{a}_B = \mathbf{a}_D + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{B/D} - {\omega_{BD}}^2 \mathbf{r}_{B/D}$$

$$= (0.00108\mathbf{i} - 0.00226\mathbf{j}) \text{ m/s}^2 + \alpha_{BD}\mathbf{k} \times (-0.05\mathbf{i} - 0.018\mathbf{j}) \text{ m}$$

$$- (-0.06 \text{ rad/s})^2 (-0.05\mathbf{i} - 0.018\mathbf{j}) \text{ m}$$

$$= (0.00126 \text{ m/s}^2 + \{0.018 \text{ m}\}\alpha_{BD})\mathbf{i}$$

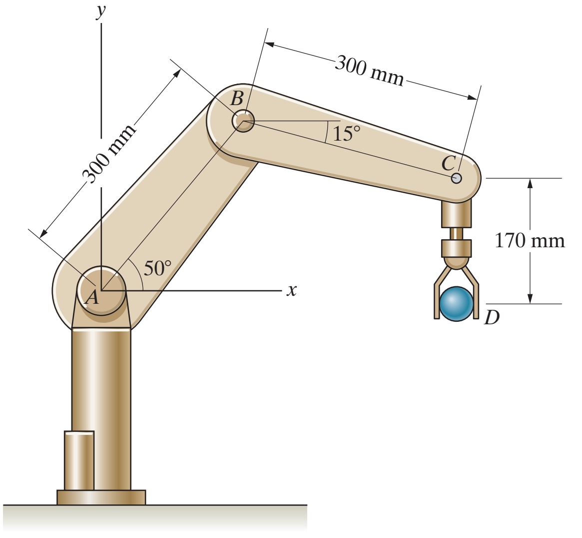
$$+ (-0.00219 \text{ m/s}^2 - \{0.05 \text{ m}\}\alpha_{BD})\mathbf{j}$$

From symmetry, B cannot accelerate in the $\mathbf{j}$ direction. Therefore

$$-0.00219 \text{ m/s}^2 - \{0.05 \text{ m}\}\alpha_{BD} = 0 \Rightarrow \alpha_{BD} = -0.0439 \text{ rad/s}^2$$

$$\boxed{\alpha_{BD} = 0.0439 \text{ rad/s}^2 \text{ CW}}$$

Problem 17.108 If arm AB has a constant clockwise angular velocity of 0.8 rad/s, arm BC has a constant angular velocity of 0.2 rad/s, and arm CD remains vertical, what is the acceleration of part D?



Solution: The constraint that the arm CD remain vertical means that the angular velocity of arm CD is zero. This implies that arm CD translates only, and in a translating, non-rotating element the velocity and acceleration at any point is the same, and the velocity and acceleration of arm CD is the velocity and acceleration of point C. The vectors:

$$\mathbf{r}_{B/A} = 300(\mathbf{i}\cos 50^\circ + \mathbf{j}\sin 50^\circ) = 192.8\mathbf{i} + 229.8\mathbf{j} \text{ (mm)}.$$

$$\mathbf{r}_{C/B} = 300(\mathbf{i}\cos 15^\circ - \mathbf{j}\sin 15^\circ) = 289.78\mathbf{i} - 77.6\mathbf{j} \text{ (mm)}.$$

The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} - \omega_{AB}^2\mathbf{r}_{A/B} = -\omega_{AB}^2(192.8\mathbf{i} + 229.8\mathbf{j}) \text{ (mm/s}^2),$$

since $\alpha_{AB} = 0$. $\mathbf{a}_B = -123.4\mathbf{i} - 147.1\mathbf{j}$ (mm/s). The acceleration of C in terms of the acceleration of B is

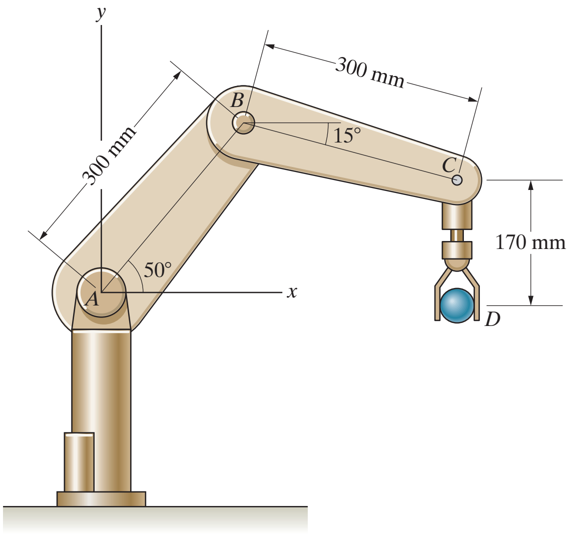
$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$

$$= -123.4\mathbf{i} - 147.1\mathbf{j} - \omega_{BC}^2(289.8\mathbf{i} - 77.6\mathbf{j}),$$

since $\boldsymbol{\alpha}_{BC} = 0$. $\mathbf{a}_C = -135\mathbf{i} - 144\mathbf{j}$ (mm/s^2). Since CD is translating:

$$\boxed{\mathbf{a}_D = \mathbf{a}_C = -135\mathbf{i} - 144\mathbf{j} \text{ (mm/s}^2)}$$

Problem 17.109 In Problem 17.108, if arm AB has a constant clockwise angular velocity of 0.8 rad/s and you want D to have zero velocity and acceleration, what are the necessary angular velocities and angular accelerations of arms BC and CD?



Solution: Except for numerical values, the solution follows the same strategy as the solution strategy for Problem 17.105. The vectors:

$$\mathbf{r}_{B/A} = 300(\mathbf{i}\cos 50^\circ + \mathbf{j}\sin 50^\circ) = 192.8\mathbf{i} + 229.8\mathbf{j} \text{ (mm)}.$$

$$\mathbf{r}_{C/B} = 300(\mathbf{i}\cos 15^\circ - \mathbf{j}\sin 15^\circ) = 289.78\mathbf{i} - 77.6\mathbf{j} \text{ (mm)},$$

$$\mathbf{r}_{C/D} = 170\mathbf{j} \text{ (mm)}.$$

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -0.8 \\ 192.8 & 229.8 & 0 \end{bmatrix}$$

$$= 183.8\mathbf{i} - 154.3\mathbf{j} \text{ (mm/s)}.$$

The velocity of C in terms of the velocity of B is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 289.8 & -77.6 & 0 \end{bmatrix}.$$

$$\mathbf{v}_C = 183.9\mathbf{i} - 154.3\mathbf{j} + \omega_{BC}(77.6\mathbf{i} + 289.8\mathbf{j}) \text{ (mm/s)}.$$

The velocity of C in terms of the velocity of point D:

$$\mathbf{v}_C = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ 0 & 170 & 0 \end{bmatrix} = -170\omega_{CD}\mathbf{i} \text{ (mm/s)}.$$

Equate the expressions for $\mathbf{v}_C$ and separate components:

$$183.9 + 77.6\omega_{BC} = -170\omega_{CD},$$

$$-154.3 + 289.8\omega_{BC} = 0.$$

Solve: $\boxed{\omega_{BC} = 0.532 \text{ rad/s}}$, $\boxed{\omega_{CD} = -1.325 \text{ rad/s}}$.

Get the angular accelerations. The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A} = -\omega_{AB}^2(192.8\mathbf{i} + 229.8\mathbf{j})$$

$$= -123.4\mathbf{i} - 147.1\mathbf{j} \text{ (mm/s}^2\text{)}.$$

The acceleration of point C in terms of the acceleration of point B:

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B} = \mathbf{a}_B - \omega_{BC}^2\mathbf{r}_{C/B}.$$

$$\mathbf{a}_C = -123.4\mathbf{i} - 147.1\mathbf{j} + 77.6\alpha_{BC}\mathbf{i} + 289.8\alpha_{BC}\mathbf{j} - 289.8\omega_{BC}^2\mathbf{i}$$

$$+ 77.6\omega_{BC}^2\mathbf{j} \text{ (mm/s}^2\text{)}.$$

The acceleration of point C in terms of the acceleration of point D:

$$\mathbf{a}_C = \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{C/D} - \omega_{CD}^2\mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ 0 & 170 & 0 \end{bmatrix} - \omega_{CD}^2(170\mathbf{j}).$$

$$\mathbf{a}_C = -170\alpha_{CD}\mathbf{i} - 170\omega_{CD}^2\mathbf{j} \text{ (mm/s}^2\text{)}.$$

Equate the expressions and separate components:

$$-123.4 + 77.6\alpha_{BC} - 289.8\omega_{BC}^2 = -170\alpha_{CD},$$

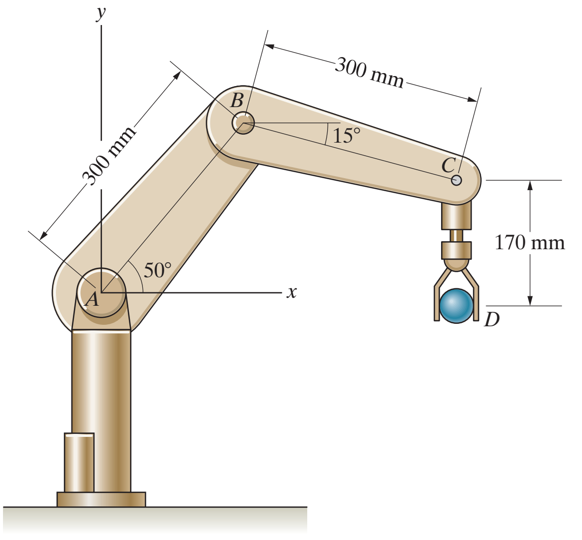
$$-147.1 + 289.8\alpha_{BC} + 77.6\omega_{BC}^2 = -170\omega_{CD}^2.$$

Solve:

$\boxed{\alpha_{BC} = -0.598 \text{ rad/s}^2}$, $\boxed{\alpha_{CD} = 1.482 \text{ rad/s}^2}$,

where the negative sign means a clockwise angular acceleration.

Problem 17.110 In Problem 17.108, if you want arm CD to remain vertical and you want part D to have velocity $\mathbf{v}_D = 1.0\mathbf{i}$ (m/s) and zero acceleration, what are the necessary angular velocities and angular accelerations of arms AB and BC?



Solution: The constraint that CD remain vertical with zero acceleration means that every point on arm CD is translating, without rotation, at a velocity of 1 m/s. This means that the velocity of point C is $\mathbf{v}_C = 1.0\mathbf{i}$ (m/s), and the acceleration of point C is zero. The vectors:

$$\mathbf{r}_{B/A} = 300(\mathbf{i}\cos 50^\circ + \mathbf{j}\sin 50^\circ) = 192.8\mathbf{i} + 229.8\mathbf{j} \text{ (mm)}.$$

$$\mathbf{r}_{C/B} = 300(\mathbf{i}\cos 15^\circ - \mathbf{j}\sin 15^\circ) = 289.78\mathbf{i} - 77.6\mathbf{j} \text{ (mm)}.$$

The angular velocities of AB and BC: The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 192.8 & 229.8 & 0 \end{bmatrix}$$

$$= \omega_{AB}(-229.8\mathbf{i} + 192.8\mathbf{j}) \text{ (mm/s)}.$$

The velocity of C in terms of the velocity of B is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 289.8 & -77.6 & 0 \end{bmatrix}.$$

$$\mathbf{v}_C = -229.8\omega_{AB}\mathbf{i} + 192.8\omega_{AB}\mathbf{j} + \omega_{BC}(77.6\mathbf{i} + 289.8\mathbf{j}) \text{ (mm/s)}.$$

The velocity of C is known, $\mathbf{v}_C = 1000\mathbf{i}$ (mm/s). Equate the expressions for $\mathbf{v}_C$ and separate components: $1000 = -229.8\omega_{AB} + 77.6\omega_{BC}$, $0 = 192.8\omega_{AB} + 289.8\omega_{BC}$. Solve:

$\boxed{\omega_{AB} = -3.55 \text{ rad/s}}$, $\boxed{\omega_{BC} = 2.36 \text{ rad/s}}$,

where the negative sign means a clockwise angular velocity.

The accelerations of AB and BC: The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 192.8 & 229.8 & 0 \end{bmatrix}$$

$$- \omega_{AB}^2(192.8\mathbf{i} + 229.8\mathbf{j} \text{ (mm/s}^2).$$

$$\mathbf{a}_B = \alpha_{AB}(-229.8\mathbf{i} + 192.8\mathbf{j}) - \omega_{AB}^2(192.8\mathbf{i} + 229.8\mathbf{j}) \text{ (mm/s}^2)$$

The acceleration of C in terms of the acceleration of B is

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$

$$= \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 289.8 & -77.6 & 0 \end{bmatrix} - \omega_{BC}^2(289.8\mathbf{i} - 77.6\mathbf{j}),$$

$$\mathbf{a}_C = \mathbf{a}_B + \alpha_{BC}(77.6\mathbf{i} + 289.8\mathbf{j}) - \omega_{BC}^2(289.8\mathbf{i} - 77.6\mathbf{j}) \text{ (mm/s}^2).$$

The acceleration of point C is known to be zero. Substitute this value for $\mathbf{a}_C$, and separate components:

$$-229.8\alpha_{AB} - 192.8\omega_{AB}^2 + 77.6\alpha_{BC} - 289.8\omega_{BC}^2 = 0,$$

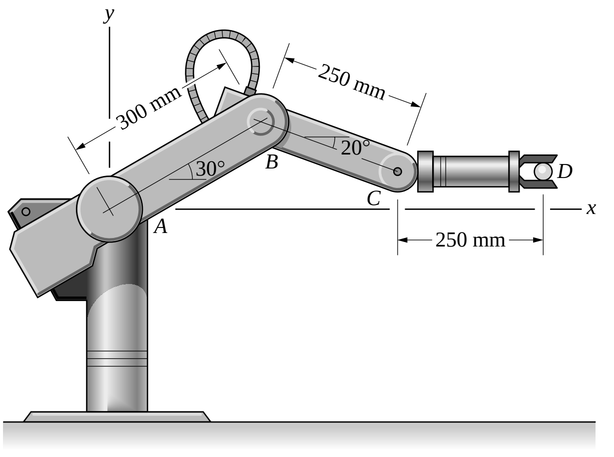
$$192.8\alpha_{AB} - 229.8\omega_{AB}^2 + 289.8\alpha_{BC} + 77.6\omega_{BC}^2 = 0.$$

Solve:

$\boxed{\alpha_{AB} = -12.1 \text{ rad/s}^2}$, $\boxed{\alpha_{BC} = 16.5 \text{ rad/s}^2}$,

where the negative sign means a clockwise angular acceleration.

Problem 17.111 Link AB of the robot's arm is rotating with a constant counterclockwise angular velocity of 2 rad/s, and link BC is rotating with a constant clockwise angular velocity of 3 rad/s. Link CD is rotating at 4 rad/s in the counterclockwise direction and has a counterclockwise angular acceleration of 6 rad/s^2. What is the acceleration of point D?



Solution: The acceleration of B is $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$. Evaluating, we get

$$\mathbf{a}_B = 0 + 0 - (2)^2(0.3\cos 30^\circ \mathbf{i} + 0.3 \sin 30^\circ \mathbf{j})$$

$$= -1.039\mathbf{i} - 0.600\mathbf{j} \ (\text{m/s}^2).$$

The acceleration of C is $\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B}$. Evaluating, we get

$$\mathbf{a}_C = -1.039\mathbf{i} - 0.600\mathbf{j} - (3)^2(0.25\cos 20^\circ \mathbf{i} - 0.25\sin 20^\circ \mathbf{j})$$

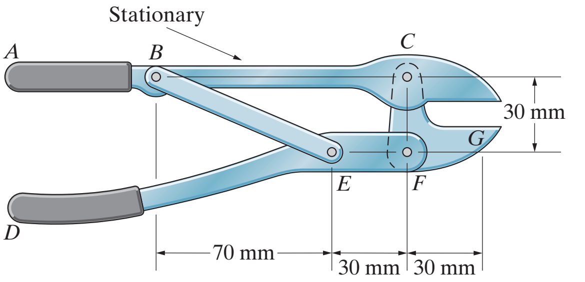
$$= -3.154\mathbf{i} + 0.170\mathbf{j} \ (\text{m/s}^2).$$

The acceleration of D is $\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{D/C} - \omega_{CD}^2 \mathbf{r}_{D/C}$. Evaluating, we get

$$\mathbf{a}_D = -3.154\mathbf{i} + 0.170\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 6 \\ 0.25 & 0 & 0 \end{vmatrix} - (4)^2(0.25\mathbf{i})$$

$$= -7.154\mathbf{i} + 1.67\mathbf{j} \ (\text{m/s}^2)$$

Problem 17.112 The upper grip and jaw of the pliers *ABC* is stationary. The lower grip *DEF* is rotating in the clockwise direction with a constant angular velocity of 0.2 rad/s. At the instant shown, what is the angular acceleration of the lower jaw *CFG*?



Solution: From 17.42 we know that $\omega_{BE} = \omega_{CFG} = 0.0857$ rad/s.

$$\mathbf{a}_E = \mathbf{a}_B + \boldsymbol{\alpha}_{BE} \times \mathbf{r}_{E/B} - \omega_{BE}{}^2 \mathbf{r}_{E/B}$$

$$= 0 + \alpha_{BE}\mathbf{k} \times (0.07\mathbf{i} - 0.03\mathbf{j})\text{ m} - (0.0857\text{ rad/s})^2(0.07\mathbf{i} - 0.03\mathbf{j})\text{ m}$$

$$= (-0.000514\text{ m/s}^2 + \{0.03\text{ m}\}\alpha_{BE})\mathbf{i}$$

$$+ (0.0002204\text{ m/s}^2 + \{0.07\text{ m}\}\alpha_{BE})\mathbf{j}$$

$$\mathbf{a}_F = \mathbf{a}_E + \boldsymbol{\alpha}_{EF} \times \mathbf{r}_{F/E} - \omega_{EF}{}^2 \mathbf{r}_{F/E}$$

$$= (-0.000514\text{ m/s}^2 + \{0.03\text{ m}\}\alpha_{BE})\mathbf{i}$$

$$+ (0.0002204\text{ m/s}^2 + \{0.07\text{ m}\}\alpha_{BE})\mathbf{j} + 0$$

$$- (0.0857\text{ rad/s})^2(0.03\text{ m})\mathbf{i}$$

$$= (-0.00171\text{ m/s}^2 + \{0.03\text{ m}\}\alpha_{BE})\mathbf{i}$$

$$+ (0.0002204\text{ m/s}^2 + \{0.07\text{ m}\}\alpha_{BE})\mathbf{j}$$

$$\mathbf{a}_C = \mathbf{a}_F + \boldsymbol{\alpha}_{CFG} \times \mathbf{r}_{C/F} - \omega_{CFG}{}^2 \mathbf{r}_{C/F}$$

$$= (-0.00171\text{ m/s}^2 + \{0.03\text{ m}\}\alpha_{BE})\mathbf{i}$$

$$+ (0.0002204\text{ m/s}^2 + \{0.07\text{ m}\}\alpha_{BE})\mathbf{j} + \alpha_{CFG}\mathbf{k} \times (0.03\text{ m})\mathbf{j}$$

$$- (0.0857\text{ rad/s})^2(0.03\text{ m})\mathbf{j}$$

$$= (-0.00171\text{ m/s}^2 + \{0.03\text{ m}\}[\alpha_{BE} - \alpha_{CFG}])\mathbf{i} + (0.07\text{ m})\alpha_{BE}\mathbf{j}$$

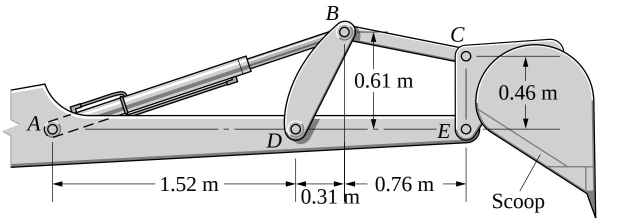
Since C is fixed we have

$$\left.\begin{array}{l} -0.00171\text{ m/s}^2 + \{0.03\text{ m}\}[\alpha_{BE} - \alpha_{CFG}] = 0 \\ (0.07\text{ m})\alpha_{BE} = 0 \end{array}\right\}$$

$$\Rightarrow \begin{array}{l} \alpha_{BE} = 0 \\ \alpha_{CFG} = -0.0571\text{ rad/s}^2 \end{array}$$

$$\boxed{\begin{array}{l} \alpha_{BE} = 0 \\ \alpha_{CFG} = 0.0571\text{ rad/s}^2\text{ CW} \end{array}}$$

Problem 17.113 The horizontal member ADE supporting the scoop is stationary. If the link BD has a clockwise angular velocity of 1 rad/s and a counterclockwise angular acceleration of 2 rad/s^2, what is the angular acceleration of the scoop?



Solution: The velocity of B is

$$\mathbf{v}_B = \mathbf{v}_D + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{B/D} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -1 \\ 0.31 & 0.61 & 0 \end{vmatrix}$$

$$= 0.61\mathbf{i} - 0.31\mathbf{j} \text{ (m/s)}.$$

The velocity of C is

$$\mathbf{v}_c = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = 0.61\mathbf{i} - 0.31\mathbf{j} + 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 0.76 & -0.15 & 0 \end{vmatrix} \quad \textbf{(1)}$$

We can also express $\mathbf{v}_c$ as

$$\mathbf{v}_c = \mathbf{v}_E + \boldsymbol{\omega}_{CE} \times \mathbf{r}_{C/E} = 0 + (\omega_{CE}\mathbf{k}) \times (0.46\,\mathbf{j}) = -0.46\omega_{CE}\mathbf{i}. \quad \textbf{(2)}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components in Equations (1) and (2) we get $0.61 + 0.15\omega_{BC} = -0.46\omega_{CE}$, and $-0.31 + 0.76\omega_{BC} = 0$. Solving, we obtain $\omega_{BC} = 0.400$ rad/s and $\omega_{CE} = -1.467$ rad/s.

The acceleration of B is

$$\mathbf{a}_B = \mathbf{a}_D + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{B/D} - \omega_{BD}^2 \mathbf{r}_{B/D},$$

or $$\mathbf{a}_B = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0.31 & 0.61 & 0 \end{vmatrix} - (1)^2(0.31\mathbf{i} + 0.61\mathbf{j})$$

$$= -1.52\,\mathbf{i} \text{ (m/s}^2).$$

The acceleration of C is

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B}$$

$$\mathbf{a}_C = -1.52\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 0.76 & -0.15 & 0 \end{vmatrix} - (0.4)^2(0.76\mathbf{i} - 0.15\mathbf{j}). \quad \textbf{(3)}$$

We can also express $\mathbf{a}_C$ as

$$\mathbf{a}_C = \mathbf{a}_E + \boldsymbol{\alpha}_{CE} \times \mathbf{r}_{C/E} - \omega_{CE}^2 \mathbf{r}_{C/E} = 0 + (\alpha_{CE}\mathbf{k})$$

$$\times (0.46\,\mathbf{j}) - (-1.467)^2(0.46\mathbf{j})$$

$$= -0.46\,\alpha_{CE}\mathbf{i} - 0.98\mathbf{j}. \quad \textbf{(4)}$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components in Equations (3) and (4), we get

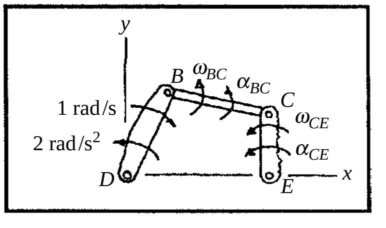
$$-1.52 + 0.15\alpha_{BC} - (0.4)^2(0.76) = -0.46\alpha_{CE},$$

and $$0.76\alpha_{BC} + (0.4)^2(0.15) = -0.98.$$

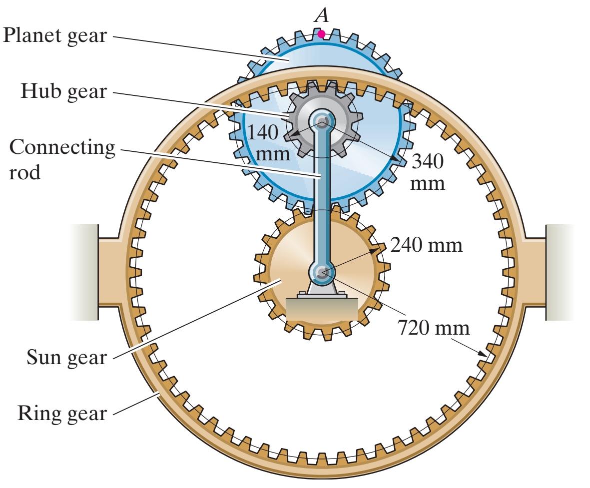
Solving, we obtain

$$\alpha_{BC} = -1.32 \text{ rad/s}^2$$

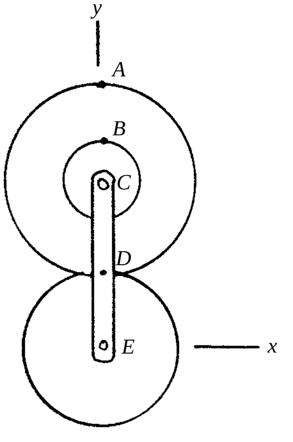
$$\alpha_{CE} = 4.04 \text{ rad/s}^2.$$



Problem 17.114 The ring gear is fixed, and the hub and planet gears are bonded together. The connecting rod has a counterclockwise angular acceleration of 10 rad/s^2. Determine the angular acceleration of the planet and sun gears.



Solution: The x components of the accelerations of pts B and C are



$$a_{Bx} = 0,$$

$$a_{Cx} = -(10 \text{ rad/s}^2)(0.58 \text{ m})$$

$$= -5.8 \text{ m/s}^2.$$

Let α_P and α_S be the angular accelerations of the planet and sun gears.

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_P \times \mathbf{r}_{B/C} - w_P^2 \mathbf{r}_{B/C}$$

$$= \mathbf{a}_C + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_P \\ 0 & 0.14 & 0 \end{vmatrix} - w_P^2(0.14\mathbf{j}).$$

The **i** component of this equation is

$$0 = -5.8 - 0.14\alpha_P.$$

We obtain

$$\underline{\alpha_P = -41.4 \text{ rad/s}^2}.$$

Also, $\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_P \times \mathbf{r}_{D/C} - \omega_P^2 \mathbf{r}_{D/C}$

$$= \mathbf{a}_C + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -41.4 \\ 0 & -0.34 & 0 \end{vmatrix} - \omega_P^2(-0.34\mathbf{j}).$$

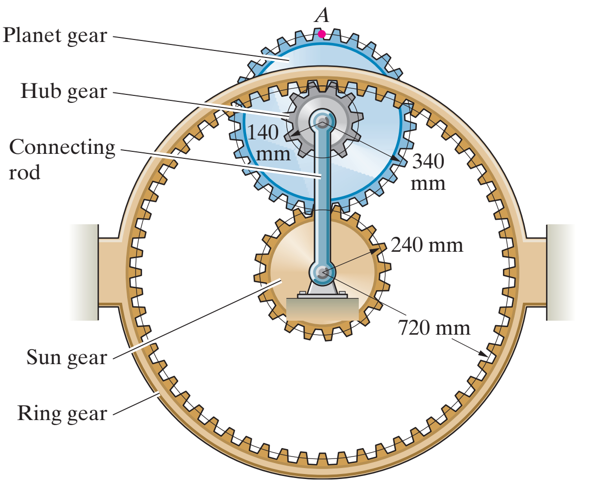
The **i** component of this equation is

$$a_{Dx} = -5.8 - (41.4)(0.34) = -19.9 \text{ m/s}^2.$$

Therefore

$$\underline{\alpha_S = \frac{19.9}{0.24} = 82.9 \text{ rad/s}^2}.$$

Problem 17.115 The connecting rod in Problem 17.114 has a counterclockwise angular velocity of 4 rad/s and a clockwise angular acceleration of 12 rad/s^2. Determine the magnitude of the acceleration at point A.



Solution: See the solution of Problem 17.114. The velocities of pts B and C are

$$\mathbf{v}_B = \mathbf{O},\ \mathbf{v}_C = -(4)(0.58)\mathbf{i} = -2.32\mathbf{i}\ \text{(m/s)}.$$

Let ω_P and ω_S be the angular velocities of the planet and sun gears.

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_P \times \mathbf{r}_{B/C}:$$

$$\mathbf{O} = -2.32\mathbf{i} + (\omega_P\mathbf{k}) \times (0.14\mathbf{j})$$

$$= (-2.32 - 0.14\omega_P)\mathbf{i}.$$

We see that $\omega_P = -16.6$ rad/s. Also,

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_P \times \mathbf{r}_{D/C}$$

$$= -2.32\mathbf{i} + (-16.6\mathbf{k}) \times (-0.34\mathbf{j})$$

$$= -7.95\mathbf{i}\ \text{(m/s)},$$

So $\omega_S = \dfrac{7.95}{0.24} = 33.1$ rad/s.

The x components of the accelerations of pts B and C are

$$a_{Bx} = 0,$$

$$a_{Cx} = (12\ \text{rad/s}^2)(0.58\ \text{m})$$

$$= 6.96\ \text{m/s}^2.$$

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_P \times \mathbf{r}_{B/C} - \omega_P^2\mathbf{r}_{B/C}$$

$$= \mathbf{a}_C + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_P \\ 0 & 0.14 & 0 \end{vmatrix} - \omega_P^2(0.14\mathbf{j}).$$

The $\mathbf{i}$ component is

$$0 = 6.96 - 0.14\alpha_P,$$

so $\alpha_P = 49.7$ rad/s^2.

The acceleration of C is

$$\mathbf{a}_C = \mathbf{a}_E + (-12\mathbf{k}) \times \mathbf{r}_{C/E} - (4)^2\mathbf{r}_{C/E}$$

$$= 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -12 \\ 0 & 0.58 & 0 \end{vmatrix} - (4)^2(0.58\mathbf{j})$$

$$= 6.96\mathbf{i} - 9.28\mathbf{j}\ (\text{m/s}^2).$$

Then the acceleration of A is

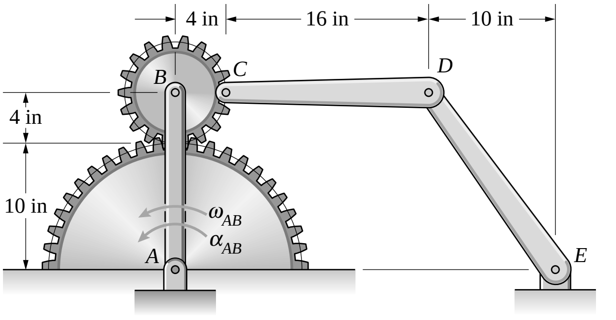
$$\mathbf{a}_A = \mathbf{a}_C + \boldsymbol{\alpha}_P \times \mathbf{r}_{A/C} - \omega_P^2\mathbf{r}_{A/C}$$

$$= 6.96\mathbf{i} - 9.28\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 49.7 \\ 0 & 0.34 & 0 \end{vmatrix} - (-16.6)^2(0.34\mathbf{j})$$

$$= -9.94\mathbf{i} - 102.65\mathbf{j}\ (\text{m/s}^2).$$

$\underline{|\mathbf{a}_A| = 103\ \text{m/s}^2}.$

Problem 17.116 The large gear is fixed. The angular velocity and angular acceleration of the bar *AB* are $\omega_{AB} = 2$ rad/s and $\alpha_{AB} = 4$ rad/s^2. Determine the angular acceleration of the bars *CD* and *DE*.



Solution: The strategy is to express vector velocity of point D in terms of the unknown angular velocities and accelerations of CD and DE, and then to solve the resulting vector equations for the unknowns. *The angular velocities* ω_{CD} and ω_{DE}. (See solution to Problem 17.51). The linear velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{AB} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0 & 14 & 0 \end{bmatrix} = -28\mathbf{i} \text{ (in/s)}.$$

The lower edge of gear B is stationary. The velocity of B is also

$$\mathbf{v}_B = \boldsymbol{\omega}_B \times \mathbf{r}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_B \\ 0 & 4 & 0 \end{bmatrix} = -4\omega_B\mathbf{i} \text{ (in/s)}.$$

Equate the velocities $\mathbf{v}_B$ to obtain the angular velocity of B:

$$\omega_B = -\frac{v_B}{4} = 7 \text{ rad/s}.$$

The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_B \times \mathbf{r}_{BC} = -28\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 7 \\ 4 & 0 & 0 \end{bmatrix} = -28\mathbf{i} + 28\mathbf{j} \text{ (in/s)}.$$

The velocity of point D is

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{CD} = -28\mathbf{i} + 28\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ 16 & 0 & 0 \end{bmatrix}$$

$$= -28\mathbf{i} + (16\omega_{CD} + 28)\mathbf{j} \text{ (in/s)}.$$

The velocity of point D is also given by

$$\mathbf{v}_D = \boldsymbol{\omega}_{DE} \times \mathbf{r}_{ED} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{DE} \\ -10 & 14 & 0 \end{bmatrix}$$

$$= -14\omega_{DE}\mathbf{i} - 10\omega_{DE}\mathbf{j} \text{ (in/s)}.$$

Equate and separate components:

$$(-28 + 14\omega_{DE})\mathbf{i} = 0, \ (16\omega_{CD} + 28 + 10\omega_{DE})\mathbf{j} = 0.$$

Solve: $\omega_{DE} = 2$ rad/s,

$\omega_{CD} = -3$ rad/s.

The negative sign means a clockwise rotation. *The angular accelerations*. The tangential acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 0 & 14 & 0 \end{bmatrix} = -56\mathbf{i} \text{ (in/s}^2\text{)}.$$

The tangential acceleration at the point of contact between the gears A and B is zero, from which

$$\mathbf{a}_B = \boldsymbol{\alpha}_{BC} \times 4\mathbf{j} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 0 & 4 & 0 \end{bmatrix} = -4\alpha_{BC}\mathbf{i} \text{ (in/s}^2\text{)},$$

from which $\alpha_{BC} = 14$ rad/s^2. The acceleration of point C in terms of the acceleration of point B is

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times 4\mathbf{i} - \omega_B^2(4\mathbf{i}) = -56\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 14 \\ 4 & 0 & 0 \end{bmatrix} - 49(4\mathbf{i})$$

$$= -252\mathbf{i} + 56\mathbf{j} \text{ (in/s}^2\text{)}.$$

The acceleration of point D in terms of the acceleration of point C is

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{CD} \times 16\mathbf{i} - \omega_{CD}^2(16\mathbf{i})$$

$$= \mathbf{a}_C + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ 16 & 0 & 0 \end{bmatrix} - \omega_{CD}^2(16\mathbf{i}),$$

$$\mathbf{a}_D = -396\mathbf{i} + (16\alpha_{CD} + 56)\mathbf{j} \text{ (in/s}^2\text{)}.$$

The acceleration of point D in terms of the acceleration of point E is

$$\mathbf{a}_D = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{DE} \\ -10 & 14 & 0 \end{bmatrix} - \omega_{DE}^2(-10\mathbf{i} + 14\mathbf{j})$$

$$= (40 - 14\alpha_{DE})\mathbf{i} - (10\alpha_{DE} + 56)\mathbf{j} \text{ (in/s}^2\text{)}$$

Equate the expressions for $\mathbf{a}_D$ and separate components:

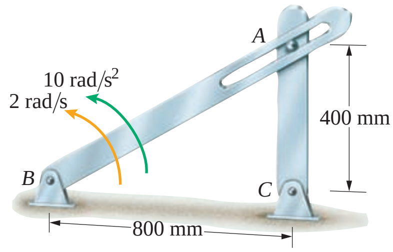
$$-396 = 40 - 14\alpha_{DE}, \ 16\alpha_{CD} + 56 = -10\alpha_{DE} - 56.$$

Solve: $\boxed{\alpha_{DE} = 31.1 \text{ rad/s}^2}$,

$\boxed{\alpha_{CD} = -26.5 \text{ rad/s}^2}$,

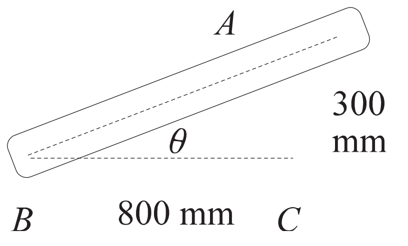
where the negative sign means a clockwise angular acceleration.

Problem 17.117 In Active Example 17.7, suppose that the distance from point C to the pin A on the vertical bar AC is 300 mm instead of 400 mm. Draw a sketch of the linkage with its new geometry. Determine the angular velocity of the bar AC and the velocity of the pin A relative to the slot in bar AB.



Solution: In the new position $\theta = \tan^{-1}\left(\dfrac{300}{800}\right) = 20.6^\circ$.

The velocity analysis:



$$\mathbf{V}_A = \mathbf{V}_C + \boldsymbol{\omega}_{AC} \times \mathbf{r}_{A/C}$$

$$= 0 + \omega_{AC}\mathbf{k} \times (0.3\mathbf{j}) = -0.3\omega_{AC}\mathbf{i}$$

$$\mathbf{V}_A = \mathbf{V}_B + \mathbf{V}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$= 0 + v_{Arel}(\cos\boldsymbol{\theta}\mathbf{i} + \sin\theta\mathbf{j})$$

$$+ (2\mathbf{k}) \times (0.8\mathbf{i} + 0.3\mathbf{j})$$

$$= (v_{Arel}\cos\theta - 0.6)\mathbf{i} + (v_{Arel}\sin\theta + 1.6)\mathbf{j}$$

Equating the components of the two expressions for $\mathbf{v}_A$ we have

$$-0.3\omega_{AC} = v_{Arel}\cos\theta - 0.6, \; 0 = v_{Arel}\sin\theta + 1.6.$$

Solving these two equations, we find

$$v_{Arel} = -4.56 \text{ m/s}, \quad \omega_{AC} = 16.2 \text{ rad/s}.$$

Thus

$$\boxed{\begin{array}{l} A \text{ is moving at 4.56 m/s from } B \text{ toward } A, \\ \omega_{AC} = 16.2 \text{ rad/s counterclockwise.} \end{array}}$$

Problem 17.118 The bar rotates with a constant counterclockwise angular velocity of 10 rad/s and sleeve A slides at a constant velocity of 4 m/s relative to the bar. Use Eq. (17.15) to determine the acceleration of A.

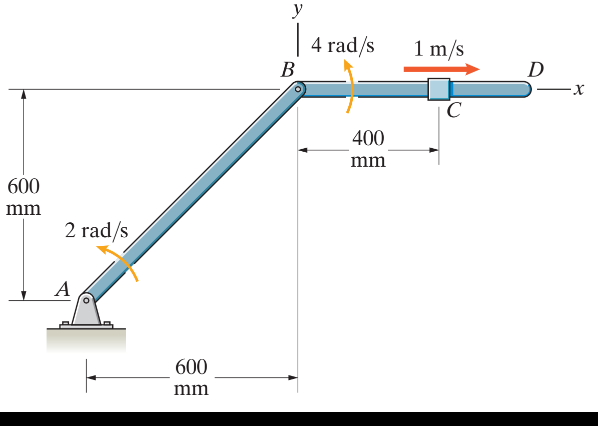
Solution: Eq. (17.15) is

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{\text{Arel}} + 2\omega \times \mathbf{v}_{\text{Arel}} + \alpha \times \mathbf{r}_{A/B} - \omega^2\mathbf{r}_{A/B}.$$

Substitute: $\mathbf{a}_A = 0 + 0 + 2\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 4 & 0 & 0 \end{bmatrix} + 0 - 100(2\mathbf{i})$

$$\boxed{= -200\mathbf{i} + 80\mathbf{j} \text{ (m/s}^2\text{)}}$$

Problem 17.119 Sleeve C slides at 1 m/s relative to bar BD. Use the body-fixed coordinate system shown to determine the velocity of C.



Solution: The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 600 & 600 & 0 \end{bmatrix} = -1200(\mathbf{i} - \mathbf{j}) \text{ (mm/s)}.$$

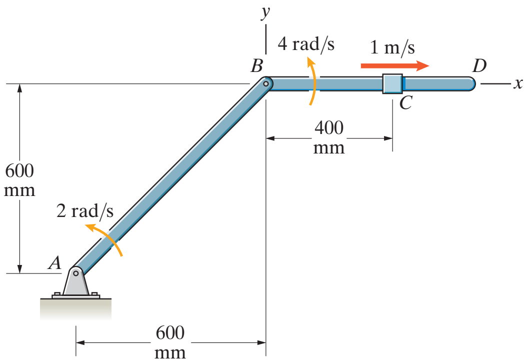
Use Eq. (17.11). The velocity of sleeve C is

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{\text{Arel}} + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{C/B}.$$

$$\mathbf{v}_C = -1200\mathbf{i} + 1200\mathbf{j} + 1000\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 400 & 0 & 0 \end{bmatrix}.$$

$$\boxed{\mathbf{v}_C = -200\mathbf{i} + 2800\mathbf{j} \text{ (mm/s)}}.$$

Problem 17.120 In Problem 17.119, the angular accelerations of the two bars are zero and the sleeve C slides at a constant velocity of 1 m/s relative to bar BD. What is the acceleration of C?



Solution: From Problem 17.119, $\omega_{AB} = 2$ rad/s, $\omega_{BC} = 4$ rad/s. The acceleration of point B is

$$\mathbf{a}_B = -\omega_{AB}^2 \mathbf{r}_{B/A} = -4(600\mathbf{i} + 600\mathbf{j})$$

$$= -2400\mathbf{i} - 2400\mathbf{j} \ (\text{mm/s}^2).$$

Use Eq. (17.15). The acceleration of C is

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C\text{rel}} + 2\boldsymbol{\omega}_{BD} \times \mathbf{v}_{C\text{rel}} + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{C/B} - \omega_{BD}^2 \mathbf{r}_{C/B}.$$

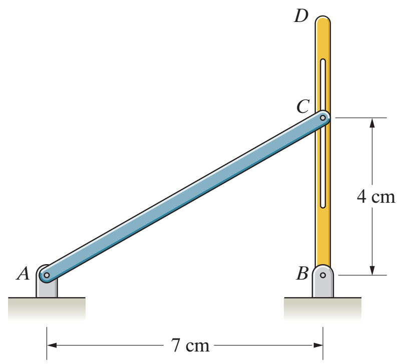
$$\mathbf{a}_C = -2400\mathbf{i} - 2400\mathbf{j} + 2 \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BD} \\ 1000 & 0 & 0 \end{bmatrix} - \omega_{BD}^2 (400\mathbf{i}),$$

$$\boxed{\mathbf{a}_C = -8800\mathbf{i} + 5600\mathbf{j} \ (\text{mm/s}^2)}$$

Problem 17.121 Bar AC has an angular velocity of 2 rad/s in the counterclockwise direction that is decreasing at 4 rad/s^2. The pin at C slides in the slot in bar BD.

(a) Determine the angular velocity of bar BD and the velocity of the pin relative to the slot.

(b) Determine the angular acceleration of bar BD and the acceleration of the pin relative to the slot.



Solution: The coordinate system is fixed with respect to the vertical bar.

(a) $$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega}_{AC} \times \mathbf{r}_{C/A} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AC} \\ 7 & 4 & 0 \end{vmatrix}. \quad \textbf{(1)}$$

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C\text{rel}} + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{C/B}$$

$$= \mathbf{0} + v_{C\text{rel}}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BD} \\ 0 & 4 & 0 \end{vmatrix}. \quad \textbf{(2)}$$

Equating **i** and **j** components in Eqs. (1) and (2),

$$-4\omega_{AC} = -4\omega_{BD}, \quad \textbf{(3)}$$

$$7\omega_{AC} = v_{C\text{rel}}, \quad \textbf{(4)}$$

We obtain $\underline{\omega_{BD} = 2 \text{ rad/s}}$, $\underline{v_{C\text{rel}} = 14 \text{ cm/s}}$.

(b) $$\boldsymbol{a}_C = \boldsymbol{a}_A + \boldsymbol{\alpha}_{AC} \times \boldsymbol{r}_{C/A} - \omega_{AC}^2 \boldsymbol{r}_{C/A}$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AC} \\ 7 & 4 & 0 \end{vmatrix} - \omega_{AC}^2 (7\mathbf{i} + 4\mathbf{j}). \quad \textbf{(5)}$$

$$\boldsymbol{a}_C = \boldsymbol{a}_B + \boldsymbol{a}_{C\text{rel}} + 2\boldsymbol{\omega}_{BD} \times \mathbf{v}_{C\text{rel}} + \boldsymbol{\alpha}_{BD} \times \boldsymbol{r}_{C/B} - \omega_{BD}^2 \boldsymbol{r}_{C/B}$$

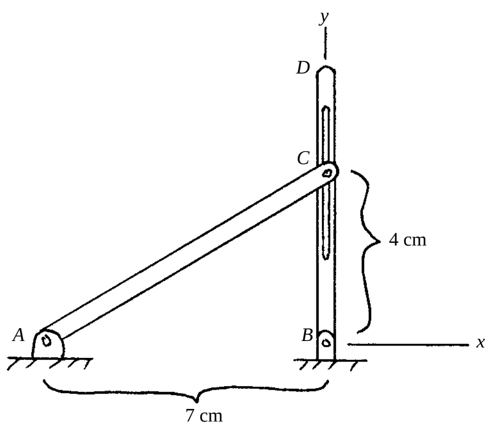
$$= \mathbf{0} + a_{C\text{rel}}\mathbf{j} + 2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BD} \\ 0 & v_{C\text{rel}} & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BD} \\ 0 & 4 & 0 \end{vmatrix} - \omega_{BD}^2 (4\mathbf{j}). \textbf{(6)}$$

Equating **i** and **j** components in Eqs. (5) and (6),

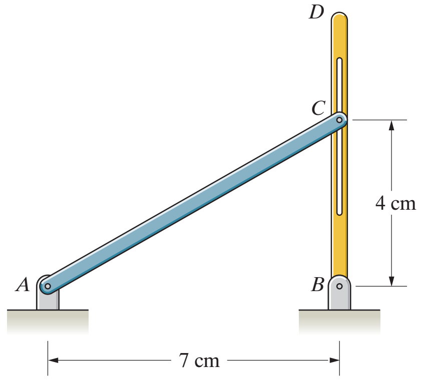
$$-4\alpha_{AC} - 7\omega_{AC}^2 = -2\omega_{BD} v_{C\text{rel}} - 4\alpha_{BD}, \quad \textbf{(7)}$$

$$7\alpha_{AC} - 4\omega_{AC}^2 = a_{C\text{rel}} - 4\omega_{BD}^2, \quad \textbf{(8)}$$

We obtain $\underline{\alpha_{BD} = -11 \text{ rad/s}^2}$, $\underline{a_{C\text{rel}} = -28 \text{ cm/s}^2}$.



Problem 17.122 In the system shown in Problem 17.121, the velocity of the pin C relative to the slot is 21 cm/s upward and is decreasing at 42 cm/s^2. What are the angular velocity and acceleration of bar AC?



Solution: See the solution of Problem 17.121. Solving Eqs. (3), (4), (7), and (8) with $v_{C\mathrm{rel}} = 21$ cm/s and $a_{C\mathrm{rel}} = -42$ cm/s^2, we obtain

$\underline{\omega_{AC} = 3 \text{ rad/s}},$

$\underline{\alpha_{AC} = -6 \text{ rad/s}^2}.$

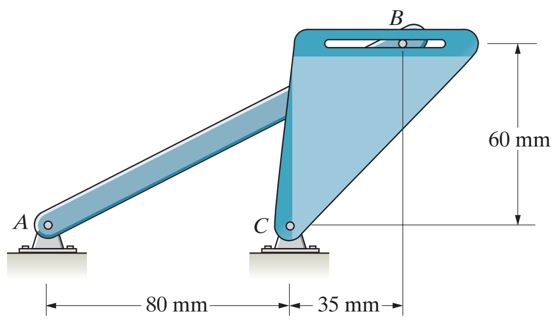
Problem 17.123 In the system shown in Problem 17.121, what should the angular velocity and acceleration of bar AC be if you want the angular velocity and acceleration of bar BD to be 4 rad/s counterclockwise and 24 rad/s^2 counterclockwise, respectively?

Solution: See the solution of Problem 17.121. Solving Eqs. (3), (4), (7), and (8) with $\omega_{BD} = 4$ rad/s^2 and $\alpha_{BD} = 24$ rad/s^2, we obtain

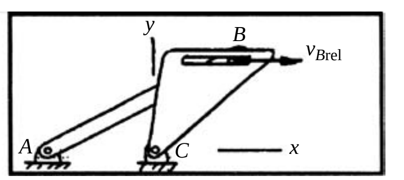
$\omega_{AC} = 4$ rad/s,

$\alpha_{AC} = 52$ rad/s^2.

Problem 17.124 Bar AB has an angular velocity of 4 rad/s in the clockwise direction. What is the velocity of pin B relative to the slot?



Solution: The velocity of point B is



$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega_{AB} \\ 115 & 60 & 0 \end{bmatrix} = 240\mathbf{i} - 460\mathbf{j} \text{ (mm/s)}.$$

The velocity of point B is also determined from bar CB

$$\mathbf{v}_B = \mathbf{v}_{B\mathrm{rel}} + \boldsymbol{\omega}_{CB} \times (35\mathbf{i} + 60\mathbf{j}),$$

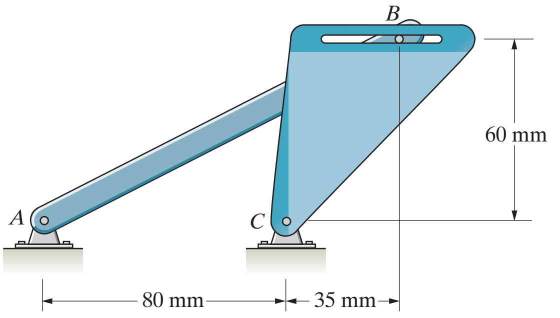
$$\mathbf{v}_B = \mathbf{v}_{B\mathrm{rel}} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CB} \\ 35 & 60 & 0 \end{bmatrix}$$

$$\mathbf{v}_B = v_{B\mathrm{rel}}\mathbf{i} - 60\omega_{CB}\mathbf{i} + 35\omega_{CB}\mathbf{j} \text{ (mm/s)}.$$

Equate like terms: $240 = v_{B\mathrm{rel}} - 60\omega_{CB}$, $-460 = 35\omega_{CB}$ from which

$\omega_{BC} = -13.14$ rad/s, $\boxed{v_{B\mathrm{rel}} = -548.6 \text{ mm/s}}$

Problem 17.125 In the system shown in Problem 17.124, the bar AB has an angular velocity of 4 rad/s in the clockwise direction and an angular acceleration of 10 rad/s^2 in the counterclockwise direction. What is the acceleration of pin B relative to the slot?



Solution: Use the solution to Problem 17.124, from which $\omega_{BC} = -13.14$ rad/s, $v_{B\text{rel}} = -548.6$ mm/s. *The angular acceleration and the relative acceleration.* The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 115 & 60 & 0 \end{bmatrix} - (16)(115\mathbf{i} + 60\mathbf{j}) \text{ (mm/s}^2\text{)},$$

$$\mathbf{a}_B = -600\mathbf{i} + 1150\mathbf{j} - 1840\mathbf{i} - 960\mathbf{j} = -2440\mathbf{i} + 190\mathbf{j} \text{ (mm/s}^2\text{)}.$$

The acceleration of pin B in terms of bar BC is

$$\mathbf{a}_B = a_{B\text{rel}}\mathbf{i} + 2\boldsymbol{\omega}_{BC} \times v_{B\text{rel}} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2 \mathbf{r}_{B/C},$$

$$\mathbf{a}_B = a_{B\text{rel}}\mathbf{i} + 2\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -13.14 \\ -548.6 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 35 & 60 & 0 \end{bmatrix} - (13.14^2)(35\mathbf{i} + 60\mathbf{j}).$$

$$\mathbf{a}_B = a_{B\text{rel}}\mathbf{i} + 14{,}419.5\mathbf{j} + 35\alpha_{BC}\mathbf{j} - 60\alpha_{BC}\mathbf{i}$$

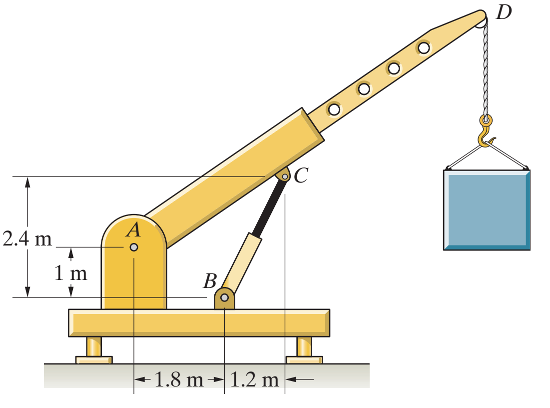
$$- 6045.7\mathbf{i} - 10,364.1\mathbf{j}.$$

Equate expressions for $\mathbf{a}_B$ and separate components: $-2440 = a_{B\text{rel}} - 60\alpha_{BC} - 6045.7$, $190 = 14{,}419.6 + 35\alpha_{BC} - 10364.1$. Solve:

$\boxed{\mathbf{a}_{B\text{rel}} = -3021\mathbf{i} \text{ (mm/s}^2\text{)}}$, $\alpha_{BC} = -110.4$ rad/s^2.

Problem 17.126 The hydraulic actuator BC of the crane is extending (increasing in length) at a constant rate of 0.2 m/s. At the instant shown, what is the angular velocity of the crane's boom AD?

Strategy: Use Eq. (17.8) to write the velocity of point C in terms of the velocity of point A, and use Eq. (17.11) to write the velocity of point C in terms of the velocity of point B. Then equate your two expressions for the velocity of point C.



Solution: Using Bar ACD

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega}_{AD} \times \mathbf{r}_{C/A} = 0 + \omega_{AD}\mathbf{k} \times (3\mathbf{i} + 1.4\mathbf{j}) \text{ m}$$

$$= \omega_{AD}(-1.4\mathbf{i} + 3\mathbf{j}) \text{ m}$$

Using cylinder BC

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{Brel} + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= 0 + (0.2 \text{ m/s})\left(\frac{1.2\mathbf{i} + 2.4\mathbf{j}}{\sqrt{1.2^2 + 2.4^2}}\right) + \omega_{BC}\mathbf{k} \times (1.2\mathbf{i} + 2.4\mathbf{j}) \text{ m}$$

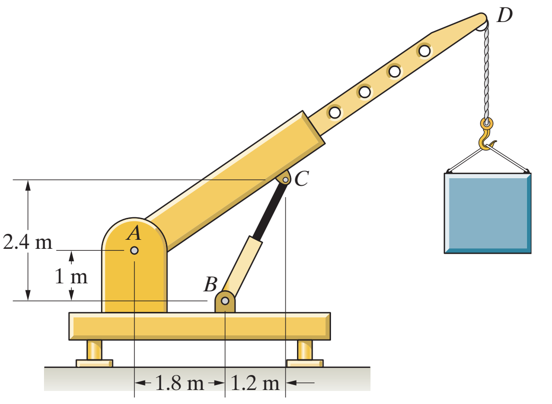
$$= (0.0894 \text{ m/s} - \{2.4 \text{ m}\}\omega_{BC})\mathbf{i} + (0.1789 \text{ m/s} + \{1.2 \text{ m}\}\omega_{BC})\mathbf{j}$$

Equating the components of the two expressions we have

$$\left.\begin{aligned} (-1.4 \text{ m})\omega_{AD} &= 0.0894 \text{ m/s} - (2.4 \text{ m})\omega_{BC} \\ (3 \text{ m})\omega_{AD} &= 0.1789 \text{ m/s} + (1.2 \text{ m})\omega_{BC} \end{aligned}\right\} \Rightarrow \begin{aligned} \omega_{BC} &= 0.0940 \text{ rad/s} \\ \omega_{AD} &= 0.0972 \text{ rad/s} \end{aligned}$$

$$\boxed{\omega_{AD} = 0.0972 \text{ rad/s CCW}}$$

Problem 17.127 In Problem 17.126, what is the angular acceleration of the crane's boom *AD* at the instant shown?



Solution: Use the angular velocities from 17.126

Bar *ACD*

$$\mathbf{a}_C = \mathbf{a}_A + \boldsymbol{\alpha}_{ACD} \times \mathbf{r}_{C/A} - \omega_{ACD}{}^2\mathbf{r}_{C/A}$$

$$= 0 + \alpha_{ACD}\mathbf{k} \times (3\mathbf{i} + 1.4\mathbf{j})\text{ m} - \omega_{ACD}{}^2(3\mathbf{i} + 1.4\mathbf{j})\text{ m}$$

$$= (-\{1.4\text{ m}\}\alpha_{ACD} - \{3\text{ m}\}\omega_{ACD}{}^2)\mathbf{i} + (\{3\text{ m}\}\alpha_{ACD}$$

$$- \{1.4\text{ m}\}\omega_{ACD}{}^2)\mathbf{j}$$

Cylinder *BC*

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{Brel} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}{}^2\mathbf{r}_{C/B} + 2\boldsymbol{\omega}_{BC} \times \mathbf{v}_{Brel}$$

$$= 0 + 0 + \alpha_{BC}\mathbf{k} \times (1.2\mathbf{i} + 2.4\mathbf{j})\text{ m} - \omega_{BC}{}^2(1.2\mathbf{i} + 2.4\mathbf{j})\text{ m}$$

$$+ 2\ \omega_{BC}\mathbf{k} \times (0.2\text{ m/s})\left(\frac{1.2\mathbf{i} + 2.4\mathbf{j}}{\sqrt{1.2^2 + 2.4^2}}\right)$$

$$= (-\{2.4\text{ m}\}\alpha_{BC} - \{1.79\text{ m/s}\}\omega_{BC} - \{1.2\text{ m}\}\omega_{BC}{}^2)\mathbf{i}$$

$$+ (\{1.2\text{ m}\}\alpha_{BC} + \{0.894\text{ m/s}\}\omega_{BC} - \{2.4\text{ m}\}\omega_{BC}{}^2)\mathbf{j}$$

Equating the two expressions for the acceleration of *C* we have

$$-\{1.4\text{ m}\}\alpha_{AD} - \{3\text{ m}\}\omega_{AD}{}^2 = -\{2.4\text{ m}\}\alpha_{BC}$$

$$- \{1.79\text{ m/s}\}\omega_{BC} - \{1.2\text{ m}\}\omega_{BC}{}^2$$

$$\{3\text{ m}\}\alpha_{AD} - \{1.4\text{ m}\}\omega_{AD}{}^2 = \{1.2\text{ m}\}\alpha_{BC}$$

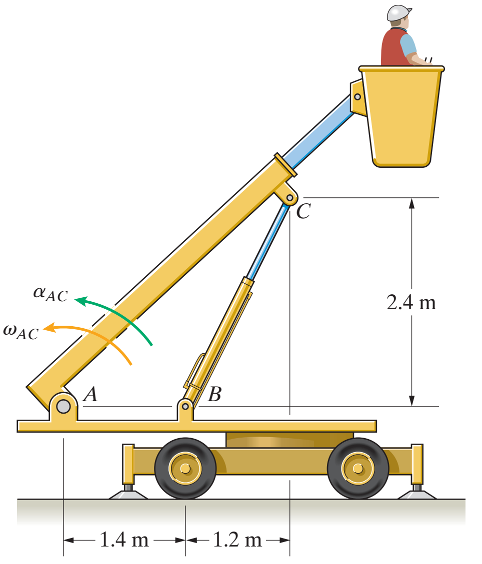
$$+ \{0.894\text{ m/s}\}\omega_{BC} - \{2.4\text{ m}\}\omega_{BC}{}^2$$

Solving (using the angular velocities from 17.126) we find

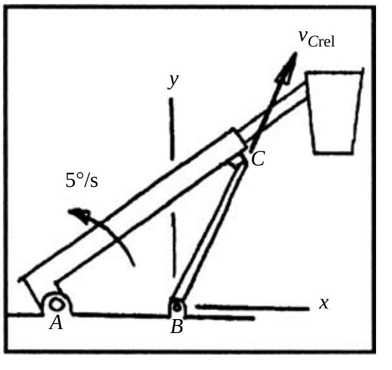
$\alpha_{BC} = -0.0624$ rad/s, $\alpha_{AD} = 0.000397$ rad/s

$$\boxed{\alpha_{AD} = 0.000397\text{ rad/s CCW}}$$

Problem 17.128 The angular velocity $\omega_{AC} = 5°$ per second. Determine the angular velocity of the hydraulic actuator BC and the rate at which the actuator is extending.



Solution: The point C effectively slides in a slot in the arm BC. The angular velocity of



$$\omega_{AC} = 5\left(\frac{\pi}{180}\right) = 0.0873 \text{ rad/s}.$$

The velocity of point C with respect to arm AC is

$$\mathbf{v}_C = \omega_{AC} \times \mathbf{r}_{C/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AC} \\ 2.6 & 2.4 & 0 \end{bmatrix}$$

$$= -0.2094\mathbf{i} + 0.2269\mathbf{j} \text{ (m/s)},$$

The unit vector parallel to the actuator BC is

$$\mathbf{e} = \frac{1.2\mathbf{i} + 2.4\mathbf{j}}{\sqrt{1.2^2 + 2.4^2}} = 0.4472\mathbf{i} + 0.8944\mathbf{j}.$$

The velocity of point C in terms of the velocity of the actuator is

$$\mathbf{v}_C = v_{C\text{rel}}\mathbf{e} + \omega_{BC} \times \mathbf{r}_{C/B}.$$

$$\mathbf{v}_C = v_{C\text{rel}}(0.4472\mathbf{i} + 0.8944\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 1.2 & 2.4 & 0 \end{bmatrix}$$

$$\mathbf{v}_C = v_{C\text{rel}}(0.4472\mathbf{i} + 0.8944\mathbf{j}) + \omega_{BC}(-2.4\mathbf{i} + 1.2\mathbf{j}).$$

Equate like terms in the two expressions:

$$-0.2094 = 0.4472 v_{C\text{rel}} - 2.4\omega_{BC},$$

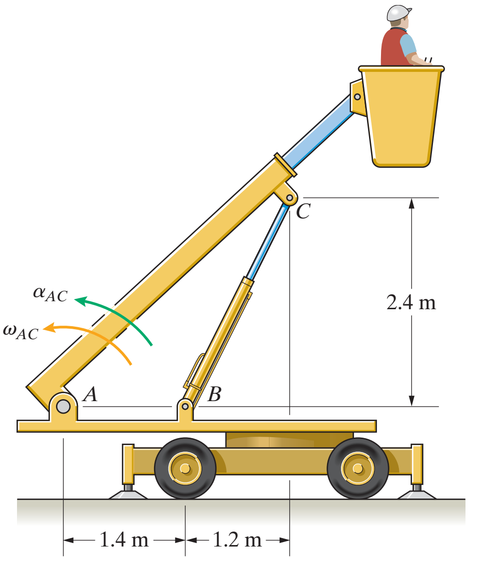
$$0.2269 = 0.8944 v_{C\text{rel}} + 1.2\omega_{BC}.$$

$$\boxed{\omega_{BC} = 0.1076 \text{ rad/s} = 6.17 \text{ deg/s}},$$

$$\boxed{v_{C\text{rel}} = 0.109 \text{ (m/s)}},$$

which is also the velocity of extension of the actuator.

Problem 17.129 In Problem 17.128, if the angular velocity $\omega_{AC} = 5^\circ$ per second and the angular acceleration $\alpha_{AC} = -2^\circ$ per second squared, determine the angular acceleration of the hydraulic actuator BC and the rate of change of the actuator's rate of extension.



Solution: Use the solution to Problem 17.128 for the velocities:

$\omega_{BC} = 0.1076$ rad/s,

$\omega_{AC} = 0.0873$ rad/s

$v_{C\text{rel}} = 0.1093$ (m/s).

The angular acceleration

$$\alpha_{AC} = -2\left(\frac{\pi}{180}\right) = -0.03491 \text{ rad/s}^2.$$

The acceleration of point C is

$$\mathbf{a}_C = \boldsymbol{\alpha}_{AC} \times \mathbf{r}_{C/A} - \omega_{AC}^2 \mathbf{r}_{C/A}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AC} \\ 2.6 & 2.4 & 0 \end{bmatrix} - \omega_{AC}^2(2.6\mathbf{i} + 2.4\mathbf{j}),$$

$$\mathbf{a}_C = \alpha_{AC}(-2.4\mathbf{i} + 2.6\mathbf{j}) - \omega_{AC}^2(2.6\mathbf{i} + 2.4\mathbf{j})$$

$$= 0.064\mathbf{i} - 0.109\mathbf{j} \text{ (m/s}^2).$$

The acceleration of point C in terms of the hydraulic actuator is

$$\mathbf{a}_C = a_{C\text{rel}}\mathbf{e} + 2\boldsymbol{\omega}_{BC} \times \mathbf{v}_{C\text{rel}} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{\mathbf{BC}}^2 \mathbf{r}_{C/B},$$

$$\mathbf{a}_C = a_{C\text{rel}}\mathbf{e} + 2\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 0.4472 v_{C\text{rel}} & 0.8944 v_{C\text{rel}} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 1.2 & 2.4 & 0 \end{bmatrix} - \omega_{BC}^2(1.2\mathbf{i} + 2.4\mathbf{j})$$

$$\mathbf{a}_C = a_{C\text{rel}}(0.4472\mathbf{i} + 0.8944\mathbf{j}) + 2\omega_{BC}(-0.0977\mathbf{i} + 0.0489\mathbf{j})$$

$$+ \alpha_{BC}(-2.4\mathbf{i} + 1.2\mathbf{j}) - \omega_{BC}^2(1.2\mathbf{i} + 2.4\mathbf{j}).$$

Equate like terms in the two expressions for $\mathbf{a}_C$.

$$0.0640 = 0.4472 a_{C\text{rel}} - 0.0139 - 2.4\alpha_{BC} - 0.0210,$$

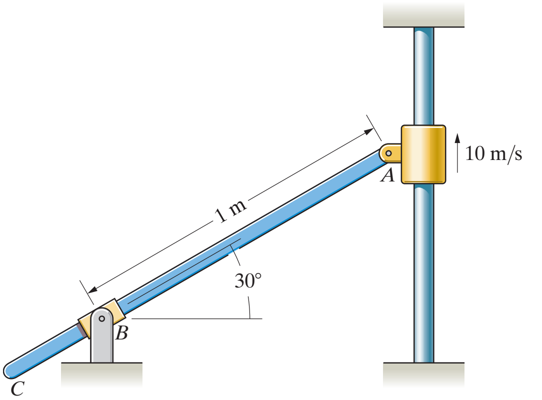
$$-0.1090 = 0.8944 a_{C\text{rel}} - 0.0278 + 1.2\alpha_{BC} + 0.0105.$$

Solve: $\boxed{a_{C\text{rel}} = -0.0378 \text{ (m/s}^2)}$,

which is the rate of change of the rate of extension of the actuator, and

$\boxed{\alpha_{BC} = -0.0483 \text{ (rad/s}^2) = -2.77 \text{ deg/s}^2}$

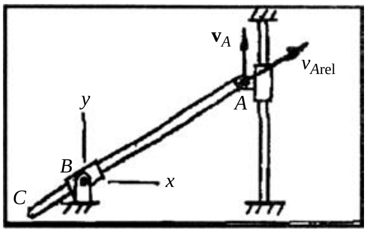
Problem 17.130 The sleeve at A slides upward at a constant velocity of 10 m/s. Bar AC slides through the sleeve at B. Determine the angular velocity of bar AC and the velocity at which the bar slides relative to the sleeve at B. (See Example 17.8.)



Solution: The velocity of the sleeve at A is given to be $\mathbf{v}_A = 10\mathbf{j}$ (m/s). The unit vector parallel to the bar (toward A) is

$$\mathbf{e} = 1(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = 0.866\mathbf{i} + 0.5\mathbf{j}.$$

Choose a coordinate system with origin at B that rotates with the bar. The velocity at A is



$$\mathbf{v}_A = \mathbf{v}_B + v_{A\mathrm{rel}}\mathbf{e} + \boldsymbol{\omega}_{AC} \times \mathbf{r}_{A/B}$$

$$= 0 + v_{A\mathrm{rel}}\mathbf{e} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 0.866 & 0.5 & 0 \end{bmatrix}$$

$$\mathbf{v}_A = (0.866\mathbf{i} + 0.5\mathbf{j})v_{A\mathrm{rel}} + \omega_{AC}(-0.5\mathbf{i} + 0.866\mathbf{j}) \text{ (m/s)}.$$

The given velocity is $\mathbf{v}_A = 10\mathbf{j}$ (m/s). Equate like components in the two expressions for $\mathbf{v}_A$:

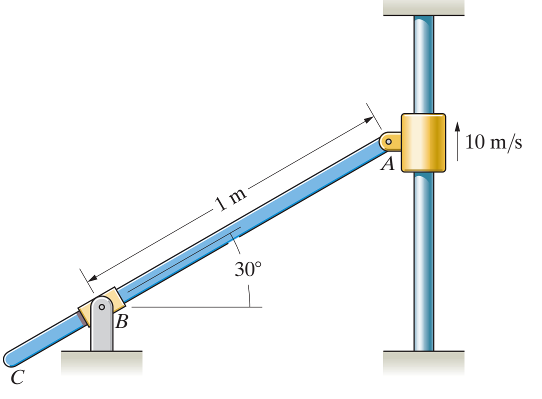
$$0 = 0.866v_{A\mathrm{rel}} - 0.5\omega_{AC},$$

$$10 = 0.5v_{A\mathrm{rel}} + 0.866\omega_{AC}.$$

Solve: $\boxed{\omega_{AC} = 8.66 \text{ rad/s}}$ (counterclockwise),

$\boxed{v_{A\mathrm{rel}} = 5 \text{ m/s}}$ from B toward A.

Problem 17.131 In Problem 17.130, the sleeve at A slides upward at a constant velocity of 10 m/s. Determine the angular acceleration of bar AC and the rate of change of the velocity at which the bar slides relative to the sleeve at B. (See Example 17.8.)



Solution: Use the solution of Problem 17.130:

$$\mathbf{e} = 0.866\mathbf{i} + 0.5\mathbf{j},$$

$$\omega_{AB} = 8.66 \text{ rad/s},$$

$$v_{A\mathrm{rel}} = 5 \text{ m/s}.$$

The acceleration of the sleeve at A is given to be zero. The acceleration in terms of the motion of the arm is

$$\mathbf{a}_A = 0 = a_{A\mathrm{rel}}\mathbf{e} + 2\boldsymbol{\omega}_{AB} \times v_{A\mathrm{rel}}\mathbf{e} + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} - \omega_{AB}^2\mathbf{r}_{A/B}.$$

$$\mathbf{a}_A = 0 = a_{A\mathrm{rel}}\mathbf{e} + 2v_{A\mathrm{rel}}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 0.866 & 0.5 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 0.866 & 0.5 & 0 \end{bmatrix} - \omega_{AB}^2(0.866\mathbf{i} + 0.5\mathbf{j})$$

$$0 = (0.866\mathbf{i} + 0.5\mathbf{j})a_{A\mathrm{rel}} - 43.3\mathbf{i} + 75\mathbf{j}$$

$$+ \alpha_{\mathbf{AB}}(-0.5\mathbf{i} + 0.866\mathbf{j}) - 64.95\mathbf{i} - 37.5\mathbf{j}.$$

Separate components:

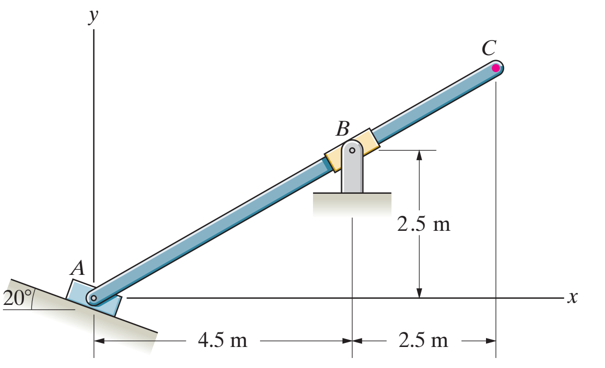
$$0 = 0.866a_{A\mathrm{rel}} - 43.3 - 0.5\alpha_{AB} - 64.95,$$

$$0 = 0.5a_{A\mathrm{rel}} + 75 + 0.866\alpha_{AB} - 37.5.$$

Solve: $\boxed{a_{A\mathrm{rel}} = 75 \text{ (m/s}^2)}$ (toward A).

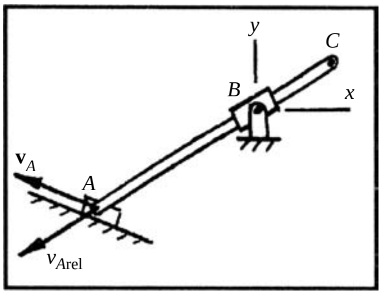
$\boxed{\alpha_{AB} = -86.6 \text{ rad/s}^2}$, (clockwise).

Problem 17.132 Block A slides up the inclined surface at 2 m/s. Determine the angular velocity of bar AC and the velocity of point C.



Solution: The velocity at A is given to be

$$\mathbf{v}_A = 2(-\mathbf{i}\cos 20^\circ + \mathbf{j}\sin 20^\circ) = -1.879\mathbf{i} + 0.6840\mathbf{j}\ \text{(m/s)}.$$



From geometry, the coordinates of point C are

$$\left(7, 2.5\left(\frac{7}{4.5}\right)\right) = (7, 3.89)\ \text{(m)}.$$

The unit vector parallel to the bar (toward A) is

$$\mathbf{e} = (7^2 + 3.89^2)^{-1/2}(-7\mathbf{i} - 3.89\mathbf{j}) = -0.8742\mathbf{i} - 0.4856\mathbf{j}.$$

The velocity at A in terms of the motion of the bar is

$$\mathbf{v}_A = v_{A\text{rel}}\mathbf{e} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B} = v_{A\text{rel}}\mathbf{e} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AC} \\ -4.5 & -2.5 & 0 \end{bmatrix},$$

$$\mathbf{v_A} = -0.8742 v_{A\text{rel}}\mathbf{i} - 0.4856 v_{A\text{rel}}\mathbf{j} + 2.5\omega_{AC}\mathbf{i} - 4.5\omega_{AC}\mathbf{j}\ \text{(m/s)}.$$

Equate the two expressions for $\mathbf{v}_A$ and separate components:

$$-1.879 = -0.8742 v_{A\text{rel}} + 2.5\omega_{AC},$$

$$0.6840 = -0.4856 v_{A\text{rel}} - 4.5\omega_{AC}.$$

Solve: $v_{A\text{rel}} = 1.311$ m/s,

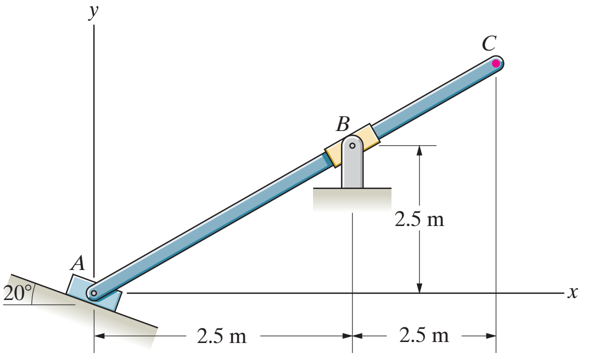
$$\boxed{\omega_{AC} = -0.293\ \text{rad/s}}\ \text{(clockwise)}.$$

Noting that $\mathbf{v}_A = 2$ m/s, the velocity at point C is

$$\mathbf{v}_C = v_A(-0.8742\mathbf{i} - 0.4856\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -0.293 \\ 2.5 & 3.89 - 2.5 & 0 \end{bmatrix},$$

$$\boxed{\mathbf{v}_C = -0.738\mathbf{i} - 1.37\mathbf{j}\ \text{(m/s)}}.$$

Problem 17.133 In Problem 17.132, block A slides up the inclined surface at a constant velocity of 2 m/s. Determine the angular acceleration of bar AC and the acceleration of point C.



Solution: *The velocities*: The velocity at A is given to be

$$\mathbf{v}_A = 2(-\mathbf{i}\cos 20^\circ + \mathbf{j}\sin 20^\circ) = -1.879\mathbf{i} + 0.6840\mathbf{j}\ (\text{m/s}).$$

From geometry, the coordinates of point C are

$$\left(7, 2.5\left(\frac{7}{4.5}\right)\right) = (7, 3.89)\ (\text{m}).$$

The unit vector parallel to the bar (toward A) is

$$\mathbf{e} = \frac{-7\mathbf{i} - 3.89\mathbf{j}}{\sqrt{7^2 + 3.89^2}} = -0.8742\mathbf{i} - 0.4856\mathbf{j}.$$

The velocity at A in terms of the motion of the bar is

$$\mathbf{v}_A = v_{A\text{rel}}\mathbf{e} + \boldsymbol{\omega}_{\mathbf{AC}} \times \mathbf{r}_{A/B} = v_{A\text{rel}}\mathbf{e} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AC} \\ -4.5 & -2.5 & 0 \end{bmatrix},$$

$$\mathbf{v}_A = -0.8742 v_{A\text{rel}}\mathbf{i} - 0.4856 v_{A\text{rel}}\mathbf{j} + 2.5\omega_{AC}\mathbf{i} - 4.5\omega_{AC}\mathbf{j}\ (\text{m/s}).$$

Equate the two expressions and separate components:

$$-1.879 = -0.8742 v_{A\text{rel}} + 2.5\omega_{AC},$$

$$0.6842 = -0.4856 v_{B\text{rel}} - 4.5\omega_{AC}.$$

Solve: $v_{A\text{rel}} = 1.311$ m/s, $\omega_{AC} = -0.293$ rad/s (clockwise).

The accelerations: The acceleration of block A is given to be zero. In terms of the bar AC, the acceleration of A is

$$\mathbf{a}_A = 0 = a_{A\text{rel}}\mathbf{e} + 2\boldsymbol{\omega}_{AC} \times v_{A\text{rel}}\mathbf{e} + \boldsymbol{\alpha}_{AC} \times \mathbf{r}_{A/B} - \omega_{AC}^2 \mathbf{r}_{A/B}.$$

$$0 = a_{A\text{rel}}\mathbf{e} + 2\omega_{AC} v_{A\text{rel}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -0.8742 & -0.4856 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AC} \\ -4.5 & -2.5 & 0 \end{bmatrix} - \omega_{AC}^2(-4.5\mathbf{i} - 2.5\mathbf{j}).$$

$$0 = a_{A\text{rel}}\mathbf{e} + 2\omega_{AC} v_{A\text{rel}}(-e_y\mathbf{i} + e_x\mathbf{j}) + \alpha_{AC}(2.5\mathbf{i} - 4.5\mathbf{j})$$

$$- \omega_{AC}^2(-4.5\mathbf{i} - 2.5\mathbf{j}).$$

Separate components to obtain:

$$0 = -0.8742 a_{A\text{rel}} - 0.3736 + 2.5\alpha_{AC} + 0.3875,$$

$$0 = -0.4856 a_{A\text{rel}} + 0.6742 - 4.5\alpha_{AC} + 0.2153.$$

Solve: $a_{A\text{rel}} = 0.4433\ (\text{m/s}^2)$ (toward A).

$\boxed{\alpha_{AC} = 0.1494\ \text{rad/s}^2}$ (counterclockwise).

The acceleration of point C is

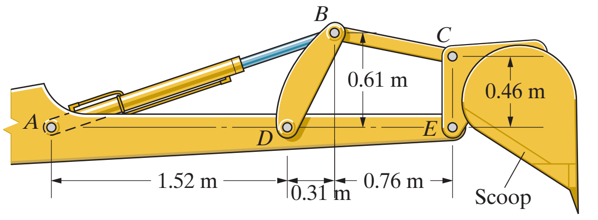
$$\mathbf{a}_C = a_{A\text{rel}}\mathbf{e} + 2\boldsymbol{\omega}_{AC} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha}_{AC} \times \mathbf{r}_{C/B} - \omega_{AC}^2 \mathbf{r}_{C/B}$$

$$\mathbf{a}_C = a_{A\text{rel}}\mathbf{e} + 2\omega_{AC} v_{A\text{rel}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ e_x & e_y & 0 \end{bmatrix}$$

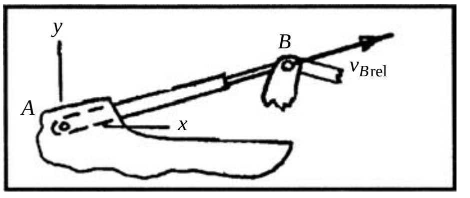
$$+ \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AC} \\ 2.5 & 3.89 - 2.5 & 0 \end{bmatrix} - \omega_{AC}^2(2.5\mathbf{i} + (3.89 - 2.5)\mathbf{j}).$$

Substitute numerical values: $\boxed{\mathbf{a}_C = -1.184\mathbf{i} + 0.711\mathbf{j}\ (\text{m/s}^2)}$

Problem 17.134 The angular velocity of the scoop is 1 rad/s clockwise. Determine the rate at which the hydraulic actuator AB is extending.



Solution: The point B slides in the arm AB. The velocity of point C is



$$\mathbf{v}_C = \boldsymbol{\omega}_{\text{scoop}} \times (0.46\mathbf{j}) = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 0.46 & 0 \end{bmatrix} = 0.46\mathbf{i} \text{ (m/s)}.$$

Point B is constrained to move normally to the arm DB: The unit vector parallel to DB is

$$\mathbf{e}_{DB} = \frac{0.31\mathbf{i} + 0.61\mathbf{j}}{\sqrt{0.31^2 + 0.61^2}} = 0.4472\mathbf{i} + 0.8944\mathbf{j}.$$

The unit vector normal to $\mathbf{e}_{DB}$ is $\mathbf{e}_{NDB} = 0.8944\mathbf{i} - 0.4472\mathbf{j}$, from which the velocity of C in terms of BC is

$$\mathbf{v}_C = v_B \mathbf{e}_{NBD} + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= v_B(0.8944\mathbf{i} - 0.4472\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 0.76 & -0.15 & 0 \end{bmatrix}.$$

$$\mathbf{v}_C = v_B(0.8944\mathbf{i} - 0.4472\mathbf{j}) + \omega_{BC}(0.15\mathbf{i} + 0.76\mathbf{j}).$$

Equate terms in $\mathbf{v}_C$, $0.46 = 0.8944 v_B + 0.15\omega_{BC}$, $O = -0.4472 v_B + 0.76\omega_{BC}$. Solve: $\omega_{BC} = 0.2727$ rad/s, $v_B = 0.465$ m/s, from which $\mathbf{v}_B = v_B \mathbf{e}_{NDB} = 1.364\mathbf{i} - 0.6818\mathbf{j}$ (m/s).

The unit vector parallel to the arm AB is

$$\mathbf{e}_{AB} = \frac{1.83\mathbf{i} + 0.61\mathbf{j}}{\sqrt{1.83^2 + 0.61^2}} = 0.9487\mathbf{i} + 0.3162\mathbf{j}$$

Choose a coordinate system with origin at A rotating with arm AB. The velocity of point B is

$$\mathbf{v}_B = v_{B\text{rel}} \mathbf{e}_{AB} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$= v_{B\text{rel}}(0.9487\mathbf{i} + 0.3162\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 1.83 & 0.61 & 0 \end{bmatrix}.$$

$$\mathbf{v}_B = v_{B\text{rel}}(0.9487\mathbf{i} + 0.3162\mathbf{j}) + \omega_{AB}(-0.61\mathbf{i} + 1.83\mathbf{j}).$$

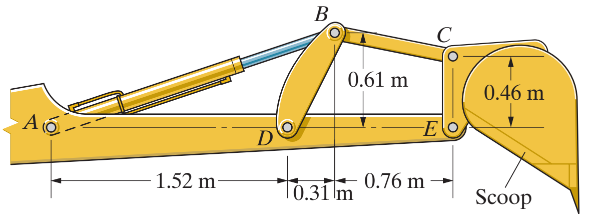
Equate the expressions and separate components:

$$0.416 = 0.9487 v_{B\text{rel}} - 0.61\omega_{AB},$$

$$-0.208 = 0.3162 v_{B\text{rel}} + 1.83\omega_{AB}.$$

Solve: $\omega_{AB} = -0.1704$ rad/s, $\boxed{v_{B\text{rel}} = 0.329 \text{ m/s}}$ which is the rate of extension of the actuator.

Problem 17.135 The angular velocity of the scoop is 1 rad/s clockwise and its angular acceleration is zero. Determine the rate of change of the rate at which the hydraulic actuator AB is extending.



Solution: Choose a coordinate system with the origin at D and the x axis parallel to ADE. The vector locations of points A, B, C, and E are $\mathbf{r}_A = -1.52\mathbf{i}$ m, $\mathbf{r}_B = 0.31\mathbf{i} + 0.61\mathbf{j}$ m, $\mathbf{r}_C = 1.07\mathbf{i} + 0.46\mathbf{j}$ m, $\mathbf{r}_E = 1.07\mathbf{i}$ m. The vector AB is

$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = 1.83\mathbf{i} + 0.61\mathbf{j} \text{ (ft)},$$

$$\mathbf{r}_{B/D} = \mathbf{r}_B - \mathbf{r}_D = 0.31\mathbf{i} + 0.61\mathbf{j} \text{ (m)}.$$

Assume that the scoop rotates at 1 rad/s about point E. The acceleration of point C is

$$\mathbf{a}_C = \boldsymbol{\alpha}_{\text{Scoop}} \times 4.57\mathbf{j} - \omega^2_{\text{scoop}}(0.46\mathbf{j}) = -0.46\mathbf{j} \text{ (m/s}^2),$$

since $\alpha_{\text{scoop}} = 0$. The vector from C to B is $\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = -0.76\mathbf{i} + 0.15\mathbf{j}$ (m). The acceleration of point B in terms of point C is

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega^2_{BC}\mathbf{r}_{B/C}$$

$$= 0.46\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -0.76 & 0.15 & 0 \end{bmatrix} - \omega^2_{BC}(-0.76\mathbf{i} + 0.15\mathbf{j}),$$

from which

$$\mathbf{a}_B = -(0.15\alpha_{BC} - 0.76\omega^2_{BC})\mathbf{i} - (0.46 + 0.76\alpha_{BC} + 0.5\omega^2_{BC})\mathbf{j}.$$

The acceleration of B in terms of D is

$$\mathbf{a}_B = \mathbf{a}_D + \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{B/D} - \omega^2_{BD}\mathbf{r}_{B/D}$$

$$= \mathbf{a}_D + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BD} \\ 0.31 & 0.61 & 0 \end{bmatrix} - \omega^2_{BD}(0.31\mathbf{i} + 0.61\mathbf{j}).$$

The acceleration of point D is zero, from which $\mathbf{a}_B = -(0.61\alpha_{BD} + 0.31\omega^2_{BD})\mathbf{i} + (0.31\alpha_{BD} - 0.61\omega^2_{BD})\mathbf{j}$. Equate like terms in the two expressions for $\mathbf{a}_B$, $-(0.15\alpha_{BC} - 0.76\omega^2_{BC}) = -(0.61\alpha_{BD} + 0.31\omega^2_{BD})$, $-(0.46 + 0.76\alpha_{BC} + 0.15\omega^2_{BC}) = (0.31\alpha_{BD} - 0.61\omega^2_{BD})$. From the solution to Problem 17.134, $\omega_{BC} = 0.2727$ rad/s, and $v_B = 0.465$ m/s. The velocity of point B is normal to the link BD, from which

$$\omega_{BD} = \frac{v_B}{\sqrt{1^2 + 2^2}} = 0.6818 \text{ rad/s}.$$

Substitute and solve for the angular accelerations: $\alpha_{BC} = -0.1026$ rad/s^2, $\alpha_{BD} = -0.3511$ rad/s^2. From which the acceleration of point B is

$$\mathbf{a}_B = -(2\alpha_{BD} + \omega^2_{BD})\mathbf{i} + (\alpha_{BD} - 2\omega^2_{BD})\mathbf{j}$$

$$= 0.072\mathbf{i} - 0.39\mathbf{j} \text{ (m/s)}.^2$$

The acceleration of point B in terms of the arm AB is

$$\mathbf{a}_B = a_{B\text{rel}}\mathbf{e}_{B/A} + 0.61\boldsymbol{\omega}_{AB} \times v_{B\text{rel}}\mathbf{e}_{B/A} + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega^2_{AB}\mathbf{r}_{B/A}.$$

From the solution to Problem 17.134: $\mathbf{e}_{B/A} = 0.9487\mathbf{i} + 0.3162\mathbf{j}$, $v_{B\text{rel}}] = 0.33$ m/s, $\omega_{AB} = -0.1705$ rad/s. From which

$$\mathbf{a}_B = a_{B\text{rel}}(0.9487\mathbf{i} + 0.3162\mathbf{j}) + 0.1162\mathbf{i} - 0.3487\mathbf{j}$$

$$+ \alpha_{AB}(-0.61\mathbf{i} + 1.83\mathbf{j}) - 0.1743\mathbf{i} - 0.0581\mathbf{j}.$$

Equate the accelerations of point B and separate components:

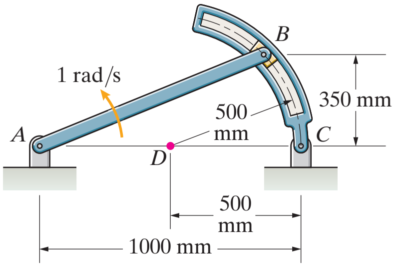
$$0.072 = 0.9487a_{B\text{rel}} - 0.61\alpha_{AB} - 0.018,$$

$$-0.39 = 0.3162a_{B\text{rel}} + 1.83\alpha_{AB} - 0.124.$$

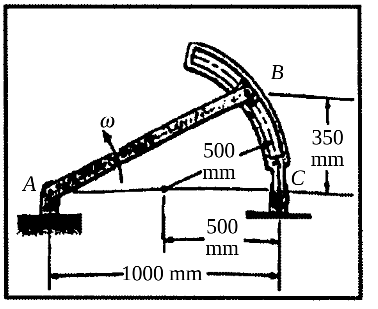
Solve: $\boxed{a_{B\text{rel}} = 0.00116 \text{ m/s}^2}$, which is the rate of change of the rate at which the actuator is extending.

Problem 17.136 Suppose that the curved bar in Example 17.9 rotates with a counterclockwise angular velocity of 2 rad/s.

(a) What is the angular velocity of bar AB?
(b) What is the velocity of block B relative to the slot?



Solution: The angle defining the position of B in the circular slot is



$$\beta = \sin^{-1}\left(\frac{350}{500}\right) = 44.4^\circ.$$

The vectors are

$$\mathbf{r}_{B/A} = (500 + 500\cos\beta)\mathbf{i} + 350\mathbf{j} = 857\mathbf{i} + 350\mathbf{j} \text{ (mm)}.$$

$$\mathbf{r}_{B/C} = (-500 + 500\cos\beta)\mathbf{i} + 350\mathbf{j} \text{ (mm)}.$$

The unit vector tangent to the slot at B is given by

$$\mathbf{e}_B = -\sin\beta\mathbf{i} + \cos\beta\mathbf{j} = -0.7\mathbf{i} + 0.714\mathbf{j}.$$

The velocity of B in terms of AB is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 857 & 350 & 0 \end{bmatrix}$$

$$= \omega_{AB}(-350\mathbf{i} + 857\mathbf{j}) \text{ (mm/s)}.$$

The velocity of B in terms of BC is

$$\mathbf{v}_B = v_{B\text{rel}}\mathbf{e}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C}$$

$$= v_{B\text{rel}}(-0.7\mathbf{i} + 0.714\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -142.9 & 350 & 0 \end{bmatrix},$$

$$\mathbf{v}_B = v_{B\text{rel}}(-0.7\mathbf{i} + 0.714\mathbf{j}) + (-700\mathbf{i} - 285.8\mathbf{j}) \text{ (mm/s)}$$

Equate the expressions for the velocity of B and separate components:

$$-350\omega_{AB} = -0.7v_{B\text{rel}} - 700,$$

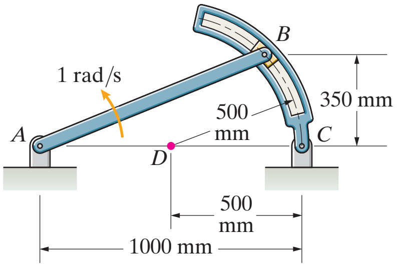
$$-857\omega_{AB} = 0.714v_{B\text{rel}} - 285.8.$$

Solve:

(a) $\boxed{\omega_{AB} = -2 \text{ rad/s}}$ (clockwise).

(b) $\boxed{v_{B\text{rel}} = -2000 \text{ mm/s}}$ (toward C).

Problem 17.137 Suppose that the curved bar in Example 17.9 has a clockwise angular velocity of 4 rad/s and a counterclockwise angular acceleration of 10 rad/s^2. What is the angular acceleration of bar AB?



Solution: Use the solution to Problem 17.118 with new data.

Get the velocities: The angle defining the position of B in the circular slot is

$$\beta = \sin^{-1}\left(\frac{350}{500}\right) = 44.4^\circ.$$

The vectors

$$\mathbf{r}_{B/A} = (500 + 500\cos\beta)\mathbf{i} + 350\mathbf{j} = 857\mathbf{i} + 350\mathbf{j} \text{ (mm)}.$$

$$\mathbf{r}_{B/C} = (-500 + 500\cos\beta)\mathbf{i} + 350\mathbf{j} \text{ (mm)}.$$

The unit vector tangent to the slot at B

$$\mathbf{e}_B = -\sin\beta\mathbf{i} + \cos\beta\mathbf{j} = -0.7\mathbf{i} + 0.714\mathbf{j}.$$

The component normal to the slot at B is

$$\mathbf{e}_{NB} = \cos\beta\mathbf{i} + \sin\beta\mathbf{j} = 0.7141\mathbf{i} + 0.7\mathbf{j}.$$

The velocity of B in terms of AB

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 857 & 350 & 0 \end{bmatrix}$$

$$= \omega_{AB}(-350\mathbf{i} + 857\mathbf{j}) \text{ (mm/s)}.$$

The velocity of B in terms of BC is

$$\mathbf{v}_B = v_{B\text{rel}}\mathbf{e}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C}$$

$$= v_{B\text{rel}}(-0.7\mathbf{i} + 0.714\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega_{BC} \\ -142.9 & 350 & 0 \end{bmatrix},$$

$$\mathbf{v}_B = v_{B\text{rel}}(-0.7\mathbf{i} + 0.714\mathbf{j}) + (1400\mathbf{i} + 571.6\mathbf{j}) \text{ (mm/s)}.$$

Equate the expressions for the velocity of B and separate components: $-350\omega_{AB} = -0.7v_{B\text{rel}} + 1400$, $857\omega_{AB} = 0.714v_{B\text{rel}} + 571.6$. Solve: $\omega_{AB} = 4$ rad/s (counterclockwise). $v_{B\text{rel}} = 4000$ mm/s (away from C).

Get the accelerations: The acceleration of point B in terms of the AB is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 857 & 350 & 0 \end{bmatrix} - \omega_{AB}^2(857\mathbf{i} + 350\mathbf{j}),$$

$$\mathbf{a}_B = \alpha_{AB}(-350\mathbf{i} + 857\mathbf{j}) - 138713\mathbf{i} - 5665\mathbf{j} \text{ (mm/s}^2).$$

The acceleration in terms of the arm BC is

$$\mathbf{a}_B = \mathbf{a}_{B\text{rel}} + 2\boldsymbol{\omega}_{BC} \times v_{B\text{rel}}\mathbf{e}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2\mathbf{r}_{B/C}.$$

Expanding term by term:

$$\mathbf{a}_{B\text{rel}} = a_{B\text{rel}}\mathbf{e}_B - \left(\frac{v_{B\text{rel}}^2}{500}\right)\mathbf{e}_{NB}$$

$$= a_{B\text{rel}}(-0.7\mathbf{i} + 0.7141\mathbf{j}) - 22{,}852.6\mathbf{i} - 22{,}400\mathbf{j}.$$

Other terms:

$$2\boldsymbol{\omega}_{BC} \times v_{B\text{rel}}\mathbf{e}_B = 22852\mathbf{i} + 22{,}400\mathbf{j},$$

$$\boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} = -3500\mathbf{i} - 1429.3\mathbf{j},$$

$$-\omega_{BC}^2\mathbf{r}_{B/C} = 2286.8\mathbf{i} - 5600\mathbf{j}.$$

Collect terms:

$$\mathbf{a}_B = a_{B\text{rel}}(-0.7\mathbf{i} + 0.7141\mathbf{j}) - 22852.6\mathbf{i} - 22400\mathbf{j} + 22852.6\mathbf{i}$$

$$+ 22400\mathbf{j} - 3500\mathbf{i} - 1429.3\mathbf{j} + 2286.9\mathbf{i} - 5600\mathbf{j}$$

$$\mathbf{a}_B = a_{B\text{rel}}(-0.7\mathbf{i} + 0.7141\mathbf{j}) - 1213\mathbf{i} - 7029.3\mathbf{j}.$$

Equate the two expressions for the acceleration of B to obtain the two equations:

$$-350\alpha_{AB} - 13{,}871 = -0.7a_{B\text{rel}} - 1213.1,$$

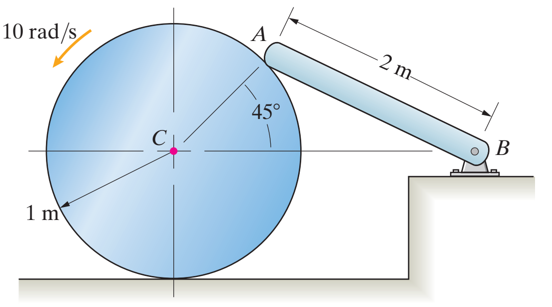
$$857\alpha_{AB} - 5665 = 0.7141a_{B\text{rel}} - 7029.3.$$

Solve:

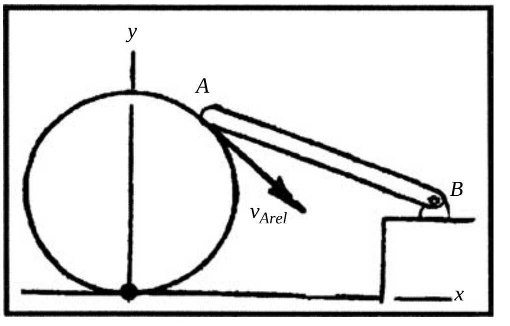
$$a_{B\text{rel}} = 29180 \text{ (mm/s}^2),$$

$$\boxed{\alpha_{AB} = 22.65 \text{ rad/s}^2} \text{ (counterclockwise).}$$

Problem 17.138* The disk rolls on the plane surface with a counterclockwise angular velocity of 10 rad/s. Bar AB slides on the surface of the disk at A. Determine the angular velocity of bar AB.



Solution: Choose a coordinate system with the origin at the point of contact between the disk and the plane surface, with the x axis parallel to the plane surface. Let A be the point of the bar in contact with the disk. The vector location of point A on the disk is



$$\mathbf{r}_A = \mathbf{i}\cos 45^\circ + \mathbf{j}(1 + \sin 45^\circ) = 0.707\mathbf{i} + 1.707\mathbf{j} \text{ (m)}.$$

The unit vector parallel to the radius of the disk is

$$\mathbf{e}_A = \cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j} = 0.707\mathbf{i} + 0.707\mathbf{j}.$$

The unit vector tangent to the surface of the disk at A is

$$\mathbf{e}_{NA} = \mathbf{i}\sin 45^\circ - \mathbf{j}\cos 45^\circ = 0.707\mathbf{i} - 0.707\mathbf{j}.$$

The angle formed by the bar AB with the horizontal is

$$\beta = \sin^{-1}\left(\frac{\sin 45^\circ}{2}\right) = 20.7^\circ.$$

The velocity of point A in terms of the motion of bar AB is

$$\mathbf{v}_A = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ -2\cos\beta & 2\sin\beta & 0 \end{bmatrix}.$$

$$\mathbf{v}_A = \omega_{AB}(-0.707\mathbf{i} - 1.871\mathbf{j}) \text{ (m/s)}.$$

The velocity of point A in terms of the point of the disk in contact with the plane surface is

$$\mathbf{v}_A = v_{\text{Arel}}\mathbf{e}_{NA} + \boldsymbol{\omega}_{\text{disk}} \times \mathbf{r}_A$$

$$= v_{\text{Arel}}(0.707\mathbf{i} - 0.707\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{\text{disk}} \\ 0.707 & 1.707 & 0 \end{bmatrix},$$

$$\mathbf{v}_A = v_{\text{Arel}}(0.707\mathbf{i} - 0.707\mathbf{j}) + (-17.07\mathbf{i} + 7.07\mathbf{j}).$$

Equate the expressions and separate components:

$$-0.707\omega_{AB} = 0.707v_{\text{Arel}} - 17.07,$$

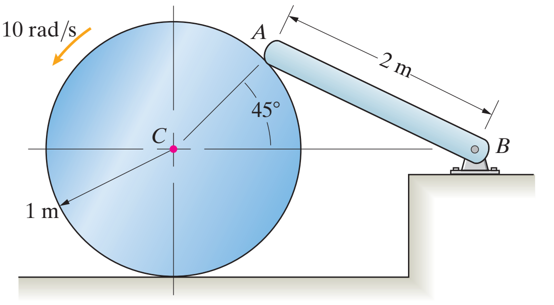
$$-1.871\omega_{AB} = -0.707v_{\text{Arel}} + 7.07.$$

Solve:

$$v_{\text{Arel}} = 20.3 \text{ m/s},$$

$\boxed{\omega_{AB} = 3.88 \text{ rad/s}}$ (counterclockwise).

Problem 17.139* In Problem 17.138, the disk rolls on the plane surface with a constant counterclockwise angular velocity of 10 rad/s. Determine the angular acceleration of bar *AB*.



Solution: Use the results of the solution to Problem 17.138. Choose a coordinate system with the origin at the point of contact between the disk and the plane surface, with the x axis parallel to the plane surface. The vector location of point A on the disk is

$$\mathbf{r}_A = \mathbf{i}\cos 45^\circ + \mathbf{j}(1 + \sin 45^\circ) = 0.707\mathbf{i} + 1.707\mathbf{j} \text{ (m)}.$$

The unit vector tangent to the surface of the disk at A is

$$\mathbf{e}_{NA} = \mathbf{i}\sin 45^\circ - \mathbf{j}\cos 45^\circ = 0.707\mathbf{i} - 0.707\mathbf{j}.$$

The angle formed by the bar AB with the horizontal is

$$\beta = \sin^{-1}(\sin 45^\circ/2) = 20.7^\circ.$$

Get the velocities: The velocity of point A in terms of the motion of bar AB is

$$\mathbf{v}_A = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ -2\cos\beta & 2\sin\beta & 0 \end{bmatrix}$$

$$= \omega_{AB}(-0.707\mathbf{i} - 1.871\mathbf{j}) \text{ (m/s)}.$$

The acceleration of the center of the disk is zero. The velocity of point A in terms of the center of the disk is

$$\mathbf{v}_A = v_{A\text{rel}}\mathbf{e}_{NA} + \boldsymbol{\omega}_{\text{disk}} \times \mathbf{r}_A$$

$$= v_{A\text{rel}}(0.707\mathbf{i} - 0.707\mathbf{j}) + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{\text{disk}} \\ 0.707 & 1.707 & 0 \end{bmatrix},$$

$$\mathbf{v}_A = v_{A\text{rel}}(0.707\mathbf{i} - 0.707\mathbf{j}) + (-17.07\mathbf{i} + 7.07\mathbf{j}).$$

Equate the expressions and separate components:

$$-0.707\omega_{AB} = 0.707v_{A\text{rel}} - 17.07, \quad -1.871\omega_{AB} = -0.707v_{A\text{rel}} + 7.07.$$

Solve:

$$v_{A\text{rel}} = 20.3 \text{ m/s},$$

$$\omega_{AB} = 3.88 \text{ rad/s} \quad \text{(counterclockwise)}.$$

Get the accelerations: The acceleration of point A in terms of the arm AB is

$$\mathbf{a}_A = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} - \omega_{AB}^2\mathbf{r}_{A/B}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ -1.87 & 0.707 & 0 \end{bmatrix} + 28.15\mathbf{i} - 10.64\mathbf{j} \text{ (m/s}^2\text{)},$$

$$\mathbf{a}_A = \alpha_{AB}(-0.707\mathbf{i} - 1.87\mathbf{j}) + 28.15\mathbf{i} - 10.64\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration of point A in terms of the disk is

$$\mathbf{a}_A = \mathbf{a}_{A\text{rel}} + 2\boldsymbol{\omega}_{\text{disk}} \times v_{A\text{rel}}\mathbf{e}_{NA} + \boldsymbol{\alpha}_{\text{disk}} \times \mathbf{r}_{A/C} - \omega_{\text{disk}}^2\mathbf{r}_{A/C}.$$

Expanding term by term: The acceleration $\mathbf{a}_{A\text{rel}}$ is composed of a tangential component and a radial component:

$$\mathbf{a}_{A\text{rel}} = a_{A\text{rel}}\mathbf{e}_{NA} - \left(\frac{v_{A\text{rel}}^2}{1}\right)\mathbf{e}_A$$

$$= a_{A\text{rel}}(0.707\mathbf{i} - 0.707\mathbf{j}) - 290.3\mathbf{i} - 290.3\mathbf{j}.$$

$$2\boldsymbol{\omega}_{\text{disk}} \times v_{A\text{rel}}\mathbf{e}_{NB} = 286.6\mathbf{i} + 286.6\mathbf{j}, \quad \boldsymbol{\alpha}_{\text{disk}} \times \mathbf{r}_A = 0,$$

since the acceleration of the disk is zero.

$$-\omega_{\text{disk}}^2\mathbf{r}_{A/C} = -70.7\mathbf{i} - 70.7\mathbf{j}.$$

Collect terms and separate components to obtain:

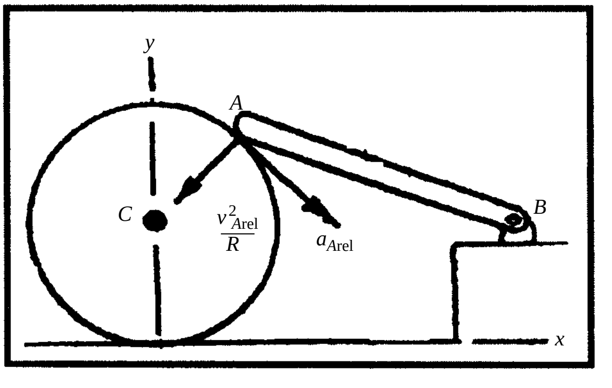
$$-0.707\alpha_{AB} + 28.15 = 0.707a_{A\text{rel}} - 290.3 + 286.6 - 70.7,$$

$$-1.87\alpha_{AB} - 10.64 = -0.707a_{A\text{rel}} - 290.3 + 286.6 - 70.7.$$

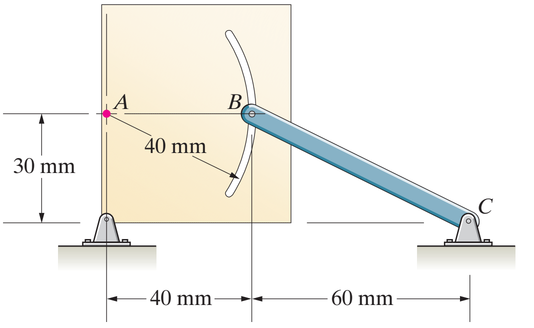
Solve:

$$a_{A\text{rel}} = 80.6 \text{ m/s}^2,$$

$$\boxed{\alpha_{AB} = 64.6 \text{ rad/s}^2} \quad \text{(counterclockwise)}.$$



Problem 17.140* Bar BC rotates with a counterclockwise angular velocity of 2 rad/s. A pin at B slides in a circular slot in the rectangular plate. Determine the angular velocity of the plate and the velocity at which the pin slides relative to the circular slot.



Solution: Choose a coordinate system with the origin O at the lower left pin and the x axis parallel to the plane surface. The unit vector parallel to AB is $\mathbf{e}_{AB} = \mathbf{i}$. The unit vector tangent to the slot at B is $\mathbf{e}_{NAB} = \mathbf{j}$. The velocity of the pin in terms of the motion of BC is $\mathbf{v}_B = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C}$.

$$\mathbf{v}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -60 & 30 & 0 \end{bmatrix} = 2(-30\mathbf{i} - 60\mathbf{j}) = -60\mathbf{i} - 120\mathbf{j} \text{ (mm/s)}.$$

The velocity of the pin in terms of the plate is

$$\mathbf{v}_B = v_{B\text{rel}}\mathbf{j} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 40 & 30 & 0 \end{bmatrix}$$

$$= v_{B\text{rel}}\mathbf{j} + \omega_{AB}(-30\mathbf{i} + 40\mathbf{j}) \text{ (mm/s)}.$$

Equate the expressions and separate components to obtain

$$-60 = -30\omega_{AB},$$

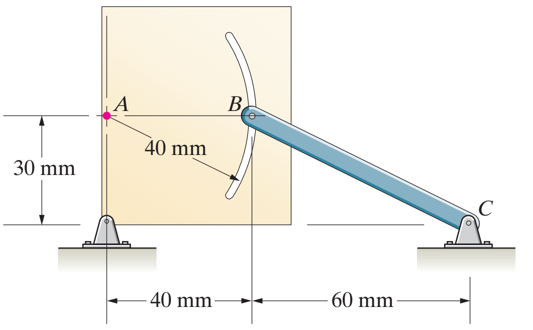
$$-120 = v_{B\text{rel}} + 40\omega_{AB}.$$

Solve:

$$\mathbf{v}_{B\text{rel}} = -200\mathbf{j} \text{ mm/s},$$

$$\boxed{\omega_{AB} = 2 \text{ rad/s}} \quad \text{(counterclockwise)}.$$

Problem 17.141* Bar BC in Problem 17.140 rotates with a constant counterclockwise angular velocity of 2 rad/s. Determine the angular acceleration of the plate.



Solution: Choose the same coordinate system as in Problem 17.140. *Get the velocities*: The unit vector parallel to AB is $\mathbf{e}_{AB} = \mathbf{i}$. The unit vector tangent to the slot at B is $\mathbf{e}_{NAB} = \mathbf{j}$. The velocity of the pin in terms of the motion of BC is

$$\mathbf{v}_B = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -60 & 30 & 0 \end{bmatrix}$$

$$= (-60\mathbf{i} - 120\mathbf{j}) \text{ (mm/s)}.$$

The velocity of the pin in terms of the plate is

$$\mathbf{v}_B = v_{B\text{rel}}\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 40 & 30 & 0 \end{bmatrix}$$

$$= v_{B\text{rel}}\mathbf{e}_{NAB} + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/O}$$

$$= v_{B\text{rel}}\mathbf{j} + \omega_{AB}(-30\mathbf{i} + 40\mathbf{j}) \text{ (mm/s)}.$$

Equate the expressions and separate components to obtain

$$-60 = -30\omega_{AB},$$

$$-120 = v_{B\text{rel}} + 40\omega_{AB}.$$

Solve:

$$\mathbf{v}_{B\text{rel}} = -200\mathbf{j} \text{ mm/s},$$

$$\omega_{AB} = 2 \text{ rad/s} \quad \text{(counterclockwise)}.$$

Get the accelerations: The acceleration of the pin in terms of the arm BC is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2\mathbf{r}_{B/C}$$

$$= 0 - 4(-60\mathbf{i} + 30\mathbf{j})$$

$$= 240\mathbf{i} - 120\mathbf{j} \text{ (mm/s}^2\text{)}.$$

The acceleration of the pin in terms of the plate AB is

$$\mathbf{a}_B = \mathbf{a}_{B\text{rel}} + 2\boldsymbol{\omega}_{AB} \times v_{B\text{rel}}\mathbf{e}_{NAB} + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/O} - \omega_{AB}^2\mathbf{r}_{B/O}.$$

Expand term by term:

$$\mathbf{a}_{B\text{rel}} = a_{B\text{rel}}\mathbf{e}_{NAB} - \left(\frac{v_{B\text{rel}}^2}{40}\right)\mathbf{e}_{AB}$$

$$= a_{B\text{rel}}\mathbf{j} - 1000\mathbf{i} \text{ (mm/s}^2\text{)},$$

$$2\boldsymbol{\omega}_{AB} \times v_{B\text{rel}}\mathbf{e}_{NAB} = 800\mathbf{i} \text{ (mm/s}^2\text{)}.$$

$$\boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 40 & 30 & 0 \end{bmatrix}$$

$$= \alpha_{BA}(-30\mathbf{i} + 40\mathbf{j}) \text{ (mm/s}^2\text{)},$$

$$-\omega_{AB}^2(40\mathbf{i} + 30\mathbf{j}) = -160\mathbf{i} - 120\mathbf{j} \text{ (mm/s}^2\text{)}.$$

Collect terms and separate components to obtain:

$$240 = -1000 + 800 - 30\alpha_{BA} - 160,$$

$$-120 = a_{B\text{rel}} + 40\alpha_{BA} - 120.$$

Solve:

$$a_{B\text{rel}} = 800 \text{ mm/s}^2 \quad \text{(upward)},$$

$$\boxed{\alpha_{AB} = -20 \text{ rad/s}^2}, \text{ (clockwise)}.$$

Problem 17.142* By taking the derivative of Eq. (17.11) with respect to time and using Eq. (17.12), derive Eq. (17.13).

Solution: Eq (17.11) is

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}.$$

Eq (17.12) is

$$\mathbf{v}_{A\text{rel}} = \left(\frac{dx}{dt}\right)\mathbf{i} + \left(\frac{dy}{dt}\right)\mathbf{j} + \mathbf{k}\left(\frac{dz}{dt}\right)\mathbf{k}.$$

Assume that the coordinate system is body fixed and that B is a point on the rigid body, (A is not necessarily a point on the rigid body), such that $\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$, where $\mathbf{r}_{A/B} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and x, y, z are the coordinates of A in body fixed coordinates. Take the derivative of both sides of Eq (17.11):

$$\frac{d\mathbf{v}_A}{dt} = \frac{d\mathbf{v}_B}{dt} + \frac{d\mathbf{v}_{A\text{rel}}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times \frac{d\mathbf{r}_{A/B}}{dt}.$$

By definition,

$$\frac{d\mathbf{v}_A}{dt} = \mathbf{a}_A, \frac{d\mathbf{v}_B}{dt} = \mathbf{a}_B, \quad \text{and} \frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\alpha}.$$

The derivative:

$$\frac{d\mathbf{v}_{A\text{rel}}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} + \frac{dx}{dt}\frac{d\mathbf{i}}{dt} + \frac{dy}{dt}\frac{d\mathbf{j}}{dt} + \frac{dz}{dt}\frac{d\mathbf{k}}{dt}.$$

Using the fact that the derivative of a unit vector represents a rotation of the unit vector,

$$\frac{d\mathbf{i}}{dt} = \boldsymbol{\omega} \times \mathbf{i}, \frac{d\mathbf{j}}{dt} = \boldsymbol{\omega} \times \mathbf{j}, \frac{d\mathbf{k}}{dt} = \boldsymbol{\omega} \times \mathbf{k}.$$

Substitute into the derivative:

$$\frac{d\mathbf{v}_{A\text{rel}}}{dt} = \mathbf{a}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}}.$$

Noting $\quad \boldsymbol{\omega} \times \dfrac{d\mathbf{r}_{A/B}}{dt} = \boldsymbol{\omega} \times \left(\dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k}\right)$

$$+ \boldsymbol{\omega} \times \left(x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}\right)$$

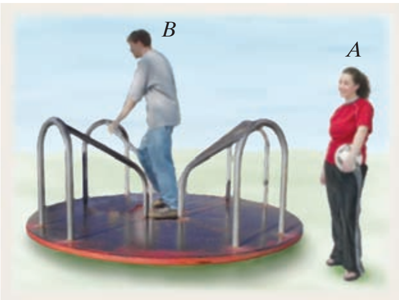
$$= \boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}).$$

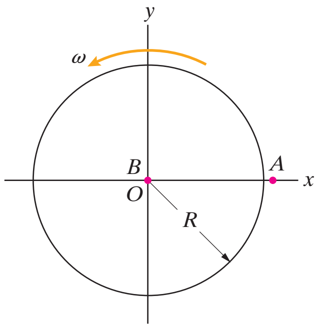
Collect and combine terms: the derivative of Eq (17.11) is

$$\boxed{\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{\text{rel}} + 2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})},$$

which is Eq (17.13).

Problem 17.143 In Active Example 17.10, suppose that the merry-go-round has counterclockwise angular velocity ω and counterclockwise angular acceleration α. The person A is standing still on the ground. Determine her acceleration relative to your reference frame at the instant shown.





Solution: First the velocity analysis

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$0 = 0 + \mathbf{v}_{Arel} + \omega\mathbf{k} \times (R\mathbf{i})$$

$$\mathbf{v}_{Arel} = -\omega R\mathbf{j}$$

Now the acceleration analysis

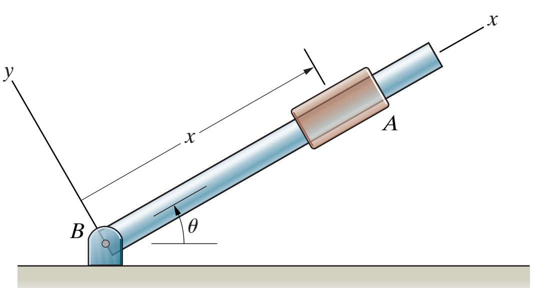
$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{Arel} + \mathbf{a} \times \mathbf{r}_{A/B} - \omega^2\mathbf{r}_{A/B} + 2\boldsymbol{\omega} \times \mathbf{v}_{Arel}$$

$$0 = 0 + \mathbf{a}_{Arel} + \alpha\mathbf{k} \times (R\mathbf{i}) - \omega^2(R\mathbf{i}) + 2(\omega\mathbf{k}) \times (-\omega R\mathbf{i})$$

$$\mathbf{a}_{Arel} = -\alpha R\mathbf{j} + \omega^2 R\mathbf{i} + 2\omega^2 R\mathbf{i}$$

$$\boxed{\mathbf{a}_{Arel} = -\omega^2 R\mathbf{i} - \alpha R\mathbf{j}}$$

Problem 17.144 The x–y coordinate system is body fixed with respect to the bar. The angle θ (in radians) is give as a function of time by $\theta = 0.2 + 0.04t^2$. The x coordinate of the sleeve A (in metre) is given as a function of time by $x = 1 + 0.03t^3$. Use Eq. (17.16) to determine the velocity of the sleeve at $t = 4$ s relative to a nonrotating reference frame with its origin at B. (Although you are determining the velocity of A relative to a nonrotating reference frame, your answer will be expressed in components in terms of the body-fixed reference frame.)



Solution: We have

$$\theta = 0.2 + 0.04t^2, \quad \dot{\theta} = 0.08t$$

$$x = 1 + 0.03t^3, \quad \dot{x} = 0.09t^2$$

At the instant $t = 4$s, $\dot{\theta} = 0.32, \quad x = 2.92, \quad \dot{x} = 1.44$

Thus

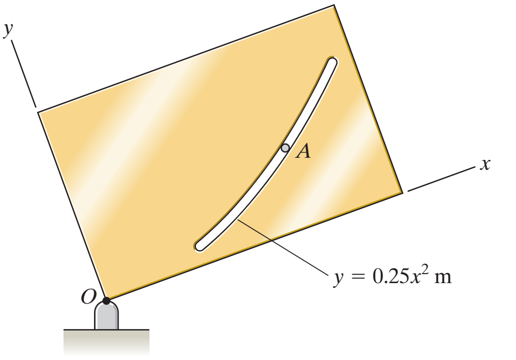
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$= 0 + \dot{x}\mathbf{i} + \dot{\theta}\mathbf{k} \times (x\mathbf{i}) = \dot{x}\mathbf{i} + x\dot{\theta}\mathbf{j}$$

$$= (1.44)\mathbf{i} + (2.92)(0.32)\mathbf{j}$$

$$\boxed{\mathbf{v}_A = (1.44\mathbf{i} + 0.934\mathbf{j})\ \text{m/s.}}$$

Problem 17.145 The metal plate is attached to a fixed ball-and-socket support at O. The pin A slides in a slot in the plate. At the instant shown, $x_A = 1$ m, $dx_A/dt = 2$ m/s, and $d^2x_A/dt^2 = 0$, and the plate's angular velocity and angular acceleration are $\boldsymbol{\omega} = 2\mathbf{k}$ (rad/s) and $\boldsymbol{\alpha} = 0$. What are the x, y, and z components of the velocity and acceleration of A relative to a nonrotating reference frame with its origin at O?



Solution: The velocity is $\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}_{A/O}$. The relative velocity is

$$\mathbf{v}_{A\text{rel}} = \left(\frac{dx}{dt}\right)\mathbf{i} + \left(\frac{dy}{dt}\right)\mathbf{j} + \left(\frac{dz}{dt}\right)\mathbf{k},$$

where $\dfrac{dx}{dt} = 2$ m/s, $\dfrac{dy}{dt} = \dfrac{d}{dt}0.25x^2 = 0.5x\dfrac{dx}{dt} = 1$ m/s, $\dfrac{dz}{dt} = 0$, and $\mathbf{r}_{A/O} = x\mathbf{i} + y\mathbf{j} + z\mathbf{j} = \mathbf{i} + 0.25\mathbf{j} + 0$, from which

$$\boxed{\begin{aligned}\mathbf{v}_A &= 2\mathbf{i} + \mathbf{j} + \omega(\mathbf{k} \times (\mathbf{i} + 0.25\mathbf{j})) = 2\mathbf{i} + \mathbf{j} + 2(-0.25\mathbf{i} + \mathbf{j})\\ &= 1.5\mathbf{i} + 3\mathbf{j} \text{ (m/s).}\end{aligned}}$$

The acceleration is

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A\text{rel}} + 2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}).$$

Noting

$$\mathbf{a}_{A\text{rel}} = \left(\frac{d^2x}{dt^2}\right)\mathbf{i} + \left(\frac{d^2y}{dt^2}\right)\mathbf{j} + \left(\frac{d^2z}{dt^2}\right)\mathbf{k},$$

where

$$\left(\frac{d^2x}{dt^2}\right) = 0, \ \frac{d^2y}{dt^2} = \frac{d^2}{dt^2}0.25x^2 = 0.5\left(\frac{dx}{dt}\right)^2 = 2, \ \left(\frac{d^2z}{dt^2}\right) = 0.$$

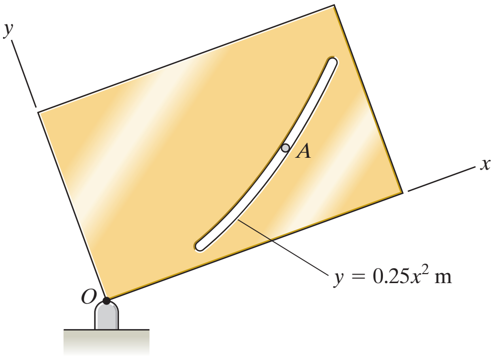
Substitute:

$$\mathbf{a}_A = 2\mathbf{j} + 2\omega(\mathbf{k} \times (2\mathbf{i} + \mathbf{j})) + \omega^2(\mathbf{k} \times (\mathbf{k} \times (\mathbf{i} + 0.25\mathbf{j})))$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 4 \\ 2 & 1 & 0 \end{bmatrix} + 4\mathbf{k} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0.25 & 0 \end{bmatrix}$$

$$\mathbf{a}_A = 2\mathbf{j} - 4\mathbf{i} + 4\omega\mathbf{j} + 4\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -0.25 & 1 & 0 \end{bmatrix} \quad \boxed{= -8\mathbf{i} + 9\mathbf{j} \text{ (m/s}^2\text{)}}$$

Problem 17.146 Suppose that at the instant shown in Problem 17.145, $x_A = 1$ m, $dx_A/dt = -3$ m/s, $d^2x_A/dt^2 = 4$ m/s^2, and the plate's angular velocity and angular acceleration are $\boldsymbol{\omega} = -4\mathbf{j} + 2\mathbf{k}$ (rad/s), and $\boldsymbol{\alpha} = 3\mathbf{i} - 6\mathbf{j}$ (rad/s^2). What are the x, y, z components of the velocity and acceleration of A relative to a nonrotating reference frame that is stationary with respect to O?



Solution: The velocity is $\mathbf{v}_A = \mathbf{v}_O + \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}_{A/O}$. The relative velocity is

$$\mathbf{v}_{A\text{rel}} = \left(\frac{dx}{dt}\right)\mathbf{i} + \left(\frac{dy}{dt}\right)\mathbf{j} + \left(\frac{dz}{dt}\right)\mathbf{k},$$

where $\dfrac{dx}{dt} = -3$ m/s, $\dfrac{dy}{dt} = \dfrac{d}{dt}0.25x^2 = 0.5x\dfrac{dx}{dt} = -15$ m/s, $\dfrac{dz}{dt} = 0$, and $\mathbf{r}_{A/O} = x\mathbf{i} + y\mathbf{j} + z\mathbf{j} = \mathbf{i} + 0.25\mathbf{j} + 0$, from which $\mathbf{v}_A = -3\mathbf{i} - 1.5\mathbf{j} + \boldsymbol{\omega} \times (\mathbf{i} + 0.25\mathbf{j})$.

$$\mathbf{v}_A = -3\mathbf{i} - 1.5\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & 2 \\ 1 & 0.25 & 0 \end{bmatrix} = -3\mathbf{i} - 1.5\mathbf{j} - 0.5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

$$\boxed{= -3.5\mathbf{i} + 0.5\mathbf{j} + 4\mathbf{k}\ (\text{m/s})}$$

The acceleration is $\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A\text{rel}} + 2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$. Noting

$$\mathbf{a}_{A\text{rel}} = \left(\frac{d^2x}{dt^2}\right)\mathbf{i} + \left(\frac{d^2y}{dt^2}\right)\mathbf{j} + \left(\frac{d^2z}{dt^2}\right)\mathbf{k},$$

where

$$\left(\frac{d^2x}{dt^2}\right) = 4\ \text{m/s}^2,$$

$$\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}0.25x^2 = 0.5\left(\frac{dx}{dt}\right)^2 + 0.5x\left(\frac{d^2x}{dt^2}\right) = 6.5\ (\text{m/s}^2),$$

$$\left(\frac{d^2z}{dt^2}\right) = 0,\ \boldsymbol{\alpha} = 3\mathbf{i} - 6\mathbf{j}\ (\text{rad/s}^2),\ \mathbf{v}_{A\text{rel}} = -3\mathbf{i} - 1.5\mathbf{j},$$

and from above: $\boldsymbol{\omega} \times \mathbf{r}_{A/O} = -0.5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Substitute:

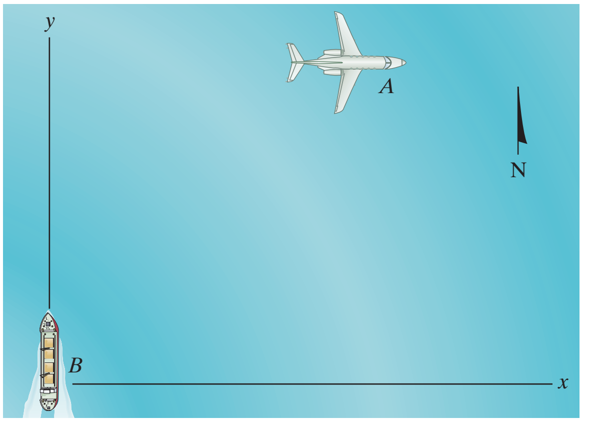
$$\mathbf{a}_A = 4\mathbf{i} + 6.5\mathbf{j} + 2\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & 2 \\ -3 & -1.5 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & 0 \\ 1 & 0.25 & 0 \end{bmatrix}$$

$$+\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & 2 \\ -0.5 & 2 & 4 \end{bmatrix}.$$

$$\mathbf{a}_A = 4\mathbf{i} + 6.5\mathbf{j} + 2(3\mathbf{i} + 6\mathbf{j} - 12\mathbf{k}) + (6.75\mathbf{k}) + (-20\mathbf{i} - \mathbf{j} - 2\mathbf{k})$$

$$\boxed{\mathbf{a}_A = -10\mathbf{i} - 6.5\mathbf{j} - 19.25\mathbf{k}\ (\text{m/s}^2)}$$

Problem 17.147 The coordinate system is fixed relative to the ship B. At the instant shown, the ship is sailing north at 5 m/s relative to the earth, and its angular velocity is 0.26 rad/s counterclockwise. Using radar, it is determined that the position of the airplane is $1080\mathbf{i} + 1220\mathbf{j} + 6300\mathbf{k}$ (m) and its velocity relative to the ship's coordinate system is $870\mathbf{i} - 45\mathbf{j} - 21\mathbf{k}$ (m/s). What is the airplane's velocity relative to the earth? (See Example 17.11.)



Solution:

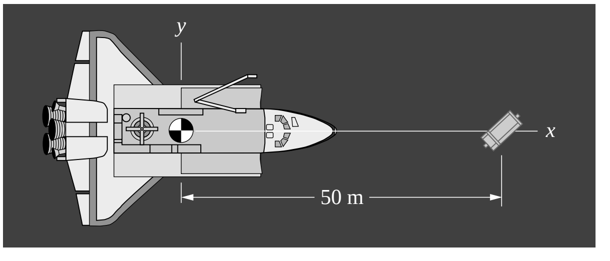
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{Arel} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$= 5\mathbf{j} + (870\mathbf{i} - 45\mathbf{j} - 21\mathbf{k}) + (0.26\mathbf{k}) \times (1080\mathbf{i} + 1220\mathbf{j} + 6300\mathbf{k})$$

$$= (870\mathbf{i} - 40\mathbf{j} - 21\mathbf{k}) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.26 \\ 1080 & 1220 & 6300 \end{vmatrix}$$

$$\boxed{\mathbf{v}_A = (553\mathbf{i} + 241\mathbf{j} - 21\mathbf{k})\ \text{m/s.}}$$

Problem 17.148 The space shuttle is attempting to recover a satellite for repair. At the current time, the satellite's position relative to a coordinate system fixed to the shuttle is $50\mathbf{i}$ (m). The rate gyros on the shuttle indicate that its current angular velocity is $0.05\mathbf{j} + 0.03\mathbf{k}$ (rad/s). The Shuttle pilot measures the velocity of the satellite relative to the body-fixed coordinate system and determines it to be $-2\mathbf{i} - 1.5\mathbf{j} + 2.5\mathbf{k}$ (rad/s). What are the x, y, and z components of the satellite's velocity relative to a nonrotating coordinate system with its origin fixed to the shuttle's center of mass?



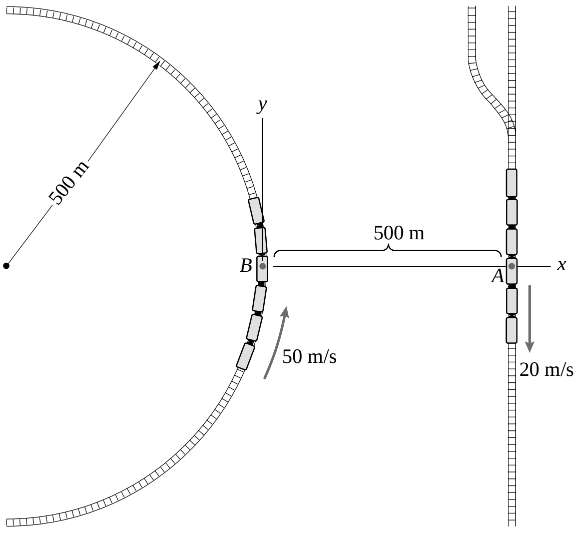
Solution: The velocity of the satellite is

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$= 0 - 2\mathbf{i} - 1.5\mathbf{j} + 2.5\mathbf{k} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.05 & 0.03 \\ 50 & 0 & 0 \end{bmatrix}$$

$$\boxed{= -2\mathbf{i} + 1.5\mathbf{j} + 2.5\mathbf{k} - 1.5\mathbf{j} - 2.5\mathbf{k} = -2\mathbf{i}\ \text{(m/s)}}$$

Problem 17.149 The train on the circular track is traveling at a constant speed of 50 m/s in the direction shown. The train on the straight track is traveling at 20 m/s in the direction shown and is increasing its speed at 2 m/s^2. Determine the velocity of passenger A that passenger B observes relative to the given coordinate system, which is fixed to the car in which B is riding.



Solution:

The angular velocity of B is $\omega = \dfrac{50}{500} = 0.1$ rad/s.

The velocity of A is $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$.

At the instant shown, $\mathbf{v}_A = -20\mathbf{j}$ (m/s), $\mathbf{v}_B = +50\mathbf{j}$ (m/s),

and $\mathbf{r}_{A/B} = 500\mathbf{i}$ (m), from which

$$\mathbf{v}_{A\text{rel}} = -20\mathbf{j} - 50\mathbf{j} - \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0.1 \\ 500 & 0 & 0 \end{bmatrix} = -20\mathbf{j} - 50\mathbf{j} - 50\mathbf{j},$$

$$\boxed{\mathbf{v}_{A\text{rel}} = -120\mathbf{j} \text{ (m/s)}}$$

Problem 17.150 In Problem 17.149, determine the acceleration of passenger A that passenger B observes relative to the coordinate system fixed to the car in which B is riding.

Solution:

Use the solution to Problem 17.149:

$\mathbf{v}_{A\text{rel}} = -120\mathbf{j}$ (m/s),

$$\omega = \frac{50}{500} = 0.1 \text{ rad/s},$$

$\mathbf{r}_{A/B} = 500\mathbf{i}$ (m).

The acceleration of A is $\mathbf{a}_A = -2\mathbf{j}$ (m/s),

The acceleration of B is

$\mathbf{a}_B = -500(\omega^2)\mathbf{i} = -5\mathbf{i}$ m/s^2,

and $\boldsymbol{\alpha} = 0$, from which

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A\text{rel}} + 2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}).$$

Rearrange:

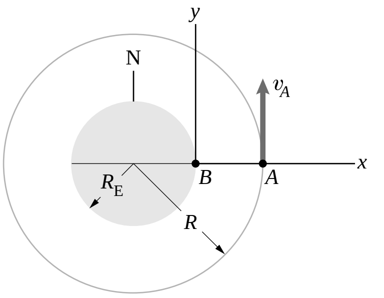
$$\mathbf{a}_{A\text{rel}} = \mathbf{a}_A - \mathbf{a}_B - 2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}).$$

$$\mathbf{a}_{\text{Re}l} = -2\mathbf{j} + 5\mathbf{i} - 2\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ 0 & -120 & 0 \end{bmatrix} - \omega^2\left(\mathbf{k} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 500 & 0 & 0 \end{bmatrix}\right).$$

$$\mathbf{a}_{\text{Re}l} = -2\mathbf{j} + 5\mathbf{i} - 2(120\omega)\mathbf{i} - \omega^2\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 500 & 0 \end{bmatrix},$$

$$\boxed{\mathbf{a}_{A\text{rel}} = -14\mathbf{i} - 2\mathbf{j} \text{ (m/s}^2)}.$$

Problem 17.151 The satellite A is in a circular polar orbit (a circular orbit that intersects the earth's axis of rotation). The radius of the orbit is R, and the magnitude of the satellite's velocity relative to a non-rotating reference frame with its origin at the center of the earth is v_A. At the instant shown, the satellite is above the equator. An observer B on the earth directly below the satellite measures its motion using the earth-fixed coordinate system shown. What are the velocity and acceleration of the satellite relative to B's earth-fixed coordinate system? The radius of the earth is R_E and the angular velocity of the earth is ω_E. (See Example 17.12.)



Solution: From the sketch, in the coordinate system shown, the location of the satellite in this system is $\mathbf{r}_A = (R - R_E)\mathbf{i}$, from which $\mathbf{r}_{A/B} = \mathbf{r}_A - 0 = (R - R_E)\mathbf{i}$. The angular velocity of the observer is $\boldsymbol{\omega}_E = -\omega_E \mathbf{k}$. The velocity of the observer is $\mathbf{v}_B = -\omega_E R_E \mathbf{k}$. The velocity of the satellite is $\mathbf{v}_A = v_A \mathbf{j}$.
The relative velocity is

$$\mathbf{v}_{A\text{rel}} = \mathbf{v}_A - \mathbf{v}_B - \boldsymbol{\omega}_E \times \mathbf{r}_{A/B},$$

$$\mathbf{v}_{A\text{rel}} = v_A \mathbf{j} + \omega_E R_E \mathbf{k} - (\omega_E) \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ R - R_E & 0 & 0 \end{bmatrix}$$

$$\boxed{= v_A \mathbf{j} + R\omega_E \mathbf{k}}.$$

From Eqs (17.26) and (17.27)

$$\mathbf{a}_{A\text{rel}} = \mathbf{a}_A - \mathbf{a}_B - 2\boldsymbol{\omega}_E \times \mathbf{v}_{A\text{rel}} - \boldsymbol{\alpha} \times \mathbf{r}_{A/B} - \boldsymbol{\omega}_E \times (\boldsymbol{\omega}_E \times \mathbf{r}_{A/B}).$$

The accelerations:

$$\mathbf{a}_A = -\omega_A^2 R\mathbf{i} = -\left(\frac{v_A^2}{R}\right)\mathbf{i},\ \mathbf{a}_B = -\omega_E^2 R_E \mathbf{i},\ \boldsymbol{\alpha} = 0,\ \text{from which}$$

$$\mathbf{a}_{A\text{rel}} = -\left(\frac{v_A^2}{R}\right) + \mathbf{i}\omega_E^2 R_E - 2\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_E & 0 \\ 0 & v_A & R\omega_E \end{bmatrix}$$

$$- \boldsymbol{\omega}_E \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_E & 0 \\ R - R_E & 0 & 0 \end{bmatrix}.$$

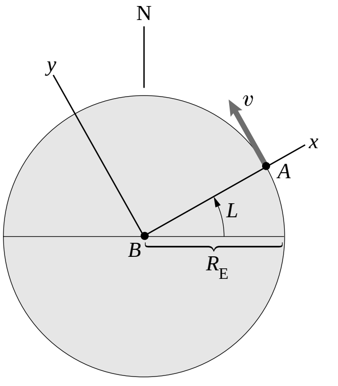
$$\mathbf{a}_{A\text{rel}} = -\omega_A^2 R\mathbf{i} + \omega_E^2 R_E \mathbf{i}$$

$$- 2\omega_E^2 R\mathbf{i} - \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_E & 0 \\ 0 & 0 & -\omega_E(R - R_E) \end{bmatrix}$$

$$= -\left(\frac{v_A^2}{R}\right)\mathbf{i} + \omega_E^2 R_E \mathbf{i} - 2\omega_E^2 R\mathbf{i} + \omega_E^2 (R - R_E)\mathbf{i}$$

$$\boxed{\mathbf{a}_{A\text{rel}} = -\left(\left(\frac{v_A^2}{R}\right) + \omega_E^2 R\right)\mathbf{i}}$$

Problem 17.152 A car A at north latitude L drives north on a north–south highway with constant speed v. The earth's radius is R_E, and the earth's angular velocity is ω_E. (The earth's angular velocity vector points north.) The coordinate system is earth fixed, and the x axis passes through the car's position at the instant shown. Determine the car's velocity and acceleration (a) relative to the earth-fixed coordinate system and (b) relative to a nonrotating reference frame with its origin at the center of the earth.

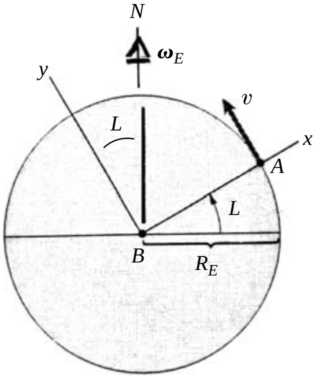


Solution:

(a) In earth fixed coords,

$$\mathbf{v}_{\text{rel}} = v\mathbf{j},$$

$$\mathbf{a}_{\text{rel}} = -v^2/R_E\mathbf{i}. \text{ (motion in a circle)}$$



(b) $\mathbf{v}_A = \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega}_E \times \mathbf{r}_{A/B} + \mathbf{v}_B (\mathbf{v}_B = 0)$

$$= v\mathbf{j} + (\omega_E \sin L\mathbf{i} + \omega_E \cos L\mathbf{j}) \times R_E\mathbf{i}$$

$$\underline{\mathbf{v}_A = v\mathbf{j} - \omega_E R_E \cos L\mathbf{k}}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A\text{rel}} + 2\boldsymbol{\omega}_E \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/B}$$

$$+ \boldsymbol{\omega}_E \times (\boldsymbol{\omega}_E \times \mathbf{r}_{A/B})$$

where $\boldsymbol{\omega}_E = \omega_E \sin L\mathbf{i} + \omega_E \cos L\mathbf{j}$

and $\mathbf{r}_{A/B} = R_E\mathbf{i}$

$$\mathbf{a}_A = 0 - \frac{v^2}{R_E}\mathbf{i} + 2v\omega_E \sin L\mathbf{k}$$

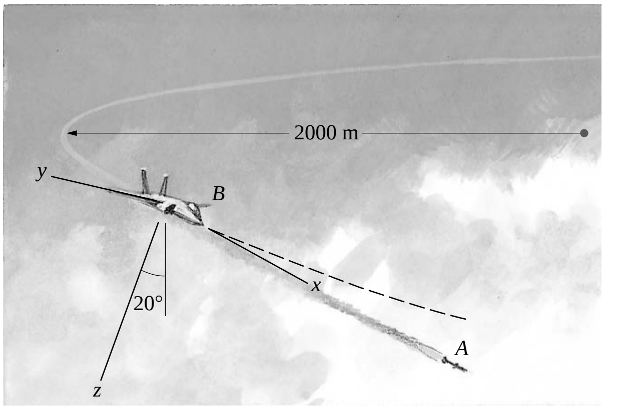
$$+ (\omega_E \sin L\mathbf{i} + \omega_E \cos L\mathbf{j}) \times (-\omega_E R_E \cos L\mathbf{k})$$

$$\mathbf{a}_A = -\left(\frac{v^2}{R_E} + \omega_E^2 R_E \cos^2 L\right)\mathbf{i}$$

$$+ (\omega_E^2 R_E \sin L \cos L)\mathbf{j}$$

$$+ 2v\omega_E \sin L\mathbf{k}$$

Problem 17.153 The airplane B conducts flight tests of a missile. At the instant shown, the airplane is traveling at 200 m/s relative to the earth in a circular path of 2000-m radius *in the horizontal plane*. The coordinate system is fixed relative to the airplane. The x axis is tangent to the plane's path and points forward. The y axis points out the plane's right side, and the z axis points out the bottom of the plane. The plane's bank angle (the inclination of the z axis from the vertical) is constant and equal to 20°. *Relative to the airplane's coordinate system*, the pilot measures the missile's position and velocity and determines them to be $\mathbf{r}_{A/B} = 1000\mathbf{i}$ (m) and $\mathbf{v}_{A/B} = 100.0\mathbf{i} + 94.0\mathbf{j} + 34.2\mathbf{k}$ (m/s).



(a) What are the x, y, and z components of the airplane's angular velocity vector?
(b) What are the x, y, and z components of the missile's velocity relative to the earth?

Solution:

(a) The bank angle is a rotation about the x axis; assume that the rotation is counterclockwise, so that the z axis is rotated toward the positive y axis. The magnitude of the angular velocity is

$$\omega = \frac{200}{2000} = 0.1 \text{ rad/s}.$$

In terms of airplane fixed coordinates,

$$\boldsymbol{\omega} = 0.1(\mathbf{i}\sin 20^\circ - \mathbf{j}\cos 20^\circ) \text{ (rad/s)}.$$

$$\boxed{\boldsymbol{\omega} = 0.03242\mathbf{j} - 0.0940\mathbf{k} \text{ rad/s}}$$

(b) The velocity of the airplane in earth fixed coordinates is

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$= 200\mathbf{i} + 100\mathbf{i} + 94.0\mathbf{j} + 34.2\mathbf{k}$$

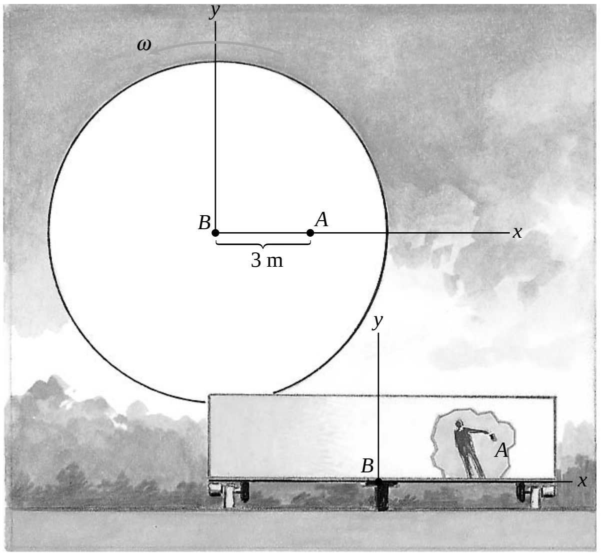
$$+ \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.0342 & -0.0940 \\ 1000 & 0 & 0 \end{bmatrix}$$

$$\boxed{\mathbf{v}_A = 300\mathbf{i} + 94.0\mathbf{j} + 34.2\mathbf{k} - 94.0\mathbf{j} - 34.2\mathbf{k} = 300\mathbf{i} \text{ (m/s)}}$$

Problem 17.154 To conduct experiments related to long-term spaceflight, engineers construct a laboratory on earth that rotates about the vertical axis at B with a constant angular velocity ω of one revolution every 6 s. They establish a laboratory-fixed coordinate system with its origin at B and the z axis pointing upward. An engineer holds an object stationary relative to the laboratory at point A, 3 m from the axis of rotation, and releases it. At the instant he drops the object, determine its acceleration relative to the laboratory-fixed coordinate system,

(a) assuming that the laboratory-fixed coordinate system is inertial and
(b) not assuming that the laboratory-fixed coordinate system is inertial, but assuming that an earth-fixed coordinate system with its origin at B is inertial.

(See Example 17.13.)



Solution: (a) If the laboratory system is inertial, Newton's second law is $\mathbf{F} = m\mathbf{a}$. The only force is the force of gravity; so that as the object free falls the acceleration is

$$\boxed{-g\mathbf{k} = -9.81\mathbf{k}\ (\text{m/s}^2)}.$$

If the earth fixed system is inertial, the acceleration observed is the centripetal acceleration and the acceleration of gravity:

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_B) - \mathbf{g},$$

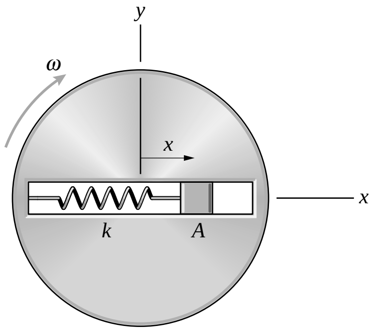
where the angular velocity is the angular velocity of the coordinate system relative to the inertial frame.

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) - \mathbf{g} = -\left(\frac{2\pi}{6}\right)^2 \left(\mathbf{k} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}\right) - 9.81\mathbf{k}$$

$$= \left(\frac{\pi^2}{3}\right)\mathbf{i} - 9.81\mathbf{k}$$

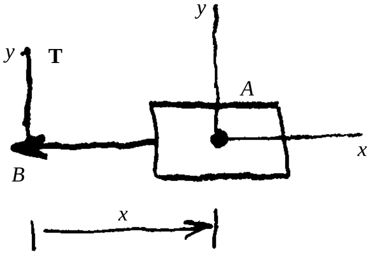
$$\boxed{= 3.29\mathbf{i} - 9.81\mathbf{k}\ (\text{m/s}^2)}$$

Problem 17.155 The disk rotates *in the horizontal plane* about a fixed shaft at the origin with constant angular velocity $w = 10$ rad/s. The 2-kg slider A moves in a smooth slot in the disk. The spring is unstretched when $x = 0$ and its constant is $k = 400$ N/m. Determine the acceleration of A relative to the body-fixed coordinate system when $x = 0.4$ m.



Strategy: Use Eq. (17.30) to express Newton's second law for the slider in terms of the body-fixed coordinate system.

Solution:



$$\mathbf{T} = -kx = (-400)(0.4)$$

$$\mathbf{T} = -160\mathbf{i} \text{ Newtons}$$

$$\sum \mathbf{F} = -160\mathbf{i} = m\mathbf{a}_{A\text{rel}} + m[\cancel{\mathbf{a}}_B^0 + 2\cancel{\boldsymbol{\omega}}^0 \times \mathbf{v}_{\text{rel}}$$

$$+ \cancel{\boldsymbol{\alpha}}^0 \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})]$$

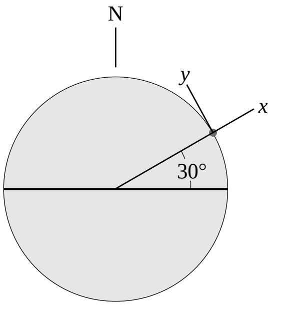
where $\boldsymbol{\omega} = -10\mathbf{k}$, $\mathbf{r}_{A/B} = 0.4\mathbf{i}$, $m = 2$

$$-160\mathbf{i} = 2\mathbf{a}_{A\text{rel}} - 80\mathbf{i}$$

$$\underline{\mathbf{a}_{A\text{rel}} = -40\mathbf{i} \text{ (m/s}^2\text{)}}.$$

Problem 17.156* Engineers conduct flight tests of a rocket at 30° north latitude. They measure the rocket's motion using an earth-fixed coordinate system with the x axis pointing upward and the y axis directed northward. At a particular instant, the mass of the rocket is 4000 kg, the velocity of the rocket relative to the engineers' coordinate system is $2000\mathbf{i} + 2000\mathbf{j}$ (m/s), and the sum of the forces exerted on the rocket by its thrust, weight, and aerodynamic forces is $400\mathbf{i} + 400\mathbf{j}$ (N). Determine the rocket's acceleration relative to the engineers' coordinate system,

(a) assuming that their earth-fixed coordinate system is inertial and
(b) not assuming that their earth-fixed coordinate system is inertial.



Solution: Use Eq. (17.22):

$$\sum \mathbf{F} - m[\mathbf{a}_B + 2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})]$$

$$= m\mathbf{a}_{A\text{rel}}.$$

(a) If the earth fixed coordinate system is assumed to be inertial, this reduces to $\sum \mathbf{F} = m\mathbf{a}_{A\text{rel}}$, from which

$$\mathbf{a}_{A\text{rel}} = \frac{1}{m}\sum \mathbf{F} = \frac{1}{4000}(400\mathbf{i} + 400\mathbf{j})$$

$$\boxed{= 0.1\mathbf{i} + 0.1\mathbf{j}\ (\text{m/s}^2)}$$

(b) If the earth fixed system is not assumed to be inertial, $\mathbf{a}_B = -R_E\omega_E^2\cos^2\lambda\mathbf{i} + R_E\omega_E^2\cos\lambda\sin\lambda\mathbf{j}$, the angular velocity of the rotating coordinate system is $\boldsymbol{\omega} = \omega_E\sin\lambda\mathbf{i} + \omega_E\cos\lambda\mathbf{j}$ (rad/s). The relative velocity in the earth fixed system is $\mathbf{v}_{A\text{rel}} = 2000\mathbf{i} + 2000\mathbf{j}$ (m/s), and $\mathbf{r}_{A/B} = R_E\mathbf{i}$ (m).

$$2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} = 2\omega_E \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin\lambda & \cos\lambda & 0 \\ 2000 & 2000 & 0 \end{bmatrix}$$

$$= 4000\omega_E(\sin\lambda - \cos\lambda)\mathbf{k}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B}) = \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_E\sin\lambda & \omega_E\cos\lambda & 0 \\ R_E & 0 & 0 \end{bmatrix}$$

$$= \boldsymbol{\omega} \times (-R_E\omega_E\cos\lambda)\mathbf{k}$$

$$\boldsymbol{\omega} \times (-R_E\omega_E\cos\lambda)\mathbf{k} = R_E\omega_E^2\cos\lambda \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= (-R_E\omega_E^2\cos^2\lambda)\mathbf{i} + (R_E\omega_E^2\cos\lambda\sin\lambda)\mathbf{j}.$$

Collect terms,

$$\mathbf{a}_{A\text{rel}} = +(R_E\omega_E^2\cos^2\lambda)\mathbf{i} - (R_E\omega_E^2\cos\lambda\sin\lambda)\mathbf{j}$$

$$- 4000\omega_E(\sin\lambda - \cos\lambda)\mathbf{k} + 0.1\mathbf{i} + 0.1\mathbf{j}.$$

Substitute values:

$$R_E = 6336 \times 10^3\ \text{m},\ \omega_E = 0.73 \times 10^{-4}\ \text{rad/s},\ \lambda = 30^\circ,$$

$$\boxed{\mathbf{a}_{A\text{rel}} = 0.125\mathbf{i} + 0.0854\mathbf{j} + 0.1069\mathbf{k}\ (\text{m/s}^2)}$$

Note: The last two terms in the parenthetic expression for $\mathbf{a}_A$ in

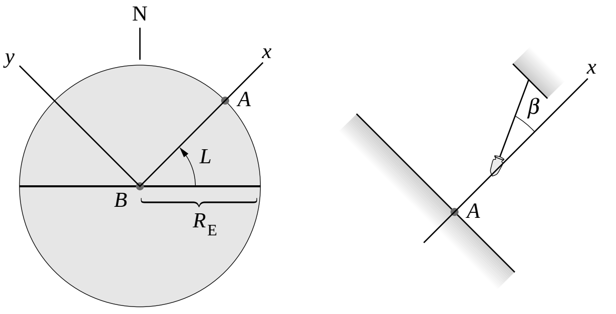
$$\sum \mathbf{F} - m[\mathbf{a}_B + 2\boldsymbol{\omega} \times \mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})]$$

$$= m\mathbf{a}_{A\text{rel}}$$

can be neglected without significant change in the answers.

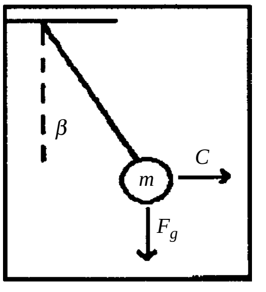
Problem 17.157* Consider a point A on the surface of the earth at north latitude L. The radius of the earth is R_E and its angular velocity is ω_E. A plumb bob suspended just above the ground at point A will hang at a small angle β relative to the vertical because of the earth's rotation. Show that β is related to the latitude by

$$\tan\beta = \frac{\omega_E^2 R_E \sin L \cos L}{g - \omega_E^2 R_E \cos^2 L}.$$



Strategy: Using the earth-fixed coordinate system shown, express Newton's second law in the form given by Eq. (17.22).

Solution: Use Eq. (17.25). $\sum \mathbf{F} - m[\mathbf{a}_B + 2\boldsymbol{\omega}\times\mathbf{v}_{A\text{rel}} + \boldsymbol{\alpha}\times\mathbf{r}_{A/B} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})] = m\mathbf{a}_{A\text{rel}}$. The bob is stationary, so that $\mathbf{v}_{A\text{rel}} = 0$. The origin of the coordinate system is stationary, so that $\mathbf{a}_B = 0$. The external force is the weight of the bob $\sum\mathbf{F} = m\mathbf{g}$. The relative acceleration is the apparent acceleration due to gravity, $m\mathbf{a}_{A\text{rel}} = m\mathbf{g}_{\text{Apparent}}$. Substitute:



$$\mathbf{g}_{\text{Apparent}} = \mathbf{g} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_{A/B})$$

$$= g\mathbf{i} - \boldsymbol{\omega}\times\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \omega_E\sin L & \omega_E\cos L & 0\\ R_E & 0 & 0\end{bmatrix}$$

$$\mathbf{g}_{\text{Apparent}} = g\mathbf{i} - \boldsymbol{\omega}\times(-R_E\omega_E\cos L)\mathbf{k}$$

$$= g\mathbf{i} + R_E\omega_E^2\cos L\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin L & \cos L & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$\mathbf{g}_{\text{Apparent}} = g\mathbf{i} - (R_E\omega_E^2\cos^2 L)\mathbf{i} + (R_E\omega_E^2\cos L\sin L)\mathbf{j}.$$

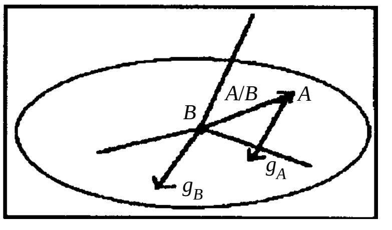
The vertical component of the apparent acceleration due to gravity is $g_{\text{vertical}} = g - R_E\omega_E^2\cos^2 L$. The horizontal component of the apparent acceleration due to gravity is $g_{\text{horizontal}} = R_E\omega_E^2\cos L\sin L$. From equation of angular motion, the moments about the bob suspension are $M_{\text{vertical}} = (\lambda\sin\beta)mg_{\text{vertical}}$ and $M_{\text{horizontal}} = (\lambda\cos\beta)mg_{\text{horizontal}}$, where λ is the length of the bob, and m is the mass of the bob. In equilibrium, $M_{\text{vertical}} = M_{\text{horizontal}}$, from which $g_{\text{vertical}}\sin\beta = g_{\text{horizontal}}\cos\beta$. Substitute and rearrange:

$$\boxed{\tan\beta = \frac{g_{\text{horizontal}}}{g_{\text{vertical}}} = \frac{R_E\omega_E^2\cos L\sin L}{g - R_E\omega_E^2\cos^2 L}}.$$

Problem 17.158* Suppose that a space station is in orbit around the earth and two astronauts on the station toss a ball back and forth. They observe that the ball appears to travel between them in a straight line at constant velocity.

(a) Write Newton's second law for the ball as it travels between the astronauts in terms of a nonrotating coordinate system with its origin fixed to the station. What is the term $\sum F$? Use the equation you wrote to explain the behavior of the ball observed by the astronauts.

(b) Write Newton's second law for the ball as it travels between the astronauts in terms of a nonrotating coordinate system with its origin fixed to the center of the earth. What is the term $\sum \mathbf{F}$? Explain the difference between this equation and the one you obtained in part (a).

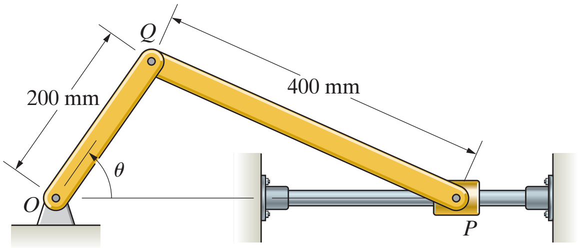


Solution: An earth-centered, non-rotating coordinate system can be treated as inertial for analyzing the motions of objects near the earth (See Section 17.2.) Let O be the reference point of this reference frame, and let B be the origin of the non-rotating reference frame fixed to the space station, and let A denote the ball. The orbiting station and its contents and the station-fixed non-rotating frame are in free fall about the earth (they accelerate relative to the earth due to the earth's gravitational attraction), so that the forces on the ball in the fixed reference frame *exclude the earth's gravitational attraction*. Let $\mathbf{g}_B$ be the station's acceleration, and let $\mathbf{g}_A$ be the ball's acceleration relative to the earth due to the earth's gravitational attraction. Let $\sum F$ be the sum of all forces on the ball, *not including the earth's gravitational attraction*. Newton's second law for the ball of mass m is $\sum \mathbf{F} + m\mathbf{g}_A = m\mathbf{a}_A = m(\mathbf{a}_B + \mathbf{a}_{A/B}) = m\mathbf{g}_B + m\mathbf{a}_{A/B}$. Since the ball is within a space station whose dimensions are small compared to the distance from the earth, $\mathbf{g}_A$ is equal to $\mathbf{g}_B$ within a close approximation, from which $\sum \mathbf{F} = m\mathbf{a}_{A/B}$. The sum of the forces on the ball *not including the force exerted by the earth's gravitational attraction* equals the mass times the ball's acceleration relative to a reference frame fixed with respect to the station. As the astronauts toss the ball back and forth, the only other force on it is aerodynamic drag. Neglecting aerodynamic drag, $\mathbf{a}_{A/B} = 0$, from which the ball will travel in a straight line with constant velocity.

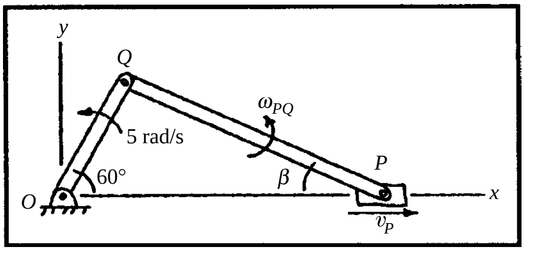
(b) Relative to the earth-centered non-rotating reference frame, Newton's second law for the ball is $\sum \mathbf{F} = m\mathbf{a}_A$ where $\sum \mathbf{F}$ is the sum of all forces on the ball, including aerodynamic drag *and the force due to the earth's gravitational attraction*. Neglect drag, from which $\mathbf{a}_A = \mathbf{g}_A$; the ball's acceleration is its acceleration due to the earth's gravitational attraction, because in this case we are determining the ball's motion relative to the earth.

Note: An obvious unstated assumption is that the time of flight of the ball as it is tossed between the astronauts is much less than the period of an orbit. Thus the very small acceleration differences $\mathbf{g}_A - \mathbf{g}_B$ will have a negligible effect on the path of the ball over the short time interval.

Problem 17.159 If $\theta = 60^\circ$ and bar OQ rotates in the counterclockwise direction at 5 rad/s, what is the angular velocity of bar PQ?



Solution: By applying the law of sines, $\beta = 25.7^\circ$. The velocity of Q is



$\mathbf{v}_Q = \mathbf{v}_0 + \boldsymbol{\omega}_{OQ} \times \mathbf{r}_{Q/O}$ or

$$\mathbf{v}_Q = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 5 \\ 0.2\cos 60^\circ & 0.2\sin 60^\circ & 0 \end{vmatrix} = -\sin 60^\circ \mathbf{i} + \cos 60^\circ \mathbf{j}.$$

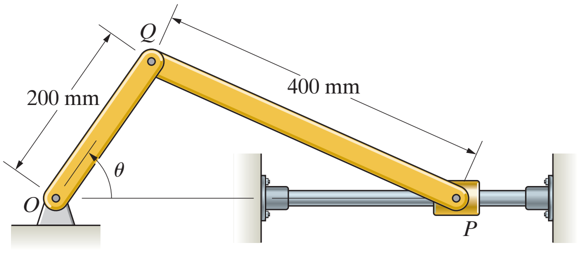
The velocity of P is

$$v_p \mathbf{i} = \mathbf{v}_Q + \boldsymbol{\omega}_{PQ} \times \mathbf{r}_{P/Q}$$

$$= -\sin 60^\circ \mathbf{i} + \cos 60^\circ \mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{PQ} \\ 0.4\cos\beta & -0.4\sin\beta & 0 \end{vmatrix}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components $v_P = -\sin 60^\circ + 0.4\omega_{PQ}\sin\beta$, and $0 = \cos 60^\circ + 0.4\omega_{PQ}\cos\beta$. Solving, we obtain $v_P = -1.11$ m/s and $\omega_{PQ} = -1.39$ rad/s.

Problem 17.160 Consider the system shown in Problem 17.159. If $\theta = 55^\circ$ and the sleeve P is moving to the left at 2 m/s, what are the angular velocities of bars OQ and PQ?



Solution: By applying the law of sines, $\beta = 24.2^\circ$ The velocity of Q is

$$\mathbf{v}_Q = \mathbf{v}_0 + \boldsymbol{\omega}_{OQ} \times \mathbf{r}_{Q/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{OQ} \\ 0.2\cos 55^\circ & 0.2\sin 55^\circ & 0 \end{vmatrix}$$

$$= -0.2\omega_{OQ}\sin 55^\circ \mathbf{i} + 0.2\omega_{OQ}\cos 55^\circ \mathbf{j} \qquad \textbf{(1)}$$

We can also express $\mathbf{v}_Q$ as

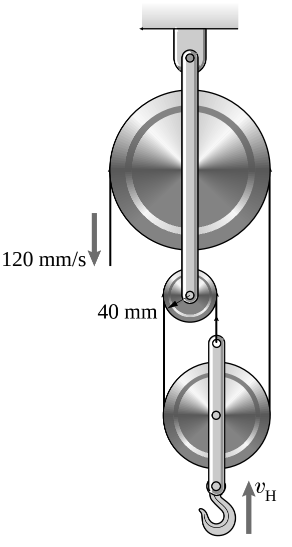
$$\mathbf{v}_Q = \mathbf{v}_P + \boldsymbol{\omega}_{PQ} \times \mathbf{r}_{Q/P}$$

$$= -2\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{PQ} \\ -0.4\cos\beta & 0.4\sin\beta & 0 \end{vmatrix}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components in Equations (1) and (2), we get $-0.2\omega_{OQ}\sin 55^\circ = -2 - 0.4\omega_{PQ}\sin\beta$, and $0.2\omega_{OQ}\cos 55^\circ = -0.4\omega_{PQ}\cos\beta$. Solving, we obtain

$$\omega_{OQ} = 9.29 \text{ rad/s} \omega_{PQ} = -2.92 \text{ rad/s}.$$

Problem 17.161 Determine the vertical velocity v_H of the hook and the angular velocity of the small pulley.



Solution: The upper pulley is fixed so that it cannot move, from which the upward velocity of the rope on the right is equal to the downward velocity on the left, and the upward velocity of the rope on the right of the lower pulley is 120 mm/s. The small pulley is fixed so that it does not move. The upward velocity on the right of the small pulley is v_H mm/s, from which the downward velocity on the left is v_H mm/s. The upward velocity of the center of the bottom pulley is the mean of the difference of the velocities on the right and left, from which

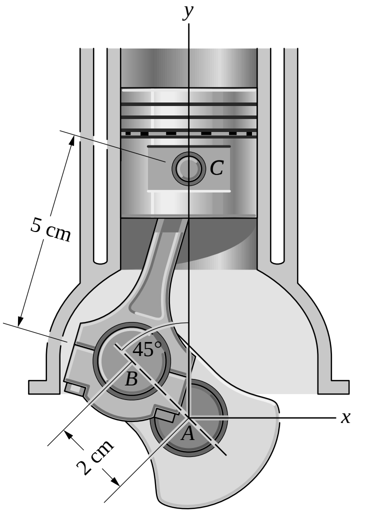
$$v_H = \frac{120 - v_H}{2}.$$

Solve, $\boxed{v_H = 40 \text{ mm/s}}$.

The angular velocity of the small pulley is

$$\boxed{\omega = \frac{v_H}{R} = \frac{40}{40} = 1 \text{ rad/s}}.$$

Problem 17.162 If the crankshaft AB is turning in the counterclockwise direction at 2000 rpm, what is the velocity of the piston?



Solution: The angle of the crank with the vertical is 45°. The angular velocity of the crankshaft is

$$\omega = 2000\left(\frac{2\pi}{60}\right) = 209.44 \text{ rad/s}.$$

The vector location of point B (the main rod bearing) $\mathbf{r}_B = 2(-\mathbf{i}\sin 45^\circ + \mathbf{j}\cos 45^\circ) = 1.414(-\mathbf{i}+\mathbf{j})$ cm. The velocity of point B (the main rod bearing) is

$$\mathbf{v}_B = \boldsymbol{\omega}\times\mathbf{r}_{B/A} = 1.414\omega\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & 1\\ -1 & 1 & 0\end{bmatrix}$$

$$= -296.2(\mathbf{i}+\mathbf{j}) \text{ (cm/s)}.$$

From the law if sines the interior angle between the connecting rod and the vertical at the piston is obtained from $\dfrac{2}{\sin\theta} = \dfrac{5}{\sin 45^\circ}$, from which

$$\theta = \sin^{-1}\left(\frac{2\sin 45^\circ}{5}\right) = 16.43^\circ.$$

The location of the piston is $\mathbf{r}_C = (2\sin 45^\circ + 5\cos\theta)\mathbf{j} = 6.21\mathbf{j}$ (cm). The vector $\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = -1.414\mathbf{i} - 4.796\mathbf{j}$ (cm). The piston is constrained to move along the y axis. In terms of the connecting rod the velocity of the point B is

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{BC}\times\mathbf{r}_{B/C} = v_C\mathbf{j} + \begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & \omega_{BC}\\ 1.414 & -4.796 & 0\end{bmatrix}$$

$$= v_C\mathbf{j} + 4.796\omega_{BC}\mathbf{i} - 1.414\omega_{BC}\mathbf{j} \text{ (cm/s)}.$$

Equate expressions for $\mathbf{v}_B$ and separate components:

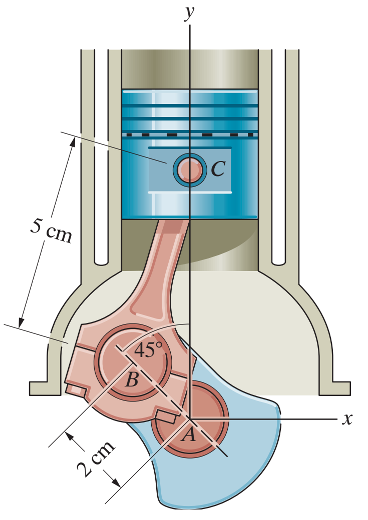
$$-296.2 = 4.796\omega_{BC},$$

$$-296.2 = v_C - 1.414\omega_{BC}.$$

Solve: $\boxed{\mathbf{v}_C = -383.5\mathbf{j} \text{ (cm/s)} = -3.835\mathbf{j} \text{ (m/s)}}$,

$$\omega_{BC} = -61.8 \text{ rad/s}.$$

Problem 17.163 In Problem 17.162, if the piston is moving with velocity $\mathbf{v}_C = 240\mathbf{j}$ (cm/s), what are the angular velocities of the crankshaft AB and the connecting rod BC?



Solution: Use the solution to Problem 17.162. The vector location of point B (the main rod bearing) $\mathbf{r}_B = 1.414(-\mathbf{i}+\mathbf{j})$ cm. From the law if sines the interior angle between the connecting rod and the vertical at the piston is

$$\theta = \sin^{-1}\left(\frac{2\sin 45^\circ}{5}\right) = 16.43^\circ.$$

The location of the piston is $\mathbf{r}_C = (2\sin 45^\circ + 5\cos\theta)\mathbf{j} = 6.21\mathbf{j}$ (cm). The piston is constrained to move along the y axis. In terms of the connecting rod the velocity of the point B is

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C} = 240\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -1.414 & -4.796 & 0 \end{bmatrix}$$

$$\mathbf{v}_B = 240\mathbf{j} + 4.796\omega_{BC}\mathbf{i} - 1.414\omega_{BC}\mathbf{j} \text{ (cm/s)}.$$

In terms of the crank angular velocity, the velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 1.414\omega_{AB}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$= -1.414\omega_{AB}(\mathbf{i}+\mathbf{j}) \text{ (cm/s)}.$$

Equate expressions and separate components:

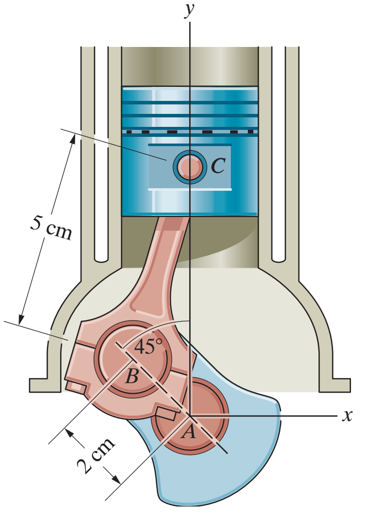
$$4.796\omega_{BC} = -1.414\omega_{BC} = -1.414\omega_{AB}.$$

Solve:

$\boxed{\omega_{BC} = 38.65 \text{ rad/s}}$ (counterclockwise).

$\boxed{\omega_{AB} = 131.1 \text{ rad/s} = -12515 \text{ rpm}}$ (clockwise).

Problem 17.164 In Problem 17.162, if the piston is moving with velocity $\mathbf{v}_C = 240\mathbf{j}$ (cm/s), and its acceleration is zero, what are the angular accelerations of crankshaft AB and the connecting rod BC?



Solution: Use the solution to Problem 17.163. $\mathbf{r}_{B/A} = 1.414(-\mathbf{i}+\mathbf{j}$ cm, $\omega_{AB} = -131.1$ rad/s, $\mathbf{r}_{B/C} = \mathbf{r}_B - \mathbf{r}_C = -1.414\mathbf{i} - 4.796\mathbf{j}$ (cm), $\omega_{BC} = 38.65$ rad/s. For point B,

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A}$$

$$= 1.414\alpha_{AB}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} - 1.414\omega_{AB}^2(-\mathbf{i}+\mathbf{j}),$$

$$\mathbf{a}_B = -1.414\alpha_{AB}(\mathbf{i}+\mathbf{j}) + 24291(\mathbf{i}-\mathbf{j}) \text{ (cm/s}^2).$$

In terms of the angular velocity of the connecting rod,

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2\mathbf{r}_{B/C},$$

$$\mathbf{a}_B = \alpha_{BC}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -1.414 & -4.796 & 0 \end{bmatrix} - \omega_{BC}^2(-1.414\mathbf{i} - 4.795\mathbf{j}) \text{ (cm/s}^2),$$

$$\mathbf{a}_B = 4.796\alpha_{BC}\mathbf{i} - 1.414\alpha_{BC}\mathbf{j} + 2112.3\mathbf{i} + 71630\mathbf{j} \text{ (cm/s}^2).$$

Equate expressions and separate components:

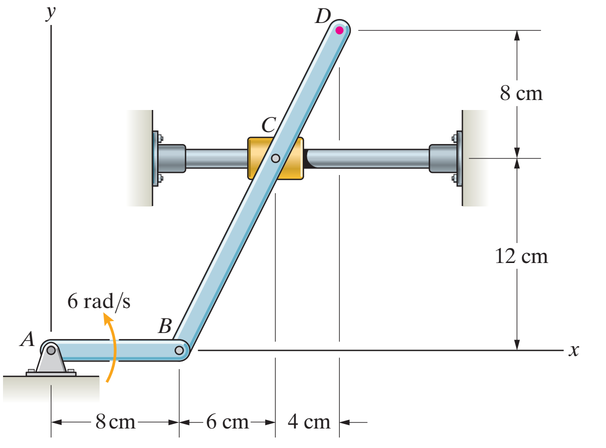
$$-1.414\alpha_{AB} + 24291 = 4.796\alpha_{BC} + 2112.3,$$

$$-1414\alpha_{AB} - 24291 = 1.414\alpha_{BC} + 7163.$$

Solve: $\boxed{\alpha_{AB} = -13{,}605 \text{ rad/s}^2}$ (clockwise).

$\boxed{\alpha_{BC} = 8636.5 \text{ rad/s}^2}$ (counterclockwise).

Problem 17.165 Bar AB rotates at 6 rad/s in the counterclockwise direction. Use instantaneous centers to determine the angular velocity of bar BCD and the velocity of point D.



Solution: The strategy is to determine the angular velocity of bar BC from the instantaneous center; using the constraint on the motion of C. The vector $\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = (8\mathbf{i} + 4\mathbf{j}) - 4\mathbf{j} = 8\mathbf{i}$ (cm). The velocity of point B is $\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \omega_{AB}(\mathbf{k} \times 8\mathbf{i}) = 48\mathbf{j}$ (cm/s). The velocity of point B is normal to the x axis, and the velocity of C is parallel to the x axis. The instantaneous center of bar BC has the coordinates (14, 0). The vector

$$\mathbf{r}_{B/IC} = \mathbf{r}_B - \mathbf{r}_{IC} = (8\mathbf{i} + 4\mathbf{j}) - (14\mathbf{i} - 4\mathbf{j}) = -6\mathbf{i} \text{ (cm)}.$$

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/IC} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -6 & 0 & 0 \end{bmatrix} = -6\omega_{BC}\mathbf{j} = 48\mathbf{j},$$

from which

$$\boxed{\omega_{BC} = -\frac{48}{6} = -8 \text{ rad/s}}.$$

The velocity of point C is

$$\mathbf{v}_C = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/IC} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 0 & 12 & 0 \end{bmatrix} = 96\mathbf{i} \text{ (cm/s)}.$$

The velocity of point D is normal to the unit vector parallel to BCD

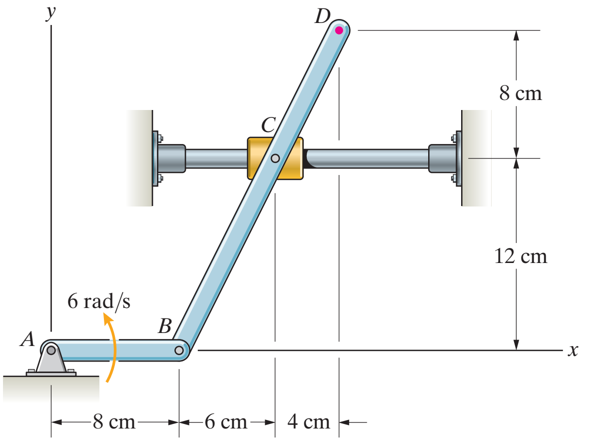
$$\mathbf{e} = \frac{6\mathbf{i} + 12\mathbf{j}}{\sqrt{6^2 + 12^{12}}} = 0.4472\mathbf{i} + 0.8944\mathbf{j}.$$

The intersection of the projection of this unit vector with the projection of the unit vector normal to velocity of C is occurs at point C, from which the coordinates of the instantaneous center for the part of the bar CD are (14, 12). The instantaneous center is translating at velocity $\mathbf{v}_C$, from which the velocity of point D is

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{D/ICD} = 96\mathbf{i} - 8(\mathbf{k} \times (4\mathbf{i} + 8\mathbf{j}))$$

$$= 160\mathbf{i} - 32\mathbf{j} \text{ (cm/s)}.$$

Problem 17.166 In Problem 17.165, bar AB rotates with a constant angular velocity of 6 rad/s in the counterclockwise direction. Determine the acceleration of point D.



Solution: Use the solution to Problem 17.165. The accelerations are determined from the angular velocity, the known accelerations of B, and the constraint on the motion of C. The vector $\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = 8\mathbf{i}$ (cm). The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{A/B} = 0 - 36(8\mathbf{i}) = -288\mathbf{i} \text{ (cm/s}^2).$$

From the solution to Problem 17.165, $\omega_{BC} = -8$ rad/s. (clockwise).

$$\mathbf{r}_{C/B} = \mathbf{r}_C - \mathbf{v}_B = (14\mathbf{i} + 12\mathbf{j}) - (8\mathbf{i}) = 6\mathbf{i} + 12\mathbf{j} \text{ (cm)}.$$

The vector $\mathbf{r}_{B/C} = -\mathbf{r}_{C/B}$. The acceleration of point B in terms of the angular acceleration of point C is

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{BC} - \omega_{BC}^2 \mathbf{r}_{B/C}$$

$$= \mathbf{a}_C\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -6 & -12 & 0 \end{bmatrix} - 64(-6\mathbf{i} - 12\mathbf{j}).$$

$$\mathbf{a}_B = a_C\mathbf{i} + 12\alpha_{BC}\mathbf{i} - 6\alpha_{BC}\mathbf{j} + 384\mathbf{i} + 768\mathbf{j}.$$

Equate the expressions and separate components:

$$-288 = a_C + 12\alpha_{BC} + 384, \quad 0 = -6\alpha_{BC} + 768.$$

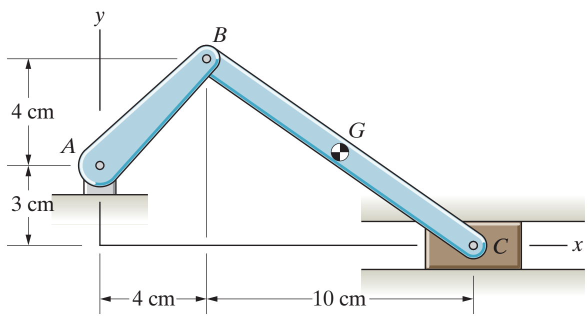
Solve $\alpha_{BC} = 128$ rad/s^2 (counterclockwise), $a_C = -2208$ cm/s^2. The acceleration of point D is

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{D/C} - \omega_{BC}^2 \mathbf{r}_{D/C}$$

$$= -2208\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 4 & 8 & 0 \end{bmatrix} - \omega_{BC}^2(4\mathbf{i} + 8\mathbf{j}).$$

$$\boxed{\begin{aligned} \mathbf{a}_D &= -2208\mathbf{i} + (128)(-8\mathbf{i} + 4\mathbf{j}) - (64)(4\mathbf{i} + 8\mathbf{j}) \\ &= -3490\mathbf{i} \text{ (cm/s}^2) \end{aligned}}.$$

Problem 17.167 Point C is moving to the right at 20 cm/s. What is the velocity of the midpoint G of bar BC?



Solution:

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{w}_{AB} \times \mathbf{r}_{B/A} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 4 & 4 & 0 \end{vmatrix}.$$

Also,

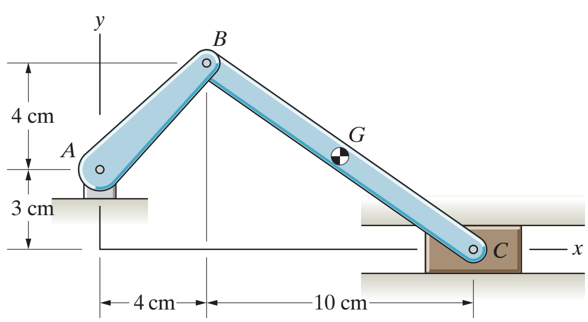
$$\mathbf{v}_B = \mathbf{v}_C + \mathbf{w}_{BC} \times \mathbf{r}_{B/C} = 20\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -10 & 7 & 0 \end{vmatrix}.$$

Equating **i** and **j** components in these two expressions, $-4\omega_{AB} = 20 - 7\omega_{BC}$, $4\omega_{AB} = -10\omega_{BC}$, and solving, we obtain $\omega_{AB} = -2.94$ rad/s, $\omega_{BC} = 1.18$ rad/s. Then the velocity of G is

$$\mathbf{v}_G = \mathbf{v}_C + \omega_{BC} \times \mathbf{r}_{G/C} = 20\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -5 & 3.5 & 0 \end{vmatrix}$$

$$= \underline{15.88\,\mathbf{i} - 5.88\,\mathbf{j}\ \text{(cm/s)}}.$$

Problem 17.168 In Problem 17.167, point C is moving to the right with a constant velocity of 20 cm/s. What is the acceleration of the midpoint G of bar BC?



Solution: See the solution of Problem 17.167.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

$$= \mathbf{O} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 4 & 4 & 0 \end{vmatrix} - \omega_{AB}^2(4\mathbf{i} + 4\mathbf{j}).$$

Also, $\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2 \mathbf{r}_{B/C}$

$$= \mathbf{O} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -10 & 7 & 0 \end{vmatrix} - \omega_{BC}^2(-10\mathbf{i} + 7\mathbf{j}).$$

Equating **i** and **j** components in these two expressions,

$$-4\alpha_{AB} - 4\omega_{AB}^2 = -7\alpha_{BC} + 10\omega_{BC}^2,$$

$$4\alpha_{AB} - 4\omega_{AB}^2 = -10\alpha_{BC} - 7\omega_{BC}^2,$$

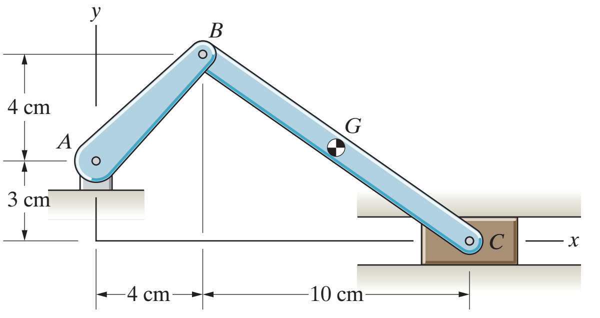
and solving yields $\alpha_{AB} = -4.56$ rad/s^2, $\alpha_{BC} = 4.32$ rad/s^2.

Then $\mathbf{a}_G = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G/C} - \omega_{BC}^2 \mathbf{r}_{G/C}$

$$= \mathbf{O} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -5 & 3.5 & 0 \end{vmatrix} - \omega_{BC}^2(-5\mathbf{i} + 3.5\mathbf{j})$$

$$= \underline{-8.18\mathbf{i} - 26.4\mathbf{j}\ (\text{cm/s}^2)}.$$

Problem 17.169 In Problem 17.167, if the velocity of point C is $\mathbf{v}_C = 1.0\mathbf{i}$ (cm/s), what are the angular velocity vectors of arms AB and BC?



Solution: Use the solution to Problem 17.167: The velocity of the point B is determined from the known velocity of point C and the known velocity of C:

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C} = 1.0\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -10 & 7 & 0 \end{bmatrix}$$

$$= 1.0\mathbf{i} - 7\omega_{BC}\mathbf{i} - 10\omega_{BC}\mathbf{j}.$$

The angular velocity of bar AB is determined from the velocity of B.

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 4 & 4 & 0 \end{bmatrix} = -4\boldsymbol{\omega}_{\mathbf{AB}}(\mathbf{i} - \mathbf{j}) \text{ (cm/s)}$$

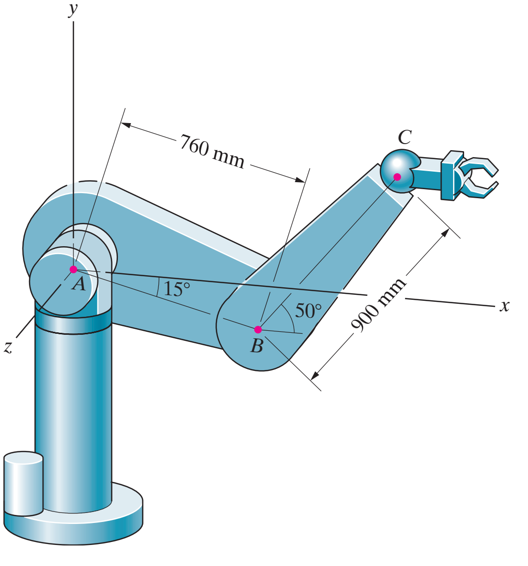
Equate expressions, separate components,

$$1.0 - 7\omega_{BC} = -4\omega_{AB}, -10\omega_{BC} = 4\omega_{AB}.$$

Solve: $\omega_{AB} = -0.147$ rad/s, $\omega_{BC} = 0.0588$ rad/s, from which

$\boxed{\boldsymbol{\omega}_{AB} = -0.147\mathbf{k} \text{ (rad/s)}}$, $\boxed{\boldsymbol{\omega}_{BC} = 0.0588\mathbf{k} \text{ (rad/s)}}$

Problem 17.170 Points B and C are in the $x-y$ plane. The angular velocity vectors of arms AB and BC are $\boldsymbol{\omega}_{AB} = -0.5\mathbf{k}$ (rad/s) and $\boldsymbol{\omega}_{BC} = -2.0\mathbf{k}$ (rad/s). Determine the velocity of point C.



Solution: The vector

$$\mathbf{r}_{B/A} = 760(\mathbf{i}\cos 15^\circ - \mathbf{j}\sin 15^\circ) = 734.1\mathbf{i} - 196.7\mathbf{j} \text{ (mm)}.$$

The vector

$$\mathbf{r}_{C/B} = 900(\mathbf{i}\cos 50^\circ + \mathbf{j}\sin 50^\circ) = 578.5\mathbf{i} + 689.4\mathbf{j} \text{ (mm)}.$$

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = -0.5 \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 734.1 & -196.7 & 0 \end{bmatrix}$$

$$\mathbf{v}_B = -98.35\mathbf{i} - 367.1\mathbf{j} \text{ (mm/s)}.$$

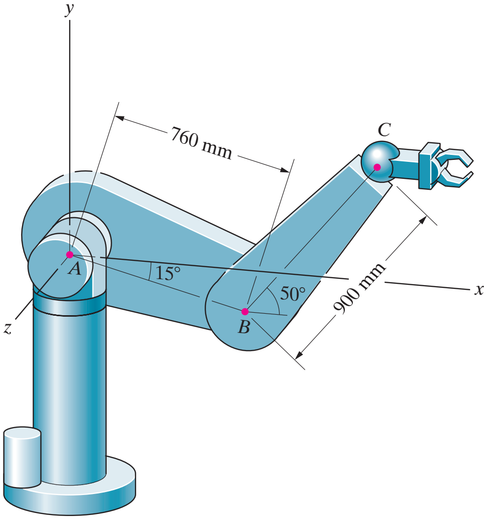
The velocity of point C

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= -98.35\mathbf{i} - 367.1\mathbf{j} + (2)(\mathbf{k} \times (578.5\mathbf{i} + 689.4\mathbf{j})),$$

$$\boxed{\begin{aligned} \mathbf{v}_C &= -98.35\mathbf{i} - 367.1\mathbf{j} - 1378.9\mathbf{i} + 1157.0\mathbf{j} \\ &= -1477.2\mathbf{i} + 790\mathbf{j} \text{ (mm/s)} \end{aligned}}$$

Problem 17.171 In Problem 17.170, if the velocity vector of point C is $\mathbf{v}_C = 1.0\mathbf{i}$ (m/s), what are the angular velocity vectors of arms AB and BC?



Solution: Use the solution to Problem 17.170.

The vector

$$\mathbf{r}_{B/A} = 760(\mathbf{i}\cos 15^\circ - \mathbf{j}\sin 15^\circ) = 734.1\mathbf{i} - 196.7\mathbf{j} \text{ (mm)}.$$

The vector

$$\mathbf{r}_{C/B} = 900(\mathbf{i}\cos 50^\circ + \mathbf{j}\sin 50^\circ) = 578.5\mathbf{i} + 689.4\mathbf{j} \text{ (mm)}.$$

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 734.1 & -196.7 & 0 \end{bmatrix}$$

$$= 196.7\omega_{AB}\mathbf{i} + 734.1\omega_{AB}\mathbf{j} \text{ (mm/s)}.$$

The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= 196.7\omega_{AB}\mathbf{i} + 734.1\omega_{AB}\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 578.5 & 687.4 & 0 \end{bmatrix},$$

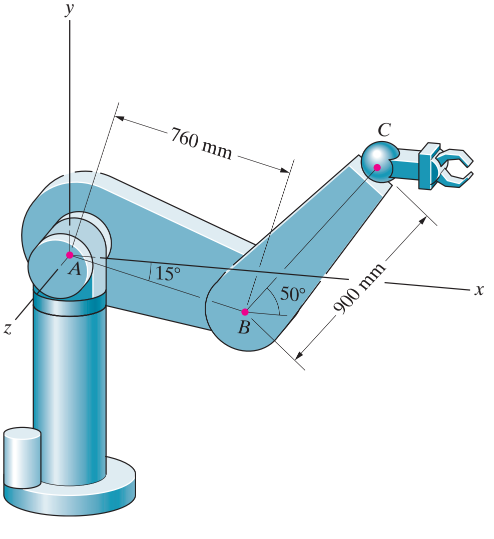
$$1000\mathbf{i} = 196.7\omega_{AB}\mathbf{i} + 734.1\omega_{AB}\mathbf{j} - 687.4\omega_{BC}\mathbf{i} + 578.5\omega_{BC}\mathbf{j} \text{ (mm/s)}.$$

Separate components:

$$1000 = 196.7\omega_{AB} - 687.4\omega_{BC},\ 0 = 734.1\omega_{AB} + 578.5\omega_{BC}.$$

Solve: $\boxed{\boldsymbol{\omega}_{AB} = 0.933\mathbf{k} \text{ (rad/s)}}$, $\boxed{\boldsymbol{\omega}_{BC} = -1.184\mathbf{k} \text{ (rad/s)}}$

Problem 17.172 The angular velocity vectors of arms AB and BC are $\boldsymbol{\omega}_{AB} = -0.5\mathbf{k}$ (rad/s) and $\boldsymbol{\omega}_{BC} = 2.0\mathbf{k}$ (rad/s), and their angular accelerations are $\boldsymbol{\alpha}_{AB} = 1.0\mathbf{k}$ (rad/s^2), and $\alpha_{BC} = 1.0\mathbf{k}$ (rad/s^2). What is the acceleration of point C?



Solution: Use the solution to Problem 17.170.

The vector

$$\mathbf{r}_{B/A} = 760(\mathbf{i}\cos 15^\circ - \mathbf{j}\sin 15^\circ) = 734.1\mathbf{i} - 196.7\mathbf{j} \text{ (mm)}.$$

The vector

$$\mathbf{r}_{C/B} = 900(\mathbf{i}\cos 50^\circ + \mathbf{j}\sin 50^\circ) = 578.5\mathbf{i} + 689.4\mathbf{j} \text{ (mm)}.$$

The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 734.1 & -196.7 & 0 \end{bmatrix}$$

$$- (0.5^2)(734.1\mathbf{i} - 196.7\mathbf{j}) \text{ (mm/s}^2).$$

$$\mathbf{a}_B = 196.7\mathbf{i} + 734.1\mathbf{j} - 183.5\mathbf{i} + 49.7\mathbf{j} = 13.2\mathbf{i} + 783.28\mathbf{j} \text{ (mm/s}^2).$$

The acceleration of point C is

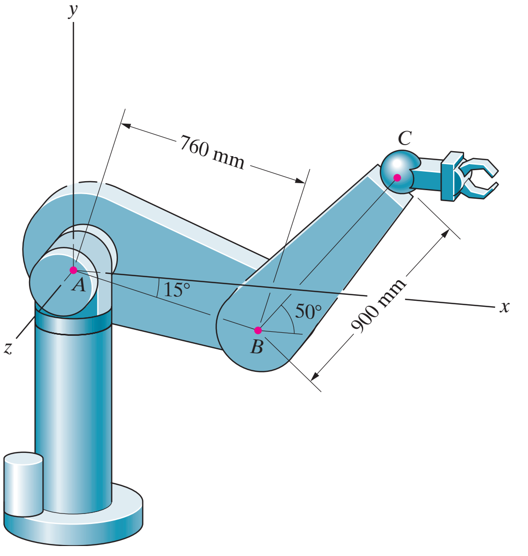
$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \boldsymbol{\omega}_{BC}^2\mathbf{r}_{C/B}$$

$$= \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 578.5 & 689.4 & 0 \end{bmatrix} - (2^2)(578.5\mathbf{i} + 689.4\mathbf{j})$$

$$\mathbf{a}_C = 13.2\mathbf{i} + 783.3\mathbf{j} - 689.4\mathbf{i} + 578.5\mathbf{j} - 2314\mathbf{i} - 2757.8\mathbf{j} \text{ (mm/s}^2).$$

$$\boxed{\mathbf{a}_C = -2990\mathbf{i} - 1396\mathbf{j} \text{ (mm/s}^2)}$$

Problem 17.173 The velocity of point C is $\mathbf{v}_C = 1.0\mathbf{i}$ (m/s) and $\mathbf{a}_C = 0$. What are the angular velocity and angular acceleration vectors of arm BC?



Solution: Use the solution to Problem 17.171. The vector $\mathbf{r}_{B/A} = 760(\mathbf{i}\cos 15^\circ - \mathbf{j}\sin 15^\circ) = 734.1\mathbf{i} - 196.7\mathbf{j}$ (mm). The vector $\mathbf{r}_{C/B} = 900(\mathbf{i}\cos 50^\circ + \mathbf{j}\sin 50^\circ) = 578.5\mathbf{i} + 689.4\mathbf{j}$ (mm). The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 734.1 & -196.7 & 0 \end{bmatrix}$$

$$= 196.7\omega_{AB}\mathbf{i} + 734.1\omega_{AB}\mathbf{j} \text{ (mm/s)}.$$

The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= 196.7\omega_{AB}\mathbf{i} + 734.1\omega_{AB}\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 578.5 & 689.4 & 0 \end{bmatrix}$$

$$1000\mathbf{i} = 196.7\omega_{AB}\mathbf{i} + 734.1\omega_{AB}\mathbf{j} - 689.4\omega_{BC}\mathbf{i}$$

$$+ 578.5\omega_{BC}\mathbf{j} \text{ (mm/s)}.$$

Separate components:

$$1000 = 196.7\omega_{AB} - 689.4\omega_{BC},$$

$$0 = 734.1\omega_{AB} + 578.5\omega_{BC}.$$

Solve: $\boldsymbol{\omega}_{AB} = 0.933\mathbf{k}$ (rad/s), $\boxed{\boldsymbol{\omega}_{BC} = -1.184\mathbf{k} \text{ (rad/s)}}$.

The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 734.1 & -196.7 & 0 \end{bmatrix}$$

$$- (\omega_{AB}^2)(734.1\mathbf{i} - 196.7\mathbf{j}) \text{ (mm/s)}^2.$$

$$\mathbf{a}_B = 196.7\alpha_{AB}\mathbf{i} + 734.1\alpha_{AB}\mathbf{j} - 639.0\mathbf{i} + 171.3\mathbf{j} \text{ (mm/s}^2)$$

The acceleration of point C is

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$$

$$= \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 578.5 & 689.4 & 0 \end{bmatrix} - (\omega_{BC}^2)(578.5\mathbf{i} + 689.4)$$

$$\mathbf{a}_C = 0 = \alpha_{AB}(196.7\mathbf{i} + 734.1\mathbf{j}) + \alpha_{BC}(-689.4\mathbf{i} + 578.5\mathbf{j})$$

$$- 811.0\mathbf{i} - 966.5\mathbf{j} - 639.0\mathbf{i} + 171.3\mathbf{j} \text{ (mm/s}^2)$$

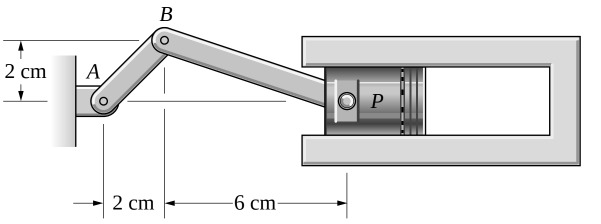
Separate components:

$$196.7\alpha_{AB} - 689.4\alpha_{BC} - 811.3 - 639.0 = 0,$$

$$734.1\alpha_{AB} + 578.5\alpha_{BC} - 966.5 + 171.3 = 0.$$

Solve: $\alpha_{AB} = 2.24$ rad/s^2, $\boxed{\alpha_{BC} = -1.465 \text{ (rad/s}^2)}$

Problem 17.174 The crank AB has a constant clockwise angular velocity of 200 rpm. What are the velocity and acceleration of the piston P?



Solution:

200 rpm $= 200(2\pi)/60 = 20.9$ rad/s.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -20.9 \\ 2 & 2 & 0 \end{vmatrix}.$$

Also, $$\mathbf{v}_B = \mathbf{v}_P + \boldsymbol{\omega}_{BP} \times \mathbf{r}_{B/P} = v_P\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BP} \\ -6 & 2 & 0 \end{vmatrix}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components in these two expressions,

$-(-20.9)(2) = v_P - 2\omega_{BP}$, $(-20.9)(2) = -6\omega_{BP}$,

we obtain $\omega_{BP} = 6.98$ rad/s and $\underline{v_P = 55.9 \text{ cm/s}}$.

$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$

$\quad = \mathbf{0} + \mathbf{0} - (-20.9)^2(2\mathbf{i} + 2\mathbf{j})$.

Also, $\mathbf{a}_B = \mathbf{a}_P + \boldsymbol{\alpha}_{BP} \times \mathbf{r}_{B/P} - \omega_{BP}^2 \mathbf{r}_{B/P}$

$$= a_P\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BP} \\ -6 & 2 & 0 \end{vmatrix} - \omega_{BP}^2(-6\mathbf{i} + 2\mathbf{j}).$$

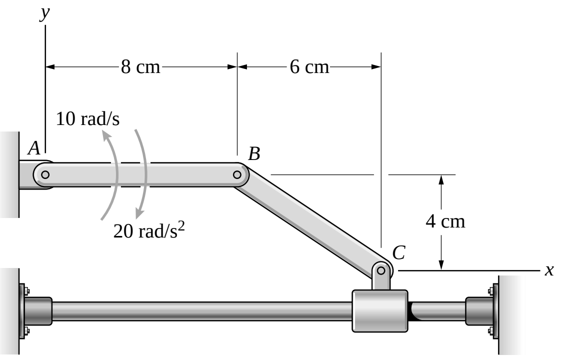
Equating $\mathbf{i}$ and $\mathbf{j}$ components,

$-2(20.9)^2 = a_P - 2\alpha_{BP} + 6\omega_{BP}^2$,

$-2(20.9)^2 = -6\alpha_{BP} - 2\omega_{BP}^2$,

and solving, we obtain $\underline{a_P = -910 \text{ cm/s}^2}$.

Problem 17.175 Bar AB has a counterclockwise angular velocity of 10 rad/s and a clockwise angular acceleration of 20 rad/s^2. Determine the angular acceleration of bar BC and the acceleration of point C.



Solution: Choose a coordinate system with the origin at the left end of the horizontal rod and the x axis parallel to the horizontal rod. The strategy is to determine the angular velocity of bar BC from the instantaneous center; using the angular velocity and the constraint on the motion of C, the accelerations are determined.

The vector $\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = (8\mathbf{i} + 4\mathbf{j}) - 4\mathbf{j} = 8\mathbf{i}$ (cm).

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 8 & 0 & 0 \end{bmatrix} = 80\mathbf{j} \text{ (cm/s)}.$$

The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{A/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -20 \\ 8 & 0 & 0 \end{bmatrix} - 100(8\mathbf{i})$$

$$= -800\mathbf{i} - 160\mathbf{j} \text{ (cm/s}^2\text{)}.$$

The velocity of point B is normal to the x axis, and the velocity of C is parallel to the x axis. The line perpendicular to the velocity at B is parallel to the x-axis, and the line perpendicular to the velocity at C is parallel to the y axis. The intercept is at (14, 4), which is the instantaneous center of bar BC. Denote the instantaneous center by C''.

The vector $\mathbf{r}_{B/C''} = \mathbf{r}_B - \mathbf{r}_{C''} = (8\mathbf{i} + 4\mathbf{j}) - (14\mathbf{i} - 4\mathbf{j}) = -6\mathbf{i}$ (cm).

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/IC} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -6 & 0 & 0 \end{bmatrix} = -6\omega_{BC}\mathbf{j} = 80\mathbf{j},$$

from which $\omega_{BC} = -\dfrac{80}{6} = -13.33$ rad/s.

The vector $\mathbf{r}_{C/B} = \mathbf{r}_C - \mathbf{r}_B = (14\mathbf{i}) - (8\mathbf{i} + 4\mathbf{j}) = 6\mathbf{i} - 4\mathbf{j}$ (cm).

The acceleration of point C is

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B}$$

$$= -800\mathbf{i} - 160\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 6 & -4 & 0 \end{bmatrix} - 1066.7\mathbf{i} + 711.1\mathbf{j} \text{ (cm/s}^2\text{)}.$$

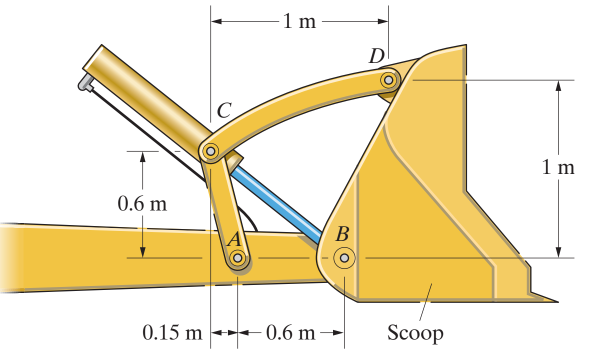
The acceleration of point C is constrained to be parallel to the x axis. Separate components:

$$a_C = -800 + 4\alpha_{BC} - 1066.7, \quad 0 = -160 + 6\alpha_{BC} + 711.1.$$

Solve:

$\boxed{\mathbf{a}_C = -2234\mathbf{i} \text{ (cm/s}^2\text{)}}$, $\boxed{\alpha_{BC} = -91.9 \text{ rad/s}^2}$ (clockwise).

Problem 17.176 The angular velocity of arm AC is 1 rad/s counterclockwise. What is the angular velocity of the scoop?



Solution: Choose a coordinate system with the origin at A and the y axis vertical. The vector locations of B, C and D are $\mathbf{r}_B = 0.6\mathbf{i}$ (m), $\mathbf{r}_C = -0.15\mathbf{i} + 0.6\mathbf{j}$ (m), $\mathbf{r}_D = (1 - 0.15)\mathbf{i} + 1\mathbf{j} = 0.85\mathbf{i} + \mathbf{j}$ (m), from which $\mathbf{r}_{D/C} = \mathbf{r}_D - \mathbf{r}_C = 1\mathbf{i} + 0.4\mathbf{j}$ (m), and $\mathbf{r}_{D/B} = \mathbf{r}_D - \mathbf{r}_B = 0.25\mathbf{i} + \mathbf{j}$ (m). The velocity of point C is

$$\mathbf{v}_C = \boldsymbol{\omega}_{AC} \times \mathbf{r}_{C/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AC} \\ -0.15 & 0.6 & 0 \end{bmatrix}$$

$$= -0.6\mathbf{i} - 0.15\mathbf{j} \text{ (m/s)}.$$

The velocity of D in terms of the velocity of C is

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C} = -0.6\mathbf{i} - 0.15\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ 1 & 0.4 & 0 \end{bmatrix}$$

$$= -0.6\mathbf{i} - 0.15\mathbf{j} + \omega_{CD}(-0.4\mathbf{i} + \mathbf{j}).$$

The velocity of point D in terms of the angular velocity of the scoop is

$$\mathbf{v}_D = \boldsymbol{\omega}_{DB} \times \mathbf{r}_{D/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{DB} \\ 0.25 & 1 & 0 \end{bmatrix} = \omega_{DB}(-\mathbf{i} + 0.25\mathbf{j}).$$

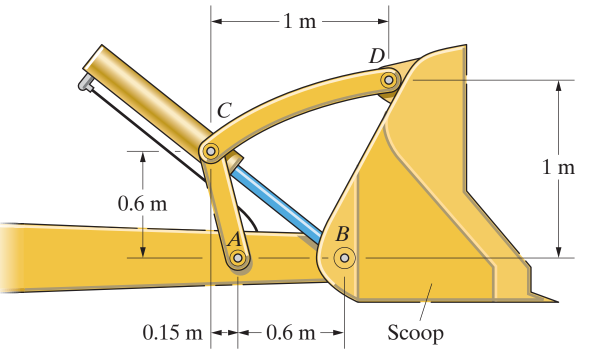
Equate expressions and separate components:

$$-0.6 - 0.4\omega_{CD} = -\omega_{DB}, \; -0.15 + \omega_{CD} = 0.25\omega_{DB}.$$

Solve:

$$\omega_{CD} = 0.333 \text{ rad/s}, \boxed{\omega_{DB} = 0.733 \text{ rad/s}} \text{ (counterclockwise)}.$$

Problem 17.177 The angular velocity of arm AC in Problem 17.176 is 2 rad/s counterclockwise, and its angular acceleration is 4 rad/s^2 clockwise. What is the angular acceleration of the scoop?



Solution: Use the solution to Problem 17.176. Choose a coordinate system with the origin at A and the y axis vertical. The vector locations of B, C and D are $\mathbf{r}_B = 0.6\mathbf{i}$ (m), $\mathbf{r}_C = -0.15\mathbf{i} + 0.6\mathbf{j}$ (m), $\mathbf{r}_D = (1 - 0.15)\mathbf{i} + 1\mathbf{j} = 0.85\mathbf{i} + \mathbf{j}$ (m), from which $\mathbf{r}_{D/C} = \mathbf{r}_D - \mathbf{r}_C = 1\mathbf{i} + 0.4\mathbf{j}$ (m), and $\mathbf{r}_{D/B} = \mathbf{r}_D - \mathbf{r}_B = 0.25\mathbf{i} + \mathbf{j}$ (m). The velocity of point C is

$$\mathbf{v}_C = \boldsymbol{\omega}_{AC} \times \mathbf{r}_{C/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AC} \\ -0.15 & 0.6 & 0 \end{bmatrix} = -1.2\mathbf{i} - 0.3\mathbf{j} \text{ (m/s)}.$$

The velocity of D in terms of the velocity of C is

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C} = -1.2\mathbf{i} - 0.3\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ 1 & 0.4 & 0 \end{bmatrix}$$

$$= -1.2\mathbf{i} - 0.3\mathbf{j} + \omega_{CD}(-0.4\mathbf{i} + \mathbf{j}).$$

The velocity of point D in terms of the angular velocity of the scoop is

$$\mathbf{v}_D = \boldsymbol{\omega}_{DB} \times \mathbf{r}_{D/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{DB} \\ 0.25 & 1 & 0 \end{bmatrix} = \omega_{DB}(-\mathbf{i} + 0.25\mathbf{j}).$$

Equate expressions and separate components:

$$-1.2 - 0.4\omega_{CD} = -\omega_{DB}, \quad -0.3 + \omega_{CD} = 0.25\omega_{DB}.$$

Solve: $\omega_{CD} = 0.667$ rad/s, $\omega_{DB} = 1.47$ rad/s. The angular acceleration of the point C is

$$\mathbf{a}_C = \boldsymbol{\alpha}_{AC} \times \mathbf{r}_{C/A} - \omega_{AC}^2 \mathbf{r}_{C/A}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \boldsymbol{\alpha}_{AC} \\ -0.15 & 0.6 & 0 \end{bmatrix} - \omega_{AC}^2(-0.15\mathbf{i} + 0.6\mathbf{j}),$$

$$\mathbf{a}_C = 2.4\mathbf{i} + 0.6\mathbf{j} + 0.6\mathbf{i} - 2.4\mathbf{j} = 3\mathbf{i} - 1.8\mathbf{j} \text{ (m/s}^2).$$

The acceleration of point D in terms of the acceleration of point C is

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{D/C} - \omega_{CD}^2 \mathbf{r}_{D/C}$$

$$= 3\mathbf{i} - 1.8\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ 1 & 0.4 & 0 \end{bmatrix} - \omega_{CD}^2(\mathbf{i} + 0.4\mathbf{j}).$$

$$\mathbf{a}_C = \alpha_{CD}(-0.4\mathbf{i} + \mathbf{j}) + 2.56\mathbf{i} - 1.98\mathbf{j} \text{ (m/s}^2).$$

The acceleration of point D in terms of the angular acceleration of point B is

$$\mathbf{a}_D = \boldsymbol{\alpha}_{BD} \times \mathbf{r}_{D/B} - \omega_{BD}^2 \mathbf{r}_{D/B}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ 0.25 & 1 & 0 \end{bmatrix} - \omega_{BD}^2(0.25\mathbf{i} + \mathbf{j}).$$

$$\mathbf{a}_D = \alpha_{BD}(-\mathbf{i} + 0.25\mathbf{j}) - 0.538\mathbf{i} - 2.15\mathbf{j}.$$

Equate expressions for $\mathbf{a}_D$ and separate components:

$$-0.4\alpha_{CD} + 2.56 = -\alpha_{BD} - 0.538,$$

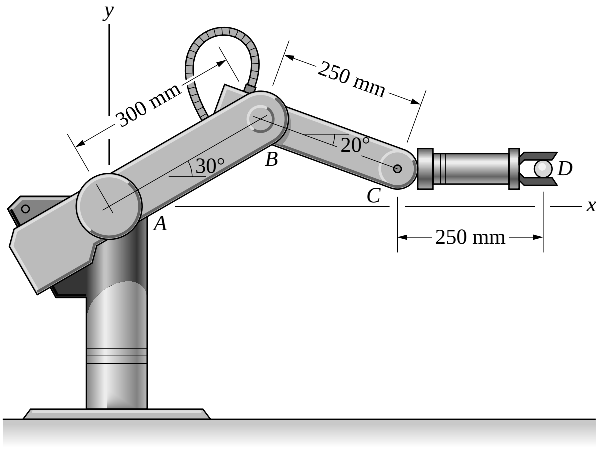
$$\alpha_{CD} - 1.98 = 0.25\alpha_{BD} - 2.15.$$

Solve:

$$\alpha_{CD} = -1.052 \text{ rad/s}^2, \ \boxed{\alpha_{BD} = -3.51 \text{ rad/s}^2},$$

where the negative sign means a clockwise acceleration.

Problem 17.178 If you want to program the robot so that, at the instant shown, the velocity of point D is $\mathbf{v}_D = 0.2\mathbf{i} + 0.8\mathbf{j}$ (m/s) and the angular velocity of arm CD is 0.3 rad/s counterclockwise, what are the necessary angular velocities of arms AB and BC?



Solution: The position vectors are:

$$\mathbf{r}_{B/A} = 300(\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ) = 259.8\mathbf{i} + 150\mathbf{j} \text{ (mm)},$$

$$\mathbf{r}_{C/B} = 250(\mathbf{i}\cos 20^\circ - \mathbf{j}\sin 20^\circ) = 234.9\mathbf{i} - 85.5\mathbf{j} \text{ (mm)},$$

$$\mathbf{r}_{C/D} = -250\mathbf{i} \text{ (mm)}.$$

The velocity of the point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 259.8 & 150 & 0 \end{bmatrix}$$

$$\mathbf{v}_B = -150\omega_{AB}\mathbf{i} + 259.8\omega_{AB}\mathbf{j}.$$

The velocity of point C in terms of the velocity of B is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 234.9 & -85.5 & 0 \end{bmatrix}$$

$$= -150\omega_{AB}\mathbf{i} + 259.8\omega_{AB}\mathbf{j} + 85.5\omega_{BC}\mathbf{i} + 234.9\omega_{BC}\mathbf{j} \text{ (mm/s)}.$$

The velocity of point C in terms of the velocity of point D is

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{C/D} = 200\mathbf{i} + 800\mathbf{j} + 0.3 \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ -250 & 0 & 0 \end{bmatrix}$$

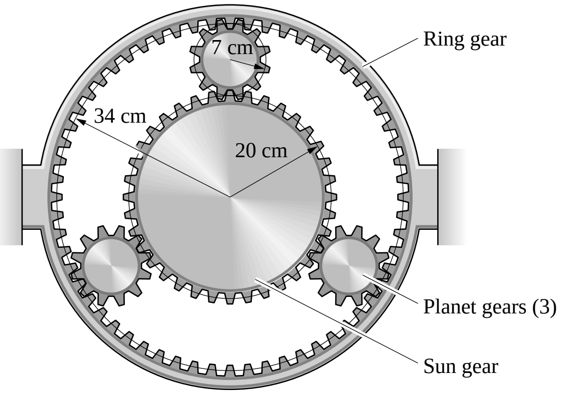
$$= 200\mathbf{i} + 725\mathbf{j} \text{ (mm/s)}.$$

Equate the expressions for $\mathbf{v}_C$ and separate components:

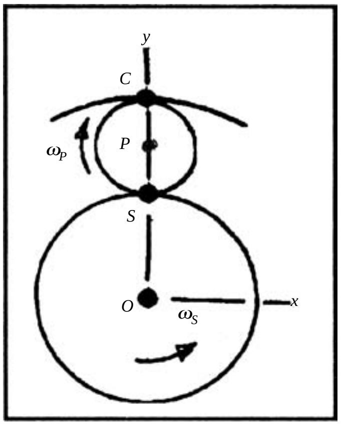
$$-150\omega_{AB} + 85.5\omega_{BC} = 200, \text{ and } 259.8\omega_{AB} + 234.9\omega_{BC} = 725.$$

Solve: $\boxed{\omega_{AB} = 0.261 \text{ rad/s}}$, $\boxed{\omega_{BC} = 2.80 \text{ rad/s}}$.

Problem 17.179 The ring gear is stationary, and the sun gear rotates at 120 rpm (revolutions per minute) in the counterclockwise direction. Determine the angular velocity of the planet gears and the magnitude of the velocity of their centerpoints.



Solution: Denote the point O be the center of the sun gear, point S to be the point of contact between the upper planet gear and the sun gear, point P be the center of the upper planet gear, and point C be the point of contact between the upper planet gear and the ring gear. The angular velocity of the sun gear is



$$\omega_S = \frac{120(2\pi)}{60} = 4\pi \text{ rad/s},$$

from which $\boldsymbol{\omega}_S = 4\pi\mathbf{k}$ (rad/s). At the point of contact between the sun gear and the upper planet gear the velocities are equal. The vectors are: from center of sun gear to S is $\mathbf{r}_{P/S} = 20\mathbf{j}$ (cm), and from center of planet gear to S is $\mathbf{r}_{S/P} = -7\mathbf{j}$ (cm). The velocities are:

$$\mathbf{v}_{S/O} = \mathbf{v}_O + \boldsymbol{\omega}_S \times (20\mathbf{j}) = 0 + \omega_S(20)(\mathbf{k} \times \mathbf{j})$$

$$\mathbf{v}_{S/O} = 20\omega_S \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = -20(\omega_S)$$

$$\mathbf{i} = -251.3\mathbf{i} \text{ (cm/s)}.$$

From equality of the velocities, $\mathbf{v}_{S/P} = \mathbf{v}_{S/O} = -251.3\mathbf{i}$ (cm/s). The point of contact C between the upper planet gear and the ring gear is stationary, from which

$$\mathbf{v}_{S/P} = -251.3\mathbf{i} = \mathbf{v}_C + \boldsymbol{\omega}_P \times \mathbf{r}_{C/S}$$

$$= 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_P \\ -14 & 0 & 0 \end{bmatrix} = 14\omega_P\mathbf{i} = -251.3\mathbf{i}$$

from which $\boxed{\omega_P = 17.95 \text{ rad/s.}}$

The velocity of the centerpoint of the top most planet gear is

$$\mathbf{v}_P = \mathbf{v}_{S/P} + \boldsymbol{\omega}_P \times \mathbf{r}_{P/S} = -251.3\mathbf{i} + (-17.95)(-7)(\mathbf{k} \times \mathbf{j})$$

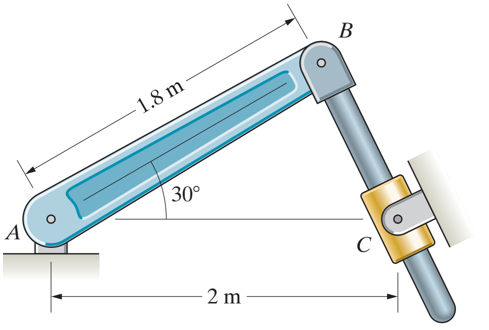
$$= -251.3\mathbf{i} + 125.65 \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{v}_P = -125.7\mathbf{i} \text{ (cm/s)}$$

The magnitude is $\boxed{v_{PO} = 125.7 \text{ cm/s}}$

By symmetry, the magnitudes of the velocities of the centerpoints of the other planetary gears is the same.

Problem 17.180 Arm AB is rotating at 10 rad/s in the clockwise direction. Determine the angular velocity of the arm BC and the velocity at which the arm slides relative to the sleeve at C.



Solution: The position vectors are

$$\mathbf{r}_{B/A} = 1.8(\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ) = 1.56\mathbf{i} + 0.9\mathbf{j} \text{ (m).}$$

$$\mathbf{r}_{B/C} = \mathbf{r}_{B/A} - 2\mathbf{i} = -0.441\mathbf{i} + 0.9\mathbf{j} \text{ (m).}$$

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -10 \\ 1.56 & 0.9 & 0 \end{bmatrix} = 9\mathbf{i} - 15.6\mathbf{j} \text{ (m/s)}$$

The unit vector from B to C is

$$\mathbf{e}_{BC} = \frac{-\mathbf{r}_{B/C}}{|\mathbf{r}_{B/C}|} = 0.4401\mathbf{i} - 0.8979\mathbf{j}.$$

The relative velocity is parallel to this vector:

$$\mathbf{v}_{C\text{rel}} = v_{C\text{rel}}\mathbf{e}_{BC} = v_{C\text{rel}}(0.4401\mathbf{i} - 0.8979\mathbf{j}) \text{ (m/s)}$$

The velocity of B in terms of the velocity of C is

$$\mathbf{v}_B = \mathbf{v}_{\text{rel}} + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C} = \mathbf{v}_{\text{rel}} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -0.441 & 0.9 & 0 \end{bmatrix},$$

$$\mathbf{v}_B = 0.4401 v_{C\text{rel}}\mathbf{i} - 0.8979 v_{C\text{rel}}\mathbf{j} - 0.9\omega_{BC}\mathbf{i} - 0.441\omega_{BC}\mathbf{j} \text{ (m/s).}$$

Equate the expressions for $\mathbf{v}_B$ and separate components:

$$9 = 0.4401 v_{C\text{rel}} - 0.9\omega_{BC}, \text{ and}$$

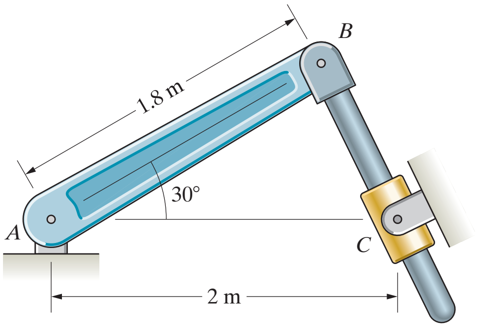
$$-15.6 = -0.8979 v_{C\text{rel}} - 0.441\omega_{BC}.$$

Solve:

$\boxed{v_{C\text{rel}} = 17.96 \text{ m/s}}$ (toward C).

$\boxed{\omega_{BC} = -1.22 \text{ rad/s}}$ (clockwise)

Problem 17.181 In Problem 17.180, arm AB is rotating with an angular velocity of 10 rad/s and an angular acceleration of 20 rad/s^2, both in the clockwise direction. Determine the angular acceleration of arm BC.



Solution: Use the solution to 17.180. The vector

$$\mathbf{r}_{B/A} = 1.8(\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ) = 1.56\mathbf{i} + 0.9\mathbf{j} \text{ (m)}.$$

$$\mathbf{r}_{B/C} = \mathbf{r}_{B/A} - 2\mathbf{i} = -0.441\mathbf{i} + 0.9\mathbf{j} \text{ (m)}.$$

The angular velocity:

$$\omega_{BC} = -1.22 \text{ rad/s},$$

and the relative velocity is $v_{C\text{rel}} = 17.96$ m/s.

The unit vector parallel to bar BC is $\mathbf{e} = 0.4401\mathbf{i} - 0.8979\mathbf{j}$

The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 1.56 & 0.9 & 0 \end{bmatrix} - \omega_{AB}^2(1.56\mathbf{i} + 0.9\mathbf{j}),$$

$$\mathbf{a}_B = 18\mathbf{i} - 31.2\mathbf{j} - 155.9\mathbf{i} - 90\mathbf{j} = -137.9\mathbf{i} - 121.2\mathbf{j} \text{ (m/s}^2).$$

The acceleration of point B in terms of the acceleration of bar BC is

$$\mathbf{a}_B = \mathbf{a}_{C\text{rel}} + 2\boldsymbol{\omega}_{BC} \times \mathbf{v}_{C\text{rel}} + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2 \mathbf{r}_{B/C}.$$

Expanding term by term:

$$\mathbf{a}_{C\text{rel}} = a_{C\text{rel}}(0.4401\mathbf{i} - 0.8979\mathbf{j}) \text{ (m/s}^2),$$

$$2\boldsymbol{\omega}_{BC} \times \mathbf{v}_{C\text{rel}} = 2v_{C\text{rel}}\omega_{BC} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0.440 & -0.8979 & 0 \end{bmatrix}$$

$$= -39.26\mathbf{i} - 19.25\mathbf{j} \text{ (m/s}^2),$$

$$\boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -0.4411 & 0.9 & 0 \end{bmatrix}$$

$$= \alpha_{BC}(-0.9\mathbf{i} - 0.4411\mathbf{j}) \text{ (m/s}^2)$$

$$- \omega_{BC}^2(-0.4411\mathbf{i} + 0.9\mathbf{j})$$

$$= 0.6539\mathbf{i} - 1.334\mathbf{j} \text{ (m/s}^2).$$

Collecting terms,

$$\mathbf{a}_B = a_{C\text{rel}}(0.4401\mathbf{i} - 0.8979\mathbf{j}) - \alpha_{BC}(0.9\mathbf{i} + 0.4411\mathbf{j})$$

$$- 38.6\mathbf{i} - 20.6\mathbf{j} \text{ (m/s}^2).$$

Equate the two expressions for $\mathbf{a}_B$ and separate components:

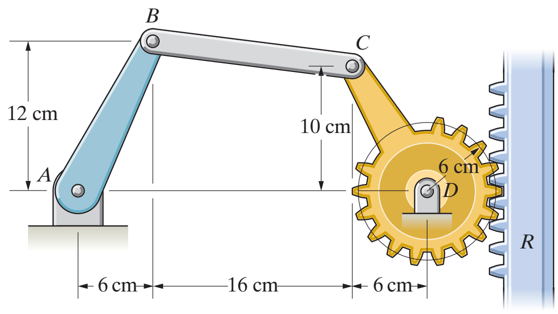
$$-137.9 = 0.4401a_{C\text{rel}} - 0.9\alpha_{BC} - 38.6,$$

and $-121.2 = -0.8979a_{C\text{rel}} - 0.4411\alpha_{BC} - 20.6.$

Solve: $a_{C\text{rel}} = 46.6$ m/s^2 (toward C)

$$\boxed{\alpha_{BC} = 133.1 \text{ rad/s}^2}$$

Problem 17.182 Arm AB is rotating with a constant counterclockwise angular velocity of 10 rad/s. Determine the vertical velocity and acceleration of the rack R of the rack-and-pinion gear.



Solution: The vectors:

$\mathbf{r}_{B/A} = 6\mathbf{i} + 12\mathbf{j}$ (cm). $\mathbf{r}_{C/B} = 16\mathbf{i} - 2\mathbf{j}$ (cm).

The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 6 & 12 & 0 \end{bmatrix} = -120\mathbf{i} + 60\mathbf{j} \text{ (cm/s)}.$$

The velocity of point C in terms of the velocity of point B is

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC} \times \mathbf{r}_{C/B} = \mathbf{v}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 16 & -2 & 0 \end{bmatrix}$$

$$= -120\mathbf{i} + 60\mathbf{j} + \omega_{BC}(2\mathbf{i} + 16\mathbf{j}) \text{ (cm/s)}$$

The velocity of point C in terms of the velocity of the gear arm CD is

$$\mathbf{v}_C = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ -6 & 10 & 0 \end{bmatrix}$$

$$= -10\omega_{CD}\mathbf{i} - 6\omega_{CD}\mathbf{j} \text{ (cm/s)}.$$

Equate the two expressions for $\mathbf{v}_C$ and separate components:

$$-120 + 2\omega_{BC} = -10\omega_{CD}, \quad 60 + 16\omega_{BC} = -6\omega_{CD}.$$

Solve: $\omega_{BC} = -8.92$ rad/s, $\omega_{CD} = 13.78$ rad/s,

where the negative sign means a clockwise rotation. The velocity of the rack is

$$\mathbf{v}_R = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{R/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CD} \\ 6 & 0 & 0 \end{bmatrix} = 6\omega_{CD}\mathbf{j},$$

$$\boxed{\mathbf{v}_R = 82.7\mathbf{j} \text{ (cm/s)} = 0.827\mathbf{j} \text{ (m/s)}}$$

The angular acceleration of point B is

$$\mathbf{a}_B = -\omega_{AB}^2\mathbf{r}_{B/A} = -100(6\mathbf{i} + 12\mathbf{j}) = -600\mathbf{i} - 1200\mathbf{j} \text{ (cm/s}^2\text{)}.$$

The acceleration of point C is

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B},$$

$$\mathbf{a}_C = \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 16 & -2 & 0 \end{bmatrix} - \omega_{BC}^2(16\mathbf{i} - 2\mathbf{j})$$

$$= \mathbf{a}_B + 2\alpha_{BC}\mathbf{i} + 16\alpha_{BC}\mathbf{j} - \omega_{BC}^2(16\mathbf{i} - 2\mathbf{j}).$$

Noting

$$\mathbf{a}_B - \omega_{BC}^2(16\mathbf{i} - 2\mathbf{j}) = -600\mathbf{i} - 1200\mathbf{j} - 1272.7\mathbf{i} + 159.1\mathbf{j}$$

$$= -1872.7\mathbf{i} - 1040.9\mathbf{j},$$

from which $\mathbf{a}_C = +\alpha_{BC}(2\mathbf{i} + 16\mathbf{j}) - 1873\mathbf{i} - 1041\mathbf{j}$ (cm/s^2)

The acceleration of point C in terms of the gear arm is

$$\mathbf{a}_C = \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{C/D} - \omega_{CD}^2\mathbf{r}_{C/D}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ -6 & 10 & 0 \end{bmatrix} - \omega_{CD}^2(-6\mathbf{i} + 10\mathbf{j}) \text{ (cm/s}^2\text{)},$$

$$\mathbf{a}_C = -10\alpha_{CD}\mathbf{i} - 6\alpha_{CD}\mathbf{j} + 1140\mathbf{i} - 1900\mathbf{j} \text{ (cm/s}^2\text{)}.$$

Equate expressions for $\mathbf{a}_C$ and separate components:

$$2\alpha_{BC} - 1873 = -10\alpha_{CD} + 1140,$$

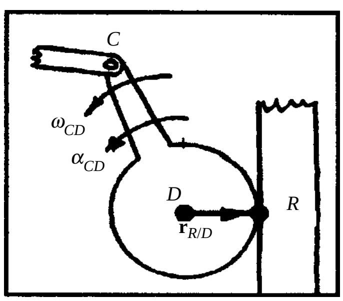
$$16\alpha_{BC} - 1041 = -6\alpha_{CD} - 1900.$$

Solve: $\alpha_{CD} = 337.3$ rad/s^2, and $\alpha_{BC} = -180.2$ rad/s^2.

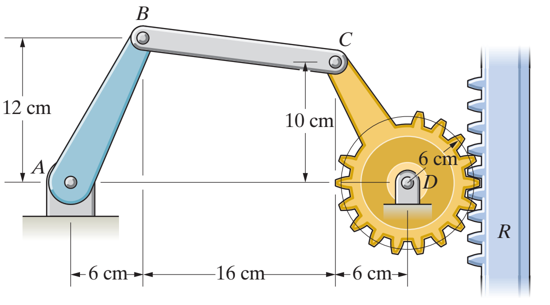
The acceleration of the rack R is the tangential component of the acceleration of the gear at the point of contact with the rack:

$$\mathbf{a}_R = \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{R/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CD} \\ 6 & 0 & 0 \end{bmatrix} = 6\alpha_{CD}\mathbf{j} \text{ (cm/s}^2\text{)}.$$

$$\boxed{\mathbf{a}_R = 2024\mathbf{j} \text{ (cm/s}^2\text{)} = 20.24\mathbf{j} \text{ (m/s}^2\text{)}}$$



Problem 17.183 The rack R of the rack-and-pinion gear is moving upward with a constant velocity of 120 cm/s. What are the angular velocity and angular acceleration of bar BC?



Solution: The constant velocity of the rack R implies that the angular acceleration of the gear is zero, and the angular velocity of the gear is $\omega_{CD} = \dfrac{120}{6} = 20$ rad/s. The velocity of point C in terms of the gear angular velocity is

$$\mathbf{v}_C = \boldsymbol{\omega}_{CD} \times \mathbf{r}_{C/D} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 20 \\ -6 & 10 & 0 \end{bmatrix} = -200\mathbf{i} - 120\mathbf{j} \text{ (cm/s)}.$$

The velocity of point B in terms of the velocity of point C is

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C} = \mathbf{v}_C + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -16 & 2 & 0 \end{bmatrix},$$

$$\mathbf{v}_B = -200\mathbf{i} - 120\mathbf{j} - 2\omega_{BC}\mathbf{i} - 16\omega_{BC}\mathbf{j} \text{ (cm/s)}.$$

The velocity of point B in terms of the angular velocity of the arm AB is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 6 & 12 & 0 \end{bmatrix}$$

$$= -12\omega_{AB}\mathbf{i} + 6\omega_{AB}\mathbf{j} \text{ (cm/s)}.$$

Equate the expressions for $\mathbf{v}_B$ and separate components

$$-200 - 2\omega_{BC} = -12\omega_{AB}, -120 - 16\omega_{BC} = 6\omega_{AB}.$$

Solve: $\omega_{AB} = 14.5$ rad/s, $\boxed{\omega_{BC} = -12.94 \text{ rad/s}}$, where the negative sign means a clockwise rotation. The angular acceleration of the point C in terms of the angular velocity of the gear is

$$\mathbf{a}_C = \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{C/D} - \omega_{CD}^2 \mathbf{r}_{C/D} = 0 - \omega_{CD}^2(-6\mathbf{i} + 10\mathbf{j})$$

$$= 2400\mathbf{i} - 4000\mathbf{j} \text{ (cm/s}^2\text{)}.$$

The acceleration of point B in terms of the acceleration of C is

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} - \omega_{BC}^2 \mathbf{r}_{B/C}$$

$$= \mathbf{a}_C + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -16 & 2 & 0 \end{bmatrix} - \omega_{BC}^2(-16\mathbf{i} + 2\mathbf{j}).$$

$$\mathbf{a}_B = \alpha_{BC}(-2\mathbf{i} - 16\mathbf{j}) + 2400\mathbf{i} - 4000\mathbf{j} + 2680\mathbf{i} - 335\mathbf{j}.$$

$$\mathbf{a}_B = -2\alpha_{BC}\mathbf{i} - 16\alpha_{BC}\mathbf{j} + 5080\mathbf{i} - 433.5\mathbf{j} \text{ (cm/s}^2\text{)}.$$

The acceleration of point B in terms of the angular acceleration of arm AB is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A}$$

$$= \alpha_{AB}(\mathbf{k} \times (6\mathbf{i} + 12\mathbf{j})) - \omega_{AB}^2(6\mathbf{i} + 12\mathbf{j}) \text{ (cm/s}^2\text{)}$$

$$\mathbf{a}_B = \alpha_{AB}(-12\mathbf{i} + 6\mathbf{j}) - 1263.2\mathbf{i} - 2526.4\mathbf{j} \text{ (cm/s}^2\text{)}.$$

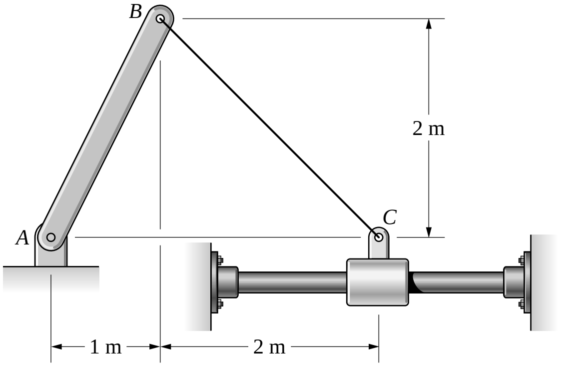
Equate the expressions for $\mathbf{a}_B$ and separate components:

$$-2\alpha_{BC} + 5080 = -12\alpha_{AB} - 1263.2,$$

$$-16\alpha_{BC} - 4335 = 6\alpha_{AB} - 2526.4.$$

Solve: $\alpha_{AB} = -515.2$ rad/s^2, $\boxed{\alpha_{BC} = 80.17 \text{ rad/s}^2}$

Problem 17.184 Bar AB has a constant counterclockwise angular velocity of 2 rad/s. The 1-kg collar C slides on the smooth horizontal bar. At the instant shown, what is the tension in the cable BC?



Solution:

$$\mathbf{v}_B = \mathbf{v}_A + \omega_{AB} \times \mathbf{r}_{B/A} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -4\mathbf{i} + 2\mathbf{j} \text{ (m/s)}.$$

$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC} \times \mathbf{r}_{C/B}$:

$$v_C\mathbf{i} = -4\mathbf{i} + 2\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 2 & -2 & 0 \end{vmatrix}.$$

From the **i** and **j** components of this equation,

$$v_C = -4 + 2\omega_{BC},$$

$$0 = 2 + 2\omega_{BC},$$

we obtain $\omega_{BC} = -1$ rad/s.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2\mathbf{r}_{B/A}$$

$$= \mathbf{0} + \mathbf{0} - (2)^2(\mathbf{i} + 2\mathbf{j})$$

$$= -4\mathbf{i} - 8\mathbf{j} \text{ (m/s}^2\text{)}.$$

$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$:

$$a_C\mathbf{i} = -4\mathbf{i} - 8\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 2 & -2 & 0 \end{vmatrix} - (-1)^2(2\mathbf{i} - 2\mathbf{j}).$$

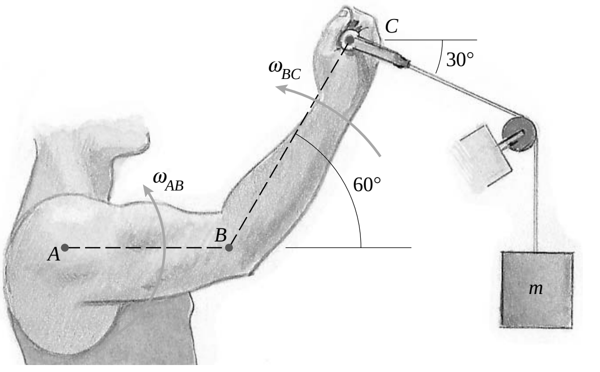
From the **i** and **j** components of this equation,

$$a_C = -4 + 2\alpha_{BC} - 2,$$

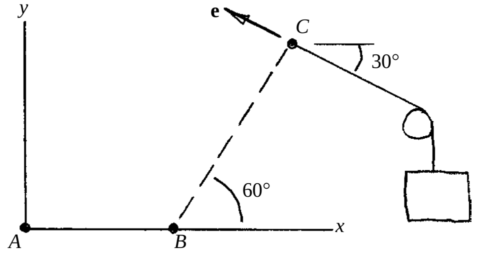
$$0 = -8 + 2\alpha_{BC} + 2,$$

we obtain $a_C = 0$. The force exerted on the collar at this instant is zero, so $\underline{T_{BC} = 0}$.

Problem 17.185 An athlete exercises his arm by raising the 8-kg mass m. The shoulder joint A is stationary. The distance AB is 300 mm, the distance BC is 400 mm, and the distance from C to the pulley is 340 mm. The angular velocities $\omega_{AB} = 1.5$ rad/s and $\omega_{BC} = 2$ rad/s are constant. What is the tension in the cable?



Solution:



$$\mathbf{a}_B = -\omega_{AB}^2 r_{B/A}\mathbf{i} = -(1.5)^2(0.3)\mathbf{i}$$

$$= -0.675\mathbf{i} \text{ (m/s}^2\text{)}.$$

$$\mathbf{a}_C = \mathbf{a}_B - \omega_{BC}^2\mathbf{r}_{C/B}$$

$$= -0.675\mathbf{i} - (2)^2(0.4\cos 60^\circ\mathbf{i} + 0.4\sin 60^\circ\mathbf{j})$$

$$= -1.475\mathbf{i} - 1.386\mathbf{j} \text{ (m/s}^2\text{)}.$$

$$\mathbf{a}_C \cdot \mathbf{e} = (-1.475)(-\cos 30^\circ) + (-1.386)(\sin 30^\circ)$$

$$= 0.585 \text{ m/s}^2.$$

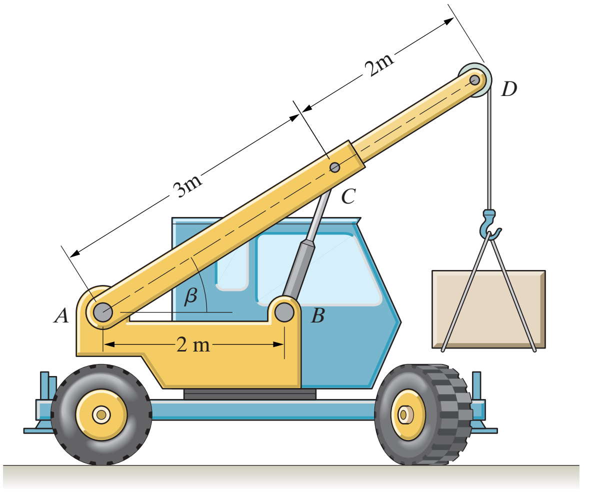
This is the upward acceleration of the mass, so

$$T - mg = m(0.585),$$

$$T = (8)(9.81 + 0.585)$$

$$= 83.2 \text{ N}.$$

Problem 17.186 The hydraulic actuator BC of the crane is extending (increasing in length) at a constant rate of 0.2 m/s. When the angle $\beta = 35°$, what is the angular velocity of the crane's boom AD?



Solution: Using AD we have

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\omega}_{AD} \times \mathbf{r}_{C/A}$$

$$= 0 + \omega_{AD}\mathbf{k} \times (3 \text{ m})(\cos 35^\circ \mathbf{i} + \sin 35^\circ \mathbf{j})$$

$$= (3 \text{ m})\omega_{AD}(-\sin 35^\circ \mathbf{i} + \cos 35^\circ \mathbf{j})$$

Using cylinder BC

$$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{Crel} + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= 0 + (0.2 \text{ m/s})\left(\frac{[3\cos 35^\circ - 2]\mathbf{i} + [3\sin 35^\circ]\mathbf{j}}{\sqrt{[3\cos 35^\circ - 2]^2 + [3\sin 35^\circ]^2}}\right)$$

$$+ \omega_{BC}\mathbf{k} \times [3\cos 35^\circ - 2]\mathbf{i} + [3\sin 35^\circ]\mathbf{j}$$

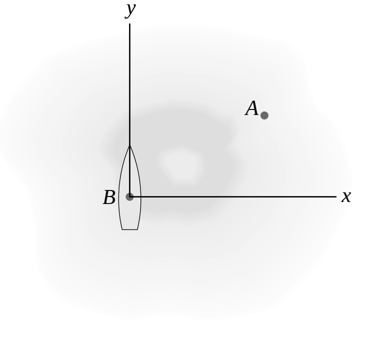
$$= (0.0514 \text{ m/s} - \{1.72 \text{ m}\}\omega_{BC})\mathbf{i} + (0.193 \text{ m/s} - \{0.457 \text{ m}\}\omega_{BC})\mathbf{j}$$

Equating components and solving we find

$\omega_{BC} = 0.133$ rad/s, $\quad \omega_{AD} = 0.103$ rad/s

$$\boxed{\omega_{AD} = 0.103 \text{ rad/s}}$$

Problem 17.187 The coordinate system shown is fixed relative to the ship B. The ship uses its radar to measure the position of a stationary buoy A and determines it to be $400\mathbf{i} + 200\mathbf{j}$ (m). The ship also measures the velocity of the buoy relative to its body-fixed coordinate system and determines it to be $2\mathbf{i} - 8\mathbf{j}$ (m/s). What are the ship's velocity and angular velocity relative to the earth? (Assume that the ship's velocity is in the direction of the y axis).



Solution:

$$\mathbf{v}_A = 0 = \mathbf{v}_B + \mathbf{v}_{A\text{rel}} + \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

$$= v_B\mathbf{j} + 2\mathbf{i} - 8\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ 400 & 200 & 0 \end{vmatrix}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components to zero, $0 = 2 - 200\omega$ $\; 0 = v_B - 8 + 400\omega$ we obtain $\omega = 0.01$ rad/s and $v_B = 4$ m/s.

Problem 18.1 A horizontal force $F = 133.4$ N is applied to the 1023 N refrigerator as shown. Friction is negligible.

(a) What is the magnitude of the refrigerator's acceleration?
(b) What normal forces are exerted on the refrigerator by the floor at A and B?

Solution: Assume that the refrigerator rolls without tipping. We have the following equations of motion.

$$\sum F_x : (133.4 \text{ N}) = \left(\frac{1023 \text{ N}}{9.81 \text{ m/s}^2}\right) a$$

$$\sum F_y : A + B - 1023 \text{ N} = 0$$

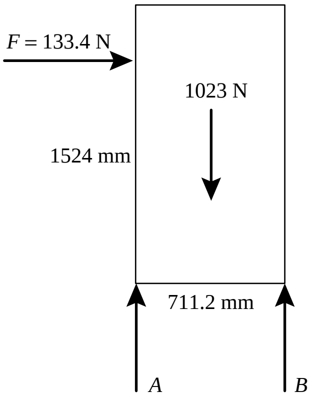
$$\sum M_G : -(133.4 \text{ N})(0.813 \text{ m.}) - A(0.356 \text{ m}) + B(0.356 \text{ m}) = 0$$

Solving we find

(a) $\boxed{a = 1.28 \text{ m/s}^2}$

(b) $\boxed{A = 359 \text{ N}, \ B = 664.1 \text{ N}}$

Since $A > 0$ and $B > 0$ then our assumption is correct.



Problem 18.2 Solve Problem 18.1 if the coefficient of kinetic friction at A and B is $\mu_k = 0.1$.

Solution: Assume sliding without tipping

$$\sum F_x : (133.4 \text{ N}) - (0.1)(A + B) = \left(\frac{1023 \text{ N}}{9.81 \text{ m/s}^2}\right) a$$

$$\sum F_y : A + B - 1023 \text{ N} = 0$$

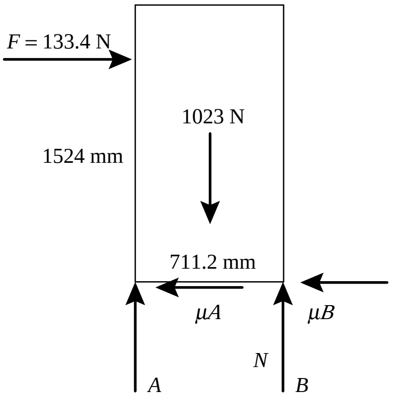
$$\sum M_G : -(133.4 \text{ N})(0.813 \text{ m}) - A(0.356 \text{ m}) + B(0.356 \text{ m})$$

$$- (0.1)(A + B)(0.711 \text{ m}) = 0$$

Solving, we find

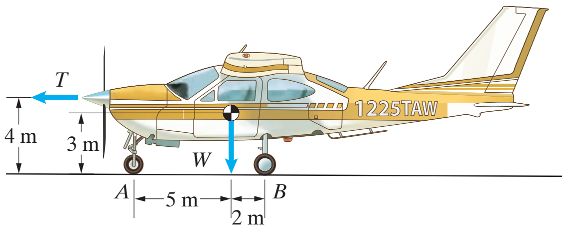
(a) $\boxed{a = 0.3 \text{ m/s}^2}$

(b) $\boxed{A = 256.6 \text{ N}, \ B = 765.1 \text{ N}}$

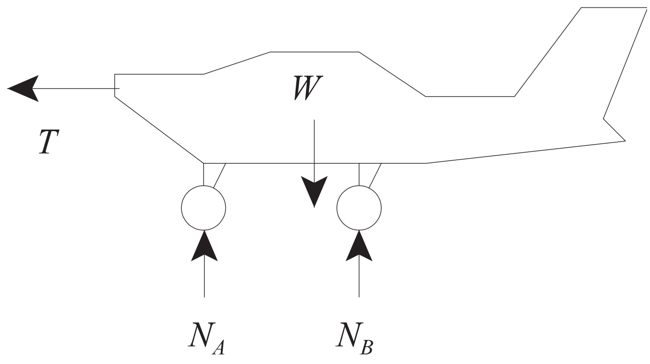


Problem 18.3 As the 2800-N airplane begins its take-off run at $t = 0$, its propeller exerts a horizontal force $T = 1000$ N. Neglect horizontal forces exerted on the wheels by the runway.

(a) What distance has the airplane moved at $t = 2$ s?
(b) what normal forces are exerted on the tires at A and B?



Solution: The unknowns are N_A, N_B, a.



The equations of motion are:

$$\Sigma F_x : -T = -\frac{W}{g}a,$$

$$\Sigma F_y : N_A + N_B - W = 0$$

$$\Sigma M_G : N_B(2\text{ m}) - N_A(5\text{ m})$$
$$+\,T(1\text{ m}) = 0$$

Putting in the numbers for T, W, and g and solving we find

$$N_A = 943\text{ N}, \quad N_B = 1860\text{ N}, \quad a = 3.5\text{ m/s}^2.$$

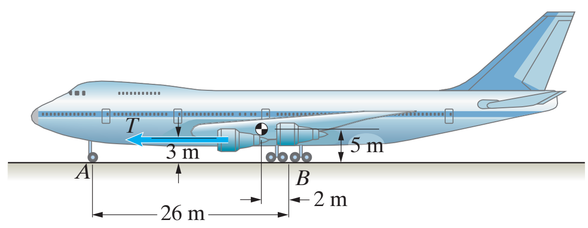
(a) The distance is given by $d = \frac{1}{2}at^2 = \frac{1}{2}(3.5\text{ m/s}^2)(2\text{ s})^2 = 7\text{ m}$

$\boxed{d = 7\text{ m}}$

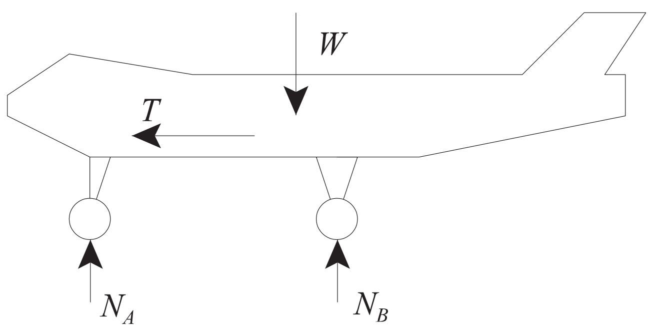
(b) The forces were found to be

$\boxed{N_A = 943\text{ N}, \quad N_B = 1860\text{ N}}$

Problem 18.4 The Boeing 747 begins its takeoff run at time $t = 0$. The normal forces exerted on its tires at A and B are $N_A = 175$ kN and $N_B = 2800$ kN. If you assume that these forces are constant and neglect horizontal forces other than the thrust T, how fast is the airplanes moving at $t = 4$ s? (See Active Example 18.1.)



Solution: The unknowns are T, W, a. The equations of motion are:



$$\Sigma F_x : -T = -\frac{W}{g}a,$$

$$\Sigma F_y : N_A + N_B - W = 0,$$

$$\Sigma M_G : N_B(2\text{ m}) - N_A(24\text{ m})$$
$$-\,T(2\text{ m}) = 0.$$

Putting in the numbers for N_A and N_B and solving, we find

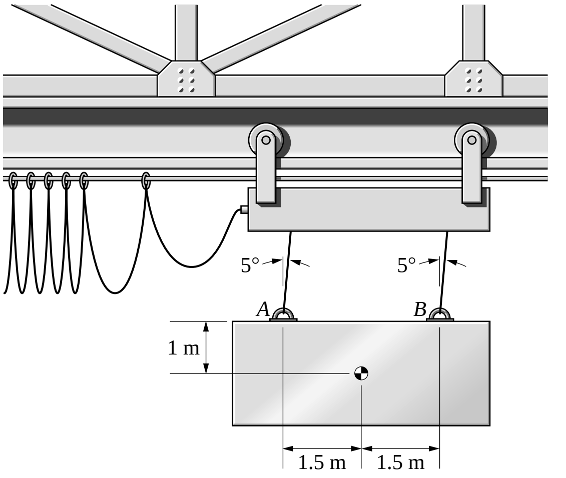
$$a = 2.31\text{ m/s}^2, \quad T = 700\text{ kN}, \quad W = 2980\text{ kN}.$$

The velocity is then given by

$$v = at = (2.31\text{ m/s}^2)(4\text{ s}) = 9.23\text{ m/s}. \quad \boxed{v = 9.23\text{ m/s.}}$$

Problem 18.5 The crane moves to the right with constant acceleration, and the 800-kg load moves without swinging.

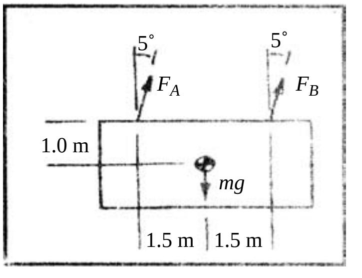
(a) What is the acceleration of the crane and load?
(b) What are the tensions in the cables attached at A and B?



Solution: From Newton's second law: $F_x = 800a$ N.
The sum of the forces on the load:

$$\sum F_x = F_A \sin 5^\circ + F_B \sin 5^\circ - 800a = 0.$$

$$\sum F_y = F_A \cos 5^\circ + F_B \cos 5^\circ - 800g = 0.$$



The sum of the moments about the center of mass:

$$\sum M_{CM} = -1.5F_A \cos 5^\circ + 1.5F_B \cos 5^\circ$$

$$- F_A \sin 5^\circ - F_B \sin 5^\circ = 0.$$

Solve these three simultaneous equations:

$\boxed{a = 0.858 \text{ m/s}^2}$,

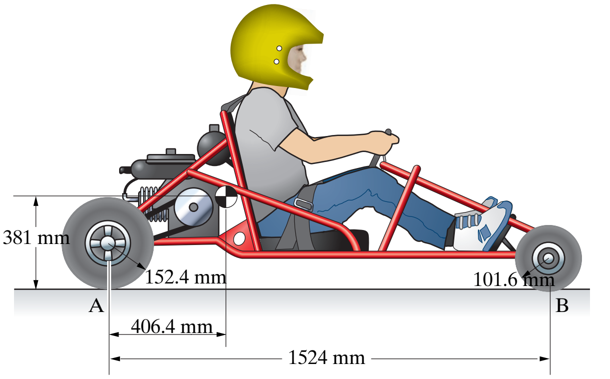
$\boxed{F_A = 3709 \text{ N}}$,

$\boxed{F_B = 4169 \text{ N}}$

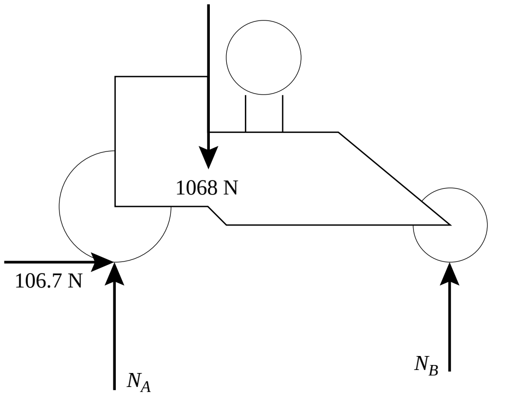
Problem 18.6 The total weight of the go-cart and driver is 1068 N. The location of their combined center of mass is shown. The rear drive wheels together exert a 106.7 N horizontal force on the track. Neglect the horizontal forces exerted on the front wheels.

(a) What is the magnitude of the go-cart's acceleration?

(b) What normal forces are exerted on the tires at A and B?



Solution:



$$\sum F_x : (106.7\ \text{N}) = \left(\frac{1068\ \text{N}}{9.81\ \text{m/s}^2}\right) a$$

$$\sum F_y : N_A + N_B - (1068\ \text{N}) = 0$$

$$\sum M_G : -N_A(0.406\ \text{m}) + N_B(1.118\ \text{m}) + (106.7\ \text{N})(0.381\ \text{m}) = 0$$

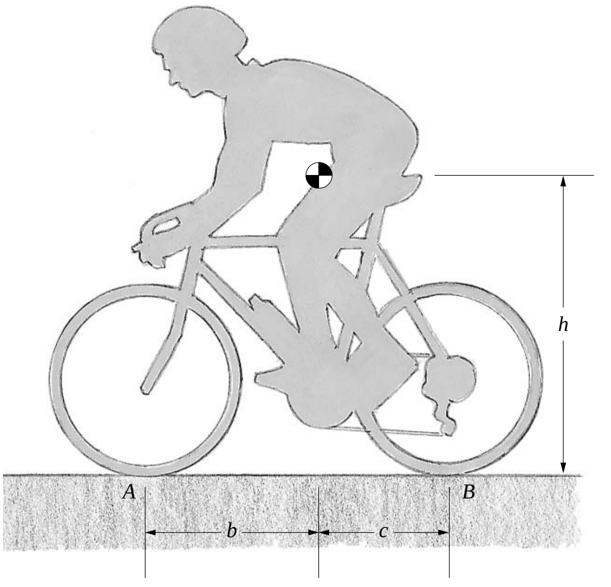
Solving we find

(a) $\boxed{a = 0.981\ \text{m/s}^2}$

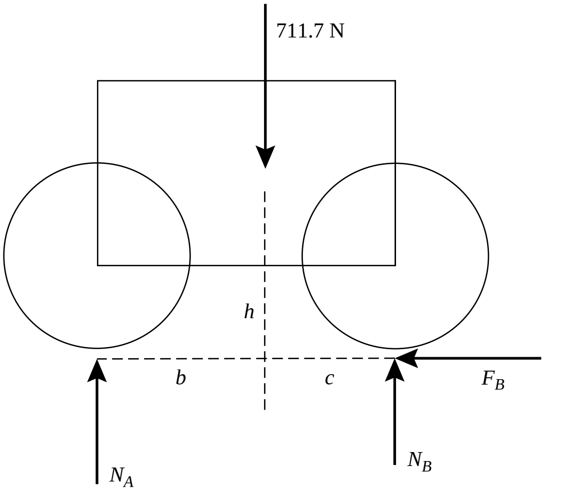
(b) $\boxed{N_A = 809.5\ \text{N},\ N_B = 258\ \text{N}}$

Problem 18.7 The total weight of the bicycle and rider is 711.7 N. The location of their combined center of mass is shown. The dimensions shown are $b = 533.4$ mm, $c = 406.4$ mm, and $h = 965$ mm What is the largest acceleration the bicycle can have without the front wheel leaving the ground? Neglect the horizontal force exerted on the front wheel by the road.

Strategy: You want to determine the value of the acceleration that causes the normal force exerted on the front wheel by the road to equal zero.



Solution: Given: $b = 0.533$ m., $c = 0.406$ m., $h = 0.965$ m.



Find: a so that $N_A = 0$

$$\sum F_x : -F_B = -\left(\frac{711.7 \text{ N}}{9.81 \text{ m/s}^2}\right) a$$

$$\sum F_y : N_A + N_B - (711.7 \text{ N}) = 0$$

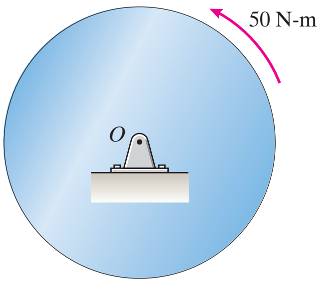
$$\sum M_G : -N_A b + N_B c - F_B h = 0$$

$N_A = 0$

Solving we find $N_B = 711.7$ N, $F_B = 300$ N, $\boxed{a = 4.15 \text{ m/s}^2}$

Problem 18.8 The moment of inertia of the disk about O is $I = 20$ kg-m^2. At $t = 0$, the stationary disk is subjected to a constant 50 N-m torque.

(a) What is the magnitude of the resulting angular acceleration of the disk?
(b) How fast is the disk rotating (in rpm) at $t = 4$ s?



Solution:

(a) $M = I\alpha \Rightarrow \alpha = \dfrac{M}{I} = \dfrac{50 \text{ N-m}}{20 \text{ kg-m}^2} = 2.5 \text{ rad/s}^2.$

$\boxed{\alpha = 2.5 \text{ rad/s}^2.}$

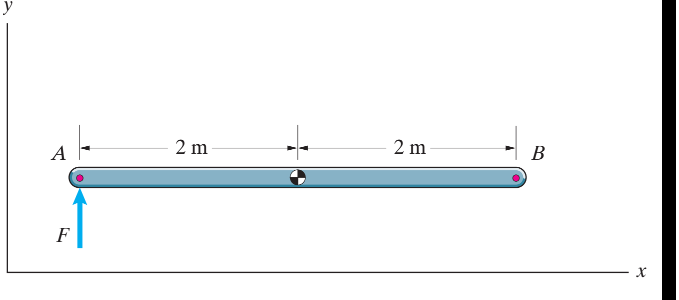
(b) The angular velocity is given by

$$\omega = \alpha t = (2.5 \text{ rad/s}^2)(4 \text{ s}) = 10 \text{ rad/s}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 95.5 \text{ rpm}.$$

$\boxed{\omega = 95.5 \text{ rpm.}}$

Problem 18.9 The 10-N bar is on a smooth horizontal table. The figure shows the bar viewed from above. Its moment of inertia about the center of mass is $I = 1$ kg–m^2. The bar is stationary when the force $F = 5$ N is applied in the direction parallel to the y axis. At that instant, determine

(a) the acceleration of the center of mass, and
(b) the acceleration of point A.



Solution:

(a) $F = ma \Rightarrow a = \dfrac{F}{m} = \dfrac{5 \text{ N}}{\left(\dfrac{10 \text{ N}}{9.81 \text{ m/s}^2}\right)} = 4.905 \text{ m/s}^2$

$\boxed{\mathbf{a}_G = (4.905 \text{ m/s}^2)\mathbf{j}.}$

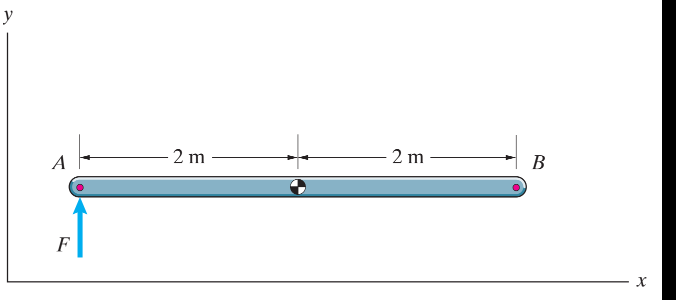
(b) $\Sigma M_G : -F\dfrac{l}{2} = I\alpha \Rightarrow \alpha = -\dfrac{Fl}{2I} = -\dfrac{(5 \text{ N})(4 \text{ m})}{2(1 \text{ kg-m}^2)} = -10 \text{ rad/s}^2.$

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G}$$

$$= (4.905 \text{ m/s}^2)\mathbf{j} + (-10 \text{ rad/s}^2)\mathbf{k} \times (-2 \text{ m})\mathbf{i}$$

$\boxed{\mathbf{a}_A = (24.9 \text{ m/s}^2)\mathbf{j}.}$

Problem 18.10 The 10-N bar is on a smooth horizontal table. The figure shows the bar viewed from above. Its moment of inertia about the center of mass is $I = 1\text{ kg-m}^2$. The bar is stationary when the force $F = 5$ N is applied in the direction parallel to the y axis. At that instant, determine the acceleration of point B.



Solution:

$$F = ma \Rightarrow a = \frac{F}{m} = \frac{5\text{ N}}{\left(\dfrac{10\text{ N}}{9.81\text{ m/s}^2}\right)} = 4.905\text{ m/s}^2$$

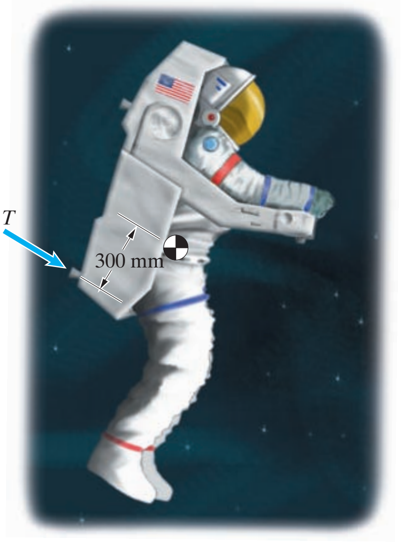
$$\Sigma M_G : -F\frac{l}{2} = I\alpha \Rightarrow \alpha = -\frac{Fl}{2I} = -\frac{(5\text{ N})(4\text{ N})}{2(1\text{ kg-m}^2)} = -10\text{ rad/s}^2.$$

$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G}$$

$$= (4.905\text{ m/s}^2)\mathbf{j} + (-10\text{ rad/s}^2)\mathbf{k} \times (2\text{ m})\mathbf{i}$$

$$\boxed{\mathbf{a}_B = (-15.1\text{ m/s}^2)\mathbf{j}.}$$

Problem 18.11 The moment of inertia of the astronaut and maneuvering unit about the axis through their center of mass perpendicular to the page is $I = 40\text{ kg-m}^2$. A thruster can exert a force $T = 10$ N. For safety, the control system of his maneuvering unit will not allow his angular velocity to exceed 15° per second. If he is initially not rotating, and at $t = 0$, he activates the thruster until he is rotating at 15° per second, through how many degrees has he rotated at $t = 10$ s?



Solution: First find the angular acceleration.

$$\Sigma M_G : Td = I\alpha$$

$$\alpha = \frac{Td}{I} = \frac{(10\text{ N})(0.3\text{ m})}{(40\text{ kg-m}^2)} = 0.075\text{ rad/s}^2.$$

To reach maximum angular velocity it takes

$$\omega = \alpha t \Rightarrow t = \frac{\omega}{\alpha} = \frac{(15^\circ/\text{s})\left(\dfrac{\pi\text{ rad}}{180^\circ}\right)}{(0.075\text{ rad/s}^2)} = 3.49\text{ s}$$

During this time, the astronaut has rotated through

$$\theta_1 = \frac{1}{2}\alpha t^2 = \frac{1}{2}(0.075\text{ rad/s}^2)(3.49\text{ s})^2 = 0.457\text{ rad}.$$

After this time, the astronauts turns at the fixed rate. He rotated an additional angle given by

$$\theta_2 = \omega t = (15^\circ/\text{s})\left(\frac{\pi\text{ rad}}{180^\circ}\right)(10\text{ s} - 3.49\text{ s}) = 1.704\text{ rad}.$$

The total rotation is then

$$\theta = \theta_1 + \theta_2 = (0.457 + 1.704)\text{ rad}\left(\frac{180^\circ}{\pi\text{ rad}}\right) = 124^\circ.$$

$$\boxed{\theta = 124^\circ.}$$

Problem 18.12 The moment of inertia of the helicopter's rotor is 420 N-m^2. The rotor starts from rest At $t = 0$, the pilot begins advancing the throttle so that the torque exerted on the rotor by the engine (in N-m) is given as a function of time in seconds by $T = 200\text{t}$.

(a) How long does it take the rotor to turn ten revolutions?
(b) What is the rotor's angular velocity (in rpm) when it has turned ten revolutions?

Solution: Find the angular acceleration

$$T = I\alpha \Rightarrow \alpha = \frac{T}{I} = \frac{200t}{420} = 0.476t$$

Now answer the kinematics questions

$$\alpha = 0.476t, \quad \omega = 0.238t^2, \quad \theta = 0.0794t^3.$$

(a) When it has turned 10 revolutions,

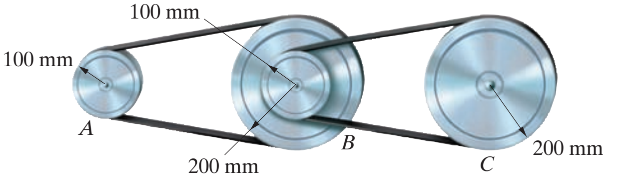
$$(10 \text{ rev})\left(\frac{2\pi \text{ rad}}{\text{rev}}\right) = 0.0794t^3 \quad \Rightarrow \quad \boxed{t = 9.25 \text{ s.}}$$

(b) The angular velocity is

$$\omega = 0.238(9.25)^2 = 20.4 \text{ rad/s}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right) = 195 \text{ rpm}$$

$$\boxed{\omega = 195 \text{ rpm.}}$$

Problem 18.13 The moments of inertia of the pulleys are $I_A = 0.0025$ kg-m^2, $I_B = 0.045$ kg-m^2, and $I_C = 0.036$ kg-m^2. A 5 N-m counterclockwise couple is applied to pulley A. Determine the resulting counterclockwise angular accelerations of the three pulleys.



Solution: The tension in each belt changes as it goes around each pulley.

The unknowns are $\Delta T_{AB}, \ \Delta T_{BC}, \ \alpha_A, \ \alpha_B, \ \alpha_C.$

We will write three dynamic equations and two constraint equations

$$\Sigma M_A : (5 \text{ N-m}) - \Delta T_{AB}(0.1 \text{ m}) = (0.0025 \text{ kg-m}^2)\alpha_A$$

$$\Sigma M_B : \Delta T_{AB}(0.2 \text{ m}) - \Delta T_{BC}(0.1 \text{ m}) = (0.045 \text{ kg-m}^2)\alpha_B$$

$$\Sigma M_C : \Delta T_{BC}(0.2 \text{ m}) = (0.036 \text{ kg-m}^2)\alpha_C$$

$$(0.1 \text{ m})\alpha_A = (0.2 \text{ m})\alpha_B$$

$$(0.1 \text{ m})\alpha_B = (0.2 \text{ m})\alpha_C.$$

Solving, we find

$$\Delta T_{AB} = 42.2 \text{ N}, \ \Delta T_{BC} = 14.1 \text{ N},$$

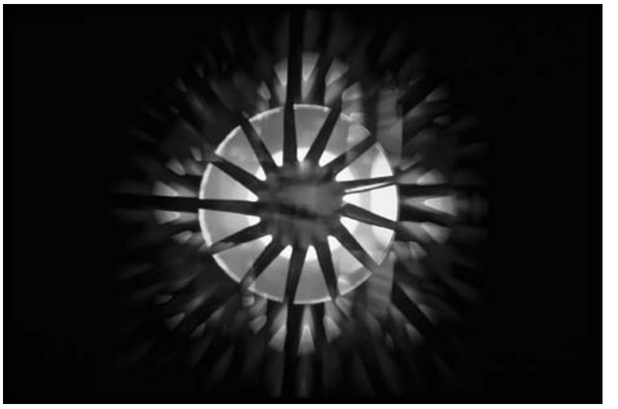
$$\boxed{\alpha_A = 313 \text{ rad/s}^2, \ \alpha_B = 156 \text{ rad/s}^2, \ \alpha_C = 78.1 \text{ rad/s}^2.}$$

Problem 18.14 The moment of inertia of the wind-tunnel fan is 225 kg-m^2. The fan starts from rest. The torque exerted on it by the engine is given as a function of the angular velocity of the fan by $T = 140 - 0.02\omega^2$ N-m.

(a) When the fan has turned 620 revolutions, what is its angular velocity in rpm (revolutions per minute)?
(b) What maximum angular velocity in rpm does the fan attain?

Strategy: By writing the equation of angular motion, determine the angular acceleration of the fan in terms of its angular velocity. Then use the chain rule:

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega.$$



Solution:

$$\sum M : (140 \text{ N-m}) - (0.02 \text{ N-m/s}^2)\omega^2 = (225 \text{ kg-m}^2)\alpha$$

$$\alpha = \left(\frac{140}{225} \text{ rad/s}^2\right) - \left(\frac{0.02}{225} \text{ rad/s}^4\right)\omega^2$$

$$= (0.622 \text{ rad/s}^2) - (0.0000889 \text{ rad/s}^4)\omega^2$$

(a) $\alpha = \omega\dfrac{d\omega}{d\theta} = (0.622 \text{ rad/s}^2) - (0.0000889 \text{ rad/s}^4)\omega^2$

$$\int_0^{\omega} \frac{\omega d\omega}{(0.622 \text{ rad/s}^2) - (0.0000889 \text{ rad/s}^4)\omega^2} = \int_0^{620(2\pi)\text{ rad}} d\theta$$

Solving we find

$$\boxed{\omega = 59.1 \text{ rad/s}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1\,\text{min}}\right) = 565 \text{ rpm}}$$

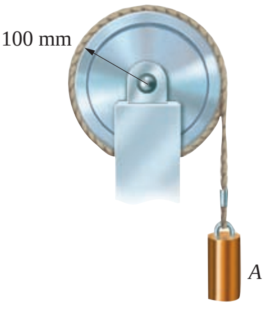
(b) The maximum angular velocity occurs when the angular acceleration is zero

$$\alpha = (0.622 \text{ rad/s}^2) - (0.0000889 \text{ rad/s}^4)\omega^2 = 0$$

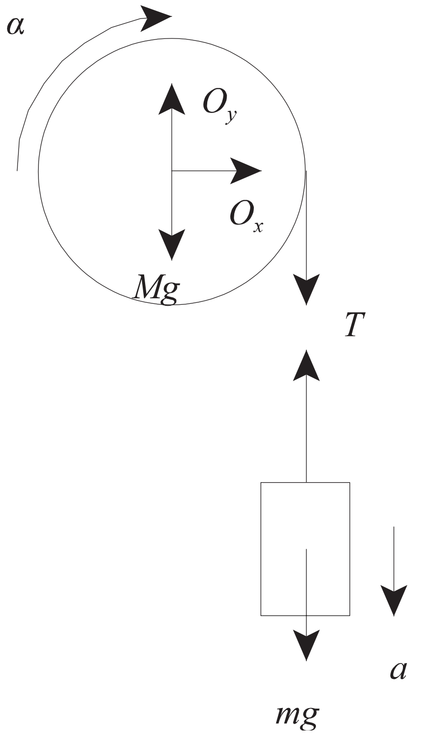
$$\boxed{\omega = 83.7 \text{ rad/s}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1\,\text{min}}\right) = 799 \text{ rpm}}$$

Problem 18.15 The moment of inertia of the pulley about its axis is $I = 0.005$ kg-m^2. If the 1-kg mass A is released from rest, how far does it fall in 0.5 s?

Strategy: Draw individual free-body diagrams of the pulley and the mass.



Solution: The two free-body diagrams are shown.



The five unknowns are $T,\ O_x,\ O_y,\ \alpha,\ a$.

We can write four dynamic equations and one constraint equation, however, we only need to write two dynamic equations and the one constraint equation.

$\Sigma M_O : -T(0.1 \text{ m}) = -(0.005 \text{ kg-m}^2)\alpha,$

$\Sigma F_y : T - (1 \text{ kg})(9.81 \text{ m/s}^2) = -(1 \text{ kg})a,$

$a = (0.1 \text{ m})\alpha.$

Solving we find

$$T = 3.27 \text{ N}, \quad a = 6.54 \text{ m/s}^2, \quad \alpha = 65.4 \text{ rad/s}^2.$$

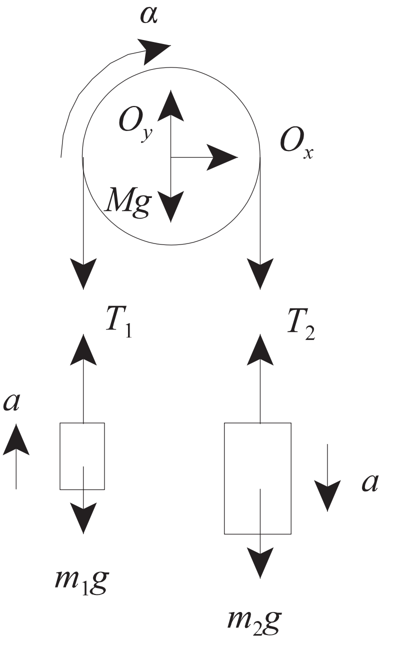
Now from kinematics we know

$d = \frac{1}{2}at^2 = \frac{1}{2}(6.54 \text{ m/s}^2)(0.5 \text{ s})^2$

$\boxed{d = 0.818 \text{ m.}}$

Problem 18.16 The radius of the pulley is 125 mm and the moment of inertia about its axis is $I = 0.05$ kg-m^2. If the system is released from rest, how far does the 20-kg mass fall in 0.5 s? What is the tension in the rope between the 20-kg mass and the pulley?

Solution: The free-body diagrams are shown.



We have six unknowns

$$T_1, T_2, O_x, O_y, a, \alpha.$$

We have five dynamic equations and one constraint equation available. We will use three dynamic equations and the one constraint equation

$\Sigma M_O : (T_1 - T_2)(0.125 \text{ m}) = -(0.05 \text{ kg-m}^2)\alpha,$

$\Sigma F_{y1} : T_1 - (4 \text{ kg})(9.81 \text{ m/s}^2) = (4 \text{ kg})a,$

$\Sigma F_{y2} : T_2 - (20 \text{ kg})(9.81 \text{ m/s}^2) = -(20 \text{ kg})a,$

$a = (0.125 \text{ m})\alpha.$

Solving we find

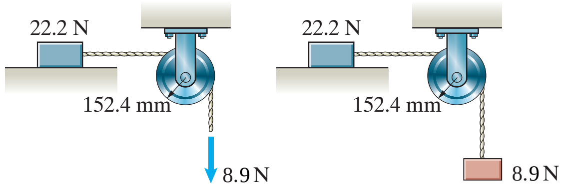
$$T_1 = 62.3 \text{ N},\ T_2 = 80.8 \text{ N},\ a = 5.77 \text{ m/s}^2,\ \alpha = 46.2 \text{ rad/s}^2.$$

From kinematics we find

$d = \frac{1}{2}at^2 = \frac{1}{2}(5.77 \text{ m/s}^2)(0.5 \text{ s})^2 = 0.721 \text{ m}.$

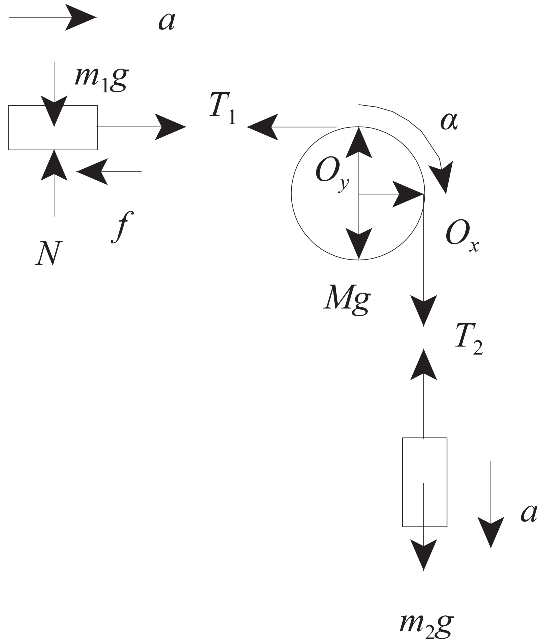
$\boxed{d = 0.721 \text{ m},\ T_2 = 80.8 \text{ N}.}$

Problem 18.17 The moment of inertia of the pulley is 0.54 kg-m^2. The coefficient of kinetic friction between the 22.2 N weight and the horizontal surface is $\mu_k = 0.2$. Determine the magnitude of the acceleration of the 22.2 N weight in each case .



(a) (b)

Solution: The free-body diagrams are shown.



(a) $T_2 = 8.9\,\text{N}$.

$$(T_1 - T_2)(0.152\ \text{m}) = -(0.54\ \text{kg-m}^2)\alpha,$$

$$T_1 - (0.2)(22.2\ \text{N}) = \left(\frac{22.2\ \text{N}}{9.18\ \text{m/s}^2}\right)a,$$

$$a = (0.152\ \text{m})\,\alpha.$$

Solving we find

$T_1 = 4.83\ \text{N},\ \alpha = 1.14\ \text{rad/s}^2,$

$\boxed{a = 0.174\ \text{m/s}^2}$

(b) $T_2 \neq 8.9\ \text{N}$.

$$(T_1 - T_2)\ (0.152\ \text{m})\ = -(0.54\ \text{kg-m}^2)\alpha,\ a = (0.152\ \text{m})\,\alpha,$$

$$T_1 - (0.2)(22.2\ \text{N}) = \left(\frac{22.2\ \text{N}}{9.81\ \text{m/s}^2}\right)a,\ T_2 - (8.9\ \text{N}) = -\left(\frac{8.9\ \text{N}}{9.81\ \text{m/s}^2}\right)a.$$

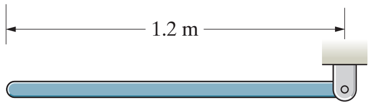
Solving we find

$T_1 = 4.8\ \text{N},\ T_2 = 8.75\ \text{N},\ \alpha = 1.10\ \text{rad/s}^2,$

$\boxed{a = 0.167\ \text{m/s}^2}$

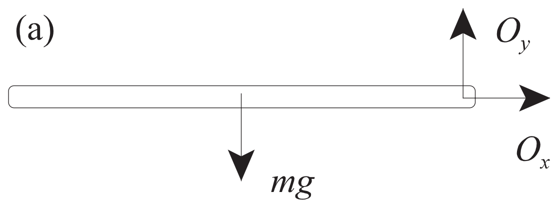
Note that (b) has more inertia than (a) and therefore has to accelerate more slowly.

Problem 18.18 The 5-kg slender bar is released from rest in the horizontal position shown. Determine the bar's counterclockwise angular acceleration (a) at the instant it is released, and (b) at the instant when it has rotated 45°.



Solution:

(a) The free-body diagram is shown.

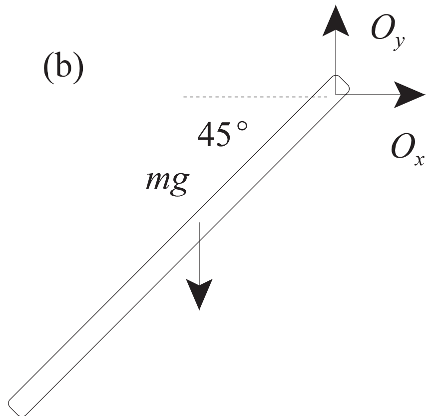


$$\Sigma M_O : mg\frac{L}{2} = \frac{1}{3}\, mL^2\alpha$$

$$\alpha = \frac{3g}{2L} = \frac{3(9.81\ \text{m/s}^2)}{2(1.2\ \text{m})} = 12.3\ \text{rad/s}^2.$$

$$\boxed{\alpha = 12.3\ \text{rad/s}^2.}$$

(b) The free-body diagram is shown.

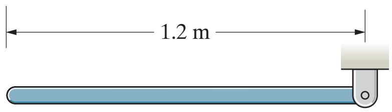


$$\Sigma M_O : mg\frac{L}{2}\cos 45^\circ = \frac{1}{3}\, mL^2\alpha$$

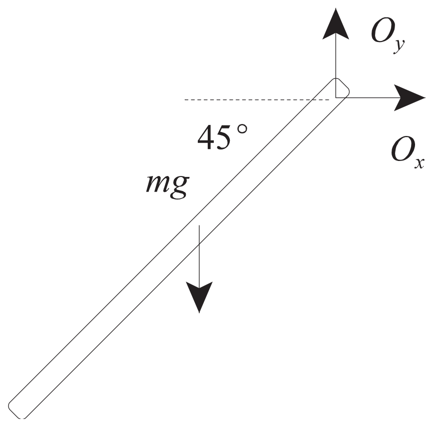
$$\alpha = \frac{3g}{2L}\cos 45^\circ = \frac{3(9.81\ \text{m/s}^2)}{2(1.2\ \text{m})}\cos 45^\circ$$

$$\boxed{\alpha = 8.67\ \text{rad/s}^2.}$$

Problem 18.19 The 5-kg slender bar is released from rest in the horizontal position shown. At the instant when it has rotated 45°, its angular velocity is 4.16 rad/s. At that instant, determine the magnitude of the force exerted on the bar by the pin support. (See Example 18.4.)



Solution: First find the angular acceleration.



$$\Sigma M_O : mg\frac{L}{2}\cos 45^\circ = \frac{1}{3}\, mL^2\alpha$$

$$\alpha = \frac{3\ g}{2L}\cos 45^\circ = \frac{3(9.81\ \text{m/s}^2)}{2(1.2\ \text{m})}\cos 45^\circ = 8.67\ \text{rad/s}^2$$

Using kinematics we find the acceleration of the center of mass.

$$\mathbf{a}_G = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{G/O} - \omega^2\mathbf{r}_{G/O}$$

$$\mathbf{a}_G = 0 + (8.67)\mathbf{k} \times (0.6)(-\cos 45^\circ\mathbf{i} - \sin 45^\circ\mathbf{j})$$

$$- (4.16)^2(0.6)(-\cos 45^\circ\mathbf{i} - \sin 45^\circ\mathbf{j})$$

$$= (11.0\mathbf{i} + 3.66\mathbf{j})\ \text{m/s}^2.$$

From Newton's second law we have

$$\Sigma F_x : O_x = ma_x = (5\ \text{kg})(11.0\ \text{m/s}^2) = 55.1\ \text{N}$$

$$\Sigma F_y : O_y - mg = ma_y$$

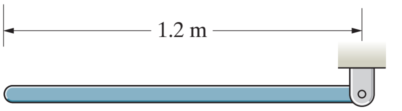
$$O_y = m(g + a_y) = (5\ \text{kg})(9.18\ \text{m/s}^2 + 3.66\ \text{m/s}^2) = 67.4\ \text{N}$$

The magnitude of the force in the pin is now

$$O = \sqrt{O_x^2 + O_y^2} = \sqrt{(55.1\ \text{N})^2 + (67.4\ \text{N})^2} = 87.0\ \text{N}.$$

$$\boxed{O = 87.0\ \text{N}.}$$

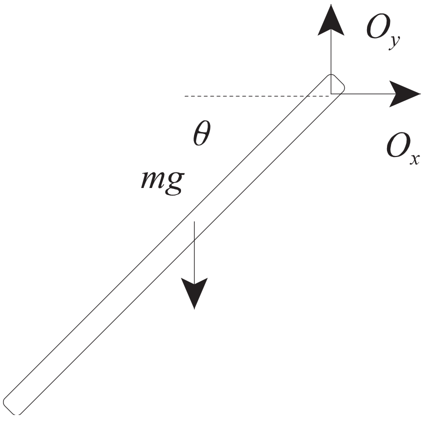
Problem 18.20 The 5-kg slender bar is released from rest in the horizontal position shown. Determine the magnitude of its angular velocity when it has fallen to the vertical position.



Strategy: : Draw the free-body diagram of the bar when it has fallen through an arbitrary angle θ and apply the equation of angular motion to determine the bar's angular acceleration as a function of θ. Then use the chain rule to write the angular acceleration as

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \frac{d\omega}{d\theta}\omega.$$

Solution: First find the angular acceleration.



$$\Sigma M_O : mg\frac{L}{2}\cos\theta = \frac{1}{3}\,mL^2\alpha \Rightarrow \alpha = \frac{3g}{2L}\cos\theta$$

Using the hint we have

$$\alpha = \omega\frac{d\omega}{d\theta} = \frac{3g}{2L}\cos\theta \Rightarrow \int_0^{\omega}\omega\,d\omega = \int_0^{90^\circ}\frac{3g}{2L}\cos\theta\,d\theta$$

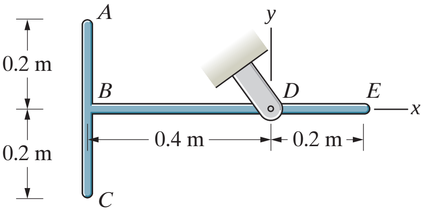
$$\frac{1}{2}\,\omega^2 = \frac{3g}{2L}\sin\theta\bigg]_0^{90^\circ} = \frac{3g}{2L}$$

$$\omega = \sqrt{\frac{3g}{L}} = \sqrt{\frac{3(9.81\ \text{m/s}^2)}{(1.2\ \text{m})}} = 4.95\ \text{rad/s}.$$

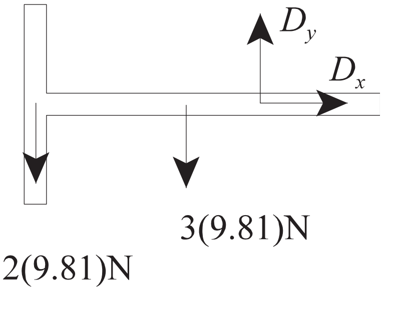
$$\boxed{\omega = 4.95\ \text{rad/s}.}$$

Problem 18.21 The object consists of the 2-kg slender bar *ABC* welded to the 3-kg slender bar *BDE*. The y axis is vertical.

(a) What is the object's moment of inertia about point *D*?
(b) Determine the object's counterclockwise angular acceleration at the instant shown.



Solution: The free-body diagram is shown



(a)

$$I_D = \frac{1}{12}(2\ \text{kg})(0.4\ \text{m})^2 + (2\ \text{kg})(0.4\ \text{m})^2$$

$$+\frac{1}{12}(3\ \text{kg})(0.6\ \text{m})^2 + (3\ \text{kg})(0.1\ \text{m})^2$$

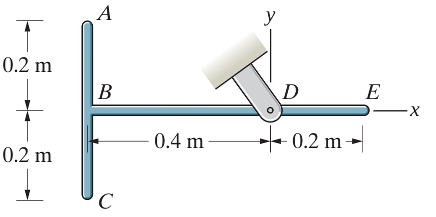
$$\boxed{I_D = 0.467\ \text{kg-m}^2.}$$

(b)

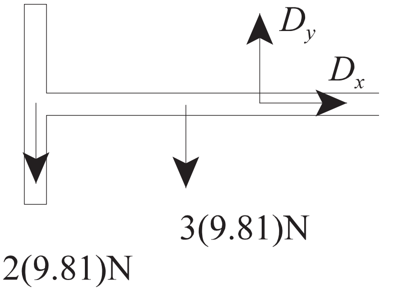
$$\Sigma M_D : [(2\ \text{kg})(0.4\ \text{m}) + (3\ \text{kg})(0.1\ \text{m})](9.81\ \text{m/s}^2) = (0.467\ \text{kg-m}^2)\alpha$$

$$\boxed{\alpha = 23.1\ \text{rad/s}^2.}$$

Problem 18.22 The object consists of the 2-kg slender bar *ABC* welded to the 3-kg slender bar *BDE*. The y axis is vertical. At the instant shown, the object has a counterclockwise angular velocity of 5 rad/s. Determine the components of the force exerted on it by the pin support.



Solution: The free-body diagram is shown.



The moment of inertia about the fixed point D is

$$I_D = \frac{1}{12}(2\text{ kg})(0.4\text{ m})^2 + (2\text{ kg})(0.4\text{ m})^2$$

$$+\frac{1}{12}(3\text{ kg})(0.6\text{ m})^2 + (3\text{ kg})(0.1\text{ m})^2$$

$$= 0.467\text{ kg-m}^2.$$

The angular acceleration is given by

$$\Sigma M_D : [(2\text{ kg})(0.4\text{ m}) + (3\text{ kg})(0.1\text{ m})](9.81\text{ m/s}^2) = (0.467\text{ kg-m}^2)\alpha$$

$$\alpha = \frac{10.8\text{ N-m}}{0.467\text{ kg-m}^2} = 23.1\text{ rad/s}^2.$$

From Newton's Second Law we have

$$\Sigma F_x : D_x = (2\text{ kg})(0.4\text{ m})(5\text{ rad/s})^2 + (3\text{ kg})(0.1\text{ m})(5\text{ rad/s})^2$$

$$\Sigma F_y : D_y - (5\text{ kg})(9.81\text{ m/s}^2) = -(2\text{ kg})(0.4\text{ m})(23.1\text{ rad/s}^2)$$

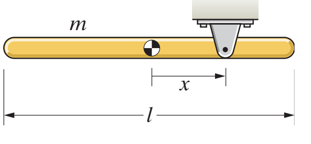
$$-(3\text{ kg})(0.1\text{ m})(23.1\text{ rad/s}^2)$$

Solving, we find $\boxed{D_x = 27.5\text{ N},\quad D_y = 23.6\text{ N}.}$

Problem 18.23 The length of the slender bar is $l = 4$ m and its mass is $m = 30$ kg. It is released from rest in the position shown.

(a) If $x = 1$ m, what is the bar's angular acceleration at the instant it is released?

(b) What value of x results in the largest angular acceleration when the bar is released? What is the angular acceleration?



Solution: The moment of inertia about the fixed point is

$$I = \frac{1}{12}ml^2 + mx^2.$$

The angular acceleration can be found

$$\Sigma M_{\text{fixed point}} : mgx = I\alpha = \frac{m}{12}(l^2 + 12x^2)\alpha \Rightarrow \alpha = \frac{12gx}{l^2 + 12x^2}$$

(a) Using the given numbers we have

$$\alpha = \frac{12(9.81\text{ m/s}^2)(1\text{ m})}{(4\text{ m})^2 + 12(1\text{ m})^2} = 4.20\text{ rad/s}^2. \quad \boxed{\alpha = 4.20\text{ rad/s}^2.}$$

(b) To find the critical value for x we differentiate and set equal to zero to get

$$\frac{d\alpha}{dx} = \frac{d}{dx}\left(\frac{12gx}{l^2 + 12x^2}\right) = \frac{12g}{l^2 + 12x^2} - \frac{288gx^2}{(l^2 + 12x^2)^2}$$

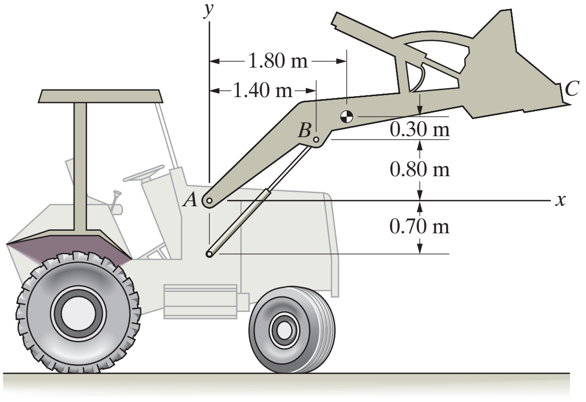
$$= \frac{12g(l^2 - 12x^2)}{(l^2 + 12x^2)^2} = 0$$

$$x = \frac{l}{\sqrt{12}} = \frac{(4\text{ m})}{\sqrt{12}} = 1.15\text{ m}. \quad \boxed{x = 1.15\text{ m}.}$$

The corresponding angular acceleration is

$$\alpha = \frac{12(9.81\text{ m/s}^2)(1.15\text{ m})}{(4\text{ m})^2 + 12(1.15\text{ m})^2} = 4.25\text{ rad/s}^2 \quad \boxed{\alpha = 4.25\text{ rad/s}^2.}$$

Problem 18.24 Model the arm ABC as a single rigid body. Its mass is 320 kg, and the moment of inertia about its center of mass is $I = 360$ kg-m^2. Point A is stationary. If the hydraulic piston exerts a 14-kN force on the arm at B what is the arm's angular acceleration?



Solution: The moment of inertia about the fixed point A is

$$I_A = I_G + md^2 = (360 \text{ kg-m}^2) + (320 \text{ kg})([1.10 \text{ m}]^2 + [1.80 \text{ m}]^2)$$

$$= 1780 \text{ kg-m}^2.$$

The angle between the force at B and the horizontal is

$$\theta = \tan^{-1}\left(\frac{1.5 \text{ m}}{1.4 \text{ m}}\right) = 47.0^\circ.$$

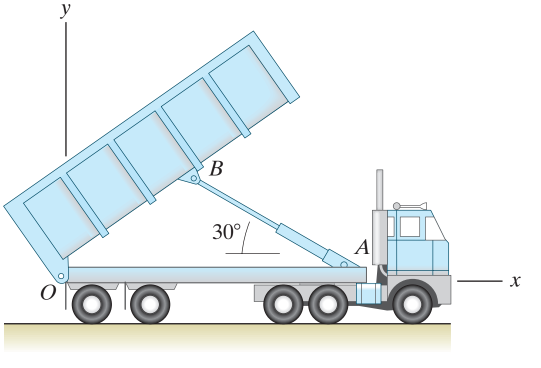
The rotational equation of motion is now

$$\Sigma M_A : (14 \text{ kN}) \sin\theta (1.4 \text{ m}) - (14 \text{ kN}) \cos\theta (0.8 \text{ m})$$

$$- (320 \text{ kg})(9.81 \text{ m/s}^2)(1.80 \text{ m}) = (1780 \text{ kg-m}^2)\,\alpha.$$

Solving, we find $\alpha = 0.581$ rad/s^2. $\boxed{\alpha = 0.581 \text{ rad/s}^2 \text{ counterclockwise.}}$

Problem 18.25 The truck's bed weighs 8000 N and its moment of inertia about O is 400000 kg-m^2. At the instant shown, the coordinates of the center of mass of the bed are (3, 4) m and the coordinates of point B are (5, 3.5) m. If the bed has a counterclockwise angular acceleration of 0.2 rad/s^2, what is the magnitude of the force exerted on the bed at B by the hydraulic cylinder AB?



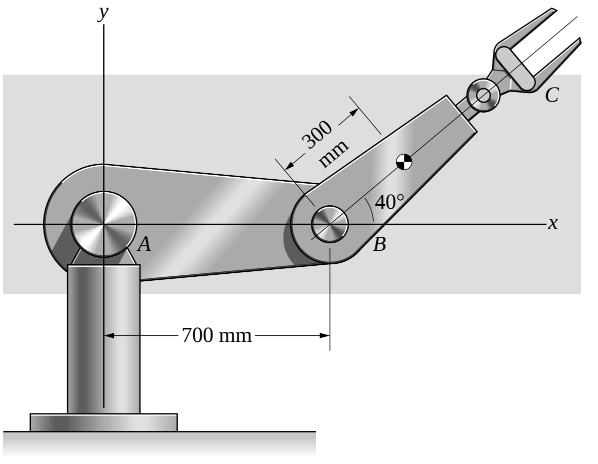
Solution: The rotational equation of motion is

$$\Sigma M_O : F \sin 30^\circ (5 \text{ m}) + F \cos 30^\circ (3.5 \text{ m}) - (8000 \text{ N})(3 \text{ m})$$

$$= (400000 \text{ kg-m}^2)(0.2 \text{ rad/s}^2)$$

Solving for F we find $\boxed{F = 18{,}807 \text{ N.}}$

Problem 18.26 Arm BC has a mass of 12 kg and the moment of inertia about its center of mass is 3 kg-m^2. Point B is stationary and arm BC has a constant counterclockwise angular velocity of 2 rad/s. At the instant shown, what are the couple and the components of force exerted on arm BC at B?



Solution: Since the angular acceleration of arm BC is zero, the sum of the moments about the fixed point B must be zero. Let $\mathbf{M}_B$ be the couple exerted by the support at B. Then

$$\mathbf{M}_B + \mathbf{r}_{CM/B} \times m\mathbf{g} = \mathbf{M}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.3\cos 40^\circ & 0.3\sin 40^\circ & 0 \\ 0 & -117.7 & 0 \end{bmatrix} = 0.$$

$\boxed{\mathbf{M}_B = 27.05\mathbf{k}\ \text{(N-m)}}$ is the couple exerted at B. From Newton's second law: $B_x = ma_x$, $B_y - mg = ma_y$ where a_x, a_y are the accelerations of the center of mass. From kinematics:

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_{CM/O} - \omega^2 \mathbf{r}_{CM/O}$$

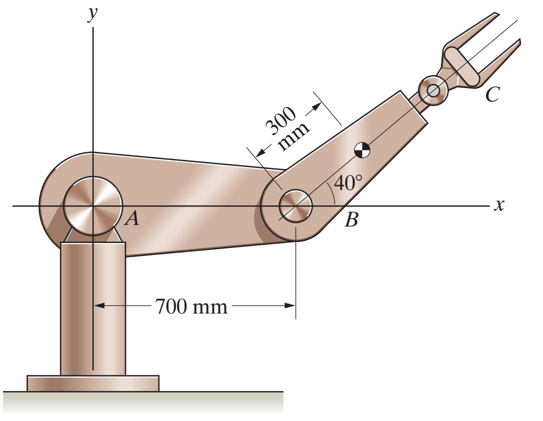
$$= -(2^2)(\mathbf{i}0.3\cos 40^\circ + \mathbf{j}0.3\sin 40^\circ)$$

$$= -0.919\mathbf{i} - 0.771\mathbf{j}\ (\text{m/s}^2),$$

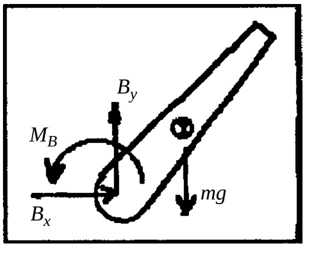
where the angular acceleration is zero from the problem statement. Substitute into Newton's second law to obtain the reactions at B:

$\boxed{B_x = -11.0\ \text{N}}$, $\boxed{B_y = 108.5\ \text{N}}$.

Problem 18.27 Arm BC has a mass of 12 kg and the moment of inertia about its center of mass is 3 kg-m^2. At the instant shown, arm AB has a constant clockwise angular velocity of 2 rad/s and arm BC has counterclockwise angular velocity of 2 rad/s and a clockwise angular acceleration of 4 rad/s^2. What are the couple and the components of force exerted on arm BC at B?



Solution: Because the point B is accelerating, the equations of angular motion must be written about the center of mass of arm BC. The vector distances from A to B and B to G, respectively, are



$$\mathbf{r}_{B/A} = \mathbf{r}_B - \mathbf{r}_A = 0.7\mathbf{i},$$

$$\mathbf{r}_{G/B} = 0.3\cos(40^\circ)\mathbf{i} + 0.3\sin(40^\circ)\mathbf{j}$$

$$= 0.2298\mathbf{i} + 0.1928\mathbf{j} \text{ (m)}.$$

The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A} = -\omega_{AB}^2(0.7\mathbf{i}) \text{ (m/s}^2).$$

The acceleration of the center of mass is

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G/B} - \omega_{BC}^2 \mathbf{r}_{G/B}$$

$$\mathbf{a}_G = -2.8\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -4 \\ 0.2298 & 0.1928 & 0 \end{bmatrix} - 0.9193\mathbf{i} - 0.7713\mathbf{j}$$

$$= -2.948\mathbf{i} - 1.691\mathbf{j} \text{ (m/s}^2).$$

From Newton's second law,

$$\boxed{B_x = ma_{Gx} = (12)(-2.948) = -35.37 \text{ N}},$$

$$B_y - mg = ma_{Gy},$$

$$\boxed{B_y = (12)(-1.691) + (12)(9.81) = 97.43 \text{ N}}$$

From the equation of angular motion, $\mathbf{M}_G = \mathbf{I}\boldsymbol{\alpha}_{BC}$. The moment about the center of mass is

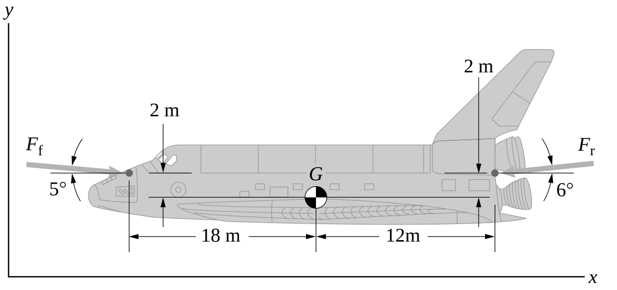
$$\mathbf{M}_G = \mathbf{M}_B + \mathbf{r}_{B/G} \times \mathbf{B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.2298 & -0.1928 & 0 \\ -35.37 & 97.43 & 0 \end{bmatrix}$$

$$= M_B\mathbf{k} - 29.21\mathbf{k} \text{ (N-m)}.$$

Note $I = 3$ kg-m^2 and $\boldsymbol{\alpha}_{BC} = -4\mathbf{k}$ (rad/s^2), from which

$$\boxed{M_B = 29.21 + 3(-4) = 17.21 \text{ N-m}}.$$

Problem 18.28 The space shuttle's attitude control engines exert two forces $F_f = 8$ kN and $F_r = 2$ kN. The force vectors and the center of mass G lie in the $x-y$ plane of the inertial reference frame. The mass of the shuttle is 54,000 kg, and its moment of inertia about the axis through the center of mass that is parallel to the z axis is 4.5×10^6 kg-m^2. Determine the acceleration of the center of mass and the angular acceleration. (You can ignore the force on the shuttle due to its weight).



Solution: Newton's second law is

$$\sum \mathbf{F} = (F_f \cos 5^\circ - F_r \cos 6^\circ)\mathbf{i} - (F_f \sin 5^\circ + F_r \sin 6^\circ)\mathbf{j} = m\mathbf{a}.$$

Setting $F_f = 8000$ N, $F_r = 2000$ N and $m = 54{,}000$ kg and solving for $\mathbf{a}$, we obtain $\mathbf{a} = 0.1108\mathbf{i} - 0.0168\mathbf{j}$ (m/s^2). The equation of angular motion is

$$\sum M = (18)(F_f \sin 5^\circ) - (2)(F_f \cos 5^\circ)$$

$$- (12)(F_r \sin 6^\circ) + (2)(F_r \cos 6^\circ) = I\alpha$$

where $I = 4.5 \times 10^6$ kg-m^2. Solving for α the counterclockwise angular acceleration is $\alpha = -0.000427$ rad/s^2.

Problem 18.29 In Problem 18.28, suppose that $F_f = 4$ kN and you want the shuttle's angular acceleration to be zero. Determine the necessary force F_r and the resulting acceleration of the center of mass.

Solution: The total moment about the center of mass must equal zero:

$$\sum M = (18)(F_f \sin 5^\circ) - (2)(F_f \cos 5^\circ)$$

$$- (12)(F_r \sin 6^\circ) + (2)(F_r \cos 6^\circ) = 0$$

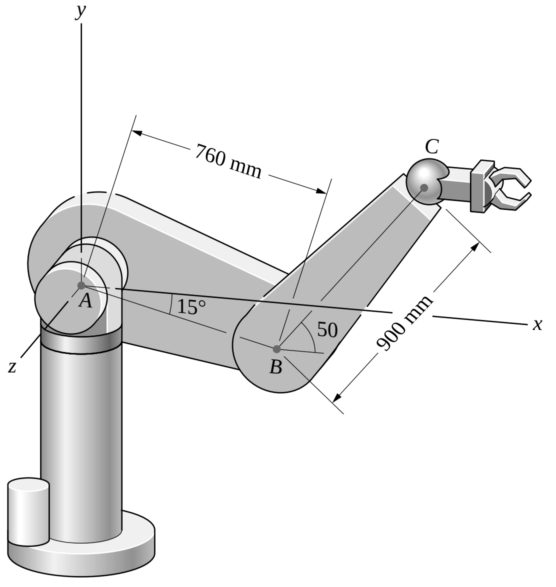
Setting $F_f = 4000$ N and solving $F_r = 2306$ N. From Newton's second law

$$\sum \mathbf{F} = (F_f \cos 5^\circ - F_r \cos 6^\circ)\mathbf{i}$$

$$- (F_f \sin 5^\circ + F_r \sin 6^\circ)\mathbf{j} = 54{,}000\mathbf{a},$$

we obtain $\mathbf{a} = 0.0313\mathbf{i} - 0.0109\mathbf{j}$ (m/s^2).

Problem 18.30 Points B and C lie in the $x-y$ plane. The y axis is vertical. The center of mass of the 18-kg arm BC is at the midpoint of the line from B to C, and the moment of inertia of the arm about the axis through the center of mass that is parallel to the z axis is 1.5 kg-m^2. At the instant shown, the angular velocity and angular acceleration vectors of arm AB are $\boldsymbol{\omega}_{AB} = 0.6\mathbf{k}$ (rad/s) and $\boldsymbol{\alpha}_{AB} = -0.3\mathbf{k}$ (rad/s^2). The angular velocity and angular acceleration vectors of arm BC are $\boldsymbol{\omega}_{BC} = 0.4\mathbf{k}$ (rad/s) and $\boldsymbol{\alpha}_{BC} = 2\mathbf{k}$ (rad/s)^2. Determine the force and couple exerted on arm BC at B.



Solution: The acceleration of point B is $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B} - \omega_{AB}^2 \mathbf{r}_{A/B}$ or

$$\mathbf{a}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -0.3 \\ 0.76\cos 15^\circ & -0.76\sin 15^\circ & 0 \end{vmatrix}$$

$$-(0.6)^2(0.76\cos 15^\circ \mathbf{i} - 0.76\sin 15^\circ \mathbf{j})$$

$$= -0.323\mathbf{i} - 0.149\mathbf{j} \ (\text{m/s}^2)$$

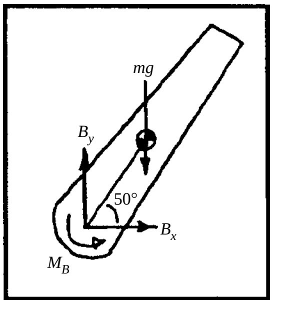
The acceleration of the center of mass G of arm BC is

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G/B} - \omega_{BC}^2 \mathbf{r}_{G/B} \quad \mathbf{a}_B = -0.323\mathbf{i} - 0.149\mathbf{j}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 2 \\ 0.45\cos 50^\circ & 0.45\sin 50^\circ & 0 \end{vmatrix}$$

$$-(0.4)^2(0.45\cos 50^\circ \mathbf{i} + 0.45\sin 50^\circ \mathbf{j}),$$

or $\mathbf{a}_G = -1.059\mathbf{i} + 0.374\mathbf{j}$ (m/s^2). The free body diagram of arm BC is:



Newton's second law is

$$\sum \mathbf{F} = B_x\mathbf{i} + (B_y - mg)\mathbf{j} = m\mathbf{a}_G:$$

$$B_x\mathbf{i} + \left[B_y - (18)(9.81)\right]\mathbf{j} = 18(-1.059\mathbf{i} + 0.374\mathbf{j}).$$

Solving, we obtain $B_x = -19.1$ N, $B_y = 183.3$ N.

The equation of angular motion is

$$\sum M_G = I_{BC}\alpha_{BC}:$$

or $\quad (0.45\sin 50^\circ)B_x - (0.45\cos 50^\circ)B_y + M_B = (1.5)(2)$

Solving for M_B, we obtain $M_B = 62.6$ N-m.

Problem 18.31 Points B and C lie in the $x-y$ plane. The y axis is vertical. The center of mass of the 18-kg arm BC is at the midpoint of the line from B to C, and the moment of inertia of the arm about the axis through the center of mass that is parallel to the z axis is 1.5 kg-m^2. At the instant shown, the angular velocity and angular acceleration vectors of arm AB are $\boldsymbol{\omega}_{AB} = 0.6\mathbf{k}$ (rad/s) and $\boldsymbol{\alpha}_{AB} = -0.3\mathbf{k}$ (rad/s^2). The angular velocity vector of arm BC is $\boldsymbol{\omega}_{BC} = 0.4\mathbf{k}$ (rad/s). If you want to program the robot so that the angular acceleration of arm BC is zero at this instant, what couple must be exerted on arm BC at B?

Solution: From the solution of Problem 18.30, the acceleration of point B is $\mathbf{a}_B = -0.323\mathbf{i} - 0.149\mathbf{j}$ (m/s^2). If $\alpha_{BC} = 0$, the acceleration of the center of mass G of arm BC is

$$\mathbf{a}_G = \mathbf{a}_B - \omega_{BC}^2 \mathbf{r}_{G/B} = -0.323\mathbf{i} - 0.149\mathbf{j}$$

$$- (0.4)^2(0.45\cos 50^\circ \mathbf{i} + 0.45 \sin 50^\circ \mathbf{j})$$

$$= -0.370\mathbf{i} - 0.205\mathbf{j} \text{ (m/s}^2\text{)}.$$

From the free body diagram of arm BC in the solution of Problem 18.30. Newton's second law is

$$\sum \mathbf{F} = B_x\mathbf{i} + (B_y - mg)\mathbf{j} = m\mathbf{a}_G:$$

$$B_x\mathbf{i} + [B_y - (18)(9.81)]\mathbf{j} = 18(-0.370\mathbf{i} - 0.205\mathbf{j}).$$

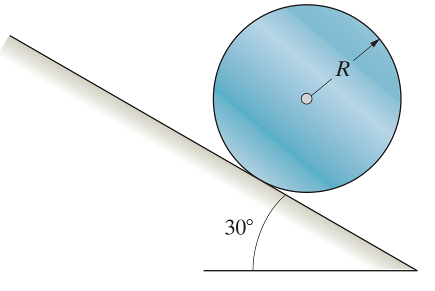
Solving, we obtain $B_x = -6.65$ N, $B_y = 172.90$ N. The equation of angular motion is

$$\sum M_G = I_{BC}\alpha_{BC} = 0:$$

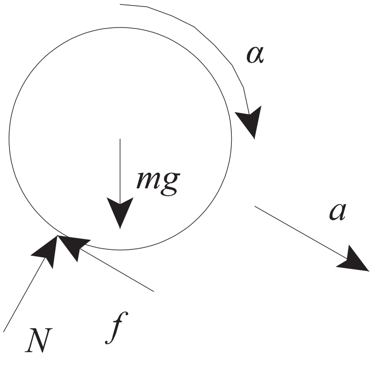
$$(0.45 \sin 50^\circ)B_x - (0.45\cos 50^\circ)B_y + M_B = 0.$$

Solving for M_B, we obtain $M_B = 52.3$ N-m.

Problem 18.32 The radius of the 2-kg disk is $R = 80$ mm. Its moment of inertia is $I = 0.0064$ kg-m^2. It rolls on the inclined surface. If the disk is released from rest, what is the magnitude of the velocity of its center two seconds later? (See Active Example 18.2).



Solution: There are four unknowns (N, f, a, α), three dynamic equations, and one constraint equation. We have



$$\Sigma M_G : -fr = -I\alpha,$$

$$\Sigma F_{\searrow} : mg \sin 30^\circ - f = ma$$

$$a = r\alpha$$

Solving, we find

$$a = \frac{mgr^2 \sin 30^\circ}{I + mr^2}$$

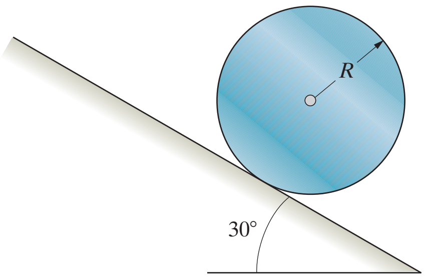
$$= \frac{(2 \text{ kg})(9.81 \text{ m/s}^2)(0.08 \text{ m})^2 \sin 30^\circ}{0.0064 \text{ kg-m}^2 + (2 \text{ kg})(0.08 \text{ m})^2}$$

$$= 3.27 \text{ m/s}^2.$$

From the kinematics we have

$$v = at = (3.27 \text{ m/s}^2)(2 \text{ s}) = 6.54 \text{ m/s}. \quad \boxed{v = 6.54 \text{ m/s.}}$$

Problem 18.33 The radius of the 2-kg disk is $R = 80$ mm. Its moment of inertia is $I = 0.0064$ kg-m^2. What minimum coefficient of static friction is necessary for the disk to roll, instead of slip, on the inclined surface? (See Active Example 18.2.)



Solution: There are five unknowns (N, f, a, α, μ_s), three dynamic equations, one constraint equation, and one friction equation. We have

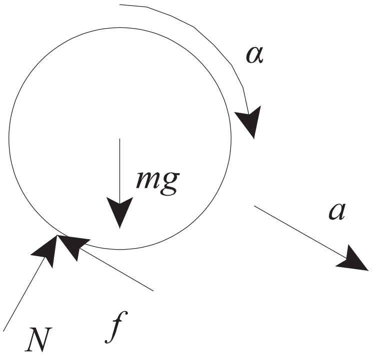
$\Sigma M_G : -fR = -I\alpha,$

$\Sigma F_{\searrow} : mg \sin 30^\circ - f = ma,$

$\Sigma F_{\nearrow} : N - mg \cos 30^\circ = 0,$

$a = R\alpha,$

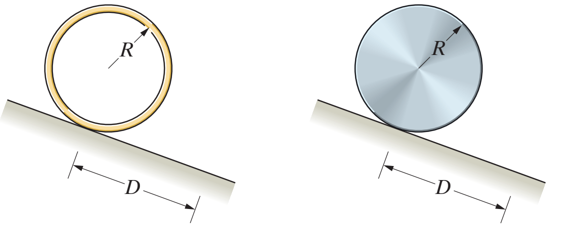
$f = \mu_s N.$



Putting in the numbers and solving, we find

$N = 17.0$ N, $f = 3.27$ N, $a = 3.27$ m/s^2, $\alpha = 40.9$ rad/s^2, $\boxed{\mu_s = 0.192.}$

Problem 18.34 A thin ring and a homogeneous circular disk, each of mass m and radius R, are released from rest on an inclined surface. Determine the ratio $v_{\text{ring}}/v_{\text{disk}}$ of the velocities of the their centers when they ave rolled a distance D.

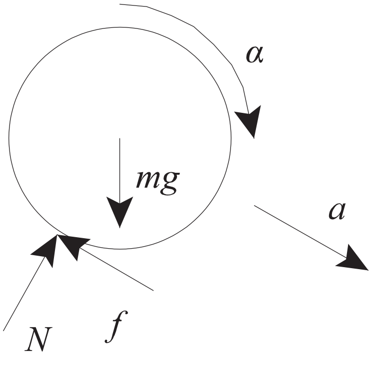


Solution: There are four unknowns (N, f, a, α), three dynamic equations, and one constraint equation. We have

$\Sigma M_G : -fR = -I\alpha,$

$\Sigma F_{\searrow} : mg \sin\theta - f = ma,$

$a = R\alpha,$



Solving, we find $a = \dfrac{mgr^2 \sin\theta}{I + mr^2}$

For the ring $I_{\text{ring}} = mr^2 \quad \Rightarrow \quad a_{\text{ring}} = \dfrac{g}{2} \sin\theta$

For the disk $I_{\text{disk}} = \dfrac{1}{2} mr^2 \quad \Rightarrow \quad a_{\text{disk}} = \dfrac{2g}{3} \sin\theta$

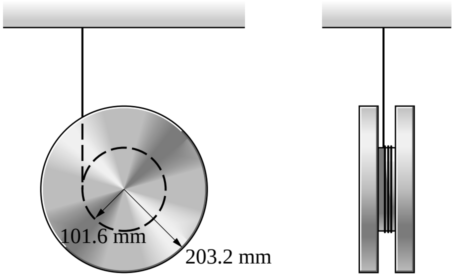
The velocities are then

$$v_{\text{ring}} = \sqrt{2a_{\text{ring}} D} = \sqrt{gD \sin\theta},\ v_{\text{disk}} = \sqrt{2a_{\text{disk}} D} = \sqrt{\frac{4}{3} gD \sin\theta}$$

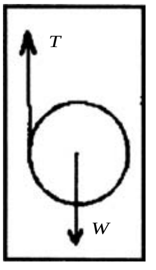
The ratio is

$$\frac{v_{\text{ring}}}{v_{\text{disk}}} = \frac{\sqrt{gD\sin\theta}}{\sqrt{\frac{4}{3} gD \sin\theta}} = \sqrt{3/4} \quad \boxed{\frac{v_{\text{ring}}}{v_{\text{disk}}} = \sqrt{3/4}}$$

Problem 18.35 The stepped disk weighs 178 N and its moment of inertia is $I = 0.27$ kg-m^2. If the disk is released from rest, how long does it take its center to fall 0.91 m? (Assume that the string remains vertical.)



Solution: The moment about the center of mass is $M = -RT$. From the equation of angular motion: $-RT = I\alpha$, from which $T = -\dfrac{I\alpha}{R}$. From the free body diagram and Newton's second law: $\sum F_y = T - W = ma_y$, where a_y is the acceleration of the center of mass. From kinematics: $a_y = -R\alpha$. Substitute and solve:



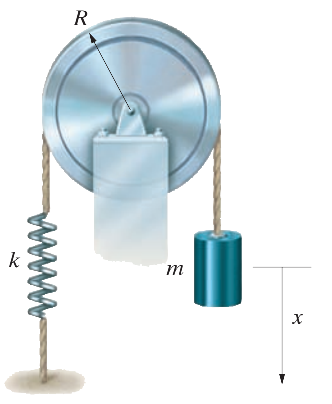
$$a_y = \frac{W}{\left(\dfrac{I}{R^2} + m\right)}.$$

The time required to fall a distance D is

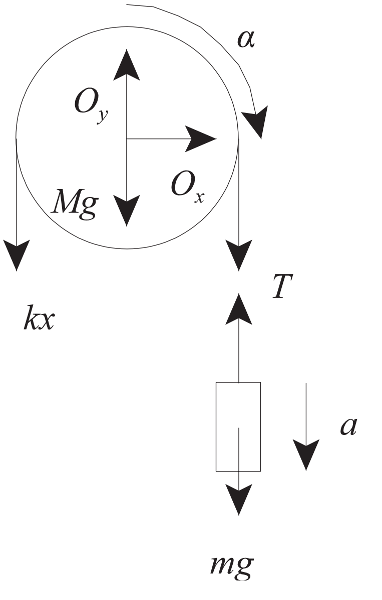
$$t = \sqrt{\frac{2D}{a_y}} = \sqrt{\frac{2D(I + R^2 m)}{R^2 W}}.$$

For $D = 0.91$ m, $R = 0.102$ m, $W = 178$ N, $m = \dfrac{W}{g} = 18.1$ kg, $I = 0.27$ kg-m^2, $t = 0.676$ s

Problem 18.36 The radius of the pulley is $R = 100$ mm and its moment of inertia is $I = 0.1$ kg-m^2. The mass $m = 5$ kg. The spring constant is $k = 135$ N/m. The system is released from rest with the spring unstretched. At the instant when the mass has fallen 0.2 m, determine (a) the angular acceleration of the pulley, and (b) the tension in the rope between the mass and the pulley.



Solution: The force in the spring is kx. There are five unknowns (O_x, O_y, T, a, α), four dynamic equations, and one constraint equation.



$\Sigma M_O : (kx)R - TR = -I\alpha,$

$\Sigma F_y : T - mg = -ma,$

$a = R\alpha$

Solving we find

(a)

$$\alpha = \frac{R(mg - kx)}{I + mR^2}$$

$$= \frac{(0.1 \text{ m})([5 \text{ kg}][9.81 \text{ m/s}^2] - [135 \text{ N/m}][0.2 \text{ m}])}{0.1 \text{ kg-m}^2 + (5 \text{ kg})(0.1 \text{ m})^2}$$

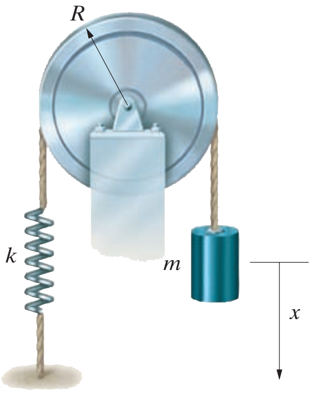
$$\boxed{\alpha = 14.7 \text{ rad/s}^2.}$$

(b)

$$T = m(g - R\alpha) = (5 \text{ kg})(9.81 \text{ m/s}^2 - [0.1 \text{ m}][14.7 \text{ rad/s}^2])$$

$$\boxed{T = 41.7 \text{ N.}}$$

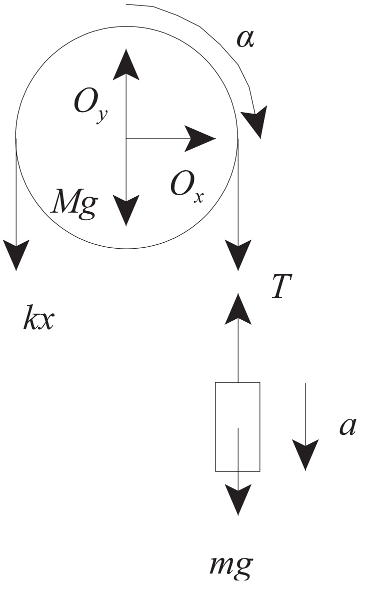
Problem 18.37 The radius of the pulley is $R = 100$ mm and its moment of inertia is $I = 0.1$ kg-m^2. The mass $m = 5$ kg. The spring constant is $k = 135$ N/m. The system is released from rest with the spring unstretched. What maximum distance does the mass fall before rebounding?



Strategy: Assume that the mass has fallen an arbitrary distance x. Write the equations of motion of the mass and the pulley and use them to determine the acceleration a of the mass as a function of x. Then apply the chain rule:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v.$$

Solution: The force in the spring is kx. There are five unknowns (O_x, O_y, T, a, α), four dynamic equations, and one constraint equation.



$\Sigma M_O : (kx)R - TR = -I\alpha,$

$\Sigma F_y : T - mg = -ma,$

$a = R\alpha$

Solving we find

$$a = \frac{R^2(mg - kx)}{I + mR^2} = v\frac{dv}{dx}$$

$$\int_0^0 v\,dv = \int_0^x \frac{R^2(mg - kx)}{I + mR^2}\,dx = \frac{R^2}{I + mR^2}\int_0^x (mg - kx)\,dx = 0$$

Thus

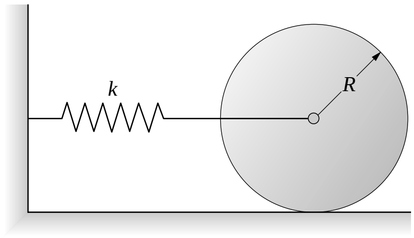
$$mgx - \frac{1}{2}kx^2 = 0 \Rightarrow x = 0 \text{ or } x = \frac{2\,mg}{k}$$

The maximum distance is

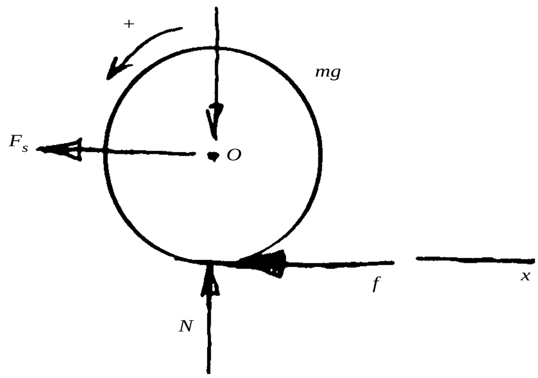
$$x = \frac{2\,mg}{k} = \frac{2(5\text{ kg})(9.81\text{ m/s}^2)}{135\text{ N/m}} = 0.727\text{ m} \quad \boxed{x = 0.727\text{ m.}}$$

Problem 18.38 The mass of the disk is 45 kg and its radius is $R = 0.3$ m. The spring constant is $k = 60$ N/m. The disk is rolled to the left until the spring is compressed 0.5 m and released from rest.

(a) If you assume that the disk rolls, what is its angular acceleration at the instant it is released?
(b) What is the minimum coefficient of static friction for which the disk will not slip when it is released?



Solution:



$x_0 = -0.5$

$k = 600$ N/m

$m = 45$ kg

$R = 0.3$ m

$I_0 = \frac{1}{2} m R^2 = 2.025$ N-m^2, $F_s = kx$

$\sum F_x: \quad -F_s - f = m a_{0x}$

$\sum F_y: \quad N - mg = 0$

$\sum M_0: \quad -fR = I_0 \alpha$

Rolling implies $a_{0x} = -R\alpha$

We have, at $x = -0.5$ m

$$-kx - f = m a_{0x}$$

$$N - mg = 0$$

$$-Rf = I_0 \alpha$$

$$a_{0x} = -R\alpha$$

Four eqns, four unknowns (a_{0x}, α, N, f)

(a) Solving $f = 100$ N, $N = 441.5$ N

$\alpha = -14.81$ rad/s^2 (clockwise)

$a_{0x} = 4.44$ m/s^2

(b) for impending slip,

$f = \mu_s$ N

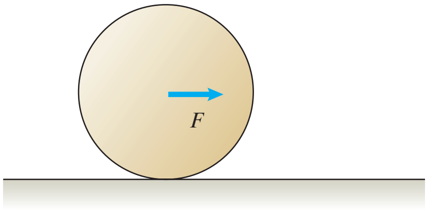
$\mu_s =$ f/N $= 100/441.5$

$\mu_s = 0.227$

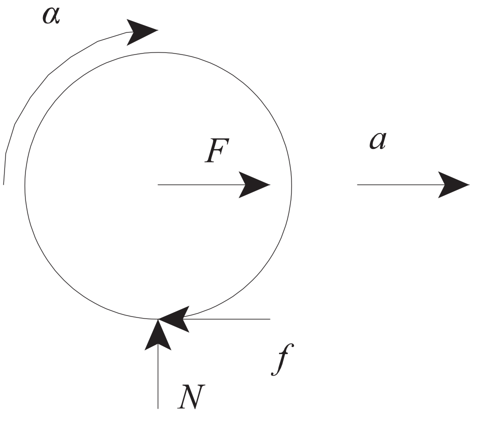
Problem 18.39 The disk weighs 12 N and its radius is 6 cm. It is stationary on the surface when the force $F = 10$ N is applied.

(a) If the disk rolls on the surface, what is the acceleration of its center?

(b) What minimum coefficient of static friction is necessary for the disk to roll instead of slipping when the force is applied?



Solution: There are five unknowns (N, f, a, α, μ_s), three dynamic equations, one constraint equation, and one friction equation.



$\Sigma F_x : F - f = ma,$

$\Sigma F_y : N - mg = 0,$

$\Sigma M_G : -fr = -\left(\frac{1}{2}mr^2\right)\alpha,$

$a = r\alpha,$

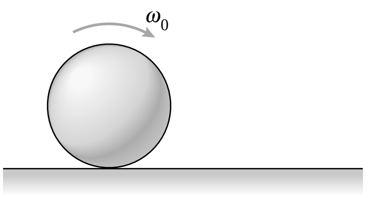
$f = \mu_s N.$

Solving, we find

(a) $a = \frac{2F}{3m} = \frac{2(10 \text{ N})}{3\left(\frac{12 \text{ N}}{9.81 \text{ m/s}^2}\right)} = 5.45 \text{ m/s}^2.$ $\boxed{a = 5.45 \text{ m/s}^2.}$

(b) $\mu_s = \frac{F}{3mg} = \frac{(10 \text{ N})}{3(12 \text{ N})} = 0.278$ $\boxed{\mu_s = 0.278.}$

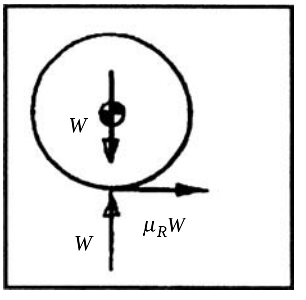
Problem 18.40 A 186.8 N sphere with radius $R = 101.6$ mm is placed on a horizontal surface with initial angular velocity $\omega_0 = 40$ rad/s. The coefficient of kinetic friction between the sphere and the surface is $\mu_k = 0.06$. What maximum velocity will the center of the sphere attain, and how long does it take to reach that velocity?



Strategy: The friction force exerted on the spinning sphere by the surface will cause the sphere to accelerate to the right. The friction force will also cause the sphere's angular velocity to decrease. The center of the sphere will accelerate until the sphere is rolling on the surface instead of slipping relative to it. Use the relation between the velocity of the center and the angular velocity of the sphere when it is rolling to determine when the sphere begins rolling.

Solution: Given

$W = 186.8$ N, $g = 9.81$ m/s^2, $m = W/g$, $R = 0.102$ m, $\mu_k = 0.06$



We have

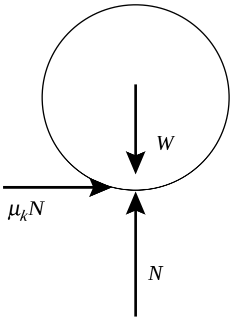
$$\sum F_x : \mu_k N = ma$$

$$\sum F_y : N - mg = 0$$

$$\sum M_G : \mu_k N R = \frac{2}{5} m R^2 a$$

Solving we find

$$\alpha = \frac{5\mu_k g}{2R} = 14.49 \text{ rad/s}^2, \; a = \mu_k g = 0.59 \text{ m/s}^2$$



From kinematics we learn that

$\alpha = 14.49$ rad/s^2, $\omega = (14.49 \text{ rad/s}^2)t - (40 \text{ rad/s})$

$a = 0.59$ m/s^2, $v = (0.59 \text{ m/s})t$

when we reach a steady motion we have

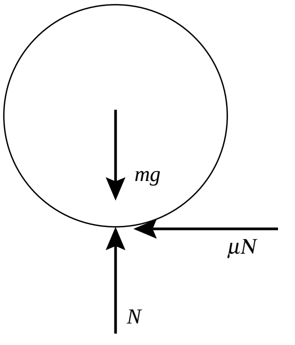
$v = -R\omega \Rightarrow (0.59 \text{ m/s}^2)t = -(0.102 \text{ m})[(14.49 \text{ rad/s}^2)t - (40 \text{ rad/s})]$

Solving for the time we find

$$\boxed{t = 1.97 \text{ s} \;\Rightarrow\; v = 1.16 \text{ m/s}}$$

Problem 18.41 A soccer player kicks the ball to a teammate 8 m away. The ball leaves the player's foot moving parallel to the ground at 6 m/s with no angular velocity. The coefficient of kinetic friction between the ball and the grass is $\mu_k = 0.32$. How long does it take the ball to reach his teammate? The radius of the ball is 112 mm and its mass is 0.4 kg. Estimate the ball's moment of inertia by using the equation for a thin spherical shell: $I = \frac{2}{3}mR^2$.

Solution: Given $\mu = 0.32$, $r = 0.112$ m, $g = 9.81$ m/s^2, $v_0 = 6$ m/s



The motion occurs in two phases.

(a) Slipping.

$$\sum F_x : -\mu N = ma$$

$$\sum F_y : N - mg = 0$$

$$\sum M_G : -\mu N R = \frac{2}{3} m R^2 \alpha$$

Solving we find

$$a = -\mu g \;\Rightarrow\; v = v_0 - \mu g t,\; s = v_0 t - \frac{1}{2}\mu g t^2$$

$$\alpha = -\frac{3\mu g}{2R} \;\Rightarrow\; \omega = -\frac{3\mu g}{2R} t$$

When it stops slipping we have

$$v = -R\omega \;\Rightarrow\; v_0 - \mu g t = \frac{3}{2}\mu g t \;\Rightarrow\; t = \frac{2v_0}{5\mu g} = 0.765 \text{ s}$$

$$v = 3.6 \text{ m/s},\; s = 3.67 \text{ m}$$

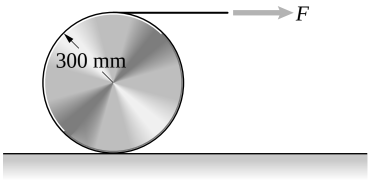
(b) Rolling — Steady motion

$$a = 0,\; v = 3.6 \text{ m/s},\; s = (3.6 \text{ m/s})(t - 0.765 \text{ s}) + 3.67 \text{ m}$$

When it reaches the teammate we have

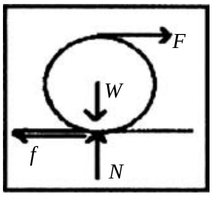
$$8 \text{ m} = (3.6 \text{ m/s})(t - 0.765 \text{ s}) + 3.67 \text{ m} \;\Rightarrow\; \boxed{t = 1.97 \text{ s}}$$

Problem 18.42 The 100-kg cylindrical disk is at rest when the force F is applied to a cord wrapped around it. The static and kinetic coefficients of friction between the disk and the surface equal 0.2. Determine the angular acceleration of the disk if (a) $F = 500$ N and (b) $F = 1000$ N.



Strategy: First solve the problem by assuming that the disk does not slip, but rolls on the surface. Determine the friction force, and find out whether it exceeds the product of the coefficient of friction and the normal force. If it does, you must rework the problem assuming that the disk slips.

Solution: Choose a coordinate system with the origin at the center of the disk in the at rest position, with the x axis parallel to the plane surface. The moment about the center of mass is $M = -RF - Rf$, from which $-RF - Rf = I\alpha$. From which



$$f = \frac{-RF - I\alpha}{R} = -F - \frac{I\alpha}{R}.$$

From Newton's second law: $F - f = ma_x$, where a_x is the acceleration of the center of mass. Assume that the disk rolls. At the point of contact $\mathbf{a}_P = 0$; from which $0 = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{P/G} - \omega^2 \mathbf{r}_{P/G}$.

$$\mathbf{a}_G = a_x\mathbf{i} = \boldsymbol{\alpha} \times R\mathbf{j} - \omega^2 R\mathbf{j}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ 0 & R & 0 \end{bmatrix} - \omega^2 R\mathbf{j} = -R\alpha\mathbf{i} - \omega^2 R\mathbf{j},$$

from which $a_y = 0$ and $a_x = -R\alpha$. Substitute for f and solve:

$$a_x = \frac{2F}{\left(m + \dfrac{I}{R^2}\right)}.$$

For a disk, the moment of inertia about the polar axis is $I = \frac{1}{2}mR^2$, from which

$$a_x = \frac{4F}{3m} = \frac{2000}{300} = 6.67 \text{ m/s}^2.$$

(a) For $F = 500$ N, the friction force is

$$f = F - ma_x = -\frac{F}{3} = -\frac{500}{3} = -167 \text{ N}.$$

Note: $-\mu_k W = -0.2$ mg $= -196.2$ N, *the disk does not slip.* The angular velocity is

$$\alpha = -\frac{a_x}{R} = -\frac{6.67}{0.3} = -22.22 \text{ rad/s}^2.$$

(b) For $F = 1000$ N the acceleration is

$$a_x = \frac{4F}{3m} = \frac{4000}{300} = 13.33 \text{ m/s}^2.$$

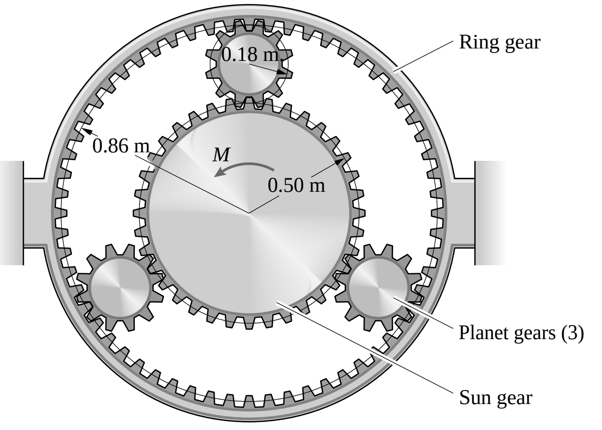
The friction force is

$$f = F - ma_x = 1000 - 1333.3 = -333.3 \text{ N}.$$

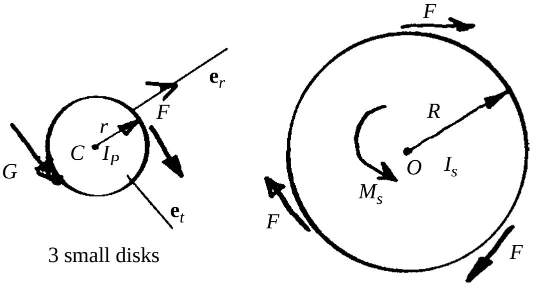
The drum slips. The moment equation for slip is $-RF + R\mu_k gm = I\alpha$, from which

$$\alpha = \frac{-RF + R\mu_k gm}{I} = -\frac{2F}{mR} + \frac{2\mu_k g}{R} = -53.6 \text{ rad/s}^2.$$

Problem 18.43 The ring gear is fixed. The mass and moment of inertia of the sun gear are $m_S = 320$ kg and $I_S = 40$ kg-m^2. The mass and moment of inertia of each planet gear are $m_P = 38$ kg and $I_P = 0.60$ kg-m^2. If a couple $M = 200$ N-m is applied to the sun gear, what is the latter's angular acceleration?



Solution:



$M_S = 200$ N-m

Sun Gear: $\sum M_0$: $M_S - 3RF = I_S\alpha_S$

Planet Gears: $\sum M_c$: $Gr - Fr = I_P\alpha_P$

$\sum F_t$: $F + G = m_P a_{ct}$

From kinematics $a_{ct} = -r\alpha_P$

$$2\alpha_P r_P = -R\alpha_S$$

We have 5 eqns in 5 unknowns. Solving, $\underline{\alpha_S = 3.95 \text{ rad/s}^2}$ (counter-clockwise)

Problem 18.44 In Problem 18.43, what is the magnitude of the tangential force exerted on the sun gear by each planet gear at their point of contact when the 200 N-m couple is applied to the sun gear?

Solution: See the solution to Problem 18.43. Solving the 5 eqns in 5 unknowns yields

$\alpha_S = 3.95$ rad/s^2,

$G = 9.63$ N,

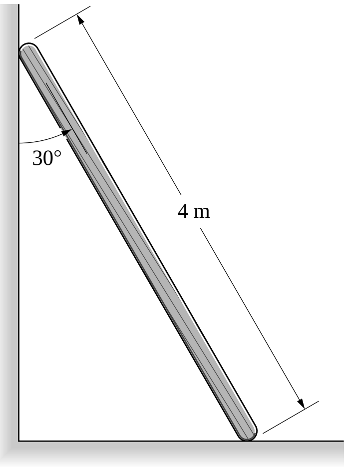
$a_{Gt} = 0.988$ m/s^2,

$\alpha_P = -5.49$ rad/s^2,

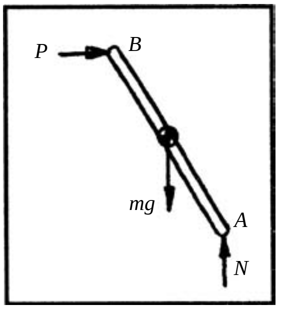
$\underline{F = 27.9 \text{ N}}$.

F is the required value.

Problem 18.45 The 18-kg ladder is released from rest in the position shown. Model it as a slender bar and neglect friction. At the instant of release, determine (a) the angular acceleration of the ladder and (b) the normal force exerted on the ladder by the floor. (See Active Example 18.3.)



Solution: The vector location of the center of mass is $\mathbf{r}_G = (L/2)\sin 30^\circ\mathbf{i} + (L/2)\cos 30^\circ\mathbf{j} = 1\mathbf{i} + 1.732\mathbf{j}$ (m). Denote the normal forces at the top and bottom of the ladder by P and N. The vector locations of A and B are $\mathbf{r}_A = L\sin 30^\circ\mathbf{i} = 2\mathbf{i}$ (m), $\mathbf{r}_B = L\cos 30^\circ\mathbf{j} = 3.46\mathbf{j}$ (m). The vectors $\mathbf{r}_{A/G} = \mathbf{r}_A - \mathbf{r}_G = 1\mathbf{i} - 1.732\mathbf{j}$ (m), $\mathbf{r}_{B/G} = \mathbf{r}_B - \mathbf{r}_G = -1\mathbf{i} + 1.732\mathbf{j}$ (m). The moment about the center of mass is



$$\mathbf{M} = \mathbf{r}_{B/G} \times \mathbf{P} + \mathbf{r}_{A/G} \times \mathbf{N},$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1.732 & 0 \\ P & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1.732 & 0 \\ 0 & N & 0 \end{bmatrix}$$

$$= (-1.732P + N)\mathbf{k} \text{ (N-m)}.$$

From the equation of angular motion: (1) $-1.732\ P + N = I\alpha$. From Newton's second law: (2) $P = ma_x$, (3) $N - mg = ma_y$, where a_x, a_y are the accelerations of the center of mass.

From kinematics: $\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A}$. The angular velocity is zero since the system was released from rest,

$$\mathbf{a}_G = a_A\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ -1 & 1.732 & 0 \end{bmatrix} = a_A\mathbf{i} - 1.732\alpha\mathbf{i} - \alpha\mathbf{j}$$

$$= (a_A - 1.732\alpha)\mathbf{i} - \alpha\mathbf{j} \text{ (m/s}^2\text{)},$$

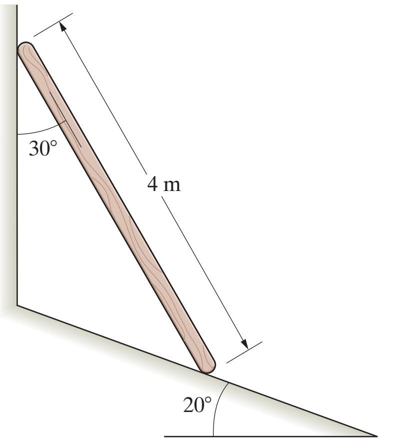
from which $a_y = -\alpha$.
Similarly,

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{G/B}, \mathbf{a}_G = \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ 1 & -1.732 & 0 \end{bmatrix}$$

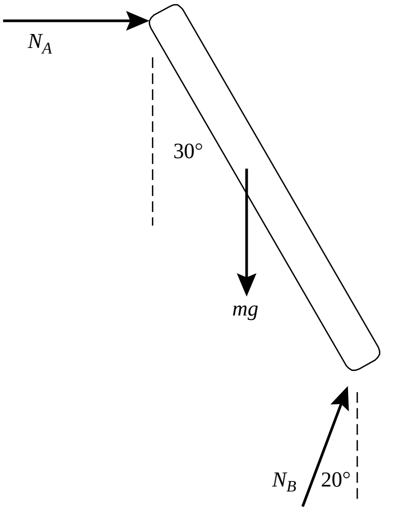
$$= a_B\mathbf{j} + 1.732\alpha\mathbf{i} + \alpha\mathbf{j},$$

from which $a_x = 1.732\alpha$. Substitute into (1), (2) and (3) to obtain three equations in three unknowns: $-1.732P + N = I\alpha$, $P = m(1.732)\alpha$, $N - mg = -m\alpha$. Solve: (a) $\alpha = 1.84$ rad/s^2, $P = 57.3$ N, (b) $N = 143.47$ N

Problem 18.46 The 18-kg ladder is released from rest in the position shown. Model it as a slender bar and neglect friction. Determine its angular acceleration at the instant of release.



Solution: Given $m = 18$ kg, $L = 4$ m, $g = 9.81$ m/s^2, $\omega = 0$



First find the kinematic constraints. We have

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G}$$

$$= a_x\mathbf{i} + a_y\mathbf{j} + \alpha\mathbf{k} \times \left(\left[-\frac{L}{2}\sin 30^\circ\right]\mathbf{i} + \left[\frac{L}{2}\cos 30^\circ\right]\mathbf{j}\right)$$

$$= \left(a_x - \alpha\frac{L}{2}\cos 30^\circ\right)\mathbf{i} + \left(a_y - \alpha\frac{L}{2}\sin 30^\circ\right)\mathbf{j}$$

$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G}$$

$$= a_x\mathbf{i} + a_y\mathbf{j} + \alpha\mathbf{k} \times \left(\left[\frac{L}{2}\sin 30^\circ\right]\mathbf{i} + \left[-\frac{L}{2}\cos 30^\circ\right]\mathbf{j}\right)$$

$$= \left(a_x + \alpha\frac{L}{2}\cos 30^\circ\right)\mathbf{i} + \left(a_y + \alpha\frac{L}{2}\sin 30^\circ\right)\mathbf{j}$$

The constraints are

$$\mathbf{a}_A \cdot \mathbf{i} = a_x - \alpha\frac{L}{2}\cos 30^\circ = 0$$

$$\mathbf{a}_B \cdot (\sin 20^\circ\mathbf{i} + \cos 20^\circ\mathbf{j})$$

$$= \left(a_x + \alpha\frac{L}{2}\cos 30^\circ\right)\sin 20^\circ + \left(a_y + \alpha\frac{L}{2}\sin 30^\circ\right)\cos 20^\circ$$

The dynamic equations:

$$\sum F_x : N_A + N_B\sin 20^\circ = ma_x$$

$$\sum F_y : N_B\cos 20^\circ - mg = ma_y$$

$$\sum M_G : -N_A\left(\frac{L}{2}\cos 30^\circ\right) + N_B\cos 20^\circ\left(\frac{L}{2}\sin 30^\circ\right)$$

$$+ N_B\sin 20^\circ\left(\frac{L}{2}\cos 30^\circ\right) = \frac{1}{12}mL^2\alpha$$

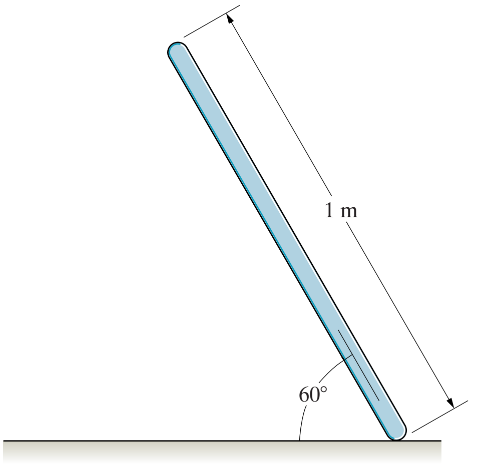
Solving five equations in five unknowns we have

$$\boxed{\alpha = 2.35 \text{ rad/s}^2 \quad CCW}$$

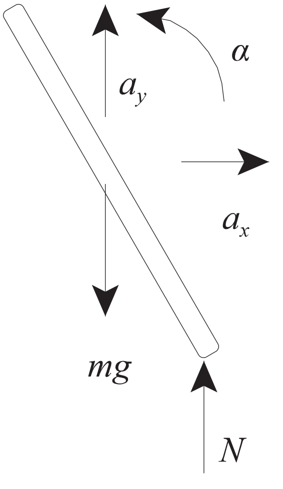
Also

$$a_x = 4.07 \text{ ft/s}^2,\ a_y = -5.31 \text{ ft/s}^2,\ N_A = 43.7 \text{ N},\ N_B = 86.2 \text{ N}$$

Problem 18.47 The 4-kg slender bar is released from rest in the position shown. Determine its angular acceleration at that instant if (a) the surface is rough and the bar does not slip, and (b) the surface is smooth.



Solution:



(a) The surface is rough. The lower end of the bar is fixed, and the bar rotates around that point.

$$\Sigma M_B : mg\frac{L}{2}\cos\theta = \frac{1}{3}\,mL^2\alpha$$

$$\alpha = \frac{3g}{2L}\cos\theta = \frac{3(9.81\ \text{m/s}^2)}{2(1\ \text{m})}\cos 60^\circ$$

$$\boxed{\alpha = 7.36\ \text{rad/s}^2.}$$

(b) The surface is smooth. There are four unknowns (N, a_x, a_y, α), three dynamic equations, and one constraint equation (the y component of the acceleration of the point in contact with the ground is zero).

$$\Sigma F_x : 0 = ma_x,$$

$$\Sigma F_y : N - mg = ma_y,$$

$$\Sigma M_G : N\frac{L}{2}\cos\theta = \frac{1}{12}\,mL^2\alpha$$

$$a_y + \alpha\frac{L}{2}\cos\theta = 0$$

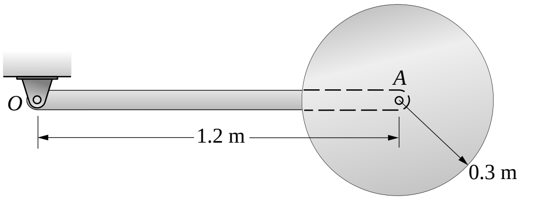
Solving, we find

$$\alpha = \frac{6g\cos\theta}{L(1+3\cos^2\theta)} = \frac{6(9.81\ \text{m/s}^2)\cos 60^\circ}{(1\ \text{m})(1+3\cos^2 60^\circ)} = 16.8\ \text{rad/s}^2.$$

$$\boxed{\alpha = 16.8\ \text{rad/s}^2.}$$

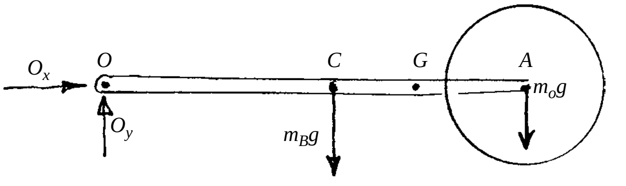
Problem 18.48 The masses of the bar and disk are 14 kg and 9 kg, respectively. The system is released from rest with the bar horizontal. Determine the bar's angular acceleration at that instant if

(a) the bar and disk are welded together at A,
(b) the bar and disk are connected by a smooth pin at A.



Strategy: In part (b), draw individual free-body diagrams of the bar and disk.

Solution:



(a) $L = 1.2$ m $\quad R = 0.3$ m

$m_B = 14$ kg $\quad m_D = 9$ kg

O is a fixed point
For the bar

$$I_G = \frac{1}{12}m_B L^2 = \frac{1}{12}(14)(1.2)^2 = 1.68 \text{ N-m}^2$$

$$I_{O_B} = I_G + m_B\left(\frac{L}{2}\right)^2$$

$$I_{O_B} = 6.72 \text{ N-m}^2$$

For the disk:

$$I_A = \tfrac{1}{2}m_D R^2 = \tfrac{1}{2}(9)(0.3)^2 = 0.405 \text{ N-m}^2$$

$$I_{OD} = I_A + m_0 L^2 = 13.37 \text{ N-m}^2$$

The total moment of inertia of the welded disk and bar about O is

$$I_T = I_{OB} + I_{OD} = 20.09 \text{ N-m}^2$$

$$\sum F_x: \quad O_x = O = m a_{G_x}$$

$$\sum F_y: \quad O_y - m_B g - m_D g = (m_B + m_D) a_{G_y}$$

$$\sum M_0: \quad -\left(\frac{L}{2}\right) m_B g - L m_D g = I_T \alpha$$

We can solve the last equation for α without finding the location and acceleration of the center of mass, G. Solving,

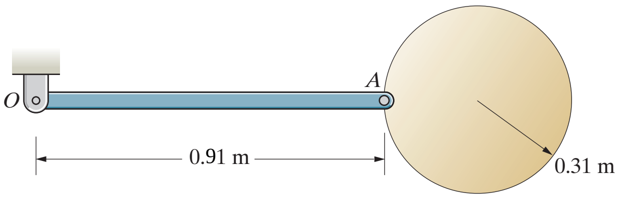
$\underline{\alpha = -9.38 \text{ rad/s}^2 \quad \text{(clockwise)}}$

(b) In this case, only the moment of inertia changes. Since the disk is on a smooth pin, it does not rotate. It acts only as a point mass at a distance L from point O.
In this case, $I'_{OD} = m_D L^2$ and $I'_T = I_{OB} + I'_{OD} = 19.68$ N-m^2
We now have

$$\sum M_0: \quad -\left(\frac{L}{2}\right) m_B g - L m_D g = I'_T \alpha'$$

Solving $\underline{\alpha' = -9.57 \text{ rad/s}^2 \quad \text{(clockwise)}}$

Problem 18.49 The 22.2 N horizontal bar is connected to the 44.5 N disk by a smooth pin at A. The system is released from rest in the position shown. What are the angular accelerations of the bar and disk at that instant?



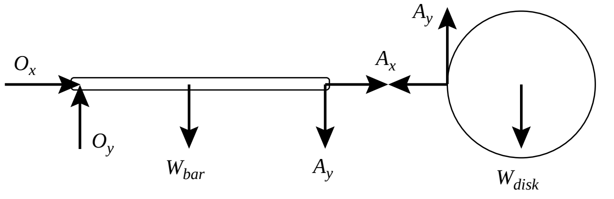
Solution: Given

$g = 9.81\ \text{m/s}^2,\ W_{bar} = 22.2\ \text{N},\ W_{disk} = 44.5\ \text{N},$

$$m_{bar} = \frac{W_{bar}}{g},\ m_{disk} = \frac{W_{\text{disk}}}{g}$$

$L = 0.91\ \text{m},\ R = 0.31\ \text{m}$

The FBDs



The dynamic equations

$$\sum M_O : -m_{bar}\,g\frac{L}{2} - A_y L = \frac{1}{3}m_{bar}L^2\alpha_{bar}$$

$$\sum M_{Gdisk} : -A_y R = \frac{1}{2}m_{disk}R^2\alpha_{disk}$$

$$\sum F_y : A_y - m_{disk}\,g = m_{disk}\,a_{ydisk}$$

Kinematic constraint

$$\alpha_{bar}L = a_{ydisk} - \alpha_{disk}R$$

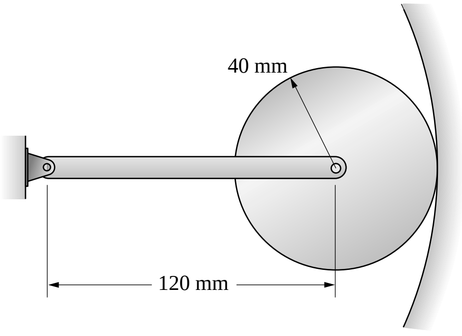
Solving we find

$$\alpha_{disk} = 3.58\ \text{rad/s}^2,\ \alpha_{bar} = -12.5\ \text{rad/s}^2,\ a_{ydisk} = -34.0\ \text{m/s}^2,$$

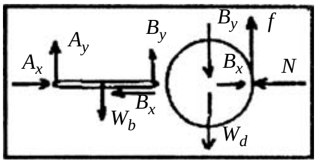
$$A_y = -0.556\ \text{N}$$

Thus $\boxed{\alpha_{disk} = 3.58\ \text{rad/s}^2\quad CCW,\ \alpha_{bar} = 12.5\ \text{rad/s}^2\quad CW}$

Problem 18.50 The 0.1-kg slender bar and 0.2-kg cylindrical disk are released from rest with the bar horizontal. The disk rolls on the curved surface. What is the bar's angular acceleration at the instant it is released?



Solution: The moment about the center of mass of the disk is $M = fR$, from the equation of angular motion, $Rf = I_d\alpha_d$. From Newton's second law: $f - B_y - W_d = m_d a_{dy}$. Since the disk rolls, the kinematic condition is $a_{dy} = -R\alpha_d$. Combine the expressions and rearrange: $f = I\alpha_d/R$, $I\alpha_d/R - B_y - W_d = m_d a_{dy}$, from which $B_y + W_d = (Rm_d + I_d/R)\alpha_d$. The moment about the center of mass of the bar is



$$M_b = -\left(\frac{L}{2}\right)A_y + \left(\frac{L}{2}\right)B_y,$$

from which

$$-\left(\frac{L}{2}\right)A_y + \left(\frac{L}{2}\right)B_y = I_b\alpha_b.$$

From Newton's second law $A_y - W_b + B_y = m_b a_{by}$, where a_{by} is the acceleration of the center of mass of the bar. The kinematic condition for the bar is

$$\mathbf{a}_{CM} = \boldsymbol{\alpha}_b \times \left(\left(\frac{L}{2}\right)\mathbf{i}\right) = \left(\frac{L}{2}\right)\alpha_b\mathbf{j},$$

from which

$$a_{by} = \left(\frac{L}{2}\right)\alpha_b.$$

Similarly, $\mathbf{a}_D = \mathbf{a}_{CM} + \boldsymbol{\alpha}_b \times ((L/2)\mathbf{i})$, from which $a_{dy} = L\alpha_b$.

From which: $\alpha_d = -L\alpha_b/R$. Substitute to obtain three equations in three unknowns:

$$B_y + W_d = \left(Rm_d + \frac{I_d}{R}\right)\left(-\frac{L}{R}\right)\alpha_b,$$

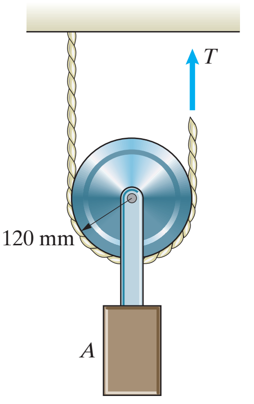
$$-\left(\frac{L}{2}\right)A_y + \left(\frac{L}{2}\right)B_y = I_b\alpha_b,$$

$$A_y - W_b + B_y = m_b\left(\frac{L}{2}\right)\alpha_b.$$

Substitute known numerical values: $L = 0.12$ m, $R = 0.04$ m, $m_b = 0.1$ kg, $W_b = m_b g = 0.981$ N, $m_d = 0.2$ kg, $W_d = m_d g = 1.962$ N, $I_b = (1/12)m_b(L^2) = 1.2 \times 10^{-4}$ kg-m^2, $I_d = (1/2)m_d R^2 = 1.6 \times 10^{-4}$ kg-m^2. Solve:

$\alpha_b = -61.3$ rad/s^2, $A_y = 0.368$ N, $B_y = 0.245$ N.

Problem 18.51 The mass of the suspended object A is 8 kg. The mass of the pulley is 5 kg, and its moment of inertia is 0.036 kg-m^2. If the force $T = 70$ N, what is the magnitude of the acceleration of A?

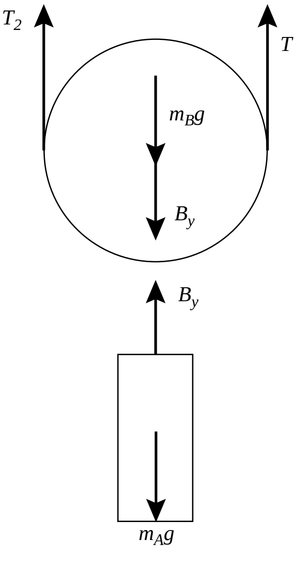


Solution: Given

$m_A = 8$ kg, $m_B = 5$ kg, $I_B = 0.036$ kg-m^2

$R = 0.12$ m, $g = 9.81$ m/s^2, $T = 70$ N

The FBDs



The dynamic equations

$$\sum F_{yB} : T_2 + T - m_B g - B_y = m_B a_{By}$$

$$\sum F_{yA} : B_y - m_A g = m_A a_{Ay}$$

$$\sum M_B : -T_2 R + T R = I_B \alpha_B$$

Kinematic constraints

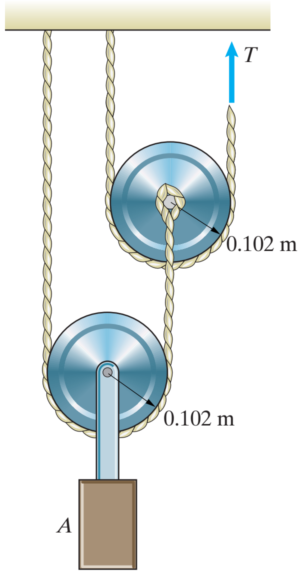
$a_{By} = a_{Ay}$, $a_{By} = R\alpha_B$

Solving we find $\boxed{a_{Ay} = 0.805 \text{ m/s}^2}$

We also have

$a_{By} = 0.805$ m/s^2, $\alpha_B = 6.70$ rad/s, $T_2 = 68.0$ N, $B_y = 84.9$ N

Problem 18.52 The suspended object A weighs 89 N. The pulleys are identical, each weighing 44.5 N and having moment of inertia 0.03 kg-m^2. If the force $T = 66.7$ N, what is the magnitude of the acceleration of A?

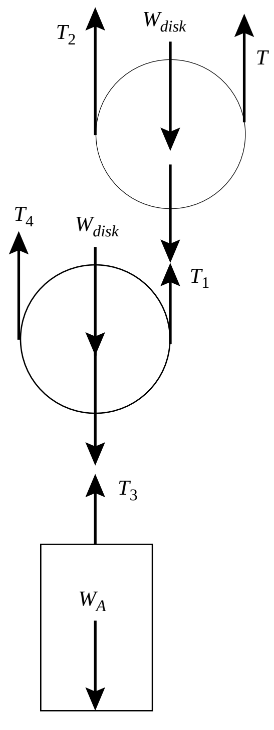


Solution: Given

$g = 9.81\ \text{m/s}^2,\ W_A = 89\ \text{N},\ W_{disk} = 44.5\ \text{N},\ I = 0.03\ \text{kg-m}^2$

$m_A = \dfrac{W_A}{g},\ m_{disk} = \dfrac{W_{disk}}{g},\ R = 0.102\ \text{m},\ T = 66.7\ \text{N}$

The FBDs



The dynamic equations

$\sum F_{y1} : T_2 + T - T_1 - m_{disk} g = m_{disk} a_1$

$\sum F_{y2} : T_4 + T_1 - T_3 - m_{disk} g = m_{disk} a_2$

$\sum F_{y3} : T_3 - m_A g = m_A a_A$

$\sum M_1 : T R - T_2 R = I \alpha_1$

$\sum M_2 : T_1 R - T_4 R = I \alpha_2$

The kinematic constraints

$a_1 = R\alpha_1,\ a_2 = R\alpha_2,\ a_1 = 2R\alpha_2,\ a_A = a_2$

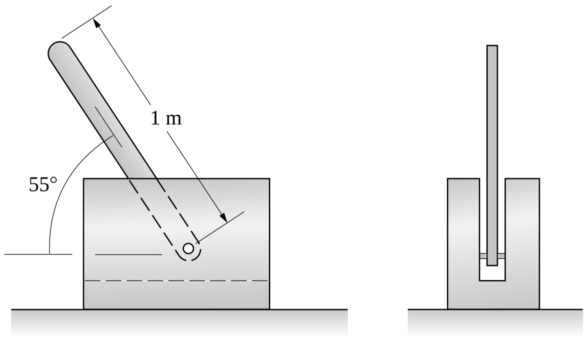
Solving we find $\boxed{a_A = 0.96\ \text{m/s}^2}$

We also have

$a_1 = 1.93\ \text{m/s}^2,\ a_2 = 0.96\ \text{m/s}^2,\ \alpha_1 = 19.0\ \text{rad/s}^2,\ \alpha_2 = 9.48\ \text{rad/s}^2$

$T_1 = 74.7\ \text{N},\ T_2 = 60.9\ \text{N},\ T_3 = 97.9\ \text{N},\ T_4 = 72.1\ \text{N}$

Problem 18.53 The 2-kg slender bar and 5-kg block are released from rest in the position shown. If friction is negligible, what is the block's acceleration at that instant? (See Example 18.5.)



Solution: $L = 1$ m, $m = 2$ kg $M = 5$ kg

Assume directions for B_x, B_y, $I_G = \frac{1}{12} m_B L^2$

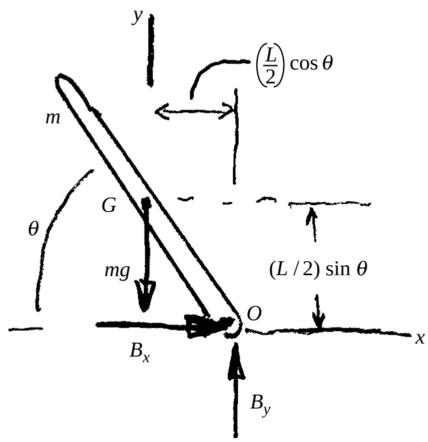
$\sum F_x$: $B_x = m a_{G_x}$ **(1)**

$\sum F_y$: $B_y - mg = m_B a_{G_y}$ **(2)**

$\sum M_G$: $\left(\frac{L}{2}\cos\theta\right) B_y + \left(\frac{L}{2}\sin\theta\right) B_x = I_G \alpha$ **(3)**

$\sum F_x$: $-B_x = M a_{0x}$ **(4)**

$\sum F_y$: $N - B_y - Mg = 0$ **(5)**



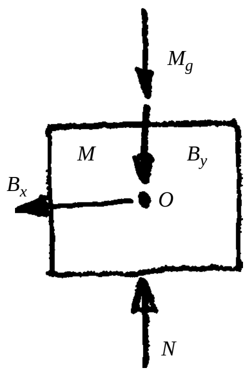
From kinematics, $\boldsymbol{\omega} = 0$ (initially)

$$\mathbf{a}_0 = \mathbf{a}_G + \alpha \mathbf{k} \times \mathbf{r}_{0/G}$$

where $\mathbf{r}_{0/G} = \frac{L}{2}\cos\theta\mathbf{i} - \frac{L}{2}\sin\theta\mathbf{j}$

From the diagram $\mathbf{a}_0 = a_{0x}\mathbf{i}$

$$\begin{cases} a_{0x} = a_{Gx} + (\alpha L/2)\sin\theta & \textbf{(6)} \\ 0 = a_{Gy} + (\alpha L/2)\cos\theta & \textbf{(7)} \end{cases}$$



We know $\theta = 55°$, $I_G = 0.167$ kg-m^2, $L = 1$ m, $m = 2$ kg, $M = 5$ kg. We have 7 eqns in 7 unknowns

$(a_{G_x}, a_{G_y}, a_{0x}, \alpha, B_x, B_y, N)$,

Solving, we get

$B_x = -5.77$ N, (opposite the assumed direction)

$B_y = 13.97$ N,

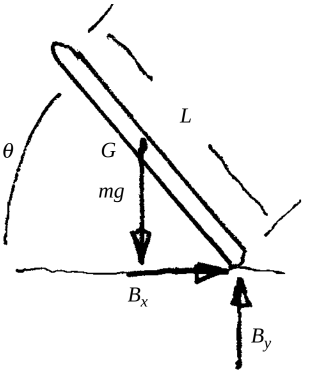
$a_{G_x} = -2.88$ m/s^2, $a_{G_y} = -2.83$ m/s^2

$\alpha = 9.86$ rad/s^2, $N = 63.0$ N

$\underline{a_{0x} = 1.15 \text{ m/s}^2}$. (to the right)

Problem 18.54 The 2-kg slender bar and 5-kg block are released from rest in the position shown. What minimum coefficient of static friction between the block and the horizontal surface would be necessary for the block not to move when the system is released? (See Example 18.5.)

Solution: This solution is very similar to that of Problem 18.53. We add a friction force $f = \mu_s N$ and set $a_{0x} = 0$.



$L = 1$ m $\quad m = 2$ kg

$M = 5$ kg

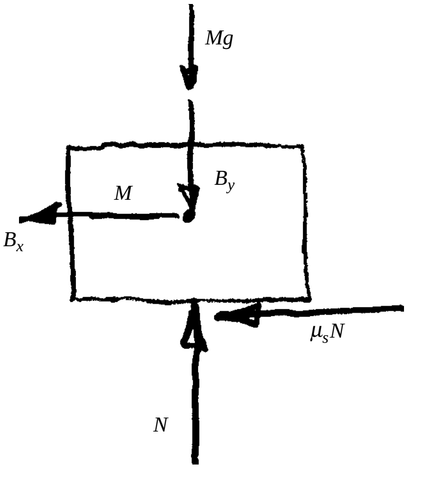
$$I_G = \frac{1}{12}mL^2 = 0.167 \text{ kg-m}^2$$

$$\sum F_x: \quad B_x = ma_{G_x} \qquad \textbf{(1)}$$

$$\sum F_y: \quad B_y - mg = ma_{G_y} \qquad \textbf{(2)}$$

$$\sum M_G: \quad \left(\frac{L}{2}\cos\theta\right)B_y + \left(\frac{L}{2}\sin\theta\right)B_x = I_G\alpha \qquad \textbf{(3)}$$

(These are the same as in Problem 18.53)
Note: In Prob 18.53, $B_x = -5.77$ N (it was in the opposite direction to that assumed). This resulted in a_{0x} to the right. Thus, friction must be to the left



$$\sum F_x: \quad -B_x - \mu_s N = ma_{0x} = 0 \qquad \textbf{(4)}$$

$$\sum F_y: \quad N - B_y - Mg = 0 \qquad \textbf{(5)}$$

From kinematics,

$$\mathbf{a}_0 = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{0/G} = 0$$

$$O = a_{Gx} + (\alpha L/2)\sin\theta \qquad \textbf{(6)}$$

$$O = a_{Gy} + (\alpha L/2)\cos\theta \qquad \textbf{(7)}$$

Solving 7 eqns in 7 unknowns, we get

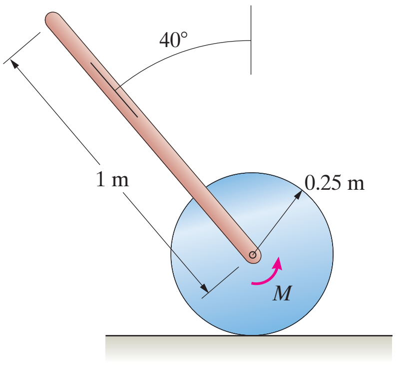
$B_x = -6.91$ N, $\quad B_y = 14.78$ N,

$a_{Gx} = -3.46$ m/s^2, $\quad a_{Gy} = -2.42$ m/s^2

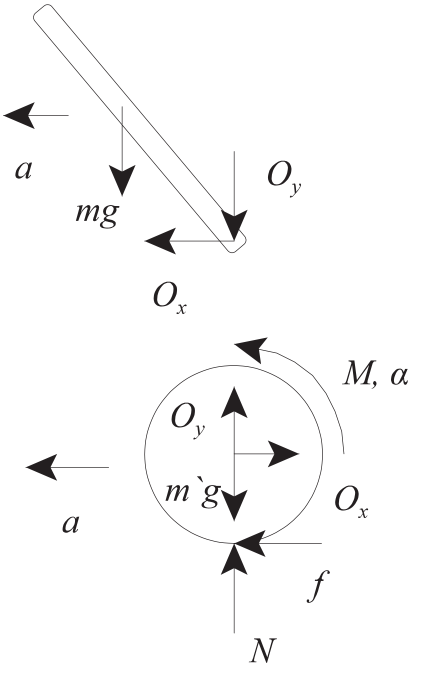
$N = 63.8$ N, $\quad \alpha = 8.44$ rad/s^2

$\underline{\mu_s = 0.108}$

Problem 18.55 As a result of the constant couple M applied to the 1-kg disk, the angular acceleration of the 0.4-kg slender bar is zero. Determine M and the counterclockwise angular acceleration of the rolling disk.



Solution: There are seven unknowns $(M, N, f, O_x, O_y, a, \alpha)$, six dynamic equations, and one constraint equation. We use the following subset of those equations.



$$\Sigma M_{G\text{ rod}} : -O_x(0.5\text{ m})\cos 40^\circ$$

$$- O_y(0.5\text{ m})\sin 40^\circ = 0,$$

$$\Sigma F_{x\text{ rod}} : -O_x = -(0.4\text{ kg})a,$$

$$\Sigma F_{y\text{ rod}} : -O_y - (0.4\text{ kg})(9.81\text{ m/s}^2) = 0,$$

$$\Sigma M_{G\text{ disk}} : M - f(0.25\text{ m})$$

$$= \frac{1}{2}(1\text{ kg})(0.25\text{ m})^2\alpha,$$

$$\Sigma F_{x\text{disk}} : O_x - f = -(1\text{ kg})a,$$

$$a = (0.25\text{ m})\alpha.$$

Solving, we find

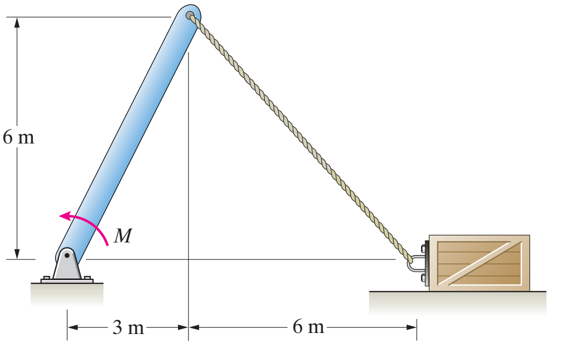
$$O_x = 3.29\text{ N},\ \ O_y = -3.92\text{ N},$$

$$f = 11.5\text{ N},\ \ a = 8.23\text{ m/s}^2,$$

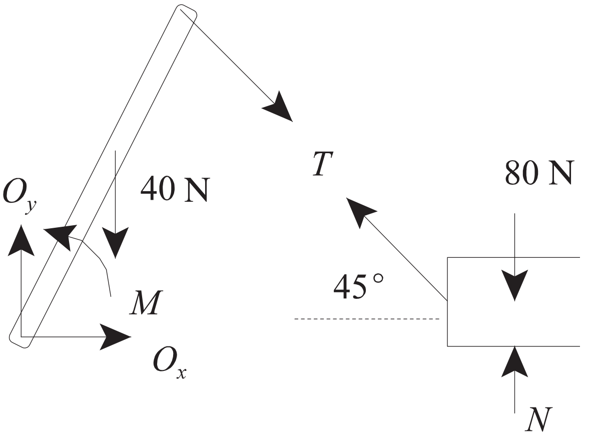
$$\alpha = 32.9\text{ rad/s}^2,\ \ M = 3.91\text{ N-m}.$$

$$\boxed{M = 3.91\text{ N-m},\ \ \alpha = 32.9\text{ rad/s}^2.}$$

Problem 18.56 The slender bar weighs 40 N and the crate weighs 80 N. At the instant shown, the velocity of the crate is zero and it has an acceleration of 14 m/s^2 toward the left. The horizontal surface is smooth. Determine the couple M and the tension in the rope.



Solution: There are six unknowns $(M, T, N, O_x, O_y, \alpha)$, five dynamic equations, and one constraint equation. We use the following subset of the dynamic equations.



$$\Sigma M_O : M - (40\text{ N})(1.5\text{ m})$$

$$- T\cos 45^\circ(6\text{ m})$$

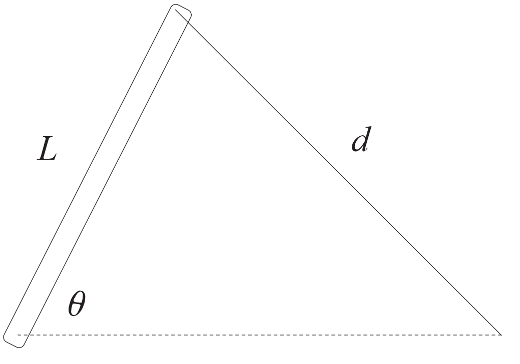
$$- T\sin 45^\circ(3\text{ m})$$

$$= \frac{1}{3}\left(\frac{40\text{ N}}{9.81\text{ m/s}^2}\right)(45\text{ m}^2)\alpha,$$

$$\Sigma F_x : -T\cos 45^\circ = -\left(\frac{80\text{ N}}{9.81\text{ m/s}^2}\right)(14\text{ m/s}^2)$$

The constraint equation is derived from the triangle shown. We have

$$L = \sqrt{45}\text{ m},\ d = 6\sqrt{2}\text{ m},\ \theta = 63.4^\circ.$$



$$x = L\cos\theta + \sqrt{d^2 - L^2\sin^2\theta}$$

$$\dot{x} = \left(-L\sin\theta - \frac{L^2\cos\theta\sin\theta}{\sqrt{d^2 - L^2\sin^2\theta}}\right)\dot{\theta}$$

Since the velocity $\dot{x} = 0$, then we know that the angular velocity $\omega = \dot{\theta} = 0$. Taking one more derivative and setting $\omega = 0$, we find

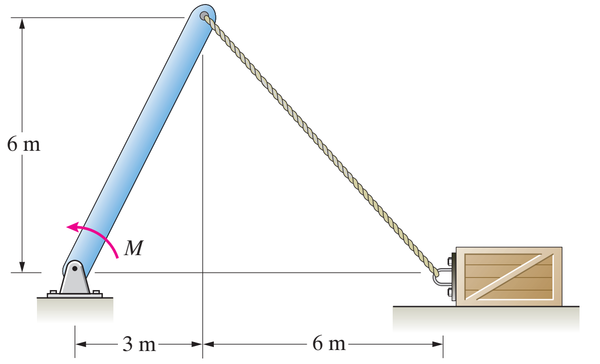
$$\ddot{x} = \left(-L\sin\theta - \frac{L^2\cos\theta\sin\theta}{\sqrt{d^2 - L^2\sin^2\theta}}\right)\ddot{\theta} \Rightarrow -(14\text{ m/s}^2)$$

$$= \left(-L\sin\theta - \frac{L^2\cos\theta\sin\theta}{\sqrt{d^2 - L^2\sin^2\theta}}\right)\alpha$$

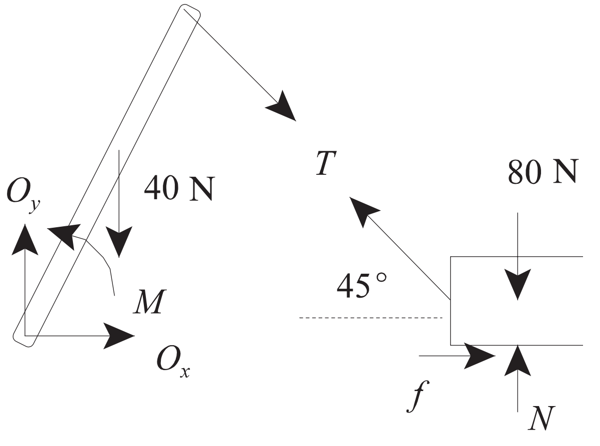
Solving these equations, we find that

$$\alpha = 1.56\text{ rad/s}^2,\quad \boxed{M = 1149\text{ N-m},\ T = 161.5\text{ N.}}$$

Problem 18.57 The slender bar weighs 40 N and the crate weighs 80 N. At the instant shown, the velocity of the crate is zero and it has an acceleration of $14\,\mathrm{m/s^2}$ toward the left. The coefficient of kinetic friction between the horizontal surface and the crate is $\mu_k = 0.2$. Determine the couple M and the tension in the rope.



Solution: There are seven unknowns $(M, T, N, O_x, O_y, \alpha, f)$, five dynamic equations, one constraint equation, and one friction equation. We use the following subset of the dynamic equations.



$$\Sigma M_O : M - (40\ \mathrm{N})(1.5\ \mathrm{m})$$

$$- T\cos 45^\circ (6\ \mathrm{m})$$

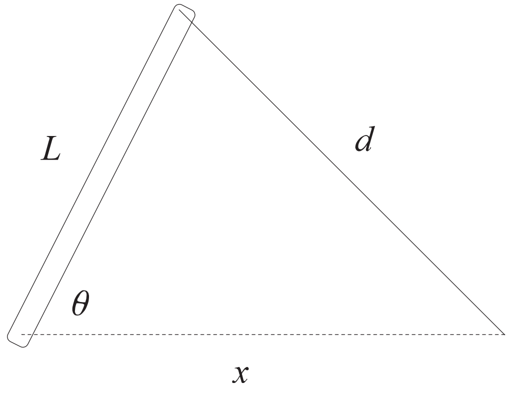
$$- T\sin 45^\circ (3\ \mathrm{m})$$

$$= \frac{1}{3}\left(\frac{40\ \mathrm{N}}{9.81\ \mathrm{m/s^2}}\right)(45\ \mathrm{m^2})\alpha,$$

$$\Sigma F_x : -T\cos 45^\circ + (0.2)N = -\left(\frac{80\ \mathrm{N}}{9.81\ \mathrm{m/s^2}}\right)(14\ \mathrm{m/s^2})$$

$$\Sigma F_y : T\sin 45^\circ + N - (80\ \mathrm{N}) = 0.$$

The constraint equation is derived from the triangle shown. We have $L = \sqrt{45}$ m, $d = 6\sqrt{2}$ m, $\theta = 63.4^\circ$.



$$x = L\cos\theta + \sqrt{d^2 - L^2\sin^2\theta}$$

$$\dot{x} = \left(-L\sin\theta - \frac{L^2\cos\theta\sin\theta}{\sqrt{d^2 - L^2\sin^2\theta}}\right)\dot{\theta}$$

Since the velocity $\dot{x} = 0$, then we know that the angular velocity $\omega = \dot{\theta} = 0$. Taking one more derivative and setting $\omega = 0$, we find

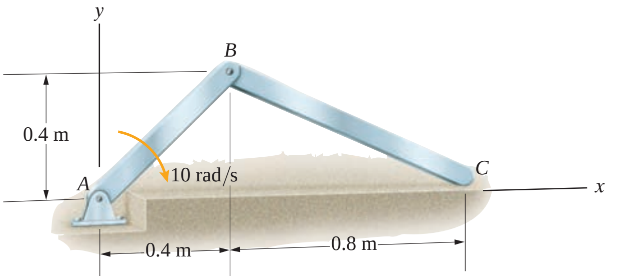
$$\ddot{x} = \left(-L\sin\theta - \frac{L^2\cos\theta\sin\theta}{\sqrt{d^2 - L^2\sin^2\theta}}\right)\ddot{\theta} \Rightarrow -(14\ \mathrm{m/s^2})$$

$$= \left(-L\sin\theta - \frac{L^2\cos\theta\sin\theta}{\sqrt{d^2 - L^2\sin^2\theta}}\right)\alpha$$

Solving these equations, we find that

$$\alpha = 1.56\ \mathrm{rad/s^2},\ N = -28.5\ \mathrm{N},\ \boxed{M = 1094\ \mathrm{N\text{-}m},\ T = 152.8\ \mathrm{N}.}$$

Problem 18.58 Bar AB is rotating with a constant clockwise angular velocity of 10 rad/s. The 8-kg slender bar BC slides on the horizontal surface. At the instant shown, determine the total force (including its weight) acting on bar BC and the total moment about its center of mass.



Solution: We first perform a kinematic analysis to find the angular acceleration of bar BC and the acceleration of the center of mass of bar BC. First the velocity analysis:

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + (-10\mathbf{k}) \times (0.4\mathbf{i} + 0.4\mathbf{j}) = (-4\mathbf{i} + 4\mathbf{j})$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = (-4\mathbf{i} + 4\mathbf{j}) + \omega_{BC}\mathbf{k} \times (0.8\mathbf{i} - 0.4\mathbf{j})$$

$$= (-4 + 0.4\,\omega_{BC})\mathbf{i} + (4 + 0.8\,\omega_{BC})\mathbf{j}$$

Since C stays in contact with the floor, we set the $\mathbf{j}$ component to zero $\Rightarrow \omega_{BC} = -5$ rad/s. Now the acceleration analysis.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}{}^2\mathbf{r}_{B/A}$$

$$= 0 + 0 - (10)^2(0.4\mathbf{i} + 0.4\mathbf{j}) = (-40\mathbf{i} - 40\mathbf{j})$$

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}{}^2\mathbf{r}_{C/B}$$

$$= (-40\mathbf{i} - 40\mathbf{j}) + \alpha_{BC}\mathbf{k} \times (0.8\mathbf{i} - 0.4\mathbf{j}) - (-5)^2(0.8\mathbf{i} - 0.4\mathbf{j})$$

$$= (-60 + 0.4\alpha_{BC})\mathbf{i} + (-30 + 0.8\alpha_{BC})\mathbf{j}$$

Since C stays in contact with the floor, we set the $\mathbf{j}$ component to zero $\Rightarrow \alpha_{BC} = 37.5$ rad/s^2. Now we find the acceleration of the center of mass G of bar BC.

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G/B} - \omega_{BC}{}^2\mathbf{r}_{G/B}$$

$$= (-40\mathbf{i} - 40\mathbf{j}) + (37.5)\mathbf{k} \times (0.4\mathbf{i} - 0.2\mathbf{j}) - (-5)^2(0.4\mathbf{i} - 0.2\mathbf{j})$$

$$= (-42.5\mathbf{i} - 20\mathbf{j}) \text{ m/s}^2.$$

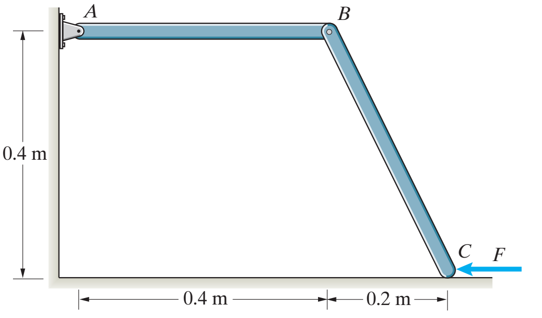
The total force and moment cause the accelerations that we just calculated. Therefore

$$\mathbf{F} = m\mathbf{a}_G = (8 \text{ kg})(-42.5\mathbf{i} - 20\mathbf{j}) \text{ m/s}^2 = (-340\mathbf{i} - 160\mathbf{j}) \text{ N},$$

$$M = I\alpha = \frac{1}{12}(8 \text{ kg})([0.8 \text{ m}]^2 + [0.4 \text{ m}]^2)(37.5 \text{ rad/s}^2) = 20 \text{ N-m}.$$

$$\boxed{\mathbf{F} = (-340\mathbf{i} - 160\mathbf{j}) \text{ N},\ \ M = 20 \text{ N-m counterclockwise.}}$$

Problem 18.59 The masses of the slender bars AB and BC are 10 kg and 12 kg, respectively. The angular velocities of the bars are zero at the instant shown and the horizontal force $F = 150$ N. The horizontal surface is smooth. Determine the angular accelerations of the bars.

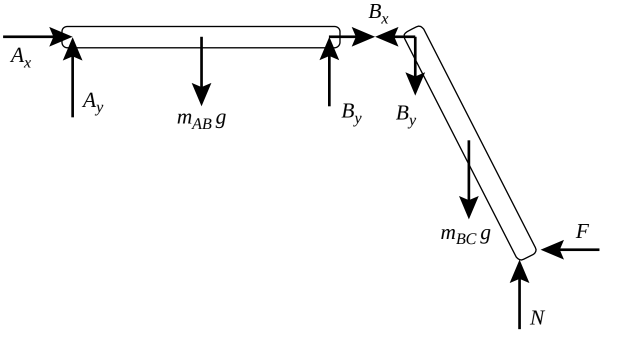


Solution: Given

$m_{AB} = 10$ kg, $m_{BC} = 12$ kg, $g = 9.81$ m/s^2

$L_{AB} = 0.4$ m, $L_{BC} = \sqrt{0.4^2 + 0.2^2}$m, $F = 150$ N

The FBDs



The dynamic equations

$$\sum M_A : -m_{AB}g\frac{L_{AB}}{2} + B_y L_{AB} = \frac{1}{3}\, m_{AB}{L_{AB}}^2\alpha_{AB}$$

$$\sum F_{BCx} : -B_x - F = m_{BC}a_{BCx}$$

$$\sum F_{BCy} : -B_y - m_{BC}g + N = m_{BC}a_{BCy}$$

$$\sum M_{BCG} : (B_x - F)(0.2 \text{ m}) + (B_y + N)(0.1 \text{ m}) = \frac{1}{12}\, m_{BC}{L_{BC}}^2\alpha_{BC}$$

The kinematic constraints

$$a_{BCy} = \alpha_{AB}L_{AB} + \alpha_{BC}(0.1 \text{ m})$$

$$a_{BCx} = \alpha_{BC}(0.2 \text{ m})$$

$$\alpha_{AB}L_{AB} + \alpha_{BC}(0.2 \text{ m}) = 0$$

Solving we find $\alpha_{AB} = 20.6$ rad/s^2, $\alpha_{BC} = -41.2$ rad/s^2

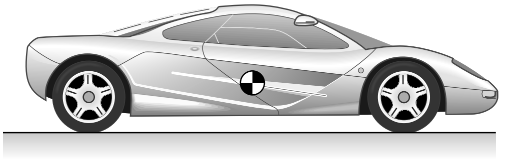
$$\boxed{\alpha_{AB} = 20.6 \text{ rad/s}^2 \quad CCW, \ \alpha_{BC} = 41.2 \text{ rad/s}^2 \quad CW}$$

We also find

$a_{BCx} = -8.23$ m/s^2, $a_{BCy} = 4.12$ m/s^2

$N = 244$ N, $B_x = -51.2$ N, $B_y = 76.5$ N,

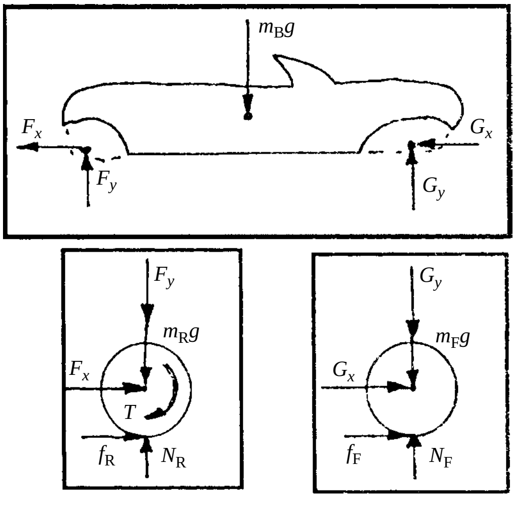
Problem 18.60 Let the total moment of inertia of the car's two rear wheels and axle be I_R, and let the total moment of inertia of the two front wheels be I_F. The radius of the tires is R, and the total mass of the car, including the wheels, is m. If the car's engine exerts a torque (couple) T on the rear wheels and the wheels do not slip, show that the car's acceleration is



$$a = \frac{RT}{R^2 m + I_R + I_F}.$$

Strategy: Isolate the wheels and draw three free-body diagrams.

Solution: The free body diagrams are as shown: We shall write three equations of motion for each wheel and two equations of motion for the body of the car: We shall sum moments about the axles on each wheel.



Rear Wheel:

$$\sum F_x = F_x + f_R = m_R a,$$

$$\sum F_y = N_R - m_R g - F_y = 0,$$

$$\sum M_{\text{Raxle}} = R f_R - T = I_R \alpha = I_R \left(-\frac{a}{R}\right)$$

Front Wheel:

$$\sum F_x = G_x + f_F = m_F a,$$

$$\sum F_y = N_F - m_F g - G_y = 0,$$

$$\sum M_{\text{Faxle}} = R f_F = I_F \alpha = I_F \left(-\frac{a}{R}\right)$$

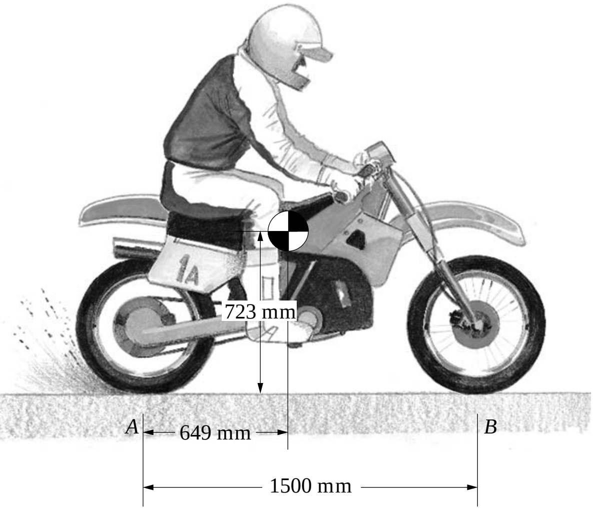
Car Body:

$$\sum F_x = -F_x - G_x = m_B a,$$

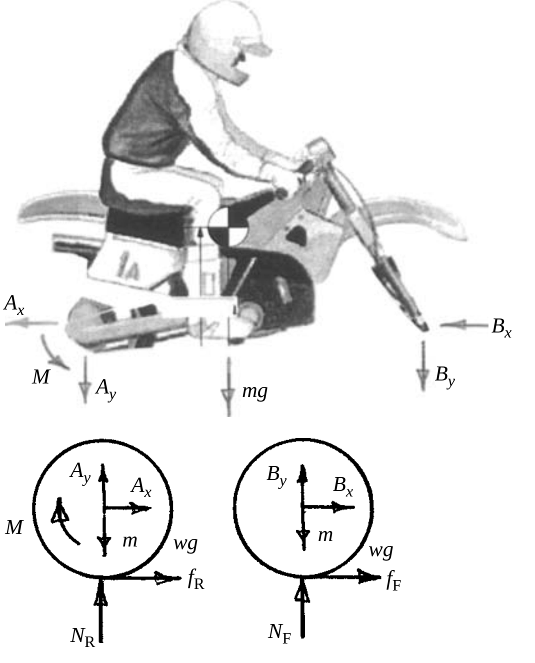
$$\sum F_y = F_y + G_y - m_B g = 0.$$

Summing the y equations for all three bodies, we get $N_R + N_F = (m_B + m_R + m_F)g = mg$. Summing the equations for all three bodies in the x direction, we get $f_R + f_F = (m_B + m_R + m_F)a = ma$. (1) From the moment equations for the wheels, we get $f_F = -I_F a/R^2$ and $f_R = -I_R a/R^2 + T/R$. Substituting these into Eq. (1), we get $a = RT/(mR^2 + I_R + I_F)$ as required.

Problem 18.61 The combined mass of the motorcycle and rider is 160 kg. Each 9-kg wheel has a 330-mm radius and a moment of inertia $I = 0.8$ kg-m^2. The engine drives the rear wheel by exerting a couple on it. If the rear wheel exerts a 400-N horizontal force on the road and you do *not* neglect the horizontal force exerted on the road by the front wheel, determine (a) the motorcycle's acceleration and (b) the normal forces exerted on the road by the rear and front wheels. (The location of the center of mass of the motorcycle *not including* its wheels, is shown.)



Solution: In the free-body diagrams shown, $m_w = 9$ kg and $m = 160 - 18 = 142$ kg. Let a be the motorcycle's acceleration to the right and let α be the wheels' clockwise angular acceleration. Note that



$a = 0.33\alpha$. **(1)**

Front Wheel:

$$\sum F_x = B_x + f_F = m_\omega a, \qquad \textbf{(2)}$$

$$\sum F_y = B_y + N_F - m_\omega g = 0, \qquad \textbf{(3)}$$

$$\sum M = -f_F(0.33) = I_\alpha. \qquad \textbf{(4)}$$

Rear Wheel:

$$\sum F_x = A_x + f_R = m_\omega a, \qquad \textbf{(5)}$$

$$\sum F_y = A_y + N_R - m_\omega g = 0, \qquad \textbf{(6)}$$

$$\sum M = M - f_R(0.33) = I\alpha. \qquad \textbf{(7)}$$

Motorcycle:

$$\sum F_x = -A_x - B_x = ma, \qquad \textbf{(8)}$$

$$\sum F_y = -A_y - B_y - mg = 0, \qquad \textbf{(9)}$$

$$\sum M = -M + (A_x + B_x)(0.723 - 0.33)$$

$$+ B_y(1.5 - 0.649) - A_y(0.649) = 0. \qquad \textbf{(10)}$$

Solving Eqs (1)–(10) with $f_R = 400$ N, we obtain

(a) $a = 2.39$ rad/s^2

and (b) $N_R = 455$ N, $N_F = 1115$ N.

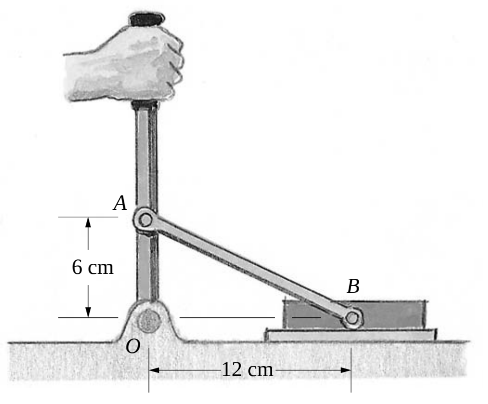
Problem 18.62 In Problem 18.61, if the front wheel lifts slightly off the road when the rider accelerates, determine (a) the motorcycle's acceleration and (b) the torque exerted by the engine on the rear wheel.

Solution: See the solution of Problem 18.61. We set $N_F = 0$ and replace Eq. (4) by $f_F = 0$. Then solving Eqs. (1)–(10), we obtain

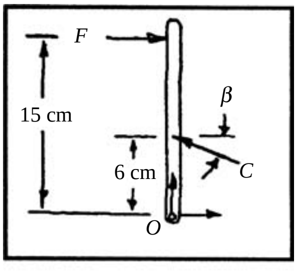
(a) $a = 9.34 \text{ m/s}^2$,

(b) $M = 516$ N-m.

Problem 18.63 The moment of inertia of the vertical handle about O is 0.16 kg-m^2. The object B weighs 66.7 N and rests on a smooth surface. The weight of the bar AB is negligible (which means that you can treat the bar as a two-force member). If the person exerts a 0.89 N horizontal force on the handle 15 cm above O, what is the resulting angular acceleration of the handle?



Solution: Let α be the clockwise angular acceleration of the handle. The acceleration of B is:

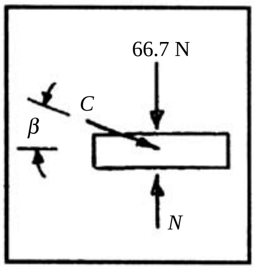


$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A}$:

$$a_B \mathbf{i} = (6/12)\alpha \mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 1 & -0.5 & 0 \end{vmatrix}$$

we see that $\alpha_{AB} = 0$ and

$a_B = (6/12)\alpha$ **(1)**.

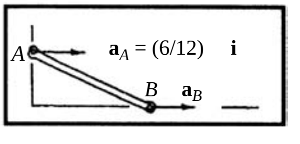


The free body diagrams of the handle and object B are as shown. Note that $\beta = \arctan(6/12) = 26.6^\circ$. Newton's second law for the object B is

$C \cos\beta = (0.15/9.81)a_B$, **(2)**

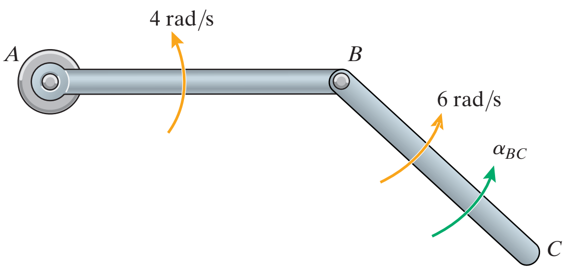
The equation of angular motion for the handle is

$(15/12)F - (6/12)C\cos\beta = (0.16)\alpha$ **(3)**.



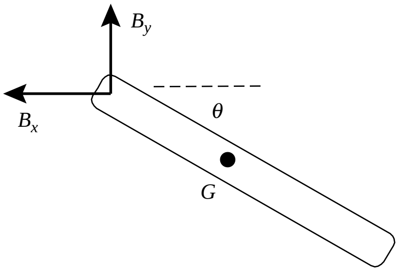
Solving Equations (1)–(3) with $F = 0.89$ N, we obtain $\alpha = 6.8 \text{ rad/s}^2$

Problem 18.64 The bars are each 1 m in length and have a mass of 2 kg. They rotate *in the horizontal plane*. Bar AB rotates with a constant angular velocity of 4 rad/s in the counterclockwise direction. At the instant shown, bar BC is rotating in the counterclockwise direction at 6 rad/s. What is the angular acceleration of bar BC?



Solution: Given $m = 2$ kg, $L = 1$ m, $\theta = 45^\circ$

The FBD



The kinematics

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}{}^2\mathbf{r}_{B/A}$$

$$= 0 + 0 - (4 \text{ rad/s})^2(1 \text{ m})\mathbf{i} = -(16 \text{ m/s}^2)\mathbf{i}$$

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G/B} - \omega_{BC}{}^2\mathbf{r}_{G/B}$$

$$= -(16 \text{ m/s}^2)\mathbf{i} + \alpha_{BC}\mathbf{k} \times (0.5 \text{ m})(\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$$

$$- (6 \text{ rad/s})^2(0.5 \text{ m})(\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$$

$$= (-16 \text{ m/s}^2 + [0.5 \text{ m}\sin\theta]\alpha_{BC} - [18 \text{ m/s}^2]\cos\theta)\mathbf{i}$$

$$+ ([0.5 \text{ m}\cos\theta]\alpha_{BC} + [18 \text{ m/s}^2]\sin\theta)\mathbf{j}$$

Our kinematic constraints are

$$a_x = -16 \text{ m/s}^2 + [0.5 \text{ m}\sin\theta]\alpha_{BC} - [18 \text{ m/s}^2]\cos\theta$$

$$a_y = [0.5 \text{ m}\cos\theta]\alpha_{BC} + [18 \text{ m/s}^2]\sin\theta$$

The dynamic equations

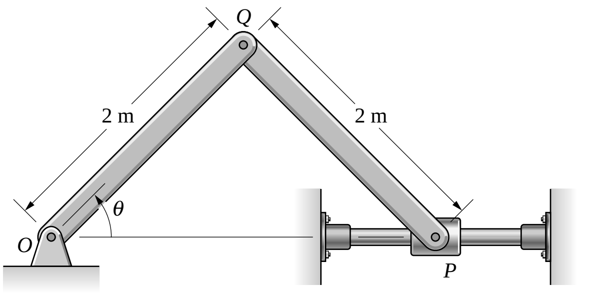
$$\sum F_x : -B_x = ma_x$$

$$\sum F_y : B_y = ma_y$$

$$\sum M_G : B_x(0.5 \text{ m})\sin\theta - B_y(0.5 \text{ m})\cos\theta = \frac{1}{12}m(1.0 \text{ m})^2\alpha_{BC}$$

Solving we find $\boxed{\alpha_{BC} = 17.0 \text{ rad/s}^2 \quad CCW}$

Problem 18.65 Bars OQ and PQ each weigh 6 N. The weight of the collar P and friction between the collar and the horizontal bar are negligible. If the system is released from rest with $\theta = 45°$, what are the angular accelerations of the two bars?



Solution: Let α_{OQ} and α_{PQ} be the clockwise angular acceleration of bar OQ and the counterclockwise angular acceleration of bar PQ. The acceleration of Q is

$$\mathbf{a}_Q = \mathbf{a}_0 + \boldsymbol{\alpha}_{0Q} \times \mathbf{r}_{Q/0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\alpha_{OQ} \\ 2\cos 45° & 2\sin 45° & 0 \end{vmatrix}$$

$$= 2\alpha_{OQ}\sin 45°\mathbf{i} - 2\alpha_{OQ}\cos 45°\mathbf{j}.$$

The acceleration of P is

$$\mathbf{a}_P = \mathbf{a}_Q + \boldsymbol{\alpha}_{PQ} \times \mathbf{r}_{P/Q}$$

$$a_P\mathbf{i} = 2\alpha_{OQ}\sin 45°\mathbf{i} - 2\alpha_{OQ}\cos 45°\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{PQ} \\ 2\cos 45° & 2\sin 45° & 0 \end{vmatrix}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components,

$$a_P = 2\alpha_{OQ}\sin 45° - 2\alpha_{PQ}\sin 45° \quad \textbf{(1)}$$

$$0 = -2\alpha_{OQ}\cos 45° + 2\alpha_{PQ}\cos 45° \quad \textbf{(2)}.$$

The acceleration of the center of mass of bar PQ is

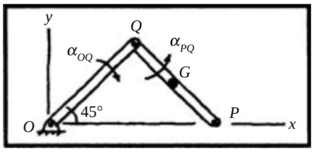
$$\mathbf{a}_G = \mathbf{a}_Q + \boldsymbol{\alpha}_{PQ} \times \mathbf{r}_{G/Q} = 2\alpha_{OQ}\sin 45°\mathbf{i}$$

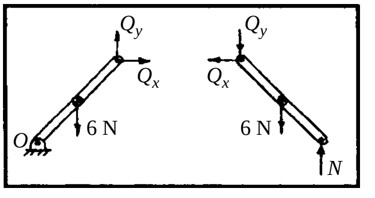
$$- 2\alpha_{OQ}\cos 45°\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{PQ} \\ \cos 45° & -\sin 45° & 0 \end{vmatrix}.$$

Hence,

$$a_{Gx} = 2\alpha_{OQ}\sin 45° + \alpha_{PQ}\sin 45° \quad \textbf{(3)};$$

$$a_{Gy} = -2\alpha_{OQ}\cos 45° + \alpha_{PQ}\cos 45° \quad \textbf{(4)}.$$





From the diagrams:

The equation of angular motion of bar OQ is $\sum M_0 = I_0\alpha_{OQ}$:

$$Q_x(2\sin 45°) - Q_y(2\cos 45°) + 6\cos 45° = \tfrac{1}{3}(6/9.81)(2)^2\alpha_{OQ} \quad \textbf{(5)}.$$

The equations of motion of bar PQ are

$$\sum F_x = -Q_x = (6/9.81)a_{Gx} \quad \textbf{(6)}$$

$$\sum F_y = N - Q_y - 6 = (6/9.81)a_{Gy} \quad \textbf{(7)}$$

$$\sum M = (N + Q_y + Q_x)(\cos 45°) = \frac{1}{12}(6/9.81)(2)^2\alpha_{PQ} \quad \textbf{(8)}.$$

Solving Equations (1)–(8), we obtain $\alpha_{OQ} = \alpha_{PQ} = 6.83\ \text{rad/s}^2$

Problem 18.66 In Problem 18.65, what are the angular accelerations of the two bars if the collar P weighs 2 N?

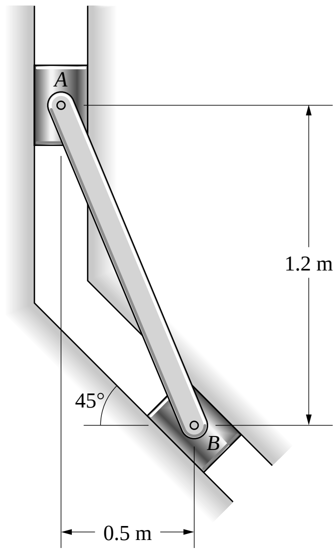
Solution: In the solution of Problem 18.65, the free body diagram of bar PQ has a horizontal component P to the left where P is the force exerted on the bar by the collar. Equations (6) and (8) become

$$\sum F_x = -Q_x - P = (6/9.81)a_{Gx}$$

$$\sum M = (N - P + Q_y + Q_x)(\cos 45^\circ) = \frac{1}{12}(6/9.81)(2)^2\alpha_{PQ}$$

and the equation of motion for the collar is $P = (2/9.81)a_P$ solving equations (1–9), we obtain $\alpha_{OQ} = \alpha_{PQ} = 4.88\ \text{rad/s}^2$.

Problem 18.67 The 4-kg slender bar is pinned to 2-kg sliders at A and B. If friction is negligible and the system is released from rest in the position shown, what is the angular acceleration of the bar at that instant?



Solution: Express the acceleration of B in terms of the acceleration of A, $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A}$:

$$a_B \cos 45^\circ \mathbf{i} - a_B \sin 45^\circ \mathbf{j} = -a_A \mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 0.5 & -1.2 & 0 \end{vmatrix},$$

or $\quad a_B \cos 45^\circ = 1.2\alpha_{AB}, \qquad$ **(1)**;

and $\quad -a_B \sin 45^\circ = -a_A + 0.5\alpha_{AB}, \qquad$ **(2)**.

We express the acceleration of G in terms of the acceleration of A, $\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{G/A}$:

$$\mathbf{a}_G = a_{Gx}\mathbf{i} + a_{Gy}\mathbf{j} = -a_A\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 0.25 & -0.6 & 0 \end{vmatrix},$$

or $\quad a_{Gx} = 0.6\alpha_{AB}, \qquad$ **(3)**;

and $\quad a_{Gy} = -a_A + 0.25\alpha_{AB}, \qquad$ **(4)**;

The free body diagrams are as shown. The equations of motion are Slider A:

$N - A_x = 0 \qquad$ **(5),**

and $\quad (2)(9.81) + A_y = 2a_A, \qquad$ **(6)**;

Slider B : $\quad P - [B_x + B_y + (2)(9.81)]\cos 45^\circ = 0, \qquad$ **(7)**;

and $\quad [(2)(9.81) - B_x + B_y]\cos 45^\circ = 2a_B, \qquad$ **(8)**;

Bar: $\quad A_x + B_k = 4a_{Gx} \qquad$ **(9)**;

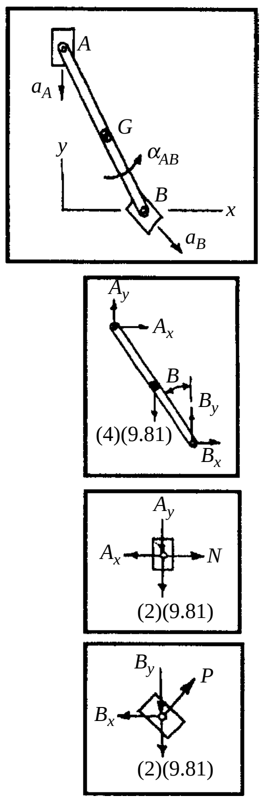
and $\quad A_y + B_y - (4)(9.81) = 4a_{Gy} \qquad$ **(10)**;

$(L/2)[(B_x - A_x)\cos\beta + (B_y - A_y)\sin\beta] = \frac{1}{12}(4)L^2\alpha_{AB} \qquad$ **(11),**

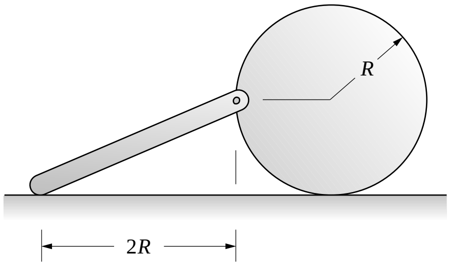
where $\quad L = \sqrt{(0.5)^2 + (1.2)^2}$ m

and $\quad \beta \arctan(0.5/1.2) = 22.6^\circ$.

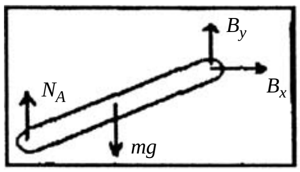
Solving Equations (1)–(11), we obtain $\alpha_{AB} = 5.18\ \text{rad/s}^2$.

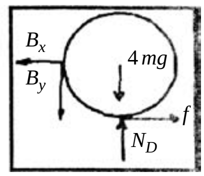


Problem 18.68 The mass of the slender bar is m and the mass of the homogeneous disk is $4m$. The system is released form rest in the position shown. If the disk rolls and the friction between the bar and the horizontal surface is negligible, show that the disk's angular acceleration is $\alpha = 6g/95R$ counterclockwise.



Solution: *For the bar*: The length of the bar is $L = \sqrt{5}R$. Apply Newton's second law to the free body diagram of the bar: $B_x = ma_{Gx}$, $B_y + N_A - mg = ma_{Gy}$, where a_{Gx}, a_{Gy} are the accelerations of the center of mass of the bar. The moment about the bar center of mass is





$$RB_y - RN_A - \frac{R}{2}B_x = I_B\alpha_{AB}.$$

For the disk: Apply Newton's second law and the equation of angular motion to the free body diagram of the disk. $f - B_x = 4ma_{Dx}$, $N_D - 4mg - B_y = 0$, $RB_y + Rf = I_D\alpha_D$

From kinematics: Since the system is released from rest, $\omega_{AB} = \omega_D = 0$. The acceleration of the center of the disk is $\mathbf{a}_D = -R\alpha_D\mathbf{i}$. The acceleration of point B in terms of the acceleration of the center of the disk is

$$\mathbf{a}_B = \mathbf{a}_D + \boldsymbol{\alpha}_D \times \mathbf{r}_{B/D} = \mathbf{a}_D + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_D \\ -R & 0 & 0 \end{bmatrix} = -R\alpha_D\mathbf{i} - R\alpha_D\mathbf{j}.$$

The acceleration of the center of mass of the bar in terms of the acceleration of B is

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{G/B} - \omega_{AB}^2\mathbf{r}_{G/B} = \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ -R & -\dfrac{R}{2} & 0 \end{bmatrix}$$

$$= \mathbf{a}_B + \frac{R\alpha_{AB}}{2}\mathbf{i} - R\alpha_{AB}\mathbf{j},$$

$$\mathbf{a}_G = -R\left(\alpha_D - \frac{\alpha_{AB}}{2}\right)\mathbf{i} - R(\alpha_D + \alpha_{AB})\mathbf{j}.$$

The acceleration of the center of mass of the bar in terms of the acceleration of A is

$$\mathbf{a}_G = \mathbf{a}_A + \mathbf{a}_{AB} \times \mathbf{r}_{G/A} = \mathbf{a}_A + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ \mathrm{R} & \dfrac{\mathrm{R}}{2} & 0 \end{bmatrix}$$

$$= \mathbf{a}_A - \frac{\mathrm{R}\alpha_{AB}}{2}\mathbf{i} + \mathrm{R}\alpha_{AB}\mathbf{j}.$$

From the constraint on the motion, $\mathbf{a}_A = a_A\mathbf{i}$. Equate the expressions for $\mathbf{a}_G$, separate components and solve: $\alpha_{AB} = -\dfrac{\alpha_D}{2}$. Substitute to obtain $a_{Gx} = -\dfrac{5R}{4}\alpha_D$, $a_{Gy} = -\dfrac{R}{2}\alpha_D$. Collect the results:

(1) $B_x = -\dfrac{5Rm}{4}\alpha_D,$

(2) $B_y + N_A - mg = -\dfrac{Rm}{2}\alpha_D,$

(3) $RB_y - RN_A - \dfrac{R}{2}B_x = -\dfrac{I_B}{2}\alpha_D,$

(4) $f - B_x = -4Rm\alpha_D,$

(5) $N_D - 4mg - B_y = 0,$

(6) $RB_y + Rf = I_D\alpha_D.$

From (1), (2), and (3)

$$B_y = \frac{mg}{2} - \left(\frac{9mR}{16} + \frac{I_B}{4R}\right)\alpha_D.$$

From (1), (4) and (6),

$$B_y = \left(\frac{I_D}{R} + \frac{21Rm}{4}\right)\alpha_D.$$

Equate the expressions for B_y and reduce to obtain

$$\alpha_D = \left(\frac{mg}{2}\right)\frac{1}{\left(\dfrac{93Rm}{16} + \dfrac{I_D}{R} + \dfrac{I_B}{4R}\right)}.$$

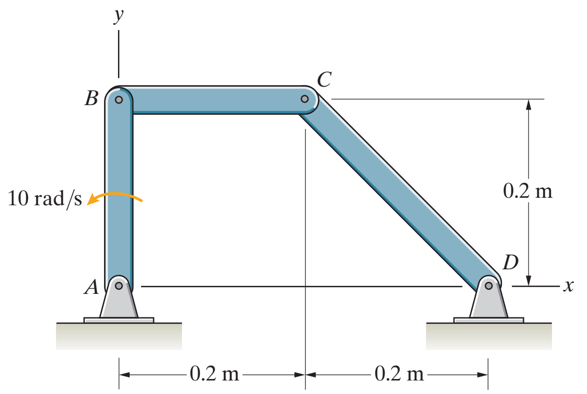
For a homogenous cylinder of mass $4m$, $I_D = 2R^2m$. For a slender bar of mass m about the center of mass,

$$I_B = \frac{1}{12}mL^2 = \frac{5}{12}mR^2.$$

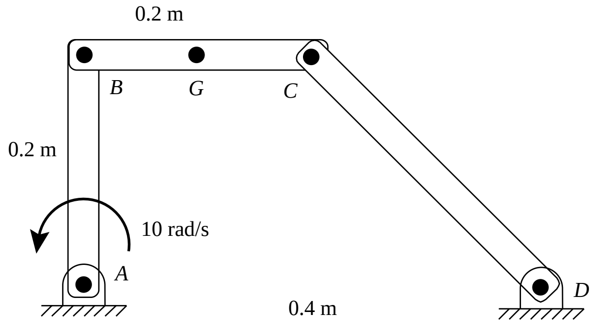
Substitute and reduce:

$$\boxed{a_D = \frac{6g}{95R}}.$$

Problem 18.69 Bar AB rotates *in the horizontal plane* with a constant angular velocity of 10 rad/s in the counterclockwise direction. The masses of the slender bars BC and CD are 3 kg and 4.5 kg, respectively. Determine the x and y components of the forces exerted on bar BC by the pins at B and C at the instant shown.



Solution: First let's do the kinematics



Velocity

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$$

$$= 0 + (10 \text{ rad/s})\mathbf{k} \times (0.2 \text{ m})\mathbf{j}$$

$$= -(2 \text{ m/s})\mathbf{i}$$

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$$

$$= -(2 \text{ m/s})\mathbf{i} + \omega_{BC}\mathbf{k} \times (0.2 \text{ m})\mathbf{i} = -(2 \text{ m/s})\mathbf{i} + (0.2 \text{ m})\omega_{BC}\mathbf{j}$$

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega}_{CD} \times \mathbf{r}_{D/C}$$

$$= -(2 \text{ m/s})\mathbf{i} + (0.2 \text{ m})\omega_{BC}\mathbf{j} + \omega_{CD}\mathbf{k} \times (0.2 \text{ m})(\mathbf{i} - \mathbf{j})$$

$$= (-[2 \text{ m/s}] + [0.2 \text{ m}]\omega_{CD})\mathbf{i} + (0.2 \text{ m})(\omega_{BC} + \omega_{CD})\mathbf{j}$$

Since D is pinned we find $\omega_{CD} = 10$ rad/s, $\omega_{BC} = -10$ rad/s

Acceleration

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - {\omega_{AB}}^2\mathbf{r}_{B/A}$$

$$= 0 + 0 - (10 \text{ rad/s})^2(0.2 \text{ m})\mathbf{j} = -(20 \text{ m/s}^2)\mathbf{j}$$

$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - {\omega_{BC}}^2\mathbf{r}_{C/B}$$

$$= -(20 \text{ m/s}^2)\mathbf{j} + \alpha_{BC}\mathbf{k} \times (0.2 \text{ m})\mathbf{i} - (-10 \text{ rad/s})^2(0.2 \text{ m})\mathbf{i}$$

$$= -(20 \text{ m/s}^2)\mathbf{i} + ([0.2 \text{ m}]\alpha_{BC} - 20 \text{ m/s}^2)\mathbf{j}$$

$$\mathbf{a}_D = \mathbf{a}_C + \boldsymbol{\alpha}_{CD} \times \mathbf{r}_{D/C} - {\omega_{CD}}^2\mathbf{r}_{D/C}$$

$$= -(20 \text{ m/s}^2)\mathbf{i} + ([0.2 \text{ m}]\alpha_{BC} - 20 \text{ m/s}^2)\mathbf{j} + \alpha_{CD}\mathbf{k}$$

$$\times (0.2 \text{ m})(\mathbf{i} - \mathbf{j}) - (10 \text{ rad/s})^2(0.2 \text{ m})(\mathbf{i} - \mathbf{j})$$

$$= (-40 \text{ m/s}^2 + [0.2 \text{ m}]\alpha_{CD})\mathbf{i} + ([0.2 \text{ m}][\alpha_{BC} + \alpha_{BC}])\mathbf{j}$$

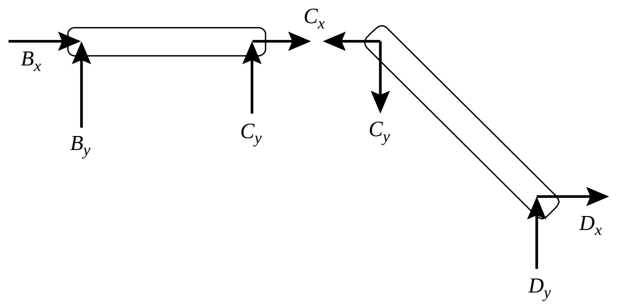
Since D is pinned we find $\alpha_{BC} = -200$ rad/s^2, $\alpha_{CD} = 200$ rad/s^2

Now find the accelerations of the center of mass G.

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{G1/B} - {\omega_{BC}}^2\mathbf{r}_{G1/B}$$

$$= -(20 \text{ m/s}^2)\mathbf{j} + (-200 \text{ rad/s}^2)\mathbf{k} \times (0.1 \text{ m})\mathbf{i} - (-10 \text{ rad/s})^2(0.1 \text{ m})\mathbf{i}$$

$$= (-10\mathbf{i} - 40\mathbf{j}) \text{ m/s}^2$$



The FBDs

The dynamics

$$\sum F_{BCx} : B_x + C_x = (3 \text{ kg})(-10 \text{ m/s}^2)$$

$$\sum F_{BCy} : B_y + C_y = (3 \text{ kg})(-40 \text{ m/s}^2)$$

$$\sum M_{G1} : (C_y - B_y)(0.1 \text{ m}) = \frac{1}{12}(3 \text{ kg})(0.2 \text{ m})^2(-200 \text{ rad/s}^2)$$

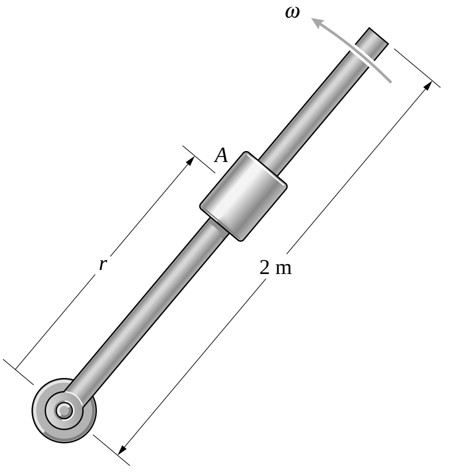
$$\sum M_D : C_x(0.2 \text{ m}) + C_y(0.2 \text{ m}) = \frac{1}{3}(4.5 \text{ kg})(\sqrt{2}[0.2 \text{ m}])^2(200 \text{ rad/s}^2)$$

Solving we find

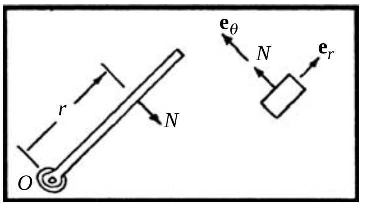
$$\boxed{\begin{array}{l} B_x = -220 \text{ N}, \; B_y = -50 \text{ N} \\ C_x = 190 \text{ N}, \; C_y = -70 \text{ N} \end{array}}$$

Problem 18.70 The 2-kg bar rotates *in the horizontal plane* about the smooth pin. The 6-kg collar A slides on the smooth bar. At the instant shown, $r = 1.2$ m, $\omega = 0.4$ rad/s, and the collar is sliding outward at 0.5 m/s relative to the bar. If you neglect the moment of inertia of the collar (that is, treat the collar as a particle), what is the bar's angular acceleration?

Strategy: Draw individual free-body diagrams of the bar and collar and write Newton's second law for the collar in terms of polar coordinates.



Solution: Diagrams of the bar and collar showing the force they exert on each other in the horizontal plane are: the bar's equation of angular motion is



$$\sum M_0 = I_0\alpha: \quad -Nr = \tfrac{1}{3}(2)(2)^2\alpha \quad \textbf{(1)}$$

In polar coordinates, Newton's second law for the collar is

$$\sum \mathbf{F} = m\mathbf{a}: \quad N\mathbf{e}_\theta = m\left[\left(\frac{d^2r}{dt^2} - r\omega^2\right)\mathbf{e}_r + \left(r\alpha + 2\frac{dr}{dt}\omega\right)\mathbf{e}_\theta\right].$$

Equating $\mathbf{e}_\theta$ components,

$$N = m\left(r\alpha + 2\frac{dr}{dt}\omega\right) = (6)[r\alpha + 2(0.5)(0.4)] \quad \textbf{(2)}.$$

Solving Equations (1) and (2) with $r = 1.2$ m gives $\alpha = -0.255$ rad/s^2

Problem 18.71 In Problem 18.70, the moment of inertia of the collar about its center of mass is 0.2 kg-m^2. Determine the angular acceleration of the bar, and compare your answer with the answer to Problem 18.70.

Solution: Let C be the couple the collar and bar exert on each other: The bar's equation of angular motion is

$$\sum M_0 = I_0\alpha: \quad -Nr - C = \tfrac{1}{3}(2)(2)^2\alpha \quad \textbf{(1)}.$$

The collar's equation of angular motion is

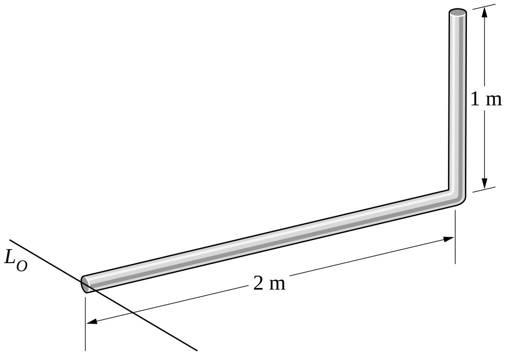
$$\sum M = I\alpha: \quad C = 0.2\alpha \quad \textbf{(2)}.$$

From the solution of Problem 18.70, the $\mathbf{e}_\theta$ component of Newton's second law for the collar is

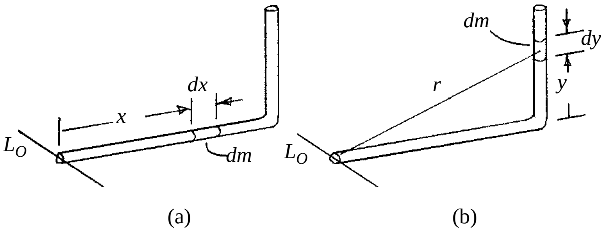
$$N = (6)[r\alpha + 2(0.5)(0.4)] \quad \textbf{(3)}$$

Solving Equations (1)–(3) with $r = 1.2$ m gives $\alpha = -0.250$ rad/s^2.

Problem 18.72 The axis L_0 is perpendicular to both segments of the L-shaped slender bar. The mass of the bar is 6 kg and the material is homogeneous. Use integration to determine the moment of inertia of the bar about L_0.



Solution: Let A be the bar's cross-sectional area. The bar's mass is $m = 6 \text{ kg} = \rho A(3 \text{ m})$, so $\rho A = 2$ kg/m.



For the horizontal part (Fig. a),

$$I_h = \int_m x^2\, dm = \int_0^2 x^2 \rho A\, dx = \frac{8}{3}\rho A = \frac{16}{3} \text{ kg-m}^2.$$

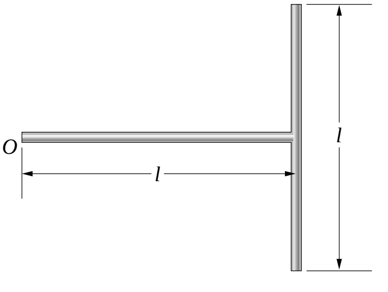
For the vertical part (Fig. b),

$$I_v = \int_m r^2\, dm = \int_0^1 (2^2 + y^2)\rho A\, dy$$

$$= \frac{13}{3}\rho A = \frac{26}{3} \text{ kg-m}^2.$$

Therefore $I_0 = I_h + I_v = 14 \text{ kg-m}^2$.

Problem 18.73 Two homogenous slender bars, each of mass m and length l, are welded together to form the T-shaped object. Use integration to determine the moment of inertia of the object about the axis through point O that is perpendicular to the bars.



Solution: Divide the object into two pieces, each corresponding to a slender bar of mass m; the first parallel to the y-axis, the second to the x-axis. By definition

$$I = \int_0^l r^2\, dm + \int_m r^2\, dm.$$

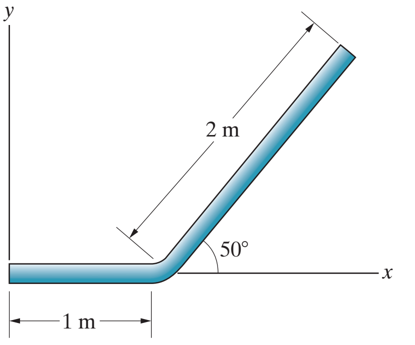
For the first bar, the differential mass is $dm = \rho A\, dr$. Assume that the second bar is very slender, so that the mass is concentrated at a distance l from O. Thus $dm = \rho A\, dx$, where x lies between the limits $-\frac{l}{2} \le x \le \frac{l}{2}$. The distance to a differential dx is $r = \sqrt{l^2 + x^2}$. Thus the definition becomes

$$I = \rho A \int_0^l r^2\, dr + \rho A \int_{-\frac{l}{2}}^{\frac{1}{2}} (l^2 + x^2)\, dx$$

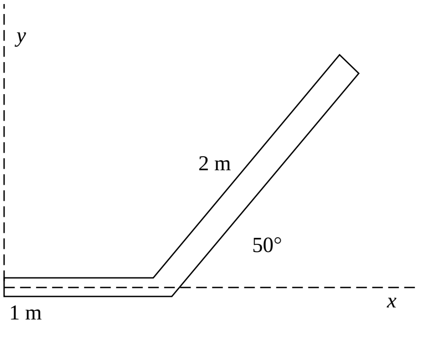
$$I = \rho A \left[\frac{r^3}{3}\right]_0^l + \rho A \left[l^2 x + \frac{x^3}{3}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= ml^2 \left(\frac{1}{3} + 1 + \frac{1}{12}\right) = \frac{17}{12}\, ml^2$$

Problem 18.74 The slender bar lies in the $x-y$ plane. Its mass is 6 kg and the material is homogeneous. Use integration to determine its moment of inertia about the z axis.



Solution: The density is $\rho = \dfrac{6\text{ kg}}{3\text{ m}} = 2\text{ kg/m}$



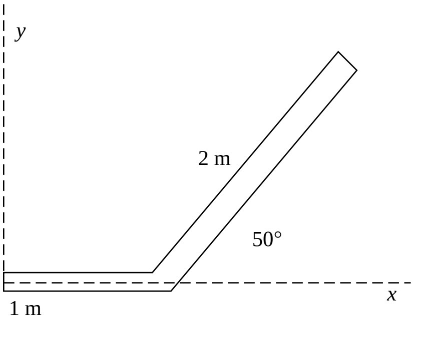
$$I_z = \int_0^{1\text{ m}} \rho x^2 dx$$

$$+ \int_0^{2\text{ m}} \rho[(1\text{ m} + s\cos 50^\circ)^2 + (s\sin 50^\circ)^2]\,ds$$

$$\boxed{I_z = 15.1\text{ kg-m}^2}$$

Problem 18.75 The slender bar lies in the $x-y$ plane. Its mass is 6 kg and the material is homogeneous. Use integration to determine its moment of inertia about the y axis.

Solution: The density is $\rho = \dfrac{6\text{ kg}}{3\text{ m}} = 2\text{ kg/m}$

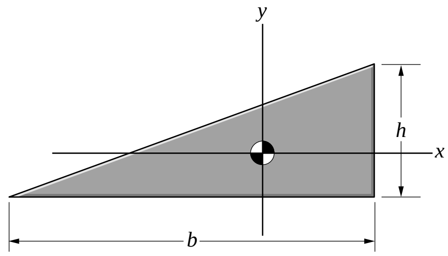


$$I_y = \int_0^{1\text{ m}} \rho x^2 dx$$

$$+ \int_0^{2\text{ m}} \rho[(1\text{ m} + s\cos 50^\circ)^2]\,ds$$

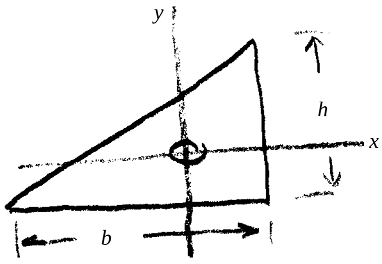
$$\boxed{I_y = 12.0\text{ kg-m}^2}$$

Problem 18.76 The homogeneous thin plate has mass $m = 12$ kg and dimensions $b = 1$ m and $h = 2$ m. Determine the mass moments of inertia of the plate about the x, y, and z axes.



Strategy: The mass moments of inertia of a thin plate of arbitrary shape are given by Eqs. (18.37)–(18.39) in terms of the moments of inertia of the cross-sectional area of the plate. You can obtain the moments of inertia of the triangular area from Appendix B.

Solution:



$m = 12$ kg

$\text{Area} = \frac{1}{2}bh$

$\rho = \text{mass/Area}$

$dm = \rho dA$

From Appendix B,

$$I_{x_A} = \frac{1}{36}bh^3 \qquad I_{y_A} = \frac{1}{36}hb^3$$

$$\text{Area} = \frac{1}{2}(1)(2) = 1 \text{ m}^2$$

$$\rho = 12 \text{ kg/m}^2$$

$$I_x = \int \rho y^2\, dA = \rho \int y^2\, dA$$

$$I_x = \rho I_{x_A},\ I_y = \rho I_{y_A}$$

$$\underline{I_x = 12\left(\frac{1}{36}\right)(1)(2)^3 = 2.667 \text{ kg-m}^2}$$

$$I_y = \rho I_{y_A} = \frac{12}{36}h \quad b^3 = \frac{1}{3}(2)(1)^3$$

$$\underline{I_y = 0.667 \text{ kg-m}^2}$$

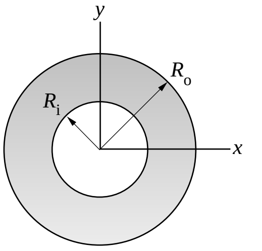
$$I_z = I_x + I_y$$

$$I_z = 2.667 + 0.667 \text{ kg-m}^2$$

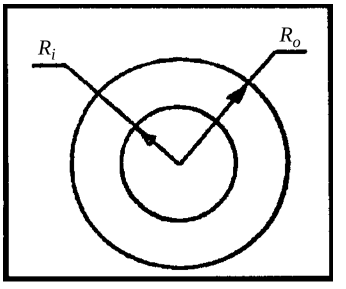
$$\underline{I_z = 3.333 \text{ kg-m}^2}$$

Problem 18.77 The brass washer is of uniform thickness and mass m.

(a) Determine its moments of inertia about the x and z axes.
(b) Let $R_i = 0$, and compare your results with the values given in Appendix C for a thin circular plate.



Solution:



(a) The area moments of inertia for a circular area are

$$I_x = I_y = \frac{\pi R^4}{4}.$$

For the plate with a circular cutout,

$$I_x = \frac{\pi}{4}(R_o^4 - R_i^4)$$

The area mass density is $\frac{m}{A}$, thus for the plate with a circular cut,

$$\frac{m}{A} = \frac{m}{\pi(R_o^2 - R_i^2)},$$

from which the moments of inertia

$$I_{(x\text{-axis})} = \frac{m(R_o^4 - R_i^4)}{4(R_o^2 - R_i^2)} = \frac{m}{4}(R_o^2 + R_i^2)$$

$$I_{(z\text{-axis})} = 2I_{(x\text{-axis})} = \frac{m}{2}(R_o^2 + R_i^2).$$

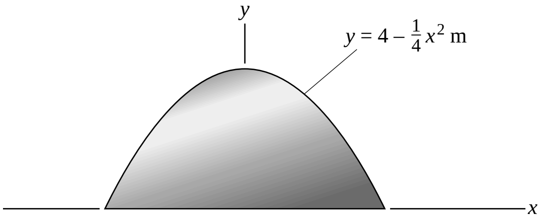
(b) Let $R_i = 0$, to obtain

$$I_{x\text{-axis}} = \frac{m}{4}R_o^2,$$

$$I_{(z\text{-axis})} = \frac{m}{2}R_o^2,$$

which agrees with table entries.

Problem 18.78 The homogenous thin plate is of uniform thickness and weighs 20 N. Determine its moment of inertia about the y axis.



Solution: The definition of the moment of inertia is

$$I = \int_m r^2\, dm.$$

The distance from the y-axis is x, where x varies over the range $-4 \le x \le 4$. Let $\tau = \frac{m}{A} = \frac{W}{gA}$ be the area mass density. The mass of an element $y\,dx$ is $dm = \frac{W}{gA} y\,dx$. Substitute into the definition:

$$I_{y\text{-axis}} = \frac{W}{gA}\int_{-4}^{4} x^2\left(4 - \frac{x^2}{4}\right) dx$$

$$= \frac{W}{gA}\left[\frac{4x^3}{3} - \frac{x^5}{20}\right]_{-4}^{+4} = \frac{W}{gA}[68.2667].$$

The area is

$$A = \int_{-4}^{4}\left(4 - \frac{x^2}{4}\right) dx = \left[4x - \frac{x^3}{12}\right]_{-4}^{4} = 21.333 \text{ m}^2$$

The moment of inertia about the y-axis is

$$I_{(y\text{-axis})} = \frac{W}{g}(3.2) = \frac{20}{9.81}(3.2) = 6.52 \text{ kg-m}^2.$$

Problem 18.79 Determine the moment of inertia of the plate in Problem 18.78 about the x axis.

Solution: The differential mass is $dm = \frac{W}{gA} dy\,dx$. The distance of a mass element from the x-axis is y, thus

$$I = \frac{W}{gA}\int_{-4}^{+4} dx \int_0^{4-\frac{x^2}{4}} y^2\, dy$$

$$= \frac{W}{3gA}\int_{-4}^{+4}\left(4 - \frac{x^2}{4}\right)^3 dx$$

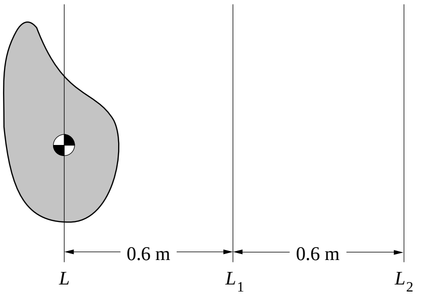
$$= \frac{W}{3gA}\left[64x - 4x^3 + \frac{3}{20}x^5 - \frac{x^7}{448}\right]_{-4}^{4}$$

$$= \frac{W}{3gA}[234.057].$$

From the solution to Problem 18.78, $A = 21.333$ ft^2. Thus the moment of inertia about the x-axis is

$$I_{x\text{-axis}} = \frac{W}{3g}\frac{(234.057)}{(21.333)} = \frac{W}{g}(3.657) = 7.46 \text{ kg-m}^2.$$

Problem 18.80 The mass of the object is 10 kg. Its moment of inertia about L_1 is 10 kg-m^2. What is its moment of inertia about L_2? (The three axes are in the same plane.)



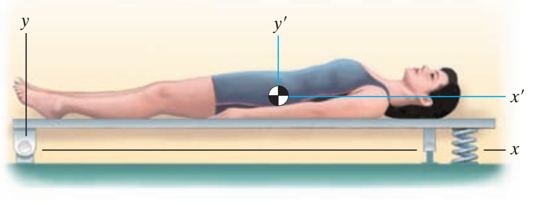
Solution: The strategy is to use the data to find the moment of inertia about L, from which the moment of inertia about L_2 can be determined.

$$\boxed{I_L = -(0.6)^2(10) + 10 = 6.4 \text{ kg-m}^2},$$

from which

$$\boxed{I_{L2} = (1.2)^2(10) + 6.4 = 20.8 \text{ kg-m}^2}$$

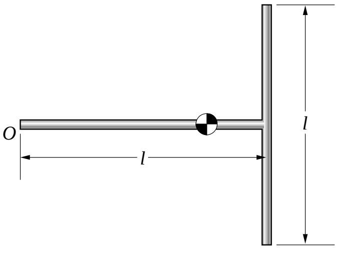
Problem 18.81 An engineer gathering data for the design of a maneuvering unit determines that the astronaut's center of mass is at $x = 1.01$ m, $y = 0.16$ m and that her moment of inertia about the z axis is 105.6 kg-m^2. The astronaut's mass is 81.6 kg. What is her moment of inertia about the z' axis through her center of mass?



Solution: The distance from the z' axis to the z axis is $d = \sqrt{x^2 + y^2} = 1.02257$ m. The moment of inertia about the z' axis is

$$\boxed{\begin{aligned} I_{z'\text{-axis}} &= -d^2 m + I_{z\text{-axis}} \\ &= -(1.0457)(81.6) + 105.6 = 20.27 \text{ kg-m}^2 \end{aligned}}$$

Problem 18.82 Two homogenous slender bars, each of mass m and length l, are welded together to form the T-shaped object. Use the parallel-axis theorem to determine the moment of inertia of the object about the axis through point O that is perpendicular to the bars.



Solution: Divide the object into two pieces, each corresponding to a bar of mass m. By definition $I = \int_0^l r^2\,dm$. For the first bar, the differential mass is $dm = \rho A dr$, from which the moment of inertia about one end is

$$I_1 = \rho A \int_0^l r^2\,dr = \rho A \left[\frac{r^3}{3}\right]_0^l = \frac{ml^2}{3}.$$

For the second bar

$$I_2 = \rho A \int_{-\frac{l}{2}}^{\frac{l}{2}} r^2\,dr = \rho A \left[\frac{r^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}} = \frac{ml^2}{12}$$

is the moment of inertia about the center of the bar. From the parallel axis theorem, the moment of inertia about O is

$$\boxed{I_0 = \frac{ml^2}{3} + l^2 m + \frac{ml^2}{12} = \frac{17}{12} ml^2}$$

Problem 18.83 Use the parallel-axis theorem to determine the moment of inertia of the T-shaped object in Problem 18.98 about the axis through the center of mass of the object that is perpendicular to the two bars.

Solution: The location of the center of mass of the object is

$$\mathbf{x} = \frac{m\left(\frac{l}{2}\right) + lm}{2m} = \frac{3}{4}l.$$

Use the results of Problem 18.98 for the moment of inertia of a bar about its center. For the first bar,

$$I_1 = \left(\frac{l}{4}\right)^2 m + \frac{ml^2}{12} = \frac{7}{48}ml^2.$$

For the second bar,

$$I_2 = \left(\frac{l}{4}\right)^2 m + \frac{ml^2}{12} = \frac{7}{48}ml^2.$$

The composite:

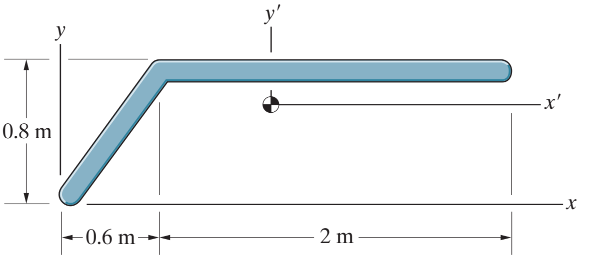
$$\boxed{I_c = I_1 + I_2 = \frac{7}{24}ml^2}$$

Check: Use the results of Problem 18.98:

$$I_c = -\left(\frac{3l}{4}\right)^2 (2m) + \frac{17}{12}ml^2$$

$$= \left(\frac{-9}{8} + \frac{17}{12}\right) ml^2 = \frac{7}{24}ml^2. \quad \textit{check}.$$

Problem 18.84 The mass of the homogeneous slender bar is 30 kg. Determine its moment of inertia about the z axis.



Solution: The density is $\rho = \dfrac{30 \text{ kg}}{3 \text{ m}} = 10 \text{ kg/m}$

$$I_z = \frac{1}{3}(10 \text{ kg})(1.0 \text{ m})^2 + \frac{1}{12}(20 \text{ kg})(2 \text{ m})^2$$

$$+ (20 \text{ kg})[(1.6 \text{ m})^2 + (0.8 \text{ m})^2]$$

$$\boxed{I_z = 74 \text{ kg-m}^2}$$

Problem 18.85 The mass of the homogeneous slender bar is 30 kg. Determine the moment of inertia of the bar about the z' axis through its center of mass.

Solution: First locate the center of mass

$$\bar{x} = \frac{(10 \text{ kg})(0.3 \text{ m}) + (20 \text{ kg})(1.6 \text{ m})}{30 \text{ kg}} = 1.167 \text{ m}$$

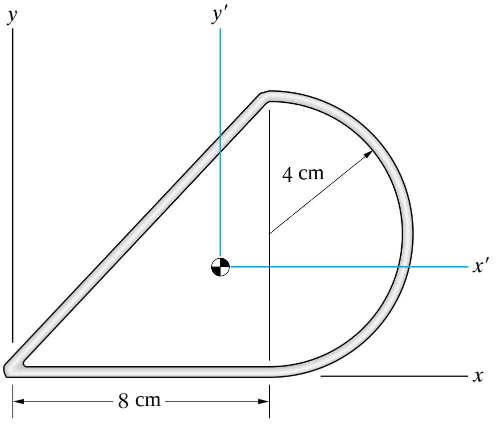
$$\bar{y} = \frac{(10 \text{ kg})(0.4 \text{ m}) + (20 \text{ kg})(0.8 \text{ m})}{30 \text{ kg}} = 0.667 \text{ m}$$

Using the answer to 18.100

$$I_{z'} = (74 \text{ kg-m}^2) - (30 \text{ kg})(1.167^2 + 0.667^2)m^2$$

$$\boxed{I_{z'} = 19.8 \text{ kg-m}^2}$$

Problem 18.86 The homogeneous slender bar weighs 1.5 N. Determine its moment of inertia about the z axis.



Solution: The Bar's mass is $m = 0.155$ kg. Its length is $L = L_1 + L_2 + L_3 = 8 + \sqrt{8^2 + 8^2} + \pi(4) = 31.9$ cm. The masses of the parts are therefore,

$$M_1 = \frac{L_1}{L}m = \left(\frac{8}{31.9}\right)(0.155) = 0.0390 \text{ kg},$$

$$M_2 = \frac{L_2}{L}m = \left(\frac{\sqrt{2(64)}}{31.9}\right)(0.155) = 0.0551 \text{ kg},$$

$$M_3 = \frac{L_3}{L}m = \left(\frac{4\pi}{31.9}\right)(0.155) = 0.0612 \text{ kg}.$$

The center of mass of part 3 is located to the right of its center C a distance $2R/\pi = 2(4)/\pi = 2.55$ cm. The moment of inertia of part 3 about C is

$$\int_{m_3} r^2\, dm = m_3 r^2 = (0.0612)(4)^2 = 0.979 \text{ kg-cm}^2.$$

The moment of inertia of part 3 about the center of mass of part 3 is therefore $I_3 = 0.979 - m_3(2.55)^2 = 0.582$ kg-cm^2. The moment of inertia of the bar about the z axis is

$$I_{(z \text{ axis})} = \frac{1}{3}m_1 L_1^2 + \frac{1}{3}m_2 L_2^2 + I_3 + m_3[(8 + 2.55)^2 + (4)^2]$$

$$= 11.6 \text{ kg-cm}^2 = 0.00116 \text{ kg-m}^2.$$

Problem 18.87 Determine the moment of inertia of the bar in Problem 18.86 about the z' axis through its center of mass.

Solution: In the solution of Problem 18.86, it is shown that the moment of inertia of the bar about the z axis is $I_{(z\ \text{axis})} = 11.6$ kg-cm^2. The x and y coordinates of the center of mass coincide with the centroid of the bar:

$$\mathbf{x} = \frac{\mathbf{x}_1 L_1 + \mathbf{x}_2 L_2 + \mathbf{x}_3 L_3}{L_1 + L_2 + L_3}$$

$$= \frac{(4)(8) + (4)\sqrt{8^2 + 8^2} + \left[8 + \dfrac{2(4)}{\pi}\right]\pi(4)}{8 + \sqrt{8^2 + 8^2} + \pi(4)} = 6.58 \text{ cm},$$

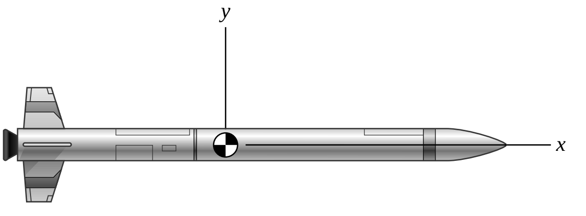
$$\mathbf{y} = \frac{\mathbf{y}_1 L_1 + \mathbf{y}_2 L_2 + \mathbf{y}_3 L_3}{L_1 + L_2 + L_3}$$

$$= \frac{0 + (4)\sqrt{8^2 + 8^2} + \pi(4)(4)}{8 + \sqrt{8^2 + 8^2} + \pi(4)} = 3.00 \text{ cm}.$$

The moment of inertia about the z' axis is

$$I_{(z'\text{axis})} = I_{(z\ \text{axis})} - (\mathbf{x}^2 + \mathbf{y}^2)\,(0.155) = 3.44 \text{ kg-cm}^2.$$

Problem 18.88 The rocket is used for atmospheric research. Its weight and its moment of inertia about the z axis through its center of mass (including its fuel) are 44480 N and 13826 kg-m2, respectively. The rocket's fuel weighs 26688 N, its center of mass is located at $x = -0.91$ m, $y = 0$, and $z = 0$, and the moment of inertia of the fuel about the axis through the fuel's center of mass parallel to z axis is 2983 kg-m^2. When the fuel is exhausted, what is the rocket's moment of inertia about the axis through its new center of mass parallel to z axis?



Solution: Denote the moment of inertia of the empty rocket as I_E about a center of mass x_E, and the moment of inertia of the fuel as I_F about a mass center x_F. Using the parallel axis theorem, the moment of inertia of the filled rocket is

$$I_R = I_E + x_E^2 m_E + I_F + x_F^2 m_F,$$

about a mass center at the origin ($x_R = 0$).

Solve: $I_E = I_R - {x_E}^2 m_E - I_F - {x_F}^2 m_F$.

The objective is to determine values for the terms on the right from the data given. Since the filled rocket has a mass center at the origin, the mass center of the empty rocket is found from

$$0 = m_E x_E + m_F x_F,$$

from which

$$x_E = -\left(\frac{m_F}{m_E}\right) x_F.$$

Using a value of $g = 9.81$ m/s^2,

$$m_F = \frac{W_F}{g} = \frac{26688}{9.81} = 2720 \text{ kg},$$

$$m_E = \frac{(W_R - W_F)}{g} = \frac{44480 - 26688}{9.81} = 1812.5 \text{ N}.$$

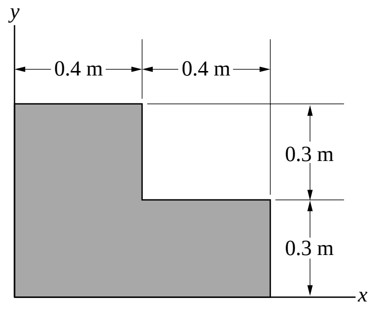
From which $x_E = -\left(\dfrac{2720}{1812.5}\right)(-0.91) = 1.37$ m

is the new location of the center of mass.

Substitute:

$$\boxed{\begin{aligned} I_E &= I_R - x_E^2 m_E - I_F - x_F^2 m_F \\ &= 13826 - 3410 - 2983 - 2273 \\ &= 5151 \text{ kg-m}^2 \end{aligned}}$$

Problem 18.89 The mass of the homogeneous thin plate is 36 kg. Determine the moment of inertia of the plate about the x axis.



Solution: Divide the plate into two areas: the rectangle 0.4 m by 0.6 m on the left, and the rectangle 0.4 m by 0.3 m on the right. The mass density is $\rho = \frac{m}{A}$.

The area is

$$A = (0.4)(0.6) + (0.4)(0.3) = 0.36 \text{ m}^2,$$

from which

$$\rho = \frac{36}{0.36} = 100 \text{ kg/m}^2.$$

The moment of inertia about the x-axis is

$$\boxed{I_{x\text{-axis}} = \rho\left(\frac{1}{3}\right)(0.4)(0.6^3) + \rho\left(\frac{1}{3}\right)(0.4)(0.3)^3 = 3.24 \text{ kg-m}^2}$$

Problem 18.90 Determine the moment of inertia of the 36-kg plate in Problem 18.89 about the z axis.

Solution: The basic relation to use is $I_{z\text{-axis}} = I_{x\text{-axis}} + I_{y\text{-axis}}$. The value of $I_{x\text{-axis}}$ is given in the solution of Problem 18.89. The moment of inertia about the y-axis using the same divisions as in Problem 8.89 and the parallel axis theorem is

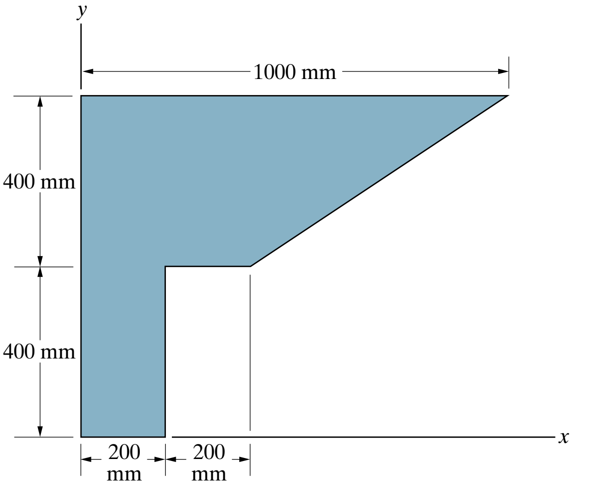
$$I_{y\text{-axis}} = \rho\left(\frac{1}{3}\right)(0.6)(0.4)^3 + \rho\left(\frac{1}{12}\right)(0.3)(0.4)^3$$

$$+ (0.6)^2\rho(0.3)(0.4) = 5.76 \text{ kg-m}^2,$$

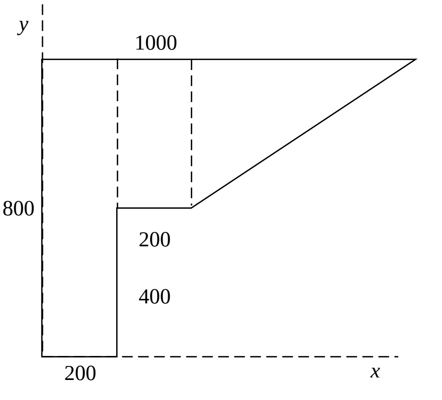
from which

$$\boxed{I_{z\text{-axis}} = I_{x\text{-axis}} + I_{y\text{-axis}} = 3.24 + 5.76 = 9 \text{ kg-m}^2}$$

Problem 18.91 The mass of the homogeneous thin plate is 20 kg. Determine its moment of inertia about the x axis.



Solution: Break the plate into the three regions shown.



$$A = (0.2\text{ m})(0.8\text{ m}) + (0.2\text{ m})(0.4\text{ m}) + \frac{1}{2}(0.4\text{ m})(0.6\text{ m}) = 0.36\text{ m}^2$$

$$\rho = \frac{20\text{ kg}}{0.36\text{ m}^2} = 55.6\text{ kg/m}^2$$

Using the integral tables we have

$$I_x = \frac{1}{3}(0.2\text{ m})(0.8\text{ m})^3 + \frac{1}{12}(0.2\text{ m})(0.4\text{ m})^3 + (0.2\text{ m})(0.4\text{ m})(0.6\text{ m})^2 + \frac{1}{36}(0.6\text{ m})(0.4\text{ m})^3 + \frac{1}{2}(0.6\text{ m})(0.4\text{ m})(0.667\text{ m})^2$$

$$= 0.1184\text{ m}^4$$

$$\boxed{I_{x-axis} = (55.6\text{ kg/m}^2)(0.1184\text{ m}^4) = 6.58\text{ kg-m}^2}$$

Problem 18.92 The mass of the homogeneous thin plate is 20 kg. Determine its moment of inertia about the y axis.

Solution: See the solution to 18.91

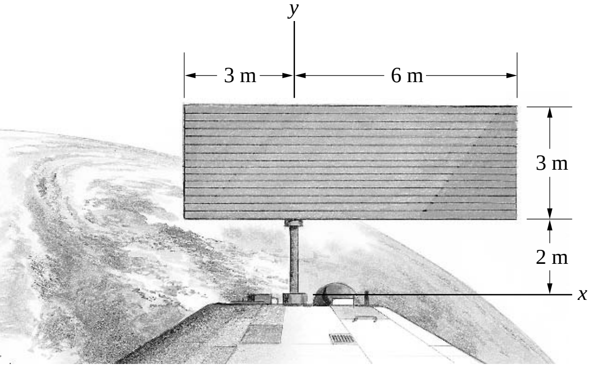
$$I_y = \frac{1}{3}(0.8\text{ m})(0.2\text{ m})^3 + \frac{1}{12}(0.4\text{ m})(0.2\text{ m})^3 + (0.2\text{ m})(0.4\text{ m})(0.3\text{ m})^2$$

$$+ \frac{1}{36}(0.4\text{ m})(0.6\text{ m})^3 + \frac{1}{2}(0.6\text{ m})(0.4\text{ m})(0.6\text{ m})^2$$

$$= 0.0552\text{ m}^4$$

$$\boxed{I_{y-axis} = (55.6\text{ kg/m}^2)(0.0552\text{ m}^4) = 3.07\text{ kg-m}^2}$$

Problem 18.93 The thermal radiator (used to eliminate excess heat from a satellite) can be modeled as a homogeneous thin rectangular plate. The mass of the radiator is 5 kg. Determine its moments of inertia about the $x, y,$ and z axes.



Solution: The area is $A = 9(3) = 27\text{ m}^2$.

The mass density is

$$\rho = \frac{m}{A} = \frac{5}{27} = 0.1852\text{ kg/m}^2.$$

The moment of inertia about the centroid of the rectangle is

$$I_{xc} = \rho\left(\frac{1}{12}\right)9(3^3) = 3.75\text{ kg-m}^2,$$

$$I_{yc} = \rho\left(\frac{1}{12}\right)3(9^3) = 33.75\text{ kg-m}^2.$$

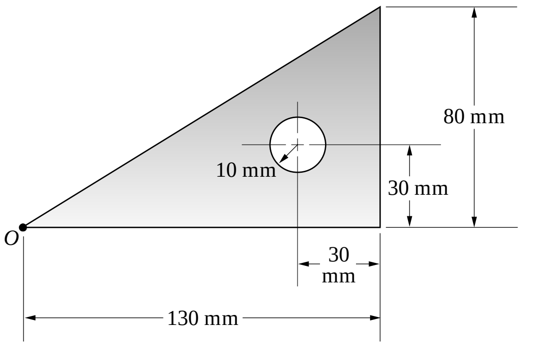
Use the parallel axis theorem:

$$I_{x\text{-axis}} = \rho A(2+1.5)^2 + I_{xc} = 65\text{ kg-m}^2,$$

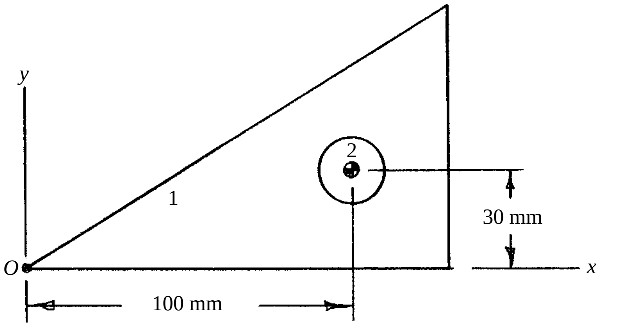
$$I_{y\text{-axis}} = \rho A(4.5-3)^2 + I_{yc} = 45\text{ kg-m}^2.$$

$$I_{z\text{-axis}} = I_{x\text{-axis}} + I_{y\text{-axis}} = 110\text{ kg-m}^2$$

Problem 18.94 The mass of the homogeneous thin plate is 2 kg. Determine the moment of inertia of the plate about the axis through point O that is perpendicular to the plate.



Solution: By determining the moments of inertia of the area about the x and y axes, we will determine the moments of inertia of the plate about the x and y axes, then sum them to obtain the moment of inertia about the z axis, which is I_0.



The areas are

$$A_1 = \tfrac{1}{2}(130)(80)\ \text{mm}^2,$$

$$A_2 = \pi(10)^2\ \text{mm}^2.$$

Using Appendix B,

$$I_x = \frac{1}{12}(130)(80)^3 - \left[\frac{1}{4}\pi(10)^4 + (30)^2 A_2\right]$$

$$= 5.26 \times 10^6\ \text{mm}^4,$$

$$I_y = \frac{1}{4}(80)(130)^3 - \left[\frac{1}{4}\pi(10)^4 + (100)^2 A_2\right]$$

$$= 40.79 \times 10^6\ \text{mm}^4.$$

Therefore

$$I_{(x\ \text{axis})} = \frac{m}{A_1 - A_2} I_x = 2150\ \text{kg-mm}^2,$$

$$I_{(y\ \text{axis})} = \frac{m}{A_1 - A_2} I_y = 16700\ \text{kg-mm}^2.$$

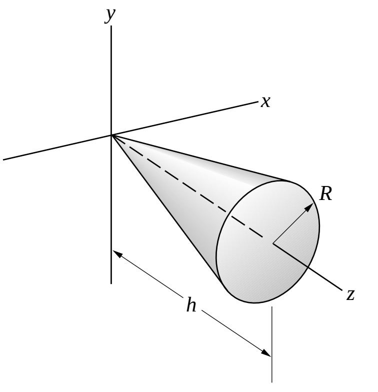
Then

$$I_{(z\ \text{axis})} = I_{(x\ \text{axis})} + I_{(y\ \text{axis})} = 18850\ \text{kg-mm}^2.$$

$$\underline{I_{(z\ \text{axis})} = 0.0188\ \text{kg-m}^2}.$$

Problem 18.95 The homogeneous cone is of mass m. Determine its moment of inertia about the z axis, and compare your result with the value given in Appendix C. (See Example 18.10.)

Strategy: Use the same approach we used in Example 18.10 to obtain the moments of inertia of a homogeneous cylinder.



Solution: The differential mass

$$dm = \left(\frac{m}{V}\right)\pi r^2 dz = \frac{3m}{R^2 h} r^2 dz.$$

The moment of inertia of this disk about the z-axis is $\frac{1}{2}mr^2$. The radius varies with z, $r = \left(\frac{R}{h}\right)z$, from which

$$\boxed{I_{z\text{-axis}} = \frac{3mR^2}{2h^5}\int_0^h z^4 dz = \frac{3mR^2}{2h^5}\left[\frac{z^5}{5}\right]_0^h = \frac{3mR^2}{10}}$$

Problem 18.96 Determine the moments of inertia of the homogeneous cone in Problem 18.95 about the x and y axes, and compare your results with the values given in Appendix C. (See Example 18.10.)

Solution: The mass density is $\rho = \frac{m}{V} = \frac{3m}{\pi R^2 h}$. The differential element of mass is $dm = \rho\pi r^2 dz.$. The moment of inertia of this elemental disk about an axis through its center of mass, parallel to the x- and y-axes, is $dI_x = \left(\frac{1}{4}\right)r^2 dm$. Use the parallel axis theorem,

$$I_x = \int_m \left(\frac{1}{4}\right) r^2 dm + \int_m z^2 dm.$$

Noting that $r = \frac{R}{h}z$, then

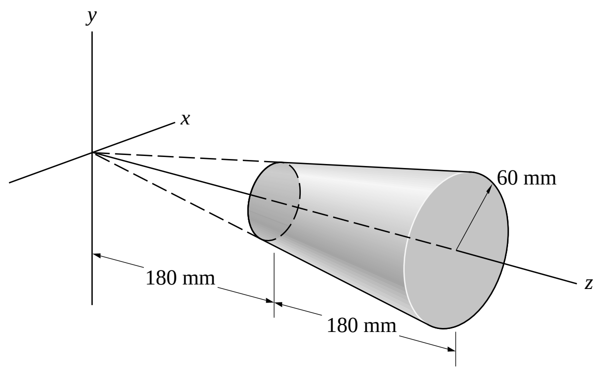
$$r^2 dm = \rho\left(\frac{\pi R^4}{h^4}\right)z^4 dz,$$

and $z^2 dm = \rho\left(\frac{\pi R^2}{h^2}\right)z^4 dz$. Substitute:

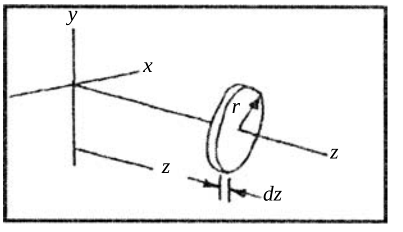
$$I_x = \rho\left(\frac{\pi R^4}{4h^4}\right)\int_0^h z^4 dz + \rho\left(\frac{\pi R^2}{h^2}\right)\int_0^h z^4 dz,$$

$$I_x = \left(\frac{3mR^2}{4h^5} + \frac{3m}{h^3}\right)\left[\frac{z^5}{5}\right]_0^h = m\left(\frac{3}{20}R^2 + \frac{3}{5}h^2\right) = I_y.$$

Problem 18.97 The homogeneous object has the shape of a truncated cone and consists of bronze with mass density $\rho = 8200$ kg/m^3. Determine the moment of inertia of the object about the z axis.



Solution: Consider an element of the cone consisting of a disk of thickness *dz*: We can express the radius as a linear function of $zr = az + b$. Using the conditions that $r = 0$ at $z = 0$ and $r = 0.06$ m at $z = 0.36$ m to evaluate a and b we find that $r = 0.167\ z$. From Appendix C, the moment of inertia of the element about the z axis is



$$(I_z)_{\text{element}} = \frac{1}{2}mr^2 = \frac{1}{2}[\rho(\pi r^2)dz]r^2 = \frac{1}{2}\rho\pi(0.167z)^4dz.$$

We integrate this result to obtain the mass moment of inertia about the *z* axis for the cone:

$$I_{(z\ axis)} = \int_{0.18}^{0.36} \frac{1}{2}\rho\pi(0.167)^4 \left[\frac{z^5}{5}\right]_{0.18}^{0.36}$$

$$= \frac{1}{2}(8200)\pi(0.167)^4 \left[\frac{z^5}{5}\right]_{0.18}^{0.36}$$

$$= 0.0116 \text{ kg-m}^2.$$

Problem 18.98 Determine the moment of inertia of the object in Problem 18.97 about the x axis.

Solution: Consider the disk element described in the solution to Problem 18.97. The radius of the laminate is $r = 0.167z$. Using Appendix C and the parallel axis theorem, the moment of inertia of the element about the x axis is

$$(I_x)_{\text{element}} = \frac{1}{4}mr^2 + mz^2 = \frac{1}{4}[\rho(\pi r^2)dz]r^2 + [\rho(\pi r^2)dz]z^2$$

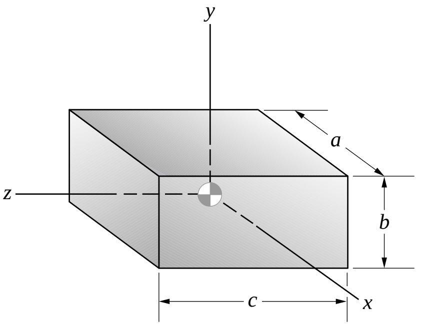
$$= \frac{1}{4}\rho\pi(0.167z)^4dz + \rho\pi(0.167z)^2z^2dz.$$

Integrating the result,

$$I_{(x\ \text{axis})} = \frac{1}{4}\rho\pi(0.167)^4\int_{0.18}^{0.36} z^4dz + \rho\pi(0.167)^2\int_{0.18}^{0.36} z^4\,dz$$

$$= 0.844 \text{ kg-m}^2.$$

Problem 18.99 The homogeneous rectangular parallelepiped is of mass m. Determine its moments of inertia about the x, y, and z axes and compare your results with the values given in Appendix C.



Solution: Consider a rectangular slice normal to the x-axis of dimensions b by c and mass dm. The area density of this slice is $\rho = \frac{dm}{bc}$. The moment of inertia about the y axis of the centroid of a thin plate is the product of the area density and the area moment of inertia of the plate: $dI_y = \rho\left(\frac{1}{12}\right)bc^3$, from which $dI_y = \left(\frac{1}{12}\right)c^2 dm$. By symmetry, the moment of inertia about the z axis is

$$dI_z = \left(\frac{1}{12}\right)b^2 dm.$$

Since the labeling of the x- y- and z-axes is arbitrary,

$$dI_x = dI_z + dI_y,$$

where the x-axis is normal to the area of the plate. Thus

$$dI_x = \left(\frac{1}{12}\right)(b^2 + c^2)\,dm,$$

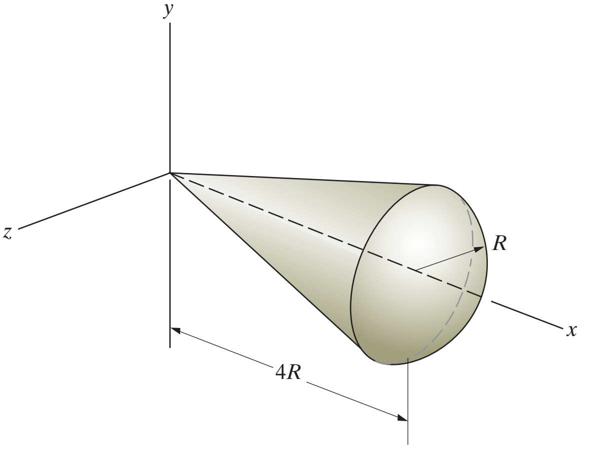
from which

$$\boxed{I_x = \left(\frac{1}{12}\right)(b^2 + c^2)\int_m dm = \frac{m}{12}(b^2 + c^2)}.$$

By symmetry, the argument can be repeated for each coordinate, to obtain

$$\boxed{I_y = \frac{m}{12}(a^2 + c^2)} \quad \boxed{I_z = \frac{m}{12}(b^2 + a^2)}$$

Problem 18.100 The sphere-capped cone consists of material with density $7800\ \text{kg/m}^3$. The radius $R = 80$ mm. Determine its moment of inertia about the x axis.



Solution: Given $\rho = 7800\ \text{kg/m}^3$, $R = 0.08$ m

Using the tables we have

$$I_x = \frac{3}{10}\left(\rho\frac{1}{3}\pi R^2[4R]\right)R^2 + \frac{2}{5}\left(\rho\frac{2}{3}\pi R^3\right)R^2$$

$$\boxed{I_x = 0.0535\ \text{kg-m}^2}$$

Problem 18.101 Determine the moment of inertia of the sphere-capped cone described in Problem 18.100 about the y axis.

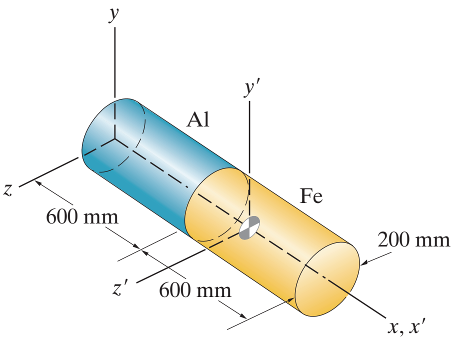
Solution: The center of mass of a half-sphere is located a distance $\frac{3R}{8}$ from the geometric center of the circle.

$$I_y = \left(\rho\frac{1}{3}\pi R^2[4R]\right)\left(\frac{3}{5}[4R]^2 + \frac{3}{20}R^2\right) + \frac{2}{5}\left(\rho\frac{2}{3}\pi R^3\right)R^2$$

$$-\left(\rho\frac{2}{3}\pi R^3\right)\left(\frac{3R}{8}\right)^2 + \left(\rho\frac{2}{3}\pi R^3\right)\left(4R + \frac{3R}{8}\right)^2$$

$$\boxed{I_y = 2.08 \text{ kg-m}^2}$$

Problem 18.102 The circular cylinder is made of aluminum (Al) with density 2700 kg/m^3 and iron (Fe) with density 7860 kg/m^3. Determine its moment of inertia about the x' axis.



Solution:

$$I_x = \frac{1}{2}[(2700 \text{ kg/m}^2)\pi(0.1 \text{ m})^2(0.6 \text{ m})](0.1 \text{ m})^2$$

$$+ \frac{1}{2}[(7860 \text{ kg/m}^2)\pi(0.1 \text{ m})^2(0.6 \text{ m})](0.1 \text{ m})^2$$

$$\boxed{I_x = 0.995 \text{ kg-m}^2}$$

Problem 18.103 Determine the moment of inertia of the composite cylinder in Problem 18.102 about the y' axis.

Solution: First locate the center of mass

$$\bar{x} = \frac{[(2700 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m})](0.3 \text{ m}) + [(7860 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m})](0.9 \text{ m})}{(2700 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m}) + (7860 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m})}$$

$$\bar{x} = 0.747 \text{ m}$$

$$I_y = [(2700 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m})]\left[\frac{1}{12}(0.6 \text{ m})^2 + \frac{1}{4}(0.1 \text{ m})^2\right]$$

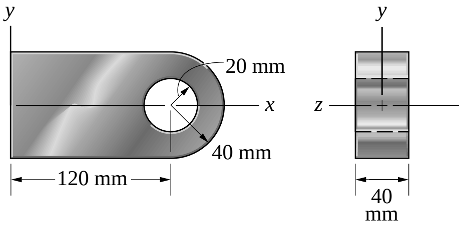
$$+ [(2700 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m})](\bar{x} - 0.3 \text{ m})^2$$

$$+ [(7680 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m})]\left[\frac{1}{12}(0.6 \text{ m})^2 + \frac{1}{4}(0.1 \text{ m})^2\right]$$

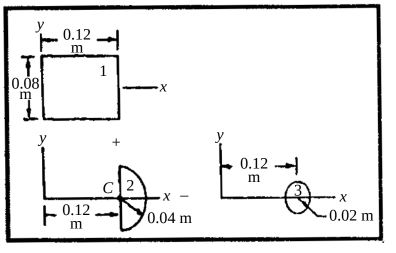
$$+ [(7680 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(0.6 \text{ m})](0.9 \text{ m} - \bar{x})^2$$

$$\boxed{I_y = 20.1 \text{ kg-m}^2}$$

Problem 18.104 The homogeneous machine part is made of aluminum alloy with mass density $\rho = 2800\ \text{kg/m}^3$. Determine the moment of inertia of the part about the z axis.



Solution: We divide the machine part into the 3 parts shown: (The dimension into the page is 0.04 m) The masses of the parts are



$$m_1 = (2800)(0.12)(0.08)(0.04) = 1.075\ \text{kg},$$

$$m_2 = (2800)\tfrac{1}{2}\pi(0.04)^2(0.04) = 0.281\ \text{kg},$$

$$m_3 = (2800)\pi(0.02)^2(0.04) = 0.141\ \text{kg}.$$

Using Appendix C and the parallel axis theorem the moment of inertia of part 1 about the z axis is

$$I_{(z\ \text{axis})_1} = \frac{1}{12}m_1[(0.08)^2 + (0.12)^2] + m_1(0.06)^2$$

$$= 0.00573\ \text{kg-m}^2.$$

The moment of inertia of part 2 about the axis through the center C that is parallel to the z axis is

$$\tfrac{1}{2}m_2R^2 = \tfrac{1}{2}m_2(0.04)^2$$

The distance along the x axis from C to the center of mass of part 2 is $4(0.04)/(3\pi) = 0.0170$ m. Therefore, the moment of inertia of part 2 about the z axis through its center of mass that is parallel to the axis is

$$\tfrac{1}{2}m_2(0.04)^2 - m_2(0.0170)^2 = 0.000144\ \text{kg-m}^2.$$

Using this result, the moment of inertia of part 2 about the z axis is

$$I_{(z\ \text{axis})2} = 0.000144 + m_2(0.12 + 0.017)^2 = 0.00544\ \text{kg-m}^2.$$

The moment of inertia of the material that would occupy the hole 3 about the z axis is

$$I_{(z\ \text{axis})3} = \tfrac{1}{2}m_3(0.02)^2 + m_3(0.12)^2 = 0.00205\ \text{kg-m}^2.$$

Therefore,

$$I_{(z\ \text{axis})} = I_{(z\ \text{axis})1} + I_{(z\ \text{axis})2} - I_{(z\ \text{axis})3} = 0.00911\ \text{kg-m}^2.$$

Problem 18.105 Determine the moment of inertia of the machine part in Problem 18.104 about the x axis.

Solution: We divide the machine part into the 3 parts shown in the solution to Problem 18.104. Using Appendix C and the parallel axis theorem, the moments of inertia of the parts about the x axis are:

$$I_{(x\ \text{axis})_1} = \frac{1}{12} m_1[(0.08)^2 + (0.04)^2]$$

$$= 0.0007168 \text{ kg-m}^2$$

$$I_{(x\ \text{axis})_2} = m_2\left[\frac{1}{12}(0.04)^2 + \frac{1}{4}(0.04)^2\right]$$

$$= 0.0001501 \text{ kg-m}^2$$

$$I_{(x\ \text{axis})_3} = m_3\left[\frac{1}{12}(0.04)^2 + \frac{1}{4}(0.02)^2\right]$$

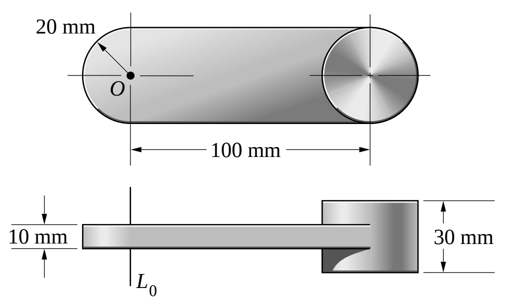
$$= 0.0000328 \text{ kg-m}^2.$$

Therefore,

$$I_{(x\text{axis})} = I_{(x\ \text{axis})_1} + I_{(x\ \text{axis})_2} - I_{(x\ \text{axis})_3}$$

$$= 0.000834 \text{ kg-m}^2.$$

Problem 18.106 The object shown consists of steel of density $\rho = 7800$ kg/m^3 of width $w = 40$ mm. Determine the moment of inertia about the axis L_0.



Solution: Divide the object into four parts:

Part (1): The semi-cylinder of radius $R = 0.02$ m, height $h_1 = 0.01$ m.
Part (2): The rectangular solid $L = 0.1$ m by $h_2 = 0.01$ m by $w = 0.04$ m.
Part (3): The semi-cylinder of radius $R = 0.02$ m, $h_1 = 0.01$ m
Part (4): The cylinder of radius $R = 0.02$ m, height $h = 0.03$ m.

Part (1) $m_1 = \dfrac{\rho\pi R^2 h_1}{2} = 0.049$ kg,

$$I_1 = \frac{m_1 R^2}{4} = 4.9 \times 10^{-6} \text{ kg-m}^2,$$

Part (2) $m_2 = \rho w L h_2 = 0.312$ kg,

$$I_2 = (1/12)m_2(L^2 + w^2) + m_2(L/2)^2$$

$$= 0.00108 \text{ kg-m}^2.$$

Part (3) $m_3 = m_1 = 0.049$ kg,

$$I_3 = -\left(\frac{4R}{3\pi}\right)^2 m_2 + I_1 + m_3\left(L - \frac{4R}{3\pi}\right)^2$$

$$= 0.00041179 \text{ kg-m}^2.$$

Part (4) $m_4 = \rho\pi R^2 h = 0.294$ kg,

$$I_4 = \left(\tfrac{1}{2}\right) m_4(R^2) + m_4 L^2 = 0.003 \text{ kg-m}^2.$$

The composite:

$$\boxed{I_{L0} = I_1 + I_2 - I_3 + I_4 = 0.00367 \text{ kg-m}^2}$$

Problem 18.107 Determine the moment of inertia of the object in Problem 18.106 about the axis through the center of mass of the object parallel to L_0.

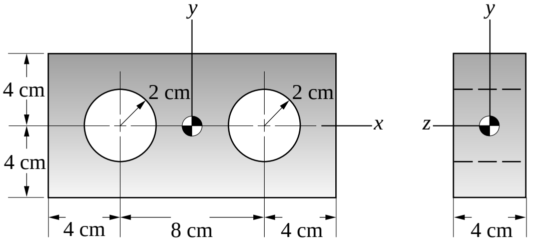
Solution: The center of mass is located relative to L_0 is given by

$$\mathbf{x} = \frac{m_1\left(-\dfrac{4R}{3\pi}\right) + m_2(0.05) - m_3\left(0.1 - \dfrac{4R}{3\pi}\right) + m_4(0.1)}{m_1 + m_2 - m_3 + m_4}$$

$= 0.066$ m,

$$\boxed{I_c = -\mathbf{x}^2 m + I_{Lo} = -0.00265 + 0.00367 = 0.00102 \text{ kg-m}^2}$$

Problem 18.108 The thick plate consists of steel of density $\rho = 7729$ kg/m^3. Determine the moment of inertia of the plate about the z axis.



Solution: Divide the object into three parts: Part (1) the rectangle 8 cm by 16 cm, Parts (2) & (3) the cylindrical cut outs.

Part (1): $m_1 = \rho 8(16)(4) = 3.96$ kg.

$$I_1 = (1/12)m_1(16^2 + 8^2) = 105.6 \text{ kg-cm}^2.$$

Part (2): $m_2 = \rho\pi(2^2)(4) = 0.388$ kg,

$$I_2 = \frac{m_2(2^2)}{2} + m_2(4^2) = 7 \text{ kg-cm}^2.$$

Part (3): $m_3 = m_2 = 0.388$ kg,

$$I_3 = I_2 = 7 \text{ kg-cm}^2.$$

The composite:

$$I_{z\text{-axis}} = I_1 - 2I_2 = 91.6 \text{ kg-cm}^2$$

$$\boxed{I_{z\text{-axis}} = 0.00916 \text{ kg-m}^2}$$

Problem 18.109 Determine the moment of inertia of the object in Problem 18.108 about the x axis.

Solution: Use the same divisions of the object as in Problem 18.108.

Part (1) : $I_{1x\text{-axis}} = \left(\dfrac{1}{12}\right) m_1(8^2 + 4^2) = 26.4$ kg-cm^2,

Part (2) : $I_{2x\text{-axis}} = (1/12)m_2(3(2^2) + 4^2) = 0.91$ kg-cm^2.

The composite:

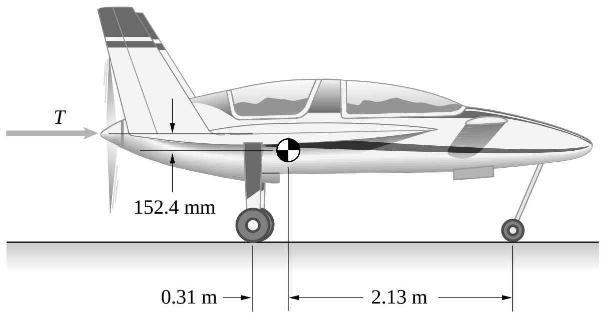
$$I_{x\text{-axis}} = I_{1x\text{-axis}} - 2I_{2x\text{-axis}} = 24.6 \text{ kg-cm}^2$$

$$\boxed{= 0.00246 \text{ kg-m}^2}$$

Problem 18.110 The airplane is at the beginning of its takeoff run. Its weight is 4448 N and the initial thrust T exerted by its engine is 1334 N. Assume that the thrust is horizontal, and neglect the tangential forces exerted on its wheels.

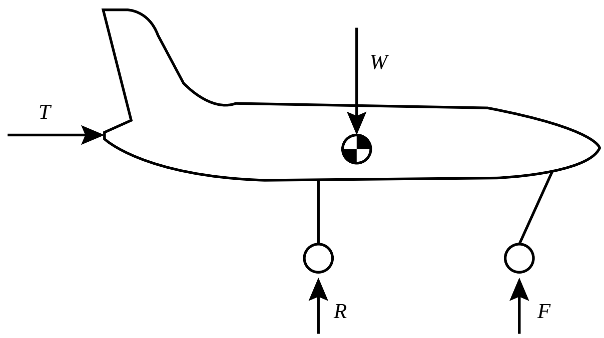
(a) If the acceleration of the airplane remains constant, how long will it take to reach its takeoff speed of 128.7 km/h

(b) Determine the normal force exerted on the forward landing gear at the beginning of the takeoff run.



Solution: The acceleration under constant thrust is

$$a = \frac{T}{m} = \frac{1334(9.81)}{4448} = 2.94 \text{ m/s}^2.$$

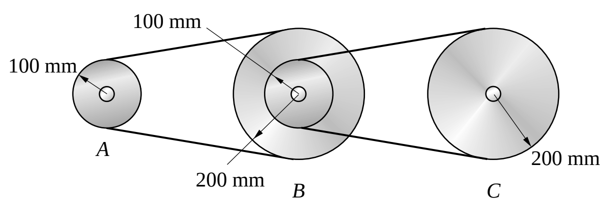


The time required to reach 128.7 km/h = 35.8 m/s is

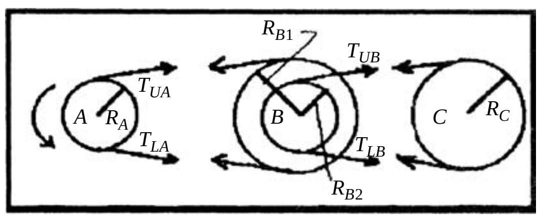
$$\boxed{t = \frac{v}{a} = \frac{35.8}{2.94} = 12.1 \text{ s}}$$

The sum of the vertical forces: $\sum F_y = R + F - W = 0$. The sum of the moments: $\sum M = 2.13F - 0.152T - 0.31R = 0$. Solve: $R = 3809$ N, $\boxed{F = 639 \text{ N}}$

Problem 18.111 The pulleys can turn freely on their pin supports. Their moments of inertia are $I_A = 0.002$ kg-m^2, $I_B = 0.036$ kg-m^2, and $I_C = 0.032$ kg-m^2. They are initially stationary, and at $t = 0$ a constant $M = 2$ N-m is applied at pulley A. What is the angular velocity of pulley C and how many revolutions has it turned at $t = 2$ s?



Solution: Denote the upper and lower belts by the subscripts U and L. Denote the difference in the *tangential* component of the tension in the belts by



$$\Delta T_A = T_{LA} - T_{UA},$$

$$\Delta T_B = T_{LB} - T_{UB}.$$

From the equation of angular motion: $M + R_A \Delta T_A = I_A \alpha_A$, $-R_{B1}\Delta T_A + R_{B2}\Delta T_B = I_B \alpha_B$, $-R_C \Delta T_B = I_C \alpha_C$. From kinematics, $R_A \alpha_A = R_{B1}\alpha_B$, $R_{B2}\alpha_B = R_C \alpha_C$, from which

$$\alpha_A = \frac{R_{B1}R_C}{R_A R_{B2}}\alpha_C = \frac{(0.2)(0.2)}{(0.1)(0.1)}\alpha_C = 4\alpha_C,$$

$$\alpha_B = \frac{R_C}{R_{B2}}\alpha_C = \frac{0.2}{0.1}\alpha_C = 2\alpha_C.$$

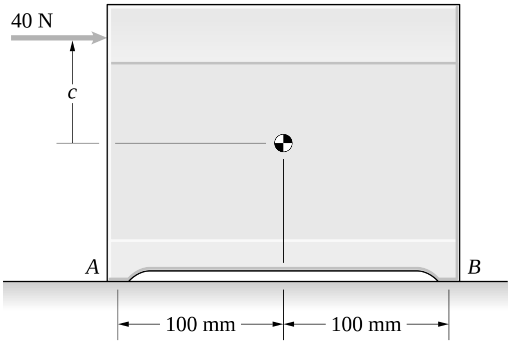
Substitute and solve: $\alpha_C = 38.5$ rad/s^2, from which

$$\boxed{\omega_C = \alpha_C t = 76.9 \text{ rad/s}}$$

$$\boxed{N = \theta\left(\frac{1}{2\pi}\right) = \frac{\alpha_C}{4\pi}(2^2) = 12.2 \text{ revolutions}}$$

Problem 18.112 A 2 kg box is subjected to a 40-N horizontal force. Neglect friction.

(a) If the box remains on the floor, what is its acceleration?
(b) Determine the range of values of c for which the box will remain on the floor when the force is applied.

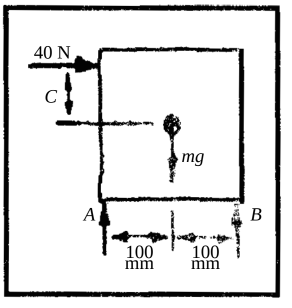


Solution:

(a) From Newton's second law, $40 = (2)a$, from which

$$\boxed{a = \frac{40}{2} = 20 \text{ m/s}^2}.$$

(b) The sum of forces: $\sum F_y = A + B - mg = 0$. The sum of the moments about the center of mass: $\sum M = 0.1B - 0.1A - 40c = 0$. Substitute the value of B from the first equation into the second equation and solve for c:



$$c = \frac{(0.1)mg - (0.2)A}{40}$$

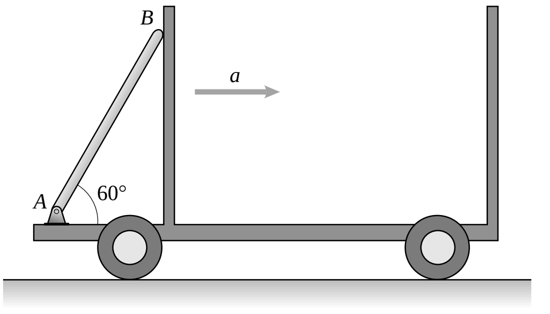
The box leg at A will leave the floor as $A \le 0$, from which

$$\boxed{c \le \frac{(0.1)(2)(9.81)}{40} \le 0.0491 \text{ m}}$$

for values of $A \ge 0$.

Problem 18.113 The slender, 2-kg bar AB is 3 m long. It is pinned to the cart at A and leans against it at B.

(a) If the acceleration of the cart is $a = 20$ m/s^2, what normal force is exerted on the bar by the cart at B?
(b) What is the largest acceleration a for which the bar will remain in contact with the surface at B?



Solution: Newton's second law applied to the center of mass of the bar yields

$$-B + A_x = ma_{Gx}, A_y - W = ma_{Gy},$$

$$-A_y\left(\frac{L\cos\theta}{2}\right) + (B + A_x)\left(\frac{L\sin\theta}{2}\right) = I_G\alpha,$$

where a_{Gx}, a_{Gy} are the accelerations of the center of mass. From kinematics,

$$\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega_{AB}^2\mathbf{r}_{G/A} = 20\mathbf{i}\ \text{m/s}^2$$

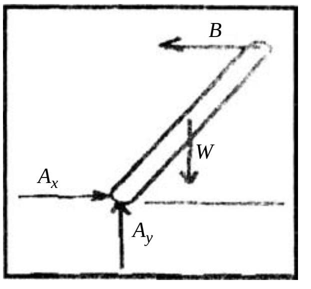
where $\alpha = 0$, $\omega_{AB} = 0$ *so long as the bar is resting on the cart at B and is pinned at A.* Substitute the kinematic relations to obtain three equations in three unknowns:

$$-B + A_x = ma, A_y - W = 0,$$

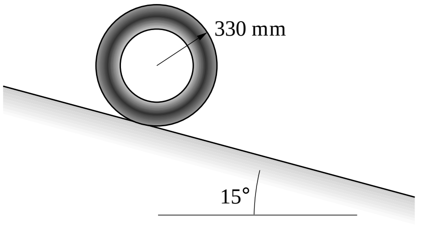
$$-A_y\left(\frac{L\cos\theta}{2}\right) + (B + A_x)\left(\frac{L\sin\theta}{2}\right) = 0.$$

Solve: $B = \dfrac{W\cot\theta}{2} - \dfrac{ma}{2}$. For $W = mg = 19.62$ N, $\theta = 60^\circ$, $m = 2$ kg, and $a = 20$ m/s^2, $B = -14.34$ N, from which *the bar has moved away from the cart at point B.* (b) The acceleration that produces a zero normal force is

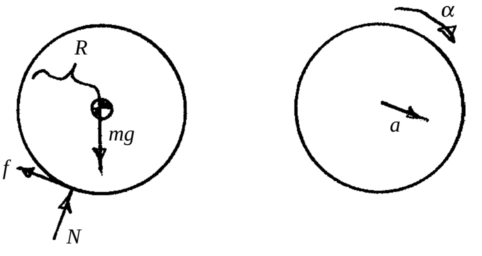
$$\boxed{a = g\cot\theta = 5.66\ \text{m/s}^2}.$$



Problem 18.114 To determine a 4.5-kg tire's moment of inertia, an engineer lets the tire roll down an inclined surface. If it takes the tire 3.5 s to start from rest and roll 3 m down the surface, what is the tire's moment of inertia about its center of mass?



Solution: From Newton's second law and the angular equation of motion,



$$mg \sin 15^\circ - f = ma,$$

$$Rf = I\alpha.$$

From these equations and the relation $a = R\alpha$, we obtain

$$a = \frac{mg \sin 15^\circ}{m + I/R^2}. \quad \textbf{(1)}$$

We can determine the acceleration from

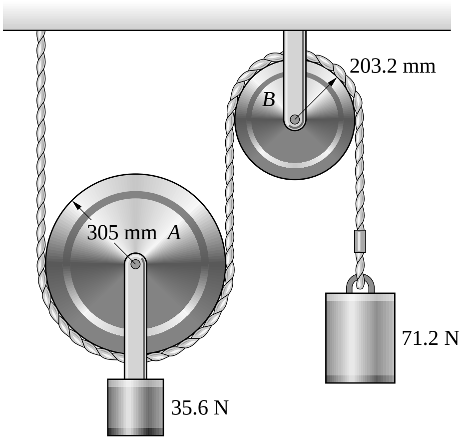
$$s = \frac{1}{2}\, at^2 :$$

$$3 = \frac{1}{2} a(3.5)^2,$$

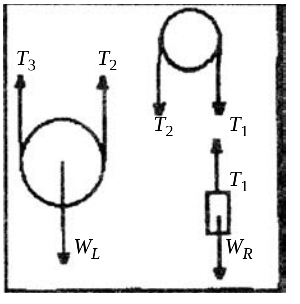
obtaining $a = 0.490\ \text{m/s}^2$. Then from Eq. (1) we obtain

$\underline{I = 2.05\ \text{kg-m}^2}$.

Problem 18.115 Pulley A weighs 17.8 N, $I_A = 0.081$ kg-m^2, and $I_B = 0.019$ kg-m^2. If the system is released from rest, what distance does the 71.2 N weight fall in 0.5 s?



Solution: The strategy is to apply Newton's second law and the equation of angular motion to the free body diagrams. Denote the rightmost weight by $W_R = 71.2$ N, the mass by $m_R = 7.26$ kg, and the leftmost weight by $W_L = 17.8 + 35.6 = 53.4$ N, and the mass by $m_L = 1.66$ kg. $R_B = 0.203$ m. is the radius of pulley B, $I_B = 0.019$ kg-m^2, and $R_A = 0.305$ m. is the radius of pulley A, and $I_A = 0.081$ kg-m^2. Choose a coordinate system with the y axis positive upward.



The 71.2 N. weight: (1) $T_1 - W_R = m_R a_{Ry}$.

Pulley B: The center of the pulley is constrained against motion, and the acceleration of the rope is equal (except for direction) on each side of the pulley. (2) $-R_B T_1 + R_B T_2 = I_B \alpha_B$. From kinematics, (3) $a_{Ry} = R_B \alpha_B$. Combine (1), (2) and (3) and reduce:

(4) $T_2 = W_R + \left(\dfrac{I_B}{R_B^2} + m_R\right) a_{Ry}$

Pulley A: (5) $T_2 + T_3 - W_L = m_L a_{Ay}$, where a_{Ay} is the acceleration of the center of the pulley. (6) $-R_A T_3 + R_A T_2 = I_A \alpha_A$. From the kinematics of pulley A, the acceleration of the left side of the pulley is zero, so that the acceleration of the right side relative to the left side is

$$\mathbf{a}_{\text{right}} = -a_{Ry}\mathbf{j} = \mathbf{a}_{\text{left}} + \boldsymbol{\alpha}_A \times (2R_A\mathbf{i})$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_A \\ 2R_A & 0 & 0 \end{bmatrix} = 0 + 2R_A\alpha_A\mathbf{j},$$

from which (7) $a_{Ry} = -2R_A\alpha_A$, where the change in direction of the acceleration of the 71.2 N. weight across pulley B is taken into account. Similarly, the acceleration of the right side relative to the acceleration of the center of the pulley is

$$\mathbf{a}_{A\text{right}} = -a_{Ry}\mathbf{j} = \mathbf{a}_A + \boldsymbol{\alpha}_A \times (R_A\mathbf{i}) = \mathbf{a}_A + R_A\alpha_A\mathbf{j},$$

from which (8) $a_{Ay} = -\dfrac{a_{Ry}}{2}$. Combine (5), (6), (7) and (8) and reduce to obtain (9) $T_2 = \dfrac{W_A}{2} - \left(\dfrac{I_A}{4R_A^2} + \dfrac{m_A}{4}\right) a_y$.

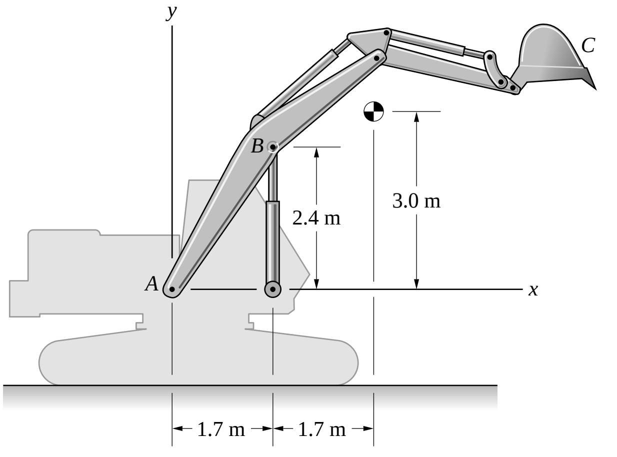
The total system: Equate (4) and (9) (the two expressions for T_2) and solve:

$$a_{Ry} = \frac{\left(\dfrac{W_L}{2} - W_R\right)}{\left(+\dfrac{I_B}{R_B^2} + m_R + \dfrac{I_A}{4R_A^2} + \dfrac{m_L}{4}\right)}.$$

Substitute numerical values: $a_{Ry} = -4.78$ m/s^2. The distance that the 71.2 N weight will fall in one-half second is

$$s = \frac{a_{Ry}}{2}t^2 = \frac{-4.78}{8} = -0.6 \text{ m}$$

Problem 18.116 Model the excavator's arm ABC as a single rigid body. Its mass is 1200 kg, and the moment of inertia *about its center of mass* is $I = 3600$ kg-m^2. If point A is stationary, the angular velocity of the arm is zero, and the angular acceleration is 1.0 rad/s^2 counterclockwise, what force does the vertical hydraulic cylinder exert on the arm at B?



Solution: The distance from A to the center of mass is

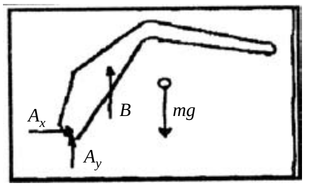
$$d = \sqrt{(3.4)^2 + (3)^2} = 4.53 \text{ m}.$$

The moment of inertia about A is

$$I_A = I + d^2 m = 28{,}270 \text{ kg-m}^2.$$

From the equation of angular motion: $1.7B - 3.4mg = I_A \alpha$.

Substitute $\alpha = 1.0$ rad/s^2, to obtain $B = 40{,}170$ N.



Problem 18.117 Model the excavator's arm ABC as a single rigid body. Its mass is 1200 kg, and the moment of inertia *about its center of mass* is $I = 3600$ kg-m^2. The angular velocity of the arm is 2 rad/s counterclockwise and its angular acceleration is 1 rad/s^2 counterclockwise. What are the components of the force exerted on the arm at A?

Solution: The acceleration of the center of mass is

$$\mathbf{a}_G = \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2 \mathbf{r}_{G/B} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ 3.4 & 3 & 0 \end{bmatrix} - \omega^2 (3.4\mathbf{i} + 3\mathbf{j})$$

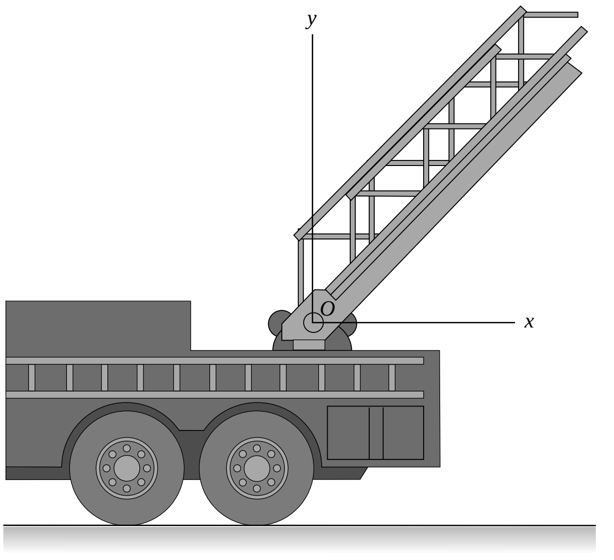
$$= -16.6\mathbf{i} - 8.6\mathbf{j} \text{ m/s}^2.$$

From Newton's second law:

$$A_x = m a_{Gx} = -19{,}900 \text{ N}, \; A_y + B - mg = m a_{Gy}.$$

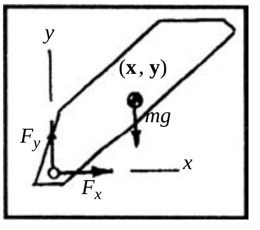
From the solution to Problem 18.132, $B = 40{,}170$ N, from which $A_y = -38{,}720$ N

Problem 18.118 To decrease the angle of elevation of the stationary 200-kg ladder, the gears that raised it are disengaged, and a fraction of a second later a second set of gears that lower it are engaged. At the instant the gears that raised the ladder are disengaged, what is the ladder's angular acceleration and what are the components of force exerted on the ladder by its support at O? The moment of inertia of the ladder about O is $I_0 = 14{,}000$ kg-m^2, and the coordinates of its center of mass at the instant the gears are disengaged are $\mathbf{x} = 3$ m, $\mathbf{y} = 4$ m.



Solution: The moment about O, $-mg\mathbf{x} = I_o\alpha$, from which

$$\alpha = -\frac{(200)(9.81)(3)}{14{,}000} = -0.420 \text{ rad/s}^2.$$



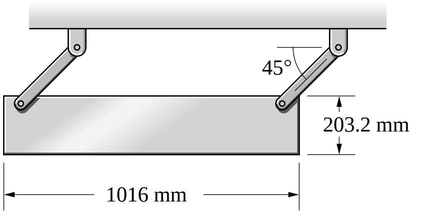
The acceleration of the center of mass is

$$\mathbf{a}_G = \boldsymbol{\alpha} \times \mathbf{r}_{G/O} - \omega^2 \mathbf{r}_{G/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ 3 & 4 & 0 \end{bmatrix} = -4\alpha\mathbf{i} + 3\alpha\mathbf{j}$$

$$\mathbf{a}_G = 1.68\mathbf{i} - 1.26\mathbf{j} \text{ (m/s}^2\text{)}.$$

From Newton's second law: $F_x = ma_{Gx} = 336$ N, $F_y - mg = ma_{Gy}$, from which $F_y = 1710$ N

Problem 18.119 The slender bars each weigh 17.8 N and are 254 mm. long. The homogenous plate weighs 44.5 N. If the system is released from rest in the position shown, what is the angular acceleration of the bars at that instant?

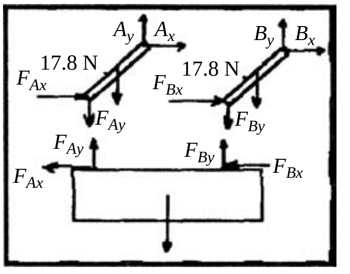


Solution: From geometry, the system is a parallelogram, so that the plate translates without rotating, so that the acceleration of every point on the plate is the same.

Newton's second law and the equation of angular motion applied to the plate: $-F_{Ax} - F_{Bx} = m_P a_{PGx}$, $F_{Ay} + F_{By} - W_p = m_P a_{PGy}$. The motion about the center of mass:

$$-F_{Ay}(0.508) + F_{Ax}(0.102) + F_{Bx}(0.102)$$

$$+F_{By}(0.508) = I_p\alpha = 0.$$



Newton's second law for the bars: $-F_{Ay} + A_y - W_B = m_B a_{BGy}$, $F_{Ax} + A_x = m_B a_{BGx}$. $-F_{By} + B_y - W_B = m_B a_{BGy}$. $F_{Bx} + B_x = m_B a_{BGx}$. The angular acceleration about the center of mass:

$$F_{Ax}\,(0.127)\,\cos\theta + F_{Ay}(0.127)\,\sin\theta - A_x\,(0.127)$$

$$\cos\theta + A_y\,(0.127)\,\sin\theta = I_B\alpha,$$

$$F_{Bx}\,(0.127)\,\cos\theta + F_{By}(0.127)\,\sin\theta - B_x\,(0.127)$$

$$\cos\theta + B_y(0.127)\,\sin\theta = I_B\alpha.$$

From kinematics: the acceleration of the center of mass of the bars in terms of the acceleration at point A is

$$\mathbf{a}_{BG} = \boldsymbol{\alpha}\times\mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ -0.127\cos\theta & -0.127\sin\theta & 0 \end{bmatrix}$$

$$= 0.127\,\sin\theta\alpha\mathbf{i} - 0.127\,\cos\alpha\mathbf{j}\ (\text{m/s}^2).$$

From which

$$a_{BGx} = (0.127)\sin\theta\alpha, \quad a_{BGy} = -(0.127)\cos\theta\alpha,$$

since $\omega = 0$ upon release.

The acceleration of the plate:

$$\mathbf{a}_P = \boldsymbol{\alpha}\times\mathbf{r}_{P/A} - \omega^2\mathbf{r}_{P/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ -0.254\cos\theta & -0.254\sin\theta & 0 \end{bmatrix}$$

$$= 0.254\,\sin\theta\alpha\mathbf{i} - 0.254\,\cos\theta\alpha\mathbf{j}\ (\text{m/s}^2).$$

From which $a_{Px} = (0.254)\sin\theta\alpha$, $a_{Py} = -(0.254)\cos\theta\alpha$.

Substitute to obtain the nine equations in nine unknowns:

(1) $-F_{Ax} - F_{Bx} = 0.254\ m_p\sin\theta\alpha$,

(2) $F_{Ay} + F_{By} - W_p = -0.254\ m_p\cos\theta\alpha$,

(3) $-0.508F_{Ay} + 0.102F_{Ax} + 0.508F_{By} + 0.102F_{Bx} = 0$,

(4) $-F_{Ay} + A_y - W_B = -(0.127)\ m_B\cos\theta\alpha$,

(5) $F_{Ax} + A_x = (0.127)\ m_B\sin\theta\alpha$,

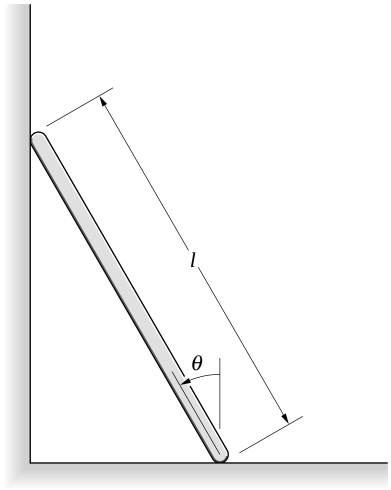
(6) $F_{Ax}\sin\theta + F_{Ay}\cos\theta - A_x\sin\theta + A_y\cos\theta = (7.87)\ I_B\alpha$,

(7) $F_{Bx} + B_x = (0.127)\ m_B\sin\theta\alpha$,

(8) $-F_{By} + B_y - W_B = -(0.127)\ m_B\cos\theta\alpha$,

(9) $F_{Bx}\cos\theta + F_{By}\sin\theta - B_x\cos\theta + B_y\sin\theta = (7.87)\ I_B\alpha$. The number of equations and number of unknowns can be reduced by combining equations, but here the choice is to solve the system by iteration using **TK Solver Plus**. The results: $F_{Ax} = -9.83$ N, $F_{Ay} = 7.47$ N, $F_{Bx} = -14.77$ N, $A_x = 14.77$ N, $A_y = 20.37$ N, $B_x = 19.7$ N, $B_y = 25.3$ N. $\alpha = 30.17$ rad/s^2.

Problem 18.120 A slender bar of mass m is released from rest in the position shown. The static and kinetic friction coefficients of friction at the floor and the wall have the same value μ. If the bar slips, what is its angular acceleration at the instant of release?



Solution: Choose a coordinate system with the origin at the intersection of wall and floor, with the x axis parallel to the floor. Denote the points of contact at wall and floor by P and N respectively, and the center of mass of the bar by G. The vector locations are

$$\mathbf{r}_N = \mathbf{i}L\sin\theta, \mathbf{r}_P = \mathbf{j}L\cos\theta, \mathbf{r}_G = \frac{L}{2}(\mathbf{i}\sin\theta + \mathbf{j}\cos\theta).$$

From Newton's second law:

$$P - \mu N = ma_{Gx}, N + \mu P - mg = ma_{Gy},$$

where a_{Gx}, a_{Gy} are the accelerations of the center of mass. The moment about the center of mass is

$$\mathbf{M}_G = \mathbf{r}_{P/G} \times (P\mathbf{i} + \mu P\mathbf{j}) + \mathbf{r}_{N/G} \times (\mathrm{N}\mathbf{j} - \mu\ \mathrm{N}\mathbf{i}):$$

$$\mathbf{M}_G = \frac{\mathrm{PL}}{2}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 1 & \mu & 0 \end{bmatrix} + \frac{\mathrm{NL}}{2}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin\theta & -\cos\theta & 0 \\ -\mu & 1 & 0 \end{bmatrix}.$$

$$\mathbf{M}_G = -\left(\frac{\mathrm{PL}}{2}\right)(\cos\theta + \mu\sin\theta)\mathbf{k} + \left(\frac{\mathrm{NL}}{2}\right)(\sin\theta - \mu\cos\theta)\mathbf{k}$$

From the equation of angular motion,

$$-\left(\frac{\mathrm{PL}}{2}\right)(\cos\theta + \mu\sin\theta) + \left(\frac{\mathrm{NL}}{2}\right)(\sin\theta - \mu\cos\theta) = \mathrm{I}_B\alpha$$

From kinematics: Assume that at the instant of slip the angular velocity $\omega = 0$. The acceleration of the center of mass in terms of the acceleration at point N is

$$\mathbf{a}_G = \mathbf{a}_N + \boldsymbol{\alpha} \times \mathbf{r}_{G/N} - \omega^2\mathbf{r}_{G/N}$$

$$= a_N\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ -\dfrac{L\sin\theta}{2} & \dfrac{L\cos\theta}{2} & 0 \end{bmatrix}$$

$$\mathbf{a}_G = \left(a_N - \frac{\alpha L\cos\theta}{2}\right)\mathbf{i} + \left(-\frac{\alpha L\sin\theta}{2}\right)\mathbf{j},$$

from which $a_{Gy} = -\dfrac{L\sin\theta}{2}\alpha$.

The acceleration of the center of mass in terms of the acceleration at point P is $\mathbf{a}_G = \mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r}_{G/P}$.

$$\mathbf{a}_G = \mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r}_{G/P} - \omega^2\mathbf{r}_{G/P}$$

$$= a_P\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ \dfrac{L\sin\theta}{2} & -\dfrac{L\cos\theta}{2} & 0 \end{bmatrix},$$

$$\mathbf{a}_G = \left(\frac{\alpha L\cos\theta}{2}\right)\mathbf{i} + \left(a_P + \frac{\alpha L\sin\theta}{2}\right)\mathbf{j},$$

from which $a_{Gx} = \dfrac{L\cos\theta}{2}\alpha$.

Substitute to obtain the three equations in three unknowns,

(1) $P - \mu N = \dfrac{mL\cos\theta}{2}\alpha,$

(2) $\mu P + N = -\dfrac{mL\sin\theta}{2}\alpha + mg.$

(3) $-\dfrac{PL}{2}(\cos\theta + \mu\sin\theta) + \dfrac{NL}{2}(\sin\theta - \mu\cos\theta) = I_B\alpha.$

Solve the first two equations for P and N:

$$P = \frac{mL}{2(1+\mu^2)}(\cos\theta - \mu\sin\theta)\alpha + \frac{\mu mg}{(1+\mu^2)}.$$

$$N = -\frac{mL}{2(1+\mu^2)}(\sin\theta + \mu\cos\theta)\alpha + \frac{mg}{(1+\mu^2)}.$$

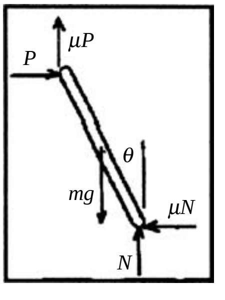
Substitute the first two equations into the third, and reduce to obtain

$$\alpha\left[I_B + \frac{mL^2}{4}\left(\frac{1-\mu^2}{1+\mu^2}\right)\right] = \frac{mgL}{2}\left(\frac{1-\mu^2}{1+\mu^2}\right)$$

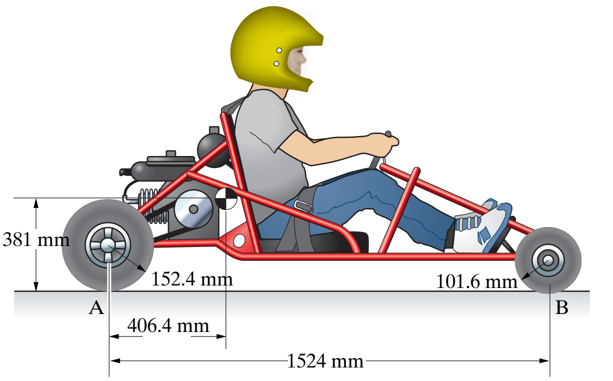
$$\sin\theta - mgL\left(\frac{\mu}{1+\mu^2}\right)\cos\theta.$$

Substitute $I_B = \left(\frac{1}{12}\right)mL^2$, reduce, and solve:

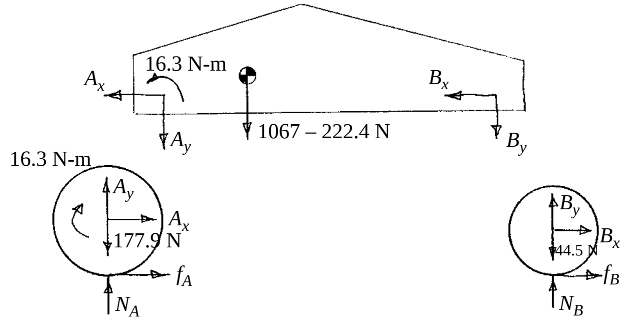
$$\alpha = \frac{(3(1-\mu^2)\sin\theta - 6\mu\cos\theta)g}{(2-\mu^2)L}$$



Problem 18.121 Each of the go-cart's front wheels weighs 22.2 N and has a moment of inertia of 0.014 kg-m^2. The two rear wheels and rear axle form a single rigid body weighing 177.9 N and having a moment of inertia of 0.136 kg-m^2. The total weight of the go-cart and driver is 1067 N. (The location of the center of mass of the go-cart and driver, *not including* the front wheels or the rear wheels and rear axle, is shown.) If the engine exerts a torque of 16.3 N-m on the rear axle, what is the go-cart's acceleration?



Solution: Let a be the cart's acceleration and α_A and α_B the wheels' angular accelerations. Note that



$a = (0.152)\alpha_A$, **(1)**

$a = (0.102)\alpha_B$. **(2)**

Front wheel:

$\sum F_x = B_x + f_B = (44.5/9.81)a$, **(3)**

$\sum F_y = B_y + N_B - 10 = 0$, **(4)**

$\sum M = -f_B(0.102) = (0.028)\alpha_B$. **(5)**

Rear Wheel:

$\sum F_x = A_x + f_A = (177.9/9.81)\,a$, **(6)**

$\sum F_y = A_y + N_A - 177.9 = 0$, **(7)**

$\sum M = 16.3 - f_A(0.152) = (0.136)\alpha_A$. **(8)**

Cart:

$\sum F_x = -A_x - B_x = (844.6/9.81)a$, **(9)**

$\sum F_y = -A_y - B_y - 844.6 = 0$, **(10)**

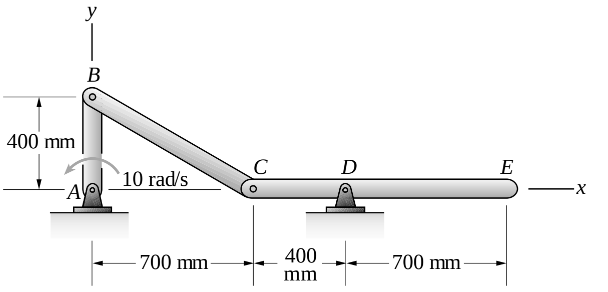
$\sum M = B_x[(0.381 - 0.102)] + B_y\,[(1.524 - 0.406)]$

$+ A_x[(0.381 - 0.152)] - A_y(0.406) - 16.3 = 0$. **(11)**

Solving Eqs. (1)–(11), we obtain

$\underline{a = 0.91 \text{ m/s}^2}$.

Problem 18.122 Bar AB rotates with a constant angular velocity of 10 rad/s in the counterclockwise direction. The masses of the slender bars BC and CDE are 2 kg and 3.6 kg, respectively. The y axis points upward. Determine the components of the forces exerted on bar BC by the pins at B and C at the instant shown.



Solution: The velocity of point B is

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 10 \\ 0 & 0.4 & 0 \end{bmatrix}$$

$$= -0.4(10)\mathbf{i} = -4\mathbf{i} \text{ (m/s)}.$$

The velocity of point C is

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B} = -4\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ 0.7 & -0.4 & 0 \end{bmatrix}$$

$$= -4\mathbf{i} + 0.4\omega_{BC}\mathbf{i} + 0.7\omega_{BC}\mathbf{j} \text{ (m/s)}.$$

From the constraint on the motion at point C, $\mathbf{v}_C = v_C\mathbf{j}$. Equate components: $0 = -4 + 0.4\omega_{BC}$, $v_C = 0.7\omega_{BC}$, from which $\omega_{BC} = 10$ rad/s, $v_C = 7$ m/s. The velocity at C in terms of the angular velocity ω_{CDE},

$$\mathbf{v}_C = \mathbf{v}_D + \boldsymbol{\omega}_{CDE} \times \mathbf{r}_{C/D}$$

$$= 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{CDE} \\ -0.4 & 0 & 0 \end{bmatrix} = -0.4\omega_{CDE}\mathbf{j},$$

from which $\omega_{CDE} = -\dfrac{7}{0.4} = -17.5$ rad/s.

The acceleration of point B is

$$\mathbf{a}_B = -\omega_{AB}^2\mathbf{r}_B = -(10^2)(0.4)\mathbf{j} = -40\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration at point C is $\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2\mathbf{r}_{C/B}$.

$$\mathbf{a}_C = -40\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 0.7 & -0.4 & 0 \end{bmatrix} - \omega_{BC}^2(0.7\mathbf{i} - 0.4\mathbf{j}) \text{ (m/s}^2\text{)}.$$

$$\mathbf{a}_C = +(0.4\alpha_{BC} - 0.7\omega_{BC}^2)\mathbf{i} + (-40 + 0.7\alpha_{BC} + 0.4\omega_{BC}^2)\mathbf{j} \text{ (m/s}^2\text{)}.$$

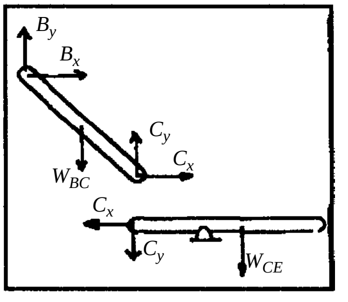
The acceleration in terms of the acceleration at D is

$$\mathbf{a}_C = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{CDE} \\ -0.4 & 0 & 0 \end{bmatrix} - \omega_{CDE}^2(-0.4\mathbf{i})$$

$$= -0.4\alpha_{CDE}\mathbf{j} + 0.4\omega_{CDE}^2\mathbf{i}.$$

Equate components and solve:

$\alpha_{BC} = 481.25$ rad/s^2, $\alpha_{CDE} = -842.19$ rad/s^2.



The acceleration of the center of mass of BC is

$$\mathbf{a}_G = -40\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ 0.35 & -0.2 & 0 \end{bmatrix} - \omega_{BC}^2(0.35\mathbf{i} - 0.2\mathbf{j}),$$

from which $\mathbf{a}_G = 61.25\mathbf{i} + 148.44\mathbf{j}$ (m/s^2)

The equations of motion: $B_x + C_x = m_{BC}a_{Gx}$, $B_y + C_y - m_{BC}g = m_{BC}a_{Gy}$, where the accelerations a_{Gx}, a_{Gy} are known. The moment equation, $0.35C_y + 0.2C_x - 0.2B_x - 0.35B_y = I_{BC}\alpha_{BC}$, where α_{BC}, is known, and

$$I_{BC} = \left(\frac{1}{12}\right)m_{BC}L_{BC}^2 = 0.1083 \text{ kg-m}^2,\ 0.4C_y - 0.15m_{CE}g = I_D\alpha_{CE},$$

where $\quad I_D = \left(\dfrac{1}{12}\right)m_{CE}L_{CE}^2 + (0.15)^2m_{CE} = 0.444$ kg-m^2,

is the moment of inertia about the pivot point D, and 0.15 m is the distance between the point D and the center of mass of bar CDE. Solve these four equations in four unknowns by iteration:

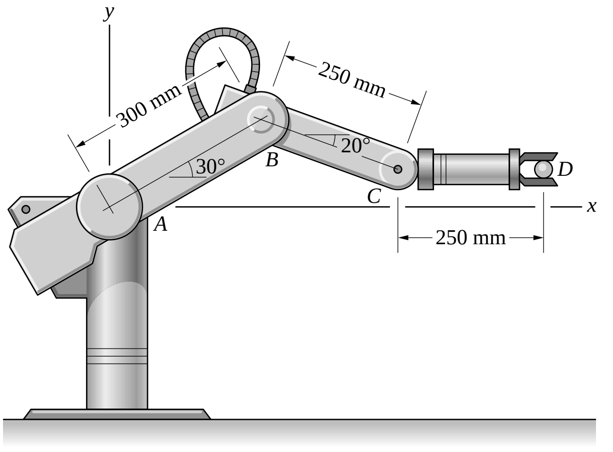
$B_x = -1959$ N,

$B_y = 1238$ N,

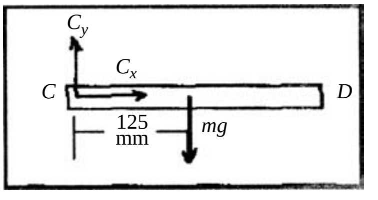
$C_x = 2081$ N,

$C_y = -922$ N.

Problem 18.123 At the instant shown, the arms of the robotic manipulator have the constant counterclockwise angular velocities $\omega_{AB} = -0.5$ rad/s, $\omega_{BC} = 2$ rad/s, and $\omega_{CD} = 4$ rad/s. The mass of arm CD is 10 kg, and the center of mass is at its midpoint. At this instant, what force and couple are exerted on arm CD at C?



Solution: The relative vector locations of B, C, and D are



$$\mathbf{r}_{B/A} = 0.3(\mathbf{i}\cos 30^\circ + \mathbf{j}\sin 30^\circ)$$

$$= 0.2598\mathbf{i} + 0.150\mathbf{j} \text{ (m)},$$

$$\mathbf{r}_{C/B} = 0.25(\mathbf{i}\cos 20^\circ - \mathbf{j}\sin 20^\circ)$$

$$= 0.2349\mathbf{i} - 0.08551\mathbf{j} \text{ (m)},$$

$$\mathbf{r}_{D/C} = 0.25\mathbf{i} \text{ (m)}.$$

The acceleration of point B is

$$\mathbf{a}_B = -\omega_{AB}^2\mathbf{r}_{B/A} = -(0.5^2)(0.3\cos 30^\circ\mathbf{i} + 0.3\sin 30^\circ\mathbf{j}),$$

$$\mathbf{a}_B = -0.065\mathbf{i} - 0.0375\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration at point C is

$$\mathbf{a}_C = \mathbf{a}_B - \omega_{BC}^2\mathbf{r}_{C/B} = \mathbf{a}_B - \omega_{BC}^2(0.2349\mathbf{i} - 0.08551\mathbf{j}).$$

$$\mathbf{a}_C = -1.005\mathbf{i} + 0.3045\mathbf{j} \text{ (m/s}^2\text{)}.$$

The acceleration of the center of mass of CD is

$$\mathbf{a}_G = \mathbf{a}_C - \omega_{CD}^2(0.125\mathbf{i}) \text{ (m/s}^2\text{)},$$

from which

$$\mathbf{a}_G = -3.005\mathbf{i} + 0.3045\mathbf{j} \text{ (m/s}^2\text{)}.$$

For the arm CD the three equations of motion in three unknowns are

$$C_y - m_{CD}g = m_{CD}a_{Gy}, C_x = m_{CD}a_{Gx}, M - 0.125C_y = 0,$$

which have the direct solution:

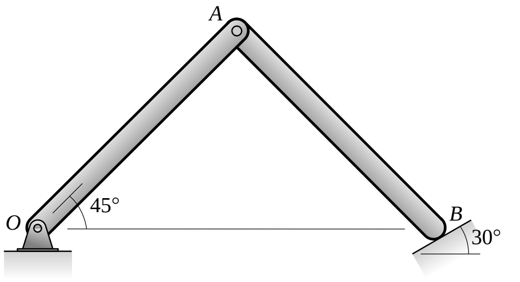
$$C_y = 101.15 \text{ N},$$

$$C_x = -30.05 \text{ N}.$$

$$M = 12.64 \text{ N-m},$$

where the negative sign means a direction opposite to that shown in the free body diagram.

Problem 18.124 Each bar is 1 m in length and has a mass of 4 kg. The inclined surface is smooth. If the system is released from rest in the position shown, what are the angular accelerations of the bars at that instant?



Solution: For convenience, denote $\theta = 45°$, $\beta = 30°$, and $L = 1$ m. The acceleration of point A is

$$\mathbf{a}_A = \boldsymbol{\alpha}_{OA} \times \mathbf{r}_{A/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{OA} \\ L\cos\theta & L\sin\theta & 0 \end{bmatrix}.$$

$$\mathbf{a}_A = \alpha_{OA}(-\mathbf{i}L\sin\theta + \mathbf{j}L\cos\theta) \text{ (m/s}^2).$$

The acceleration of A is also given by

$$\mathbf{a}_A = \mathbf{a}_B + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B}.$$

$$\mathbf{a}_A = \mathbf{a}_B + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ -L\cos\theta & L\sin\theta & 0 \end{bmatrix}.$$

$$\mathbf{a}_A = \mathbf{a}_B - \mathbf{i}\alpha_{AB}L\sin\theta - \mathbf{j}\alpha_{AB}L\cos\theta \text{ (m/s}^2).$$

From the constraint on the motion at B, Equate the expressions for the acceleration of A to obtain the two equations:

(1) $-\alpha_{OA}L\sin\theta = a_B\cos\beta - \alpha_{AB}L\sin\theta$,

and (2) $\alpha_{OA}L\cos\theta = a_B\sin\beta - \alpha_{AB}L\cos\theta$.

The acceleration of the center of mass of AB is

$$\mathbf{a}_{GAB} = \mathbf{a}_A + \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{GAB/A}$$

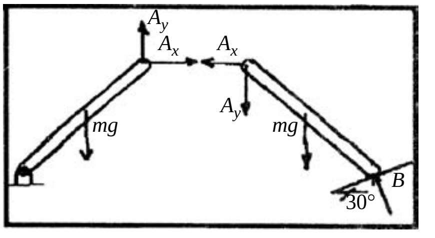
$$= \mathbf{a}_A + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ \dfrac{L\cos\theta}{2} & -\dfrac{L\sin\theta}{2} & 0 \end{bmatrix},$$

$$\mathbf{a}_{GAB} = \mathbf{a}_A + \frac{L\alpha_{AB}}{2}\sin\theta\mathbf{i} + \frac{\alpha_{AB}L}{2}\cos\theta\mathbf{j} \text{ (m/s}^2),$$

from which

(3) $a_{GABx} = -\alpha_{OA}L\sin\theta + \dfrac{L\alpha_{AB}}{2}\sin\theta$ (m/s^2),

(4) $a_{GABy} = \alpha_{OA}L\cos\theta + \dfrac{L\alpha_{AB}}{2}\cos\theta$.



The equations of motion for the bars: for the pin supported left bar:

(5) $A_yL\cos\theta - A_xL\sin\theta - mg\left(\dfrac{L}{2}\right)\cos\theta = I_{OA}\alpha_{OA}$,

where $I_{OA} = \left(\dfrac{mL^2}{3}\right) = \dfrac{4}{3}$ kg-m^2.

The equations of motion for the right bar:

(6) $-A_x - B\sin\beta = ma_{GABx}$,

(7) $-A_y - mg + B\cos\beta = ma_{GABy}$,

(8) $A_y\left(\dfrac{L}{2}\right)\cos\theta + A_x\left(\dfrac{L}{2}\right)\sin\theta + B\left(\dfrac{L}{2}\right)\sin\theta\cos\beta$

$$- B\left(\frac{L}{2}\right)\cos\theta\sin\beta = I_{CAB}\alpha_{AB},$$

where $I_{GAB} = \left(\dfrac{1}{12}\right)mL^2 = \left(\dfrac{1}{3}\right)$ kg-m^2.

These eight equations in eight unknowns are solved by iteration: $A_x = -19.27$ N, $A_y = 1.15$ N, $\alpha_{OA} = 0.425$ rad/s^2, $\alpha_{AB} = -1.59$ rad/s^2, $B = 45.43$ N, $a_{GABx} = -0.8610$ m/s^2, $a_{GABy} = -0.2601$ m/s^2

Problem 18.125 Each bar is 1 m in length and has a mass of 4 kg. The inclined surface is smooth. If the system is released from rest in the position shown, what is the magnitude of the force exerted on bar OA by the support at O at that instant?

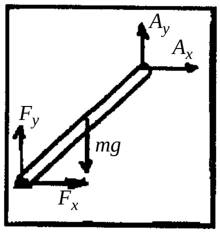
Solution: The acceleration of the center of mass of the bar OA is

$$\mathbf{a}_{GOA} = \boldsymbol{\alpha}_{OA} \times \mathbf{r}_{G/A} = \mathbf{a}_A + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{OA} \\ \dfrac{L\cos\theta}{2} & \dfrac{L\sin\theta}{2} & 0 \end{bmatrix},$$

$$\mathbf{a}_{GOA} = -\frac{L\sin\theta}{2}\alpha_{OA}\mathbf{i} + \frac{L\cos\theta}{2}\alpha_{OA}\mathbf{j}\ (\text{m/s}^2).$$

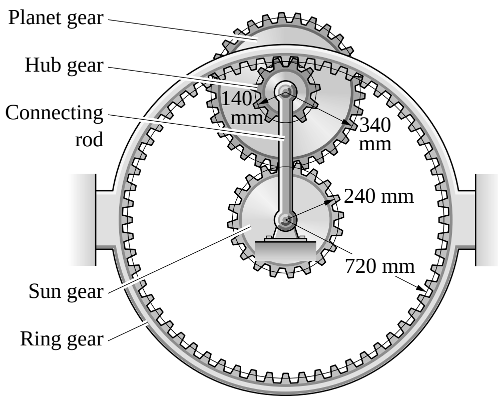
The equations of motion:

$$F_x + A_x = ma_{GOAx},\ F_y + A_y - mg = ma_{GOAy}.$$



Use the solution to Problem 18.140: $\theta = 45^\circ$, $\alpha_{GA} = 0.425$ rad/s^2, $A_x = -19.27$ N, $m = 4$ kg, from which $F_x = 18.67$ N, $F_y = 38.69$ N, from which $|\mathbf{F}| = \sqrt{F_x^2 + F_y^2} = 42.96$ N

Problem 18.126 The fixed ring gear lies *in the horizontal plane*. The hub and planet gears are bonded together. The mass and moment of inertia of the combined hub and planet gears are $m_{HP} = 130$ kg and $I_{HP} = 130$ kg-m^2. The moment of inertia of the sun gear is $I_s = 60$ kg-m^2. The mass of the connecting rod is 5 kg, and it can be modeled as a slender bar. If a 1 kN-m counterclockwise couple is applied to the sun gear, what is the resulting angular acceleration of the bonded hub and planet gears?



Solution: The moment equation *for the sun gear* is

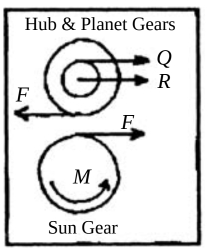
(1) $M - 0.24F = I_s\alpha_s$.

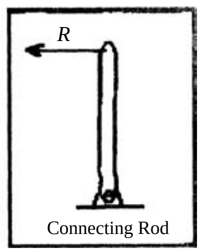
For the hub and planet gears:

(2) $(0.48)\alpha_{HP} = -0.24\alpha_s$,

(3) $F - Q - R = m_{HP}(0.14)(-\alpha_{HP})$,

(4) $(0.14)Q + 0.34F - I_{HP}(-\alpha_{HP})$.





For the connecting rod:

(5) $(0.58)R = I_{CR}\alpha_{CR}$,

where $I_{CR} = \left(\dfrac{1}{3}\right)m_{GR}(0.58^2) = 0.561$ kg-m^2.

(6) $(0.58)\alpha_{CR} = -(0.14)\alpha_{HP}$.

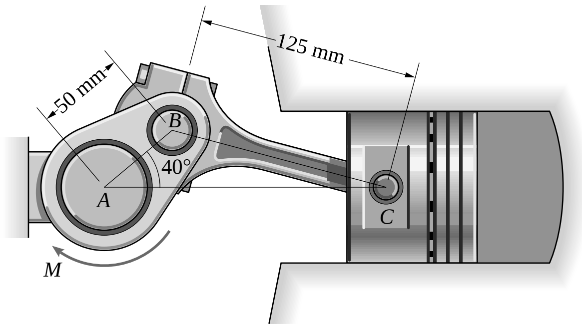
These six equations in six unknowns are solved by iteration:

$F = 1482.7$ N, $\alpha_s = 10.74$ rad/s^2,

$\alpha_{HP} = -5.37$ rad/s^2, $Q = 1383.7$ N,

$R = 1.25$ N, $\alpha_{CR} = 1.296$ rad/s^2.

Problem 18.127 The system is stationary at the instant shown. The net force exerted on the piston by the exploding fuel-air mixture and friction is 5 kN to the left. A clockwise couple $M = 200$ N-m acts on the crank AB. The moment of inertia of the crank about A is 0.0003 kg-m^2. The mass of the connecting rod BC is 0.36 kg, and its center of mass is 40 mm from B on the line from B to C. The connecting rod's moment of inertia about its center of mass is 0.0004 kg-m^2. The mass of the piston is 4.6 kg. What is the piston's acceleration? (Neglect the gravitational forces on the crank and connecting rod.)



Solution: From the law of sines:

$$\frac{\sin\beta}{0.05} = \frac{\sin 40^\circ}{0.125},$$

from which $\beta = 14.9^\circ$. The vectors

$$\mathbf{r}_{B/A} = 0.05(\mathbf{i}\cos 40^\circ + \mathbf{j}\sin 40^\circ)\mathbf{r}_{B/A}$$

$$= 0.0383\mathbf{i} + 0.0321\mathbf{j} \text{ (m)}.$$

$$\mathbf{r}_{B/C} = 0.125(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta) \text{ (m)}.$$

$$\mathbf{r}_{B/C} = -0.121\mathbf{i} + 0.0321, \text{ (m)}.$$

The acceleration of point B is

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{B/A} - \omega_{AB}^2 \mathbf{r}_{B/A},$$

$$\mathbf{a}_B = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{AB} \\ 0.0383 & 0.0321 & 0 \end{bmatrix}$$

$$- \omega_{AB}^2(0.0383\mathbf{i} + 0.0321\mathbf{j}) \text{ (m/s}^2).$$

The acceleration of point B in terms of the acceleration of point C is

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{B/C} = a_C\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -0.121 & 0.0321 & 0 \end{bmatrix}$$

$$- \omega_{BC}^2(-0.121\mathbf{i} + 0.0321\mathbf{j}) \text{ (m/s}^2).$$

Equate the two expressions for the acceleration of point B, note $\omega_{AB} = \omega_{BC} = 0$, and separate components:

(1) $-0.05\alpha_{AB}\sin 40^\circ = a_C - 0.125\alpha_{BC}\sin\beta$,

(2) $0.05\alpha_{AB}\cos 40^\circ = -0.125\alpha_{BC}\cos\beta$.

The acceleration of the center of mass of the connecting rod is

$$\mathbf{a}_{GCR} = \mathbf{a}_C + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{GCR/C} - \omega_{BC}^2 \mathbf{r}_{GCR/C},$$

$$\mathbf{a}_{GCR} = a_C\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha_{BC} \\ -0.085\cos\beta & 0.085\sin\beta & 0 \end{bmatrix}$$

$$- \omega_{BC}^2(-0.085\cos\beta\mathbf{i} + 0.085\sin\beta\mathbf{j}) \text{ (m/s}^2),$$

from which

(3) $a_{GCRx} = a_C - 0.085\alpha_{BC}\sin\beta$ (m/s^2),

(4) $a_{GCRy} = -0.085\alpha_{BC}\cos\beta$ (m/s^2).

The equations of motion *for the crank*:

(5) $B_y(0.05\cos 40^\circ) - B_x(0.05\sin 40^\circ) - M = I_A\alpha_{AB}$

For the connecting rod:

(6) $-B_x + C_x = m_{CR}a_{GCRx}$

(7) $-B_y + C_y = m_{CR}a_{GCRy}$

(8) $C_y(0.085\cos\beta) + C_x(0.085\sin\beta) + B_x(0.04\sin\beta)$

$+ B_y(0.04\cos\beta) = I_{GCR}\alpha_{BC}$

For the piston:

(9) $-C_x - 5000 = m_P a_C$.

These nine equations in nine unknowns are solved by iteration:

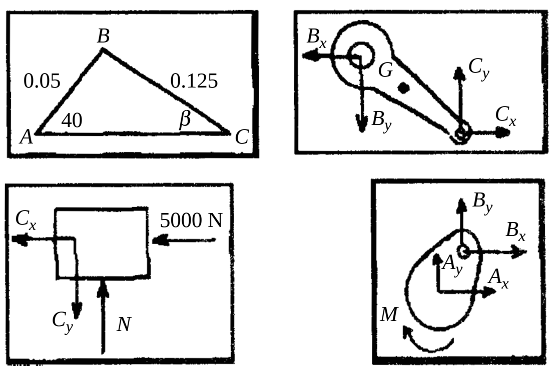
$\alpha_{AB} = 1255.7$ rad/s^2, $\alpha_{BC} = -398.2$ rad/s^2,

$a_{GCRx} = -44.45$ m/s^2, $a_{GCRy} = 32.71$ m/s^2,

$B_y = 1254.6$ N, $B_x = -4739.5$ N,

$C_x = -4755.5$ N, $C_y = 1266.3$ N,

$a_C = -53.15$ m/s^2.



Problem 18.128 If the crank AB in Problem 18.127 has a counterclockwise angular velocity of 2000 rpm at the instant shown, what is the piston's acceleration?

Solution: The angular velocity of AB is

$$\omega_{AB} = 2000\left(\frac{2\pi}{60}\right) = 209.44 \text{ rad/s}.$$

The angular velocity of the connecting rod BC is obtained from the expressions for the velocity at point B and the known value of ω_{AB}:

$$\mathbf{v}_B = \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB} \\ 0.05\cos 40^\circ & 0.05\sin 40^\circ & 0 \end{bmatrix}.$$

$$\mathbf{v}_B = -0.05\sin 40^\circ \omega_{AB}\mathbf{i} + 0.05\cos 40^\circ \omega_{AB}\mathbf{j} \text{ (m/s)}.$$

$$\mathbf{v}_B = v_C\mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -0.125\cos\beta & 0.125\sin\beta & 0 \end{bmatrix},$$

$$\mathbf{v}_B = v_C\mathbf{i} - 0.125\sin\beta\omega_{BC}\mathbf{i} - 0.125\cos\beta\omega_{BC}\mathbf{j} \text{ (m/s)}.$$

From the $\mathbf{j}$ component, $0.05\cos 40^\circ\omega_{AB} = -0.125\cos\beta\omega_{BC}$, from which $\omega_{BC} = -66.4$ rad/s. The nine equations in nine unknowns obtained in the solution to Problem 18.127 are

(1) $-0.05\alpha_{AB}\sin 40^\circ - 0.05\omega_{AB}^2\cos 40^\circ$

$= a_C - 0.125\alpha_{BC}\sin\beta + 0.125\omega_{BC}^2\cos\beta,$

(2) $0.05\alpha_{AB}\cos 40^\circ - 0.05\omega_{AB}^2\sin 40^\circ$

$= -0.125\alpha_{BC}\cos\beta - 0.125\omega_{BC}^2\sin\beta,$

(3) $a_{GCRx} = a_C - 0.085\alpha_{BC}\sin\beta + 0.085\omega_{BC}^2\cos\beta \text{ (m/s}^2),$

(4) $a_{GCRy} = -0.085\alpha_{BC}\cos\beta - 0.085\omega_{BC}^2\sin\beta \text{ (m/s}^2),$

(5) $B_y(0.05\cos 40^\circ) - B_x(0.05\sin 40^\circ) - M = I_A\alpha_{AB},$

(6) $-B_x + C_x = m_{CR}a_{GCRx},$

(7) $-B_y + C_y = m_{CR}a_{GCRy},$

(8) $C_y(0.085\cos\beta) + C_x(0.085\sin\beta)$

$+ B_x(0.04\sin\beta) + B_y(0.04\cos\beta) = I_{GCR}\alpha_{BC}.$

(9) $-C_x - 5000 = m_P a_C.$

These nine equations in nine unknowns are solved by iteration:

$\alpha_{AB} = -39,386.4 \text{ rad/s}^2 \alpha_{BC} = 22{,}985.9 \text{ rad/s}^2,$

$a_{GCRx} = -348.34 \text{ m/s}^2, a_{GCRy} = -1984.5 \text{ m/s}^2,$

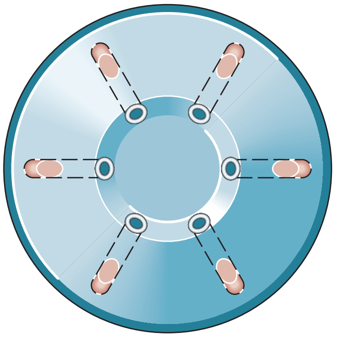
$B_y = 1626.7 \text{ N}, B_x = -3916.7 \text{ N},$

$C_x = -4042.1 \text{ N}, C_y = 912.25 \text{ N},$

$\boxed{a_c = -208.25 \text{ (m/s}^2)}$

Problem 19.1 The moment of inertia of the rotor of the medical centrifuge is $I = 0.2$ kg-m^2. The rotor starts from rest and the motor exerts a constant torque of 0.8 N-m on it.

(a) How much work has the motor done on the rotor when the rotor has rotated through four revolutions?
(b) What is the rotor's angular velocity (in rpm) when it has rotated through four revolutions?



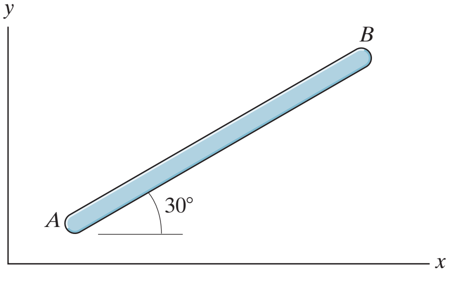
Solution:

(a) $W = (0.8 \text{ N-m})(4 \text{ rev})\left(\dfrac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 20.1 \text{ N-m}$ $\boxed{W = 20.1 \text{ N-m}}$

(b) $\dfrac{1}{2}I\omega^2 = W, \quad \dfrac{1}{2}(0.2 \text{ kg-m}^2)\omega^2 = 20.1 \text{ N-m} \quad \Rightarrow \quad \omega = \sqrt{\dfrac{2(20.1 \text{ N-m})}{0.2 \text{ kg-m}^2}}$

$\boxed{\omega = 14.2 \text{ rad/s}}$

Problem 19.2 The 17.8 N slender bar is 0.61 m in length. It started from rest in an initial position relative to the inertial reference frame. When it is in the position shown, the velocity of the end A is $6.71\mathbf{i} + 4.27\mathbf{j}$ (m/s) and the bar has a counterclockwise angular velocity of 12 rad/s. How much work was done on the bar as it moved from its initial position to its present position?



Solution: Work = Change in kinetic energy. To calculate the kinetic energy we will first need to find the velocity of the center of mass

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A}$$

$$= (6.71\mathbf{i} + 4.27\mathbf{j})(\text{m/s}) + (12 \text{ rad/s})\mathbf{k} \times (0.305 \text{ m})(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j})$$

$$= (-4.27\mathbf{i} + 8.5\mathbf{j}) \text{ m/s}$$

Now we can calculate the work, which is equal to the kinetic energy.

$$W = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

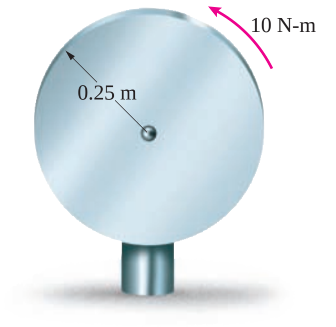
$$= \frac{1}{2}\left(\frac{17.8 \text{ N}}{9.81 \text{ m/s}^2}\right)[(-4.27 \text{ m/s})^2 + (8.5 \text{ m/s})^2]$$

$$+ \frac{1}{2}\left[\frac{1}{12}\left(\frac{17.8 \text{ N}}{9.81 \text{ m/s}^2}\right)(0.61 \text{ m})^2\right](12 \text{ rad/s})^2$$

$$= 86.1 \text{ N-m}.$$

$\boxed{W = 86.1 \text{ N-m.}}$

Problem 19.3 The 20-kg disk is at rest when the constant 10 N-m counterclockwise couple is applied. Determine the disk's angular velocity (in rpm) when it has rotated through four revolutions (a) by applying the equation of angular motion $\Sigma M = 1\alpha$, and (b) by applying the principle of work and energy.



Solution:

(a) First we find the angular acceleration

$$\Sigma M = I\alpha$$

$$(10\ \text{N-m}) = \frac{1}{2}(20\ \text{kg})(0.25\ \text{m})^2\alpha \Rightarrow \alpha = 16\ \text{rad/s}$$

Next we will integrate the angular acceleration to find the angular velocity.

$$\omega\frac{d\omega}{d\theta} = \alpha \Rightarrow \int_0^\omega \omega\, d\omega = \int_0^\theta \alpha\, d\theta \Rightarrow \frac{\omega^2}{2} = \alpha\theta$$

$$\omega = \sqrt{2\alpha\theta} = \sqrt{2(16\ \text{rad/s}^2)(4\ \text{rev})\left(\frac{\text{rev}}{2\pi\ \text{rad}}\right)}\left(\frac{60\ \text{s}}{\text{min}}\right) = 271\frac{\text{rev}}{\text{min}}$$

$$\boxed{\omega = 271\ \text{rpm.}}$$

(b) Applying the principle of work energy

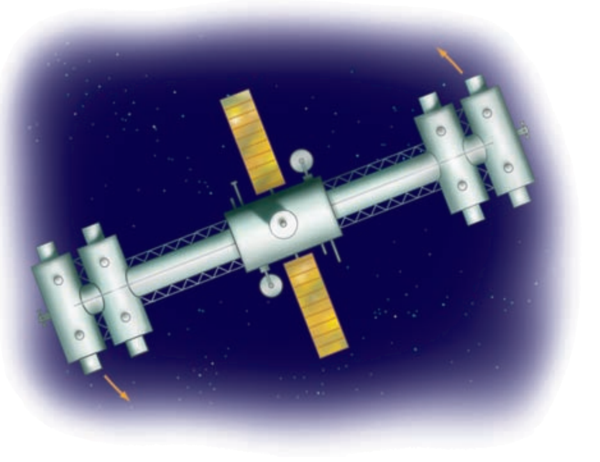
$$W = \frac{1}{2}I\omega^2$$

$$(10\ \text{N-m})(4\ \text{rev})\left(\frac{2\pi\ \text{rad}}{\text{rev}}\right) = \frac{1}{2}\left[\frac{1}{2}(20\ \text{kg})(0.25\ \text{m})^2\right]\omega^2$$

$$\omega = 28.4\ \text{rad/s}\left(\frac{\text{rev}}{2\pi\ \text{rad}}\right)\left(\frac{60\ \text{s}}{\text{min}}\right) = 271\ \text{rpm}$$

$$\boxed{\omega = 271\ \text{rpm.}}$$

Problem 19.4 The space station is initially not rotating. Its reaction control system exerts a constant couple on it until it has rotated 90°, then exerts a constant couple of the same magnitude in the opposite direction so that its angular velocity has decreased to zero when it has undergone a total rotation of 180°. The maneuver takes 6 hours. The station's moment of inertia about the axis of rotation is $I = 1.5 \times 10^{10}$ kg-m^2. How much work is done in performing this maneuver? In other words, how much energy had to be expended in the form of reaction control fuel?



Solution: We need to solve for the moment that causes a 90° rotation in a 3 hr time period. We will use $\Sigma M = I\alpha$ and the principle of work energy.

$$M = I\alpha = I\frac{d\omega}{dt} \Rightarrow \frac{d\omega}{dt} = \frac{M}{I} \Rightarrow \omega = \frac{M}{I}t$$

$$W = M\theta = \frac{1}{2}I\omega^2 \Rightarrow \omega = \sqrt{\frac{2M\theta}{I}}$$

Solving these two equations together, we find

$$M = \frac{2\theta I}{t^2} = \frac{2(90^\circ)\left(\dfrac{\pi \text{ rad}}{180^\circ}\right)(1.5 \times 10^{10} \text{ kg-m}^2)}{\left[(3 \text{ hr})\left(\dfrac{60 \text{ min}}{\text{hr}}\right)\left(\dfrac{60 \text{ min}}{\text{s}}\right)\right]^2} = 404 \text{ N-m}$$

The total work to accomplish the entire maneuver is then

$$W = (404 \text{ N-m})(180^\circ)\left(\frac{\pi \text{ rad}}{180^\circ}\right) = 1270 \text{ N-m}.$$

$\boxed{W = 1270 \text{ N-m.}}$

Problem 19.5 The helicopter's rotor starts from rest. Suppose that its engine exerts a constant 1627 N-m couple on the rotor and aerodynamic drag is negligible. The rotor's moment of inertia is $I = 542$ kg-m^2.

(a) Use work and energy to determine the magnitude of the rotor's angular velocity when it has rotated through five revolutions.
(b) What average power is transferred to the rotor while it rotates through five revolutions?

Solution:

(a) $U = (1627 \text{ N-m})(10\pi \text{ rad}) = \dfrac{1}{2}(542 \text{ N-s}^2\text{-m})\omega^2$

$\boxed{\omega = 13.7 \text{ rad/s}}$

(b) To find the average power we need to know the time

$1627 \text{ N-m} = (542 \text{ N-s}^2\text{-m})\alpha$

$\alpha = 3 \text{ rad/s}^2, \quad \omega = (3 \text{ rad/s}^2)t, \quad \theta = \dfrac{1}{2}(3 \text{ rad/s}^2)t^2$

$10\pi \text{ rad} = \dfrac{1}{2}(3 \text{ rad/s}^2)t^2 \Rightarrow t = 4.58 \text{ s}$

$$\boxed{\begin{aligned}\text{Power} &= \frac{U}{t} = \frac{(1627 \text{ N})(10\pi \text{ rad})}{4.58 \text{ s}} \\ &= 11171 \text{ N-m/s} = 15.0 \text{ hp}\end{aligned}}$$

Problem 19.6 The helicopter's rotor starts from rest. The moment exerted on it (in N-m) is given as a function of the angle through which it has turned in radians by $M = 6500 - 20\theta$. The rotor's moment of inertia is $I = 540$ kg-m^2. Determine the rotor's angular velocity (in rpm) when it has turned through 10 revolutions.

Solution: We will integrate to find the work.

$$W = \int M\, d\theta$$

$$= \int_0^{10(2\pi)} (6500 - 20\,\theta)\, d\theta = [6500\,\theta - 10\,\theta^2]_0^{10(2\pi)} = 3.69 \times 10^5 \text{ N-m}$$

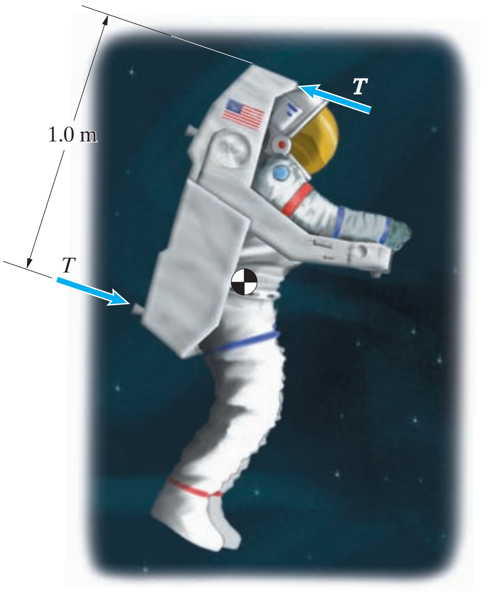
Using the principle of work energy we can find the angular velocity.

$$W = \frac{1}{2}I\omega^2 \Rightarrow \omega = \sqrt{\frac{2W}{I}} = \sqrt{\frac{2(3.69 \times 10^5 \text{ N-m})}{540 \text{ kg-m}^2}} = 37.0 \text{ rad/s}$$

$$\omega = 37.0 \text{ rad/s}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{\text{min}}\right) = 353 \text{ rpm.}$$

$\boxed{\omega = 353 \text{ rpm.}}$

Problem 19.7 During extravehicular activity, an astronaut's angular velocity is initially zero. She activates two thrusters of her maneuvering unit, exerting equal and opposite forces $T = 2$ N. The moment of inertia of the astronaut and her equipment about the axis of rotation is 45 kg-m^2. Use the principle of work and energy to determine the angle through which she has rotated when her angular velocity reaches 15° per second.



Solution: The moment that is exerted on the astronaut is

$$M = Td = (2\text{ N})(1\text{ m}) = 2\text{ N-m}.$$

Using work energy we have

$$M\theta = \frac{1}{2}I\omega^2 \Rightarrow \theta = \frac{I\omega^2}{2M} = \frac{(45\text{ kg-m}^2)\left(\dfrac{15^\circ}{180^\circ}\pi\text{ rad/s}\right)^2}{2(2\text{ N-m})} = 0.771\text{ rad}.$$

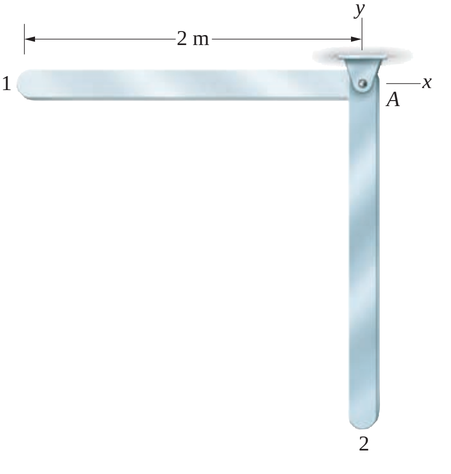
Thus

$$\theta = 0.771\text{ rad}\left(\frac{180^\circ}{\pi\text{ rad}}\right) = 44.2^\circ$$

$\boxed{\theta = 44.2^\circ}$

Problem 19.8 The 8-kg slender bar is released from rest in the horizontal position 1 and falls to position 2.

(a) How much work is done by the bar's weight as it falls from position 1 to position 2?
(b) How much work is done by the force exerted on the bar by the pin support as the bar falls from position 1 to position 2?
(c) Use conservation of energy to determine the bar's angular velocity when it is in position 2.



Solution:

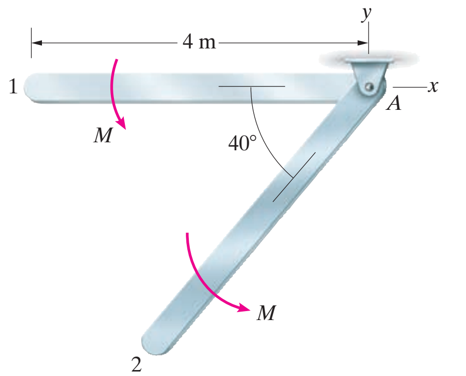
(a) $W = mgh = (8\text{ kg})(9.81\text{ m/s}^2)(1\text{ m}) = 78.5\text{ N-m}.$ $\boxed{W = 78.5\text{ N-m.}}$
(b) The pin force is applied to point A, which does not move. Therefore $\boxed{W = 0.}$
(c) Using conservation of energy, we have

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\omega^2 - mgh \Rightarrow \omega = \frac{\sqrt{6gh}}{L} = \frac{\sqrt{6(9.81\text{ m/s}^2)(1\text{ m})}}{2\text{ m}}$$

$\boxed{\omega = 3.84\text{ rad/s}}$

Problem 19.9 The 20-N bar is released from rest in the horizontal position 1 and falls to position 2. In addition to the force exerted on it by its weight, it is subjected to a constant counterclockwise couple $M = 30$ N-m. Determine the bar's counterclockwise angular velocity in position 2.



Solution: We will use the energy equation in the form

$$T_1 + V_1 + U_{12} = T_2 + V_2$$

$$0 + 0 + M\theta = \frac{1}{2}\left(\frac{1}{3}\,mL^2\right)\omega^2 - mg\frac{L}{2}\sin\theta$$

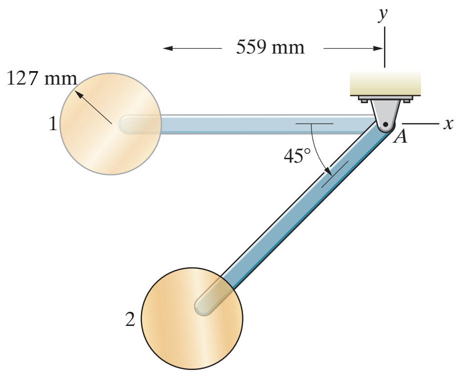
$$\Rightarrow \omega = \sqrt{\frac{6M\theta}{mL^2} + \frac{3g}{L}\sin\theta}$$

Thus

$$\omega = \sqrt{\frac{6(30\text{ N-m})\left(\dfrac{40^\circ}{180^\circ}\pi\text{ rad}\right)}{\left(\dfrac{20\text{ N}}{9.81\text{ m/s}^2}\right)(4\text{ m})^2} + \frac{3(9.81\text{ m/s}^2)}{4\text{ m}}\sin(40^\circ)} = 2.93\text{ rad/s}.$$

$$\boxed{\omega = 2.93\text{ rad/s.}}$$

Problem 19.10 The object consists of an 35.6 N slender bar welded to a circular disk. When the object is released from rest in position 1, its angular velocity in position 2 is 4.6 rad/s. What is the weight of the disk?



Solution: Using conservation of energy we have

$$T_1 + V_1 = T_2 + V_2 \Rightarrow 0 + 0 = T_2 + V_2$$

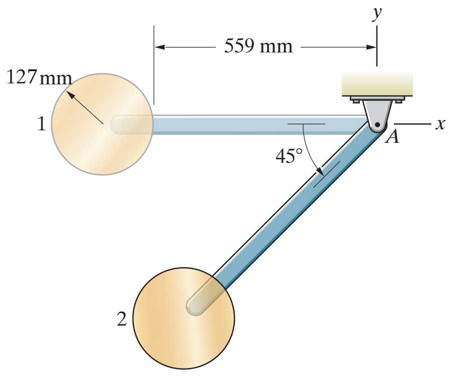
$$0 = \frac{1}{2}\left(\frac{1}{3}\left[\frac{35.6\text{ N}}{9.81\text{ m/s}^2}\right]\left[0.559\text{ m}\right]^2 + \frac{1}{2}\left[\frac{W\text{ s}^2}{9.81\text{ m}}\right]\left[0.127\text{ m}\right]^2 + \left[\frac{W\text{ s}^2}{9.81\text{ m}}\right]\left[0.686\right]^2\right)(4.6\text{ rad/s})^2$$

$$-(35.6\text{ N})(0.280\text{ m})\sin(45^\circ) - W(0.686)\sin(45^\circ)$$

Solving for W we find

$$W = 98.7\text{ N.}$$

Problem 19.11 The object consists of an 35.6 N slender bar welded to a 53.4 N circular disk The object is released from rest in position 1. Determine the x and y components of force exerted on the object by the pin support when it is in position 2.



Solution: We first determine the moment of inertia about the fixed point A and the distance from A to the center of mass.

$$I_A = \frac{1}{3}\left(\frac{35.6\ \text{N-s}^2}{9.81\ \text{m}}\right)(0.559)^2 + \frac{1}{2}\left(\frac{53.4\ \text{N-s}^2}{9.81\ \text{m}}\right)(0.127\,\text{m})^2 + \left(\frac{53.4\ \text{N-s}^2}{9.81\ \text{m}}\right)(0.686\ \text{m})^2$$

$$= 2.98\ \text{N-s}^2\text{-m}$$

$$d = \frac{(35.6\ \text{N})(0.280\ \text{m}) + (53.4\ \text{N})(0.686\ \text{m})}{(35.6\ \text{N}) + (53.4\ \text{N})} = 0.523\ \text{m}.$$

We now write the equations of motion and the work energy equation

$$\Sigma F_x : A_x = \left(\frac{89\ \text{N-s}^2}{9.81\ \text{m}}\right)(\alpha d \sin 45^\circ + \omega^2 d \cos 45^\circ),$$

$$\Sigma F_y : A_y - (89\ \text{N}) = \left(\frac{89\ \text{N-s}^2}{9.81\ \text{m}}\right)(-\alpha d \cos 45^\circ + \omega^2 d \sin 45^\circ),$$

$$\Sigma M_A : (89\ \text{N})\, d \cos 45^\circ = I_A \alpha,$$

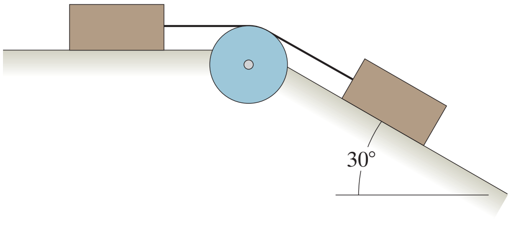
$$0 + 0 = -(89\ \text{N})\, d \sin 45^\circ + \frac{1}{2} I_A \omega^2.$$

Solving we have

$$\omega = 4.70\ \text{rad/s},\ \ \alpha = 11.0\ \text{rad/s}^2,\ \ A_x = 111.2\ \text{N},\ \ A_y = 125.9\ \text{N}.$$

$$\boxed{A_x = 111.2\ \text{N},\ A_y = 125.9\ \text{N}}$$

Problem 19.12 The mass of each box is 4 kg. The radius of the pulley is 120 mm and its moment of inertia is 0.032 kg-m^2. The surfaces are smooth. If the system is released from rest, how fast are the boxes moving when the left box has moved 0.5 m to the right?



Solution: Use conservation of energy. The angular velocity of the pulley is v/r.

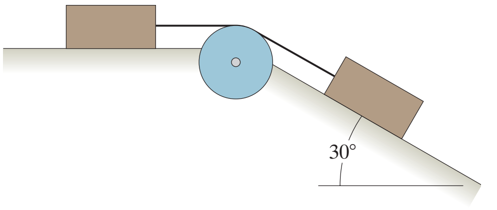
$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = \frac{1}{2}(4\ \text{kg})v^2 + \frac{1}{2}(4\ \text{kg})v^2 + \frac{1}{2}(0.032\ \text{kg-m}^2)\left(\frac{v}{0.12\ \text{m}}\right)^2$$

$$- (4\ \text{kg})(9.81\ \text{m/s}^2)(0.5\ \text{m})\sin 30^\circ$$

Solving we have $\boxed{v = 1.39\ \text{m/s.}}$

Problem 19.13 The mass of each box is 4 kg. The radius of the pulley is 120 mm and its moment of inertia is 0.032 kg-m^2. The coefficient of kinetic friction between the boxes and the surfaces is $\mu_k = 0.12$. If the system is released from rest, how fast are the boxes moving when the left box has moved 0.5 m to the right?



Solution: Use work energy. The angular velocity of the pulley is v/r.

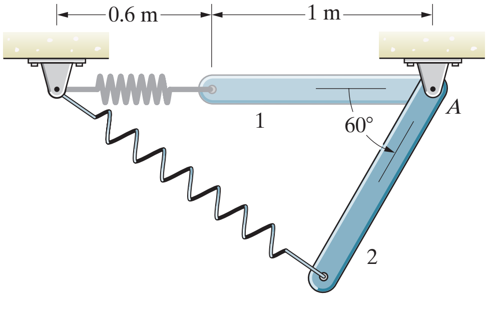
$$T_1 + V_1 + U_{12} = T_2 + V_2$$

$$0 + 0 - (0.12)(8\text{ kg})(9.81\text{ m/s}^2)(0.5\text{ m}) = \frac{1}{2}(8\text{ kg})v^2 + \frac{1}{2}(0.032\text{ kg-m}^2)\left(\frac{v}{0.12\text{ m}}\right)^2$$

$$- (4\text{ kg})(9.81\text{ m/s}^2)(0.5\text{ m})\sin 30^\circ$$

Solving we have $\boxed{v = 1.00\text{ m/s.}}$

Problem 19.14 The 4-kg bar is released from rest in the horizontal position 1 and falls to position 2. The unstretched length of the spring is 0.4 m and the spring constant is $k = 20$ N/m. What is the magnitude of the bar's angular velocity when it is in position 2.



Solution: In position 1 the spring is stretched a distance

$$d_1 = 0.6\text{ m} - 0.4\text{ m} = 0.2\text{ m},$$

while in position 2 the spring is stretched a distance

$$d_2 = \sqrt{(1.6\text{ m})^2 + (1\text{ m})^2 - 2(1.6\text{ m})(1\text{ m})\cos 60^\circ} - 0.4\text{ m} = 1.0\text{ m}$$

Using conservation of energy we have

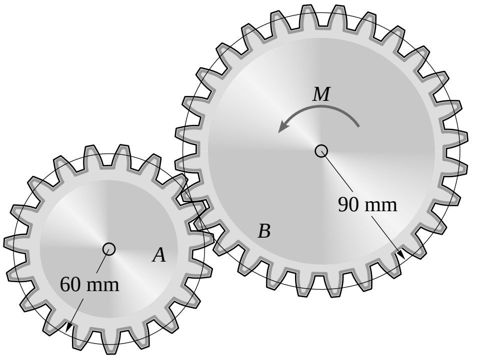
$$T_1 + V_1 = T_2 + V_2$$

$$0 + \frac{1}{2}kd_1^2 = \frac{1}{2}\left[\frac{1}{3}(4\text{ kg})(1\text{ m})^2\right]\omega^2$$

$$- (4\text{ kg})(9.81\text{ m/s}^2)(0.5\text{ m})\sin 60^\circ + \frac{1}{2}kd_2^2$$

Solving we find $\boxed{\omega = 3.33\text{ rad/s.}}$

Problem 19.15 The moments of inertia of gears that can turn freely on their pin supports are $I_A = 0.002$ kg-m^2 and $I_B = 0.006$ kg-m^2. The gears are at rest when a constant couple $M = 2$ N-m is applied to gear B. Neglecting friction, use principle of work and energy to determine the angular velocities of the gears when gear A has turned 100 revolutions.



Solution: From the principle of work and energy: $U = T_2 - T_1$, where $T_1 = 0$ since the system starts from rest. The work done is

$$U = \int_0^{\theta_B} M d\theta = M\theta_B.$$

Gear B rotates in a positive direction; gear A rotates in a negative direction, $\theta_A = -2\pi(100) = -200\pi$ rad. The angle traveled by the gear B is

$$\theta_B = -\frac{r_A}{r_B}\theta_A = -\left(\frac{0.06}{0.09}\right)(-200\pi) = 418.9 \text{ rad},$$

from which $U = M\theta_B = 2(418.9) = 837.76$ N-m.

The kinetic energy is $T_2 = \left(\frac{1}{2}\right) I_A\omega_A^2 + \left(\frac{1}{2}\right) I_B\omega_B^2$,

where $\omega_B = -\left(\frac{r_A}{r_B}\right)\omega_A$, from which

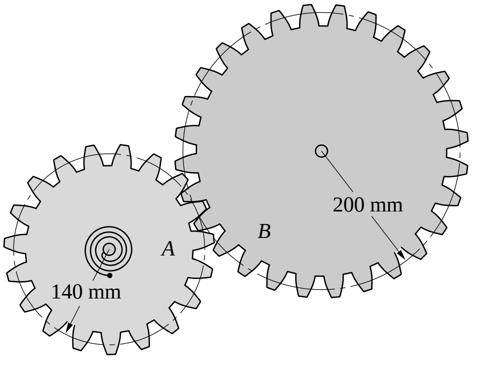
$$U = \left(\frac{1}{2}\right)\left(0.002 + \left(\frac{0.06}{0.09}\right)^2 (0.006)\right)\omega_A^2 = 837.76 \text{ N-m},$$

from which

$$\boxed{\omega_A = \pm\sqrt{359{,}039} = -599.2 \text{ rad/s}},$$

and $\boxed{\omega_B = -\left(\frac{0.06}{0.09}\right)\omega_A = 399.5 \text{ rad/s}}$

Problem 19.16 The moments of inertia of gears A and B are $I_A = 0.02$ kg-m^2 and $I_B = 0.09$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 12$ N-m/rad. If gear B is given an initial counterclockwise angular velocity of 10 rad/s with the torsional spring unstretched, through what maximum counterclockwise angle does gear B rotate?



Solution: The counterclockwise velocity of B and clockwise velocity of A are related by $0.2\omega_B = 0.14\omega_A$.

Applying conservation of energy,

$$\frac{1}{2}I_A\omega_{A1}^2 + \frac{1}{2}I_B\omega_{B1}^2 = \frac{1}{2}k\theta_{A2}^2:$$

$$\frac{1}{2}(0.02)\left[\left(\frac{0.2}{0.14}\right)(10)\right]^2 + \frac{1}{2}(0.09)(10)^2 = \frac{1}{2}(12)\theta_{A2}^2$$

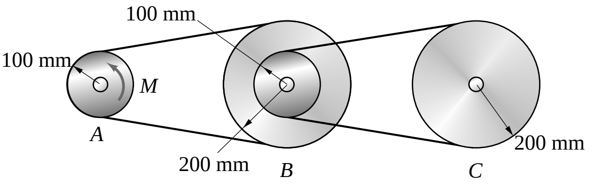
Solving, we obtain

$$\theta_{A2} = 1.04 \text{ rad}$$

so $\theta_{B2} = \dfrac{0.14}{0.2}\theta_{A2}$

$$= 0.731 \text{ rad} = 41.9^\circ.$$

Problem 19.17 The moments of inertia of three pulleys that can turn freely on their pin supports are $I_A = 0.002$ kg-m^2, $I_B = 0.036$ kg-m^2, and $I_C = 0.032$ kg-m^2. They are stationary when a constant couple $M = 2$ N-m is applied to pulley A. What is the angular velocity of pulley A when it has turned 10 revolutions?



Solution: All pulleys rotate in a positive direction:

$$\omega_C = \left(\frac{0.1}{0.2}\right)\omega_B,$$

$$\omega_B = \left(\frac{0.1}{0.2}\right)\omega_A,$$

from which

$$\omega_C = \left(\frac{0.1}{0.2}\right)^2 \omega_A.$$

From the principle of work and energy: $U = T_2 - T_1$, where $T_1 = 0$ since the pulleys start from a stationary position. The work done is

$$U = \int_0^{\theta_A} M\,d\theta = 2(2\pi)(10) = 40\pi \text{ N-m}.$$

The kinetic energy is

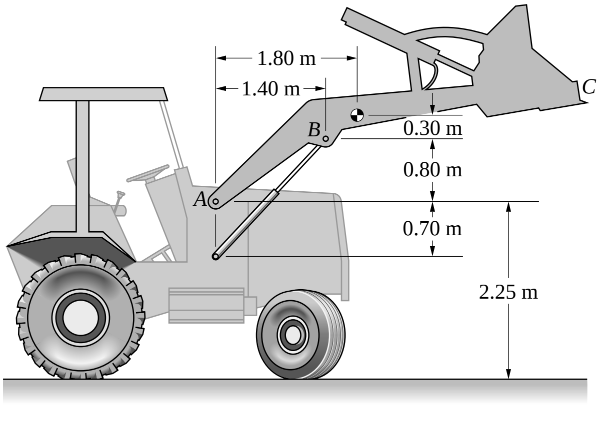
$$T_2 = \left(\frac{1}{2}\right) I_A\omega_A^2 + \left(\frac{1}{2}\right) I_B\omega_B^2 + \left(\frac{1}{2}\right) I_C\omega_C^2$$

$$= \left(\frac{1}{2}\right)\left(I_A + \left(\frac{0.1}{0.2}\right)^2 I_B + \left(\frac{0.1}{0.2}\right)^4 I_C\right)\omega_A^2,$$

from which $T_2 = 0.0065\omega_A^2$, and $U = 40\pi = 0.0065\omega_A^2$.

Solve: $\boxed{\omega_A = \sqrt{\dfrac{40\pi}{0.0065}} = 139.0 \text{ rad/s}}$

Problem 19.18 Model the arm ABC as a single rigid body. Its mass is 300 kg, and the moment of inertia about its center of mass is $I = 360$ kg-m^2. Starting from rest with its center of mass 2 m above the ground (position 1), the ABC is pushed upward by the hydraulic cylinders. When it is in the position shown (position 2), the arm has a counterclockwise angular velocity of 1.4 rad/s. How much work do the hydraulic cylinders do on the arm in moving it from position 1 to position 2?



Solution: From the principle of work and energy: $U = T_2 - T_1$, where $T_1 = 0$ since the system starts from rest. The work done is $U = U_{\text{cylinders}} - mg(h_2 - h_1)$, and the kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) I_A\omega^2,$$

from which

$$U_{\text{cylinders}} - mg(h_2 - h_1) = \left(\frac{1}{2}\right) I_A\omega^2$$

In position 1, $h_1 = 2$ m above the ground. In position 2, $h_2 = 2.25 + 0.8 + 0.3 = 3.35$ m. The distance from A to the center of mass is

$$d = \sqrt{1.8^2 + 1.1^2} = 2.11 \text{ m},$$

from which

$$I_A = I + md^2 = 1695 \text{ kg-m}^2.$$

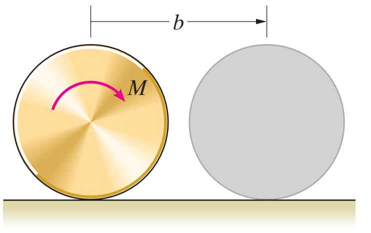
Substitute: $U_{\text{cylinders}} - 3973.05 = 1661.1$, from which

$\boxed{U_{\text{cylinders}} = 5630 \text{ N-m}}$

Problem 19.19 The mass of the circular disk is 5 kg and its radius is $R = 0.2$ m. The disk is stationary when a constant clockwise couple $M = 10$ N-m is applied to it, causing the disk to roll toward the right. Consider the disk when its center has moved a distance $b = 0.4$ m.

(a) How much work has the couple M done on the disk?
(b) How much work has been done by the friction force exerted on the disk by the surface?
(c) What is the magnitude of the velocity of the center of the disk?

(See Active Example 19.1.)



Solution:

(a) $U_{12} = M\theta = M\left(\frac{b}{R}\right) = (10 \text{ N-m})\left(\frac{0.4 \text{ m}}{0.2 \text{ m}}\right) = 20 \text{ N-m}.$ $\boxed{U_{12} = 20 \text{ N-m.}}$

(b) The friction force is a workless constraint force and does no work. $\boxed{U_{12} = 0}$.

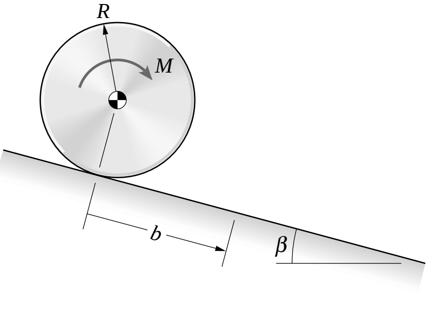
(c) Use work energy

$$U_{12} = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2$$

$$20 \text{ N-m} = \frac{1}{2}(5 \text{ kg})v^2 + \frac{1}{2}\left(\frac{1}{2}[5 \text{ kg}][0.2 \text{ m}]^2\right)\left(\frac{v}{0.2 \text{ m}}\right)^2$$

Solving we find $\boxed{v = 2.31 \text{ m/s.}}$

Problem 19.20 The mass of the homogeneous cylindrical disk is $m = 5$ kg and its radius is $R = 0.2$ m. The angle $\beta = 15^\circ$. The disk is stationary when a constant clockwise couple $M = 10$ N-m is applied to it. What is the velocity of the center of the disk when it has moved a distance $b = 0.4$ m? (See Active Example 19.1.)



Solution: The angle through which the disk rolls is $\theta = b/R$. The work done by the couple and the disk's weight is

$$v_{12} = M\left(\frac{b}{R}\right) + mgb\sin\beta.$$

Equating the work to the disk's kinetic energy

$$M\left(\frac{b}{R}\right) + mgb\sin\beta = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

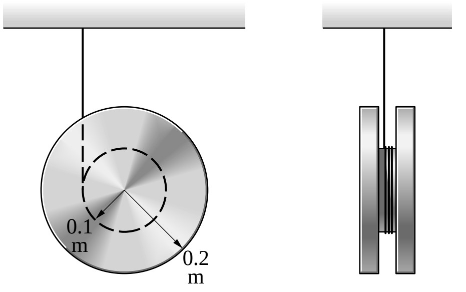
and using the relation $\omega = v/R$, we obtain

$$v = 2\sqrt{\frac{b}{3}\left(\frac{M}{mR} + g\sin\beta\right)}$$

$$= 2\sqrt{\frac{0.4}{3}\left[\frac{10}{(5)(0.2)} + 9.81\sin 15^\circ\right]}$$

$$= 2.59 \text{ m/s.}$$

Problem 19.21 The mass of the stepped disk is 18 kg and its moment of inertia is 0.28 kg-m^2. If the disk is released from rest, what is its angular velocity when the center of the disk has fallen 1 m?



Solution: The work done by the disk's weight is

$$v_{12} = mgh = (18 \text{ kg})\ (9.81 \text{ m/s}^2)\ (1 \text{ m})$$

$$= 176.6 \text{ N-m}.$$

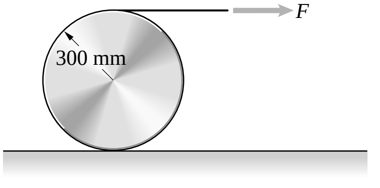
We equate the work to the final kinetic energy,

$$176.6 = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}(18)v^2 + \frac{1}{2}(0.28)\omega^2.$$

Using the relation $v = (0.1)\omega$ and solving for ω, we obtain $\omega = 27.7$ rad/s.

Problem 19.22 The 100-kg homogenous cylindrical disk is at rest when the force $F = 500$ N is applied to a cord wrapped around it, causing the disk to roll. Use the principle of work and energy to determine the angular velocity of the disk when it has turned one revolution.



Solution: From the principle of work and energy: $U = T_2 - T_1$, where $T_1 = 0$ since the disk is at rest initially. The distance traveled in one revolution by the center of the disk is $s = 2\pi R = 0.6\pi$ m. As the cord unwinds, the force F acts through a distance of 2 s.
The work done is

$$U = \int_0^{2s} F\,ds = 2\ F(0.6\pi) = 1884.96 \text{ N-m}.$$

The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) I\omega^2 + \left(\frac{1}{2}\right) mv^2,$$

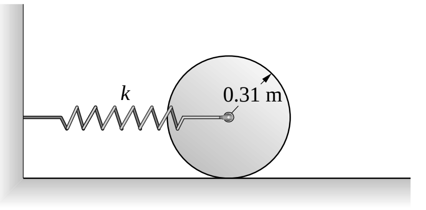
where $I = \frac{1}{2}mR^2$, and $v = R\omega$, from which

$$T_2 = \frac{3}{4}mR^2\omega^2 = 6.75\omega^2.$$

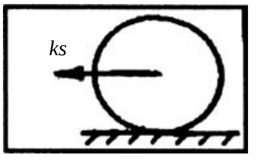
$$U = T_2,\ 1884.96 = 6.75\omega^2,$$

from which $\omega = -16.7$ rad/s (clockwise).

Problem 19.23 The 15 kg homogenous cylindrical disk is given a clockwise angular velocity of 2 rad/s with the spring unstretched. The spring constant is $k = 43.8$ N/m. If the disk rolls, how far will its center move to the right?



Solution: From the principle of work and energy: $U = T_2 - T_1$, where $T_2 = 0$ since the disk comes to rest at point 2. The work done by the spring is



$$U = -\frac{1}{2}kS^2.$$

The kinetic energy is

$$T_1 = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2.$$

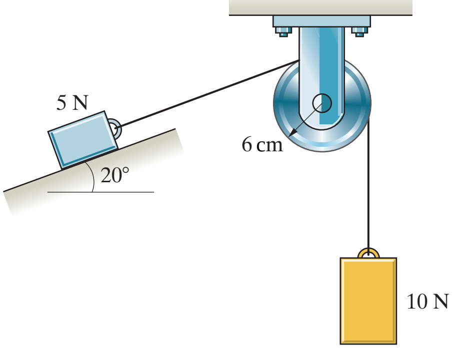
By inspection $v = \omega R$, from which

$$T_1 = \left(\frac{1}{2}\right)\left(\frac{m}{2}R^2 + mR^2\right)\omega^2 = \frac{3}{4}mR^2\omega^2 = 0.75(2^2) = 4.07 \text{ N-m},$$

$$U = -T_1, - \ (43.8)\ S^2 = -4.07,$$

from which $S = \sqrt{0.186} = 0.43$ m.

Problem 19.24 The system is released from rest. The moment of inertia of the pulley is 0.04 kg-m^2. The slanted surface is smooth. Determine the magnitude of the velocity of the 10 N weight when it has fallen 2 m.



Solution: Use conservation of energy

$$T_1 = 0$$

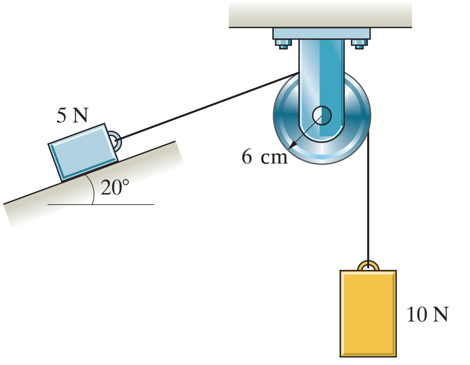
$$V_1 = 0$$

$$T_2 = \frac{1}{2}\left(\frac{15 \text{ N-s}^2}{9.81 \text{ m}}\right)v^2 + \frac{1}{2}(0.04 \text{ kg-m}^2)\left(\frac{v}{0.06 \text{ m}}\right)^2$$

$$V_2 = -(10 \text{ N})(2 \text{ m}) + (5 \text{ N})(2 \text{ m})\sin 20^\circ$$

$$T_1 + V_1 = T_2 + V_2 \Rightarrow \boxed{v = 1.62 \text{ m/s.}}$$

Problem 19.25 The system is released from rest. The moment of inertia of the pulley is 0.04 kg-m^2. The coefficient of kinetic friction between the 5-N weight and the slanted surface is $\mu_k = 0.3$. Determine the magnitude of the velocity of the 10-N weight when it has fallen 2 m.



Solution: Use work energy.

$$T_1 = 0$$

$$V_1 = 0$$

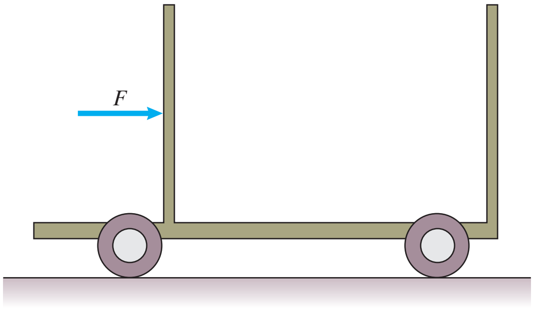
$$U_{12} = -(0.3)(5\text{ N})\cos 20^\circ(2\text{ m})$$

$$T_2 = \frac{1}{2}\left(\frac{15\text{ N-s}^2}{9.81\text{ m}}\right)v^2 + \frac{1}{2}(0.04\text{ kg-m}^2)\left(\frac{v}{0.06\text{ m}}\right)^2$$

$$V_2 = -(10\text{ N})(2\text{ m}) + (5\text{ N})(2\text{ m})\sin 20^\circ$$

$$T_1 + V_1 + U_{12} = T_2 + V_2 \Rightarrow \boxed{v = 1.48\text{ m/s.}}$$

Problem 19.26 Each of the cart's four wheels weighs 10 N, has a radius of 5 cm, and has moment of inertia $I = 0.02$ kg-m^2. The cart (not including its wheels) weighs 80 N. The cart is stationary when the constant horizontal force $F = 40$ N is applied. How fast is the cart going when it has moved 2 m to the right?



Solution: Use work energy

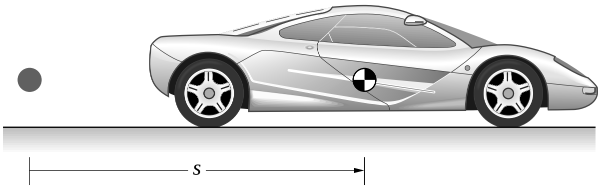
$$U_{12} = Fd = (40\text{ N})(2\text{ m}) = 80\text{ N-m}$$

$$T_1 = 0$$

$$T_2 = \frac{1}{2}\left(\frac{80\text{ N-s}^2}{9.81\text{ m}}\right)v^2 + 4\left[\frac{1}{2}\left(\frac{10\text{ N-s}^2}{9.81\text{ m}}\right)v^2 + \frac{1}{2}(0.02\text{ kg-m}^2)\left(\frac{v}{0.05\text{ m}}\right)^2\right]$$

$$T_1 + U_{12} = T_2 \Rightarrow \boxed{v = 1.91\text{ m/s.}}$$

Problem 19.27 The total moment of inertia of car's two rear wheels and axle is I_R, and the total moment of inertia of the two front wheels is I_F. The radius of the tires is R, and the total mass of the car, including the wheels, is m. The car is moving at velocity v_0 when the driver applies the brakes. If the car's brakes exert a constant retarding couple M on each wheel and the tires do not slip, determine the car's velocity as a function of the distance s from the point where the brakes are applied.



Solution: When the car rolls a distance s, the wheels roll through an angle s/R so the work done by the brakes is $V_{12} = -4M(s/R)$. Let m_F be the total mass of the two front wheels, m_R the mass of the rear wheels and axle, and m_c the remainder of the car's mass. When the car is moving at velocity v its total kinetic energy is

$$T = \frac{1}{2}m_c v^2 + \frac{1}{2}m_R v^2 + \frac{1}{2}I_R(v/R)^2 + \frac{1}{2}I_F(v/R)^2$$

$$= \frac{1}{2}[m + (I_R + I_F)/R^2]v^2.$$

From the principle of work and energy, $V_{12} = T_2 - T_1$:

$$-4M(s/R) = \frac{1}{2}[m + (I_R + I_F)/R^2]v^2 - \frac{1}{2}[m + (I_R + I_F)/R^2]v_0^2$$

Solving for v, we get $v = \sqrt{v_0^2 - 8Ms/[Rm + (I_R + I_F)/R]}$.

Problem 19.28 The total moment of inertia of the car's two rear wheels and axle is 0.24 kg-m^2. The total moment of inertia of the two front wheels is 0.2 kg-m^2. The radius of the tires is 0.3 m. The mass of the car, including the wheels, is 1480 kg. The car is moving at 100 km/h. If the car's brakes exert a constant retarding couple of 650 N-m on each wheel and the tires do not slip, what distance is required for the car to come to a stop? (See Example 19.2.)

Solution: From the solution of Problem 19.27, the car's velocity when it has moved a distance s is

$$v = \sqrt{v_0^2 - 8Ms/[Rm + (I_R + I_F)/R]}.$$

Setting $v = 0$ and solving for s we obtain

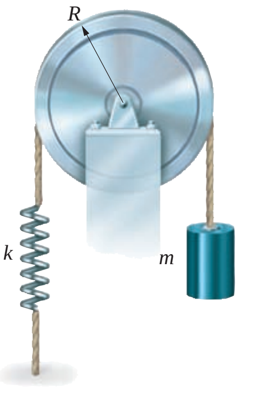
$$s = [Rm + (I_R + I_F)/R]v_0^2/(8M).$$

The car's initial velocity is

$$v_0 = 100{,}000/3600 = 27.8 \text{ m/s}$$

So, $s = [(0.3)(1480) + (0.24 + 0.2)/0.3](27.8)^2/[8(650)] = 66.1$ m.

Problem 19.29 The radius of the pulley is $R = 100$ mm and its moment of inertia is $I = 0.1$ kg-m^2. The mass $m = 5$ kg. The spring constant is $k = 135$ N/m. The system is released from rest with the spring unstretched. Determine how fast the mass is moving when it has fallen 0.5 m.



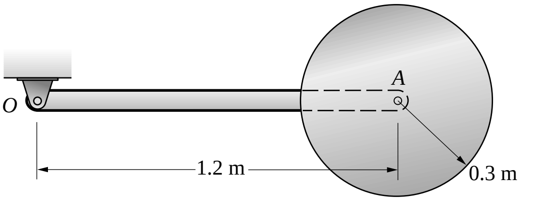
Solution:

$$T_1 = 0,\ V_1 = 0,\ T_2 = \frac{1}{2}(5\text{ kg})v^2 + \frac{1}{2}(0.1\text{ kg-m}^2)\left(\frac{v}{0.1\text{ m}}\right)^2$$

$$V_2 = -(5\text{ kg})(9.81\text{ m/s}^2)(0.5\text{ m}) + \frac{1}{2}(135\text{ N/m})(0.5\text{ m})^2$$

$$T_1 + V_1 = T_2 + V_2 \Rightarrow \boxed{v = 1.01\text{ m/s}}$$

Problem 19.30 The masses of the bar and disk are 14 kg and 9 kg, respectively. The system is released from rest with the bar horizontal. Determine the angular velocity of the bar when it is vertical if the bar and disk are welded together at A.



Solution: The work done by the weights of the bar and disk as they fall is

$$U_{12} = m_{\text{bar}}g(0.6\text{ m}) + m_{\text{disk}}g(1.2\text{ m})$$

$$= (14)(9.81)(0.6) + (9)(9.81)(1.2)$$

$$= 188.4\text{ N-m}.$$

The bar's moment of inertia about the pinned end is

$$I_0 = \tfrac{1}{3}m_{\text{bar}}l^2 = \tfrac{1}{3}(14)(1.2)^2$$

$$= 6.72\text{ kg-m}^2,$$

So the bar's final kinetic energy is

$$T_{\text{bar}} = \tfrac{1}{2}I_0\omega^2$$

$$= 3.36\omega^2.$$

The moment of inertia of the disk about A is

$$I_A = \tfrac{1}{2}m_{\text{disk}}R^2 = \tfrac{1}{2}(9)(0.3)^2$$

$$= 0.405\text{ kg-m}^2,$$

so the disk's final kinetic energy is

$$T_{\text{disk}} = \tfrac{1}{2}m_{\text{disk}}(l\omega)^2 + \tfrac{1}{2}I_A\omega^2$$

$$= \tfrac{1}{2}[(9)(1.2)^2 + 0.405]\omega^2$$

$$= 6.68\omega^2.$$

Equating the work to the final kinetic energy,

$$U_{12} = T_{\text{bar}} + T_{\text{disk}}$$

$$188.4 = 3.36\omega^2 + 6.68\omega^2,$$

we obtain

$$\omega = 4.33\text{ rad/s}.$$

Problem 19.31 The masses of the bar and disk are 14 kg and 9 kg, respectively. The system is released from rest with the bar horizontal. Determine the angular velocity of the bar when it is vertical if the bar and disk are connected by a smooth pin at A.

Solution: See the solution of Problem 19.30. In this case the disk does not rotate, so its final kinetic energy is

$$T_{\text{disk}} = \tfrac{1}{2}m_{\text{disk}}(l\omega)^2$$

$$= \tfrac{1}{2}(9)(1.2)^2\omega^2$$

$$= 6.48\omega^2.$$

Equating the work to the final kinetic energy,

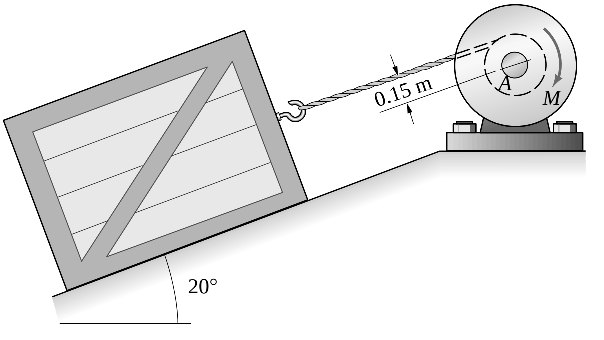
$$U_{12} = T_{\text{bar}} + T_{\text{disk}}:$$

$$188.4 = 3.36\omega^2 + 6.48\omega^2,$$

we obtain

$$\omega = 4.38 \text{ rad/s}.$$

Problem 19.32 The 45-kg crate is pulled up the inclined surface by the winch. The coefficient of kinetic friction between the crate and the surface is $\mu_k = 0.4$. The moment of inertia of the drum on which the cable is being wound is $I_A = 4$ kg-m^2. The crate starts from rest, and the motor exerts a constant couple $M = 50$ N-m on the drum. Use the principle of work and energy to determine the magnitude of the velocity of the crate when it has moved 1 m.

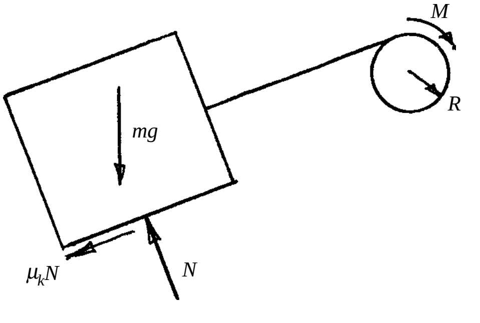


Solution: The normal force is

$$N = mg\cos 20^\circ.$$

As the crate moves 1 m, the drum rotates through an angle $\theta = (1 \text{ m})/R$, so the work done is

$$U_{12} = M\left(\frac{1}{R}\right) - mg\sin 20^\circ(1) - \mu_k(mg\cos 20^\circ)(1).$$



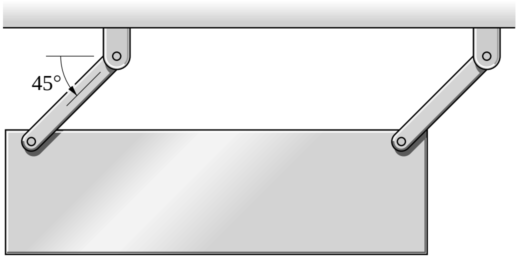
The final kinetic energy is

$$T = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2.$$

Equating the work to the final kinetic energy and using the relation $v = R\omega$, we solve for v, obtaining

$$v = 0.384 \text{ m/s}.$$

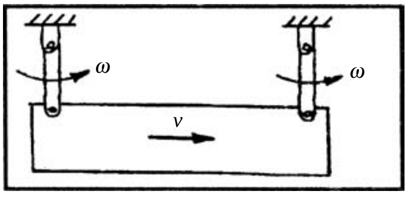
Problem 19.33 The 0.61 m slender bars each weigh 17.8 N, and the rectangular plate weighs 89 N. If the system is released from rest in the position shown, what is the velocity of the plate when the bars are vertical?



Solution: The work done by the weights:

$U = 2W_{\text{bar}}h + W_{\text{plate}}2h,$

where $h = \frac{L}{2}(1 - \cos 45^\circ) = 0.089$ m



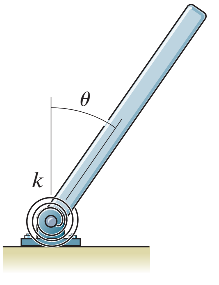
is the change in height, from which $U = 19.06$ N-m. The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right)\left(\frac{W_{\text{plate}}}{g}\right)v^2 + 2\left[\frac{1}{2}\left(\frac{W_{\text{bar}}L^2}{3g}\right)\right]\omega^2 = 5.1v^2,$$

where $\omega = \frac{v}{L}$ has been used.

Substitute into $U = T_2$ and solve: $v = 1.93$ m/s.

Problem 19.34 The mass of the 2-m slender bar is 8 kg. A torsional spring exerts a counterclockwise couple $k\theta$ on the bar, where $k = 40$ N-m/rad and θ is in radians. The bar is released from rest with $\theta = 5^\circ$. What is the magnitude of the bar's angular velocity when $\theta = 60^\circ$?



Solution: We will use conservation of energy

$T_1 = 0$

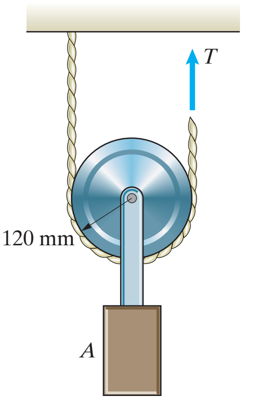
$$V_1 = (8\text{ kg})(9.81\text{ m/s}^2)(1\text{ m})\cos 5^\circ + \frac{1}{2}\left(40\frac{\text{N-m}}{\text{rad}}\right)\left(\frac{5}{180}\pi\text{ rad}\right)^2$$

$$T_2 = \frac{1}{2}\left[\frac{1}{3}(8\text{ kg})(2\text{ m})^2\right]\omega^2$$

$$V_2 = (8\text{ kg})(9.81\text{ m/s}^2)(1\text{ m})\cos 60^\circ + \frac{1}{2}\left(40\frac{\text{N-m}}{\text{rad}}\right)\left(\frac{60}{180}\pi\text{ rad}\right)^2$$

$T_1 + V_1 = T_2 + V_2 \Rightarrow \boxed{\omega = 1.79\text{ rad/s.}}$

Problem 19.35 The mass of the suspended object A is 8 kg. The mass of the pulley is 5 kg, and its moment of inertia is 0.036 kg-m^2. If the force $T = 70$N is applied to the stationary system, what is the magnitude of the velocity of A when it has risen 0.2 m?



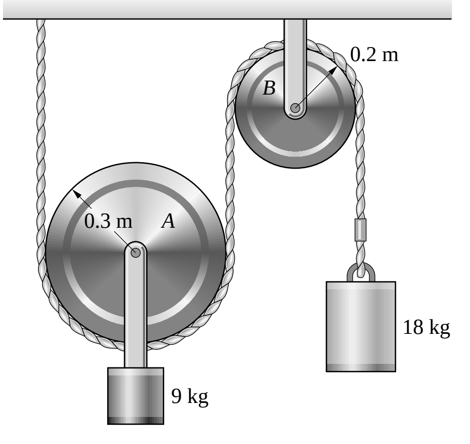
Solution: When the mass A rises 0.2 m, the end of the rope rises 0.4 m.

$$T_1 = 0,\ \ V_1 = 0,\ \ T_2 = \frac{1}{2}(13\text{ kg})v^2 + \frac{1}{2}(0.036\text{ kg-m}^2)\left(\frac{v}{0.12\text{ m}}\right)^2$$

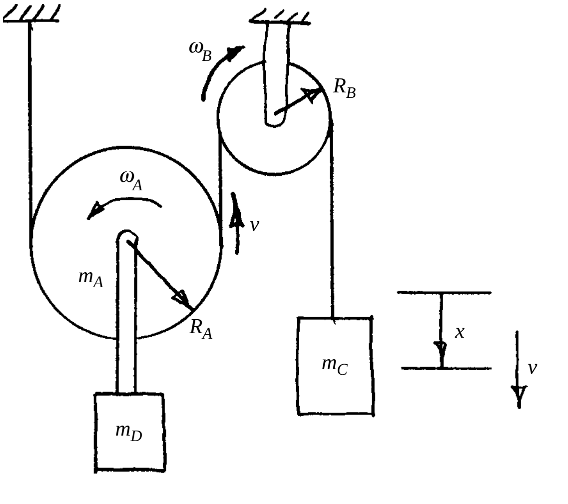
$$V_2 = (13\text{ kg})(9.81\text{ m/s}^2)(0.2\text{ m}),\ \ U = (70\text{ N})(0.4\text{ m})$$

$$T_1 + V_1 + U = T_2 + V_2\ \ \Rightarrow\ \ v = \boxed{0.568\text{ m/s}}$$

Problem 19.36 The mass of the left pulley is 7 kg, and its moment of inertia is 0.330 kg-m^2. The mass of the right pulley is 3 kg, and its moment of inertia is 0.054 kg-m^2. If the system is released from rest, how fast is the 18-kg mass moving when it has fallen 0.1 m?



Solution: When mass C falls a distance x, the center of pulley A rises $\frac{1}{2}x$. The potential energy is



$$v = -m_c g x + (m_A + m_D) g \tfrac{1}{2} x.$$

The angular velocity of pulley B is $w_B = v/R_B$, and the angular velocity of pulley A is $w_A = v/2R_A$. The velocity of the center of pulley A is $w_A R_A = v/2$. The total kinetic energy is

$$T = \tfrac{1}{2}m_c v^2 + \tfrac{1}{2}I_B(v/R_B)^2 + \tfrac{1}{2}(m_A + m_D)(v/2)^2 + \tfrac{1}{2}I_A(v/2R_A)^2.$$

Applying conservation of energy to the initial and final positions,

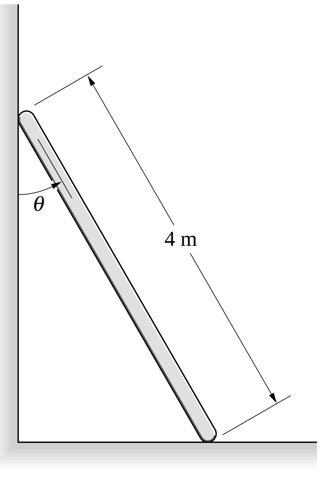
$$O = -m_c g(0.1) + (m_A + m_D) g \tfrac{1}{2}(0.1) + \tfrac{1}{2}m_c v^2 + \tfrac{1}{2}I_B(v/R_B)^2$$

$$+ \tfrac{1}{2}(m_A + m_D)(v/2)^2 + \tfrac{1}{2}I_A(v/2R_A)^2.$$

Solving for v, we obtain

$$v = 0.899\text{ m/s}.$$

Problem 19.37 The 18-kg ladder is released from rest with $\theta = 10^\circ$. The wall and floor are smooth. Modeling the ladder as a slender bar, use conservation of energy to determine the angular velocity of the bar when $\theta = 40^\circ$.



Solution: Choose the datum at floor level. The potential energy at the initial position is

$$V_1 = mg\left(\frac{L}{2}\right)\cos 10^\circ.$$

At the final position,

$$V_2 = mg\left(\frac{L}{2}\right)\cos 40^\circ.$$

The instantaneous center of rotation has the coordinates $(L\sin\theta, L\cos\theta)$, where $\theta = 40^\circ$ at the final position. The distance of the center of rotation from the bar center of mass is $\frac{L}{2}$. The angular velocity about this center is $\omega = \left(\frac{2}{L}\right)v$, where v is the velocity of the center of mass of the ladder. The kinetic energy of the ladder is

$$T_2 = \left(\frac{1}{2}\right)mv^2 + \left(\frac{1}{2}\right)\left(\frac{mL^2}{12}\right)\omega^2 = \left(\frac{mL^2}{6}\right)\omega^2,$$

where $v = \left(\frac{L}{2}\right)\omega$ has been used.

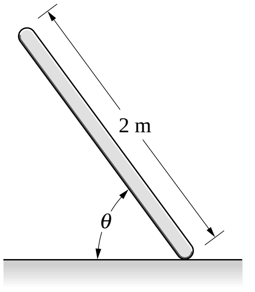
From the conservation of energy,

$$V_1 = V_2 + T_2, \text{ from which } mg\left(\frac{L}{2}\right)\cos 10^\circ$$

$$= mg\left(\frac{L}{2}\right)\cos 40^\circ + \left(\frac{mL^2}{6}\right)\omega^2.$$

Solve: $\omega = 1.269$ rad/s.

Problem 19.38 The 8-kg slender bar is released from rest with $\theta = 60^\circ$. The horizontal surface is smooth. What is the bar's angular velocity when $\theta = 30^\circ$.



Solution: The bar's potential energy is

$$V = mg\tfrac{1}{2}l\sin\theta.$$

No horizontal force acts on the bar, so its center of mass will fall straight down.

The bar's kinetic energy is

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}Iw^2,$$

where $I = \frac{1}{12}ml^2$. Applying conservation of energy to the initial and final states, we obtain

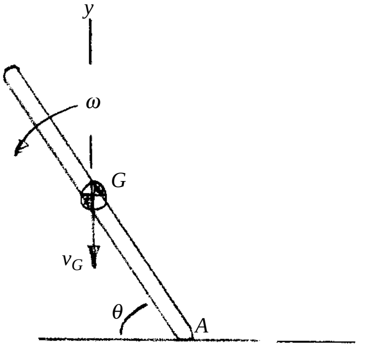
$$mg\tfrac{1}{2}l\sin 60^\circ = mg\tfrac{1}{2}l\sin 30^\circ + \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}Iw^2. \quad \textbf{(1)}$$

To complete the solution, we must determine v_G in terms of ω.

We write the velocity of pt. A in terms of the velocity of pt. G.

$$\mathbf{v}_A = \mathbf{v}_G + \omega \times \mathbf{r}_{A/G}:$$

$$v_A\mathbf{i} = -v_G\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ \dfrac{l}{2}\cos\theta & -\dfrac{l}{2}\sin\theta & 0 \end{vmatrix}.$$

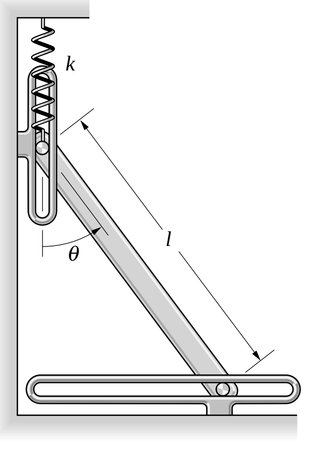


The j component of this equation is

$$0 = -v_G + \omega\tfrac{l}{2}\cos\theta.$$

Setting $\theta = 30^\circ$ in this equation and solving it together with Eq. (1), we obtain $\omega = 2.57$ rad/s.

Problem 19.39 The mass and length of the bar are $m = 4$ kg and $l = 1.2$ m. The spring constant is $k = 180$ N/m. If the bar is released from rest in the position $\theta = 10^\circ$, what is its angular velocity when it has fallen to $\theta = 20^\circ$?



Solution: If the spring is unstretched when $\theta = 0$, the stretch of the spring is

$$S = l(1 - \cos\theta).$$

The total potential energy is

$$v = mg(l/2)\cos\theta + \tfrac{1}{2}kl^2(1 - \cos\theta)^2.$$

From the solution of Problem 19.37, the bar's kinetic energy is

$$T = \frac{1}{6}ml^2\omega^2.$$

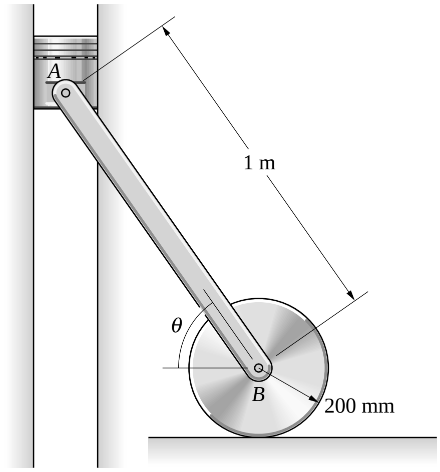
We apply conservation of energy. $T_1 + V_1 = T_2 + V_2$:

$$0 + mg(l/2)\cos 10^\circ + \frac{1}{2}kl^2(1 - \cos 10^\circ)^2$$

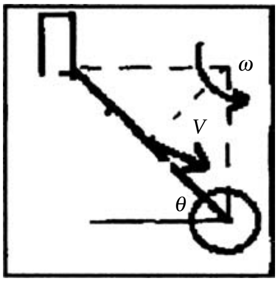
$$= \frac{1}{6}ml^2\omega_2^2 + mg(l/2)\cos 20^\circ + \frac{1}{2}kl^2(1 - \cos 20^\circ)^2.$$

Solving for ω_2, we obtain $\omega_2 = 0.804$ rad/s.

Problem 19.40 The 4-kg slender bar is pinned to a 2-kg slider at A and to a 4-kg homogenous cylindrical disk at B. Neglect the friction force on the slider and assume that the disk rolls. If the system is released from rest with $\theta = 60^\circ$, what is the bar's angular velocity when $\theta = 0$? (See Example 19.3.)



Solution: Choose the datum at $\theta = 0$. The instantaneous center of the bar has the coordinates $(L\cos\theta, L\sin\theta)$ (see figure), and the distance from the center of mass of the bar is $\frac{L}{2}$, from which the angular velocity about the bar's instantaneous center is



$$v = \left(\frac{L}{2}\right)\omega,$$

where v is the velocity of the center of mass. The velocity of the slider is

$$v_A = \omega L\cos\theta,$$

and the velocity of the disk is

$$v_B = \omega L\sin\theta$$

The potential energy of the system is

$$V_1 = m_A g L\sin\theta_1 + mg\left(\frac{L}{2}\right)\sin\theta_1.$$

At the datum, $V_2 = 0$. The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right)m_A v_A^2 + \left(\frac{1}{2}\right)mv^2 + \left(\frac{1}{2}\right)\left(\frac{mL^2}{12}\right)\omega^2 + \left(\frac{1}{2}\right)m_B v_B^2$$

$$+\left(\frac{1}{2}\right)\left(\frac{m_B R^2}{2}\right)\left(\frac{v_B}{R}\right)^2,$$

where at the datum

$$v_A = \omega L\cos 0^\circ = \omega L,$$

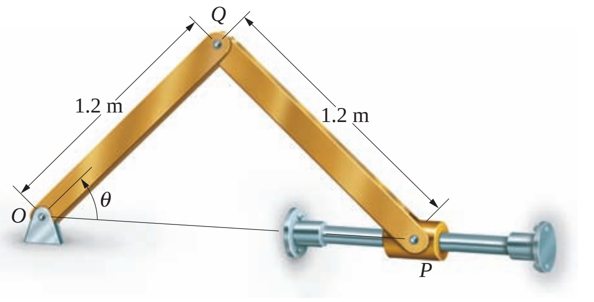
$$v_B = \omega L\sin 0^\circ = 0,$$

$$v = \omega\left(\frac{L}{2}\right).$$

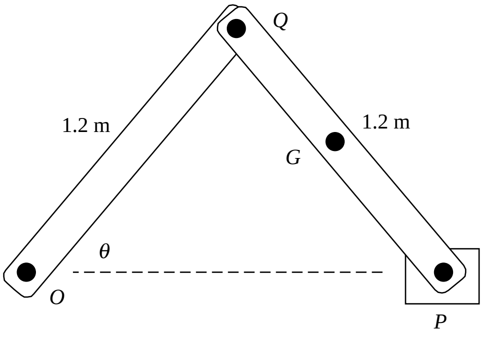
From the conservation of energy: $V_1 = T_2$.

Solve: $\omega = 4.52$ rad/s.

Problem 19.41* The sleeve P slides on the smooth horizontal bar. The mass of each bar is 4 kg and the mass of the sleeve P is 2 kg. If the system is released from rest with $\theta = 60^\circ$, what is the magnitude of the velocity of the sleeve P when $\theta = 40^\circ$?



Solution: We have the following kinematics



$$\mathbf{v}_Q = \boldsymbol{\omega}_{OQ}k \times \mathbf{r}_{Q/O}$$

$$= \omega_{OQ}\mathbf{k} \times (1.2 \text{ m})(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= \omega_{OQ}(1.2 \text{ m})(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})$$

$$\mathbf{v}_P = \mathbf{v}_Q + \boldsymbol{\omega}_{PQ} \times \mathbf{r}_{P/Q}$$

$$= \omega_{OQ}(1.2 \text{ m})(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + \omega_{PQ}\mathbf{k}$$

$$\times (1.2 \text{ m})(\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$$

$$= [(\omega_{PQ} - \omega_{OQ})(1.2 \text{ m})\sin\theta]\mathbf{i} + [(\omega_{PQ} + \omega_{OQ})(1.2 \text{ m})\cos\theta]\mathbf{j}$$

Point P is constrained to horizontal motion. We conclude that

$$\omega_{PQ} = -\omega_{OQ} \equiv \omega, \;\; v_P = 2\omega(1.2 \text{ m})\sin\theta$$

We also need the velocity of point G

$$\mathbf{v}_G = \mathbf{v}_Q + \boldsymbol{\omega}_{PQ} \times \mathbf{r}_{G/Q}$$

$$= -\omega(1.2 \text{ m})(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) + \omega\mathbf{k} \times (0.6 \text{ m})(\cos\theta\mathbf{i} - \sin\theta\mathbf{j})$$

$$= [(1.8 \text{ m})\omega\sin\theta]\mathbf{i} - [(0.6 \text{ m})\omega\cos\theta]\mathbf{j}$$

Now use the energy methods

$$T_1 = 0, \;\; V_1 = 2(4 \text{ kg})(9.81 \text{ m/s}^2)(0.6 \text{ m})\sin 60^\circ;$$

$$T_2 = \frac{1}{2}\left(\frac{1}{3}[4 \text{ kg}][1.2 \text{ m}]^2\right)\omega^2 + \frac{1}{2}\left(\frac{1}{12}[4 \text{ kg}][1.2 \text{ m}]^2\right)\omega^2$$

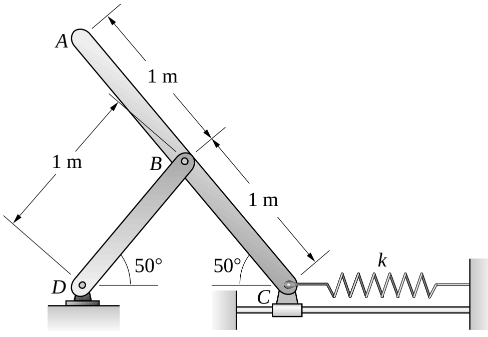
$$+ \frac{1}{2}(4 \text{ kg})([(1.8 \text{ m})\omega\sin 40^\circ]^2 + [(0.6 \text{ m})\omega\cos 40^\circ]^2)$$

$$+ \frac{1}{2}(2 \text{ kg})(2\omega[1.2 \text{ m}]\sin 40^\circ)^2$$

$$V_2 = 2(4 \text{ kg})(9.81 \text{ m/s}^2)(0.6 \text{ m})\sin 40^\circ;$$

Solving $T_1 + V_1 = T_2 + V_2 \Rightarrow \omega = 1.25 \text{ rad/s} \Rightarrow \boxed{v_P = 1.94 \text{ m/s}}$

Problem 19.42* The system is in equilibrium in the position shown. The mass of the slender bar ABC is 6 kg, the mass of the slender bar BD is 3 kg, and the mass of the slider at C is 1 kg. The spring constant is $k = 200$ N/m. If a constant 100-N downward force is applied at A, what is the angular velocity of the bar ABC when it has rotated 20° from its initial position?



Solution: Choose a coordinate system with the origin at D and the x axis parallel to DC. The equilibrium conditions for the bars: for bar BD,

$$\sum F_x = -B_x + D_x = 0,$$

$$\sum F_y = -B_y - m_{BD}g + D_y = 0.$$

$$\sum M_D = B_x \sin 50^\circ - \left(B_y + \frac{M_{BD}g}{2}\right)\cos 50^\circ = 0.$$

For the bar ABC,

$$\sum F_x = B_x - F = 0, \sum F_y = -F_A + C - m_{ABC}g + B_y = 0.$$

$$\sum M_C = (2F_A - B_y + m_{ABC}g)\cos 50^\circ - B_x \sin 50^\circ = 0.$$

At the initial position $F_A = 0$. The solution:

$$B_x = 30.87 \text{ N}, \; D_x = 30.87 \text{ N},$$

$$B_y = 22.07 \text{ N}, \; D_y = 51.5 \text{ N},$$

$$F = 30.87 \text{ N}, \; C = 36.79 \text{ N}.$$

[*Note*: Only the value $F = 30.87$ N is required for the purposes of this problem.] The initial stretch of the spring is

$$S_1 = \frac{F}{k} = \frac{30.87}{200} = 0.154 \text{ m}.$$

The distance D to C is $2\cos\theta$, so that the final stretch of the spring is

$$S_2 = S_1 + (2\cos 30^\circ - 2\cos 50^\circ) = 0.601 \text{ m}.$$

From the principle of work and energy: $U = T_2 - T_1$,

where $T_1 = 0$ since the system starts from rest. The work done is

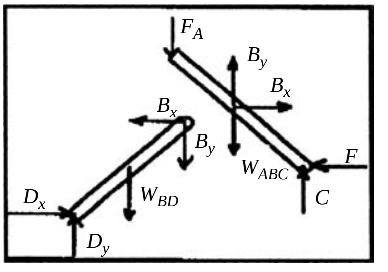
$$U = U_{\text{force}} + U_{ABC} + U_{BD} + U_{\text{spring}}.$$

The height of the point A is $2\sin\theta$, so that the change in height is $h = 2(\sin 50^\circ - \sin 30^\circ)$, and the work done by the applied force is

$$U_{\text{force}} = \int_0^h F_A\,dh = 100(2\sin 50^\circ - 2\sin 30^\circ) = 53.2 \text{ N-m}.$$

The height of the center of mass of bar BD is $\frac{\sin\theta}{2}$, so that the work done by the weight of bar BD is

$$U_{BD} = \int_0^h -m_{BD}g\,dh = -\frac{m_{BD}g}{2}(\sin 30^\circ - \sin 50^\circ) = 3.91 \text{ N-m}.$$



The height of the center of mass of bar ABC is $\sin\theta$, so that the work done by the weight of bar ABC is

$$U_{ABC} = \int_0^h -m_{ABC}g\,dh = -m_{ABC}g(\sin 30^\circ - \sin 50^\circ)$$

$$= 15.66 \text{ N-m}.$$

The work done by the spring is

$$U_{\text{spring}} = \int_{S_1}^{S_2} -ks\,ds$$

$$= -\frac{k}{2}(S_2^2 - S_1^2) = -33.72 \text{ N-m}.$$

Collecting terms, the total work:

$$U = 39.07 \text{ N-m}.$$

The bars form an isosceles triangle, so that the changes in angle are equal; by differentiating the changes, it follows that the angular velocities are equal. The distance D to C is $x_{DC} = 2\cos\theta$, from which $v_C = -2\sin\theta\omega$, since D is a stationary. The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) I_{BD}\omega^2 + \left(\frac{1}{2}\right) I_{ABC}\omega^2 + \left(\frac{1}{2}\right) m_{ABC}v_{ABC}^2$$

$$+ \left(\frac{1}{2}\right) m_C v_C^2 = 5\omega^2,$$

where $I_{BD} = \frac{m_{BD}}{3}(1^2)$,

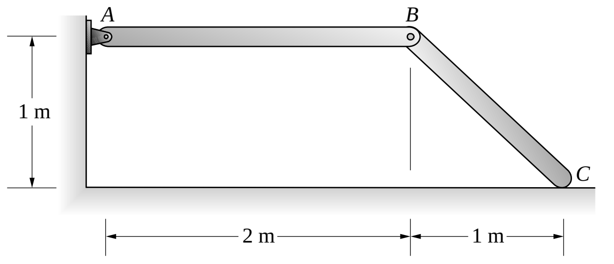
$$I_{ABC} = \frac{m_{ABC}}{12}(2^2),$$

$$v_{ABC} = (1)\omega,$$

$$v_C = -2\sin\theta\omega.$$

Substitute into $U = T_2$ and solve: $\boxed{\omega = 2.80 \text{ rad/s}}$

Problem 19.43* The masses of bars AB and BC are 5 kg and 3 kg, respectively. If the system is released from rest in the position shown, what are the angular velocities of the bars at the instant before the joint B hits the smooth floor?



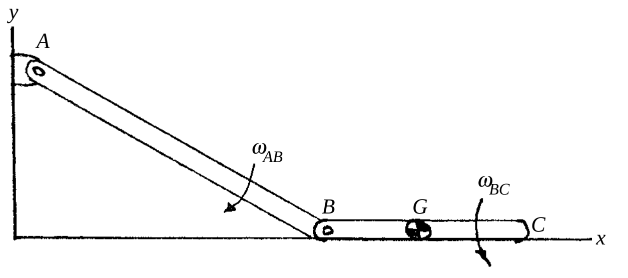
Solution: The work done by the weights of the bars as they fall is

$$v_{12} = m_{AB}g(0.5\text{ m}) + m_{BC}g(0.5\text{ m})$$

$$= (5+3)(9.81)(0.5)$$

$$= 39.24\text{ N-m}.$$

Consider the two bars just before B hits.



The coordinates of pt B are $(\sqrt{3}, 0)$m.

The velocity of pt B is

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{w}_{AB} \times \mathbf{r}_{B/A}$$

$$= 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega_{AB} \\ \sqrt{3} & -1 & 0 \end{vmatrix}$$

$$= -\omega_{AB}\mathbf{i} - \sqrt{3}\omega_{AB}\mathbf{j}.$$

In terms of pt. C,

$$\mathbf{v}_B = \mathbf{v}_C + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{B/C}$$

$$= v_C\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -\sqrt{2} & 0 & 0 \end{vmatrix}$$

$$= v_C\mathbf{i} - \sqrt{2}\omega_{BC}\mathbf{j}.$$

Equating the two expressions for $\mathbf{v}_B$, we obtain

$$\mathbf{v}_C = -\omega_{AB}\mathbf{i},$$

$$\omega_{BC} = \sqrt{\frac{3}{2}}\omega_{AB}. \quad \textbf{(1)}$$

The velocity of pt G is

$$\mathbf{v}_G = \mathbf{v}_C + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{G/C}$$

$$= -\omega_{AB}\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ -\frac{1}{2}\sqrt{2} & 0 & 0 \end{vmatrix}$$

$$= -\omega_{AB}\mathbf{i} - \frac{1}{2}\sqrt{2}\omega_{BC}\mathbf{j}$$

$$= -\omega_{AB}\mathbf{i} - \frac{1}{2}\sqrt{3}\omega_{AB}\mathbf{j}. \qquad \textbf{(2)}$$

We equate the work done to the final kinetic energy of the bars:

$$U_{12} = \frac{1}{2}\left[\frac{1}{3}m_{AB}(2)^2\right]\omega_{AB}^2$$

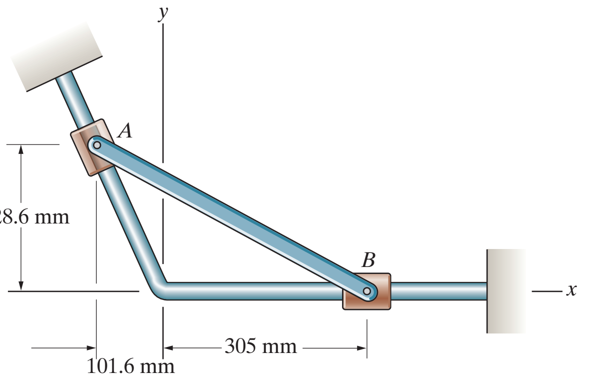
$$+ \frac{1}{2}m_{BC}|\mathbf{v}_G|^2 + \frac{1}{2}\left[\frac{1}{12}m_{BC}(\sqrt{2})^2\right]\omega_{BC}^2. \quad \textbf{(3)}$$

Substituting Eqs. (1) and (2) into this equation and solving for ω_{AB}, we obtain

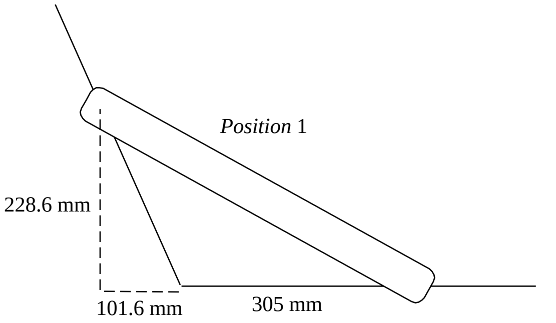
$$\omega_{AB} = 2.49\text{ rad/s}.$$

Then, from Eq. (1), $\omega_{BC} = 3.05$ rad/s.

Problem 19.44* Bar AB weighs 22.2 N. Each of the sleeves A and B weighs 8.9 N. The system is released from rest in the position shown. What is the magnitude of the angular velocity of the bar when sleeve B has moved 76.2 mm to the right?



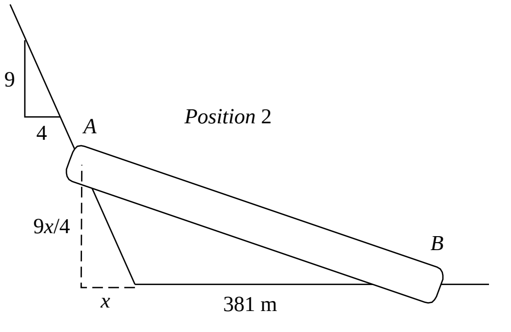
Solution: Find the geometry in position 2



$$(0.407\text{ m})^2 + (0.229\text{ m})^2 = (0.381\text{ m} + x)^2 + \left(\frac{0.229x}{0.102}\right)^2$$

$x = 0.063$ m

Now do the kinematics in position 2



$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$v_B\mathbf{i} = v_A\left(\frac{0.102}{0.25}\mathbf{i} - \frac{0.229}{0.25}\mathbf{j}\right)$$

$$+\omega\mathbf{k} \times (0.445\mathbf{i} - 0.142\mathbf{j})\text{ m}$$

$$= (0.406v_A + [0.142\text{ in}]\omega)\mathbf{i} + (-0.914v_A + [0.445\text{ m}]\omega)\mathbf{j}$$

Equating components we find $v_A = (0.485\text{ m})\omega,\ v_B = (0.34\text{ m})\omega$

Now find the velocity of the center of mass G

$$\mathbf{v}_G = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r}_{G/B}$$

$$= (0.34\text{ m})\omega\mathbf{i} + \omega\mathbf{k} \times (-0.222\mathbf{i} + 0.071\mathbf{j})\text{ m}$$

$$= \omega(0.269\mathbf{i} - 0.222\mathbf{j})\text{ m}$$

Now we can do work energy

$$T_1 = 0,\ V_1 = (22.2\text{ N})(0.114\text{ m}) + (8.9\text{ N})(0.229\text{ m}),$$

$$V_2 = (22.2\text{ N})(0.071\text{ m}) + (8.9\text{ N})(0.142\text{ m})$$

$$T_2 = \frac{1}{2}\left(\frac{1}{12}\left[\frac{22.2\text{ N}}{9.81\text{ m/s}^2}\right]\left[(0.229\text{ m})^2 + (0.407\text{ m})^2\right]\right)\omega^2$$

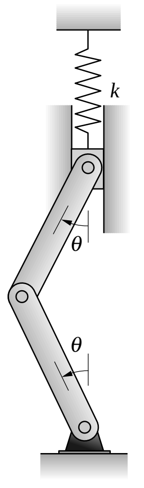
$$+\frac{1}{2}\left(\frac{8.9\text{ N}}{9.81\text{ m/s}^2}\right)(0.485\text{ m})^2\omega^2$$

$$+\frac{1}{2}\left(\frac{22.2\text{ N}}{9.81\text{ m/s}^2}\right)\left[(0.269\text{ m})^2 + (0.222\text{ m})^2\right]\omega^2$$

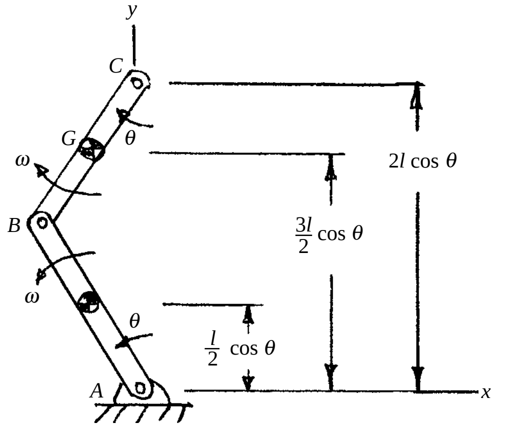
$$+\frac{1}{2}\left(\frac{8.9\text{ N}}{9.81\text{ m/s}^2}\right)(0.34\text{ m})^2\omega^2$$

Thus $T_1 + V_1 = T_2 + V_2 \Rightarrow \boxed{\omega = 2.34\text{ rad/s}}$

Problem 19.45* Each bar has a mass of 8 kg and a length of l m. The spring constant is $k = 100$ N/m, and the spring is unstretched when $\theta = 0$. If the system is released from rest with the bars vertical, what is the magnitude of the angular velocity of the bars when $\theta = 30°$?



Solution: The stretch of the spring is $2l(1 - \cos\theta)$, so the potential energy is



$$v = \frac{1}{2}k[2l(1-\cos\theta)]^2 + mg\frac{l}{2}\cos\theta + mg\frac{3l}{2}\cos\theta$$

$$= 2kl^2(1-\cos\theta)^2 + 2mgl\cos\theta.$$

The velocity of pt B is

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ -l\sin\theta & l\cos\theta & 0 \end{vmatrix}$$

$$= -\omega l\cos\theta\mathbf{i} - \omega l\sin\theta\mathbf{j}.$$

The velocity of pt. G is

$$\mathbf{v}_G = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{G/B}$$

$$= -\omega l\cos\theta\mathbf{i} - \omega l\sin\theta\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega \\ \frac{l}{2}\sin\theta & \frac{l}{2}\cos\theta & 0 \end{vmatrix}$$

$$= -\omega\frac{l}{2}\cos\theta\mathbf{i} - \omega\frac{3l}{2}\sin\theta\mathbf{j}.$$

From this expression,

$$|\mathbf{v}_G|^2 = (1 + 8\sin^2\theta)\frac{1}{4}l^2\omega^2.$$

Applying conservation of energy to the initial and final positions,

$$2mgl = 2kl^2(1-\cos 30°) + 2mgl\cos 30°$$

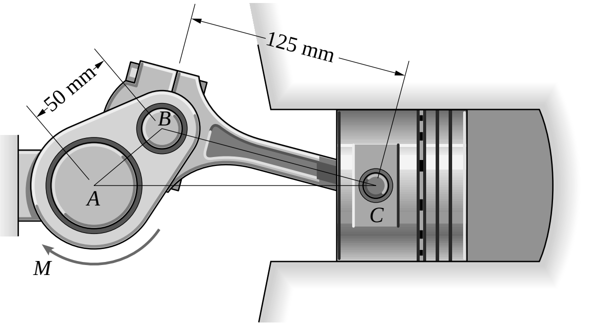
$$+ \frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2 + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\omega^2$$

$$+ \frac{1}{2}m(1 + 8\sin^2 30°)\frac{1}{4}l^2\omega^2.$$

Solving for ω, we obtain

$$\omega = 1.93 \text{ rad/s}.$$

Problem 19.46* The system starts from rest with the crank AB vertical. A constant couple M exerted on the crank causes it to rotate in the clockwise direction, compressing the gas in the cylinder. Let s be the displacement (in meters) of the piston to the right relative to its initial position. The net force toward the left exerted on the piston by atmospheric pressure and the gas in the cylinder is $350/(1-10s)$ N. The moment of inertia of the crank about A is 0.0003 kg-m^2. The mass of the connecting rod BC is 0.36 kg, and the center of mass of the rod is at its midpoint. The connecting rod's moment of inertia about its center of mass is 0.0004 kg-m^2. The mass of the piston is 4.6 kg. If the clockwise angular velocity of the crank AB is 200 rad/s when it has rotated 90° from its initial position, what is M? (Neglect the work done by the weights of the crank and connecting rod).



Solution: As the crank rotates through an angle θ the work done by the couple is $M\theta$. As the piston moves to the right a distance s, the work done on the piston by the gas is

$$-\int_0^s \frac{350\,ds}{1-10s}.$$

Letting $1-10s=10z$, this is

$$\int_{0.1}^{0.1-s} 35\frac{dz}{z} = 35ln(1-10s).$$

The total work is $U_{12} = M\theta + 35ln(1-10s)$.

From the given dimensions of the crank and connecting rod, the vector components are

$$r_{B/Ax} = 0.05\sin\theta, \quad \textbf{(1)}$$

$$r_{B/Ay} = 0.05\cos\theta \quad \textbf{(2)};$$

$$r_{C/Bx} = \sqrt{(0.125)^2 - (0.05\cos\theta)^2} \quad \textbf{(3)};$$

$$r_{C/By} = -0.05\cos\theta \quad \textbf{(4)}.$$

The distance s that the piston moves to the right is

$$s = r_{B/Ax} + r_{C/Bx} - \sqrt{(0.125)^2 - (0.05)^2} \quad \textbf{(5)}.$$

The velocity of B is

$$\mathbf{v}_B = \mathbf{v}_A + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega \\ r_{B/Ax} & r_{B/Ay} & 0 \end{vmatrix} = \omega r_{B/Ay}\mathbf{i} - \omega r_{B/Ax}\mathbf{j}.$$

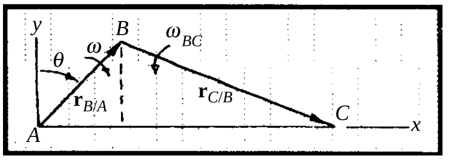
The velocity of C is $\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$:

$$v_C\mathbf{i} = \omega r_{B/Ay}\mathbf{i} - \omega r_{B/Ax}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ r_{c/Bx} & r_{c/By} & 0 \end{vmatrix}.$$

Equating $\mathbf{i}$ and $\mathbf{j}$ components, we can solve for v_c and ω_{BC} in terms of ω:

$$v_C = [r_{B/Ay} - r_{C/By}(r_{B/Ax}/r_{C/Bx})]\omega \quad \textbf{(6)};$$

$$\omega_{BC} = (r_{B/Ax}/r_{C/Bx})\omega \quad \textbf{(7)}.$$



The velocity of the center of mass G (the midpoint) of the connecting rod BC is

$$\mathbf{v}_G = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \tfrac{1}{2}\mathbf{r}_{C/B} = \omega r_{B/Ay}\mathbf{i} - \omega r_{B/Ax}\mathbf{j}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{BC} \\ \frac{1}{2}r_{C/Bx} & \frac{1}{2}r_{C/By} & 0 \end{vmatrix}, \text{ or}$$

$$v_C = \left(\omega r_{B/Ay} - \tfrac{1}{2}\omega_{BC}r_{C/By}\right)\mathbf{i} - \left(\omega r_{B/Ax} - \tfrac{1}{2}\omega_{BC}r_{C/Bx}\right)\mathbf{j} \quad \textbf{(8)}.$$

Let I_A be the moment of inertia of the crank about A, I_{BC} the moment of inertia of the connecting rod about its center of mass, m_{BC} the mass of the connecting rod, and m_p the mass of the piston. The principle of work and energy is: $U_{12} = T_2 - T_1$:

$$M\theta + 35ln(1-10s) = \tfrac{1}{2}I_A\omega^2 + \tfrac{1}{2}I_{BC}\omega_{BC}^2 + \tfrac{1}{2}m_{BC}|\mathbf{v}_G|^2$$

$$+ \tfrac{1}{2}m_p v_C^2 \quad \textbf{(9)}.$$

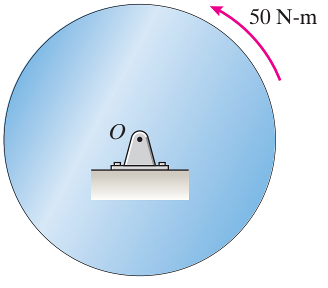
Solving this equation for $\boldsymbol{M}$ and using Equations (1)–(8) with $\omega = 200$ rad/s and $\theta = \pi/2$ rad, we obtain $M = 28.2$ N-m.

Problem 19.47* In Problem 19.46, if the system starts from rest with the crank AB vertical and the couple $M = 40$ N-m, what is the clockwise angular velocity of AB when it has rotated 45° from its initial position?

Solution: In the solution to Problem 19.46, we substitute Equations (1)–(8) into Equation (9), set $M = 40$ N-m and $\theta = \pi/4$ rad and solve for ω obtaining $\omega = 49.6$ rad/s.

Problem 19.48 The moment of inertia of the disk about O is 22 kg-m². At $t = 0$, the stationary disk is subjected to a constant 50 N-m torque.

(a) Determine the angular impulse exerted on the disk from $t = 0$ to $t = 5$ s.
(b) What is the disk's angular velocity at $t = 5$ s?



Solution:

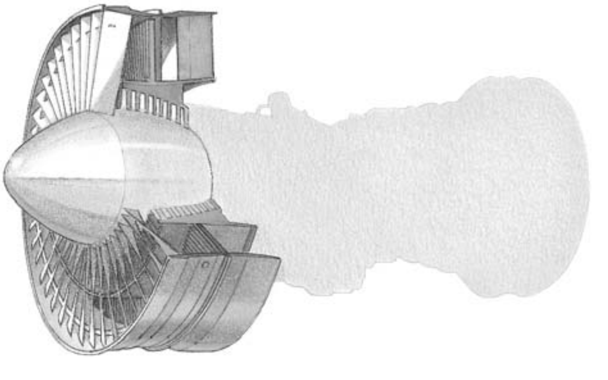
(a) $$\boxed{\begin{aligned}\text{Angular Impulse} &= \int_0^{5\text{ s}} \sum M\,dt = (50\text{ N-m})(5\text{ s}) \\ &= 250\text{ N-m-s CCW}\end{aligned}}$$

(b) $250\text{ N-m-s} = (22\text{ kg-m}^2)\omega$

$$\Rightarrow \boxed{\omega = 11.4\text{ rad/s counterclockwise}}$$

Problem 19.49 The moment of inertia of the jet engine's rotating assembly is 400 kg-m². The assembly starts from rest. At $t = 0$, the engine's turbine exerts a couple on it that is given as a function of time by $M = 6500 - 125t$ N-m.

(a) What is the magnitude of the angular impulse exerted on the assembly from $t = 0$ to $t = 20$ s?
(b) What is the magnitude of the angular velocity of the assembly (in rpm) at $t = 20$ s?

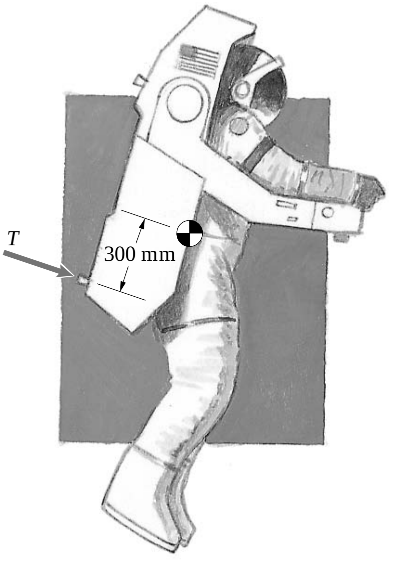


Solution:

(a) $$\boxed{\begin{aligned}\text{Angular Impulse} &= \int_0^{20\text{ s}} \sum M\,dt \\ &= \int_0^{20} (6500 - 125t)\text{ N-m}\,dt = 105\text{ kN-m-s}\end{aligned}}$$

(b) $105\text{ kN-m-s} = (400\text{ kg-m}^2)\,\omega \;\Rightarrow\; \boxed{\omega = 263\text{ rad/s (2510 rpm)}}$

Problem 19.50 An astronaut fires a thruster of his maneuvering unit, exerting a force $T = 2\,(1 + t)$ N, where t is in seconds. The combined mass of the astronaut and his equipment is 122 kg, and the moment of inertia about their center of mass is 45 kg-m^2. Modeling the astronaut and his equipment as a rigid body, use the principle of angular impulse and momentum to determine how long it takes for his angular velocity to reach 0.1 rad/s.



Solution: From the principle of impulse and angular momentum,

$$\int_{t_1}^{t_2} \sum M\, dt = I\omega_2 - I\omega_1,$$

where $\omega_1 = 0$, since the astronaut is initially stationary. The normal distance from the thrust line to the center of mass is $R = 0.3$ m, from which

$$\int_0^{t_2} 2(1 + t)(R)\, dt = I\omega_2.$$

$$0.6\left(t_2 + \frac{t_2^2}{2}\right) = 45(0.1).$$

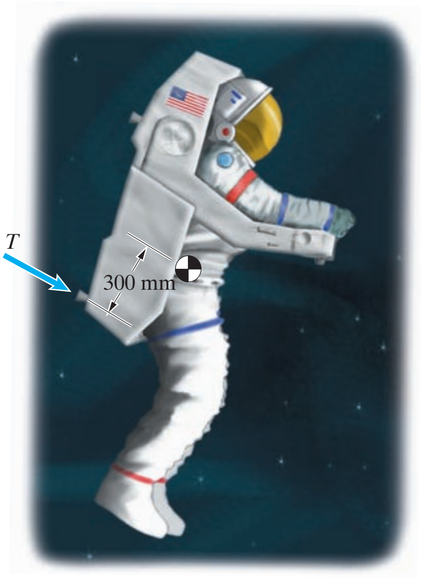
Rearrange: $t_2^2 + 2bt_2 + c = 0$, where $b = 1$, $c = -15$. Solve:

$$t_2 = -b \pm \sqrt{b^2 - c} = 3 \text{ or } -5.$$

Since the negative solution has no meaning here, $\boxed{t_2 = 3 \text{ s}}$

Problem 19.51 The combined mass of the astronaut and his equipment is 122 kg, and the moment of inertia about their center of mass is 45 kg-m^2. The maneuvering unit exerts an impulsive force T of 0.2-s duration, giving him a counterclockwise angular velocity of 1 rpm.

(a) What is the average magnitude of the impulsive force?
(b) What is the magnitude of the resulting change in the velocity of the astronaut's center of mass?



Solution:

(a) From the principle of moment impulse and angular momentum,

$$\int_{t_1}^{t_2} \sum M\,dt = I\omega_2 - I\omega_1,$$

where $\omega_1 = 0$ since the astronaut is initially stationary. The angular velocity is

$$\omega_2 = \frac{2\pi}{60} = 0.1047 \text{ rad/s}$$

from which $\int_0^{0.2} TR\,dt = I\omega_2$,

from which $T(0.3)(0.2) = 45(\omega_2)$,

$$\boxed{T = \frac{45\omega_2}{0.06} = 78.54 \text{ N}}$$

(b) From the principle of impulse and linear momentum

$$\int_{t_1}^{t_2} \sum F\,dt = m(v_2 - v_1)$$

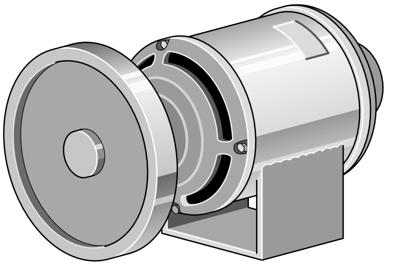
where $v_1 = 0$ since the astronaut is initially stationary.

$$\int_0^{0.2} T\,dt = mv_2,$$

from which $\boxed{v_2 = \frac{T(0.2)}{m} = 0.129 \text{ m/s}}$

Problem 19.52 A flywheel attached to an electric motor is initially at rest. At $t = 0$, the motor exerts a couple $M = 200e^{-0.1t}$ N-m on the flywheel. The moment of inertia of the flywheel is 10 kg-m^2.

(a) What is the flywheel's angular velocity at $t = 10$ s?
(b) What maximum angular velocity will the flywheel attain?



Solution:

(a) From the principle of moment impulse and angular momentum,

$$\int_{t_1}^{t_2} \sum M\,dt = I\omega_2 - I\omega_1,$$

where $\omega_1 = 0$, since the motor starts from rest.

$$\int_0^{10} 200e^{-0.1t}\,dt = \frac{200}{0.1}\left[-e^{-0.1t}\right]_0^{10} = 2000\left[1 - e^{-1}\right],$$

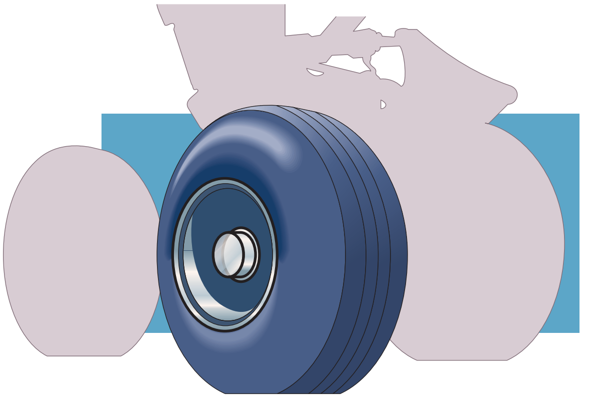
$$= 1264.2 \text{ N-m-s.}$$

From which $\boxed{\omega_2 = \dfrac{1264.24}{10} = 126 \text{ rad/s.}}$

(b) An inspection of the angular impulse function shows that the angular velocity of the flywheel is an increasing monotone function of the time, so that the greatest value occurs as $t \to \infty$.

$$\boxed{\omega_{2\max} = \lim_{t\to\infty} \frac{200}{(10)(0.1)}\left[1 - e^{-0.1t}\right] \to 200 \text{ rad/s}}$$

Problem 19.53 A main landing gear wheel of a Boeing 777 has a radius of 0.62 m and its moment of inertia is 24 kg-m^2. After the plane lands at 75 m/s, the skid marks of the wheel's tire is measured and determined to be 18 m in length. Determine the average friction force exerted on the wheel by the runway. Assume that the airplane's velocity is constant during the time the tire skids (slips) on the runway.

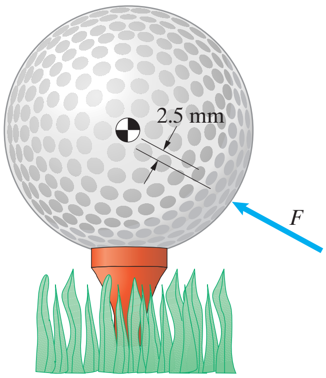


Solution: The tire skids for $t = \dfrac{18 \text{ m}}{75 \text{ m/s}} = 0.24$ s.

When the skidding stops the tire is turning at the rate $\omega = \dfrac{75 \text{ m/s}}{0.62 \text{ m}} =$ 121 rad/s

$$Frt = I\omega \Rightarrow F = \frac{I\omega}{rt} = \frac{(24 \text{ kg-m}^2)(121 \text{ rad/s})}{(0.62 \text{ m})(0.24 \text{ s})} \quad \boxed{F = 19.5 \text{ kN}}$$

Problem 19.54 The force a club exerts on a 0.045-kg golf ball is shown. The ball is 42 mm in diameter and can be modeled as a homogeneous sphere. The club is in contact with the ball for 0.0006 s, and the magnitude of the velocity of the ball's center of mass after the ball is hit is 36 m/s. What is the magnitude of the ball's angular velocity after it is hit?



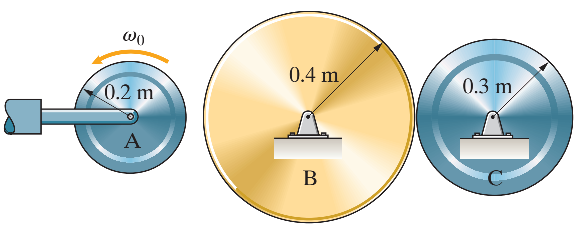
Solution: Linear momentum: $Ft = mv \Rightarrow F = \frac{mv}{t}$

Angular momentum

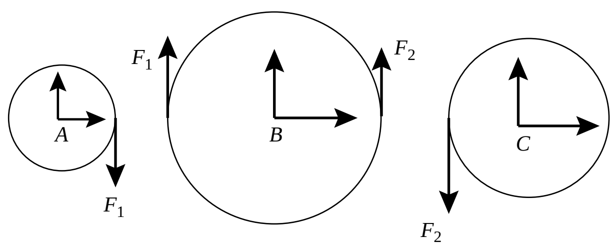
$$Ftd = I\omega \Rightarrow \omega = \frac{Ftd}{I} = \frac{mvd}{I} = \frac{(0.045 \text{ kg})(36 \text{ m/s})(0.0025 \text{ m})}{2/5\ (0.045 \text{ kg})(0.021 \text{ m})^2}$$

Solving $\boxed{\omega = 510 \text{ rad/s}}$

Problem 19.55 Disk A initially has a counterclockwise angular velocity $\omega_0 = 50$ rad/s. Disks B and C are initially stationary. At $t = 0$, disk A is moved into contact with disk B. Determine the angular velocities of the three disks when they have stopped slipping relative to each other. The masses of the disks are $m_A = 4$ kg, $m_B = 16$ kg, and $m_C = 9$ kg. (See Active Example 19.4.)



Solution: The FBDs



Given:

$\omega_0 = 50$ rad/s

$m_A = 4 \text{ kg},\ r_A = 0.2 \text{ m},\ I_A = 1/2\ m_A r_A{}^2 = 0.08 \text{ kg-m}^2$

$m_B = 16 \text{ kg},\ r_B = 0.4 \text{ m},\ I_B = 1/2\ m_B r_B{}^2 = 1.28 \text{ kg-m}^2$

$m_C = 9 \text{ kg},\ r_C = 0.3 \text{ m},\ I_C = 1/2\ m_C r_C{}^2 = 0.405 \text{ kg-m}^2$

Impulse Momentum equations:

$$-F_1 t r_A = I_A\omega_A - I_A\omega_0,\ \ F_1 t r_B - F_2 t r_B = I_B\omega_B,\ \ F_2 t r_C = I_C\omega_C$$

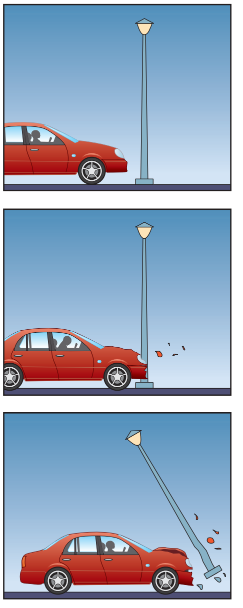
Constraints when it no longer slips:

$$\omega_A r_A = \omega_B r_B = \omega_C r_C$$

We cannot solve for the slipping time, however, treating $F_1 t$ and $F_2 t$ as two unknowns we have

$$\boxed{\begin{array}{l}\omega_A = 6.90 \text{ rad/s counterclockwise}\\ \omega_B = 3.45 \text{ rad/s clockwise}\\ \omega_C = 4.60 \text{ rad/s counterclockwise}\end{array}}$$

Problem 19.56 In Example 19.5, suppose that in a second test at a higher velocity the angular velocity of the pole following the impact is $\omega = 0.81$ rad/s, the horizontal velocity of its center of mass is $v = 7.3$ m/s, and the duration of the impact is $\Delta t = 0.009$ s. Determine the magnitude of the average force the car exerts on the pole in shearing off the supporting bolts. Do so by applying the principle of angular impulse and momentum in the form given by Eq. (19.32).



Solution: Using the data from Example 19.5 we write the linear and angular impulse momentum equations for the pole. F is the force of the car and S is the shear force in the bolts

$$(F - S)\Delta t = mv$$

$$F\Delta t\left(\frac{L}{2} - h\right) - S\Delta t\frac{L}{2} = \frac{1}{12}\, mL^2\, \omega$$

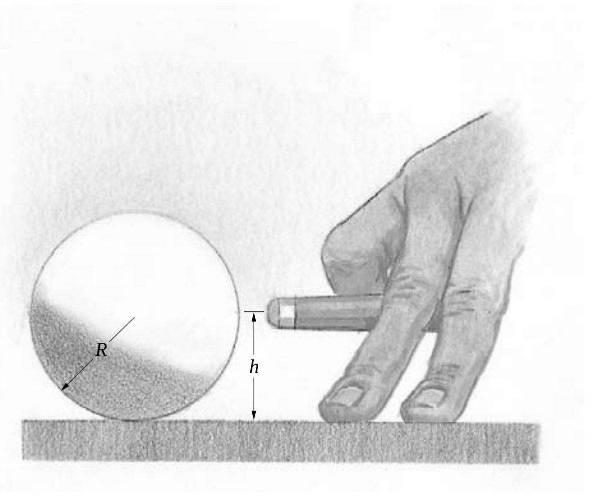
Putting in the numbers we have

$$(F - S)(0.009 \text{ s}) = (70 \text{ kg})(7.3 \text{ m/s})$$

$$F(0.009 \text{ s})(2.5 \text{ m}) - S(0.009 \text{ s})(43 \text{ m}) = \frac{1}{12}(70 \text{ kg})(6 \text{ m})^2\, \omega$$

Solving we find $F = 303$ N, $\boxed{S = 246 \text{ N.}}$

Problem 19.57 The force exerted on the cue ball by the cue is horizontal. Determine the value of h for which the ball rolls without slipping. (Assume that the frictional force exerted on the ball by the table is negligible.)



Solution: From the principle of moment impulse and angular momentum,

$$\int_{t_1}^{t_2} (h - R)F\,dt = I(\omega_2 - \omega_1),$$

where $\omega_1 = 0$ since the ball is initially stationary. From the principle of impulse and linear momentum

$$\int_{t_1}^{t_2} F\,dt = m(v_2 - v_1)$$

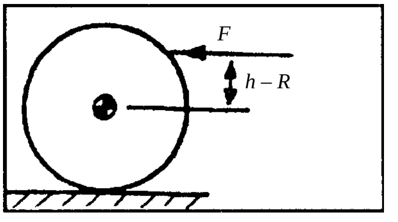
where $v_1 = 0$ since the ball is initially stationary. *Since the ball rolls*, $v_2 = R\omega_2$, from which the two equations:

$$(h - R)F(t_2 - t_2) = I\omega_2,$$

$$F(t_2 - t_1) = mR\omega_2.$$

The ball is a homogenous sphere, from which

$I = \frac{2}{5}mR^2$.



Substitute:

$$(h - R)mR\omega_2 = \frac{2mR^2}{5}\omega_2.$$

Solve: $\boxed{h = \left(\frac{7}{5}\right)R}$

Problem 19.58 In Example 19.6, we neglected the moments of inertia of the two masses m about the axes through their centers of mass in calculating the total angular momentum of the person, platform, and masses. Suppose that the moment of inertia of each mass about the vertical axis through its center of mass is $I_M = 0.001$ kg-m^2. If the person's angular velocity with her arms extended to $r_1 = 0.6$ m is $\omega_1 = 1$ revolution per second, what is her angular velocity ω_2 when she pulls the masses inward to $r_2 = 0.2$ m? Compare your result to the answer obtained in Example 19.6.

Solution: Using the numbers from Example 19.6, we conserve angular momentum

$$H_{O1} = (I_P + 2mr_1^2 + 2I_M)\omega_1$$

$$= (0.4 \text{ kg-m}^2 + 2[4 \text{ kg}][0.6 \text{ m}]^2 + 2[0.001 \text{ kg-m}^2])\left(1\ \frac{\text{rev}}{\text{s}}\right)$$

$$H_{O2} = (I_P + 2mr_2^2 + 2I_M)\omega_2$$

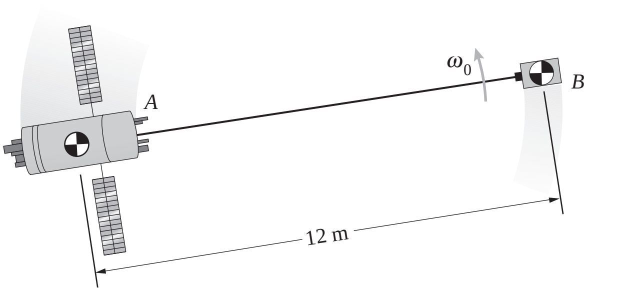
$$= (0.4 \text{ kg-m}^2 + 2[4 \text{ kg}][0.2 \text{ m}]^2 + 2[0.001 \text{ kg-m}^2])\omega_2$$

$$H_{O1} = H_{O2} \Rightarrow \boxed{\omega = 4.55\ \frac{\text{rev}}{\text{s}}.}$$

If we include the moments of inertia of the weights then $\omega = 4.55\ \frac{\text{rev}}{\text{s}}$.

Without the moments of inertia (Example 19.6) we found $\omega = 4.56\ \frac{\text{rev}}{\text{s}}$.

Problem 19.59 Two gravity research satellites ($m_A = 250$ kg, $I_A = 350$ kg-m^2; $m_B = 50$ kg, $I_B = 16$ kg-m^2) are tethered by a cable. The satellites and cable rotate with angular velocity $\omega_0 = 0.25$ rpm. Ground controllers order satellite A to slowly unreel 6 m of additional cable. What is the angular velocity afterward?



Solution: The initial distance from A to the common center of mass is

$$x_0 = \frac{(0)(250) + (12)(50)}{250 + 50} = 2 \text{ m}.$$

The final distance from A to the common center of mass is

$$x = \frac{(0)(250) + (18)(50)}{250 + 50} = 3 \text{ m}.$$

The total angular momentum about the center of mass is conserved.

$$x_0 m_A(x_0\omega_0) + I_A\omega_0 + (12 - x_0)m_B[(12 - x_0)\omega_0] + I_B\omega_0$$

$$= xm_A(x_\omega) + I_A\omega + (18 - x)m_B[(18 - x)\omega] + I_B\omega;$$

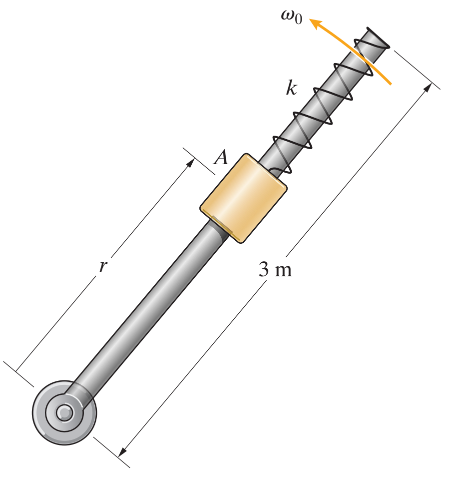
or

$$(2)(250)(2)\omega_0 + 350\omega_0 + (10)(50)(10)\omega_0 + 16\omega_0$$

$$= (3)(250)(3)\omega + 350\omega + (15)(50)(15)\omega + 16\omega.$$

we obtain $\omega = 0.459\omega_0 = 0.115$ rpm.

Problem 19.60 The 2-kg bar rotates *in the horizontal plane* about the smooth pin. The 6-kg collar A slides on the smooth bar. Assume that the moment of inertia of the collar A about its center of mass is negligible; that is, treat the collar as a particle. At the instant shown, the angular velocity of the bar is $\omega_0 = 60$ rpm and the distance from the pin to the collar is $r = 1.8$ m. Determine the bar's angular velocity when $r = 2.4$ m.



Solution: Angular momentum is conserved

$$\left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(1.8\text{ m})^2\right](60\text{ rpm})$$

$$= \left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(2.4\text{ m})^2\right]\omega_2$$

Solving we find $\boxed{\omega_2 = 37.6\text{ rpm}}$

Problem 19.61 The 2-kg bar rotates *in the horizontal plane* about the smooth pin. The 6-kg collar A slides on the smooth bar. The moment of inertia of the collar A about its center of mass is 0.2 kg-m^2. At the instant shown, the angular velocity of the bar is $\omega_0 = 60$ rpm and the distance from the pin to the collar is $r = 1.8$ m. Determine the bar's angular velocity when $r = 2.4$ m and compare your answer to that of Problem 19.60.

Solution: Angular momentum is conserved

$$\left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(1.8\text{ m})^2 + (0.2\text{ kg-m}^2)\right](60\text{ rpm})$$

$$= \left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(2.4\text{ m})^2 + (0.2\text{ kg-m}^2)\right]\omega_2$$

Solving we find $\boxed{\omega_2 = 37.7\text{ rpm}}$

Problem 19.62* The 2-kg bar rotates *in the horizontal plane* about the smooth pin. The 6-kg collar A slides on the smooth bar. The moment of inertia of the collar A about its center of mass is 0.2 kg-m^2. The spring is unstretched when $r = 0$, and the spring constant is $k =$ 10 N-m. At the instant shown, the angular velocity of the bar is $\omega_0 = 2$ rad/s, the distance from the pin to the collar is $r = 1.8$ m, and the radial velocity of the collar is zero. Determine the radial velocity of the collar when $r = 2.4$ m.

Solution: Angular momentum is conserved

$$\left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(1.8\text{ m})^2 + (0.2\text{ kg-m}^2)\right](2\text{ rad/s})$$

$$= \left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(2.4\text{ m})^2 + (0.2\text{ kg-m}^2)\right]\omega_2$$

Thus $\omega_2 = 1.258$ rad/s

Energy is conserved

$$T_1 = \frac{1}{2}\left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(1.8\text{ m})^2 + (0.2\text{ kg-m}^2)\right](2\text{ rad/s})^2$$

$$V_1 = \frac{1}{2}(10\text{ N/m})(1.8\text{ m})^2$$

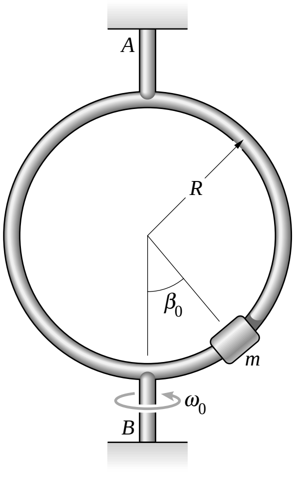
$$T_2 = \frac{1}{2}\left[\frac{1}{3}(2\text{ kg})(3\text{ m})^2 + (6\text{ kg})(2.4\text{ m})^2 + (0.2\text{ kg-m}^2)\right]{\omega_2}^2$$

$$+ \frac{1}{2}(6\text{ kg}){v_r}^2$$

$$V_2 = \frac{1}{2}(10\text{ N/m})(2.4\text{ m})^2$$

Solving we find that $\boxed{v_r = 1.46\text{ m/s}}$.

Problem 19.63 The circular bar is welded to the vertical shafts, which can rotate freely in bearings at A and B. Let I be the moment of inertia of the circular bar and shafts about the vertical axis. The circular bar has an initial angular velocity ω_0, and the mass m is released in the position shown with no velocity relative to the bar. Determine the angular velocity of the circular bar as a function of the angle β between the vertical and the position of the mass. Neglect the moment of inertia of the mass about its center of mass; that is, treat the mass as a particle.



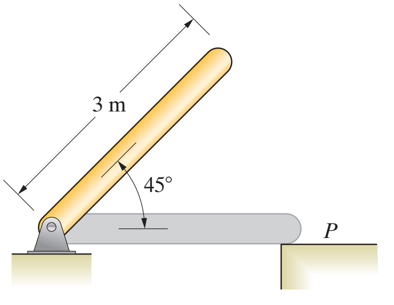
Solution: Angular momentum about the vertical axis is conserved:

$$I\omega_0 + (R \sin\beta_0)m(R\sin\beta_0)\omega_0 = I\omega + (R\sin\beta)m(R\sin\beta)\omega.$$

Solving for

$$\omega,\ \omega = \left(\frac{I + mR^2\sin^2\beta_0}{I + mR^2\sin^2\beta}\right)\omega_0.$$

Problem 19.64 The 10-N bar is released from rest in the 45° position shown. It falls and the end of the bar strikes the horizontal surface at P. The coefficient of restitution of the impact is $e = 0.6$. When the bar rebounds, through what angle relative to the horizontal will it rotate?



Solution: We solve the problem in three phases.
We start with a work energy analysis to find out how fast the bar is rotating just before the collision

$$T_1 + V_1 = T_2 + V_2,$$

$$0 + (10\text{ N})(1.5\text{ m})\sin 45^\circ = \frac{1}{2}\left[\frac{1}{3}\left(\frac{10\text{ N-s}^2}{9.81\text{ m}}\right)(3\text{ m})^2\right]\omega_2^2 + 0,$$

$$\omega_2 = 2.63\text{ rad/s}.$$

Next we do an impact analysis to take it through the collision

$$e\omega_2 L = \omega_3 L$$

$$0.6(2.63\text{ rad/s})(3\text{ m}) = \omega_3(3\text{ m}) \Rightarrow \omega_3 = 1.58\text{ rad/s}.$$

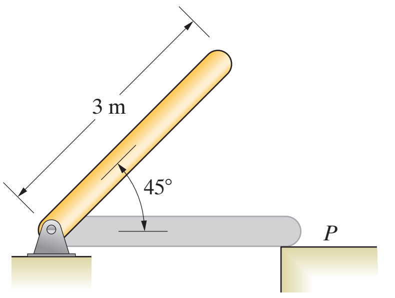
Finally we do another work energy analysis to find the rebound angle.

$$T_3 + V_3 = T_4 + V_4,$$

$$\frac{1}{2}\left[\frac{1}{3}\left(\frac{10\text{ N}}{9.81\text{ m/s}^2}\right)(3\text{ m})^2\right]\omega_3^2 + 0 = 0 + (10\text{ N})(1.5\text{ m})\sin\theta,$$

$$\boxed{\theta = 14.7^\circ.}$$

Problem 19.65 The 10-N bar is released from rest in the 45° position shown. It falls and the end of the bar strikes the horizontal surface at P. The bar rebounds to a position 10° relative to the horizontal. If the duration of the impact is 0.01 s, what is the magnitude of the average vertical force the horizontal surface exerted on the bar at P?



Solution: We solve the problem in three phases.

We start with a work energy analysis to find out how fast the bar is rotating just before the collision

$T_1 + V_1 = T_2 + V_2,$

$$0 + (10\ \text{N})(1.5\ \text{m})\sin 45^\circ = \frac{1}{2}\left[\frac{1}{3}\left(\frac{10\ \text{N-s}^2}{9.81\ \text{m}}\right)(3\ \text{m})^2\right]\omega_2^2 + 0,$$

$\omega_2 = 2.63$ rad/s.

Next we do another work energy analysis to find out how fast the bar is rotating just after the collision.

$T_3 + V_3 = T_4 + V_4,$

$$\frac{1}{2}\left[\frac{1}{3}\left(\frac{10\ \text{N-s}^2}{9.81\ \text{m}}\right)(3\ \text{m})^2\right]\omega_3^2 + 0 = 0 + (10\ \text{N})(1.5\ \text{m})\sin 10^\circ,$$

$\omega_3 = 1.31$ rad/s.

Now we can use the angular impulse momentum equation about the pivot point to find the force.

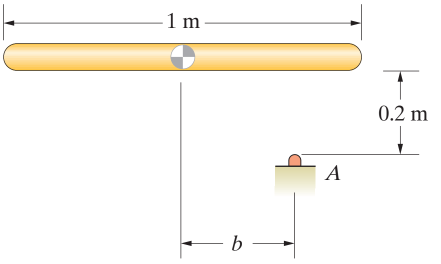
$$-I\omega_2 - W\Delta t\frac{L}{2} + F\Delta t L = I\omega_3,$$

$$-\frac{1}{3}\left(\frac{10\ \text{N-s}^2}{9.81\ \text{m}}\right)(3\ \text{m})^2(2.63\ \text{rad/s}) - (10\ \text{N})(0.01\ \text{s})(1.5\ \text{m})$$

$$+F(0.01\ \text{s})(3\ \text{m}) = \frac{1}{3}\left(\frac{10\ \text{N-s}^2}{9.81\ \text{m}}\right)(3\ \text{m})^2(1.31\ \text{rad/s})$$

Solving we find $\boxed{F = 406.6\ \text{N.}}$

Problem 19.66 The 4-kg bar is released from rest in the horizontal position above the fixed projection at A. The distance $b = 0.35$ m. The impact of the bar with the projection is plastic; that is, the coefficient of restitution of the impact is $e = 0$. What is the bar's angular velocity immediately after the impact?



Solution: First we do work energy to find the velocity of the bar just before impact.

$$T_1 + V_1 = T_2 + V_2$$

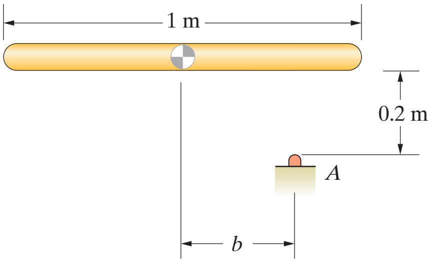
$$0 + mgh = \frac{1}{2}\, mv_2^2 + 0 \Rightarrow v_2 = \sqrt{2gh} = \sqrt{2(9.81\ \text{m/s}^2)(0.2\ \text{m})} = 1.98\ \text{m/s}$$

Angular momentum is conserved about the impact point A. After the impact, the bar pivots about the fixed point A.

$$mv_2 b = m\left(\frac{L^2}{12} + b^2\right)\omega_3 \Rightarrow \omega_3 = \frac{12bv_2}{L^2 + 12b^2} = \frac{12(0.35\ \text{m})(1.98\ \text{m/s})}{(1\ \text{m})^2 + 12(0.35\ \text{m})^2}$$

$$\boxed{\omega_3 = 3.37\ \text{rad/s.}}$$

Problem 19.67 The 4-kg bar is released from rest in the horizontal position above the fixed projection at A. The coefficient of restitution of the impact is $e = 0.6$. What value of the distance b would cause the velocity of the bar's center of mass to be zero immediately after the impact? What is the bar's angular velocity immediately after the impact?



Solution: First we do work energy to find the velocity of the bar just before impact.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + mgh = \frac{1}{2}\, mv_2^2 + 0 \Rightarrow v_2 = \sqrt{2gh} = \sqrt{2(9.81\ \text{m/s}^2)(0.2\ \text{m})} = 1.98\ \text{m/s}$$

Angular momentum is conserved about point A. The coefficient of restitution is used to relate the velocities of the impact point before and after the collision.

$$mv_2 b = mv_3 b + \frac{1}{12}\, mL^2\, \omega_3,\ ev_2 = \omega_3 b - v_3,\ v_3 = 0,$$

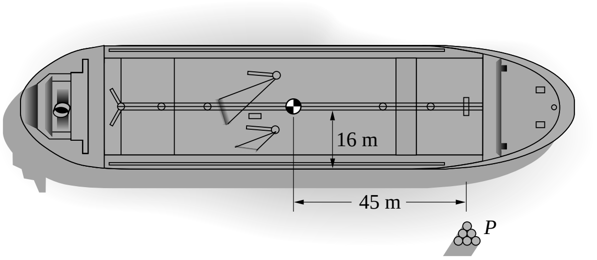
Solving we have

$$b = \sqrt{\frac{e}{12}}L = \sqrt{\frac{0.6}{12}}(1\ \text{m}) = 0.224\ \text{m},$$

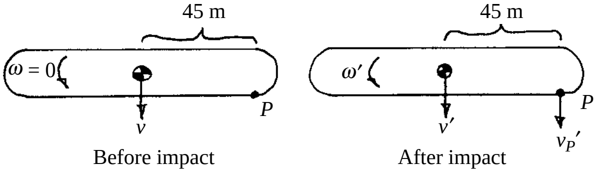
$$\omega_3 = \frac{12v_2 b}{L^2} = \frac{12(1.98\ \text{m/s})(0.224\ \text{m})}{(1\ \text{m})^2} = 5.32\ \text{rad/s.}$$

$$\boxed{b = 0.224\ \text{m},\ \ \omega_3 = 5.32\ \text{rad/s.}}$$

Problem 19.68 The mass of the ship is 544,000 kg, and the moment of inertia of the vessel about its center of mass is 4×10^8 kg-m^2. Wind causes the ship to drift sideways at 0.1 m/s and strike the stationary piling at P. The coefficient of restitution of the impact is $e = 0.2$. What is the ship's angular velocity after the impact?



Solution: Angular momentum about P is conserved



$$(45)(mv) = (45)(mv') + I\omega'. \quad \textbf{(1)}$$

The coefficient of restitution is

$$e = \frac{-v'_P}{v}, \quad \textbf{(2)}$$

where v'_P is the vertical component of the velocity of P after the impact. The velocities v'_P and v' are related by

$$v' = v'_P + (45)\omega'. \quad \textbf{(3)}$$

Solving Eqs. (1)–(3), we obtain $\underline{\omega' = 0.00196 \text{ rad/s}}$, $v' = 0.0680$ m/s, and $v'_P = -0.02$ m/s.

Problem 19.69 In Problem 19.68, if the duration of the ship's impact with the piling is 10 s, what is the magnitude of the average force exerted on the ship by the impact?

Solution: See the solution of Problem 19.68. Let F_p be the average force exerted on the ship by the piling. We apply linear impulse and momentum.

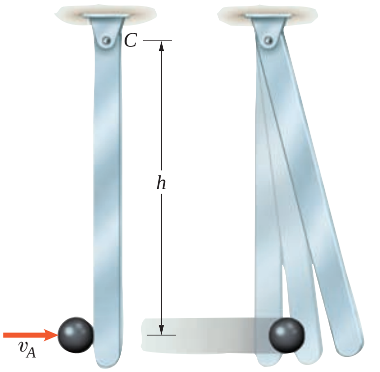
$$-F_p \Delta t = mv' - mv:$$

$$-F_p(10) = (544{,}000)(0.0680 - 0.1).$$

Solving, we obtain

$$F_p = 1740 \text{ N}.$$

Problem 19.70 In Active Example 19.7, suppose that the ball A weighs 2 N, the bar B weighs 6 N, and the length of the bar is 1 m. The ball is translating at $v_A = 3$ m/s before the impact and strikes the bar at $h = 0.6$ m. What is the angular velocity of the bar after the impact if the ball adheres to the bar?



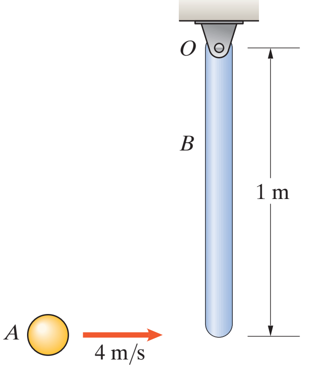
Solution: Angular momentum is conserved about point C.

$$m_A v_A h = \left(\frac{1}{3} m_B L^2 + m_A h^2\right)\omega$$

$$\omega = \frac{m_A v_A h}{\frac{1}{3} m_B L^2 + m_A h^2} = \frac{(2\ \text{N})(3\ \text{m/s})(0.6\ \text{m})}{\frac{1}{3}(6\ \text{N})(1\ \text{m})^2 + (2\ \text{N})(0.6\ \text{m})^2} = 1.32\ \text{rad/s}.$$

$\boxed{\omega = 1.32\ \text{rad/s}.}$

Problem 19.71 The 2-kg sphere A is moving toward the right at 4 m/s when it strikes the end of the 5-kg slender bar B. Immediately after the impact, the sphere A is moving toward the right at 1 m/s. What is the angular velocity of the bar after the impact?



Solution: Angular momentum is conserved about point O.

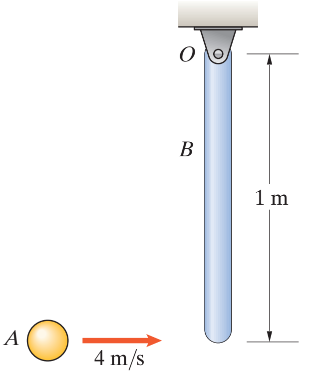
$$m_A v_{A1} L = m_A v_{A2} L + \frac{1}{3} m_B L^2\ \omega_2$$

$$\omega_2 = \frac{3 m_A (v_{A1} - v_{A2})}{m_B L}$$

$$\omega_2 = \frac{3(2\ \text{kg})(4\ \text{m/s} - 1\ \text{m/s})}{(5\ \text{kg})(1\ \text{m})} = 3.6\ \text{m/s}.$$

$\boxed{\omega_2 = 3.6\ \text{m/s counterclockwise.}}$

Problem 19.72 The 2-kg sphere A is moving toward the right at 4 m/s when it strikes the end of the 5-kg slender bar B. The coefficient of restitution is $e = 0.4$. The duration of the impact is 0.002 seconds. Determine the magnitude of the average horizontal force exerted on the bar by the pin support as a result of the impact.



Solution: System angular momentum is conserved about point O. The coefficient of restitution is used to relate the relative velocities before and after the impact. We also use the linear impulse momentum equation for the ball and for the bar.

$$m_A v_{A1} L = m_A v_{A2} L + \frac{1}{3} m_B L^2 \omega_2, \quad e v_{A1} = \omega_2 L - v_{A2},$$

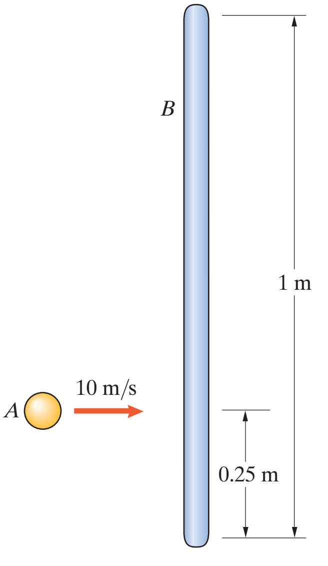
$$m_A v_{A1} - F\Delta t = m_A v_{A2}, \quad F\Delta t + R\Delta t = m_B \omega_2 \frac{L}{2}.$$

Solving we find

$$R = \frac{(1+e) m_A m_B v_{A1}}{2(3m_A + m_B)\Delta t} = \frac{1.4(2 \text{ kg})(5 \text{ kg})(4 \text{ m/s})}{2(11 \text{ kg})(0.002 \text{ s})} = 1270 \text{ N}$$

$\boxed{R = 1.27 \text{ kN.}}$

Problem 19.73 The 2-kg sphere A is moving toward the right at 10 m/s when it strikes the unconstrained 4-kg slender bar B. What is the angular velocity of the bar after the impact if the sphere adheres to the bar?



Solution: The coefficient of restitution is $e = 0$. Angular momentum for the system is conserved about the center of mass of the bar. The coefficient of restitution is used to relate the relative velocities before and after the impact. Linear momentum is conserved for the system.

$$m_A v_{A1} \left(\frac{L}{2} - h\right) = m_A v_{A2} \left(\frac{L}{2} - h\right) + \frac{1}{12} m_B L^2 \omega_2,$$

$$e v_{A1} = 0 = v_{B2} + \omega_2 \left(\frac{L}{2} - h\right) - v_{A2},$$

$$m_A v_{A1} = m_A v_{A2} + m_B v_{B2}.$$

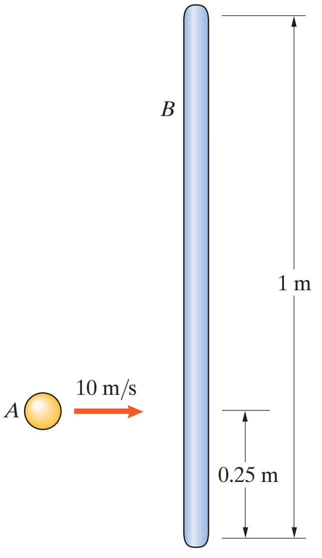
Solving we find

$$\omega_2 = \frac{6(L - 2h) m_A v_{A1}}{12h^2 m_A - 12hL m_A + L^2(4m_A + m_B)}$$

$$= \frac{6(1 \text{ m} - 2[0.25 \text{ m}])(2 \text{ kg})(10 \text{ m/s})}{12(0.25 \text{ m})^2(2 \text{ kg}) - 12(0.25 \text{ m})(1 \text{ m})(2 \text{ kg}) + (1 \text{ m})^2(12 \text{ kg})}$$

$\boxed{\omega_2 = 8 \text{ rad/s.}}$

Problem 19.74 The 2-kg sphere A is moving to the right at 10 m/s when it strikes the unconstrained 4-kg slender bar B. The coefficient of restitution of the impact is $e = 0.6$. What are the velocity of the sphere and the angular velocity of the bar after the impact?



Solution: Angular momentum for the system is conserved about the center of mass of the bar. The coefficient of restitution is used to relate the relative velocities before and after the impact. Linear momentum is conserved for the system.

$$m_A v_{A1}\left(\frac{L}{2} - h\right) = m_A v_{A2}\left(\frac{L}{2} - h\right) + \frac{1}{12} m_B L^2\ \omega_2,$$

$$e v_{A1} = v_{B2} + \omega_2 \left(\frac{L}{2} - h\right) - v_{A2},$$

$$m_A v_{A1} = m_A v_{A2} + m_B v_{B2}.$$

Solving we find

$$\omega_2 = \frac{6(1+e)(L-2h)m_A v_{A1}}{12h^2 m_A - 12hLm_A + L^2(4m_A + m_B)},$$

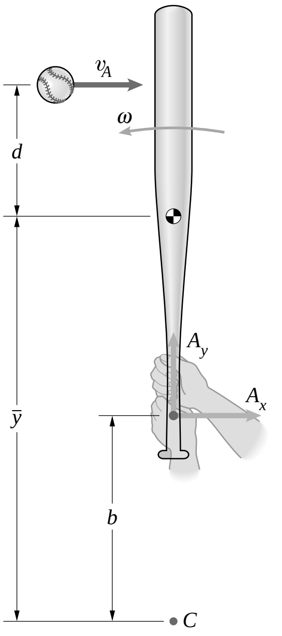
$$= \frac{6(1.6)(1\text{ m} - 2[0.25\text{ m}])(2\text{ kg})(10\text{ m/s})}{12(0.25\text{ m})^2(2\text{ kg}) - 12(0.25\text{ m})(1\text{ m})(2\text{ kg}) + (1\text{ m})^2(12\text{ kg})},$$

$$v_{A2} = \frac{(12h[h-L]m_A + L^2[4m_A - em_B])v_{A1}}{12h^2 m_A - 12hLm_A + L^2(4m_A + m_B)},$$

$$= \frac{(12[0.25\text{ m}][-0.75\text{ m}](2\text{ kg}) + [1\text{ m}]^2[5.6\text{ kg}])(10\text{ m/s})}{12(0.25\text{ m})^2(2\text{ kg}) - 12(0.25\text{ m})(1\text{ m})(2\text{ kg}) + (1\text{ m})^2(12\text{ kg})}.$$

$$\boxed{\begin{aligned} \omega_2 &= 12.8\text{ rad/s counterclockwise,} \\ v_{A2} &= 1.47\text{ m/s to the right.} \end{aligned}}$$

Problem 19.75 The 1.4 N ball is translating with velocity $v_A = 24.4$ m/s perpendicular to the bat just before impact. The player is swinging the 8.6 N bat with angular velocity $\omega = 6\pi$ rad/s before the impact. Point C is the bat's instantaneous center both before and after the impact. The distances $b = 355.6$ mm and $\overline{y} = 660.4$ mm. The bat's moment of inertia about its center of mass is $I_B = 0.045$ kg-m^2. The coefficient of restitution is $e = 0.6$, and the duration of the impact is 0.008 s. Determine the magnitude of the velocity of the ball after the impact and the average force A_x exerted on the bat by the player during the impact if (a) $d = 0$, (b) $d = 76.2$ mm, and (c) $d = 203$ mm.



Solution: By definition, the coefficient of restitution is

$$(1) \quad e = \frac{v'_P - v'_A}{v_A - v_P}.$$

The angular momentum about A is conserved:

$$(2) \quad m_A v_A (d + \overline{y} - b) + m_B v_B (\overline{y} - b) - I_B \omega$$

$$= m_A v'_A (d + \overline{y} - b) + m_B v'_B (\overline{y} - b) - I_B \omega'.$$

From kinematics, the velocities about the instantaneous center:

$$(3) \quad v_P = -\omega(\overline{y} + d),$$

$$(4) \quad v_B = -\omega\overline{y},$$

$$(5) \quad v'_P = -\omega'(\overline{y} + d),$$

$$(6) \quad v'_B = -\omega\overline{y}.$$

Since ω, $\overline{y}$, and d are known, v_B and v_P are determined from (3) and (4), and these six equations in six unknowns reduce to four equations in four unknowns. v'_P, v'_B, v'_A, and ω'. Further reductions may be made by substituting (5) and (6) into (1) and (2); however here the remaining four unknowns were solved by iteration for values of $d = 0$, $d = 0.076$ m, $d = 0.203$ m. The reaction at A is determined from the principle of angular impulse-momentum applied about the point of impact:

$$(7) \quad \int_{t_1}^{t_2} A_x (d + \overline{y} - b)\, dt = (d m_B v'_B + I_B \omega')$$

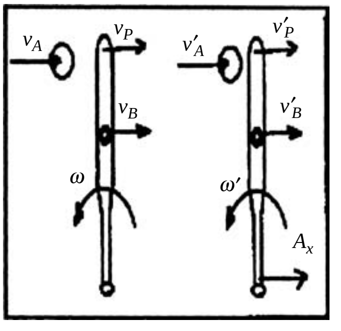
$$- (d m_B v_B + I_B \omega),$$

where $t_2 - t_1 = 0.08$ s. Using (4) and (5), the reaction is

$$A_x = \frac{(I_B - d m_B \mathbf{y})}{(d + \overline{y} - b)(t_2 - t_1)} (\omega' - \omega),$$

where the unknown, ω', is determined from the solution of the first six equations. The values are tabulated:

d, m	$v_A{}'$, m/s	A_x, N	$v_P{}'$, m/s	$v_B{}'$, m/s	ω', rad/s
0	−27.83	−186.7	−5.73	−5.73	8.672
0.076	−27.53	−2.12	−4.57	−4.09	6.20
0.203	−26.43	297.3	−2.02	−1.55	2.34



Only the values in the first two columns are required for the problem; the other values are included for checking purposes. *Note:* The reaction reverses between $d = 0.076$ m and $d = 0.203$ m, which means that the point of zero reaction occurs in this interval.

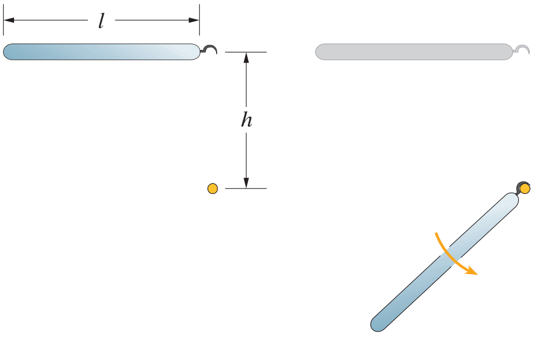
Problem 19.76 In Problem 19.75, show that the force A_x is zero if $d = I_B/(m_B\overline{y})$, where m_B is the mass of the bat.

Solution: From the solution to Problem 19.75, the reaction is

$$A_x = \frac{(I_B - dm_B\overline{y})}{(d + \overline{y} - b)(t_2 - t_1)}(\omega' - \omega).$$

Since $(\omega' - \omega) \neq 0$, the condition for zero reaction is $I_B - dm_B\overline{y} = 0$, from which $d = \dfrac{I_B}{\overline{y}m_B}$.

Problem 19.77 A 10-N slender bar of length $l = 2$ m is released from rest in the horizontal position at a height $h = 2$ m above a peg (Fig. a). A small hook at the end of the bar engages the peg, and the bar swings from the peg (Fig. b). What it the bar's angular velocity immediately after it engages the peg?



(a) (b)

Solution: Work energy is used to find the velocity just before impact.

$$T_1 + V_1 = T_2 + V_2$$

$$0 + mgh = \frac{1}{2}mv_2^2 + 0 \Rightarrow v_2 = \sqrt{2gh}$$

Angular momentum is conserved about the peg.

$$mv_2\frac{l}{2} = \frac{1}{3}ml^2\omega_3,$$

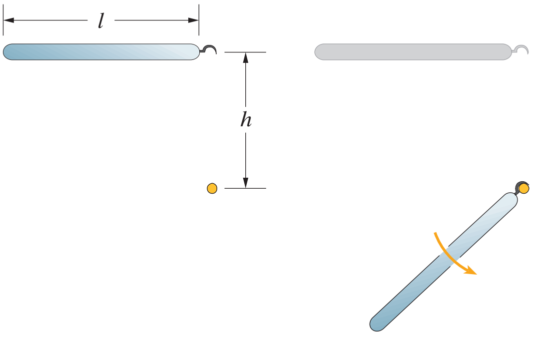
$$\omega_3 = \frac{3v_2}{2l} = \frac{3\sqrt{2gh}}{2l} = \frac{3\sqrt{2(9.81\text{ m/s}^2)(2\text{ m})}}{2(2\text{ m})} = 4.7\text{ rad/s}.$$

$$\boxed{\omega_3 = 4.7\text{ rad/s.}}$$

Problem 19.78 A 10-N slender bar of length $l = 2$ m is released from rest in the horizontal position at a height $h = 1$ m above a peg (Fig. a). A small hook at the end of the bar engages the peg, and the bar swings from the peg (Fig. b).

(a) Through what maximum angle does the bar rotate relative to its position when it engages the peg?

(b) At the instant when the bar has reached the angle determined in part (a), compare its gravitational potential energy to the gravitational potential energy the bar had when it was released from rest. How much energy has been lost?



(a) (b)

Solution: Work energy is used to find the velocity just before impact.

$$T_1 + V_1 = T_2 + V_2,\; 0 + mgh = \frac{1}{2}\,mv_2^2 + 0 \Rightarrow v_2 = \sqrt{2gh}$$

Angular momentum is conserved about the peg.

$$mv_2\frac{l}{2} = \frac{1}{3}\,ml^2\,\omega_3,\;\; \omega_3 = \frac{3v_2}{2l} = \frac{3\sqrt{2gh}}{2l}$$

Work energy is used to find the angle through which the bar rotates

$$T_3 + V_3 = T_4 + V_4,\; \frac{1}{2}\left(\frac{1}{3}\,ml^2\right)\omega_3^2 - mgh = 0 - mg\left(h + \frac{l}{2}\sin\theta\right)$$

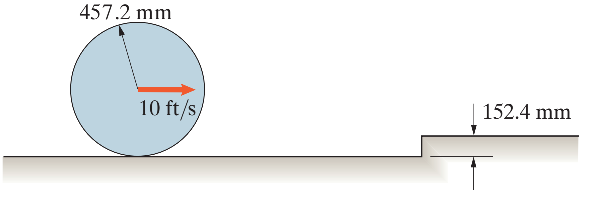
$$\theta = \sin^{-1}\left(-\frac{l\omega_3^2}{3g}\right) = \sin^{-1}\left(-\frac{3h}{2l}\right) = \sin^{-1}\left(-\frac{3}{4}\right) = 229^\circ.$$

The energy that is lost is the difference in potential energies

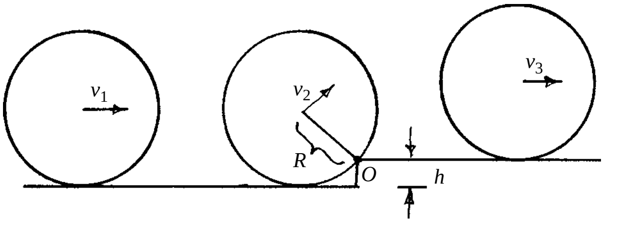
$$V_1 - V_4 = 0 - \left[-mg\left(h + \frac{l}{2}\sin\theta\right)\right] = (10\text{ N})\left(1\text{ m} + \frac{2\text{ m}}{2}\sin[229^\circ]\right) = 2.5\text{ N-m}.$$

$\boxed{(a)\ \theta = 229^\circ,\ (b)\ 2.5\text{ N-m.}}$

Problem 19.79 The 14.6 kg disk rolls at velocity $v = 3.05$ m/s toward a 152.4 mm step. The wheel remains in contact with the step and does not slip while rolling up onto it. What is the wheel's velocity once it is on the step?



Solution: We apply conservation of angular momentum about 0 to analyze the impact with the step.



$$(R - h)mv_1 + I\left(\frac{v_1}{R}\right) = Rmv_2 + I\left(\frac{v_2}{R}\right). \quad \textbf{(1)}$$

Then we apply work and energy to the "climb" onto the step.

$$-mgh = \left[\frac{1}{2}mv_3^2 + \frac{1}{2}I\left(\frac{v_3}{R}\right)^2\right] - \left[\frac{1}{2}mv_2^2 + \frac{1}{2}I\left(\frac{v_2}{R}\right)^2\right]. \quad \textbf{(2)}$$

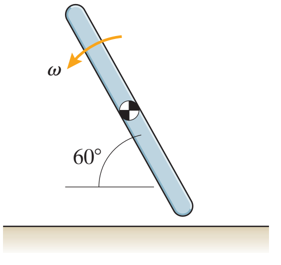
Solving Eqs. (1) and (2) with $v_1 = 3.05$ m/s and $I = \frac{1}{2}\,mR^2$, we obtain $v_3 = 1.91$ m/s.

Problem 19.80 The 14.6 kg disk rolls toward a 152.4 mm step. The wheel remains in contact with the step and does not slip while rolling up onto it. What is the minimum velocity v the disk must have in order to climb up onto the step?

Solution: See the solution of Problem 19.79 solving Eqs. (1) and (2) with $v_3 = 0$, we obtain

$\underline{v_1 = 1.82 \text{ m/s}}$.

Problem 19.81 The length of the bar is 1 m and its mass is 2 kg. Just before the bar hits the floor, its angular velocity is $\omega = 0$ and its center of mass is moving downward at 4 m/s. If the end of the bar adheres to the floor, what is the bar's angular velocity after the impact?



Solution: Given: $\omega = 0$, $L = 1$ m, $m = 2$ kg, $v_G = 4$ m/s, sticks to floor

Angular momentum about the contact point

$$mv_G \frac{L}{2} \cos 60^\circ = \frac{1}{3} m L^2 \omega' \quad \boxed{\omega' = 3 \text{ rad/s counterclockwise}}$$

Problem 19.82 The length of the bar is 1 m and its mass is 2 kg. Just before the bar hits the *smooth* floor, its angular velocity is $\omega = 0$ and its center of mass is moving downward at 4 m/s. If the coefficient of restitution of the impact is $e = 0.4$, what is the bar's angular velocity after the impact?

Solution: Given $\omega = 0$, $L = 1$ m, $m = 2$ kg, $v_G = 4$ m/s, $e = 0.4$, smooth floor.

Angular momentum about the contact point

$$mv_G \frac{L}{2} \cos 60^\circ = \frac{1}{12} m L^2 \omega' + m v_G' \frac{L}{2} \cos 60^\circ$$

Coefficient of restitution

$$e v_G = \omega \left(\frac{L}{2} \cos 60^\circ \right) - v_G'$$

Solving we find $\boxed{\omega' = 9.6 \text{ rad/s counterclockwise}}$

Problem 19.83 The length of the bar is 1 m and its mass is 2 kg. Just before the bar hits the *smooth* floor, it has angular velocity ω and its center of mass is moving downward at 4 m/s. The coefficient of restitution of the impact is $e = 0.4$. What value of ω would cause the bar to have no angular velocity after the impact?

Solution: Given $L = 1$ m, $m = 2$ kg, $v_G = 4$ m/s, $e = 0.4$, $\omega' = 0$ smooth floor.

Angular momentum about the contact point

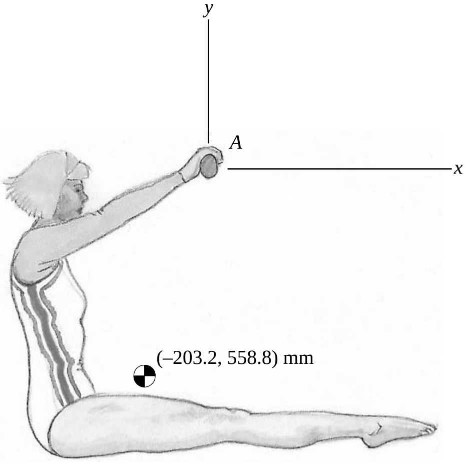
$$mv_G \frac{L}{2}\cos 60^\circ + \frac{1}{12}mL^2\omega = mv_G{}'\frac{L}{2}\cos 60^\circ$$

Coefficient of restitution

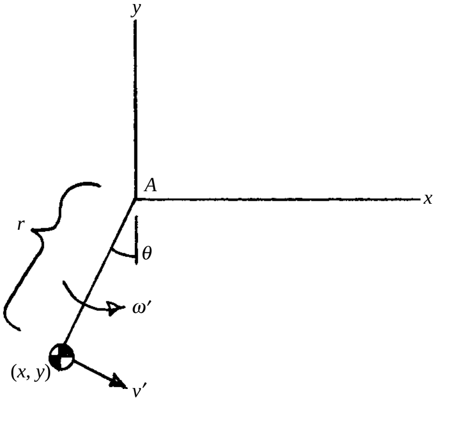
$$e\left(v_G - \omega\frac{L}{2}\cos 60^\circ\right) = 0 - v_G{}'$$

Solving we find $\omega = -24$ rad/s $\boxed{\omega = 24 \text{ rad/s clockwise}}$

Problem 19.84 During her parallel-bars routine, the velocity of the 400 N gymnast's center of mass is $1.2\mathbf{i} - 3.05\mathbf{j}$ (m/s) and her angular velocity is zero just before she grasps the bar at A. In the position shown, her moment of inertia about her center of mass is 2.44 kg-m^2. If she stiffens her shoulders and legs so that she can be modeled as a rigid body, what is the velocity of her center of mass and her angular velocity just after she grasps the bar?



Solution: Let v' and ω' be her velocity and angular velocity after she grasps the bar. The angle $\theta = 20.0^\circ$ and $r = 0.59$ m. Conservation of angular momentum about A is



$$\mathbf{k}\cdot(\mathbf{r}\times m\mathbf{v}) = rmv' + I\omega':$$

$$\mathbf{k}\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & 0\\ 1.2m & -3.05m & 0\end{vmatrix} = rmv' + I\omega',$$

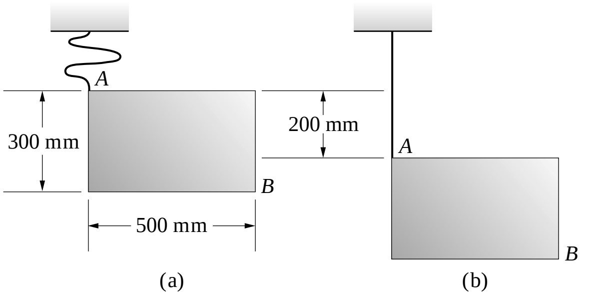
$$-3.05mx - 1.2my = rmv' + I\omega'.$$

Solving this equation together with the relation $v' = r\omega'$, we obtain $\underline{\omega' = 3.15 \text{ rad/s}}$, $v' = 1.87$ m/s. Her velocity is

$$\mathbf{v}' = v'\cos\theta\mathbf{i} - v'\sin\theta\mathbf{j}$$

$$= 1.76\mathbf{i} - 0.64\mathbf{j} \text{ (m/s)}.$$

Problem 19.85 The 20-kg homogenous rectangular plate is released from rest (Fig. a) and falls 200 mm before coming to the end of the string attached at the corner A (Fig. b). Assuming that the vertical component of the velocity of A is zero just after the plate reaches the end of the string, determine the angular velocity of the plate and the magnitude of the velocity of the corner B at that instant.

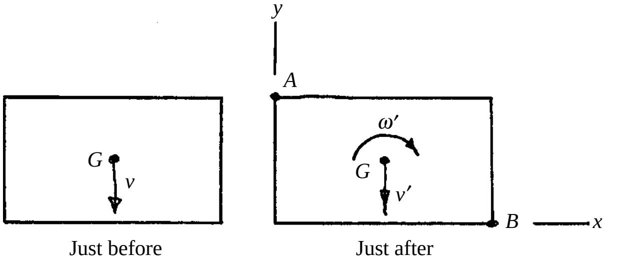


Solution: We use work and energy to determine the plate's downward velocity just before the string becomes taut.

$$mg(0.2) = \tfrac{1}{2}mv^2.$$

Solving, $v = 1.98$ m/s. The plate's moment of inertia is

$$I = \frac{1}{12}(20)[(0.3)^2 + (0.5)^2] = 0.567 \text{ kg-m}^2.$$



Angular momentum about A is conserved:

$$0.25(mv) = 0.25(mv') + I\omega'. \quad \textbf{(1)}$$

The velocity of A just after is

$$\mathbf{v}'_A = \mathbf{v}'_G + \boldsymbol{\omega}' \times \mathbf{r}_{A/G}$$

$$= -v'\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega' \\ -0.25 & 0.15 & 0 \end{vmatrix}.$$

The $\mathbf{j}$ component of $\mathbf{v}'_A$ is zero:

$$-v' + 0.25\omega' = 0. \quad \textbf{(2)}$$

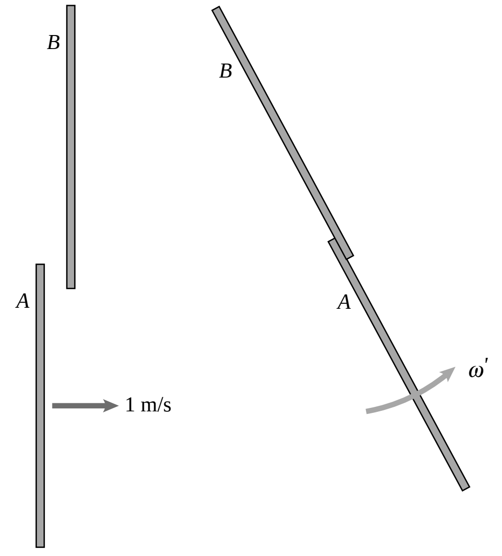
Solving Eqs. (1) and (2), we obtain $v' = 1.36$ m/s and $\underline{\omega' = 5.45 \text{ rad/s}}$. The velocity of B is

$$\mathbf{v}'_B = \mathbf{v}'_G + \boldsymbol{\omega}' \times \mathbf{r}_{B/G}$$

$$= -v'\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\omega' \\ 0.25 & -0.15 & 0 \end{vmatrix}$$

$$\underline{\mathbf{v}'_B = -0.818\mathbf{i} - 2.726\mathbf{j} \text{ (m/s)}}.$$

Problem 19.86* Two bars A and B are each 2 m in length, and each has a mass of 4 kg. In Fig. (a), bar A has no angular velocity and is moving to the right at 1 m/s, and bar B is stationary. If the bars bond together on impact, (Fig. b), what is their angular velocity ω' after the impact?



(a) (b)

Solution: Linear momentum is conserved:

$$mv_A = mv_A' + mv_B'. \quad \textbf{(1)}$$

Angular momentum about any point is conserved. About P,

$$mv_A\left(\frac{l}{2}\right) = mv_A'\left(\frac{l}{2}\right) - mv_B'\left(\frac{l}{2}\right) + 2I\omega', \quad \textbf{(2)}$$

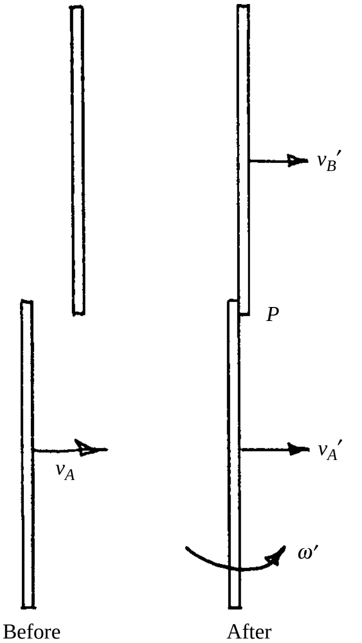
where $I = \dfrac{1}{12}ml^2$.

The velocities are related by

$$v_A' = v_B' + l\omega'. \quad \textbf{(3)}$$

Solving Eqs. (1)–(3), we obtain

$$\omega' = 0.375 \text{ rad/s}.$$



Before After

Problem 19.87* In Problem 19.86, if the bars do not bond together on impact and the coefficient of restitution is $e = 0.8$, what are the angular velocities of the bars after the impact?

Solution: Linear momentum is conserved:

$$mv_A = mv'_A + mv'_B. \quad \textbf{(1)}$$

Angular momentum of *each* bar about the point of contact is conserved:

$$mv_A\left(\frac{l}{2}\right) = mv'_A\left(\frac{l}{2}\right) + I\omega'_A, \qquad \textbf{(2)}$$

$$0 = -mv'_B\left(\frac{l}{2}\right) + I\omega'_B. \quad \textbf{(3)}$$

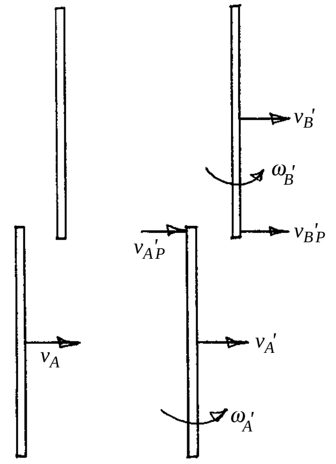
Coefficient of restitution:

$$e = \frac{v'_{BP} - v'_{AP}}{v_A}. \quad \textbf{(4)}$$

The velocities are related by

$$v'_A = v'_{AP} + \left(\frac{l}{2}\right)\omega'_A, \quad \textbf{(5)}$$

$$v'_{BP} = v'_B + \left(\frac{l}{2}\right)\omega'_B. \quad \textbf{(6)}$$

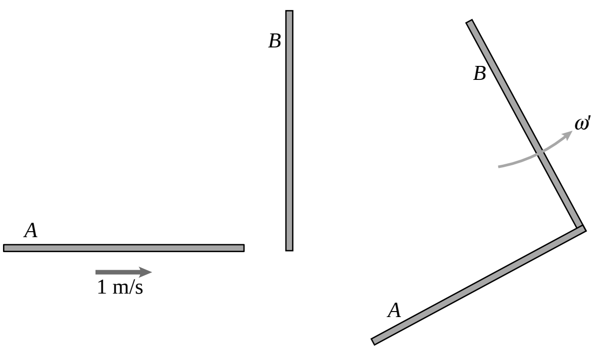


Before After

Solving Eqs. (1)–(6), we obtain

$\underline{\omega'_A = \omega'_B = 0.675 \text{ rad/s}.}$

Problem 19.88* Two bars A and B are each 2 m in length, and each has a mass of 4 kg. In Fig. (a), bar A has no angular velocity and is moving to the right at 1 m/s, and B is stationary. If the bars bond together on impact (Fig. b), what is their angular velocity ω' after the impact?



(a) (b)

Solution: Linear momentum is conserved:

$$x - \text{DIR:} \quad m_A v_A = m_A v'_{Ax} + m_B v'_{Bx}, \quad \textbf{(1)}$$

$$y - \text{DIR:} \quad 0 = m_A v'_{Ay} + m_B v'_{By}. \quad \textbf{(2)}$$

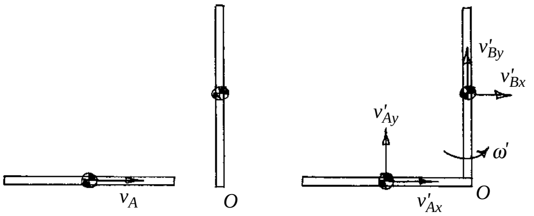
Total angular momentum is conserved. Calculating it about 0,

$$0 = -\frac{l}{2} m_A v'_{Ay} + I_A\omega' - \frac{l}{2} m_B v'_{Bx} + I_B\omega'. \quad \textbf{(3)}$$

We also have the kinematic relation

$$\mathbf{v}'_B = \mathbf{v}'_A + \boldsymbol{\omega}' \times \mathbf{r}_{B/A}:$$

$$v'_{Bx}\mathbf{i} + v'_{By}\mathbf{j} = v'_{Ax}\mathbf{i} + v'_{Ay}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega' \\ \dfrac{l}{2} & \dfrac{l}{2} & 0 \end{vmatrix},$$



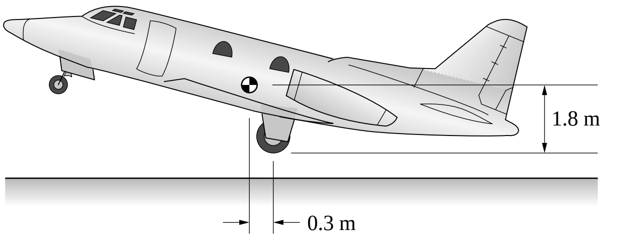
Before impact After impact

$$v'_{Bx} = v'_{Ax} - \frac{l}{2}\omega', \quad \textbf{(4)}$$

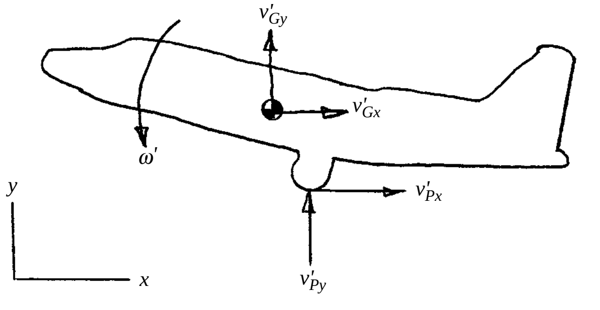
$$v'_{By} = v'_{Ay} + \frac{l}{2}\omega'. \quad \textbf{(5)}$$

Solving Eqs. (1)–(5), we obtain $\omega' = 0.3$ rad/s.

Problem 19.89* The horizontal velocity of the landing airplane is 50 m/s, its vertical velocity (rate of descent) is 2 m/s, and its angular velocity is zero. The mass of the airplane is 12 Mg, and the moment of inertia about its center of mass is 1×10^5 kg-m^2. When the rear wheels touch the runway, they remain in contact with it. Neglecting the horizontal force exerted on the wheels by the runway, determine the airplane's angular velocity just after it touches down. (See Example 19.8.)



Solution: Use a reference frame that moves to the left with the airplane's horizontal velocity. Before touchdown, the velocity of the center of mass is $\mathbf{v}_G = -2\mathbf{j}$ (m/s). Because there is no horizontal force, $v'_{Gx} = 0$ (see Fig.), and it is given that $v'_{Py} = 0$, where P is the point of contact.



Angular momentum about P is conserved:

$$m(2 \text{ m/s})(0.3 \text{ m}) = -mv'_{Gy}(0.3 \text{ m}) + I\omega'. \quad \textbf{(1)}$$

The velocities are related by

$$\mathbf{v}'_G = \mathbf{v}'_P + \boldsymbol{\omega}' \times \mathbf{r}_{G/P}:$$

$$v'_{Gy}\mathbf{j} = v'_{Px}\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega' \\ -0.3 & 1.8 & 0 \end{vmatrix}. \quad \textbf{(2)}$$

The $\mathbf{j}$ component of this equation is

$$v'_{Gy} = -0.3\omega'. \quad \textbf{(3)}$$

Solving Eqs. (1) and (3), we obtain

$\underline{\omega' = 0.0712 \text{ rad/s.}}$

Problem 19.90* Determine the angular velocity of the airplane in Problem 19.89 just after it touches down if its wheels don't stay in contact with the runway and the coefficient of restitution of the impact is $e = 0.4$. (See Example 19.8.)

Solution: See the solution of Problem 19.89. In this case v'_{Py} is not zero but is determined by

$$e = -\frac{v'_{Py}}{v_{Py}}:$$

$$0.4 = -\frac{v'_{Py}}{(-2)}$$

We see that $v'_{Py} = 0.8$ m/s. From Eqs. (1) and (2) of the solution of Problem 19.89,

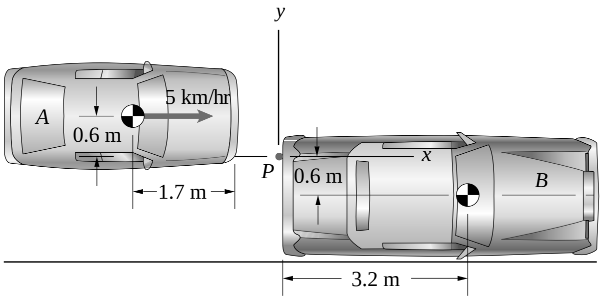
$$0.6m = -0.3mv'_{Gy} + I\omega',$$

$$v'_{Gy} = v'_{Py} - 0.3\omega'.$$

Solving these two equations, we obtain

$\underline{\omega' = 0.0997 \text{ rad/s}}.$

Problem 19.91* While attempting to drive on an icy street for the first time, a student skids his 1260-kg car (A) into the university president's unoccupied 2700-kg Rolls-Royce Corniche (B). The point of impact is P. Assume that the impacting surfaces are smooth and parallel to the y axis, and the coefficient of restitution of the impact is $e = 0.5$. The moments of inertia of the cars about their centers of mass are $I_A = 2400$ kg-m^2 and $I_B = 7600$ kg-m^2. Determine the angular velocities of the cars and the velocities of their centers of mass after the collision. (See Example 19.9.)



Solution: Car A's initial velocity is

$$v_A = \frac{5000}{3600} = 1.39 \text{ m/s}$$

The force of the impact is parallel to the x-axis so the cars are moving in the x direction after the collision. Linear momentum is conserved:

$$m_A v_A = m_A v'_A + m_B v'_B \quad \textbf{(1)}$$

The angular momentum of each car about the point of impact is conserved.

$$-0.6 m_A v_A = -0.6 m_A v'_A + I_A \omega'_A \quad \textbf{(2)}$$

$$0 = 0.6 m_B v'_B + I_B \omega'_B \quad \textbf{(3)}.$$

The velocity of car A at the point of impact after the collision is

$$\mathbf{v}'_{AP} = \mathbf{v}'_A + \boldsymbol{\omega}'_A \times \mathbf{r}_{P/A} = v'_A \mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega'_A \\ 1.7 & -0.6 & 0 \end{vmatrix}$$

$$\mathbf{v}'_{AP} = (v'_A + 0.6\omega'_A)\mathbf{i} + 1.7\omega'_A \mathbf{j}.$$

The corresponding equation for car B is

$$\mathbf{v}'_{BP} = \mathbf{v}'_B + \boldsymbol{\omega}'_B \times \mathbf{r}_{P/B} = v'_B \mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega'_B \\ -3.2 & 0.6 & 0 \end{vmatrix}, \quad \text{or}$$

$$v'_{BP} = (v'_B - 0.6\omega'_B)\mathbf{i} - 3.2\omega'_B \mathbf{j}$$

The x-components of the velocities at P are related by the coefficient of restitution:

$$e = \frac{(v'_B - 0.6\omega'_B) - (v'_A + 0.6\omega'_A)}{v_A} \quad \textbf{(4)}.$$

Solving Equations (1)–(4), we obtain

$v'_A = 0.174$ m/s, $v'_B = 0.567$ m/s

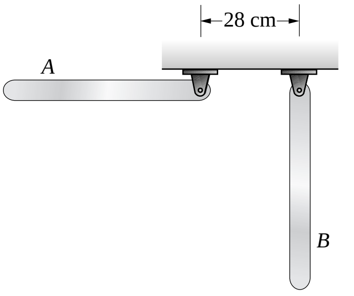
$\omega'_A = -0.383$ rad/s, $\omega'_B = -0.12$ rad/s.

Problem 19.92* The student in Problem 19.91 claimed he was moving at 5 km/h prior to the collision, but police estimate that the center of mass of the Rolls-Royce was moving at 1.7 m/s after the collision. What was the student's actual speed? (See Example 19.9.)

Solution: Setting $v'_B = 1.7$ m/s in Equations (1)–(4) of the solution of Problem 19.91 and treating v_A as an unknown, we obtain

$\underline{v_A = 4.17 \text{ m/s} = 15.0 \text{ km/h.}}$

Problem 19.93 Each slender bar is 48 cm long and weighs 20 N. Bar A is released in the horizontal position shown. The bars are smooth, and the coefficient of restitution of their impact is $e = 0.8$. Determine the angle through which B swings afterward.



Solution: Choose a coordinate system with the x axis parallel to bar A in the initial position, and the y axis positive upward. The strategy is (a) from the principle of work and energy, determine the angular velocity of bar A the instant before impact with B; (b) from the definition of the coefficient of restitution, determine the value of e in terms of the angular velocities of the two bars at the instant after impact; (c) from the principle of angular impulse-momentum, determine the relation between the angular velocities of the two bars after impact; (d) from the principle of work and energy, determine the angle through which bar B swings.

The angular velocity of bar A before impact: The angle of swing of bar A is

$$\beta = \sin^{-1}\left(\frac{28}{48}\right) = 35.69^\circ.$$

Denote the point of impact by P. The point of impact is

$$-h = -L\cos\beta = -3.25 \text{ cm}$$

The change in height of the center of mass of bar A is

$$-\frac{L}{2}\cos\beta = -\frac{h}{2}.$$

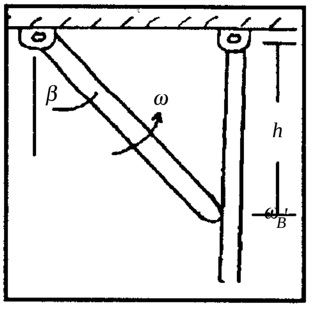
From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$, since the bar is released from rest. The work done by the weight of bar A is

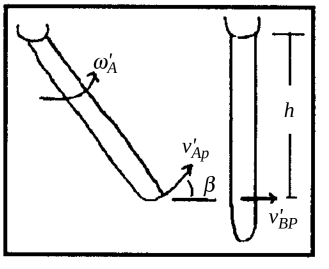
$$U = \int_0^{-\frac{h}{2}} -W\,dh = \frac{Wh}{2}.$$

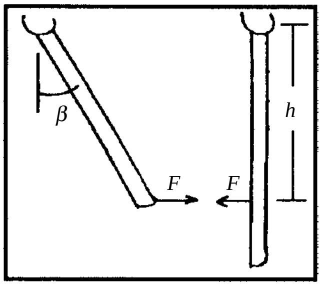
The kinetic energy the bar is $T_2 = \left(\frac{1}{2}\right) I_A \omega_A^2$,

from which $\omega_A = \sqrt{\dfrac{Wh}{I_A}} = \sqrt{\dfrac{3g\cos\beta}{L}} = 4.43$ rad/s,

where $I_A = \dfrac{mL^2}{3}$.







The angular velocities at impact: By definition, the coefficient of restitution is

$$e = \frac{v'_{BPx} - v'_{APx}}{v_{APx} - v_{BPx}}.$$

Bar B is at rest initially, from which

$$v_{BPx} = 0,\ v'_{BPx} = v'_{BP}, \text{ and}$$

$$v'_{APx} = v'_{AP}\cos\beta,\ v_{APx} = v_{AP}\cos\beta,$$

from which

$$e = \frac{v'_{BP} - v'_{AP}\cos\beta}{v_{AP}\cos\beta}.$$

The angular velocities are related by

$$v'_{BP} = h\omega'_B,\ v'_{AP} = L\omega'_A,\ v_{AP} = L\omega_A,$$

from which

$$e = \frac{h\omega'_B - (L\cos\beta)\omega'_A}{(L\cos\beta)\omega_A} = \frac{\omega_B - \omega'_A}{\omega_A},$$

from which

(1) $\omega'_B - \omega'_A = e\omega_A$

The force reactions at P: From the principle of angular impulse-momentum,

$$\int_{t_1}^{t_2} Fh\,dt = Fh(t_2 - t_1) = I_B(\omega'_B - 0), \quad \text{and}$$

$$\int_{t_1}^{t_2} -F(L\cos\beta)\,dt = -F(L\cos\beta)(t_2 - t_1)$$

$$= I_A(\omega'_A - \omega_A).$$

Divide the second equation by the first:

$$\frac{-L\cos\beta}{h} = -1 = \frac{\omega'_A - \omega_A}{\omega'_B},$$

from which

(2) $\omega'_B + \omega'_A = \omega_A.$

Solve (1) and (2):

$$\omega'_A = \frac{(1-e)}{2}\omega_A,$$

$$\omega'_B = \frac{(1+e)}{2}\omega_A.$$

The principle of work and energy: From the principle of work and energy, $U = T_2 - T_1$, where $T_2 = 0$, since the bar comes to rest after rotating through an angle γ. The work done by the weight of bar B as its center of mass rotates through the angle γ is

$$U = \int_{-\frac{L}{2}}^{-\frac{L\cos\gamma}{2}} -W_B\,dh = -W\left(\frac{L}{2}\right)(1 - \cos\gamma).$$

The kinetic energy is

$$T_1 = \left(\frac{1}{2}\right) I_B\omega_B'^2 = \left(\frac{1}{8}\right) I_B(1+e)^2\omega_A^2$$

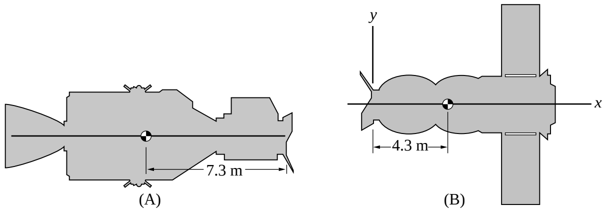
$$= \frac{I_B(1+e)^2(3g\cos\beta)}{8L} = \frac{WL(1+e)^2\cos\beta}{8}.$$

Substitute into $U = -T_1$ to obtain

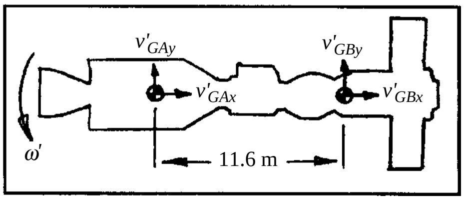
$$\cos\gamma = 1 - \frac{(1+e)^2\cos\beta}{4} = 0.3421,$$

from which $\boxed{\gamma = 70^\circ}$

Problem 19.94* The *Apollo* CSM (A) approaches the *Soyuz* Space Station (B). The mass of the *Apollo* is $m_A = 18$ Mg, and the moment of inertia about the axis through the center of mass parallel to the z axis is $I_A = 114$ Mg-m^2. The mass of the *Soyuz* is $m_B = 6.6$ Mg, and the moment of inertia about the axis through its center of mass parallel to the z axis is $I_B = 70$ Mg-m^2. The *Soyuz* is stationary relative to the reference frame shown and the CSM approaches with velocity $\mathbf{v}_A = 0.2\mathbf{i} + 0.05\mathbf{j}$ (m/s) and no angular velocity. What is the angular velocity of the attached spacecraft after docking?



Solution: The docking port is at the origin on the *Soyuz*, and the configuration the instant after contact is that the centers of mass of both spacecraft are aligned with the x axis. Denote the docking point of contact by P. (P is a point on each spacecraft, and by assumption, lies on the x-axis.) The linear momentum is conserved:



$$m_A \mathbf{v}_{GA} = m_A \mathbf{v}'_{GA} + m_B v'_{GB},$$

from which

$$m_A(0.2) = m_A v'_{GAx} + m_B v'_{GBx},$$

and

(1) $$m_A(0.05) = m_A v'_{GAy} + m_B v'_{GBy}.$$

Denote the vectors from P to the centers of mass by

$\mathbf{r}_{P/GA} = -7.3\mathbf{i}$ (m), and

$\mathbf{r}_{P/GB} = +4.3\mathbf{i}$ (m).

The angular momentum about the origin is conserved:

$$\mathbf{r}_{P/GA} \times m_A \mathbf{v}_{GA} = \mathbf{r}_{P/GA} \times m_A \mathbf{v}'_{GA} + I_A \boldsymbol{\omega}'_A$$

$$+ \mathbf{r}_{P/GB} \times m_B \mathbf{v}'_{GB}.$$

Denote the vector distance from the center of mass of the *Apollo* to the center of mass of the *Soyuz* by

$\mathbf{r}_{B/A} = 11.6\mathbf{i}$ (m).

From kinematics, the instant after contact:

$$\mathbf{v}'_{GB} = \mathbf{v}'_{GA} + \boldsymbol{\omega}' \times \mathbf{r}_{B/A}.$$

Reduce:

$$\mathbf{v}'_{GB} = \mathbf{v}'_{GA} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega' \\ 11.6 & 0 & 0 \end{bmatrix}$$

$$= v'_{GAx}\mathbf{i} + (v'_{GAy} + 11.6\omega')\mathbf{j},$$

from which $v'_{GBx} = v'_{GAx}$,

(2) $$v'_{GBy} = v'_{GAy} + 11.6\omega' \cdot \mathbf{r}_{P/GA} \times m_A \mathbf{v}_{GA}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7.3 & 0 & 0 \\ 0.21m_A & 0.05m_A & 0 \end{bmatrix} = (-0.365m_A)\mathbf{k},$$

$$\mathbf{r}_{P/GA} \times m_A \mathbf{v}'_{GA} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7.3 & 0 & 0 \\ m_A v'_{GAx} & m_A v'_{GAy} & 0 \end{bmatrix}$$

$$= (-7.3m_A v'_{GAy})\mathbf{k}.$$

$$\mathbf{r}_{P/GB} \times m_B \mathbf{v}'_{GB} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4.3 & 0 & 0 \\ m_B v'_{GAx} & m_A(v'_{GAy} + 11.6\omega') & 0 \end{bmatrix}$$

$$= 4.3m_B(v'_{GAy} + 11.6\omega')\mathbf{k}.$$

Collect terms and substitute into conservation of angular momentum expression, and reduce:

$$-0.365m_A = (-7.3m_A + 4.3m_B)v'_{GAy}$$

$$+ (49.9m_B + I_A + I_B)\omega'.$$

From (1) and (2),

$$v'_{GAy} = \frac{0.05m_A - 11.6m_B\omega'}{m_A + m_B}.$$

These two equations in two unknowns can be further reduced by substitution and algebraic reduction, but here they have been solved by iteration:

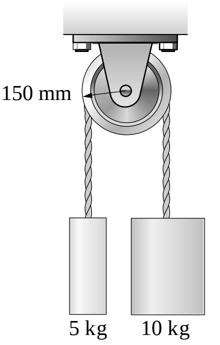
$\boxed{\omega' = -0.003359 \text{ rad/s}}$,

$v'_{GAy} = 0.04704$ m/s,

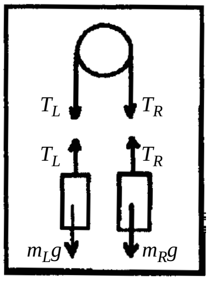
from which

$$v'_{GBy} = v'_{GAy} + 11.6\omega' = 0.00807 \text{ m/s}$$

Problem 19.95 The moment of inertia of the pulley is 0.2 kg-m^2. The system is released from rest. Use the principle of work and energy to determine the velocity of the 10-kg cylinder when it has fallen 1 m.



Solution: Choose a coordinate system with the y axis positive upward. Denote $m_L = 5$ kg, $m_R = 10$ kg, $h_R = -1$ m, $R = 0.15$ m. From the principle of work and energy, $U = T_2 - T_1$ where $T_1 = 0$ since the system is released from rest. The work done by the left hand weight is



$$U_L = \int_0^{h_L} -m_L g\, dh = -m_L g h_L.$$

The work done by the right hand weight is

$$U_L = \int_0^{h_R} -m_R g\, dh = -m_R g h_R.$$

Since the pulley is one-to-one, $h_L = -h_R$, from which

$$U = U_L + U_R = (m_L - m_R) g h_R.$$

The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) I_P \omega^2 + \left(\frac{1}{2}\right) m_L v_L^2 + \left(\frac{1}{2}\right) m_R v_R^2.$$

Since the pulley is one-to-one, $v_L = -v_R$. From kinematics

$$\omega = \frac{v_R}{R},$$

from which

$$T_2 = \left(\frac{1}{2}\right)\left(\frac{I_P}{R^2} + m_L + m_R\right) v_R^2.$$

Substitute and solve:

$$\boxed{v_R = \sqrt{\frac{2(m_L - m_R) g h_R}{\left(\frac{I_P}{R^2} + m_L + m_R\right)}} = 2.026\ldots = 2.03 \text{ m/s}}$$

Problem 19.96 The moment of inertia of the pulley is 0.2 kg-m^2. The system is released from rest. Use momentum principles to determine the velocity of the 10-kg cylinder 1 s after the system is released.

Solution: Use the coordinate system and notations of Problem 19.95. From the principle of linear impulse-momentum for the left hand weight:

$$\int_{t_1}^{t_2} (T_L - m_L g)\, dt = m_L(v_{L2} - v_{L1}) = m_L v_{L2},$$

since $v_{L1} = 0$, from which

(1) $T_L(t_2 - t_1) = m_L g(t_2 - t_1) + m_L v_2.$

For the right hand weight,

$$\int_{t_1}^{t_2} (T_R - m_R g)\, dt = m_R v_{R2},$$

from which

(2) $T_R(t_2 - t_1) = m_R g(t_2 - t_1) + m_R v_{R2}.$

From the principle of angular impulse-momentum for the pulley:

$$\int_{t_1}^{t_2} (T_L - T_R) R\, dt = I_P \omega_2,$$

from which

(3) $(T_L - T_R)(t_2 - t_1) = \dfrac{I_P}{R}\omega_2.$

Substitute (1) and (2) into (3):

$$(m_L - m_R)g(t_2 - t_1) + m_L v_{L2} - m_R v_{R2} = \frac{I_P}{R}\omega_2.$$

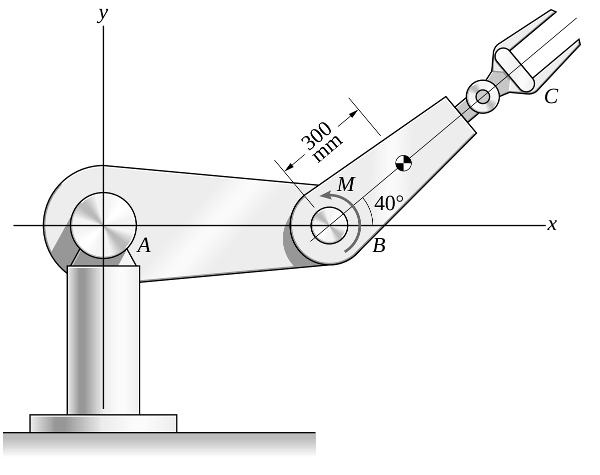
Since the pulley is one to one, $v_{L2} = -v_{R2}$. From kinematics:

$$\omega_2 = \frac{v_{R2}}{R},$$

from which

$$\boxed{v_{R2} = -\frac{(m_R - m_L)g(t_2 - t_1)}{\dfrac{I_P}{R^2} + m_L + m_R} = -2.05 \text{ m/s}}$$

Problem 19.97 Arm BC has a mass of 12 kg, and the moment of inertia about its center of mass is 3 kg-m^2. Point B is stationary. Arm BC is initially aligned with the (horizontal) x axis with zero angular velocity, and a constant couple M applied at B causes the arm to rotate upward. When it is in the position shown, its counterclockwise angular velocity is 2 rad/s. Determine M.



Solution: Assume that the arm BC is initially stationary. Denote $R = 0.3$ m. From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$. The work done is

$$U = \int_0^{\theta} M\, d\theta + \int_0^{R\sin\theta} -mg\, dh = M\theta - mgR\sin\theta.$$

The angle is

$$\theta = 40\left(\frac{\pi}{180}\right) = 0.6981 \text{ rad}.$$

The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) mv^2 + \left(\frac{1}{2}\right) I_{BC}\omega^2.$$

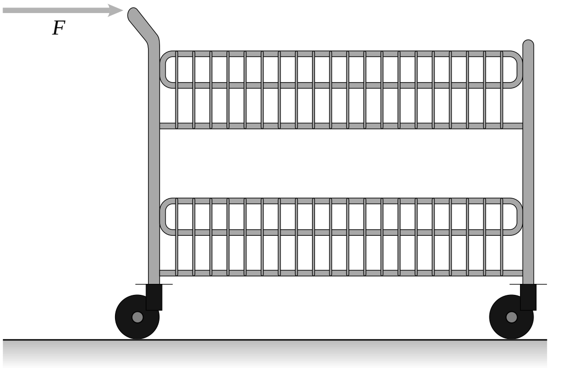
From kinematics, $v = R\omega$, from which

$$T_2 = \left(\frac{1}{2}\right)(mR^2 + I_{BC})\omega^2.$$

Substitute into $U = T_2$ and solve:

$$\boxed{M = \frac{(mR^2 + I_{BC})\omega^2 + 2mgR\sin\theta}{2\theta} = 44.2 \text{ N-m}}$$

Problem 19.98 The cart is stationary when a constant force F is applied to it. What will the velocity of the cart be when it has rolled a distance b? The mass of the body of the cart is m_C, and each of the four wheels has mass m, radius R, and moment of inertia I.



Solution: From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$. The work done is

$$U = \int_0^b F\,dx = Fb.$$

The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) m_C v^2 + 4\left(\frac{1}{2}\right) m v^2 + 4\left(\frac{1}{2}\right) I\omega^2.$$

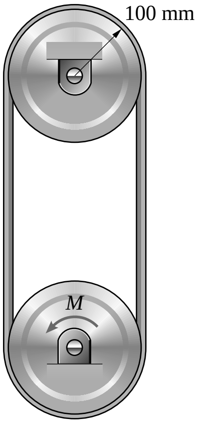
From kinematics, $v = R\omega$, from which

$$T_2 = \left(\frac{m_C}{2} + 2m + 2\frac{I}{R^2}\right) v^2.$$

Substitute into $U = T_2$ and solve:

$$\boxed{v = \sqrt{\frac{2Fb}{m_C + 4m + 4\left(\frac{I}{R^2}\right)}}}$$

Problem 19.99 Each pulley has moment of inertia $I = 0.003$ kg-m^2, and the mass of the belt is 0.2 kg. If a constant couple $M = 4$ N-m is applied to the bottom pulley, what will its angular velocity be when it has turned 10 revolutions?



Solution: Assume that the system is initially stationary. From the principle of work $U = T_2 - T_1$, where $T_1 = 0$. The work done is

$$U = \int_0^\theta M\,d\theta = M\theta,$$

where the angle is $\theta = 10(2\pi) = 62.83$ rad. The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) m_{\text{belt}} v^2 + 2\left(\frac{1}{2}\right) I_{\text{pulley}}\omega^2.$$

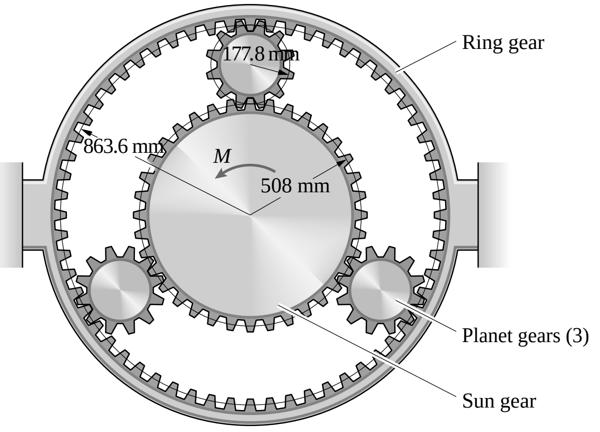
From kinematics, $v = R\omega$, where $R = 0.1$ m, from which

$$T_2 = \left(\frac{1}{2}\right)(R^2 m_{\text{belt}} + 2I_{\text{pulley}})\omega^2.$$

Substitute into $U = T_2$ and solve:

$$\boxed{\omega = \sqrt{\frac{2M\theta}{(R^2 m_{\text{belt}} + 2I_{\text{pulley}})}} = 250.7 \text{ rad/s}}$$

Problem 19.100 The ring gear is fixed. The mass and moment of inertia of the sun gear are $m_S = 321$ kg and $I_S = 5962$ kg-m^2. The mass and moment of inertia of each planet gear are $m_P = 39.4$ kg and $I_P = 88.1$ kg-m^2. A couple $M = 814$ N-m is applied to the sun gear. Use work and energy to determine the angular velocity of the sun gear after it has turned 100 revolutions.



Solution: Denote the radius of planetary gear, $R_P = 0.178$ m, the radius of sun gear $R_S = 0.508$ m, and angular velocities of the sun gear and planet gear by ω_S, ω_P. Assume that the system starts from rest. From the principle of work and energy $U = T_2 - T_1$, where $T_1 = 0$. The work done is

$$U = \int_0^\theta M d\theta = M\theta,$$

where the angle is

$$\theta = (100)(2\pi) = 200\pi = 628.3 \text{ rad}.$$

The kinetic energy is

$$T_2 = \left(\tfrac{1}{2}\right) I_S \omega_S^2 + 3\left(\left(\tfrac{1}{2}\right) m_P v_P^2 + \left(\tfrac{1}{2}\right) I_P \omega_P^2\right).$$

The velocity of the outer radius of the sun gear is $v_S = R_S\omega_S$. The velocity of the center of mass of the planet gears is the average velocity of the velocity of the sun gear contact and the ring gear contact,

$$v_P = \frac{v_S + v_R}{2} = \frac{v_S}{2},$$

since $v_R = 0$. The angular velocity of the planet gears is

$$\omega_P = \frac{v_P}{R_P}.$$

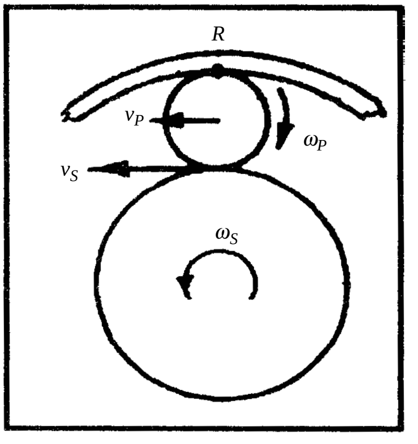
Collect terms:

$$\omega_P = \left(\frac{R_S}{R_P}\right)\frac{\omega_S}{2},$$

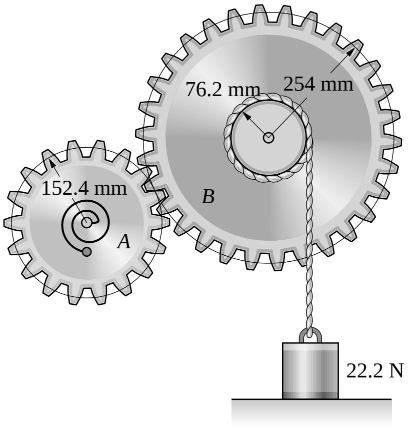
$$v_P = \frac{R_S}{2}\omega_S.$$

Substitute into $U = T_2$ and solve:

$$\boxed{\omega_S = \sqrt{\frac{2M\theta}{\left(I_S + \frac{3}{4}\left(m_P R_S^2 + I_P\left(\frac{R_S}{R_P}\right)^2\right)\right)}} = 12.5 \text{ rad/s}}$$



Problem 19.101 The moments of inertia of gears A and B are $I_A = 0.019$ kg-m^2, and $I_B = 0.136$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 0.27$ N-m/rad. If the spring is unstretched and the surface supporting the 22.2 N weight is removed, what is the velocity of the weight when it has fallen 76.2 mm?



Solution: Denote $W = 22.2$ N, $s = 0.762$ m is the distance the weight falls, $r_B = 0.254$ m, $r_A = 0.152$ m, $r'_B = 0.762$ m, are the radii of the gears and pulley. Choose a coordinate system with y positive upward. From the conservation of energy $T + V = \text{const.}$ Choose the datum at the initial position, such that $V_1 = 0$, $T_1 = 0$, from which $T_2 + V_2 = 0$ at any position 2. The gear B rotates in a negative direction and the gear A rotates in a positive direction. By inspection,

$$\theta_B = -\frac{s}{r'_B} = -\frac{0.762}{0.762} = -1 \text{ rad},$$

$$\theta_A = -\left(\frac{r_B}{r_A}\right)\theta_B = 1.667 \text{ rad},$$

$$v = r'_B\omega_B = 0.762\omega_B,$$

$$\omega_A = -\left(\frac{r_B}{r_A}\right)\omega_B = 1.667\omega_B.$$

The moment exerted by the spring is negative, from which the potential energy in the spring is

$$V_{\text{spring}} = -\int_0^{\theta_A} M\,d\theta = \int_0^{\theta_A} k\theta\,d\theta = \frac{1}{2}k\theta_A^2 = 0.377 \text{ N-m}.$$

The force due to the weight is negative, from which the potential energy of the weight is

$$V_{\text{weight}} = -\int_0^{-s}(-W)\,dy = -Ws = -1.695 \text{ N-m}.$$

The kinetic energy of the system is

$$T_2 = \left(\frac{1}{2}\right)I_A\omega_A^2 + \left(\frac{1}{2}\right)I_B\omega_B^2 + \left(\frac{1}{2}\right)\left(\frac{W}{g}\right)v^2.$$

Substitute: $T_2 = 17.3v^2$, from which

$$V_{\text{spring}} + V_{\text{weight}} + 17.3\,v^2 = 0.$$

Solve $\boxed{v = 0.276 \text{ m/s}}$ downward.

Problem 19.102 Consider the system in Problem 19.101.

(a) What maximum distance does the 22.2 N weight fall when the supporting surface is removed?
(b) What maximum velocity does the weight achieve?

Solution: Use the solution to Problem 19.101:

$$V_{\text{spring}} + V_{\text{weight}} + 17.3v^2 = 0.$$

$$V_{\text{spring}} = -\int_0^{\theta_A} M\,d\theta = \int_0^{\theta_A} k\theta\,d\theta = \frac{1}{2}k\theta_A^2 = 64.5s^2,$$

$$V_{\text{weight}} = -\int_0^{-s}(-W)\,dy = -Ws,$$

from which $64.5s^2 - 22.2s + 17.3v^2 = 0$.

(a) The maximum travel occurs when $v = 0$, from which

$$\boxed{s_{\max} = \frac{22.2}{64.5} = 0.344 \text{ m}}$$

(where the other solution $s_{\max} = 0$ is meaningless here).

(b) The maximum velocity occurs at

$$\frac{dv^2}{ds} = 0 = 2\frac{64.5}{17.3}s - \frac{22.2}{17.3} = 0,$$

from which

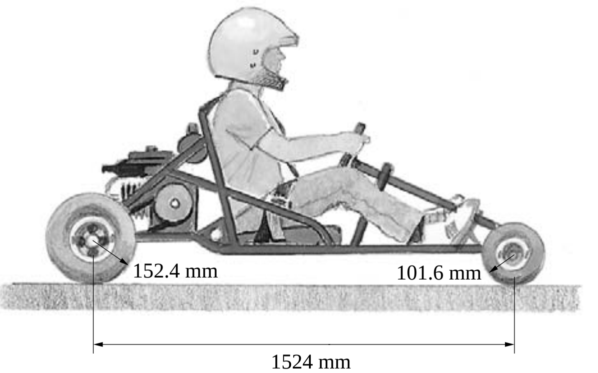
$$s_{v-\max} = 0.17 \text{ m}.$$

This is indeed a maximum, since

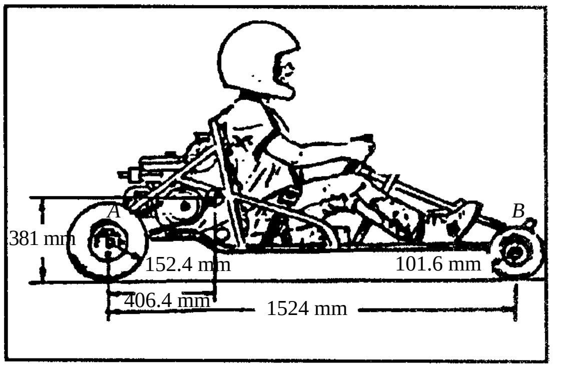
$$\frac{d^2(v^2)}{ds^2} = 2\left(\frac{64.5}{17.3}\right) > 0,$$

and $\boxed{|v_{\max}| = 0.332 \text{ m/s}}$ (downward).

Problem 19.103 Each of the go-cart's front wheels weighs 22.2 N and has a moment of inertia of 0.0136 kg-m^2. The two rear wheels and rear axle form a single rigid body weighing 178 N and having a moment of inertia of 0.136 kg-m^2. The total weight of the rider and go-cart, including its wheels, is 1067 N. The go-cart starts from rest, its engine exerts a constant torque of 20.3 N-m on the rear axle, and its wheels do not slip. Neglecting friction and aerodynamic drag, how fast is the go-cart moving when it has traveled 15.2 m?



Solution: From the principle of work and energy: $U = T_2 - T_1$, where $T_1 = 0$, since the go-cart starts from rest. Denote the rear and front wheels by the subscripts A and B, respectively. The radius of the rear wheels $r_A = 0.152$ m. The radius of the front wheels is $r_B = 0.102$ m. The rear wheels rotate through an angle $\theta_A = \dfrac{15.2}{0.152} = 100$ rad, from which the work done is



$$U = \int_0^{\theta_A} M d\theta = M\theta_A = 20.3(100) \text{ N-m}.$$

The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right)\frac{W}{g}v^2 + \left(\frac{1}{2}\right)I_A\omega_A^2 + 2\left(\frac{1}{2}\right)I_B\omega_B^2$$

(for two front wheels). The angular velocities are related to the go-cart velocity by $\omega_A = \dfrac{v}{r_A} = 2v$, $\omega_B = \dfrac{v}{r_B} = 3v$, from which $T_2 = 1.23v^2$ N-m. Substitute into $U = T_2$ and solve: $v = 5.89$ m/s.

Problem 19.104 Determine the maximum power and the average power transmitted to the go-cart in Problem 19.103 by its engine.

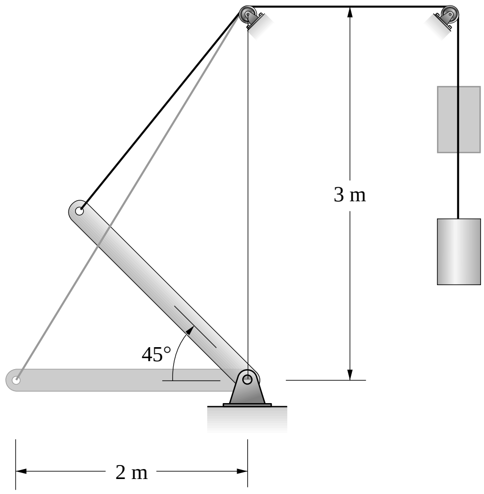
Solution: The maximum power is $P_{\max} = M\omega_{A\max}$, where $\omega_{A\max} = \dfrac{v_{\max}}{R}$. From which $P_{\max} = \dfrac{Mv_{\max}}{R}$. Under constant torque, the acceleration of the go-cart is constant, from which the maximum velocity is the greatest value of the velocity, which will occur at the end of the travel. From the solution to Problem 19.103, $v_{\max} = 19.32$ ft/s, from which

$$P_{\max} = \frac{Mv_{\max}}{r_A} = \frac{20.3(5.89)}{0.152} = 786.6 \text{ N-m/s}.$$

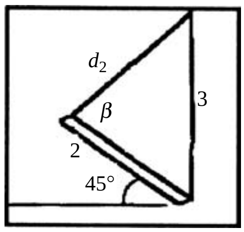
The average power is $P_{ave} = \dfrac{U}{t}$. From the solution to Problem 19.103, $U = 2030$ N-m. Under constant acceleration, $v = at$, and $s = \frac{1}{2}at^2$, from which $s = \frac{1}{2}vt$, and $t = \dfrac{2s}{v} = \dfrac{30.48}{5.89} = 5.177$ s, from which

$$P_{ave} = \frac{U}{t} = \frac{2030}{5.177} = 392 \text{ N-m/s}.$$

Problem 19.105 The system starts from rest with the 4-kg slender bar horizontal. The mass of the suspended cylinder is 10 kg. What is the angular velocity of the bar when it is in the position shown?



Solution: From the principle of work and energy: $U = T_2 - T_1$, where $T_1 = 0$ since the system starts from rest. The change in height of the cylindrical weight is found as follows: By inspection, the distance between the end of the bar and the pulley when the bar is in the horizontal position is $d_1 = \sqrt{2^2 + 3^2} = 3.61$ m. The law of cosines is used to determine the distance between the end of the bar and the pulley in the position shown:



$$d_2 = \sqrt{2^2 + 3^2 - 2(2)(3)\cos 45^\circ} = 2.125 \text{ m},$$

from which $h = d_1 - d_2 = 1.481$ m. The work done by the cylindrical weight is

$$U_{\text{cylinder}} = \int_0^{-h} -m_c g\, ds = m_c g h = 145.3 \text{ N-m}.$$

The work done by the weight of the bar is

$$U_{\text{bar}} = \int_0^{\cos 45^\circ} -m_b g\, dh = -m_b g \cos 45^\circ = -27.75 \text{ N-m},$$

from which $U = U_{\text{cylinder}} + U_{\text{bar}} = 117.52$ N-m. From the sketch, (which shows the final position) the component of velocity normal to the bar is $v \sin\beta$, from which $2\omega = v \sin\beta$. From the law of sines:

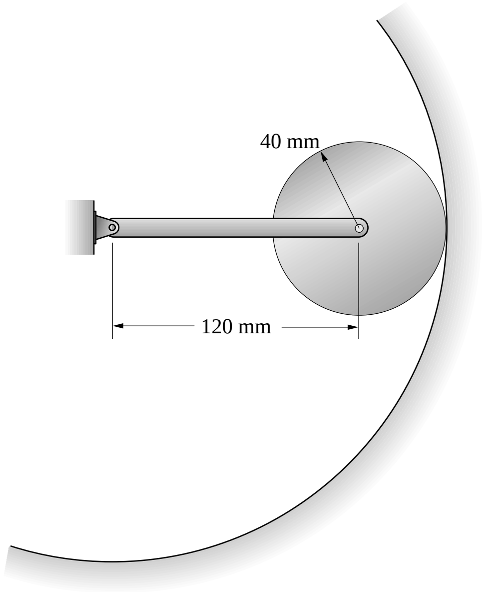
$$\sin\beta = \left(\frac{3}{d_2}\right)\sin 45^\circ = 0.9984.$$

The kinetic energy is

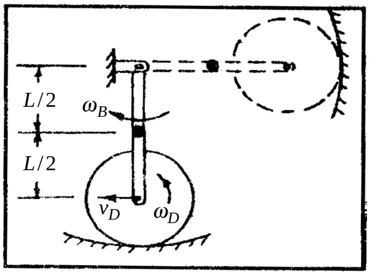
$$T_2 = \left(\frac{1}{2}\right) m_c v^2 + \left(\frac{1}{2}\right) I\omega_b^2,$$

from which $T_2 = 22.732\omega^2$, where $v = \left(\dfrac{2}{\sin\beta}\right)\omega$ and $I = \dfrac{m(2^2)}{3}$ has been used. Substitute into $U = T_2$ and solve: $\omega = 2.274$ rad/s.

Problem 19.106 The 0.1-kg slender bar and 0.2-kg cylindrical disk are released from rest with the bar horizontal. The disk rolls on the curved surface. What is the angular velocity of the bar when it is vertical?



Solution: From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$. Denote $L = 0.12$ m, $R = 0.04$ m, the angular velocity of the bar by ω_B, the velocity of the disk center by v_D, and the angular velocity of the disk by ω_D. The work done is



$$U = \int_0^{-\frac{L}{2}} -m_B g\,dh + \int_0^{-L} -m_D g\,dh = \left(\frac{L}{2}\right) m_B g + L m_D g.$$

From kinematics, $v_D = L\omega_B$, and $\omega_D = \dfrac{v_D}{R}$. The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) I_B \omega_B^2 + \left(\frac{1}{2}\right) m_D v_D^2 + \left(\frac{1}{2}\right) I_D \omega_D^2.$$

Substitute the kinematic relations to obtain

$$T_2 = \left(\frac{1}{2}\right)\left(I_B + m_D L^2 + I_D \left(\frac{L}{R}\right)^2\right)\omega_B^2,$$

where $I_B = \dfrac{m_B L^2}{3}$,

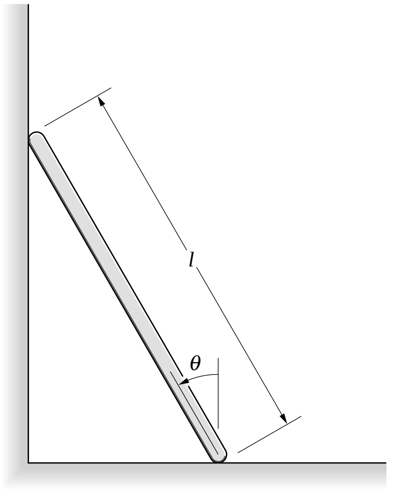
$$I_D = \frac{m_D R^2}{2},$$

from which $T_2 = \left(\dfrac{1}{2}\right)\left(\dfrac{m_B}{3} + \dfrac{3m_D}{2}\right) L^2 \omega_B^2$.

Substitute into $U = T_2$ and solve:

$$\boxed{\omega_B = \sqrt{\frac{6\text{ g}(m_B + 2m_D)}{(2m_B + 9m_D)L}} = 11.1 \text{ rad/s.}}$$

Problem 19.107 A slender bar of mass m is released from rest in the vertical position and allowed to fall. Neglecting friction and assuming that it remains in contact with the floor and wall, determine the bar's angular velocity as a function of θ.

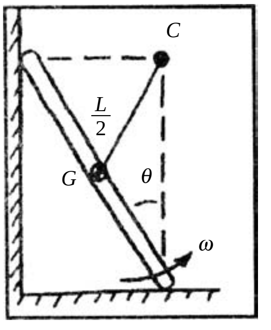


Solution: The strategy is

(a) to use the kinematic relations to determine the relation between the velocity of the center of mass and the angular velocity about the instantaneous center, and

(b) the principle of work and energy to obtain the angular velocity of the bar. *The kinematics*: Denote the angular velocity of the bar about the instantaneous center by ω. The coordinates of the instantaneous center of rotation of the bar are $(L\sin\theta, L\cos\theta)$. The coordinates of the center of mass of the bar are

$$\left(\frac{L}{2}\sin\theta, \frac{L}{2}\cos\theta\right).$$



The vector distance from the instantaneous center to the center of mass is

$$\mathbf{r}_{G/C} = -\frac{L}{2}(\mathbf{i}\sin\theta + \mathbf{j}\cos\theta).$$

The velocity of the center of mass is

$$\mathbf{v}_G = \boldsymbol{\omega} \times \mathbf{r}_{G/C} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ -\frac{L}{2}\sin\theta & -\frac{L}{2}\cos\theta & 0 \end{bmatrix}$$

$$= \frac{\omega L}{2}(\mathbf{i}\cos\theta - \mathbf{j}\sin\theta),$$

from which $|\mathbf{v}_G| = \dfrac{\omega L}{2}$.

The principle of work and energy: $U = T_2 - T_1$ where $T_1 = 0$. The work done by the weight of the bar is

$$U = mg\left(\frac{L}{2}\right)(1 - \cos\theta).$$

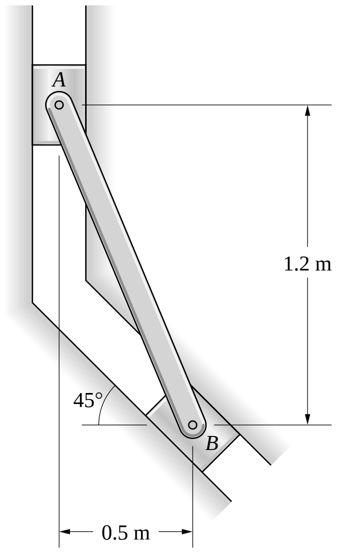
The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right)mv_G^2 + \left(\frac{1}{2}\right)I_B\omega^2.$$

Substitute $v_G = \dfrac{\omega L}{2}$ and $I_B = \dfrac{mL^2}{12}$ to obtain $T_2 = \dfrac{mL^2\omega^2}{6}$. Substitute into $U = T_2$ and solve:

$$\boxed{\omega = \sqrt{\frac{3g(1-\cos\theta)}{L}}.}$$

Problem 19.108 The 4-kg slender bar is pinned to 2-kg sliders at A and B. If friction is negligible and the system starts from rest in the position shown, what is the bar's angular velocity when the slider at A has fallen 0.5 m?

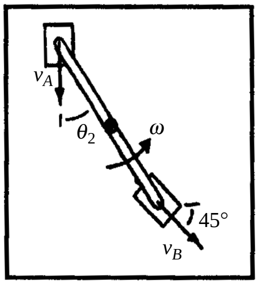


Solution: Choose a coordinate system with the origin at the initial position of A and the y axis positive upward. The strategy is

(a) to determine the distance that B has fallen and the center of mass of the bar has fallen when A falls 0.5 m,

(b) use the coordinates of A, B, and the center of mass of the bar and the constraints on the motion of A and B to determine the kinematic relations, and

(c) use the principle of work and energy to determine the angular velocity of the bar.



The displacement of B: Denote the length of the bar by $L = \sqrt{1.2^2 + 0.5^2} = 1.3$ m. Denote the horizontal and vertical displacements of B when A falls 0.5 m by d_x and d_y, which are in the ratio $\dfrac{d_y}{d_x} = \tan 45° = 1$, from which $d_x = d_y = d$. The vertical distance between A and B is reduced by the distance 0.5 m and increased by the distance d_y, and the horizontal distance between A and B is increased by the distance d_x, from which $L^2 = (1.2 - 0.5 + d_y)^2 + (0.5 + d_x)^2$. Substitute $d_x = d_y = d$ and $L = 1.3$ m and reduce to obtain $d^2 + 2bd + c = 0$, where $b = 0.6$, and $c = -0.475$. Solve: $d_{1,2} = -b \pm \sqrt{b^2 - c} = 0.3138$ m, or $= -1.514$ m, from which only the positive root is meaningful.

The final position coordinates: The coordinates of the initial position of the center of mass of the bar are

$$(x_{G1}, y_{G1}) = \left(\frac{L}{2}\sin\theta_1, -\frac{L}{2}\cos\theta_1\right) = (0.25, -0.6) \text{ (m)},$$

where $\theta_1 = \sin^{-1}\left(\dfrac{0.5}{L}\right) = 22.61°$

is the angle of the bar relative to the vertical. The coordinates of the final position of the center of mass of the bar are

$$(x_{G2}, y_{G2}) = \left(\frac{L}{2}\sin\theta_2, -0.5 - \frac{L}{2}\cos\theta_2\right) = (0.4069, -1.007),$$

where $\theta_2 = \sin^{-1}\left(\dfrac{0.5 + d}{L}\right) = 38.75°$.

The vertical distance that the center of mass falls is $h = y_{G2} - y_{G1} = -0.4069$ m. The coordinates of the final positions of A and B are, respectively $(x_{A2}, y_{A2}) = (0, -0.5)$, and $(x_{B2}, y_{B2}) = (0.5 + d, -(1.2 + d)) = (0.8138, -1.514)$. The vector distance from A to B is

$$\mathbf{r}_{B/A} = (x_{B2} - x_{A2})\mathbf{i} + (y_{B2} - y_{A2})\mathbf{j}$$

$$= 0.8138\mathbf{i} - 1.014\mathbf{j} \text{ (m)}.$$

Check: $|\mathbf{r}_{B/A}| = L = 1.3$ m. *check*. The vector distance from A to the center of mass is

$$\mathbf{r}_{G/A} = (x_{G2} - x_{A2})\mathbf{i} + (y_{G2} - y_{A2})\mathbf{j}$$

$$= 0.4069\mathbf{i} - 0.5069\mathbf{j} \text{ (m)}.$$

Check: $\mathbf{r}_{G/A} = \dfrac{L}{2} = 0.65$ m. *check*.

The kinematic relations: From kinematics, $\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$. The slider A is constrained to move vertically, and the slider B moves at a 45° angle, from which $\mathbf{v}_A = -v_A\mathbf{j}$ (m/s), and

$$\mathbf{v}_B = (v_B \cos 45^\circ)\mathbf{i} - (v_B \sin 45^\circ)\mathbf{j}.$$

$$\mathbf{v}_B = \mathbf{v}_A + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ 0.8138 & -1.014 & 0 \end{bmatrix}$$

$$= \mathbf{i}(1.014\omega) + \mathbf{j}(-v_A + 0.8138\omega),$$

from which (1) $v_B = \left(\dfrac{1.014}{\cos 45^\circ}\right)\omega = 1.434\omega$,

and (2) $v_A = v_B \sin 45^\circ + 0.8138\omega = 1.828\omega$.

The velocity of the center of mass of the bar is

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = -v_A\mathbf{j} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ 0.4069 & -0.5069 & 0 \end{bmatrix}$$

$$= (0.5069\omega)\mathbf{i} + (-v_A + 0.4069\omega)\mathbf{j},$$

$$\mathbf{v}_G = 0.5069\omega\mathbf{i} - 1.421\omega\mathbf{j} \text{ (m/s)},$$

from which (3) $v_G = 1.508\omega$

The principle of work and energy: From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$. The work done is

$$U = \int_0^{-d_1} -m_B g\, dh + \int_0^{-0.5} -m_A g\, dh + \int_0^{-h} -m_{\text{bar}} g\, dh,$$

$$U = m_B g(0.3138) + m_A g(0.5) + m_{\text{bar}} g(0.4069) = 31.93 \text{ N-m}.$$

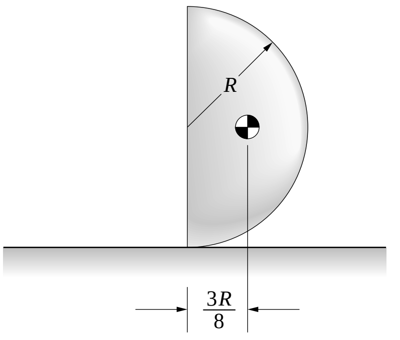
The kinetic energy is

$$T_2 = \left(\frac{1}{2}\right) m_B v_B^2 + \left(\frac{1}{2}\right) m_A v_A^2 + \left(\frac{1}{2}\right) m_{\text{bar}} v_G^2 + \left(\frac{1}{2}\right) I_{\text{bar}} \omega^2,$$

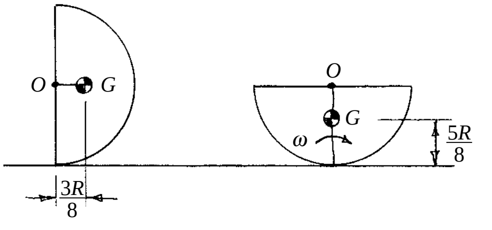
substitute $I_{\text{bar}} = \frac{m_{\text{bar}} L^2}{12}$, and (1), (2) and (3) to obtain $T_2 = 10.23\omega^2$. Substitute into $U = T_2$ and solve:

$$\boxed{\omega = \sqrt{\frac{31.93}{10.23}} = 1.77 \text{ rad/s}}$$

Problem 19.109 A homogeneous hemisphere of mass m is released from rest in the position shown. If it rolls on the horizontal surface, what is its angular velocity when its flat surface is horizontal?



Solution: The hemisphere's moment of inertia about O is $\frac{2}{5}mR^2$, so its moment of inertia about G is



$$I = \frac{2}{5}mR^2 - \left(\frac{3}{8}R\right)^2 m = \frac{83}{320}mR^2.$$

The work done is

$$mg\left(R - \frac{5}{8}R\right) = \frac{3}{8}mgR.$$

Work and energy is

$$\frac{3}{8}mgR = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

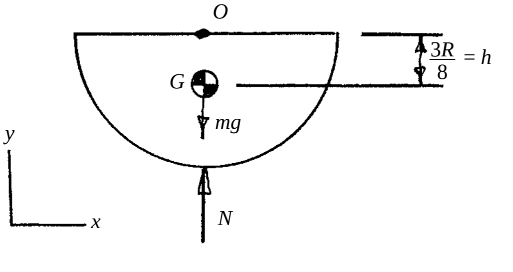
$$= \frac{1}{2}m\left[\left(\frac{5R}{8}\right)\omega\right]^2 + \frac{1}{2}\left(\frac{83}{320}mR^2\right)\omega^2.$$

Solving for ω, we obtain

$$\omega = \sqrt{\frac{15g}{13R}}.$$

Problem 19.110 The homogeneous hemisphere of mass m is released from rest in the position shown. It rolls on the horizontal surface. What normal force is exerted on the hemisphere by the horizontal surface at the instant the flat surface of the hemisphere is horizontal?

Solution: See the solution of Problem 19.109. The acceleration of G is



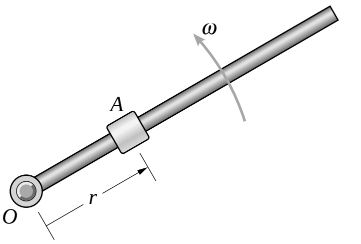
$$\mathbf{a}_G = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{G/O} - \omega^2\mathbf{r}_{G/O}$$

$$= a_O\mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \alpha \\ 0 & -h & 0 \end{vmatrix} - \omega^2(-h\mathbf{j}),$$

so $a_{Gy} = \omega^2 h$. Therefore $N - mg = ma_{Gy} = m\omega^2 h$. Using the result from the solution of Problem 19.109,

$$N = mg + m\left(\frac{15g}{13R}\right)\left(\frac{3R}{8}\right) = 1.433 \text{ mg}.$$

Problem 19.111 The slender bar rotates freely *in the horizontal plane* about a vertical shaft at O. The bar weighs 89 N and its length is 1.83 m. The slider A weighs 8.9 N. If the bar's angular velocity is $\omega = 10$ rad/s and the radial component of the velocity of A is zero when $r = 0.31$ m, what is the angular velocity of the bar when $r = 1.22$ m? (The moment of inertia of A about its center of mass is negligible; that is, treat A as a particle.)



Solution: From the definition of angular momentum, only *the radial position* of the slider need be taken into account in applying the principle of the conservation of angular momentum; that is, *the radial velocity* of the slider at $r = 1.22$ m does not change the angular momentum of the bar. From the conservation of angular momentum:

$$I_{\text{bar}}\omega_1 + r_{1A}^2\left(\frac{W_A}{g}\right)\omega_1 = I_{\text{bar}}\omega_2 + r_{2A}^2\left(\frac{W_A}{g}\right)\omega_2.$$

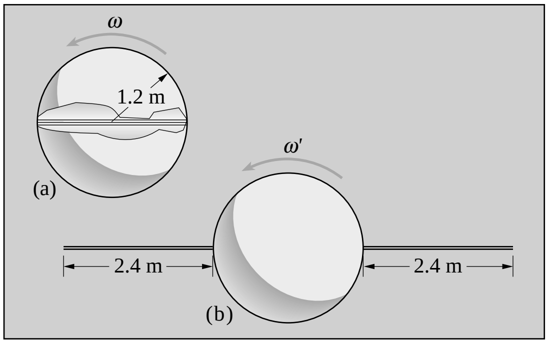
Substitute numerical values:

$$I_{\text{bar}} + r_{1A}^2\left(\frac{W_A}{g}\right) = \frac{W_{\text{bar}}}{3g}L^2 + r_{1A}^2\left(\frac{W_A}{g}\right) = 10.2 \text{ kg-m}^2$$

$$I_{\text{bar}} + r_{21A}^2\left(\frac{W_A}{g}\right) = \frac{W_{\text{bar}}}{3g}L^2 + r_{2A}^2\left(\frac{W_A}{g}\right) = 11.5 \text{ kg-m}^2,$$

from which $\boxed{\omega_2 = \left(\frac{10.2}{11.5}\right)\omega_1 = 8.90 \text{ rad/s}}$

Problem 19.112 A satellite is deployed with angular velocity $\omega = 1$ rad/s (Fig. a). Two internally stored antennas that span the diameter of the satellite are then extended, and the satellite's angular velocity decreases to ω' (Fig. b). By modeling the satellite as a 500-kg sphere of 1.2-m radius and each antenna as a 10-kg slender bar, determine ω'.



Solution: Assume (I) in configuration (a) the antennas are folded inward, each lying on a line passing so near the center of the satellite that the distance from the line to the center can be neglected; (II) when extended, the antennas are entirely external to the satellite. Denote $R = 1.2$ m, $L = 2R = 2.4$ m. The moment of inertia of the antennas about the center of mass of the satellite in configuration (a) is

$$I_{\text{ant-folded}} = 2\left(\frac{mL^2}{12}\right) = 9.6 \text{ kg-m}^2.$$

The moment of inertia of the antennas about the center of mass of the satellite in configuration (b) is

$$I_{\text{ant-ext}} = 2\left(\frac{mL^2}{12}\right) + 2\left(R + \frac{L}{2}\right)^2 m = 124.8 \text{ kg-m}^2.$$

The moment of inertia of the satellite is

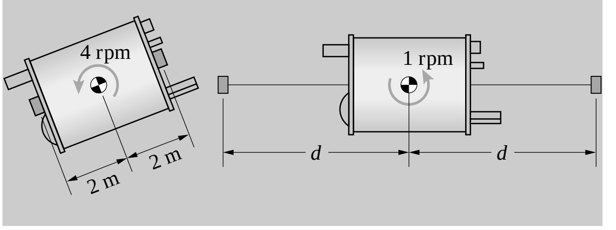
$$I_{\text{sphere}} = \frac{2}{5}m_{\text{sphere}}R^2 = 288 \text{ kg-m}^2.$$

The angular momentum is conserved,

$$I_{\text{sphere}}\omega + I_{\text{ant-folded}}\omega = I_{\text{sphere}}\omega' + I_{\text{ant-ext}}\omega',$$

where $I_{\text{ant-folded}}, I_{\text{ant-ext}}$ *are for both antennas*, from which $297.6\omega = 412.8\omega'$. Solve $\boxed{\omega' = 0.721 \text{ rad/s}}$

Problem 19.113 An engineer decides to control the angular velocity of a satellite by deploying small masses attached to cables. If the angular velocity of the satellite in configuration (a) is 4 rpm, determine the distance d in configuration (b) that will cause the angular velocity to be 1 rpm. The moment of inertia of the satellite is $I = 500$ kg-m^2 and each mass is 2 kg. (Assume that the cables and masses rotate with the same angular velocity as the satellite. Neglect the masses of the cables and the mass moments of inertia of the masses about their centers of mass.)

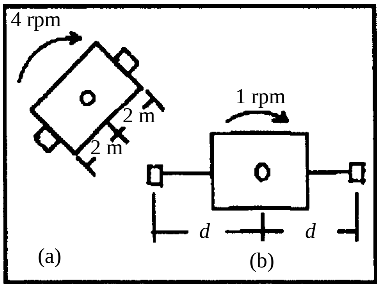


(a) (b)

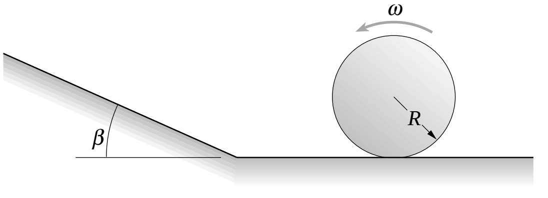
Solution: From the conservation of angular momentum,

$$(I + 2m(2^2))\omega_1 = (I + 2md^2)\omega_2.$$

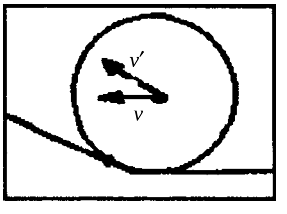
Solve: $$\boxed{d = \sqrt{\frac{(I+8m)\left(\dfrac{\omega_1}{\omega_2}\right) - I}{2m}} = 19.8 \text{ m}}$$



Problem 19.114 The homogenous cylindrical disk of mass m rolls on the horizontal surface with angular velocity ω. If the disk does not slip or leave the slanted surface when it comes into contact with it, what is the angular velocity ω' of the disk immediately afterward?



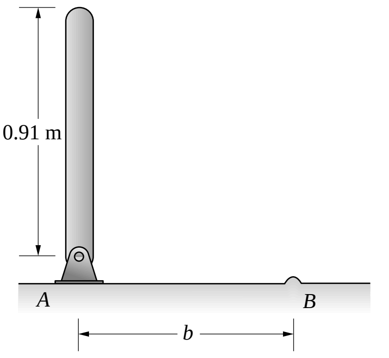
Solution: The velocity of the center of mass of the disk is parallel to the surface before and after contact. The angular momentum about the point of contact is conserved $mvR\cos\beta + I\omega = mv'R + I\omega'$. From kinematics, $v = R\omega$ and $v' = R\omega'$. Substitute into the angular moment condition to obtain:



$$(mR^2\cos\beta + I)\omega = (mR^2 + I)\omega'.$$

Solve: $$\boxed{\omega' = \frac{(2\cos\beta + 1)}{3}\omega}$$

Problem 19.115 The 44.5 N slender bar falls from rest in the vertical position and hits the smooth projection at B. The coefficient of restitution of the impact is $e = 0.6$, the duration of the impact is 0.1 s, and $b = 0.31$ m. Determine the average force exerted on the bar at B as a result of the impact.



Solution: Choose a coordinate system with the origin at A and the x axis parallel to the plane surface, and y positive upward. The strategy is to

(a) use the principle of work and energy to determine the velocity before impact,

(b) the coefficient of restitution to determine the velocity after impact,

(c) and the principle of angular impulse-momentum to determine the average force of impact.

From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$. The center of mass of the bar falls a distance $h = \frac{L}{2}$. The work done by the weight of the bar is $U = mg\left(\frac{L}{2}\right)$. The kinetic energy is $T_2 = \left(\frac{1}{2}\right) I\omega^2$, where $I = \frac{mL^2}{3}$. Substitute into $U = T_2$ and solve: $\omega = -\sqrt{\frac{3g}{L}}$, where the negative sign on the square root is chosen to be consistent with the choice of coordinates. By definition, the coefficient of restitution is $e = \frac{v'_B - v'_A}{v_A - v_B}$, where v_A, v'_A are *the velocities of the bar at a distance b from A* before and after impact. Since the projection B is stationary before and after the impact, $v_B = v'_B = 0$, from which $v'_A = -ev_A$. From kinematics, $v_A = b\omega$, and $v'_A = b\omega'$, from which $\omega' = -e\omega$. The principle of angular impulse-momentum about the point A is

$$\int_0^{t_2-t_1} bF_B\,dt = (I\omega' - I\omega) = \frac{mL^2}{3}(\omega' - \omega),$$

where F_B is the force exerted on the bar by the projection at B, from which

$$bF_B(t_2 - t_1) = -\frac{mL^2}{3}(1+e)\omega.$$

Solve: $\boxed{F_B = \frac{mL^2(1+e)}{3b(t_2-t_1)}\sqrt{\frac{3g}{L}} = 376.4 \text{ N.}}$

Problem 19.116 The 44.5 N bar falls from rest in the vertical position and hits the smooth projection at B. The coefficient of restitution of the impact is $e = 0.6$ and the duration of the impact is 0.1 s. Determine the distance b for which the average force exerted on the bar by the support A as a result of the impact is zero.

Solution: From the principle of linear impulse-momentum,

$$\int_{t_1}^{t_2} \sum F\,dt = m(v'_G - v_G),$$

where v_G, v'_G are the velocities of the center of mass of the bar before and after impact, and $\sum F = F_A + F_B$ are the forces exerted on the bar at A and B. From kinematics, $v'_G = \left(\frac{L}{2}\right)\omega'$, $v_G = \left(\frac{L}{2}\right)\omega$. From the solution to Problem 19.115, $\omega' = -e\omega$, from which

$$F_A + F_B = -\frac{mL(1+e)}{2(t_2-t_1)}\omega.$$

If the reaction at A is zero, then

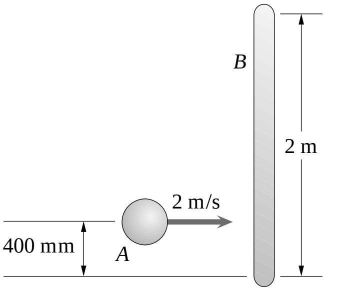
$$F_B = -\frac{mL(1+e)}{2(t_2-t_1)}\omega.$$

From the solution to Problem 19.115,

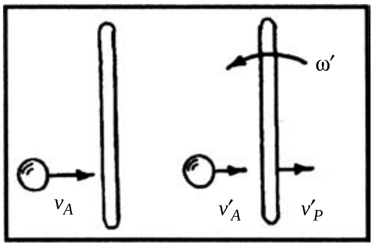
$$F_B = -\frac{mL^2(1+e)}{3b(t_2-t_1)}\omega.$$

Substitute and solve: $\boxed{b = \frac{2}{3}L = 0.61 \text{ m}}$

Problem 19.117 The 1-kg sphere A is moving at 2 m/s when it strikes the end of the 2-kg stationary slender bar B. If the velocity of the sphere after the impact is 0.8 m/s to the right, what is the coefficient of restitution?



Solution: Denote the distance of the point of impact P from the end of the bar by $d = 0.4$ m. The linear momentum is conserved: (1) $m_A v_A = m_A v'_A + m_B v'_{CM}$, where v'_{CM} is the velocity of the center of mass of the bar after impact, and v_A, v'_A are the velocities of the sphere before and after impact. By definition, the coefficient of restitution is $e = \dfrac{v'_p - v'_A}{v_A - v_p}$, where v_p, v'_p are the velocities of the bar at the point of impact. The point P is stationary before impact, from which (2) $v'_p - v'_A = e v_A$. From kinematics,



(3) $v'_{CM} = v'_P - \left(\dfrac{L}{2} - d\right)\omega'$.

Substitute (2) into (3) to obtain

(4) $v'_{CM} = v'_A + e v_A - \left(\dfrac{L}{2} - d\right)\omega'$.

Substitute (4) into (1) to obtain

(5) $(m_A - e m_B)v_A = (m_A + m_B)v'_A - \left(\dfrac{L}{2} - d\right) m_B \omega'$.

The angular momentum of the bar about the point of impact is conserved:

(6) $0 = -\left(\dfrac{L}{2} - d\right) m_B v'_{CM} + I_{CM}\omega'$.

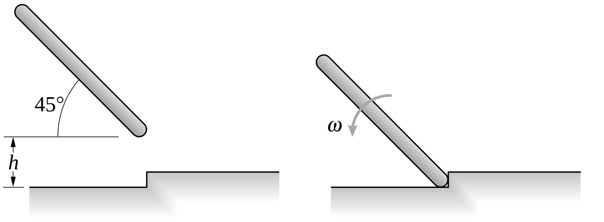
Substitute (3) into (6) to obtain,

(7) $\left(\dfrac{L}{2} - d\right) m_B (v'_A + e v_A) = \left(I_{CM} + \left(\dfrac{L}{2} - d\right)^2 m_B\right)\omega'$,

where $I_{CM} = \dfrac{m_B L^2}{12}$.

Solve the two equations (5) and (7) for the two unknowns to obtain: $\omega' = 1.08$ rad/s and $\boxed{e = 0.224.}$

Problem 19.118 The slender bar is released from rest in the position shown in Fig (a) and falls a distance $h =$ 0.31 mm. When the bar hits the floor, its tip is supported by a depression and remains on the floor (Fig. b). The length of the bar is 0.31 m and its weight is 1.1 N. What is angular velocity ω of the bar just after it hits the floor?



(a) (b)

Solution: Choose a coordinate system with the x axis parallel to the surface, with the y axis positive upward. The strategy is to use the principle of work and energy to determine the velocity just before impact, and the conservation of angular momentum to determine the angular velocity after impact. From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$ since the bar is released from rest. The work done is $U = -mgh$, where $h = 0.31$ m. The kinetic energy is $T_2 = \left(\frac{1}{2}\right) mv^2$. Substitute into $U = T_2 - T_1$ and solve: $v = \sqrt{-2gh} = 2.44$ m/s. The conservation of angular momentum about the point of impact is $(\mathbf{r} \times m\mathbf{v}) = (\mathbf{r} \times m\mathbf{v}') + I_B\boldsymbol{\omega}'$. At the instant before the impact, the perpendicular distance from the point of impact to the center of mass velocity vector is $\left(\frac{L}{2}\right)\cos 45°$. After impact, the center of mass moves in an arc of radius $\left(\frac{L}{2}\right)$ about the point of impact so that the perpendicular distance from the point of impact to the velocity vector is $\left(\frac{L}{2}\right)$. From the definition of the cross product, for motion in the x, y plane,

$$(\mathbf{r} \times m\mathbf{v}) \cdot \mathbf{k} = \left(\frac{L}{2}\cos 45°\right) mv,$$

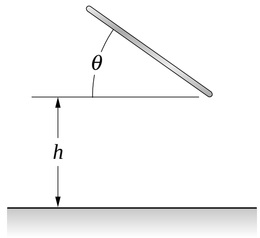
$$(\mathbf{r} \times m\mathbf{v}') \cdot \mathbf{k} = \left(\frac{L}{2}\right) mv',$$

and $I_B\boldsymbol{\omega}' \cdot \mathbf{k} = I_B\omega'$.

From kinematics, $v' = \frac{L}{2}\omega'$. Substitute to obtain:

$$\boxed{\omega' = \frac{mv(\cos 45°)}{\left(\frac{mL}{2} + \frac{2I_B}{L}\right)} = \frac{3v}{2\sqrt{2}L} = \frac{3\sqrt{-gh}}{2L} = 8.51 \text{ rad/s}}$$

Problem 19.119 The slender bar is released from rest with $\theta = 45^\circ$ and falls a distance $h = 1$ m onto the smooth floor. The length of the bar is 1 m and its mass is 2 kg. If the coefficient of restitution of the impact is $e = 0.4$, what is the angular velocity of the bar just after it hits the floor?



Solution: Choose a coordinate system with the x axis parallel to the plane surface and y positive upward. The strategy is to

(a) use the principle of work and energy to obtain the velocity the instant before impact,

(b) use the definition of the coefficient of restitution to find the velocity just after impact,

(c) get the angular velocity-velocity relations from kinematics and

(d) use the principle of the conservation of angular momentum about the point of impact to determine the angular velocity.

From the principle of work and energy, $U = T_2 - T_1$, where $T_1 = 0$. The center of mass of the bar also falls a distance h before impact. The work done is $U = \int_0^{-h} -mg\,dh = mgh$. The kinetic energy is $T_2 = \left(\frac{1}{2}\right) m v_{Gy}^2$. Substitute into $U = T_2$ and solve: $v_{Gy} = -\sqrt{2gh}$, where the negative sign on the square root is chosen to be consistent with the choice of coordinates.

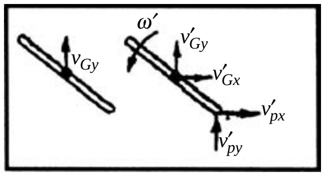
Denote the point of impact on the bar by P. From the definition of the coefficient of restitution,

$$e = \frac{v'_{Py} - v'_{By}}{v_{By} - v_{Py}}.$$

Since the floor is stationary before and after impact, $v_B = v'_B = 0$, and (1) $v'_{Py} = -ev_{Py} = -ev_{Gy}$. From kinematics: $\mathbf{v}'_P = \mathbf{v}'_G + \boldsymbol{\omega}' \times \mathbf{r}_{P/G}$, where $\mathbf{r}_{P/G} = \left(\frac{L}{2}\right)(\mathbf{i}\cos\theta - \mathbf{j}\sin\theta)$, from which

$$\text{(2) } \mathbf{v}'_{Py} = \mathbf{v}'_{Gy} + \left(\frac{L}{2}\right)\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega' \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

$$= v'_{Gy}\mathbf{j} + \left(\frac{L}{2}\right)(\mathbf{i}\omega'\sin\theta + \mathbf{j}\omega'\cos\theta).$$



Substitute (1) into (2) to obtain

$$\text{(3) } v'_{Gy} = -ev_{Gy} - \left(\frac{L}{2}\right)\omega'\cos\theta.$$

The angular momentum is conserved about the point of impact:

$$\text{(4) } -\left(\frac{L}{2}\right)\cos\theta\, m v_{Gy} = -\left(\frac{L}{2}\right)\cos\theta\, m v'_{Gy} + I_G\omega'.$$

Substitute (3) into (4) to obtain

$$-\frac{\cdot L}{2} m\cos\theta(1+e)v_G = \left(I_G + \left(\frac{L}{2}\right)^2 m\cos^2\theta\right)\omega'.$$

Solve:

$$\omega' = -\frac{mL(1+e)\cos\theta}{2\left(I_G + \left(\frac{L\cos\theta}{2}\right)^2 m\right)}v_{Gy},$$

$$\boxed{\omega' = -\frac{6(1+e)\cos\theta}{(1+3\cos^2\theta)}(-\sqrt{2gh}) = 10.52 \text{ rad/s}},$$

where $I_G = \dfrac{mL^2}{12}$ has been used.

Problem 19.120 The slender bar is released from rest and falls a distance $h = 1$ m onto the smooth floor. The length of the bar is 1 m and its mass is 2 kg. The coefficient of restitution of the impact is $e = 0.4$. Determine the angle θ for which the angular velocity of the bar after it hits the floor is a maximum. What is the maximum angular velocity?

Solution: From the solution to Problem 19.119,

$$\omega' = \frac{6(1+e)\cos\theta}{(1+3\cos^2\theta)}\sqrt{2gh}.$$

Take the derivative:

$$\frac{d\omega'}{d\theta} = 0 = \frac{-6(1+e)\sin\theta}{(1+3\cos^2\theta)}\sqrt{2gh} + \frac{6(1+e)(6)\cos^2\theta\sin\theta}{(1+3\cos^2\theta)^2}\sqrt{2gh}$$

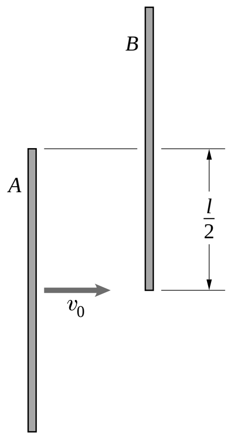
$$= 0,$$

from which $3\cos^2\theta_{\max} - 1 = 0$,

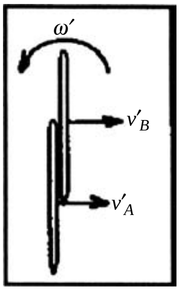
$$\cos\theta_{\max} = \sqrt{\frac{1}{3}},\ \boxed{\theta_{\max} = 54.74^\circ},$$

and $\boxed{\omega'_{\max} = 10.7 \text{ rad/s}}$.

Problem 19.121 A nonrotating slender bar A moving with velocity v_0 strikes a stationary slender bar B. Each bar has mass m and length l. If the bars adhere when they collide, what is their angular velocity after the impact?



Solution: From the conservation of linear momentum, $mv_0 = mv'_A + mv'_B$. From the conservation of angular momentum about the mass center of A



$$0 = I_B \boldsymbol{\omega}'_A + I_B \boldsymbol{\omega}'_B + (\mathbf{r} \times m\mathbf{v}'_B)$$

$$= (I_B \omega'_A + I_B \omega'_B)\mathbf{k} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \dfrac{L}{2} & 0 \\ v'_B & 0 & 0 \end{bmatrix},$$

$$0 = (I_B \omega'_A + I_B \omega'_B)\mathbf{k} - m\left(\frac{L}{2}\right) v'_B)\mathbf{k}.$$

Since the bars adhere, $\omega'_A = \omega'_B$, from which $2I_B\omega' = m\left(\dfrac{L}{2}\right) v'_B$. From kinematics

$$\mathbf{v}'_B = \mathbf{v}'_A + \boldsymbol{\omega}' \times \mathbf{r}_{AB} = v'_A \mathbf{i} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega' \\ 0 & \dfrac{L}{2} & 0 \end{bmatrix}$$

$$= \left(v'_A - \frac{L}{2}\omega'\right)\mathbf{i},$$

from which $v'_B = v'_A - \left(\dfrac{L}{2}\right)\omega'$.

Substitute into the expression for conservation of linear momentum to obtain

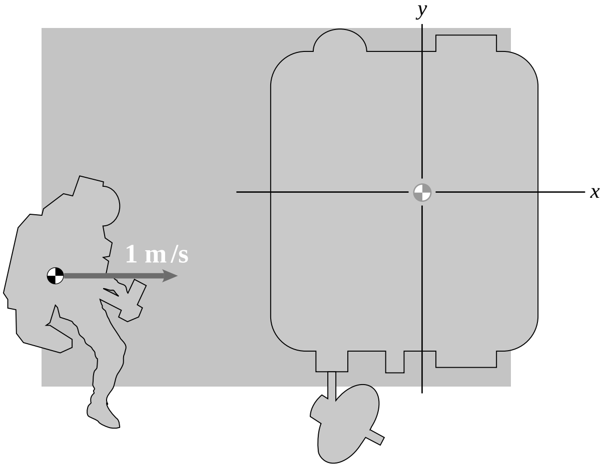
$$v'_B = \frac{v_0 - \left(\dfrac{L}{2}\right)\omega'}{2}.$$

Substitute into the expression for the conservation of angular momentum to obtain:

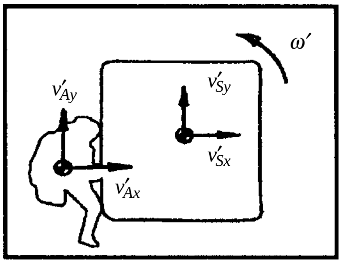
$$\omega' = \frac{\left(\dfrac{mL}{2}\right)v_0}{4I_B + \dfrac{mL^2}{4}} \cdot I_B = \frac{mL^2}{12},$$

from which $\boxed{\omega' = \dfrac{6v_0}{7L}}$

Problem 19.122 An astronaut translates toward a non-rotating satellite at $1.0\mathbf{i}$ (m/s) relative to the satellite. Her mass is 136 kg, and the moment of inertia about the axis through her center of mass parallel to the z axis is 45 kg-m^2. The mass of the satellite is 450 kg and its moment of inertia about the z axis is 675 kg-m^2. At the instant the astronaut attaches to the satellite and begins moving with it, the position of her center of mass is (−1.8, −0.9, 0) m. The axis of rotation of the satellite after she attaches is parallel to the z axis. What is their angular velocity?



Solution: Choose a coordinate system with the origin at the center of mass of the satellite, and the y axis positive upward. The linear momentum is conserved:



(1) $m_A v_{Ax} = m_A v'_{Ax} + m_S v'_{Sx}$,

(2) $0 = m_A v'_{Ay} + m_S v'_{Sy}$. The angular momentum about the center of mass of the satellite is conserved: $\mathbf{r}_{A/S} \times m_A \mathbf{v}_A = \mathbf{r}_{A/S} \times m_A \mathbf{v}'_A + (I_A + I_S)\boldsymbol{\omega}'$, where $\mathbf{r}_{A/S} = -1.8\mathbf{i} - 0.9\mathbf{j}$ (m), and $\mathbf{v}_A = 1.0\mathbf{i}$ (m/s).

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1.8 & -0.9 & 0 \\ m_A v_{Ax} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1.8 & -0.9 & 0 \\ m_A v'_{Ax} & m_A v'_{Ay} & 0 \end{bmatrix}$$

$$+ (I_A + I_S)\boldsymbol{\omega}', \text{ from which}$$

(3) $0.9 m_A v_{Ax} = -1.8 m_A v'_{Ay} + 0.9 m_A v'_{Ax} + (I_A + I_S)\omega'$.

From kinematics:

$$\mathbf{v}'_A = \mathbf{v}'_S + \boldsymbol{\omega}' \times \mathbf{r}_{A/S} = \mathbf{v}_S + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega' \\ -1.8 & -0.9 & 0 \end{bmatrix}$$

$$= (v'_{Sx} + 0.9\omega')\mathbf{i} + (v'_{Sy} - 1.8\omega')\mathbf{j} \text{ from which}$$

(4) $v'_{Ax} = v'_{Sx} + 0.9\omega'$,

(5) $v'_{Ay} = v'_{Sy} - 1.8\omega'$.

With $v_{Ax} = 1.0$ m/s, these are five equations in five unknowns. The number of equations can be reduced further, but here they are solved by iteration (using **TK Solver Plus**) to obtain: $v'_{Ax} = 0.289$ m/s, $v'_{Sx} = 0.215$ m/s, $v'_{Ay} = -0.114$ m/s, $v'_{Sy} = 0.0344$ m/s, $\boxed{\omega' = 0.0822 \text{ rad/s}}$

Problem 19.123 In Problem 19.122, suppose that the design parameters of the satellite's control system require that the angular velocity of the satellite not exceed 0.02 rad/s. If the astronaut is moving parallel to the x axis and the position of her center of mass when she attaches is $(-1.8, -0.9, 0)$ m, what is the maximum relative velocity at which she should approach the satellite?

Solution: From the solution to Problem 19.122, the five equations are:

(1) $m_A v_{Ax} = m_A v'_{Ax} + m_S v'_S$,

(2) $0 = m_A v'_{Ay} + m_S v'_{Sy}$

(3) $0.9 m_A v_{Ax} = -1.8 m_A v'_{Ay} + 0.9 m_A v'_{Ax} + (I_A + I_S)\omega'$

(4) $v'_{Ax} = v'_{Sx} + 0.9\omega'$,

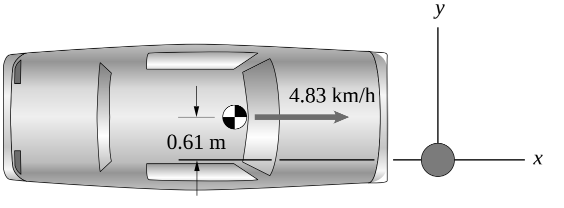
(5) $v'_{Ay} = v'_{Sy} - 1.8\omega'$.

With $\omega' = 0.02$ rad/s, these five equations have the solutions:

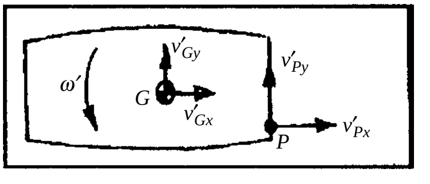
$\boxed{v_{Ax} = 0.243 \text{ m/s}}$, $v'_{Ax} = 0.070$ m/s, $v'_{Sx} = 0.052$ m/s,

$v'_{Ay} = -0.028$ m/s, $v'_{Sy} = 0.008$ m/s.

Problem 19.124 A 12454 N car skidding on ice strikes a concrete abutment at 4.83 km/h. The car's moment of inertia about its center of mass is 2439 kg-m^2. Assume that the impacting surfaces are smooth and parallel to the y axis and that the coefficient of restitution of the impact is $e = 0.8$. What are the angular velocity of the car and the velocity of its center of mass after the impact?



Solution: Let P be the point of impact on with the abutment. (P is located on the vehicle.) Denote the vector from P to the center of mass of the vehicle by $\mathbf{r} = a\mathbf{i} + 0.61\mathbf{j}$, where a is unknown. The velocity of the vehicle is



$v_{Gx} = 4.83$ km/h $= 1.34$ m/s.

From the conservation of linear momentum in the y direction, $m v_{Gy} = m v'_{Gy}$, from which $v'_{Gy} = v_{Gy} = 0$. Similarly, $v_{Ay} = v'_{Ay} = 0$. By definition,

$$e = \frac{v'_{Px} - v'_{Ax}}{v_{Ax} - v_{Px}}.$$

Assume that the abutment does not yield under the impact, $v_{Ax} = v'_{Ax} = 0$, from which

$$v'_{Px} = -e v_{Px} = -e v_{Gx} = -(0.8)(1.34) = -1.07 \text{ m/s}$$

The conservation of angular momentum about P is $\mathbf{r} \times m\mathbf{v}_G = \mathbf{r} \times m\mathbf{v}'_G + I\boldsymbol{\omega}'$. From kinematics, $\mathbf{v}'_G = \mathbf{v}'_P + \boldsymbol{\omega}' \times \mathbf{r}$. Reduce:

$$\mathbf{v}'_G = v'_{Px}\mathbf{i} + \boldsymbol{\omega}' \times \mathbf{r} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega' \\ a & 0.61 & 0 \end{bmatrix}$$

$$= (v'_{Px} - 0.61\omega')\mathbf{i} + a\omega'\mathbf{j} = (-e v_{Gx} - 0.61\omega')\mathbf{i} + a\omega'\mathbf{j},$$

from which $v'_{Gx} = -(e v_{Gx} + 0.61\,\omega')$, $\mathbf{r} \times m\mathbf{v}_G = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0.61 & 0 \\ 1.34m & 0 & 0 \end{bmatrix}$

$$= -0.82m\mathbf{k}.$$

$$\mathbf{r} \times m\mathbf{v}'_G = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0.61 & 0 \\ -(e v_{Gx} + 0.61\,\omega')m & 0 & 0 \end{bmatrix}$$

$$= 0.61\,(e v_{Gx} + 0.61\omega')m\mathbf{k},\ I\boldsymbol{\omega}' = I\omega'\mathbf{k}.$$

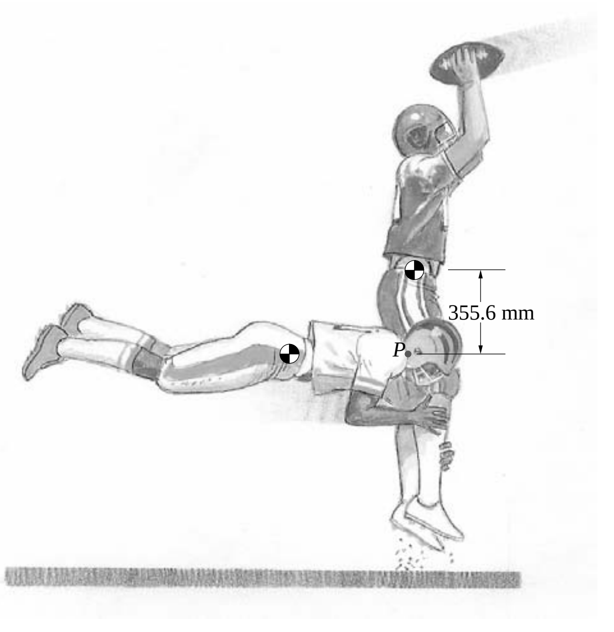
Substitute into the expression for the conservation of angular momentum to obtain

$$\boxed{\omega' = \frac{-1.47}{\left(0.61^2 + \dfrac{I}{m}\right)} = \frac{-1.47}{(0.372 + 1.92)} = -0.641 \text{ rad/s}}$$

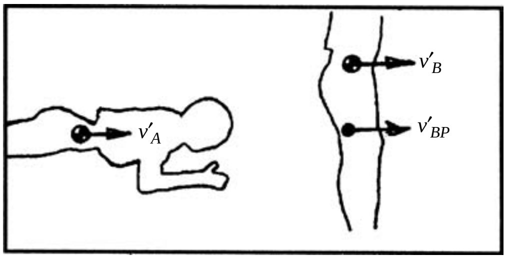
The velocity of the center of mass is

$$\mathbf{v}'_G = v'_{Gx}\mathbf{i} = -(e v_{Gx} + 0.61\omega')\mathbf{i} = -2.24\mathbf{i} \text{ (m/s)}$$

Problem 19.125 A 756 N receiver jumps vertically to receive a pass and is stationary at the instant he catches the ball. At the same instant, he is hit at P by a 801 N linebacker moving horizontally at 4.6 m/s. The wide receiver's moment of inertia about his center of mass is 9.5 kg-m^2. If you model the players as rigid bodies and assume that the coefficient of restitution is $e = 0$, what is the wide receiver's angular velocity immediately after the impact?



Solution: Denote the receiver by the subscript B, and the tacker by the subscript A. Denote $d = 0.356$ m. The conservation of linear momentum: (1) $m_A v_A = m_A v'_A + m_B v'_B$. From the definition, $e = \dfrac{v'_{BP} - v'_A}{v_A - v_{BP}}$. Since $v_{BP} = 0$, (2) $ev_A = v'_{BP} - v'_A$. The angular momentum is conserved about point P: (3) $0 = -m_B\, dv'_B + I_B\omega'_B$. From kinematics: (4) $v'_{BP} = v'_B + d\omega'_B$.



The solution: For $e = 0$, from (1),

$$v'_A = v_A - \left(\frac{m_B}{m_A}\right) v'_B.$$

From (2) and (4) $v'_A = v'_B + d\omega'_B$. From

$$(3)\ v'_B = \left(\frac{I_B}{dm_B}\right)\omega'.$$

Combine these last two equations and solve:

$$\boxed{\omega'_B = \frac{v_A}{\dfrac{I_B}{dm_B}\left(1 + \dfrac{m_B}{m_A}\right) + d} = 4.445\ldots = 4.45 \text{ rad/s.}}$$

Problem 20.1 The airplane's angular velocity relative to an earth-fixed reference frame, expressed in terms of the body-fixed coordinate system shown, is $\boldsymbol{\omega} = 0.62\mathbf{i} + 0.45\mathbf{j} - 0.23\mathbf{k}$ (rad/s). The coordinates of point A of the airplane are (3.6, 0.8, −1.2) m. What is the velocity of point A relative to the velocity of the airplane's center of mass?

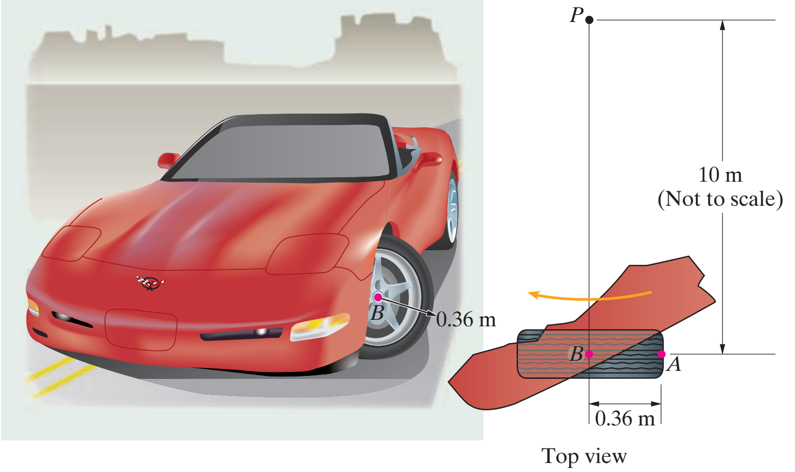
Solution:

$$\mathbf{v}_{A/G} = \boldsymbol{\omega} \times \mathbf{r}_{A/G}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.62 & 0.45 & -0.23 \\ 3.6 & 0.8 & -1.2 \end{vmatrix} \text{ (m/s)}$$

$$\boxed{\mathbf{v}_{A/G} = (-0.356\mathbf{i} - 0.084\mathbf{j} - 1.12\mathbf{k}) \text{ m/s.}}$$

Problem 20.2 In Active Example 20.1, suppose that the center of the tire moves at a constant speed of 5 m/s as the car turns. (As a result, when the angular velocity of the tire relative to an earth-fixed reference frame is expressed *in terms of components in the secondary reference frame*, $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$, the components ω_x, ω_y, and ω_z are constants.) What is the angular acceleration $\boldsymbol{\alpha}$ of the tire relative to an earth-fixed reference frame?



Top view

Solution: The angular velocity of the secondary coordinate system is

$$\boldsymbol{\Omega} = -\frac{v}{R}\mathbf{k} = -\frac{(5 \text{ m/s})}{(10 \text{ m})}\mathbf{k} = -(0.5 \text{ rad/s})\mathbf{k}$$

The angular velocity of the wheel with components in the secondary coordinate system is

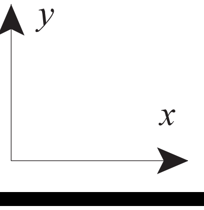
$$\boldsymbol{\omega} = \boldsymbol{\Omega} - \frac{v}{r}\mathbf{j} = -(0.5 \text{ rad/s})\mathbf{k} - \frac{(5 \text{ m/s})}{(0.36 \text{ m})}\mathbf{j}$$

$$= (-13.9\mathbf{j} - 0.5\mathbf{k}) \text{ rad/s}$$

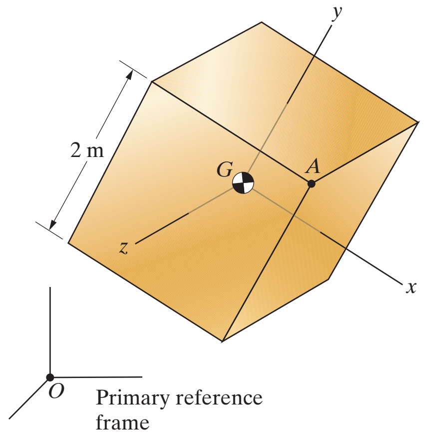
The angular acceleration is then

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = -(0.5 \text{ rad/s})\mathbf{k} \times (-13.9\mathbf{j} - 0.5\mathbf{k}) \text{ rad/s}$$

$$\boxed{\alpha = -6.94\mathbf{i} \text{ rad/s}^2.}$$



Problem 20.3 The angular velocity of the cube relative to the primary reference frame, expressed in terms of the body-fixed coordinate system shown is $\boldsymbol{\omega} = -6.4\mathbf{i} + 8.2\mathbf{j} + 12\mathbf{k}$ (rad/s). The velocity of the center of mass G of the cube relative to the primary reference frame at the instant shown is $\mathbf{v}_G = 26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}$ (m/s). What is the velocity of point A of the cube relative to the primary reference frame at the instant shown?



Solution: The vector from G to A is

$$\mathbf{r}_{G/A} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ m}.$$

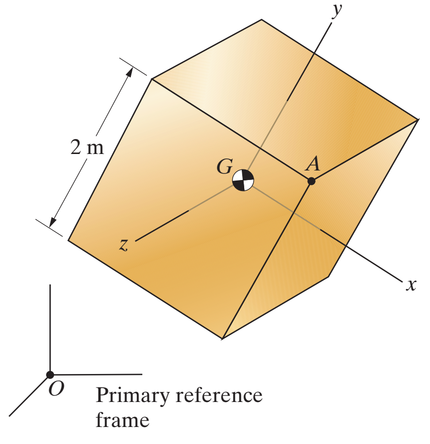
The velocity of point A is

$$\mathbf{v}_A = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}_{G/A}$$

$$= (26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}) \text{ m/s} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6.4 & 8.2 & 12 \\ 1 & 1 & 1 \end{vmatrix} \text{ m/s}$$

$$\boxed{\mathbf{v}_A = (22.2\mathbf{i} + 32.4\mathbf{j} + 17.4\mathbf{k}) \text{ m/s}.}$$

Problem 20.4 The coordinate system shown is fixed with respect to the cube. The angular velocity of the cube relative to the primary reference frame, $\boldsymbol{\omega} = -6.4\mathbf{i} + 8.2\mathbf{j} + 12\mathbf{k}$ (rad/s), is constant. The acceleration of the center of mass G of the cube relative to the primary reference frame at the instant shown is $\mathbf{a}_G = 136\mathbf{i} + 76\mathbf{j} - 48\mathbf{k}$ (m/s^2). What is the acceleration of point A of the cube relative to the primary reference frame at the instant shown?



Solution: The vector from G to A is

$$\mathbf{r}_{G/A} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \text{ m}.$$

The accleration of point A is

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/A})$$

$$= (136\mathbf{i} + 76\mathbf{j} - 48\mathbf{k}) \text{ m/s}^2 + (-6.4\mathbf{i} + 8.2\mathbf{j} + 12\mathbf{k}) \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6.4 & 8.2 & 12 \\ 1 & 1 & 1 \end{vmatrix} \text{ m/s}^2$$

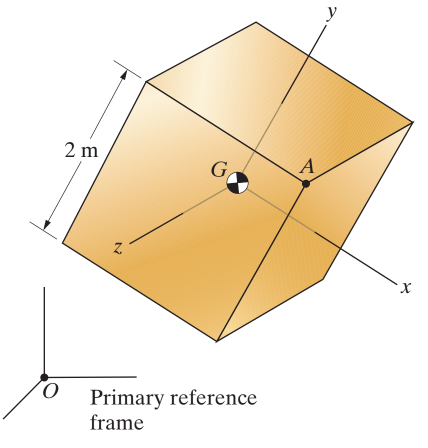
Carrying out the vector algebra, we have

$$\boxed{\mathbf{a}_A = (-205\mathbf{i} - 63.0\mathbf{j} - 135\mathbf{k}) \text{ m/s}^2.}$$

Problem 20.5 The origin of the secondary coordinate system shown is fixed to the center of mass G of the cube. The velocity of the center of mass G of the cube relative to the primary reference frame at the instant shown is $\mathbf{v}_G = 26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}$ (m/s). The cube is rotating relative to the secondary coordinate system with angular velocity $\boldsymbol{\omega}_{\text{rel}} = 6.2\mathbf{i} - 5\mathbf{j} + 8.8\mathbf{k}$ (rad/s). The secondary coordinate system is rotating relative to the primary reference frame with angular velocity $\boldsymbol{\Omega} = 2.2\mathbf{i} + 4\mathbf{j} - 3.6\mathbf{k}$ (rad/s).

(a) What is the velocity of point A of the cube relative to the primary reference frame at the instant shown?

(b) If the components of the vectors $\boldsymbol{\omega}_{\text{rel}}$ and $\boldsymbol{\Omega}$ are constant, what is the cube's angular acceleration relative to the primary reference frame?



Solution:

(a) $\mathbf{v}_A = \mathbf{v}_G + (\boldsymbol{\omega}_{\text{rel}} + \boldsymbol{\Omega}) \times \mathbf{r}_{A/G}$

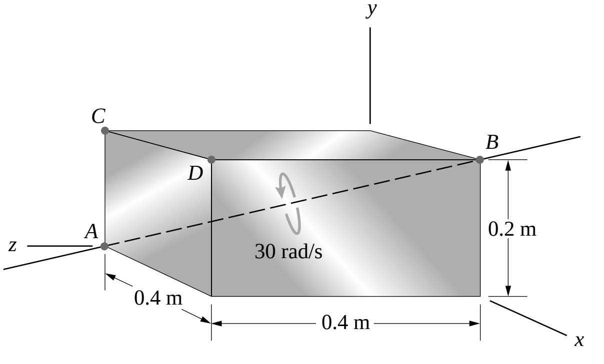
$= (26\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}) + (8.4\mathbf{i} - \mathbf{j} + 5.2\mathbf{k}) \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\boxed{\mathbf{v}_A = (19.8\mathbf{i} + 10.8\mathbf{j} + 41.4\mathbf{k}) \text{ m/s.}}$

(b) $\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (2.2\mathbf{i} + 4\mathbf{j} - 3.6\mathbf{k}) \times (6.2\mathbf{i} - 5\mathbf{j} + 8.8\mathbf{k})$

$\boxed{\boldsymbol{\alpha} = (17.2\mathbf{i} - 41.7\mathbf{j} - 35.8\mathbf{k}) \text{ rad/s}^2.}$

Problem 20.6 Relative to an earth-fixed reference frame, points A and B of the rigid parallelepiped are fixed and it rotates about the axis AB with an angular velocity of 30 rad/s. Determine the velocities of points C and D relative to the earth-fixed reference frame.



Solution: Given

$$\boldsymbol{\omega} = (30 \text{ rad/s})\frac{(0.4\mathbf{i} + 0.2\mathbf{j} - 0.4\mathbf{k})}{0.6} = (20\mathbf{i} + 10\mathbf{j} - 20\mathbf{k}) \text{ rad/s}$$

$\mathbf{r}_{C/A} = (0.2 \text{ m})\mathbf{j}$, $\mathbf{r}_{D/A} = (0.4\mathbf{i} + 0.2\mathbf{j})$ m

$\boxed{\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/A} = (4\mathbf{i} + 4\mathbf{k}) \text{ m/s}, \ \mathbf{v}_D = \boldsymbol{\omega} \times \mathbf{r}_{D/A} = (4\mathbf{i} - 8\mathbf{j}) \text{ m/s}}$

Problem 20.7 Relative to the xyz coordinate system shown, points A and B of the rigid parallelepiped are fixed and the parallelepiped rotates about the axis AB with an angular velocity of 30 rad/s. Relative to an earth-fixed reference frame, point A is fixed and the xyz coordinate system rotates with angular velocity $\boldsymbol{\Omega} = -5\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ (rad/s). Determine the velocities of points C and D relative to the earth-fixed reference frame.

Solution: Given

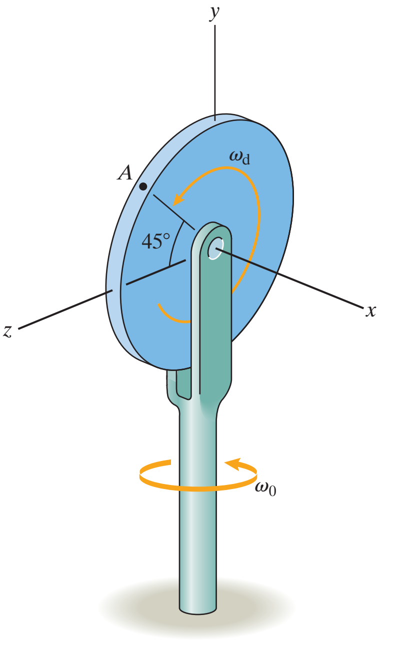
$$\boldsymbol{\omega} = (30 \text{ rad/s})\frac{(0.4\mathbf{i} + 0.2\mathbf{j} - 0.4\mathbf{k})}{0.6} = (20\mathbf{i} + 10\mathbf{j} - 20\mathbf{k}) \text{ rad/s}$$

$\boldsymbol{\Omega} = (-5\mathbf{i} + 8\mathbf{j} + 6\mathbf{k})$ rad/s, $\mathbf{r}_{C/A} = (0.2 \text{ m})\mathbf{j}$,

$\mathbf{r}_{D/A} = (0.4\mathbf{i} + 0.2\mathbf{j})$ m

$$\boxed{\begin{aligned} \mathbf{v}_C &= \mathbf{v}_A + (\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \mathbf{r}_{C/A} = (2.8\mathbf{i} + 3.0\mathbf{k}) \text{ m/s} \\ \mathbf{v}_D &= \mathbf{v}_A + (\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \mathbf{r}_{D/A} = (2.8\mathbf{i} - 5.6\mathbf{j} - 4.2\mathbf{k}) \text{ m/s} \end{aligned}}$$

Problem 20.8 Relative to an earth-fixed reference frame, the vertical shaft rotates about its axis with angular velocity $\omega_0 = 4$ rad/s. The secondary xyz coordinate system is *fixed with respect to the shaft* and its origin is stationary. Relative to the secondary coordinate system, the disk (radius = 8 cm) rotates with constant angular velocity $\omega_d = 6$ rad/s. At the instant shown, determine the velocity of pint A (a) relative to the secondary reference frame, and (b) relative to the earth-fixed reference frame.



Solution:

(a) Relative to the secondary system

$$\mathbf{v}_A = \boldsymbol{\omega}_{\text{rel}} \times \mathbf{r}_A = \omega_d \mathbf{i} \times r(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

$$= (6\mathbf{i}) \times (8)(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

$$= (-33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}$$

$$\boxed{\mathbf{v}_A = (-33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}}$$

(b) Relative to the earth-fixed reference frame

$$\mathbf{v}_A = (\boldsymbol{\omega}_{\text{rel}} + \boldsymbol{\Omega}) \times \mathbf{r}_A = (\omega_d \mathbf{i} + \omega_0 \mathbf{j}) \times r(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

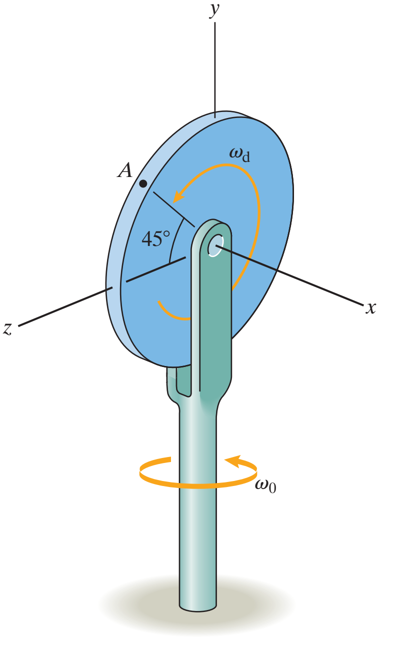
$$= (6\mathbf{i} + 4\mathbf{j}) \times (8)(\sin 45^\circ \mathbf{j} + \cos 45^\circ \mathbf{k})$$

$$= (22.6\mathbf{i} - 33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}$$

$$\boxed{\mathbf{v}_A = (22.6\mathbf{i} - 33.9\mathbf{j} + 33.9\mathbf{k}) \text{ cm/s.}}$$

Problem 20.9 Relative to an earth-fixed reference frame, the vertical shaft rotates about its axis with angular velocity $\omega_0 = 4$ rad/s. The secondary xyz coordinate system is *fixed with respect to the shaft* and its origin is stationary. Relative to the secondary coordinate system, the disk (radius = 8 cm) rotates with constant angular velocity $\omega_d = 6$ rad/s.

(a) What is the angular acceleration of the disk relative to the earth-fixed reference frame?
(b) At the instant shown, determine the acceleration of point A relative to the earth-fixed reference frame.



Solution:

(a) The angular acceleration

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = \omega_0 \mathbf{j} \times \omega_d \mathbf{i}$$

$$= -\omega_0 \omega_d \mathbf{k} = -(4)(6)\mathbf{k} = -24\mathbf{k}$$

$$\boxed{\boldsymbol{\alpha} = -24\mathbf{k} \text{ rad/s}^2.}$$

(b) The acceleration of point A.

$$\mathbf{a}_A = \boldsymbol{\alpha} \times \mathbf{r}_A + (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times [(\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times \mathbf{r}_A]$$

$$= (-24\mathbf{k}) \times (8 \sin 45^\circ \mathbf{j} + 8 \cos 45^\circ \mathbf{k})$$

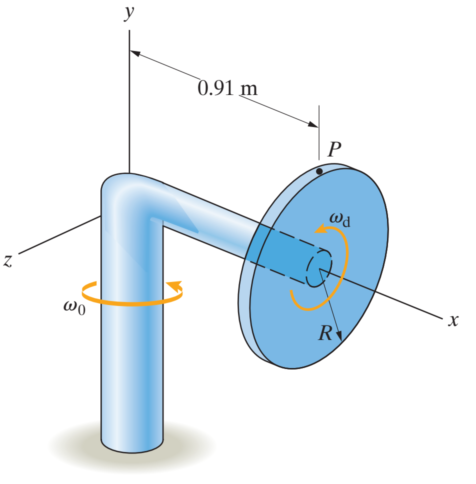
$$+ (6\mathbf{i} + 4\mathbf{j}) \times [(6\mathbf{i} + 4\mathbf{j}) \times (8 \sin 45^\circ \mathbf{j} + 8 \cos 45^\circ \mathbf{k})]$$

$$\boxed{\mathbf{a}_A = (272\mathbf{i} - 204\mathbf{j} - 294\mathbf{k}) \text{ cm/s}^2.}$$

Problem 20.10 The radius of the disk is $R = 0.61$ m. It is perpendicular to the horizontal part of the shaft and rotates relative to it with constant angular velocity $\omega_d = 36$ rad/s. Relative to an earth-fixed reference frame, the shaft rotates about the vertical axis with constant angular velocity $\omega_0 = 8$ rad/s.

(a) Determine the velocity relative to the earth-fixed reference frame of point P, which is the uppermost point of the disk.
(b) Determine the disk's angular acceleration vector $\boldsymbol{\alpha}$ relative to the earth-fixed reference frame.

(See Example 20.2.)



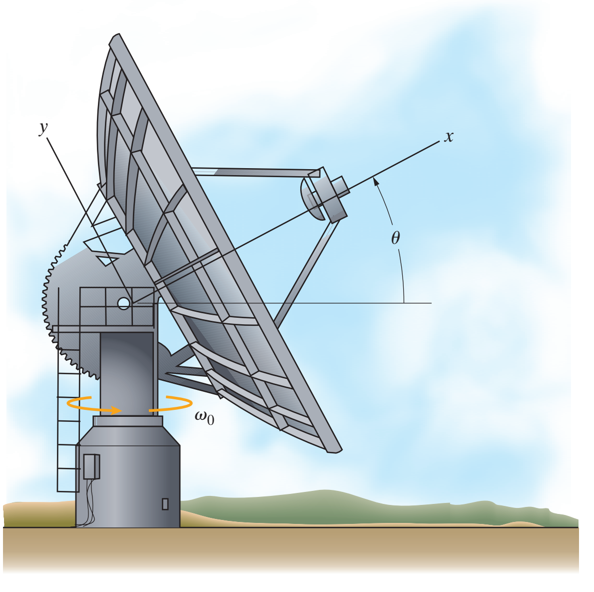
Solution:

(a) $$\boxed{\begin{aligned}\mathbf{v}_P &= (8\text{ rad/s})\mathbf{j} \times (0.91\text{ m})\mathbf{i} + [(36\mathbf{i} + 8\mathbf{j})\text{ rad/s}] \times (0.61\text{ m})\,\mathbf{j}\\ &= (14.7\text{ m/s})\mathbf{k}\end{aligned}}$$

(b) $$\boxed{\boldsymbol{\alpha} = (8\text{ rad/s})\mathbf{j} \times [(36\mathbf{i} + 8\mathbf{j})\text{ rad/s}] = -(288\text{ rad/s}^2)\mathbf{k}}$$

Problem 20.11 The vertical shaft supporting the disk antenna is rotating with a constant angular velocity $\omega_0 = 0.2$ rad/s. The angle θ from the horizontal to the antenna's axis is 30° at the instant shown and is increasing at a constant rate of 15° per second. The secondary xyz coordinate system shown is fixed with respect to the dish.

(a) What is the dish's angular velocity relative to an earth-fixed reference frame?
(b) Determine the velocity of the point of the antenna with coordinates (4,0,0) m relative to an earth-fixed reference frame.



Solution: The relative angular velocity is

$$\omega_{rel} = (15^\circ/\text{s})\left(\frac{\pi\text{ rad}}{180^\circ}\right) = \frac{\pi}{12}\text{ rad/s.}$$

(a) $$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{rel} = (0.2\sin 30^\circ\mathbf{i} + 0.2\cos 30^\circ\mathbf{j}) + \left(\frac{\pi}{12}\right)\mathbf{k}$$

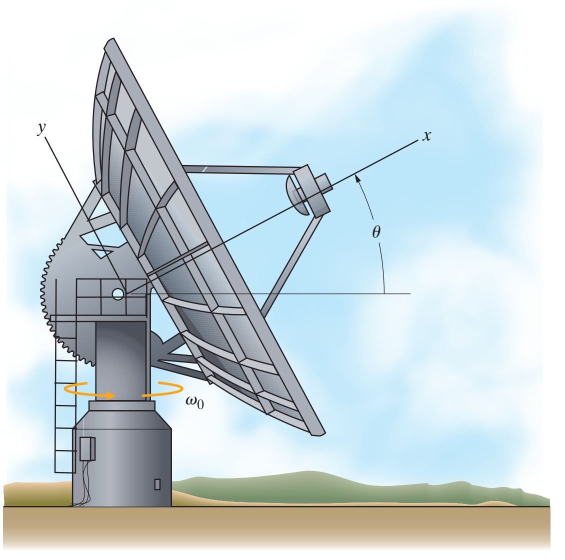
$$\boxed{\boldsymbol{\omega} = (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k})\text{ rad/s.}}$$

(b) $$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \times (4\mathbf{i})$$

$$\boxed{\mathbf{v} = (1.05\mathbf{j} - 0.693\mathbf{k})\text{ m/s.}}$$

Problem 20.12 The vertical shaft supporting the disk antenna is rotating with a constant angular velocity $\omega_0 = 0.2$ rad/s. The angle θ from the horizontal to the antenna's axis is 30° at the instant shown and is increasing at a constant rate of 15° per second. The secondary xyz coordinate system shown is fixed with respect to the dish.

(a) What is the dish's angular acceleration relative to an earth-fixed reference frame?
(b) Determine the acceleration of the point of the antenna with coordinates (4, 0, 0) m relative to an earth-fixed reference frame.



Solution: The angular velocity is

$$\omega_{\text{rel}} = (15^\circ/\text{s})\left(\frac{\pi \text{ rad}}{180^\circ}\right) = \frac{\pi}{12} \text{ rad/s.}$$

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = (0.2\sin 30^\circ\mathbf{i} + 0.2\cos 30^\circ\mathbf{j}) + \left(\frac{\pi}{12}\right)\mathbf{k}$$

$$= (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \text{ rad/s}$$

(a) The angular acceleration is

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (0.2\sin 30^\circ\mathbf{i} + 0.2\cos 30^\circ\mathbf{j}) \times \left(\frac{\pi}{12}\mathbf{k}\right)$$

$$\boxed{\boldsymbol{\alpha} = (0.0453\mathbf{i} - 0.0262\mathbf{j}) \text{ rad/s}^2.}$$

(b) The acceleration of the point

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

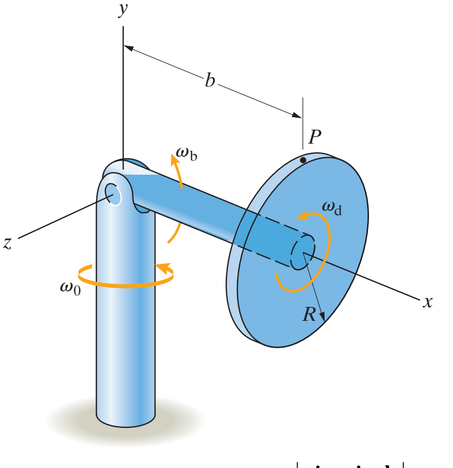
$$= (0.0453\mathbf{i} - 0.0262\mathbf{j}) \times (4\mathbf{i})$$

$$+ (0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \times [(0.1\mathbf{i} + 0.173\mathbf{j} + 0.262\mathbf{k}) \times (4\mathbf{i})]$$

$$\boxed{\mathbf{a} = (-0.394\mathbf{i} + 0.0693\mathbf{j} + 0.209\mathbf{k}) \text{ m/s}^2.}$$

Problem 20.13 The radius of the circular disk is $R = 0.2$ m, and $b = 0.3$ m. The disk rotates with angular velocity $\omega_d = 6$ rad/s relative to the horizontal bar. The horizontal bar rotates with angular velocity $\omega_b = 4$ rad/s relative to the vertical shaft, and the vertical shaft rotates with angular velocity $\omega_0 = 2$ rad/s relative to an earth-fixed reference frame. Assume that the secondary reference frame shown is fixed with respect to the horizontal bar.

(a) What is the angular velocity vector $\boldsymbol{\omega}_{\text{rel}}$ of the disk relative to the secondary reference frame?
(b) Determine the velocity relative to the earth-fixed reference frame of point P, which is the uppermost point of the disk.



Solution:

(a) The angular velocity of the disk relative to the secondary reference frame is

$$\boldsymbol{\omega}_{\text{rel}} = \omega_d\mathbf{i} = 6\mathbf{i} \text{ (rad/s).}$$

(b) The angular velocity of the reference frame is

$$\boldsymbol{\Omega} = \omega_0\mathbf{j} + \omega_b\mathbf{k} = 2\mathbf{j} + 4\mathbf{k} \text{ (rad/s),}$$

so the disk's angular velocity is

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = 6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \text{ (rad/s).}$$

Let O be the origin and C the center of the disk. The velocity of C is

$$\mathbf{v}_c = \mathbf{v}_0 + \boldsymbol{\Omega} \times \mathbf{r}_{C/O} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 4 \\ 0.3 & 0 & 0 \end{vmatrix}$$

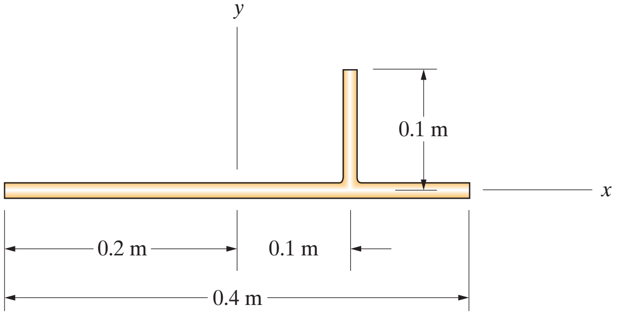
$$= 1.2\mathbf{j} - 0.6\mathbf{k} \text{ (m/s).}$$

The velocity of P is

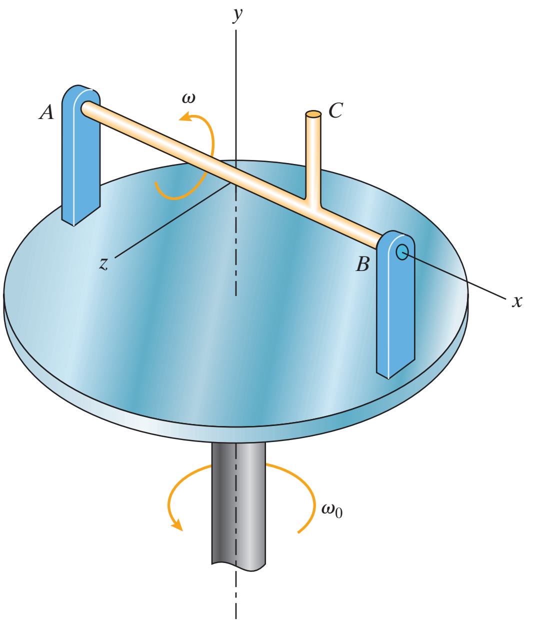
$$\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{P/C} = 1.2\mathbf{j} - 0.6\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 4 \\ 0 & 0.2 & 0 \end{vmatrix}$$

$$= -0.8\mathbf{i} + 1.2\mathbf{j} + 0.6\mathbf{k} \text{ (m/s).}$$

Problem 20.14 The Object in Fig. a is supported by bearings at A and B in Fig. b. The horizontal circular disk is supported by a vertical shaft that rotates with angular velocity $\omega_0 = 6$ rad/s. The horizontal bar rotates with angular velocity $\omega = 10$ rad/s. At the instant shown, what is the velocity relative to an earth-fixed reference frame of the end C of the vertical bar?



(a)



(b)

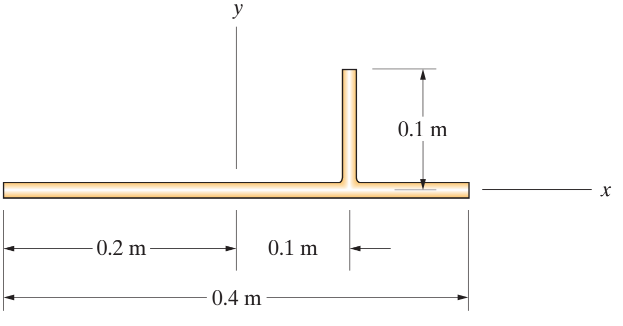
Solution:

$\mathbf{v}_c = (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times \mathbf{r} = (10\mathbf{i} + 6\mathbf{k}) \times (0.1\mathbf{i} + 0.1\mathbf{j})$

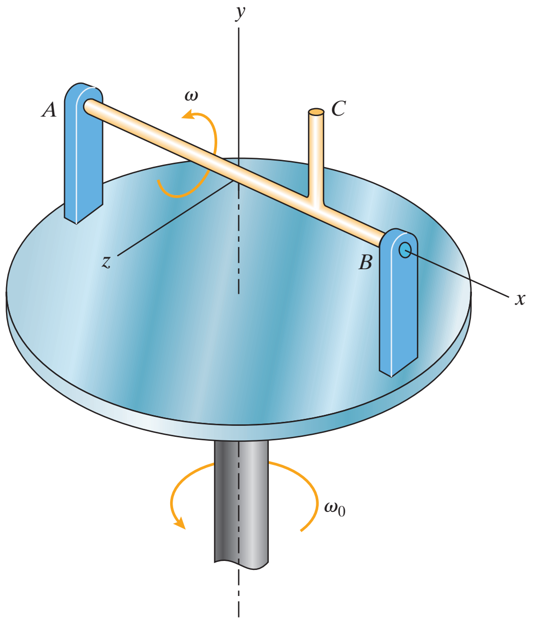
$\boxed{\mathbf{v}_c = (0.4 \text{ m/s})\mathbf{k}}$

Problem 20.15 The object in Fig. a is supported by bearings at A and B in Fig. b. The horizontal circular disk is supported by a vertical shaft that rotates with angular velocity $\omega_0 = 6$ rad/s. The horizontal bar rotates with angular velocity $\omega = 10$ rad/s.

(a) What is the angular acceleration of the object relative to an earth-fixed reference frame?
(b) At the instant shown, what is the acceleration relative to an earth-fixed reference frame of the end C of the vertical bar?



(a)



(b)

Solution:

(a) $\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (6\mathbf{j}) \times (10\mathbf{i}) = -60\mathbf{k}$ $\boxed{\boldsymbol{\alpha} = -(60 \text{ rad/s}^2)\mathbf{k}}$

(b)

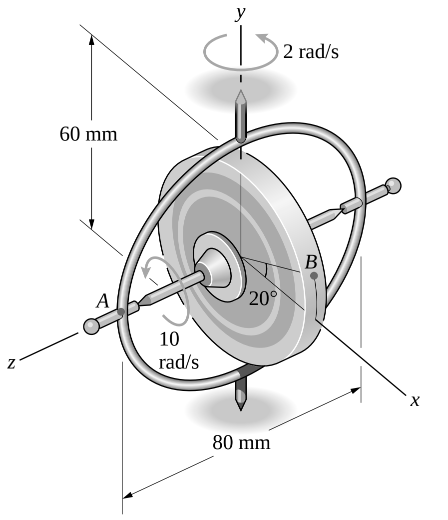
$$\mathbf{a}_c = \boldsymbol{\alpha} \times \mathbf{r} + (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times [(\Omega + \boldsymbol{\omega}_{\text{rel}}) \times \mathbf{r}]$$

$$= (-60\mathbf{k}) \times (0.1\mathbf{i} + 0.1\mathbf{j}) + (10\mathbf{i} + 6\mathbf{j}) \times [(10\mathbf{i} + 6\mathbf{j}) \times (0.1\mathbf{i} + 0.1\mathbf{j})]$$

$$= (8.4\mathbf{i} - 10\mathbf{j}) \text{ m/s}^2.$$

$$\boxed{\mathbf{a}_c = (8.4\mathbf{i} - 10\mathbf{j}) \text{ m/s}^2.}$$

Problem 20.16 Relative to a primary reference frame, the gyroscope's circular frame rotates about the vertical axis at 2 rad/s. The 60-nm diameter wheel rotates at 10 rad/s relative to the frame. Determine the velocities of points A and B relative to the primary reference frame.



Solution: Let the secondary reference frame shown be fixed with respect to the gyroscope's frame. The angular velocity of the SRF is $\mathbf{\Omega} = 2\mathbf{j}$ (rad/s). The angular velocity of the wheel relative to the SRF is $\boldsymbol{\omega}_{\text{rel}} = 10\mathbf{k}$ (rad/s), so the wheel's angular velocity is

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{\text{rel}} = 2\mathbf{j} + 10\mathbf{k} \ \text{(rad/s)}.$$

Let O denote the origin. The velocity of pt. A is

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ 0 & 0 & 40 \end{vmatrix} = 80\mathbf{i} \ \text{(mm/s)}.$$

The velocity of pt. B is

$$\mathbf{v}_B = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{B/O}$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ 30\cos 20^\circ & 30\sin 20^\circ & 0 \end{vmatrix}$$

$$= -102.6\mathbf{i} + 281.9\mathbf{j} - 56.4\mathbf{k} \ \text{(mm/s)}.$$

Problem 20.17 Relative to a primary reference frame, the gyroscope's circular frame rotates about the vertical axis with a constant angular velocity of 2 rad/s. The 60-mm diameter wheel rotates with a constant angular velocity of 10 rad/s relative to the frame. Determine the accelerations of points A and B relative to the primary reference frame.

Solution: See the solution of Problem 20.16. From Eq. (20.4), the wheel's angular acceleration is

$$\boldsymbol{\alpha} = \mathbf{\Omega} \times \boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ 0 & 2 & 10 \end{vmatrix} = 20\mathbf{i} \ \text{(rad/s}^2\text{)}.$$

The acceleration of pt. A is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & 0 & 0 \\ 0 & 0 & 40 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ 80 & 0 & 0 \end{vmatrix}$$

$$= -160\mathbf{k} \ \text{(mm/s}^2\text{)}.$$

The acceleration of pt. B is

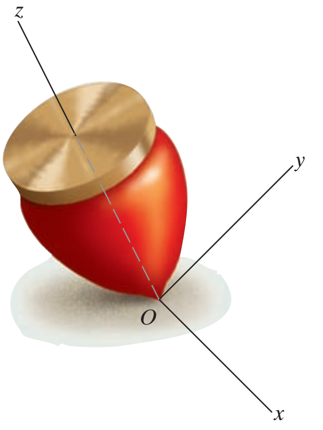
$$\mathbf{a}_B = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{B/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/O})$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & 0 & 0 \\ 30\cos 20^\circ & 30\sin 20^\circ & 0 \end{vmatrix}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 10 \\ -102.6 & 281.9 & -56.4 \end{vmatrix}$$

$$= -2932\mathbf{i} - 1026\mathbf{j} + 410\mathbf{k} \ \text{(mm/s}^2\text{)}.$$

Problem 20.18 The point of the spinning top remains at a fixed point on the floor, which is the origin O of the secondary reference frame shown. The top's angular velocity relative to the secondary reference frame, $\boldsymbol{\omega}_{\text{rel}} = 50\mathbf{k}$ (rad/s), is constant. The angular velocity of the secondary reference frame relative to an earth-fixed primary reference frame is $\boldsymbol{\Omega} = 2\mathbf{j} + 5.6\mathbf{k}$ (rad/s). The components of this vector are constant. (Notice that it is expressed in terms of the secondary reference frame.) Determine the velocity relative to the earth-fixed reference frame of the point of the top with coordinates (0, 20, 30) mm.



Solution:

$$\mathbf{v} = (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times \mathbf{r} = (2\mathbf{j} + 55.6\mathbf{k}) \times (0.02\mathbf{j} + 0.03\mathbf{k})$$

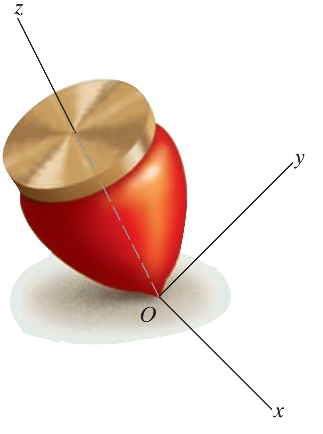
$$= (-1.05 \text{ m/s})\mathbf{i}$$

$$\boxed{\mathbf{v} = (-1.05 \text{ m/s})\mathbf{i}}$$

Problem 20.19 The point of the spinning top remains at a fixed point on the floor, which is the origin O of the secondary reference frame shown. The top's angular velocity relative to the secondary reference frame, $\boldsymbol{\omega}_{\text{rel}} = 50\mathbf{k}$ (rad/s), is constant. The angular velocity of the secondary reference frame relative to an earth-fixed primary reference frame is $\boldsymbol{\Omega} = 2\mathbf{j} + 5.6\mathbf{k}$ (rad/s). The components of this vector are constant. (Notice that it is expressed in terms of the secondary reference frame.)

(a) What is the top's angular acceleration relative to the earth-fixed reference frame?

(b) Determine the acceleration relative to the earth-fixed reference frame of the point of the top with coordinates (0, 20, 30) mm.



Solution:

(a) $\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega}_{\text{rel}} = (2\mathbf{j} + 5.6\mathbf{k}) \times (50\mathbf{k}) = 100\mathbf{i}$ $\boxed{\boldsymbol{\alpha} = (100 \text{ rad/s}^2)\mathbf{i}}$

(b) $\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + (\boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}}) \times [(\boldsymbol{\Omega} + \omega_{\text{rel}}) \times \mathbf{r}]$

$$= (100\mathbf{i}) \times (0.02\mathbf{j} + 0.03\mathbf{k})$$

$$+ (2\mathbf{j} + 55.6\mathbf{k}) \times [(2\mathbf{j} + 55.6\mathbf{k}) \times (0.02\mathbf{j} + 0.03\mathbf{k})]$$

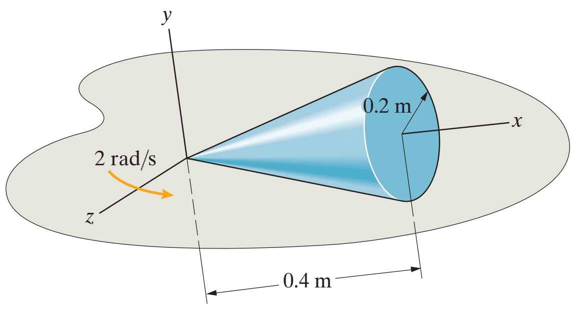
$$= (-61.5\mathbf{j} + 4.10\mathbf{k})$$

$$\boxed{\mathbf{a} = (-61.5\mathbf{j} + 4.10\mathbf{k}) \text{ m/s}^2.}$$

Problem 20.20* The cone rolls on the horizontal surface, which is fixed with respect to an earth-fixed reference frame. The x axis of the secondary reference frame remains coincident with the cone's axis, and the z axis remains horizontal. As the cone rolls, the z axis rotates in the horizontal plane with an angular velocity of 2 rad/s.

(a) What is the angular velocity vector $\mathbf{\Omega}$ of the secondary reference frame?
(b) What is the angular velocity vector $\boldsymbol{\omega}_{\text{rel}}$ of the cone relative to the secondary reference frame?

(See Example 20.3.)



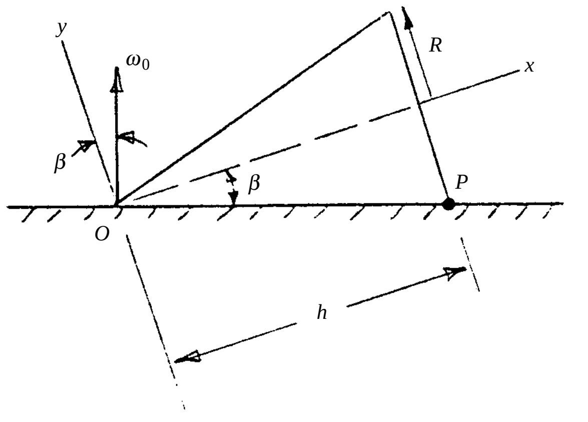
Strategy: To solve part (b), use the fact that the velocity relative to the earth-fixed reference frame of points of the cone in contact with the surface is zero.

Solution:

(a) The angle $\beta = \arctan(R/h) = \arctan(0.2/0.4) = 26.6^\circ$. The angular velocity of the secondary reference frame is

$$\mathbf{\Omega} = \omega_0 \sin\beta \mathbf{i} + \omega_0 \cos\beta \mathbf{j} = 2(\sin 26.6^\circ \mathbf{i} + \cos 26.6^\circ \mathbf{j})$$

$$= 0.894\mathbf{i} + 1.789\mathbf{j} \text{ (rad/s)}.$$



(b) The cone's angular velocity relative to the secondary reference frame can be written $\boldsymbol{\omega}_{\text{rel}} = \omega_{\text{rel}}\mathbf{i}$, so the cone's angular velocity is

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{\text{rel}}$$

$$= (0.894 + \omega_{\text{rel}})\mathbf{i} + 1.789\mathbf{j} \text{ (rad/s)}.$$

To determine ω_{rel}, we use the fact that the point P in contact with the surface has zero velocity:

$$\mathbf{v}_P = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{P/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.894 + \omega_{\text{rel}} & 1.789 & 0 \\ 0.4 & -0.2 & 0 \end{vmatrix} = 0.$$

Solving, we obtain $\omega_{\text{rel}} = -4.47$ (rad/s), so $\boldsymbol{\omega}_{\text{rel}} = -4.47\mathbf{i}$ (rad/s).

Problem 20.21* The cone rolls on the horizontal surface, which is fixed with respect to an earth-fixed reference frame. The x axis of the secondary reference frame remains coincident with the cone's axis, and the z axis remains horizontal. As the cone rolls, the z axis rotates in the horizontal plane with an angular velocity of 2 rad/s. Determine the velocity relative to the earth-fixed reference frame of the point of the base of the cone with coordinates $x = 0.4$ m, $y = 0$, $z = 0.2$ m. (See Example 20.3.)

Solution: See the solution of Problem 20.20. The cone's angular velocity is

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{\text{rel}} = (0.894\mathbf{i} + 1.789\mathbf{j}) - 4.472\mathbf{i}$$

$$= -3.578\mathbf{i} + 1.789\mathbf{j} \text{ (rad/s)}.$$

Let A denote the pt with coordinates (0.4, 0, 0.2) m. Its velocity is

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3.578 & 1.789 & 0 \\ 0.4 & 0 & 0.2 \end{vmatrix}$$

$$= 0.358\mathbf{i} + 0.716\mathbf{j} - 0.716\mathbf{k} \text{ (m/s)}.$$

Problem 20.22* The cone rolls on the horizontal surface, which is fixed with respect to an earth-fixed reference frame. The x axis of the secondary reference frame remains coincident with the cone's axis, and the z axis remains horizontal. As the cone rolls, the z axis rotates in the horizontal plane with a constant angular velocity of 2 rad/s. Determine the acceleration relative to the earth-fixed reference frame of the point of the base of the cone with coordinates $x = 0.4$ m, $y = 0$, $z = 0.2$ m. (See Example 20.3.)

Solution: See the solutions of Problems 20.20 and 20.21. The cone's angular acceleration is

$$\boldsymbol{\alpha} = \boldsymbol{\Omega} \times \boldsymbol{\omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.894 & 1.789 & 0 \\ -3.578 & 1.789 & 0 \end{vmatrix} = 8.000\mathbf{k} \text{ (rad/s}^2).$$

The acceleration of the point is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

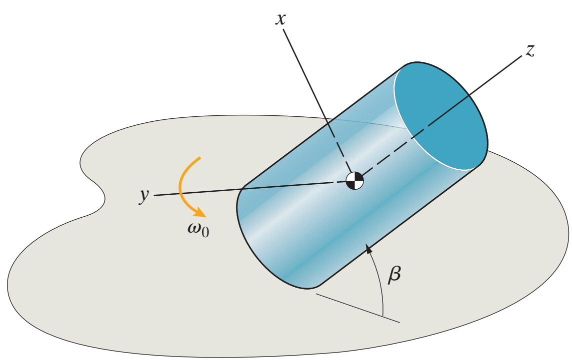
$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 8 \\ 0.4 & 0 & 0.2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3.578 & 1.789 & 0 \\ 0.358 & 0.716 & -0.716 \end{vmatrix}$$

$$= -1.28\mathbf{i} + 0.64\mathbf{j} - 3.20\mathbf{k} \text{ (m/s}^2).$$

Problem 20.23* The radius and length of the cylinder are $R = 0.1$ m and $l = 0.4$ m. The horizontal surface is fixed with respect to an earth-fixed reference frame. One end of the cylinder rolls on the surface while its center, the origin of the secondary reference frame, remains stationary. The angle $\beta = 45^\circ$. The z axis of the secondary reference frame remains coincident with the cylinder's axis, and the y axis remains horizontal. As the cylinder rolls, the y axis rotates in a horizontal plane with angular velocity $\omega_0 = 2$ rad/s.

(a) What is the angular velocity vector $\boldsymbol{\Omega}$ of the secondary reference frame?

(b) What is the angular velocity vector $\boldsymbol{\omega}_{\text{rel}}$ of the cylinder relative to the secondary reference frame?



Solution:

(a) The angular velocity of the secondary reference frame is

$$\boldsymbol{\Omega} = \omega_0 \sin 45^\circ \mathbf{i} + \omega_0 \cos 45^\circ \mathbf{k}$$

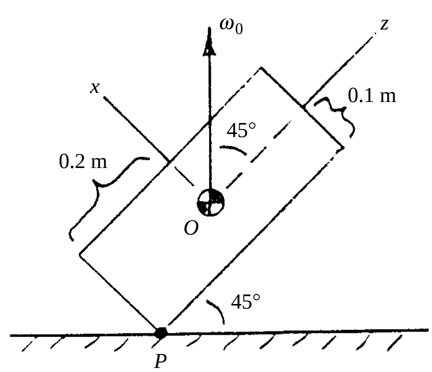
$$= (2) \sin 45^\circ \mathbf{i} + (2) \cos 45^\circ \mathbf{k}$$

$$= 1.414\mathbf{i} + 1.414\mathbf{k} \text{ (rad/s)}.$$

(b) The cylinder's angular velocity relative to the SRF can be written $\boldsymbol{\omega}_{\text{rel}} = \omega_{\text{rel}}\mathbf{k}$, so $\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = 1.414\mathbf{i} + (1.414 + \omega_{\text{rel}})\mathbf{k}$. We determine ω_{rel} from the condition that the velocity of pt. P is zero:

$$\mathbf{v}_P = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{P/O} = 0 + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.414 & 0 & 1.414 + \omega_{\text{rel}} \\ -0.1 & 0 & -0.2 \end{vmatrix} = 0.$$

Solving, we obtain $\omega_{\text{rel}} = 1.414$ rad/s, so $\boldsymbol{\omega}_{\text{rel}} = 1.414\mathbf{k}$ (rad/s).



Problem 20.24* The radius and length of the cylinder are $R = 0.1$ m and $l = 0.4$ m. The horizontal surface is fixed with respect to an earth-fixed reference frame. One end of the cylinder rolls on the surface while its center, the origin of the secondary reference frame, remains stationary. The angle $\beta = 45^\circ$. The z axis of the secondary reference frame remains coincident with the cylinder's axis, and the y axis remains horizontal. As the cylinder rolls, the y axis rotates in a horizontal plane with angular velocity $\omega_0 = 2$ rad/s. Determine the velocity relative to the earth-fixed reference frame of the point of the upper end of the cylinder with coordinates $x = 0.1$ m, $y = 0$, $z = 0.2$ m.

Solution: See the solution of Problem 20.23. The cylinder's angular velocity is

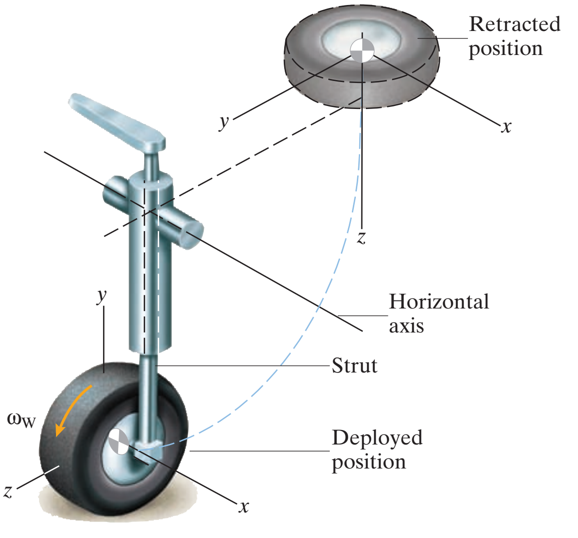
$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{\text{rel}} = (1.414\mathbf{i} + 1.414\mathbf{k}) + 1.414\mathbf{k}$$

$$= 1.414\mathbf{i} + 2.828\mathbf{k} \text{ (rad/s)}.$$

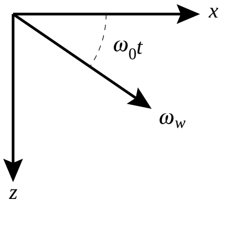
Let A devote the pt with coordinates (0.1, 0, 0.2) m. Its velocity is

$$\mathbf{v}_A = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_{A/O} = \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1.414 & 0 & 2.828 \\ 0.1 & 0 & 0.2 \end{vmatrix} = \mathbf{0}.$$

Problem 20.25* The landing gear of the P-40 airplane used in World War II retracts by rotating 90° about the horizontal axis toward the rear of the airplane. As the wheel retracts, a linkage rotates the strut supporting the wheel 90° about the strut's longitudinal axis so that the wheel is horizontal in the retracted position. (Viewed from the horizontal axis toward the wheel, the strut rotates in the clockwise direction.) The x axis of the coordinate system shown remains parallel to the horizontal axis and the y axis remains parallel to the strut as the wheel retracts. Let ω_W be the magnitude of the wheel's angular velocity when the airplane lifts off, and assume that it remains constant. Let ω_0 be the magnitude of the constant angular velocity of the strut about the horizontal axis as the landing gear is retracted. The magnitude of the angular velocity of the strut about its longitudinal axis also equals ω_0. The landing gear begins retracting at $t = 0$. Determine the wheel's angular velocity relative to the airplane as a function of time.



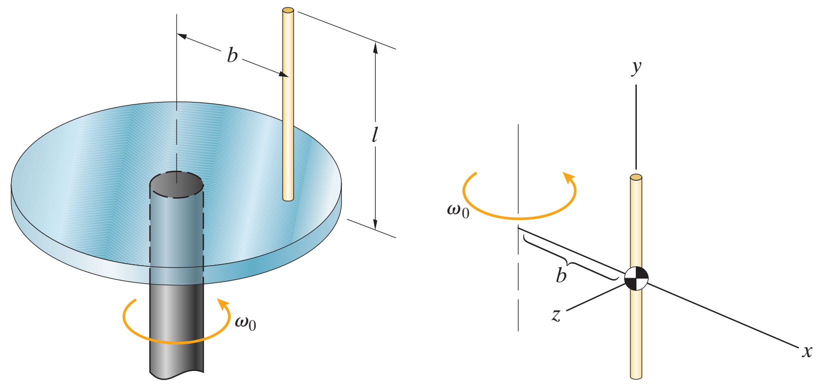
Solution: The angular velocity is given by



$$\boldsymbol{\omega} = \omega_0\mathbf{i} + \omega_0\mathbf{j} + \omega_W[(\cos\omega_0 t)\mathbf{i} + (\sin\omega_0 t)\mathbf{k}]$$

$$\boxed{\boldsymbol{\omega} = (\omega_0 + \omega_W \cos\omega_0 t)\mathbf{i} + \omega_0\mathbf{j} + (\omega_0 + \omega_W \cos\omega_0 t)\mathbf{k}}$$

Problem 20.26 In Active Example 20.4, suppose that the shaft supporting the disk is initially stationary, and at $t = 0$ it is subjected to a constant angular acceleration α_0 in the counterclockwise direction viewed from above the disk. Determine the force and couple exerted on the bar by the disk at that instant.



Solution: We have Use Newton's Second Law for rigid bodies to find the force **F** at the base of the rod

$$\mathbf{F} = mg\mathbf{j} = m\mathbf{a} = m(-b\alpha_0\mathbf{k}) \quad \Rightarrow \quad \boxed{\mathbf{F} = mg\mathbf{j} - mb\alpha_0\mathbf{k}}$$

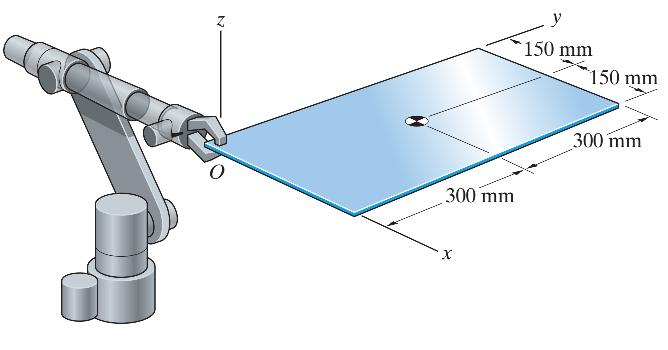
Now use Euler's equations to find the moment **C** exerted by the disk on the base of the rod. Note that the angular velocity is zero, and the only nonzero inertias are $I_{xx} = I_{zz} = ml^2/12$.

$$\mathbf{C} + \left(-\frac{1}{2}\mathbf{j}\right) \times \mathbf{F} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \alpha_0 \\ 0 \end{Bmatrix} = 0$$

$$\mathbf{C} = \frac{1}{2}\mathbf{j}(mg\mathbf{j} - mb\alpha_0\mathbf{k})$$

$$\boxed{\mathbf{C} = -\frac{1}{2}mbl\alpha_0\mathbf{i}}$$

Problem 20.27 In Example 20.5, suppose that the horizontal plate is initially stationary, and at $t = 0$ the robotic manipulator exerts a couple **C** on the plate at the fixed point O such that the plate's angular acceleration at this instant is $\boldsymbol{\alpha} = 150\mathbf{i} + 320\mathbf{j} + 25\mathbf{k}$ (rad/s^2). Determine **C**.



Solution: The mass of the plate is 4 kg. Point O is a fixed point. The nonzero inertias are

$$I_{xx} = \frac{1}{3}(4 \text{ kg})(0.6 \text{ m})^2 = 0.48 \text{ kg-m}^2,$$

$$I_{yy} = \frac{1}{3}(4 \text{ kg})(0.3 \text{ m})^2 = 0.12 \text{ kg-m}^2$$

$$I_{zz} = I_{xx} + I_{yy} = 0.60 \text{ kg-m}^2,$$

$$I_{xy} = (4 \text{ kg})(0.15 \text{ m})(0.3 \text{ m}) = 0.18 \text{ kg-m}^2.$$

Euler's equations can be written as (note that the angular velocity is zero)

$$\mathbf{C} + (0.15\mathbf{i} + 0.3\mathbf{j}) \times (-[4][9.81]\mathbf{k}) = [I]\boldsymbol{\alpha}$$

$$\mathbf{C} = -(0.15\mathbf{i} + 0.3\mathbf{j}) \times (-[4][9.81]\mathbf{k}) + [I]\boldsymbol{\alpha}$$

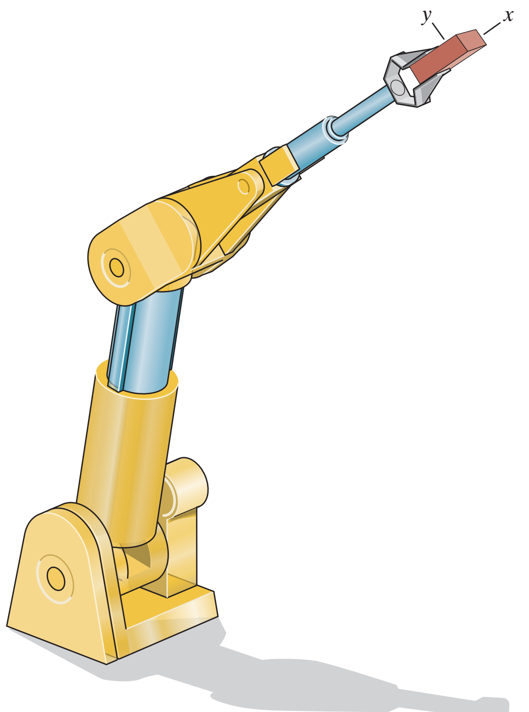
In Matrix Form this is

$$\begin{Bmatrix} C_x \\ C_y \\ C_z \end{Bmatrix} = \begin{Bmatrix} 11.8 \\ -5.89 \\ 0 \end{Bmatrix} + \begin{bmatrix} 0.48 & -0.18 & 0 \\ -0.18 & 0.12 & 0 \\ 0 & 0 & 0.60 \end{bmatrix} \begin{Bmatrix} 150 \\ 320 \\ 25 \end{Bmatrix}$$

Solving we find $C_x = 26.2, \quad C_y = 5.51, \quad C_z = 15.$

Thus $\boxed{\mathbf{C} = (26.2\mathbf{i} + 5.51\mathbf{j} + 15\mathbf{k}) \text{ N-m.}}$

Problem 20.28 A robotic manipulator moves a casting. The inertia matrix of the casting in terms of a body-fixed coordinate system with its origin at the center of mass is shown. At the present instant, the angular velocity and angular acceleration of the casting are $w = 1.2\mathbf{i} + 0.8\mathbf{j} - 0.4\mathbf{k}$ (rad/s) and $\boldsymbol{\alpha} = 0.26\mathbf{i} - 0.07\mathbf{j} + 0.13\mathbf{k}$ (rad/s^2). What moment is exerted about the center of mass of the casting by the manipulator?



$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \text{ kg-m}^2.$$

Solution:

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \begin{Bmatrix} 0.26 \\ -0.07 \\ 0.13 \end{Bmatrix} \text{ N-m}$$

$$+ \begin{bmatrix} 0 & 0.4 & 0.8 \\ -0.4 & 0 & -1.2 \\ -0.8 & 1.2 & 0 \end{bmatrix} \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix}$$

$$\times \begin{Bmatrix} 1.2 \\ 0.8 \\ -0.4 \end{Bmatrix} \text{ N-m}$$

$$\boxed{\mathbf{M} = (M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}) = (0.0135\mathbf{i} - 0.0086\mathbf{j} + 0.01\mathbf{k}) \text{ N-m}}$$

Problem 20.29 A robotic manipulator holds a casting. The inertia matrix of the casting in terms of a body-fixed coordinate system with its origin at the center of mass is shown. At the present instant, the casting is stationary. If the manipulator exerts a moment $\Sigma\mathbf{M} = 0.042\mathbf{i} + 0.036\mathbf{j} + 0.066\mathbf{k}$ (N-m) about the center of mass, what is the angular acceleration of the casting at that instant?

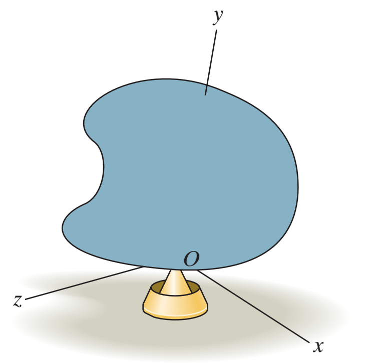
Solution:

$$\begin{Bmatrix} 0.042 \\ 0.036 \\ 0.066 \end{Bmatrix} \text{ N-m} = \begin{bmatrix} 0.05 & -0.03 & 0 \\ -0.03 & 0.08 & 0 \\ 0 & 0 & 0.04 \end{bmatrix} \text{ kg-m}^2 \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix}$$

Solving we find

$$\boxed{\boldsymbol{\alpha} = (\alpha_x\mathbf{i} + \alpha_y\mathbf{j} + \alpha_z\mathbf{k}) = (1.43\mathbf{i} + 0.987\mathbf{j} + 1.65\mathbf{k}) \text{ rad/s}^2}$$

Problem 20.30 The rigid body rotates about the fixed point O. Its inertia matrix in terms of the body-fixed coordinate system is shown. At the present instant, the rigid body's angular velocity is $\boldsymbol{\omega} = 6\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ (rad/s) and its angular acceleration is zero. What total moment about O is being exerted on the rigid body?



$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \text{ kg-m}^2.$$

Solution:

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} 0 & 4 & 6 \\ -4 & 0 & -6 \\ -6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} 6 \\ 6 \\ -4 \end{Bmatrix} \text{ N-m}$$

$$\boxed{\mathbf{M} = (M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k}) = (-76\mathbf{i} + 36\mathbf{j} - 60\mathbf{k}) \text{ N-m}}$$

Problem 20.31 The rigid body rotates about the fixed point O. Its inertia matrix in terms of the body-fixed coordinate system is shown. At the present instant, the rigid body's angular velocity is $\boldsymbol{\omega} = 6\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ (rad/s). The total moment about O due to the forces and couples acting on the rigid body is zero. What is its angular acceleration?

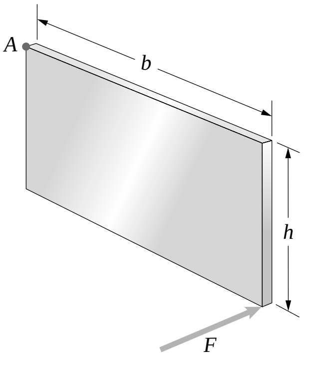
Solution:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \text{ kg-m}^2 \begin{Bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{Bmatrix}$$

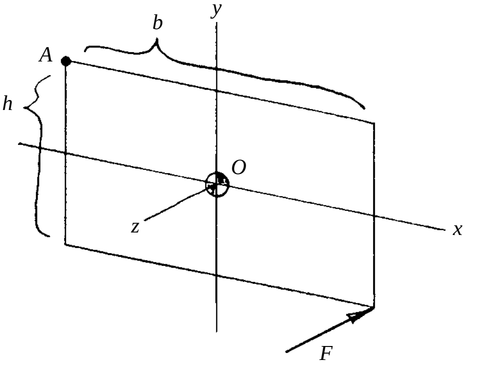
$$+ \begin{bmatrix} 0 & 4 & 6 \\ -4 & 0 & -6 \\ -6 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{Bmatrix} 6 \\ 6 \\ -4 \end{Bmatrix} \text{ N-m}$$

Solving we find $\boxed{\boldsymbol{\alpha} = (16.2\mathbf{i} - 5.56\mathbf{j} + 13.1\mathbf{k}) \text{ rad/s}^2}$

Problem 20.32 The dimensions of the 20-kg thin plate are $h = 0.4$ m and $b = 0.6$ m. The plate is stationary relative to an inertial reference frame when the force $F = 10$ N is applied in the direction perpendicular to the plate. No other forces or couples act on the plate. At the instant F is applied, what is the magnitude of the acceleration of point A relative to the inertial reference frame?



Solution: From Appendix C, the inertia matrix in terms of the body-fixed reference frame shown is



$$[I] = \begin{bmatrix} \frac{1}{12}mh^2 & 0 & 0 \\ 0 & \frac{1}{12}mb^2 & 0 \\ 0 & 0 & \frac{1}{12}m(b^2+h^2) \end{bmatrix}$$

$$= \begin{bmatrix} 0.267 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.867 \end{bmatrix} \text{ kg-m}^2.$$

The moment of the force about the center of mass is

$$\mathbf{M} = \left(\frac{b}{2}\mathbf{i} - \frac{h}{2}\mathbf{j}\right) \times (-F\mathbf{k}) = 2\mathbf{i} + 3\mathbf{j} \text{ (N-m)}.$$

From Eq. (20.19) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.267 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.367 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}$$

Solving, we obtain $\boldsymbol{\alpha} = 7.5\mathbf{i} + 5\mathbf{j}$ (m/s^2). From Newton's second law, $\sum \mathbf{F} = -F\mathbf{k} = m\mathbf{a}_0$, the acceleration of the center of mass is $\mathbf{a}_0 = -\dfrac{F}{m}\mathbf{k} = -0.5\mathbf{k}$ (m/s^2). The acceleration of pt A is

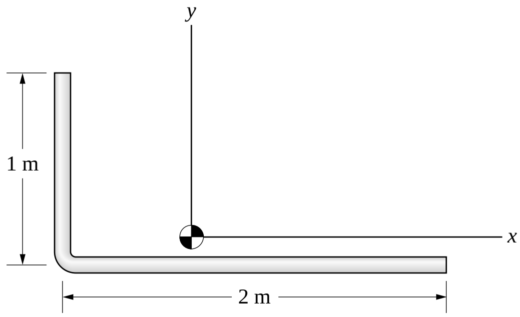
$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= -0.5\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7.5 & 5 & 0 \\ -0.3 & 0.2 & 0 \end{vmatrix} + \mathbf{0}$$

$$= 2.5\mathbf{k} \text{ (m/s}^2\text{)}.$$

We see that $|\mathbf{a}_A| = 2.5$ m/s^2.

Problem 20.33 In terms of the coordinate system shown, the inertia matrix of the 6-kg slender bar is



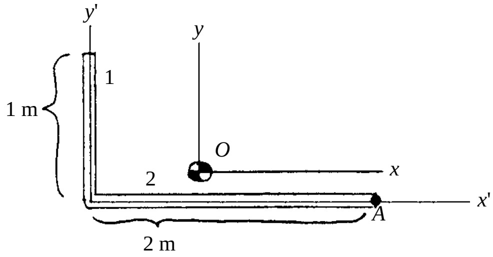
$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 0.500 & 0.667 & 0 \\ 0.667 & 2.667 & 0 \\ 0 & 0 & 3.167 \end{bmatrix} \text{ kg-m}^2.$$

The bar is stationary relative to an inertial reference frame when the force $\mathbf{F} = 12\mathbf{k}$ (N) is applied at the right end of the bar. No other forces or couples act on the bar. Determine

(a) the bar's angular acceleration relative to the inertial reference frame and
(b) the acceleration of the right end of the bar relative to the inertial reference frame at the instant the force is applied.

Solution:



(a) In terms of the primed reference frame shown, the coordinates of the center of mass are

$$\mathbf{x}' = \frac{\mathbf{x}'_1 m_1 + \mathbf{x}'_2 m_2}{m_1 + m_2} = \frac{(0)\frac{1}{3}(6) + (1)\frac{2}{3}(6)}{\frac{1}{3}(6) + \frac{2}{3}(6)}$$

$$= 0.667 \text{ m},$$

$$\mathbf{y}' = \frac{\mathbf{y}'_1 m_1 + \mathbf{y}'_2 m_2}{m_1 + m_2} = \frac{(0.5)\frac{1}{3}(6) + (0)\frac{2}{3}(6)}{\frac{1}{3}(6) + \frac{2}{3}(6)}$$

$$= 0.167 \text{ m}.$$

The moment of $\mathbf{F}$ about the center of mass is

$$\mathbf{M} = (1.333\mathbf{i} - 0.167\mathbf{j}) \times 12\mathbf{k}$$

$$= -2\mathbf{i} - 16\mathbf{j} \text{ (N-m)}.$$

From Eq. (20.19) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} -2 \\ -16 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.667 & 0 \\ 0.667 & 2.667 & 0 \\ 0 & 0 & 3.167 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

Solving, we obtain $\boldsymbol{\alpha} = 6.01\mathbf{i} - 7.50\mathbf{j}$ (rad/s^2).

(b) From Newton's second law, $\sum \mathbf{F} = 12\mathbf{k} = (6)\mathbf{a}_0$, the acceleration of the center of mass is $\mathbf{a}_0 = 2\mathbf{k}$ (m/s^2). The acceleration of pt A is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

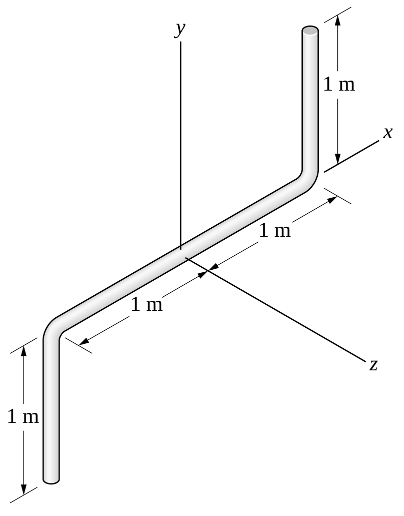
$$= 2\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6.01 & -7.50 & 0 \\ 1.333 & -0.167 & 0 \end{vmatrix} + \mathbf{0}$$

$$= 11.0\mathbf{k} \text{ (m/s}^2).$$

Problem 20.34 In terms of the coordinate system shown, the inertia matrix of the 12-kg slender bar is

$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix} \text{ kg-m}^2.$$

The bar is stationary relative to an inertial reference frame when a force $\mathbf{F} = 20\mathbf{i} + 40\mathbf{k}$ (N) is applied at the point $x = 1$ m, $y = 1$ m. No other forces or couples act on the bar. Determine (a) the bar's angular acceleration and (b) the acceleration of the point $x = -1$ m, $y = -1$ m, relative to the inertial reference frame at the instant the force is applied.



Solution:

(a) The moment of the force about the center of mass is

$$\mathbf{M} = (\mathbf{i} + \mathbf{j}) \times (20\mathbf{i} + 40\mathbf{k})$$

$$= 40\mathbf{i} - 40\mathbf{j} - 20\mathbf{k} \text{ (N-m)}.$$

From Eq. (20.19) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} 40 \\ -40 \\ -20 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

Solving, we obtain

$$\boldsymbol{\alpha} = 28.57\mathbf{i} + 5.71\mathbf{j} - 2\mathbf{k} \text{ (rad/s}^2).$$

From Newton's second law,

$$\sum \mathbf{F} = 20\mathbf{i} + 40\mathbf{k} = (12)\mathbf{a}_0,$$

the acceleration of the center of mass is

$$\mathbf{a}_0 = 1.67\mathbf{i} + 3.33\mathbf{k} \text{ (N/s}^2).$$

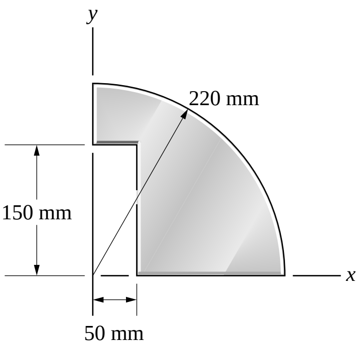
The acceleration of the pt with coordinates $(-1, -1, 0)$ is

$$\mathbf{a}_A = \mathbf{a}_0 + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times r_{A/O})$$

$$= 1.67\mathbf{i} + 3.33\mathbf{k} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 28.57 & 5.71 & -2 \\ -1 & -1 & 0 \end{vmatrix} + \mathbf{0}$$

$$= -0.333\mathbf{i} + 2.000\mathbf{j} - 19.524\mathbf{k} \text{ (m/s}^2)$$

Problem 20.35 The inertia matrix of the 2.4-kg plate in terms of the given coordinate system is shown. The angular velocity of the plate is $\boldsymbol{\omega} = 6.4\mathbf{i} + 8.2\mathbf{j} + 14\mathbf{k}$ (rad/s), and its angular acceleration is $\boldsymbol{\alpha} = 60\mathbf{i} + 40\mathbf{j} - 120\mathbf{k}$ (rad/s^2). What are the components of the total moment exerted on the plate about its center of mass?



Solution: In the solution of Problem 20.87, the location of the center of mass, $\mathbf{x} = 0.1102$ (m), $\mathbf{y} = 0.0979$ (m) and the moments of inertia in terms of a parallel coordinate system with its origin at the center of mass are determined:

$$I_{x'x'} = 0.00876 \text{ (kg-m}^2), \; I_{y'y'} = I_{yy} = 0.00655 \text{ (kg-m}^2),$$

$$I_{z'z'} = 0.01531 \text{ (kg-m}^2), \; I_{x'y'} = -0.00396 \text{ (kg-m}^2), \; I_{y'z'} = I_{z'x'} = 0.$$

The components of the total moment are given by Equation (20.19) with $\boldsymbol{\Omega} = \boldsymbol{\omega}$:

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0.00876 & 0.00396 & 0 \\ 0.00396 & 0.00655 & 0 \\ 0 & 0 & 0.01531 \end{bmatrix} \begin{bmatrix} 60 \\ 40 \\ -120 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -14.0 & 8.2 \\ 14.0 & 0 & -6.4 \\ -8.2 & 6.4 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.00876 & 0.00396 & 0 \\ 0.00396 & 0.00655 & 0 \\ 0 & 0 & 0.01531 \end{bmatrix} \begin{bmatrix} 6.4 \\ 8.2 \\ 14.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.335 \\ 0.367 \\ -2.057 \end{bmatrix} \text{ (N-m).}$$

Problem 20.36 The inertia matrix of the 2.4-kg plate in terms of the given coordinate system is shown. At $t = 0$, the plate is stationary and is subjected to a force $\mathbf{F} = -10\mathbf{k}$ (N) at the point with coordinates (220,0,0) mm. No other forces or couples act on the plate. Determine (a) the acceleration of the plate's center of mass and (b) the plate's angular acceleration at the instant the force is applied.

Solution:

(a) From Newton's second law, $\sum \mathbf{F} = m\mathbf{a}$: $\quad -10\mathbf{k} = 2.4\mathbf{a}$, and the acceleration of the center of mass is $\mathbf{a} = -4.17\mathbf{k}$ (m/s^2).

(b) From the solution of Problem 20.87, the center of mass is at $\mathbf{x} = 0.1102$ (m), $\mathbf{y} = 0.0979$ (m). Therefore, the moment of the force about the center of mass is

$$\sum \mathbf{M} = [(0.22 - 0.1102)\mathbf{i} - 0.0979\mathbf{j}] \times (-10\mathbf{k})$$

$$= 0.979\mathbf{i} + 1.098\mathbf{j} \text{ (N-m).}$$

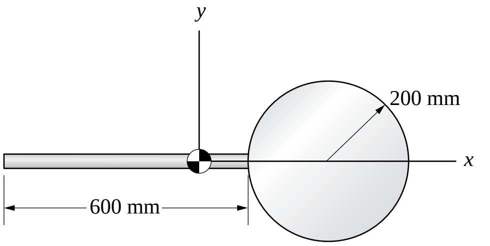
Equation (20.19) is

$$\begin{bmatrix} 0.979 \\ 1.098 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.00876 & 0.00396 & 0 \\ 0.00396 & 0.00655 & 0 \\ 0 & 0 & 0.01531 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

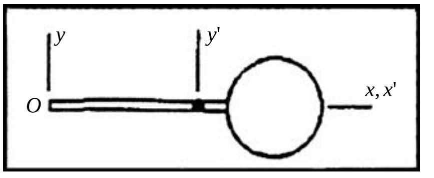
Solving these equations, we obtain

$$\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt = 49.5\mathbf{i} + 137.7\mathbf{j} \text{ (rad/s}^2).$$

Problem 20.37 A 3-kg slender bar is rigidly attached to a 2-kg thin circular disk. In terms of the body-fixed coordinate system shown, the angular velocity of the composite object is $\boldsymbol{\omega} = 100\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ (rad/s) and its angular acceleration is zero. What are the components of the total moment exerted on the object about its center of mass?



Solution: Choose an x, y, z coordinate system with the origin at O and the x axis parallel to the slender rod, as shown. From the solution to Problem 20.92, the coordinates of the center of mass in the x, y, z system are (0.5, 0, 0), and the inertia matrix about a parallel coordinate system with origin at the center of mass is:



$$[I]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \text{ kg-m}^2.$$

Since the coordinate system is body fixed, $\boldsymbol{\Omega} = \boldsymbol{\omega}$, and Eq. (20.19) reduces to

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & -6 & -4 \\ 6 & 0 & -100 \\ 4 & 100 & 0 \end{bmatrix} \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \begin{bmatrix} 100 \\ -4 \\ 6 \end{bmatrix},$$

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & -2.46 & -1.72 \\ 0.12 & 0 & -43 \\ 0.08 & 41 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ -4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -0.48 \\ -246 \\ -156 \end{bmatrix} \text{ N-m},$$

$$\mathbf{M}_0 = -0.48\mathbf{i} - 246\mathbf{j} - 156\mathbf{k} \text{ N-m}.$$

Problem 20.38 A 3-kg slender bar is rigidly attached to a 2-kg thin circular disk. At $t = 0$, the composite object is stationary and is subjected to the moment $\Sigma\mathbf{M} = -10\mathbf{i} + 10\mathbf{j}$ (N-m) about its center of mass. No other forces or couples act on the object. Determine the object's angular acceleration at $t = 0$.

Solution: From the solution to Problem 20.92, the inertia matrix in terms of the parallel coordinate system with origin at the center of mass is

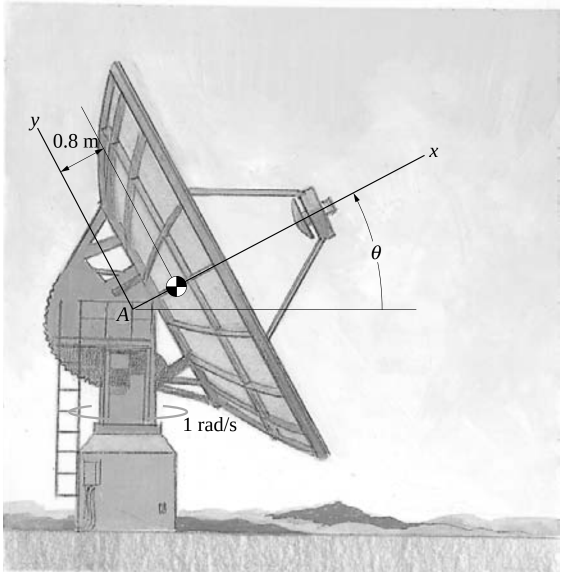
$$[I]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \text{ kg-m}^2.$$

Since the coordinates are body-fixed and the object is stationary at $t = 0$, $\boldsymbol{\Omega} = \boldsymbol{\omega} = 0$, and Eq. (20.19) reduces to:

$$\begin{bmatrix} -10 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} 0.02\alpha_x \\ 0.41\alpha_y \\ 0.43\alpha_z \end{bmatrix}.$$

Solve: $\boldsymbol{\alpha} = -500\mathbf{i} + 24.4\mathbf{j}$ (rad/s^2).

Problem 20.39 The vertical shaft supporting the dish antenna is rotating with a constant angular velocity of 1 rad/s. The angle $\theta = 30^\circ$, $d\theta/dt = 20^\circ/\text{s}^2$, and $d^2\theta/dt^2 = -40^\circ/\text{s}^2$. The mass of the antenna is 280 kg, and its moments and products of inertia, in kg-m^2, are $I_{xx} = 140$, $I_{yy} = I_{zz} = 220$, $I_{xy} = I_{yz} = I_{zx} = 0$. Determine the couple exerted on the antenna by its support at A at the instant shown.



Solution: The reactions at the support arise from (a) the Euler moments about the point A, and (b) the weight unbalance due to the offset center of mass. *The Euler Equations*: Express the reactions in the x, y, z system. The angular velocity in the x, y, z system

$$\boldsymbol{\omega} = \mathbf{i}\sin\theta + \mathbf{j}\cos\theta + (d\theta/dt)\mathbf{k}$$

$$= 0.5\mathbf{i} + 0.866\mathbf{j} + 0.3491\mathbf{k} \text{ (rad/s).}$$

The angular acceleration is

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = (\mathbf{i}\cos\theta - \mathbf{j}\sin\theta)\frac{d\theta}{\text{dt}} + \frac{d^2\theta}{\text{dt}^2}\mathbf{k}$$

$$= 0.3023\mathbf{i} - 0.1745\mathbf{j} - 0.6981\mathbf{k} \text{ (rad/s}^2\text{).}$$

Since the coordinates are body fixed $\boldsymbol{\Omega} = \boldsymbol{\omega}$, and Eq. (20.13) is

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 140 & 0 & 0 \\ 0 & 220 & 0 \\ 0 & 0 & 220 \end{bmatrix} \begin{bmatrix} 0.302 \\ -0.1745 \\ -0.6981 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -0.3491 & 0.866 \\ 0.3491 & 0 & -0.5 \\ -0.866 & 0.5 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 140 & 0 & 0 \\ 0 & 220 & 0 \\ 0 & 0 & 220 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.866 \\ 0.3491 \end{bmatrix}.$$

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 42.32 \\ -38.39 \\ -153.6 \end{bmatrix} + \begin{bmatrix} 0 & -76.79 & 190.52 \\ 48.87 & 0 & -110 \\ -121.2 & 110 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.5 \\ 0.866 \\ 0.3491 \end{bmatrix}$$

$$= \begin{bmatrix} 42.32 \\ -38.39 \\ -153.6 \end{bmatrix} + \begin{bmatrix} 0 \\ -13.96 \\ 34.64 \end{bmatrix} = \begin{bmatrix} 42.32 \\ -52.36 \\ -118.9 \end{bmatrix},$$

$$\mathbf{M}_0 = 42.32\mathbf{i} - 52.36\mathbf{j} - 118.9\mathbf{k} \text{ N-m}$$

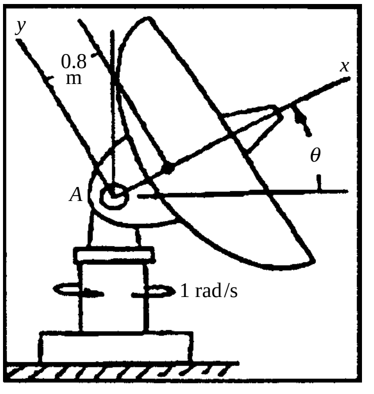
The unbalance exerted by the offset center of mass: The weight of the antenna acting through the center of mass in the x, y, z system is

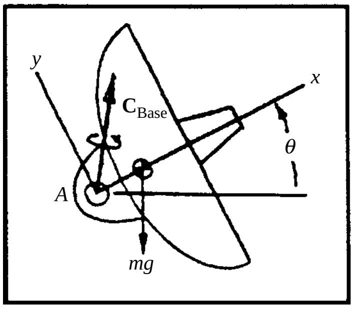
$$\mathbf{W} = \text{mg}(-\mathbf{i}\sin\theta - \mathbf{j}\cos\theta) = -1373.4\mathbf{i} - 2378.8\mathbf{j} \text{ (N).}$$

The vector distance to the center of mass is in the x, y, z system is $\mathbf{r}_{G/O} = 0.8\mathbf{i}$ (m). The moment exerted by the weight is

$$\mathbf{M}_W = \mathbf{r}_{G/O} \times \mathbf{W} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.8 & 0 & 0 \\ -1373.4 & -2378.8 & 0 \end{bmatrix}$$

$$= 1903.0\mathbf{k}.$$





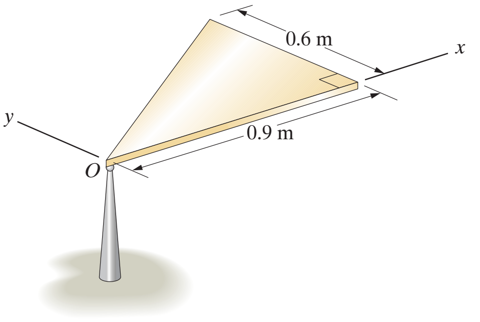
The couple exerted by the base:

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1903.0 \end{bmatrix}$$

$$= \begin{bmatrix} 42.32 \\ -52.36 \\ -118.9 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ +1903.0 \end{bmatrix} = \begin{bmatrix} 42.32 \\ -52.36 \\ 1784.1 \end{bmatrix} \text{ (N-m),}$$

$$\mathbf{C}_{\text{Base}} = 42.35\mathbf{i} - 52.36\mathbf{j} + 1784.1\mathbf{k} \text{ (N-m)}$$

Problem 20.40 The 5-kg triangular plate is connected to a ball-and-socket support at O. If the plate is released from rest in the horizontal position, what are the components of its angular acceleration at that instant?



Solution: From the appendix to Chapter 20 on moments of inertia, the inertia matrix in terms of the reference frame shown is

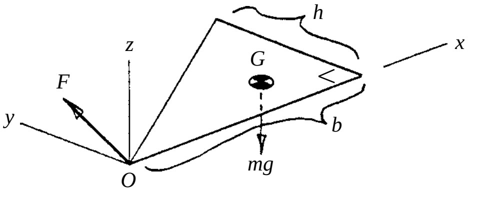
$$[I] = \begin{bmatrix} \frac{m}{A}\left(\frac{1}{12}bh^3\right) & -\frac{m}{A}\left(\frac{1}{8}b^2h^2\right) & 0 \\ -\frac{m}{A}\left(\frac{1}{8}b^2h^2\right) & \frac{m}{A}\left(\frac{1}{4}hb^3\right) & 0 \\ 0 & 0 & \frac{m}{A}\left(\frac{1}{12}bh^3 + \frac{1}{4}hb^3\right) \end{bmatrix}$$

$$= \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \text{ kg-m}^2.$$

The moment exerted by the weight about the fixed pt. 0 is

$$\sum \mathbf{m}_0 = \left(\frac{2}{3}b\mathbf{i} + \frac{1}{3}h\mathbf{j}\right) \times (-mg\mathbf{k})$$

$$= -9.81\mathbf{i} + 29.43\mathbf{j} \text{ (N-m)}.$$



From Eq. (20.13) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = \mathbf{0}$,

$$\begin{bmatrix} -9.81 \\ 29.43 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \begin{bmatrix} d\omega_x/dt \\ d\omega_y/dt \\ d\omega_z/dt \end{bmatrix}.$$

Solving, we obtain $\boldsymbol{\alpha} = 14.5\ \mathbf{j}$ (rad/s^2).

Problem 20.41 If the 5-kg plate is released from rest in the horizontal position, what force is exerted on it by the ball-and-socket support at that instant?

Solution: See the solution of Problem 20.40. Let G denote the center of mass and Let $\mathbf{F}$ be the force exerted by the support.

The acceleration of the center of mass is

$$\mathbf{a}_G = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{G/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/O})$$

$$= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 14.5 & 0 \\ \frac{2}{3}b & \frac{1}{3}h & 0 \end{vmatrix} + \mathbf{0}$$

$$= -8.72\ \mathbf{k} \text{ (m/s}^2)$$

From Newton's second law,

$\Sigma\mathbf{F} = m\mathbf{a}_G$: $\quad \mathbf{F} - (5)(9.81)\mathbf{k} = (5)(-8.72\mathbf{k})$,

we obtain $\mathbf{F} = 5.45\mathbf{k}$ (N).

Problem 20.42 The 5-kg triangular plate is connected to a ball-and-socket support at O. If the plate is released in the horizontal position with angular velocity $\boldsymbol{\omega} = 4\mathbf{i}$ (rad/s), what are the components of its angular acceleration at that instant?

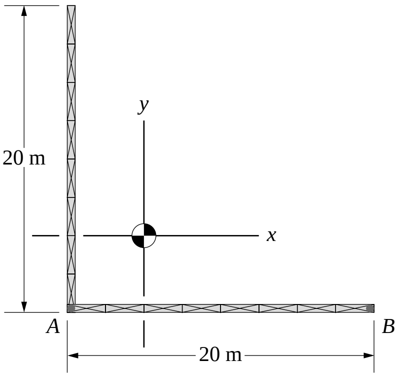
Solution: See the solution of Problem 20.40. From Eq. (20.13) with $\boldsymbol{\omega} = \boldsymbol{\Omega} = 4\mathbf{i}$ (rad/s);

$$\begin{bmatrix} -9.81 \\ 29.43 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \begin{bmatrix} \mathrm{d}\omega_x/\mathrm{d}t \\ \mathrm{d}\omega_y/\mathrm{d}t \\ \mathrm{d}\omega_z/\mathrm{d}t \end{bmatrix}$$

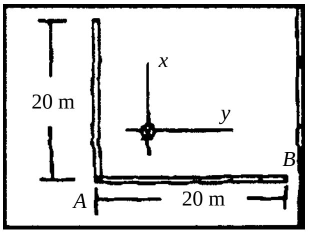
$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0.300 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}.$$

Solving, we obtain $\boldsymbol{\alpha} = 14.53\mathbf{j} + 4.65\mathbf{k}$ (rad/s^2).

Problem 20.43 A subassembly of a space station can be modeled as two rigidly connected slender bars, each with a mass of 5000 kg. The subassembly is not rotating at $t = 0$, when a reaction control motor exerts a force $\mathbf{F} = 400\mathbf{k}$ (N) at B. What is the acceleration of point A relative to the center of mass of the subassembly at $t = 0$?



Solution: Choose a x', y', z' coordinate system with the origin at A and the x' axis parallel to the horizontal bar, and a parallel x, y, z system with origin at the center of mass.



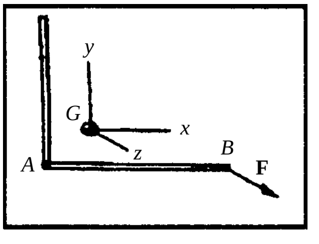
The Euler Equations: The center of mass in the x', y', z' system has the coordinates

$$x_G = \frac{10(5000) + 0(5000)}{10000} = 5 \text{ m},$$

$$y_G = \frac{10(5000) + 0(5000)}{10000} = 5 \text{ m},$$

$$z_G = 0,$$

from which $(d_x, d_y, d_z) = (5, 5, 0)$ m.



From Appendix C, the moments and products of inertia of each bar about A are

$$I_{xx}^A = I_{yy}^A = \frac{mL^2}{3},$$

$$I_{zz}^A = I_{xx}^A + I_{yy}^A = \frac{2mL^2}{3}$$

$$I_{xy}^A = I_{xz}^A = I_{yz}^A = 0,$$

where $m = 5000$ kg, and $L = 20$ m. The moment of inertia matrix is

$$[I^A] = \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 0.6667 & 0 \\ 0 & 0 & 1.333 \end{bmatrix} \text{ Mg-m}^2.$$

From the parallel axis theorem, Eq. (20.42), the moments and products of inertia about the center of mass are:

$$I_{xx} = I_{xx}^A - (d_z^2 + d_y^2)(2 \text{ m}) = 0.4167 \text{ Mg-m}^2,$$

$$I_{yy} = I_{yy}^A - (d_x^2 + d_z^2)(2 \text{ m}) = 0.4167 \text{ Mg-m}^2,$$

$$I_{zz} = I_{zz}^A - (d_x^2 + d_y^2)(2 \text{ m}) = 0.8333 \text{ Mg-m}^2.$$

$$I_{xy} = I_{xy}^A - d_x d_y (2 \text{ m}) = -0.2500 \text{ Mg-m}^2,$$

from which the inertia matrix is

$$[I] = \begin{bmatrix} 0.4167 & 0.2500 & 0 \\ 0.2500 & 0.4167 & 0 \\ 0 & 0 & 0.8333 \end{bmatrix} \text{ Mg-m}^2.$$

The vector distance from the center of mass to the point B is

$$\mathbf{r}_{B/G} = (20 - 5)\mathbf{i} + (0 - 5)\mathbf{j} = 15\mathbf{i} - 5\mathbf{j} \text{ (m)}.$$

The moment about the center of mass is

$$\mathbf{M}_G = \mathbf{r}_{B/G} \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 15 & -5 & 0 \\ 0 & 0 & 400 \end{bmatrix}$$

$$= -2000\mathbf{i} - 6000\mathbf{j} \text{ (N-m)}.$$

The coordinates are body-fixed, and the object is initially stationary, from which $\mathbf{\Omega} = \boldsymbol{\omega} = 0$, and Eq. (20.19) reduces to

$$\begin{bmatrix} -2000 \\ -6000 \\ 0 \end{bmatrix} = [I]$$

$$= \begin{bmatrix} 4.167 \times 10^5 & 2.5 \times 10^5 & 0 \\ 2.5 \times 10^5 & 4.167 \times 10^5 & 0 \\ 0 & 0 & 8.333 \times 10^5 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}.$$

Carry out the matrix multiplication to obtain:

$$4.167 \times 10^5 \alpha_x + 2.5 \times 10^5 \alpha_y = -2000,$$

$$2.5 \times 10^5 \alpha_x + 4.167 \times 10^5 \alpha_y = -6000,$$

and $\alpha_z = 0$. Solve: $\boldsymbol{\alpha} = 0.006\mathbf{i} - 0.018\mathbf{j}$ (rad/s^2).

Newton's second law: The acceleration of the center of mass of the object from Newton's second law is

$$\mathbf{a}_G = \left(\frac{1}{2 \text{ m}}\right)\mathbf{F} = 0.04\mathbf{k} \text{ (m/s}^2\text{)}.$$

The acceleration of point A: The vector distance from the center of mass to the point A is $\mathbf{r}_{A/G} = -5\mathbf{i} - 5\mathbf{j}$ (m). The acceleration of point A is

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/G}).$$

Since the object is initially stationary, $\boldsymbol{\omega} = 0$.

$$\mathbf{a}_A = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} = 0.04\mathbf{k} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.006 & -0.018 & 0 \\ -5 & -5 & 0 \end{bmatrix}$$

$$= -0.08\mathbf{k} \text{ (m/s}^2\text{)}, \quad \mathbf{a}_A = -0.08\mathbf{k} \text{ (m/s}^2\text{)}$$

Problem 20.44 A subassembly of a space station can be modeled as two rigidly connected slender bars, each with a mass of 5000 kg. If the subassembly is rotating about the x axis at a constant rate of 1 revolution every 10 minutes, what is the magnitude of the couple its reaction control system is exerting on it?

Solution: (See Figure in solution to Problem 20.100.) The angular acceleration of the disk is given by

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt}(\omega_d \mathbf{i} + \omega_O \mathbf{j}) + \boldsymbol{\omega}_O \times \boldsymbol{\omega}_d = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_O & 0 \\ \omega_d & 0 & 0 \end{bmatrix}$$

$$= -\omega_O \omega_d \mathbf{k}.$$

The velocity of point A relative to O is

$$\mathbf{a}_{A/O} = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= (-\omega_O \omega_d)(\mathbf{k} \times \mathbf{r}_{A/O}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}).$$

Term by term:

$$-\omega_O \omega_d (\mathbf{k} \times \mathbf{r}_{A/O}) = -\omega_O \omega_d \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \omega_O \omega_d R \sin\theta \mathbf{i} - \omega_O \omega_d b \mathbf{j},$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}) = \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ -R\omega_O \cos\theta & R\omega_d \cos\theta & R\omega_d \sin\theta - b\omega_O \end{bmatrix}$$

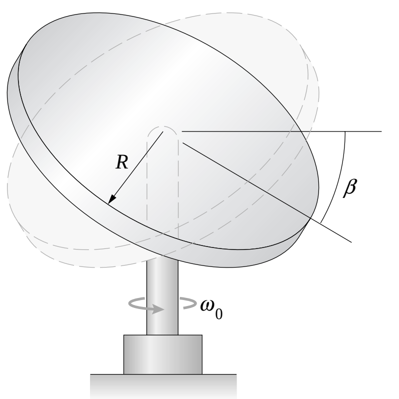
$$= (R\omega_d \sin\theta - b\omega_O)(\omega_o \mathbf{i} - \omega_d \mathbf{j}) + (R\cos\theta)(\omega_d^2 + \omega_O^2)\mathbf{k}.$$

Collecting terms:

$$\mathbf{a}_{A/O} = (2R\omega_O \omega_d \sin\theta - b\omega_O^2)\mathbf{i} - (R\omega_d^2 \sin\theta)\mathbf{j}$$

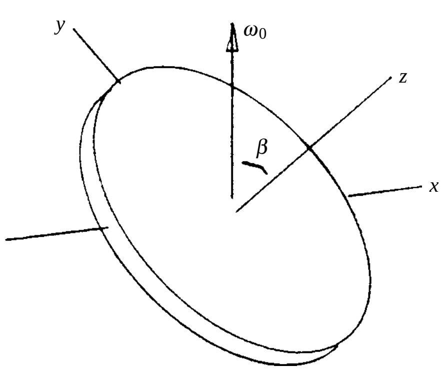
$$+ (R\omega_d^2 \cos\theta + R\omega_O^2 \cos\theta)\mathbf{k}.$$

Problem 20.45 The thin circular disk of radius $R = 0.2$ m and mass $m = 4$ kg is rigidly attached to the vertical shaft. The plane of the disk is slanted at an angle $\beta = 30^\circ$ relative to the horizontal. The shaft rotates with constant angular velocity $\omega_0 = 25$ rad/s. Determine the magnitude of the couple exerted on the disk by the shaft.



Solution: In terms of the body-fixed reference frame shown, the disk's inertia matrix is

$$[I] = \begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix} = \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \text{ kg-m}^2.$$



The disk's angular velocity is

$$\boldsymbol{\omega} = \boldsymbol{\Omega} = \omega_0 \sin\beta \mathbf{j} + \omega_0 \cos\beta \mathbf{k}$$

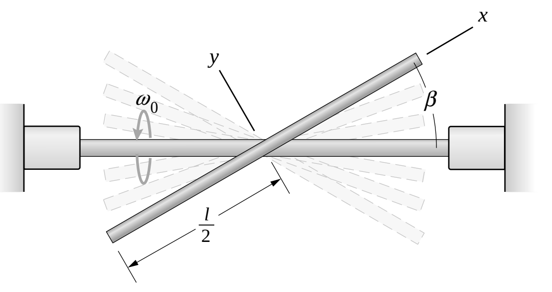
$$= 12.50\mathbf{j} + 21.65\mathbf{k} \text{ (rad/s)}.$$

From Eq. (20.19) with $d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0$,

$$\begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} = \begin{bmatrix} 0 & -21.65 & 12.5 \\ 21.65 & 0 & 0 \\ -12.5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}$$

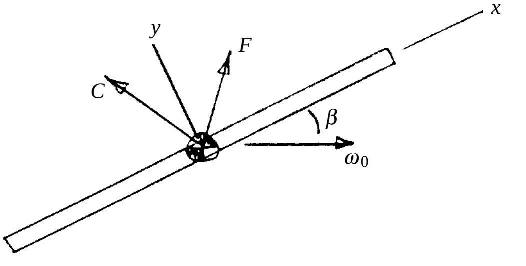
$$\times \begin{bmatrix} 0 \\ 12.5 \\ 21.65 \end{bmatrix} = \begin{bmatrix} 10.8 \\ 0 \\ 0 \end{bmatrix} \text{ N-m}.$$

Problem 20.46 The slender bar of mass $m = 8$ kg and length $l = 1.2$ m is welded to a horizontal shaft that rotates with constant angular velocity $\omega_0 = 25$ rad/s. The angle $\beta = 30^\circ$. Determine the magnitudes of the force **F** and couple **C** exerted on the bar by the shaft. (Write the equations of angular motion in terms of the body-fixed coordinate system shown.)



Solution: In terms of the body-fixed reference frame shown, the inertia matrix is

$$[I] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.96 \end{bmatrix} \text{ kg-m}^2.$$



The bar's angular velocity is

$$\boldsymbol{\omega} = \omega_0 \cos\beta \mathbf{i} - \omega_0 \sin\beta \mathbf{j}$$

$$= 21.65\mathbf{i} - 12.50\mathbf{j} \text{ (rad/s)}.$$

The acceleration of the center of mass is zero, so the force **F** must be equal and opposite to the force exerted by the bar's weight. Therefore $|\mathbf{F}| = mg = 78.5$ N. From Eq. (20.19) with $d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0$,

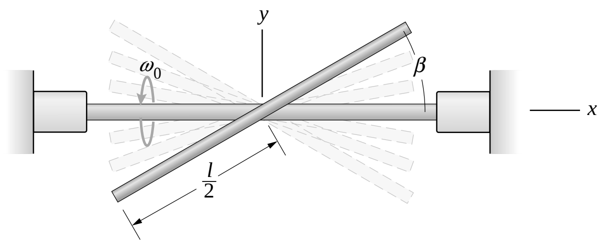
$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -12.5 \\ 0 & 0 & -21.65 \\ 12.5 & 21.65 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.96 \end{bmatrix} \begin{bmatrix} 21.65 \\ -12.5 \\ 0 \end{bmatrix}$$

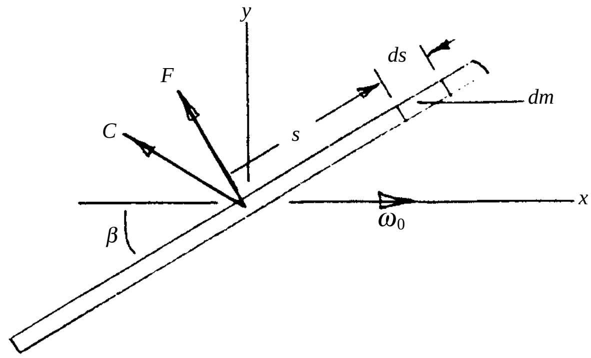
$$= \begin{bmatrix} 0 \\ 0 \\ -260 \end{bmatrix} \text{ N-m}$$

We see that $|\mathbf{C}| = 260$ N-m.

Problem 20.47 The slender bar of mass $m = 8$ kg and length $l = 1.2$ m is welded to a horizontal shaft that rotates with constant angular velocity $\omega_0 = 25$ rad/s. The angle $\beta = 30^\circ$. Determine the magnitudes of the force **F** and couple **C** exerted on the bar by the shaft. (Write the equations of angular motion in terms of the body-fixed coordinate system shown. See Problem 20.98.)



Solution: Let ρ be the bar's density and A its cross-sectional area. The mass dm is $dm = \rho A\, ds$. The bar's moment of inertia about the x axis is



$$I_{xx} = \int_m y^2\, dm = \int_{-l/2}^{l/2} (s \sin\beta)^2 \rho A\, ds$$

$$= \rho A \sin^2\beta \left[\frac{5^3}{3}\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{1}{12}\rho A\, l^3 \sin^2\beta$$

$$= \frac{1}{12} m l^2 \sin^2\beta.$$

The moment of inertia about the y axis is

$$I_{yy} = \int_m x^2\, dm = \int_{-l/2}^{l/2} (s\cos\beta)^2 \rho A\, ds$$

$$= \frac{1}{12} m l^2 \cos^2\beta,$$

and the product of inertia I_{xy} is

$$I_{xy} = \int_m xy\, dm = \int_{-l/2}^{l/2} (s^2 \sin\beta\cos\beta)\rho A\, ds$$

$$= \frac{1}{12} m l^2 \sin\beta\cos\beta.$$

The inertia matrix is

$$[I] = \frac{1}{12} m l^2 \begin{bmatrix} \sin^2\beta & -\sin\beta\cos\beta & 0 \\ -\sin\beta\cos\beta & \cos^2\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.240 & -0.416 & 0 \\ -0.416 & 0.720 & 0 \\ 0 & 0 & 0.960 \end{bmatrix} \text{kg-m}^2.$$

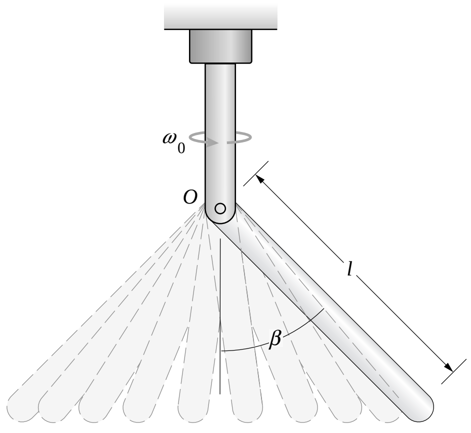
The bar's angular velocity is $\boldsymbol{\omega} = \omega_0 \mathbf{i}$. The acceleration of the center of mass is zero, so the force **F** must be equal and opposite to the force exerted by the bar's weight. Therefore $|\mathbf{F}| = mg = 78.5$ N. From Eq. (20.19) with $d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0$,

$$\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -25 \\ 0 & 25 & 0 \end{bmatrix} \begin{bmatrix} 0.240 & -0.416 & 0 \\ -0.416 & 0.720 & 0 \\ 0 & 0 & 0.960 \end{bmatrix} \begin{bmatrix} 25 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -260 \end{bmatrix} \text{N-m}.$$

We see that $|\mathbf{C}| = 260$ N-m.

Problem 20.48 The slender bar of length l and mass m is pinned to the vertical shaft at O. The vertical shaft rotates with a constant angular velocity ω_0. Show that the value of ω_0 necessary for the bar to remain at a constant angle β relative to the vertical is $\omega_0 = \sqrt{3g/2l\cos\beta}$.



Solution: This is motion about a fixed point so Eq. (20.13) is applicable. Choose a body-fixed x, y, z coordinate system with the origin at O, the positive x axis parallel to the slender bar, and z axis out of the page. The angular velocity of the vertical shaft is

$$\mathbf{\Omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta).$$

The vector from O to the center of mass of the bar is $\mathbf{r}_{G/O} = (L/2)\mathbf{i}$. The weight is

$$\mathbf{W} = mg(\mathbf{i}\cos\beta - \mathbf{j}\sin\beta).$$

The moment about the point O is

$$\mathbf{M}_G = \mathbf{r}_{G/O} \times \mathbf{W} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{L}{2} & 0 & 0 \\ mg\cos\beta & -mg\sin\beta & 0 \end{bmatrix}$$

$$= \left(-\frac{mgL}{2}\sin\beta\right)\mathbf{k}$$

The moments and products of inertia about O in the x, y, z system are

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = mL^2/3,$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The body-fixed coordinate system rotates with angular velocity

$$\mathbf{\Omega} = \boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta).$$

Eq. (20.13) reduces to

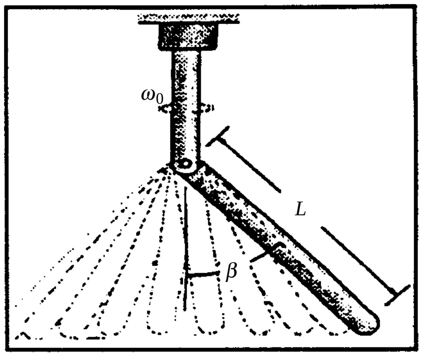
$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0\sin\beta \\ 0 & 0 & \omega_0\cos\beta \\ -\omega_0\sin\beta & -\omega_0\cos\beta & 0 \end{bmatrix}$$

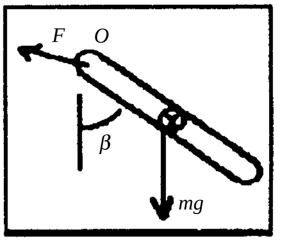
$$\times \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{mL^2}{3} & 0 \\ 0 & 0 & \dfrac{mL^2}{3} \end{bmatrix} \begin{bmatrix} -\omega_0\cos\beta \\ \omega_0\sin\beta \\ 0 \end{bmatrix}.$$

Carry out the matrix multiplication,

$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dfrac{\omega_0 mL^2\sin\beta}{3} \\ 0 & 0 & \dfrac{\omega_0 mL^2\cos\beta}{3} \\ 0 & -\dfrac{\omega_0 mL^2\cos\beta}{3} & 0 \end{bmatrix} \begin{bmatrix} -\omega_0\cos\beta \\ \omega_0\sin\beta \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\dfrac{\omega_0^2 mL^2\cos\beta\sin\beta}{3} \end{bmatrix}.$$





Equate the z components:

$$M_{Oz} = -\frac{\omega_0^2 mL^2\cos\beta\sin\beta}{3}.$$

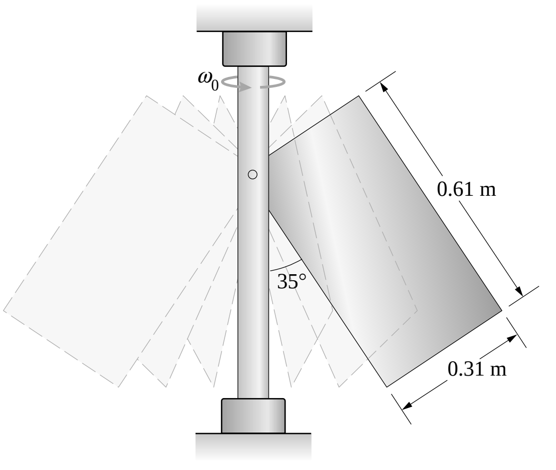
The pin-supported joint at O cannot support a couple, $\mathbf{C} = 0$, from which

$$\mathbf{M}_O = \mathbf{M}_G. - \frac{mgL}{2}\sin\beta = -\frac{\omega_0^2 mL^2\cos\beta\sin\beta}{3}$$

Assume that $\beta \neq 0$, from which $\sin\beta \neq 0$, and the equation can be solved for

$$\boxed{\omega_0 = \sqrt{\frac{3g}{2l\cos\beta}}}$$

Problem 20.49 The vertical shaft rotates with constant angular velocity ω_0. The 35° angle between the edge of the 44.5 N thin rectangular plate pinned to the shaft and the shaft remains constant. Determine ω_0.



Solution: This is motion about a fixed point, and Eq. (20.13) is applicable. Choose an x, y, z coordinate system with the origin at the pinned joint O and the x axis parallel to the lower edge of the plate, and the y axis parallel to the upper narrow edge of the plate. Denote $\beta = 35^\circ$. The plate rotates with angular velocity

$$\boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta) = \omega_0(-0.8192\mathbf{i} + 0.5736\mathbf{j}) \text{ (rad/s)}.$$

The vector from the pin joint to the center of mass of the plate is

$$\mathbf{r}_{G/O} = 0.31\mathbf{i} + (0.152)\mathbf{j} \text{ (m)}.$$

The weight of the plate is

$$\mathbf{W} = 44.5\,(\mathbf{i}\cos\beta - \mathbf{j}\sin\beta)$$

$$= 36.4\mathbf{i} - 25.5\mathbf{j} \text{ (N)}.$$

The moment about the center of mass is

$$\mathbf{M}_G = \mathbf{r}_{G/O} \times \mathbf{W} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.31 & 0.152 & 0 \\ 36.4 & -25.5 & 0 \end{bmatrix}$$

$$= -13.33 \text{ N-m}.$$

From Appendix C, the moments and products of inertia of a thin plate about O are

$$I_{xx} = \frac{mh^2}{3} = 0.14 \text{ kg-m}^2,$$

$$I_{yy} = \frac{mb^2}{3} = 0.56 \text{ kg-m}^2,$$

$$I_{zz} = \frac{m}{3}(h^2 + b^2) = 0.7 \text{ kg-m}^2,$$

$$I_{xy} = \frac{mbh}{4} = 0.21 \text{ kg-m}^2,$$

$$I_{xz} = I_{yz} = 0.$$

At a constant rate of rotation, the angle $\beta = 35^\circ = \text{const}$, $\boldsymbol{\alpha} = 0$. The body-fixed coordinate system rotates with angular velocity

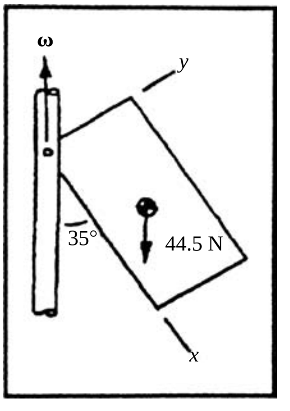
$$\boldsymbol{\Omega} = \boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta)$$

$$= -3.64\omega_0\mathbf{i} + 2.55\omega_0\mathbf{j} \text{ (rad/s)},$$

and Eq. (20.13) reduces to:

$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \omega_0^2 \begin{bmatrix} 0 & 0 & 2.55 \\ 0 & 0 & 3.64 \\ -2.55 & -3.64 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.14 & -0.21 & 0 \\ -0.21 & 0.56 & 0 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} -3.64 \\ 2.55 \\ 0 \end{bmatrix}.$$



Carry out the matrix operations to obtain:

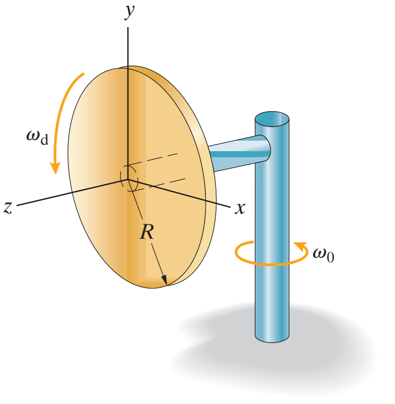
$$\begin{bmatrix} M_{Ox} \\ M_{Oy} \\ M_{Oz} \end{bmatrix} = \omega_0^2 \begin{bmatrix} 0 \\ 0 \\ -0.27 \end{bmatrix}$$

The pin support cannot support a couple, from which

$$M_{Oz} = M_{Gz}, \quad -13.33 = -\omega_0^2(0.27),$$

from which $\boxed{\omega_0 = 7.025 \text{ rad/s}}$

Problem 20.50 The radius of the 100 N thin circular disk is $R = 0.5$ m. The disk is mounted on the horizontal shaft and rotates with constant angular velocity $\omega_d = 10$ rad/s relative to the shaft. The horizontal shaft is 1 m in length. The vertical shaft rotates with constant angular velocity $\omega_0 = 4$ rad/s. Determine the force and couple exerted at the center of the disk by the horizontal shaft.



Solution: Using Newton's Second Law

$$\mathbf{F} - mg\mathbf{j} = m\mathbf{a} = -mr\omega_0{}^2\mathbf{k}$$

$$\mathbf{F} = (100 \text{ N})\mathbf{j} - \left(\frac{100 \text{ N}}{9.81 \text{ m/s}^2}\right)(1 \text{ m})(4 \text{ rad/s})^2\mathbf{k}$$

$$\boxed{\mathbf{F} = (100\mathbf{j} - 163\mathbf{k}) \text{ N.}}$$

In preparation to use Euler's Equations we have

$$\boldsymbol{\omega} = \omega_0\mathbf{j} + \omega_d\mathbf{k} = (4\mathbf{j} + 10\mathbf{k}) \text{ rad/s}$$

$$\boldsymbol{\alpha} = \omega_0\mathbf{j} \times \omega_d\mathbf{k} = (40 \text{ rad/s}^2)\mathbf{i}$$

$$[I] = \begin{bmatrix} \frac{1}{4}mR^2 & 0 & 0 \\ 0 & \frac{1}{4}mR^2 & 0 \\ 0 & 0 & \frac{1}{2}mR^2 \end{bmatrix} = \begin{bmatrix} 0.637 & 0 & 0 \\ 0 & 0.637 & 0 \\ 0 & 0 & 1.274 \end{bmatrix} \text{ kg-m}^2$$

Euler's Equations are now

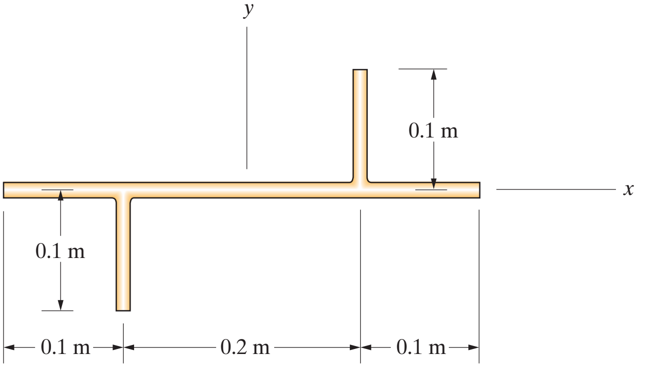
$$\mathbf{M} = [I]\boldsymbol{\alpha} + \boldsymbol{\omega} \times [I]\boldsymbol{\omega}$$

$$\mathbf{M} = \begin{bmatrix} 0.637 & 0 & 0 \\ 0 & 0.637 & 0 \\ 0 & 0 & 1.274 \end{bmatrix} \begin{Bmatrix} 40 \\ 0 \\ 0 \end{Bmatrix}$$

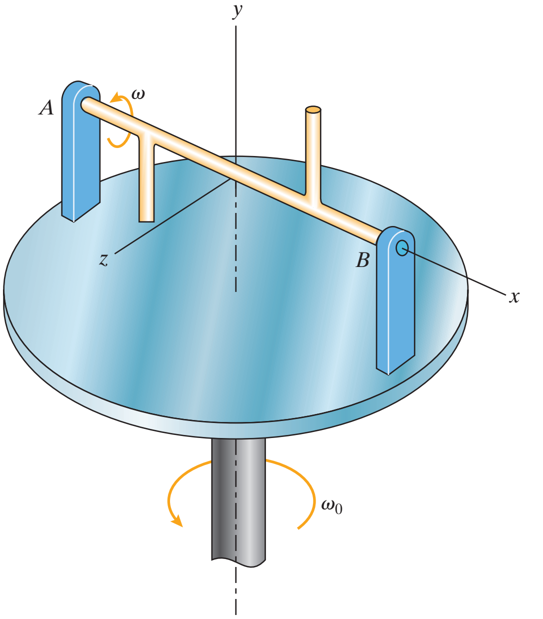
$$+ \begin{bmatrix} 0 & -10 & 4 \\ 10 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.637 & 0 & 0 \\ 0 & 0.637 & 0 \\ 0 & 0 & 1.274 \end{bmatrix} \begin{Bmatrix} 0 \\ 4 \\ 10 \end{Bmatrix}$$

$$\boxed{\mathbf{M} = 25.5\mathbf{i} \text{ N-m.}}$$

Problem 20.51 The object shown in Fig. a consists of two 1-kg vertical slender bars welded to the 4-kg horizontal slender bar. In Fig. b, the object is supported by bearings at A and B. The horizontal circular disk is supported by a vertical shaft that rotates with constant angular velocity $\omega_0 = 6$ rad/s. The horizontal bar rotates with constant angular velocity $\omega = 10$ rad/s. At the instant shown, determine the y and z components of the forces exerted on the object at A and B.



(a)



(b)

Solution: The nonzero inertias are

$$I_{xx} = 2\frac{1}{3}(1 \text{ kg})(0.1 \text{ m})^2 = 0.00667 \text{ kg-m}^2,$$

$$I_{yy} = \frac{1}{12}(4 \text{ kg})(0.4 \text{ m})^2 + 2(1 \text{ kg})(0.1 \text{ m})^2 = 0.0733 \text{ kg-m}^2,$$

$$I_{zz} = I_{xx} + I_{yy} = 0.08 \text{ kg-m}^2,$$

$$I_{xy} = 2(1 \text{ kg})(0.1 \text{ m})(0.05 \text{ m}) = 0.01 \text{ kg-m}^2.$$

Newton's Second Law gives (the acceleration of the center of mass is zero).

$$\Sigma F_y : A_y + B_y - (6 \text{ kg})(9.81 \text{ m/s}^2) = 0,$$

$$\Sigma F_z : A_z + B_z = 0.$$

The angular velocity and angular acceleration are

$$\boldsymbol{\omega} = (10\mathbf{i} + 6\mathbf{j}) \text{ rad/s}$$

$$\boldsymbol{\alpha} = 6\mathbf{j} \times 10\mathbf{i} = -60\mathbf{k} \text{ rad/s}^2.$$

The moment about the center of mass is

$$M_x = 0,$$

$$M_y = B_z(0.2 \text{ m}) - A_z(0.2 \text{ m}),$$

$$M_z = B_y(0.2 \text{ m}) - A_y(0.2 \text{ m}).$$

Euler's equations are now

$$\begin{Bmatrix} 0 \\ B_z - A_z \\ B_y - A_y \end{Bmatrix}(0.2)$$

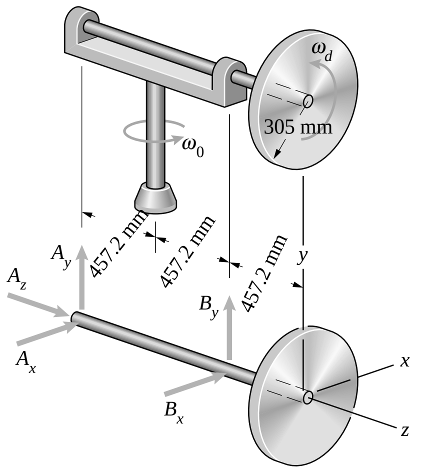
$$= \begin{bmatrix} 0.00667 & -0.01 & 0 \\ -0.01 & 0.0733 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -60 \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & 10 \\ 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} 0.00667 & -0.01 & 0 \\ -0.01 & 0.0733 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \begin{Bmatrix} 10 \\ 6 \\ 0 \end{Bmatrix}$$

Solving Euler's equations along with Newton's Second Law we find

$$\boxed{A_y = 33.0 \text{ N}, \quad A_z = 0, \quad B_y = 25.8 \text{ N}, \quad B_z = 0.}$$

Problem 20.52 The 44.5 N thin circular disk is rigidly attached to the 53.4 N slender horizontal shaft. The disk and horizontal shaft rotate about the axis of the shaft with constant angular velocity $\omega_d = 20$ rad/s. The entire assembly rotates about the vertical axis with constant angular velocity $\omega_0 = 4$ rad/s. Determine the components of the force and couple exerted on the horizontal shaft by the disk.



Solution: The shaft is $L = 3(0.457) = 1.37$ m long. The mass of the disk is

$$m_D = \frac{44.5}{9.81} = 4.54 \text{ kg}.$$

The reaction of the shaft to the disk: The moments and products of inertia of the disk are:

$$I_{xx} = I_{yy} = \frac{m_D R^2}{4} = 0.105 \text{ kg-m}^2,$$

$$I_{zz} = \frac{m_D R^2}{2} = 0.211 \text{ kg-m}^2$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The rotation rate is constant,

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = 0.$$

The body-fixed coordinate system rotates with angular velocity $\boldsymbol{\Omega} = \omega_0 \mathbf{j}$ (rad/s), and $\boldsymbol{\omega} = \omega_0 \mathbf{j} + \omega_d \mathbf{k}$ (rad/s). Eq. (20.19) reduces to:

$$\begin{bmatrix} M_{dx} \\ M_{dy} \\ M_{dz} \end{bmatrix} = \omega_d \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_0 \\ \omega_d \end{bmatrix}$$

$$= \omega_0 \omega_d \begin{bmatrix} I_{zz} \\ 0 \\ 0 \end{bmatrix}.$$

The total moment exerted by the disk is

$$\mathbf{M}_d = \omega_0 \omega_d \frac{m_d R^2}{2} \mathbf{i} = 16.85\mathbf{i} \text{ (N-m)}.$$

The reaction on the shaft by the disk is

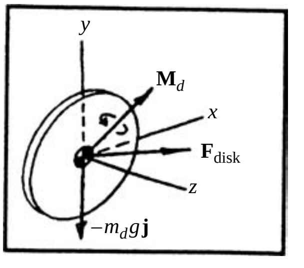
$$\boxed{\mathbf{M}_s = -\mathbf{M}_d = -16.85\mathbf{i} \text{ N-m}}$$

The reaction of the shaft to the acceleration of the disk: The attachment point of the column to the shaft has the coordinates $(0, 0, -0.91)$ m, from which the vector distance from the attachment point to the disk is $\mathbf{r}_{D/P} = 0.91\mathbf{k}$ m. The acceleration of the disk is

$$\mathbf{a}_D = \mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r}_{D/P} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/D}) = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{P/D})$$

$$\mathbf{a}_D = \boldsymbol{\Omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_0 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_0 & \\ 0.91\omega_0 & 0 & 0 \end{bmatrix}$$

$$= -0.91\omega_0^2 \mathbf{k} = -14.6\mathbf{k} \text{ (m/s}^2\text{)}.$$



From Newton's second law,

$$m_d \mathbf{a}_D = \mathbf{F}_{\text{disk}} + \mathbf{W} = \mathbf{F}_{\text{disk}} - W_d \mathbf{j},$$

from which the external force on the disk is:

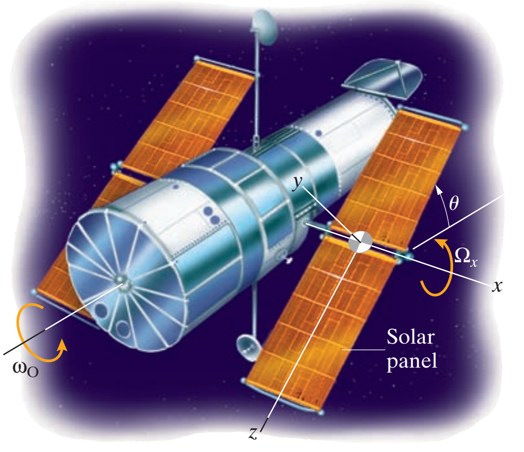
$$\mathbf{F}_{\text{disk}} = m_d g\mathbf{j} - 0.91 m_d \omega_0^2 \mathbf{k} = 44.5\mathbf{j} - 66.4\mathbf{k} \text{ (N)}.$$

The external force on the shaft is

$$\boxed{\mathbf{F}_{\text{shaft}} = -\mathbf{F}_{\text{disk}} = -44.5\mathbf{j} + 66.4\mathbf{k} \text{ (N)}}$$

Problem 20.53 The Hubble telescope is rotating about its longitudinal axis with constant angular velocity ω_0. The coordinate system is fixed with respect to the solar panel. Relative to the telescope, the solar panel rotates about the x axis with constant angular velocity ω_x. Assume that the moments of inertia I_{xx}, I_{yy}, and I_{zz} are known, and $I_{xy} = I_{yz} = I_{zx} = 0$. Show that the moment about the x axis the servomechanisms must exert on the solar panel is

$$\Sigma M_x = (I_{zz} - I_{yy})\omega_0^2 \sin\theta \cos\theta.$$



Solution: We have

$$\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_0(\sin\theta \mathbf{j} + \cos\theta \mathbf{k})$$

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = 0\mathbf{i} + \left(\omega_0 \cos\theta \frac{d\theta}{dt}\right)\mathbf{j} - \left(\omega_0 \sin\theta \frac{d\theta}{dt}\right)\mathbf{k}$$

$$\boldsymbol{\alpha} = \omega_0\omega_x \cos\theta \mathbf{j} - \omega_0\omega_x \sin\theta \mathbf{k}$$

Thus

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_0\omega_x \cos\theta \\ -\omega_0\omega_x \sin\theta \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_0 \cos\theta & \omega_0 \sin\theta \\ \omega_0 \cos\theta & 0 & \omega_x \\ -\omega_0 \sin\theta & -\omega_x & 0 \end{bmatrix}$$

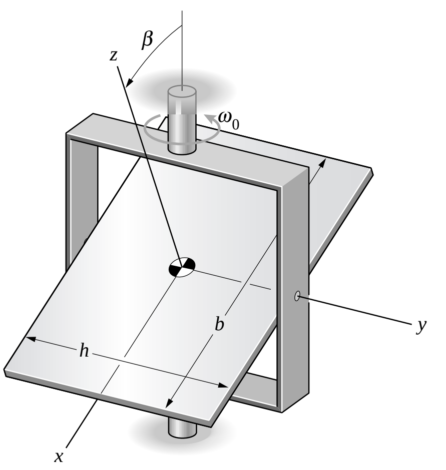
$$\times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_0 \sin\theta \\ \omega_0 \cos\theta \end{Bmatrix}$$

Solving we find

$$\boxed{M_x = (I_{zz} - I_{yy})\omega_0{}^2 \sin\theta \cos\theta}$$

Problem 20.54 The thin rectangular plate is attached to the rectangular frame by pins. The frame rotates with constant angular velocity ω_0. Show that

$$\frac{d^2\beta}{dt^2} = -\omega_0^2 \sin\beta \cos\beta.$$



Solution: Assume that the only external moment applied to the object is the moment required to maintain a constant rotation ω_0 about the axis of rotation. Denote this moment by $\mathbf{M}_0$. In the x, y, z system $\mathbf{M}_0 = M_0(-\mathbf{i}\sin\beta + \mathbf{k}\cos\beta)$, from which

$$M_x = -M_0 \sin\beta,$$

$$M_y = 0,$$

$$M_z = M_0 \cos\beta.$$

From Appendix C, in the x, y, z system the moments and products of inertia of the plate are

$$I_{xx} = \frac{mh^2}{12},$$

$$I_{yy} = \frac{mb^2}{12},$$

$$I_{zz} = \frac{m}{12}(h^2 + b^2),$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The plate is attached to the frame by pins, so the assumption is that the plate is free to rotate about the y-axis. The body-fixed coordinate system rotates with angular velocity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} = -\mathbf{i}\omega_0 \sin\beta + \mathbf{j}\left(\frac{d\beta}{dt}\right) + \mathbf{k}\omega_0\cos\beta \text{ (rad/s)},$$

where $\frac{d\beta}{dt}$ is the angular velocity about the y-axis. For $\omega_0 = \text{const.}$ *for all time*, the derivative

$$\frac{d\omega_0}{dt} = 0,$$

and the acceleration is

$$\boldsymbol{\alpha} = -\mathbf{i}\omega_0\cos\beta\left(\frac{d\beta}{dt}\right) + \mathbf{j}\left(\frac{d^2\beta}{dt^2}\right) - \mathbf{k}\omega_0\sin\beta\left(\frac{d\beta}{dt}\right).$$

Eq. (20.19) becomes

$$\begin{bmatrix} -M_0\sin\beta \\ 0 \\ M_0\cos\beta \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_0\cos\beta & \frac{d\beta}{dt} \\ \omega_0\cos\beta & 0 & \omega_0\sin\beta \\ -\frac{d\beta}{dt} & -\omega_0\sin\beta & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} -\omega_0\sin\beta \\ \frac{d\beta}{dt} \\ \omega_0\cos\beta \end{bmatrix}$$

$$\begin{bmatrix} -M_0\sin\beta \\ 0 \\ M_0\cos\beta \end{bmatrix} = \begin{bmatrix} I_{xx}\alpha_x \\ I_{yy}\alpha_y \\ I_{zz}\alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_0 I_{yy}\cos\beta & I_{zz}\frac{d\beta}{dt} \\ \omega_0 I_{xx}\cos\beta & 0 & \omega_0 I_{zz}\sin\beta \\ -I_{xx}\frac{d\beta}{dt} & -\omega_0 I_{yy}\sin\beta & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} -\omega_0\sin\beta \\ \frac{d\beta}{dt} \\ \omega_0\cos\beta \end{bmatrix},$$

$$\begin{bmatrix} -M_0\sin\beta \\ 0 \\ M_0\cos\beta \end{bmatrix} = \begin{bmatrix} -\omega_0 I_{xx}\cos\beta\left(\frac{d\beta}{dt}\right) \\ I_{yy}\left(\frac{d\beta}{dt}\right) \\ -\omega_0 I_{zz}\sin\beta\left(\frac{d\beta}{dt}\right) \end{bmatrix}$$

$$+ \begin{bmatrix} \omega_0 I_{xx}\cos\beta\left(\frac{d\beta}{dt}\right) \\ \omega_0^2 I_{yy}\cos\beta\sin\beta \\ \omega_0(I_{xx} - I_{yy})\sin\beta\left(\frac{d\beta}{dt}\right) \end{bmatrix}$$

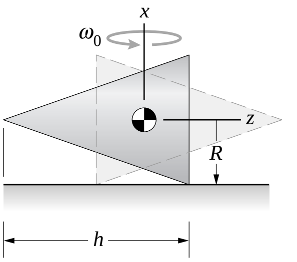
$$= \begin{bmatrix} 0 \\ I_{yy}\alpha_y + \omega_0^2 I_{yy}\cos\beta\sin\beta \\ -2\omega_0 I_{yy}\sin\beta\left(\frac{d\beta}{dt}\right) \end{bmatrix}$$

where $I_{zz} = I_{xx} + I_{yy}$ has been used. The y-component is

$$I_{yy}\alpha_y + \omega_0^2 I_{yy}\cos\beta\sin\beta = 0,$$

from which $\boxed{\frac{d^2\beta}{dt^2} = -\omega_0^2\cos\beta\sin\beta}$.

Problem 20.55* The axis of the right circular cone of mass m, height h, and radius R spins about the vertical axis with constant angular velocity ω_0. The center of mass of the cone is stationary, and its base rolls on the floor. Show that the angular velocity necessary for this motion is $\omega_0 = \sqrt{10g/3R}$. (See Example 20.6.)



Strategy: Let the z axis remain aligned with the axis of the cone and the x remain vertical.

Solution: This a problem of general motion, and Eq. (20.19) applies. The vector distance from the center of mass to the base of the cone is

$$\mathbf{r}_{B/G} = \frac{h}{4}\mathbf{k}$$

(see Appendix C). The angular velocity of rotation of the body fixed coordinate system is $\boldsymbol{\Omega} = \omega_0\mathbf{i}$. The velocity of the center of the base is

$$\mathbf{v}_B = \boldsymbol{\Omega} \times \mathbf{r}_{B/G} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_0 & 0 & 0 \\ 0 & 0 & \frac{h}{4} \end{bmatrix} = -\frac{\omega_0 h}{4}\mathbf{j}.$$

Let the spin rate about the z axis be $\dot{\phi}$, so that the angular velocity, from which $\boldsymbol{\omega} = \boldsymbol{\Omega} + \dot{\phi}\mathbf{k}$. The point of contact with the surface is stationary, and the velocity of the center of the base of the cone is

$$\mathbf{v} = -\frac{\omega_0 h}{4}\mathbf{j},$$

from which $0 = \mathbf{v} + \boldsymbol{\omega} \times (-R\mathbf{i}) = \left(-\frac{\omega_0 h}{4} - R\dot{\phi}\right)\mathbf{j} = 0,$

from which $\dot{\phi} = -\dfrac{\omega_0 h}{4R}.$

The center of mass of the cone is at a zero distance from the axis of rotation, from which the acceleration of the center of mass is zero. The angular velocity about the z-axis,

$$\boldsymbol{\omega} = \omega_0\mathbf{i} - \frac{\omega_0 h}{4R}\mathbf{k} \text{ (rad/s)}.$$

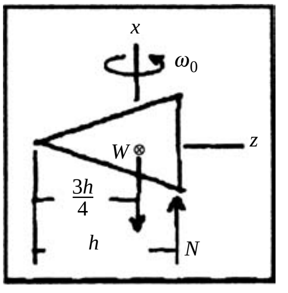
The weight of the cone is $\mathbf{W} = -mg\mathbf{i}$. The reaction of the floor on the cone is $\mathbf{N} = -\mathbf{W}$. The moment about the center of mass exerted by the weight is

$$\mathbf{M}_G = \mathbf{r}_{B/G} \times \mathbf{N} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \frac{h}{4} \\ mg & 0 & 0 \end{bmatrix} = +\left(\frac{mgh}{4}\right)\mathbf{j}.$$

The moments and products of inertia of a cone about its center of mass in the x, y, z system are, from Appendix C,

$$I_{xx} = I_{yy} = m\left(\frac{3}{80}h^2 + \frac{3}{20}R^2\right),$$

$$I_{zz} = \frac{3mR^2}{10},\ I_{xy} = I_{xz} = I_{yz} = 0.$$



Since the rotation rate is constant, and the z axis remains horizontal, the angular acceleration is zero,

$$\frac{d\boldsymbol{\omega}_0}{dt} = 0.$$

The body-fixed coordinate system rotates with angular velocity $\boldsymbol{\Omega} = \omega_0\mathbf{i}$, and

$$\boldsymbol{\omega} = \omega_0\mathbf{i} - \frac{h\omega_0}{4R}\mathbf{k}.$$

Eq. (9.26) becomes:

$$\begin{bmatrix} M_{Gx} \\ M_{Gy} \\ M_{Gz} \end{bmatrix} = \omega_0^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -\frac{h}{4R} \end{bmatrix}$$

$$= \omega_0^2 \begin{bmatrix} 0 \\ \frac{hI_{zz}}{4R} \\ 0 \end{bmatrix}.$$

For equilibrium, $M_{Oy} = M_{Gy}$, from which

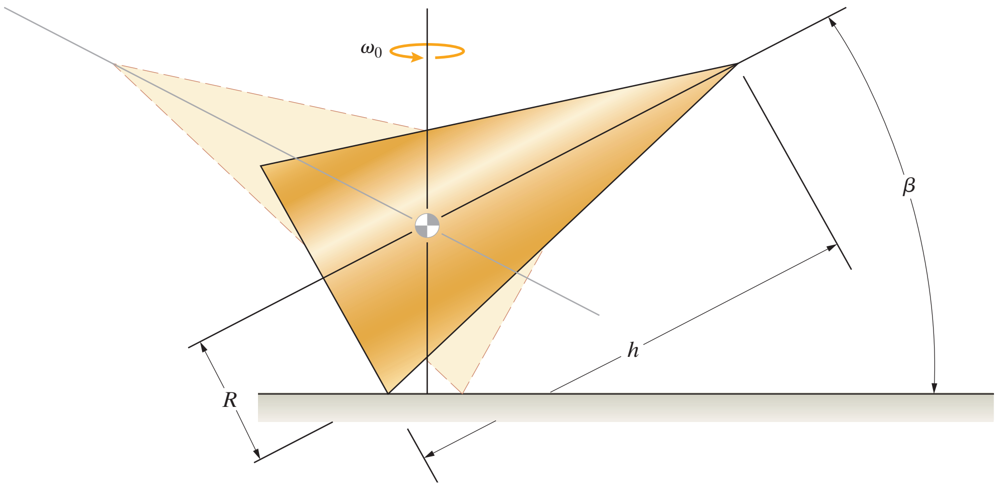
$$\frac{mgh}{4} = \frac{3mhR}{40}\omega_0^2.$$

Solve $\boxed{\omega_0 = \sqrt{\dfrac{10\text{ g}}{3R}}}$

Problem 20.56 The titled homogeneous cone undergoes a steady motion in which its flat end rolls on the floor while the center of mass remains stationary. The angle β between the axis and the horizontal remains constant, and the axis rotates about the vertical axis with constant angular velocity ω_0. The cone has mass m, radius R, and height h. Show that the angular velocity ω_0 necessary for this motion satisfies (see Example 20.6)

$$\omega_0{}^2 = \frac{g(R\sin\beta - \frac{1}{4}h\cos\beta)}{\frac{3}{20}(R^2 + \frac{1}{4}h^2)\sin\beta\cos\beta - \frac{3}{40}hR\cos^2\beta}$$

(See Example 20.6.)



Solution: Following Example 20.6, we use a coordinate system with the z axis pointing along the cone axis, the y axis remains horizontal (out of the paper) and the x axis completes the set

$$\mathbf{\Omega} = \omega_0\cos\beta\mathbf{i} + \omega_0\sin\beta\mathbf{k}$$

$$\boldsymbol{\omega} = \mathbf{\Omega} + \omega_{\text{rel}}\mathbf{k} = \omega_0\cos\beta\mathbf{i} + (\omega_0\sin\beta + \omega_{\text{rel}})\mathbf{k}$$

The point P of the cone that is in contact with the ground does not move, therefore

$$\mathbf{v}_P = \mathbf{v}_C + \boldsymbol{\omega}\times\mathbf{r}_{P/C}$$

$$0 = 0 + [\omega_0\cos\beta\mathbf{i} + (\omega_0\sin\beta + \omega_{\text{rel}})\mathbf{k}]\times[-R\mathbf{i} - \tfrac{1}{4}h\mathbf{k}]$$

$$= [\tfrac{1}{4}h\omega_0\cos\beta - R(\omega_0\sin\beta + \omega_{\text{rel}})]\mathbf{j}.$$

Solving yields

$$\omega_{\text{rel}} = \left[\frac{h}{4R}\cos\beta - \sin\beta\right]\omega_0, \quad \boldsymbol{\omega} = \omega_0\cos\beta\mathbf{i} + \frac{h}{4R}\omega_0\cos\beta\mathbf{k}.$$

Since the center of mass is stationary, the floor exerts no horizontal force, and the vertical force is equal to the weight ($N = mg$). The moment about the center of mass due to the normal force is

$$\mathbf{M} = mg(R\sin\beta - \tfrac{1}{4}h\cos\beta)\mathbf{j}$$

The moments and products of inertia for the cone are

$$[I] = \begin{bmatrix} \frac{3}{20}mR^2 + \frac{3}{80}mh^2 & 0 & 0 \\ 0 & \frac{3}{20}mR^2 + \frac{3}{80}mh^2 & 0 \\ 0 & 0 & \frac{3}{10}mR^2 \end{bmatrix}$$

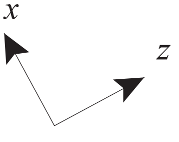
Substituting these expressions into Eq. (20.19), and evaluating the matrix products, we obtain

$$mg(R\sin\beta - \tfrac{1}{4}h\cos\beta) = [(\tfrac{3}{80}h^2 + \tfrac{3}{20}R^2)\cos\beta\sin\beta$$

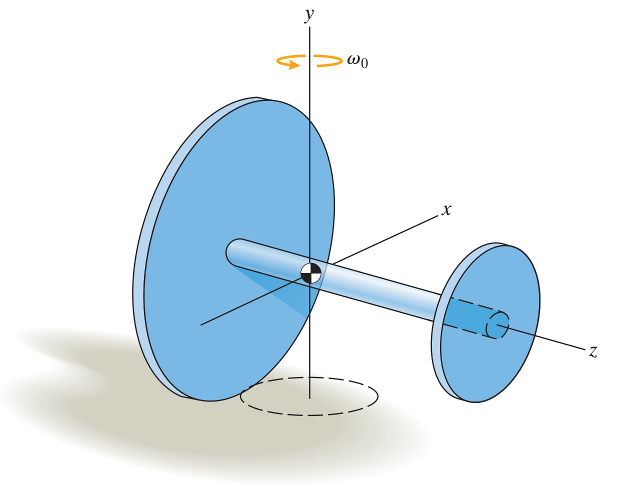
$$- \tfrac{3}{40}hR\cos^2\beta]m\omega_0{}^2$$

Solving, we find that

$$\boxed{\omega_0{}^2 = \frac{g(R\sin\beta - \frac{1}{4}h\cos\beta)}{\frac{3}{20}(R^2 + \frac{1}{4}h^2)\sin\beta\cos\beta - \frac{3}{40}hR\cos^2\beta}.}$$



Problem 20.57 The two thin disks are rigidly connected by a slender bar. The radius of the large disk is 200 mm and its mass is 4 kg. The radius of the small disk is 100 mm and its mass is 1 kg. The bar is 400 mm in length and its mass is negligible. The composite object undergoes a steady motion in which it spins about the vertical y axis through its center of mass with angular velocity ω_0. The bar is horizontal during this motion and the large disk rolls on the floor. What is ω_0?



Solution: The z axis remains aligned with the bar and the y axis remains vertical.

The center of mass (measured from the large disk) is located a distance

$$d = \frac{(1 \text{ kg})(0.4 \text{ m})}{(5 \text{ kg})} = 0.08 \text{ m}$$

The inertias are

$$I_{zz} = \frac{1}{2}(4 \text{ kg})(0.2 \text{ m})^2 + \frac{1}{2}(1 \text{ kg})(0.1 \text{ m})^2 = 0.085 \text{ kg-m}^2$$

$$I_{xx} = I_{yy} = \frac{1}{4}(4 \text{ kg})(0.2 \text{ m})^2 + (4 \text{ kg})(0.08 \text{ m}^2)$$

$$+ \frac{1}{4}(1 \text{ kg})(0.1 \text{ m})^2 + (1 \text{ kg})(0.32 \text{ m})^2 = 0.1705 \text{ kg-m}^2$$

$$I_{xy} = I_{xz} = I_{yz} = 0$$

The angular velocity of the coordinate system is $\mathbf{\Omega} = \omega_0\mathbf{j}$

Define ω_z to be the rate of rotation of the object about the z axis. Thus $\boldsymbol{\omega} = \omega_0\mathbf{j} + \omega_z\mathbf{k}$

To find ω_z, require that the velocity of the point in contact with the floor be zero

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r} = (\omega_0\mathbf{j} + \omega_z\mathbf{k}) \times (-0.08\mathbf{k} - 0.2\mathbf{j}) \text{ m}$$

$$= [(0.2 \text{ m})\omega_z - (0.08 \text{ m})\omega_0]\mathbf{i} = 0 \;\Rightarrow\; \omega_z = 0.4\omega_0$$

Since the center of mass does not move, the normal force on the contact point is equal to the weight. Therefore the moment about the center o mass is given by

$$\mathbf{M} = [(-0.2 \text{ m})\mathbf{j} - (0.08 \text{ m})\mathbf{k}] \times [(5 \text{ kg})(9.81 \text{ m/s}^2)\mathbf{j}] = (3.924 \text{ N})\mathbf{i}$$

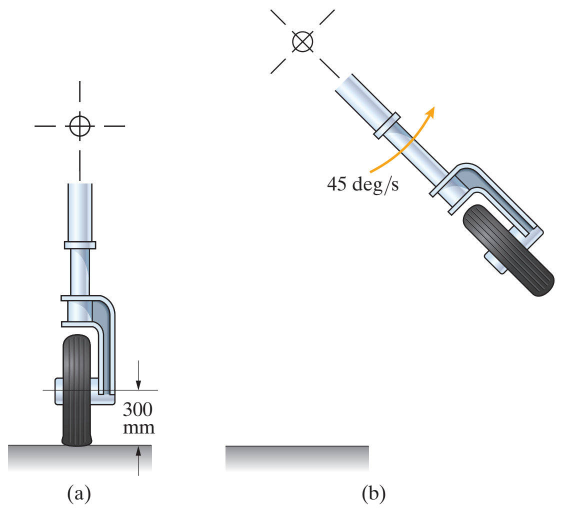
Equation 20.19 now gives $(d\omega_x/dt = d\omega_y/dt = d\omega_z/dt = 0)$

$$\begin{Bmatrix} 3.924 \text{ N} \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0 \\ 0 & 0 & 0 \\ -\omega_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_0 \\ 0.4\omega_0 \end{Bmatrix}$$

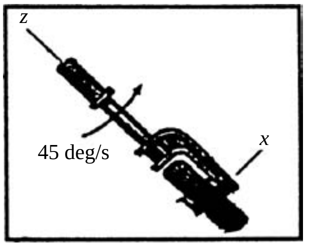
Putting in the values and solving we find

$$\boxed{\omega_0 = 10.7 \text{ rad/s}}$$

Problem 20.58 The view of an airplane's landing gear as seen looking from behind the airplane is shown in Fig. (a). The radius of the wheel is 300 mm, and its moment of inertia is 2 kg-m^2. The airplane takes off at 30 m/s. After takeoff, the landing gear retracts by rotating toward the right side of the airplane, as shown in Fig. (b). Determine the magnitude of the couple exerted by the wheel on its support. (Neglect the airplane's angular motion.)



Solution: Choose a coordinate system with the origin at the center of mass of the wheel and the z axis aligned with the carriage, as shown. Assume that the angular velocities are constant, so that the angular accelerations are zero. The moments and products of inertia of the wheel are $I_{xx} = mR^2/2 = 2$ kg-m^2, from which $m = 44.44$ kg.



$I_{yy} = I_{zz} = mR^2/4 = 1$ kg-m^2.

The angular velocities are

$\mathbf{\Omega} = -(45(\pi/180))\mathbf{j} = -0.7853\mathbf{j}$ rad/s.

$$\boldsymbol{\omega} = -\left(\frac{v}{R}\right)\mathbf{i} + \mathbf{\Omega} = -\left(\frac{30}{0.3}\right)\mathbf{i} - 0.7853\mathbf{j} = -100\mathbf{i} - 0.7853\mathbf{j}.$$

Eq. (20.19) becomes

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & \Omega_y \\ 0 & 0 & 0 \\ -\Omega_y & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ 0 \end{bmatrix}$$

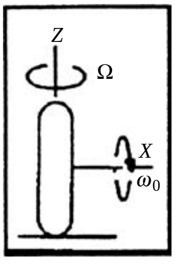
$$= \begin{bmatrix} 0 \\ 0 \\ -\Omega_y \omega_x I_{xx} \end{bmatrix},$$

from which $\mathbf{M}_0 = -\Omega_y \omega_x I_{xx} \mathbf{k}$. Substitute:

$$\boxed{|\mathbf{M}| = (0.7854)(100)2 = 157 \text{ N-m}}$$

Problem 20.59 If the rider turns to his left, will the couple exerted on the motorcycle by its wheels tend to cause the motorcycle to lean toward the rider's left side or his right side?

Solution: Choose a coordinate system as shown in the front view, with y positive into the paper. The Eqs. (20.19) in condensed notation are



$$\sum \mathbf{M} = \frac{d\mathbf{H}}{dt} + \boldsymbol{\Omega} \times \mathbf{H}.$$

For $\dfrac{d\mathbf{H}}{dt} = 0$,

$$\sum \mathbf{M} = \boldsymbol{\Omega} \times \mathbf{H}.$$

If the rider turns to his left, the angular velocity is $\boldsymbol{\Omega} = +\Omega\mathbf{k}$ rad/s. The angular momentum is $\mathbf{H} = \mathrm{H}_x\mathbf{i} + \mathrm{H}_z\mathbf{k}$, where $H_x > 0$. The cross product

$$\boldsymbol{\Omega} \times \mathbf{H} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & +\Omega \\ \mathrm{H}_x & 0 & \mathrm{H}_z \end{bmatrix} = +\Omega \mathrm{H}_x \mathbf{j}.$$

For a left turn the moment about y is positive, causing the cycle *to lean to the left.*

Problem 20.60* By substituting the components of H_O from Eqs. (20.9) into the equation

$$\Sigma \mathbf{M}_O = \frac{dH_{Ox}}{dt}\mathbf{i} + \frac{dH_{Oy}}{dt}\mathbf{j} + \frac{dH_{Oz}}{dt}\mathbf{k} + |\mathbf{\Omega}| \times \mathbf{H}_O$$

derive Eqs. (20.12).

Solution:

$$\sum \mathbf{M}_0 = \frac{dH_{Ox}}{dt}\mathbf{i} + \frac{dH_{Oy}}{dt}\mathbf{j} + \frac{dH_{Oz}}{dt}\mathbf{k}$$

$$+ \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Omega_x & \Omega_y & \Omega_z \\ H_{Ox} & H_{Oy} & H_{Oz} \end{vmatrix}.$$

The components of this equation are

$$\sum M_{Ox} = \frac{dH_{Ox}}{dt} + \Omega_y H_{Oz} - \Omega_z H_{Oy},$$

$$\sum M_{Oy} = \frac{dH_{Oy}}{dt} - \Omega_x H_{Oz} + \Omega_z H_{Ox},$$

$$\sum M_{Oz} = \frac{dH_{Oz}}{dt} + \Omega_x H_{Oy} - \Omega_y H_{Ox}.$$

Substituting Eqs. (20.9) and *assuming that the moments and products of inertia are constants*, we obtain Eqs. (20.12):

$$\sum M_{Ox} = I_{xx}\frac{d\omega_x}{dt} - I_{xy}\frac{d\omega_y}{dt} - I_{xz}\frac{d\omega_z}{dt}$$

$$+ \Omega_y(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)$$

$$- \Omega_z(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z),$$

$$\sum M_{Oy} = -I_{yx}\frac{d\omega_x}{dt} + I_{yy}\frac{d\omega_y}{dt} - I_{yz}\frac{d\omega_z}{dt}$$

$$- \Omega_x(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)$$

$$+ \Omega_z(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z),$$

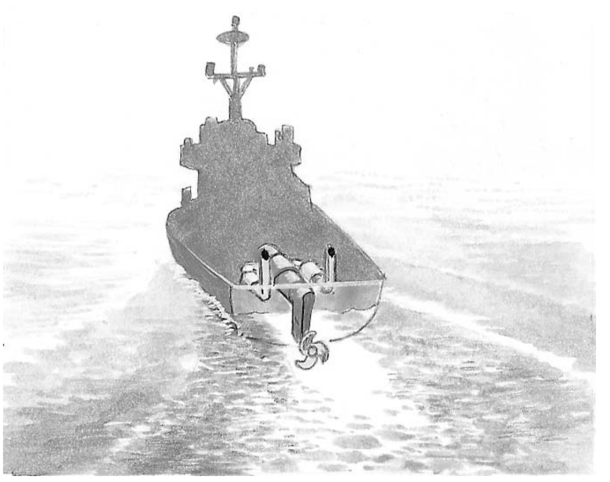
$$\sum M_{Oz} = -I_{zx}\frac{d\omega_x}{dt} - I_{zy}\frac{d\omega_y}{dt} + I_{zz}\frac{d\omega_z}{dt}$$

$$+ \Omega_x(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)$$

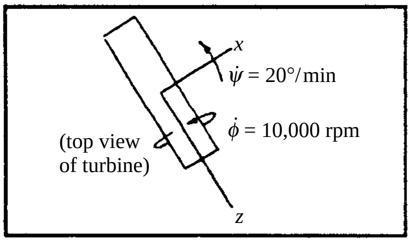
$$- \Omega_y(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z).$$

Problem 20.61 A ship has a turbine engine. The spin axis of the axisymmetric turbine is horizontal and aligned with the ship's longitudinal axis. The turbine rotates at 10,000 rpm. Its moment of inertia about its spin axis is 1000 kg-m^2. If the ship turns at a constant rate of 20 degrees per minute, what is the magnitude of the moment exerted on the ship by the turbine?

Strategy: Treat the turbine's motion as steady precession with nutation angle $\theta = 90^\circ$.



Solution: Choose a coordinate system with the z axis parallel to the axis of the turbine, and y positive upward. From Eq. (20.29),



$$\sum M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta,$$

where $I_{xx} = \frac{1}{2}I_{zz} = 500$ kg-m^2

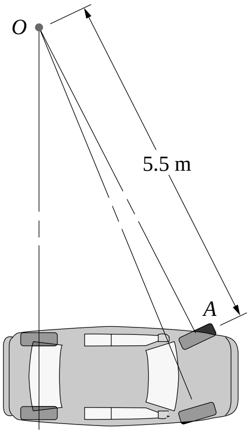
$$\dot{\psi} = 20\left(\frac{\pi}{180}\right)\left(\frac{1}{60}\right) = 0.005818 \text{ (rad/s)},$$

$$\dot{\phi} = 10000(2\pi/60) = 1047.2 \text{ rad/s},$$

$$\theta = 90^\circ.$$

$$M_x = 6092 \text{ N-m}$$

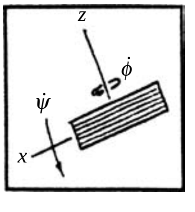
Problem 20.62 The center of the car's wheel A travels in a circular path about O at 24.1 km/h. The wheel's radius is 0.31 m, and the moment of inertia of the wheel about its axis of rotation is 1.08 kg-m^2. What is the magnitude of the total external moment about the wheel's center of mass?



Strategy: Treat the wheel's motion as steady precession with nutation angle $\theta = 90^\circ$.

Solution: From Eq. (20.29)

$$\sum M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta,$$



where the spin is

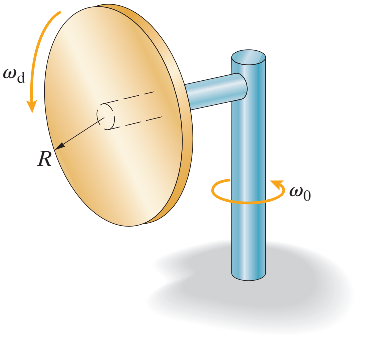
$$\dot{\phi} = \frac{v}{R} = \left(\frac{24.1 \times 1000}{5.5 \times 3600}\right) = 1.222 \text{ rad/s}$$

the precession rate is

$$\dot{\psi} = \frac{v}{R_w} = \frac{6.7}{0.31} = 22 \text{ rad/s},$$

and the nutation angle is $\theta = 90^\circ$. Using $I_{xx} = \frac{1}{2}I_{zz} = 0.542$ kg-m^2, from which $M_x = 29.2$ N-m

Problem 20.63 The radius of the 5-kg disk is $R = 0.2$ m. The disk is pinned to the horizontal shaft and rotates with constant angular velocity $\omega_d = 6$ rad/s relative to the shaft. The vertical shaft rotates with constant angular velocity $\omega_0 = 2$ rad/s. By treating the motion of the disk as steady precession, determine the magnitude of the couple exerted on the disk by the horizontal shaft.



Solution: In Problem 20.50 a thin circular disk of mass m is mounted on a horizontal shaft and rotates relative to the shaft with constant angular velocity ω_d. The horizontal shaft is rigidly attached to the vertical shaft rotating with constant angular velocity ω_0. The magnitude of the couple exerted on the disk by the horizontal shaft is to be determined. The nutation angle is $\theta = 90^\circ$. The precession rate is $\dot{\psi} = \omega_0$, and the spin rate is $\dot{\phi} = \omega_d$. The moments and products of inertia of the disk:

$$I_{xx} = I_{zz} = \frac{mR^2}{2},$$

$$I_{yy} = \frac{mR^2}{4},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

Eq. (20.29) is

$$M_y = (I_{xx} - I_{yy})\dot{\psi}^2 \sin\theta \cos\theta + I_{xx}\dot{\psi}\dot{\phi}\sin\theta,$$

from which

$$M_y = \frac{mR^2}{2}\omega_0\omega_d = 1.2 \text{ N-m}.$$

Problem 20.64 The helicopter is stationary. The z axis of the body-fixed coordinate system points downward and is coincident with the axis of the helicopter's rotor. The moment of inertia of the rotor about the z axis is 8600 kg-m^2. Its angular velocity is $-258\mathbf{k}$ (rpm). If the helicopter begins a pitch maneuver during which its angular velocity is $0.02\mathbf{j}$ (rad/s), what is the magnitude of the gyroscopic moment exerted on the helicopter by the rotor? Does the moment tend to cause the helicopter to roll about the x axis in the clockwise direction (as seen in the photograph) or the counterclockwise direction?

Solution: The spin rate is $\dot{\phi} = -258$ rpm $= -27.0$ rad/s.

The pitch rate is $\dot{\psi} = 0.02$ rad/s.

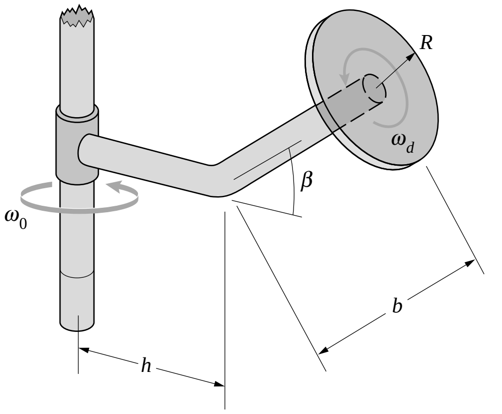
In eq. 20.29, the moment exerted *on the rotor* is

$$M = I_{zz}\dot{\phi}\dot{\psi} = (8600 \text{ kg-m}^2)(-27.0 \text{ rad/s})(0.02 \text{ rad/s}) = -4650 \text{ N-m}$$

The motor exerts a moment on the helicopter in the opposite direction which tends to roll the helicopter in the counterclockwise direction.

Answer: $\boxed{M = 4650 \text{ N-m counterclockwise}}$

Problem 20.65 The bent bar is rigidly attached to the vertical shaft, which rotates with constant angular velocity ω_0. The disk of mass m and radius R is pinned to the bent bar and rotates with constant angular velocity ω_d relative to the bar. Determine the magnitudes of the force and couple exerted on the disk by the bar.



Solution:

(a) The center of mass of the disk moves in a horizontal circular path of radius $h + b\cos\beta$ with angular velocity ω_0. The acceleration normal to the circular path is $a_N = \omega_0^2(h + b\cos\beta)$, so the bar exerts a horizontal force of magnitude $ma_N = m\omega_0^2(h + b\cos\beta)$. The bar also exerts on upward force equal to the weight of the disk, so the magnitude of the total force is

$$\sqrt{(ma_N)^2 + (mg)^2} = m\sqrt{\omega_0^4(h + b\cos\beta)^2 + g^2}.$$

(b) By orienting a coordinate system as shown, with the z axis normal to the disk and the x axis horizontal, the disk is in steady precession with precession rate $\dot{\psi} = \omega_0$, spin rate $\dot{\phi} = \omega_d$, and nutation angle

$$\theta = \frac{\pi}{2} - \beta.$$

The plate's moments of inertia are

$$I_{xx} = I_{yy} = \tfrac{1}{4}mR^2,$$

$$I_{zz} = \frac{1}{2}mR^2.$$

From Equation (20.29), the magnitude of the moment is

$$(I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\phi}\dot{\psi}\sin\theta$$

$$= \frac{1}{4}mR^2\omega_0^2 \sin\left(\frac{\pi}{2} - \beta\right)\cos\left(\frac{\pi}{2} - \beta\right)$$

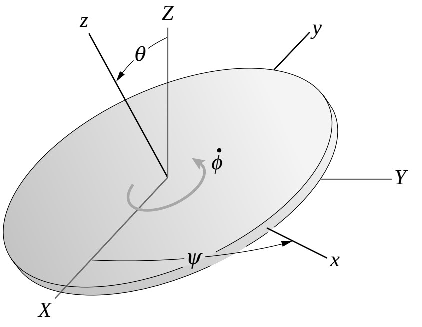
$$+ \frac{1}{2}mR^2\omega_d\omega_0 \sin\left(\frac{\pi}{2} - \beta\right)$$

$$= R^2\omega_0 m\left(\frac{1}{4}\omega_0\cos\beta\sin\beta + \frac{1}{2}\omega_d\cos\beta\right)$$

Problem 20.66 The bent bar is rigidly attached to the vertical shaft, which rotates with constant angular velocity ω_0. The disk of mass m and radius R is pinned to the bent bar and rotates with constant angular velocity ω_d relative to the bar. Determine the value of ω_d for which no couple is exerted on the disk by the bar.

Solution: From the result for the magnitude of the moment in the solution of Problem 20.65 the moment equals zero if $\frac{1}{4}\omega_0\sin\beta + \frac{1}{2}\omega_d = 0$, so $\omega_d = -\frac{1}{2}\omega_0\sin\beta$.

Problem 20.67 A thin circular disk undergoes moment-free steady precession. The z axis is perpendicular to the disk. Show that the disk's precession rate is $\dot{\psi} = -2\dot{\phi}/\cos\theta$. (Notice that when the nutation angle is small, the precession rate is approximately two times the spin rate.)



Solution: Moment free steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\dot{\phi}$ is the spin rate, and θ is the nutation angle. For a thin circular disk, the moments and products of inertia are

$$I_{xx} = I_{yy} = \frac{mR^2}{4},$$

$$I_{zz} = \frac{mR^2}{2},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

Substitute:

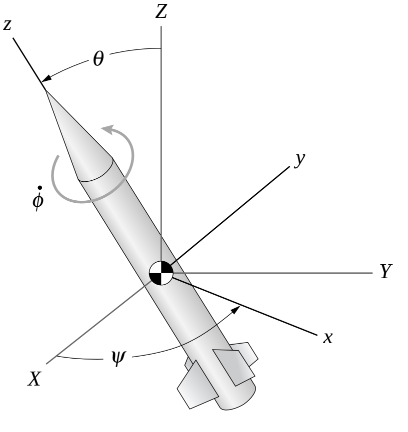
$$mR^2\left(\frac{1}{2} - \frac{1}{4}\right)\dot{\psi}\cos\theta + \left(\frac{mR^2}{2}\right)\dot{\phi} = 0.$$

Reduce, to obtain

$$\dot{\psi} = -\frac{2\dot{\phi}}{\cos\theta}.$$

When the nutation angle is small, $\theta \to 0$, $\cos\theta \to 1$, and $\dot{\psi} \cong -2\dot{\phi}$.

Problem 20.68 The rocket is in moment-free steady precession with nutation angle $\theta = 40^\circ$ and spin rate $\dot{\phi} = 4$ revolutions per second. Its moments of inertia are $I_{xx} = 10{,}000$ kg-m^2 and $I_{zz} = 2000$ kg-m^2. What is the rocket's precession rate $\dot{\psi}$ in revolutions per second?



Solution: Moment-free steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\dot{\phi}$ is the spin rate, and θ is the nutation angle. Solve for the precession rate:

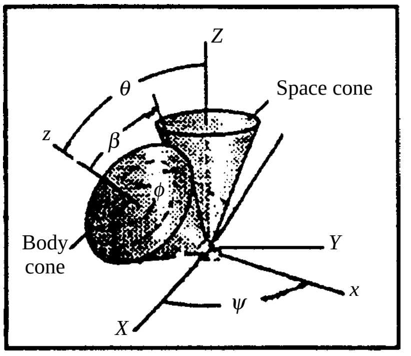
$$\dot{\psi} = \frac{I_{zz}\dot{\phi}}{(I_{xx} - I_{zz})\cos\theta} = 1.31 \text{ rev/s}.$$

Problem 20.69 Sketch the body and space cones for the motion of the rocket in Problem 20.68.

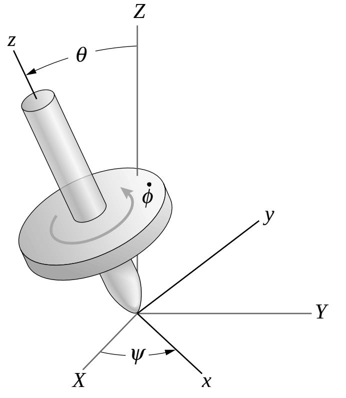
Solution: The angle $\theta = 40^\circ$. The angle β defined by

$$\beta = \tan^{-1}\left[\left(\frac{I_{zz}}{I_{xx}}\right)\tan\theta\right] = 9.53^\circ$$

satisfies the condition $\beta < \theta$. The body cone with an axis along the z axis, rolls on a space cone with axis on the Z axis. The result is shown.

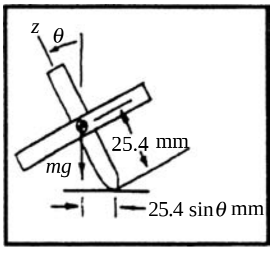


Problem 20.70 The top is in steady precession with nutation angle $\theta = 15^\circ$ and precession rate $\dot{\psi} = 1$ revolution per second. The mass of the top is 0.012 kg, its center of mass is 25.4 mm from the point, and its moments of inertia are $I_{xx} = 8.13 \times 10^{-6}$ kg-m^2 and $I_{zz} = 2.71 \times 10^{-6}$ kg-m^2. What is the spin rate ϕ of the top in revolutions per second?



Solution: The steady precession is not moment-free, since the weight of the top exerts a moment

$$M_x = 0.0254 \text{ mg} \sin\theta.$$



The motion of a spinning top is described by Eq. (20.32),

$$mgh = (I_{zz} - I_{xx})\dot{\psi}^2\cos\theta + I_{zz}\dot{\psi}\dot{\phi},$$

where $\dot{\psi}$ is the rate of precession, $\dot{\phi}$ is the spin rate, and θ is the nutation angle and

$$h = 0.0254 \text{ m}$$

is the distance from the point to the center of mass. Solve:

$$\dot{\phi} = \frac{0.0254 \text{ mg} - (I_{zz} - I_{xx})\dot{\psi}^2\cos\theta}{I_{zz}\dot{\psi}}.$$

Substitute numerical values (using $\dot{\psi} = 2\pi$ rad/s for dimensional consistency) to obtain $\dot{\phi} = 182.8$ rad/s, from which $\dot{\phi} = 29.1$ rev/s.

Problem 20.71 Suppose that the top described in Problem 20.70 has a spin rate $\dot{\phi} = 15$ revolutions per second. Draw a graph of the precession rate (in revolutions per second) as a function of the nutation angle θ for values of θ from zero to 45°.

Solution: The behavior of the top is described in Eq. (20.32),

$$mgh = (I_{xx} - I_{yy})\dot{\psi}^2 \cos\theta + I_{xx}\dot{\psi}\dot{\phi},$$

where $\dot{\psi}$ is the rate of precession, $\dot{\phi}$ is the spin rate, and θ is the nutation angle and $h = 0.254$ m is the distance from the point to the center of mass. Rearrange: $(I_{zz} - I_{xx})\dot{\psi}^2 \cos\theta + I_{zz}\dot{\psi}\dot{\phi} - mgh = 0$. The velocity of the center of the base is

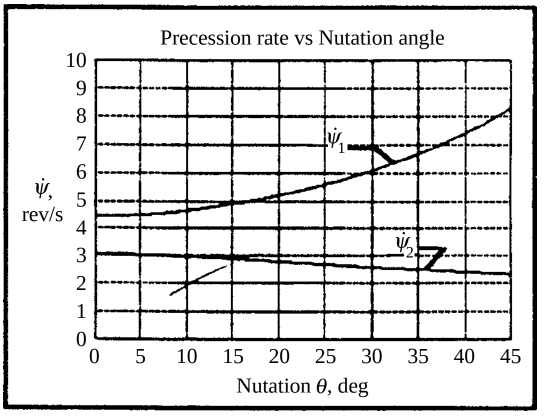
$$v = -\frac{\omega_0 h}{4},$$

from which the spin axis is the z axis and the spin *rate is*

$$\dot{\phi} = \frac{v}{R} = -\frac{\omega_0 h}{4R}.$$

The solution, $\dot{\psi}_{1,2} = -b \pm \sqrt{b^2 - c}$.

The two solutions, which are real over the interval, are graphed as a function of θ over the range $0 \le \theta \le 45°$. The graph is shown.



Problem 20.72 The rotor of a tumbling gyroscope can be modeled as being in moment-free steady precession. The moments of inertia of the gyroscope are $I_{xx} = I_{yy} = 0.04$ kg-m^2 and $I_{zz} = 0.18$ kg-m^2. The gyroscope's spin rate is $\dot{\phi} = 1500$ rpm and its nutation angle is $\theta = 20°$.

(a) What is the precession rate of the gyroscope in rpm?
(b) Sketch the body and space cones.

Solution:

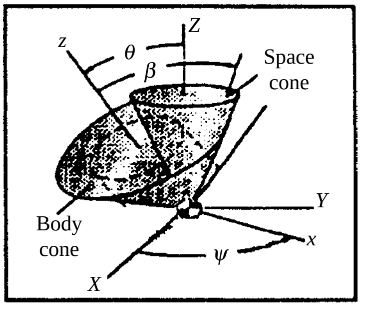
(a) The motion in moment-free, steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\dot{\phi} = 1500$ rpm is the spin rate, and $\theta = 20°$ is the nutation angle.

Solve: $\dot{\psi} = -\dfrac{I_{zz}\dot{\phi}}{(I_{zz} - I_{xx})\cos\theta} = -2050$ rpm.

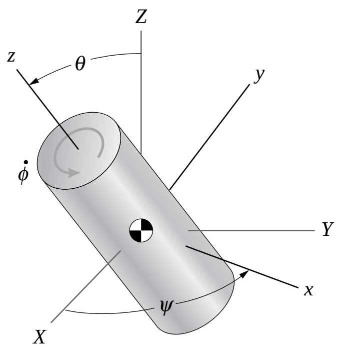
(b) The apex angle for the body cone is given by

$$\tan\beta = \left(\frac{I_{zz}}{I_{xx}}\right)\tan\theta,$$

from which $\beta = 58.6°$. Since $\beta > \theta$, the space cone lies inside the body cone as the figure.



Problem 20.73 A satellite can be modeled as an 800-kg cylinder 4 m in length and 2 m in diameter. If the nutation angle is $\theta = 20^\circ$ and the spin rate $\dot{\phi}$ is one revolution per second, what is the satellite's precession rate $\dot{\psi}$ in revolutions per second?



Solution: From Appendix C, the moments and products of inertia of a homogenous cylinder are

$$I_{xx} = I_{yy} = m\left(\frac{L^2}{12} + \frac{R^2}{4}\right) = 1267 \text{ kg-m}^2,$$

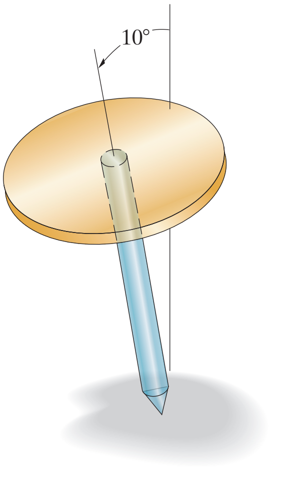
$$I_{zz} = mR^2/2 = 400 \text{ kg-m}^2.$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The angular motion of an axisymmetric moment-free object in steady precession is described by Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, where $\dot{\psi}$ is the precession rate, $\theta = 20^\circ$ is the nutation angle, and $\dot{\phi} = 1$ rps is the spin rate. Solve:

$$\dot{\psi} = -\frac{I_{zz}\dot{\phi}}{(I_{zz} - I_{xx})\cos\theta} = 0.49 \text{ rps}.$$

Problem 20.74 The top consists of a thin disk bonded to a slender bar. The radius of the disk is 30 mm and its mass is 0.008 kg. The length of the bar is 80 mm and its mass is negligible compared to the disk. When the top is in steady precession with a nutation angle of 10°, the precession rate is observed to be 2 revolutions per second in the same direction the top is spinning. What is the top's spin rate?



Solution: The inertias about the base are

$$I_{zz} = \frac{1}{2}(0.008 \text{ kg})(0.03 \text{ m})^2 = 3.6 \times 10^{-6} \text{ kg-m}^2$$

$$I_{xx} = I_{yy} = \frac{1}{2}(0.008 \text{ kg})(0.03 \text{ m})^2$$

$$+ (0.008 \text{ kg})(0.08 \text{ m})^2 = 53 \times 10^{-6} \text{ kg-m}^2$$

The precession rate is

$$\dot{\psi} = 2(2\pi) = 12.6 \text{ rad/s}$$

The moment about the base is

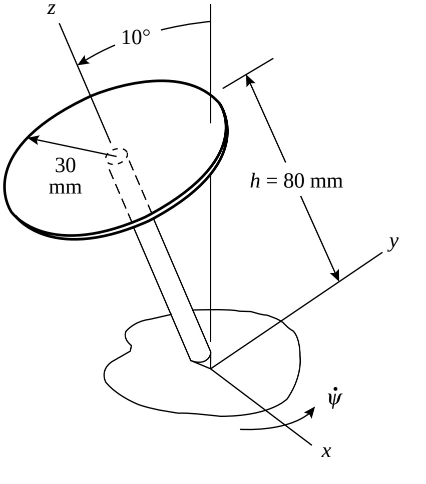
$$\mathbf{M} = M_x\mathbf{i} = (0.008 \text{ kg})(9.81 \text{ m/s}^2)(0.08 \text{ m})\mathbf{i} = (6.28 \times 10^{-3} \text{ N-m})\mathbf{i}$$

Eq. 20.32 is

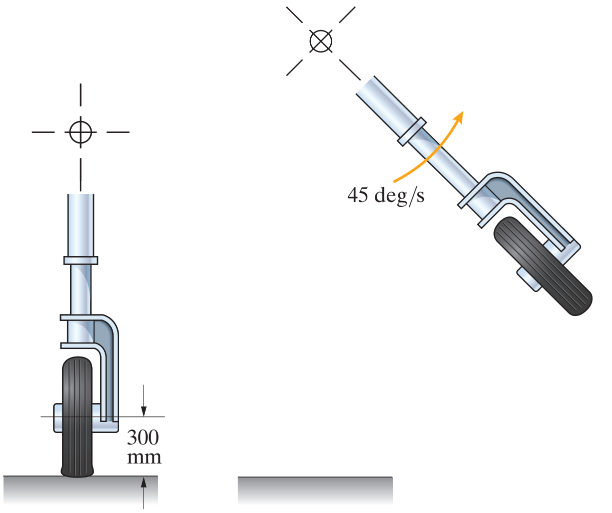
$$M_x = (I_{zz} - I_{xx})\dot{\psi}^2\cos 10^\circ + I_{zz}\dot{\psi}\dot{\phi}$$

Solving we find

$$\boxed{\dot{\phi} = 309 \text{ rad/s} \quad (49.1 \text{ rev/s})}$$



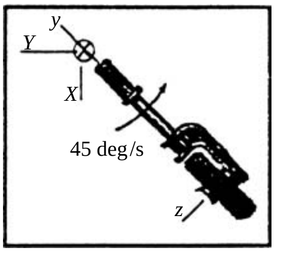
Problem 20.75 Solve Problem 20.58 by treating the motion as steady precession.



(a) (b)

Solution: The view of an airplane's landing gear looking from behind the airplane is shown in Fig. (a). The radius of the wheel is 300 mm and its moment of inertia is 2 kg-m^2. The airplane takes off at 30 m/s. After takeoff, the landing gear retracts by rotating toward the right side of the airplane as shown in Fig. (b). The magnitude of the couple exerted by the wheel on its support is to be determined.

Choose X, Y, Z with the Z axis parallel to the runway, X perpendicular to the runway, and Y parallel to the runway. Choose the x, y, z coordinate system with the origin at the center of mass of the wheel and the z axis aligned with the direction of the axis of rotation of the wheel and the y axis positive upward. The Eq. (20.29) is



$$\sum M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta \cos\theta + I_{zz}\dot{\phi}\dot{\psi} \sin\theta.$$

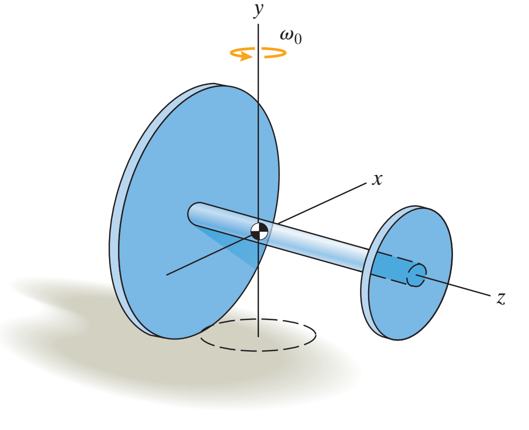
The nutation angle and rates of precession: The nutation angle is the angle between Z and z, $\theta = 90^\circ$. The precession angle is the angle between the X and x, which is increasing in value, from which $\dot{\psi} = 45^\circ/\text{s} = 0.7853$ rad/s. The spin vector is aligned with the z axis, from which

$$\dot{\phi} = \left(\frac{v}{R}\right) = \left(\frac{30}{0.3}\right) = 100 \text{ rad/s}$$

The moments and products of inertia of the wheel are $I_{zz} = mR^2/2 = 2$ kg-m^2. The moment is

$$M_x = I_{zz}\dot{\psi}\dot{\phi} \sin 90^\circ = 2(0.7854)(100) = 157 \text{ N-m}.$$

Problem 20.76* The two thin disks are rigidly connected by a slender bar. The radius of the large disk is 200 mm and its mass is 4 kg. The radius of the small disk is 100 mm and its mass is 1 kg. The bar is 400 mm in length and its mass is negligible. The composite object undergoes a steady motion in which it spins about the vertical y axis through its center of mass with angular velocity ω_0. The bar is horizontal during this motion and the large disk rolls on the floor. Determine ω_0 by treating the motion as steady precession.



Solution: Use the data from 20.57

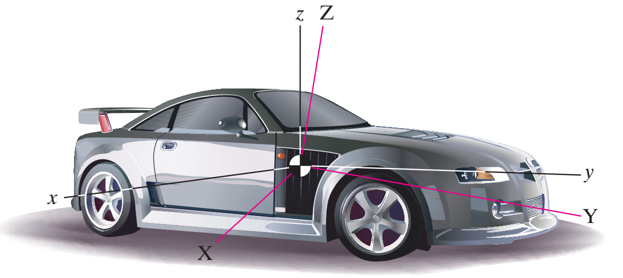
Set the nutation angle to 90° and the precession rate to ω_0

$$3.924 \text{ N-m} = (0.085 \text{ kg-m}^2)(0.4\omega_0)\omega_0$$

Solving we obtain

$$\boxed{\omega_0 = 10.7 \text{ rad/s}}$$

Problem 20.77* Suppose that you are testing a car and use accelerometers and gyroscopes to measure its Euler angles and their derivatives relative to a reference coordinate system. At a particular instant, $\psi = 15^\circ$, $\theta = 4^\circ$, $\phi = 15^\circ$, the rates of change of the Euler angles are zero, and their second derivatives with respect to time are $\ddot{\psi} = 0$, $\ddot{\theta} = 1$ rad/s^2, and $\ddot{\phi} = -0.5$ rad/s^2. The car's principal moments of inertia, in kg-m^2, are $I_{xx} = 2200$, $I_{yy} = 480$, and $I_{zz} = 2600$. What are the components of the total moment about the car's center of mass?



Solution: The description of the motion of an arbitrarily shaped object is given by the Eqs. (20.36):

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi$$

$$+ \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi)$$

$$- (I_{yy} - I_{zz})(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi$$

$$- \dot{\psi}\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi)$$

$$- (I_{zz} - I_{xx})(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta) - (I_{xx} - I_{yy})(\dot{\psi}\sin\theta\sin\phi$$

$$+ \dot{\theta}\cos\phi)(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi).$$

Substitute $\ddot{\psi} = 0$, $\dot{\psi} = \dot{\theta} = \dot{\phi} = 0$, to obtain $M_x = I_{xx}\ddot{\theta}\sin\phi$, $M_y = -I_{yy}\ddot{\theta}\sin\phi$, $M_z = I_{zz}\ddot{\phi}$. Substitute values:

$M_x = 2125$ N-m,

$M_y = -124.2$, $M_z = -1300$ N-m

Problem 20.78* If the Euler angles and their second derivatives for the car described in Problem 20.77 have the given values, but their rates of change are $\dot{\psi} = 0.2$ rad/s, $\dot{\theta} = -2$ rad/s, and $\dot{\phi} = 0$, what are the components of the total moment about the car's center of mass?

Solution: Use Eqns. (20.36)

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi$$

$$+ \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi)$$

$$- (I_{yy} - I_{zz})(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi$$

$$- \dot{\psi}\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi)$$

$$- (I_{zz} - I_{xx})(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta) - (I_{xx} - I_{yy})(\dot{\psi}\sin\theta\sin\phi$$

$$+ \dot{\theta}\cos\phi)(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)$$

Substitute values:

$M_x = 2123$ N-m,

$M_y = -155.4$ N-m,

$M_z = 534$ N-m.

Problem 20.79* Suppose that the Euler angles of the car described in Problem 20.77 are $\psi = 40^\circ$, $\theta = 20^\circ$, and $\phi = 5^\circ$, their rates of change are zero, and the components of the total moment about the car's center of mass are

$$\sum M_x = -400 \text{ N-m},$$

$$\sum M_y = 200 \text{ N-m},$$

$$\sum M_z = 0.$$

What are the x, y, and z components of the car's angular acceleration?

Solution: Eq. (20.36) simplifies when $\dot{\psi} = \dot{\phi} = \dot{\theta} = 0$ to

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi}).$$

These three simultaneous equations have the solutions,

$$\ddot{\theta} = \frac{M_x}{I_{xx}}\cos\phi - \frac{M_y}{I_{yy}}\sin\phi = -0.2174 \text{ rad/s}^2,$$

$$\ddot{\psi} = \left(\frac{M_{xx}}{I_{xx}}\right)\frac{\sin\phi}{\sin\theta} + \left(\frac{M_y}{I_{yy}}\right)\frac{\cos\phi}{\sin\theta} = 1.167 \text{ rad/s}^2,$$

$$\ddot{\phi} = (M_z/I_{zz}) - \ddot{\psi}\cos\theta = -1.097 \text{ rad/s}^2.$$

From Eq. (20.35), when

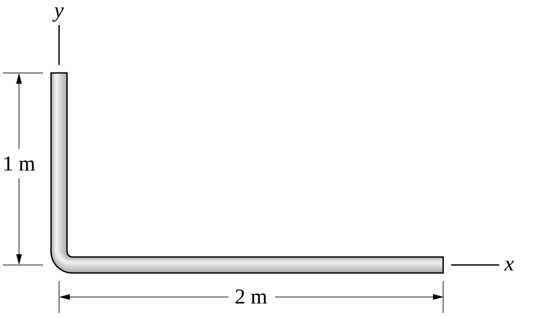
$$\dot{\psi} = \dot{\phi} = \dot{\theta} = 0:$$

$$\frac{d\omega_x}{dt} = \ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi = -0.1818 \text{ rad/s}^2,$$

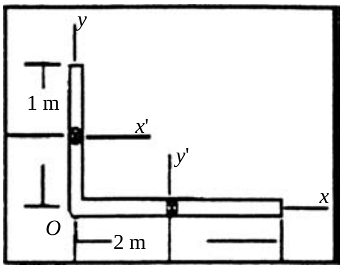
$$\frac{d\omega_y}{dt} = \ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi = 0.417 \text{ rad/s},$$

$$\frac{d\omega_z}{dt} = \ddot{\psi}\cos\phi + \ddot{\phi} = 0.$$

Problem 20.80 The mass of the bar is 6 kg. Determine the moments and products of inertia of the bar in terms of the coordinate system shown.



Solution: One strategy (not the simplest, see *Check* note following the solution) is to determine the moment of inertia matrix for each element of the bar, and then to use the parallel axis theorem to transfer each to the coordinate system shown.



(a) *The vertical element Oy of the bar*. The mass density per unit volume is $\rho = \dfrac{6}{3A}$ kg/m^3, where A is the (unknown) cross section of the bar, from which $\rho A = 2$ kg/m. The element of mass is $dm = \rho A dL$, where dL is an element of length. The mass of the vertical element is $m_v = \rho A L_v = 2$ kg, where $L_v = 1$ m. From Appendix C the moment of inertia about an x' axis passing through the center of mass is

$$I^{(1)}_{x'x'} = \frac{m_v L_v^2}{12} = 0.1667 \text{ kg-m}^2.$$

Since the bar is slender, $I^{(1)}_{y'y'} = 0$.

$$I^{(1)}_{z'z'} = \frac{m_v L_v^2}{12} = 0.1667 \text{ kg-m}^2.$$

Since the bar is slender, the products of inertia vanish:

$$I^{(1)}_{x'y'} = \int_m x'y'dm = 0,\ I^{(1)}_{x'z'} = \int_m x'z'dm = 0,$$

$$I^{(1)}_{y'z'} = \int_m y'z'dm = 0,$$

from which the inertia matrix for the element Oy about the x' axis is

$$[I^{(1)}] = \begin{bmatrix} 0.1667 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1667 \end{bmatrix} \text{ kg-m}^2.$$

(b) *The horizontal element Ox of the bar*. The mass of the horizontal element is $m_h = \rho A L_h = 4$ kg, where $L_h = 2$ m. From Appendix C the moments and products of inertia about the y' axis passing through the center of mass of the horizontal element are:

$$I^{(2)}_{x'x'} = 0,$$

$$I^{(2)}_{y'y'} = \frac{m_h L_h^2}{12} = 1.333 \text{ kg-m}^2,$$

$$I^{(2)}_{z'z'} = \frac{m_h L_h^2}{12} = 1.333 \text{ kg-m}^2.$$

Since the bar is slender, the cross products of inertia about the y' axis through the center of mass of the horizontal element of the bar vanish:

$$I^{(2)}_{x'y'} = 0,\ I^{(2)}_{x'z'} = 0,\ I^{(2)}_{y'z'} = 0.$$

The inertia matrix is

$$[I^{(2)}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.333 & 0 \\ 0 & 0 & 1.333 \end{bmatrix} \text{ kg-m}^2.$$

Use the parallel axis theorem to transfer the moment of inertia matrix to the origin O: For the vertical element the coordinates of the center of mass O are $(d_x, d_y, d_z) = (0, 0.5, 0)$ m. Use the parallel axis theorem (see Eq. (20.42)).

$$I_{xx}^{(1)} = I_{x'x'}^{(1)} + (d_y^2 + d_z^2)m_v = \frac{m_v L_v^2}{12} + m_v(0.5^2) = 0.6667 \text{ kg-m}^2.$$

$$I_{yy}^{(1)} = I_{y'y'}^{(1)} + (d_x^2 + d_z^2)m_v = 0.$$

$$I_{zz}^{(1)} = I_{z'z'}^{(1)} + (d_x^2 + d_y^2)mv = 0.6667 \text{ kg-m}^2.$$

The products of inertia are

$$I_{xy}^{(1)} = I_{x'y'}^{(1)} + d_x d_y \rho A(1) = 0,$$

$$I_{xz}^{(1)} = I_{x'z'}^{(1)} + d_x d_z \rho A(1) = 0,$$

$$I_{yz}^{(1)} = I_{y'z'}^{(1)} + d_y d_z \rho A(1) = 0.$$

The inertia matrix for the vertical element:

$$[I^{(1)}] = \begin{bmatrix} \frac{\rho A}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\rho A}{3} \end{bmatrix}.$$

For the horizontal element, the coordinates of the center of mass relative to O are $(d_x, d_y, d_z) = (1, 0, 0)$ m. From the parallel axis theorem,

$$I_{xx}^{(2)} = I_{x'x'}^{(2)} + (d_y^2 + d_z^2)m_h = 0.$$

$$I_{yy}^{(2)} = I_{y'y'}^{(2)} + (d_x^2 + d_z^2)m_h = 5.333 \text{ kg-m}^2.$$

$$I_{zz}^{(2)} = I_{z'z'}^{(2)} + (d_x^2 + d_y^2)m_h = 5.333 \text{ kg-m}^2.$$

By inspection, the products of inertia vanish. The inertia matrix is

$$[I^{(2)}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5.333 & 0 \\ 0 & 0 & 5.333 \end{bmatrix}.$$

Sum the two inertia matrices:

$$\boxed{[I]_O = [I^{(1)}] + [I^{(2)}] = \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 5.333 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ kg-m}^2.}$$

[*Check*: The moment of inertia in the coordinate system shown can be derived by insepection by taking the moment of inertia of each element about the origin: From Appendix C the moments of inertia about the origin of the slender bars are

$$I_{xx} = \frac{m_v L_v^3}{3}, I_{yy} = \frac{m_h L_h^2}{3},$$

and $I_{zz} = I_{xx} + I_{yy}$, where the subscripts v and h denote the vertical and horizontal bars respectively. Noting that the masses are

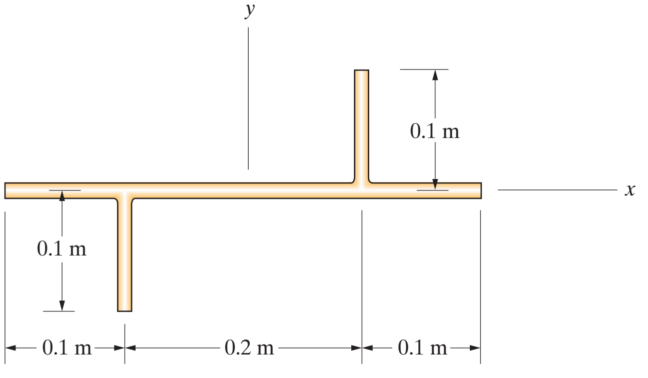
$$m_v = \frac{mL_v}{L_v + L_h}, m_h = \frac{mL_h}{L_v + L_h},$$

the moment of inertia matrix becomes:

$$[I] = \begin{bmatrix} \frac{6(1)^3}{3(3)} & 0 & 0 \\ 0 & 0 & \frac{6(2)^3}{3(3)} \\ 0 & 0 & 6.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 5.333 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ kg-m}^2. \textit{ check.}$$

Problem 20.81 The object consists of two 1-kg vertical slender bars welded to a 4-kg horizontal slender bar. Determine its moments and products of inertia in terms of the coordinate system shown.



Solution:

$$I_{xx} = 2\frac{1}{3}(1\text{ kg})(0.1\text{ m})^2 = 0.00667\text{ kg-m}^2,$$

$$I_{yy} = \frac{1}{12}(4\text{ kg})(0.4\text{ m})^2 + 2(1\text{ kg})(0.1\text{ m})^2 = 0.0733\text{ kg-m}^2,$$

$$I_{zz} = I_{yy} + I_{zz} = 0.08\text{ kg-m}^2,$$

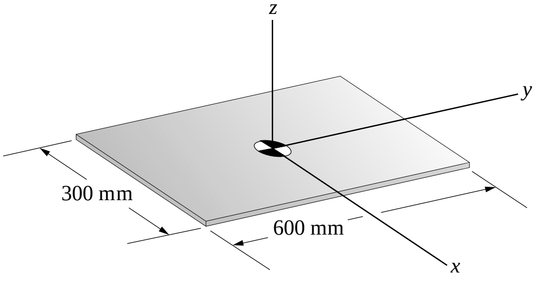
$$I_{xy} = 2(1\text{ kg})(0.1\text{ m})(0.05\text{ m}) = 0.01\text{ kg-m}^2,$$

$$I_{xz} = 0,$$

$$I_{yz} = 0.$$

$$\boxed{\begin{array}{lll} I_{xx} = 0.00667\text{ kg-m}^2, & I_{yy} = 0.0733\text{ kg-m}^2, & I_{zz} = 0.08\text{ kg-m}^2, \\ I_{xy} = 0.01\text{ kg-m}^2, & I_{xz} = 0, & I_{yz} = 0. \end{array}}$$

Problem 20.82 The 4-kg thin rectangular plate lies in the x-y plane. Determine the moments and products of inertia of the plate in terms of the coordinate system shown.



Solution: From Appendix B, the moments of inertia of the plate's area are

$$I_x = \frac{1}{12}(0.3)(0.6)^3 = 0.00540\text{ m}^4,$$

$$I_y = \frac{1}{12}(0.6)(0.3)^3 = 0.00135\text{ m}^4,$$

$$I_{xy}^A = 0.$$

Therefore the moments of inertia of the plate are

$$I_{xx} = \frac{m}{A}I_x = \frac{4}{(0.3)(0.6)}(0.00540)$$

$$= 0.12\text{ kg-m}^2,$$

$$I_{yy} = \frac{m}{A}I_y = \frac{4}{(0.3)(0.6)}(0.00135)$$

$$= 0.03\text{ kg-m}^2,$$

$$I_{zz} = I_x + I_y = 0.15\text{ kg-m}^2,$$

$$I_{xy} = I_{yz} = I_{zx} = 0.$$

Problem 20.83 If the 4-kg plate is rotating with angular velocity $\boldsymbol{\omega} = 6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

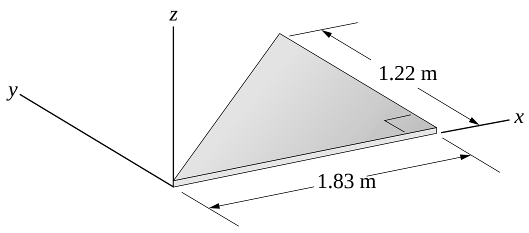
Solution: Angular momentum is

$$[H] = \begin{bmatrix} 0.12 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.72 \\ 0.12 \\ -0.3 \end{bmatrix}\text{ kg-m}^2\text{/s},$$

from which $\boxed{\mathbf{H} = 0.72\mathbf{i} + 0.12\mathbf{j} - 0.3\mathbf{k}\ (\text{kg-m}^2/\text{s})}$

Problem 20.84 The 133.4 N triangular plate lies in the $x-y$ plane. Determine the moments and products of inertia of the plate in terms of the coordinate system shown.



Solution: From Appendix B, the moments of inertia of the plate's area are

$$I_x = \frac{1}{12}(1.83)(1.22)^3 = 0.276 \text{ m}^4,$$

$$I_y = \frac{1}{4}(1.22)(1.83)^3 = 1.86 \text{ m}^4,$$

$$I_{xy}^A = \frac{1}{8}(1.83)^2(1.22)^2 = 0.62 \text{ m}^4.$$

The plate's area and mass are

$$A = \frac{1}{2}(1.83)(1.22) = 1.11 \text{ m}^2$$

and $m = 133.4/9.81 = 13.6$ kg. The plate's moments and products of inertia are

$$I_{xx} = \frac{m}{A}I_x = 3.36 \text{ kg-m}^2,$$

$$I_{yy} = \frac{m}{A}I_y = 22.73 \text{ kg-m}^2,$$

$$I_{xy} = \frac{m}{A}I_{xy}^A = 7.58 \text{ kg-m}^2,$$

$$I_{yz} = I_{zx} = O,$$

$$I_{zz} = I_{xx} + I_{yy} = 26.1 \text{ kg-m}^2.$$

Problem 20.85 The 133.4 N triangular plate lies in the $x-y$ plane.

(a) Determine its moments and products of inertia in terms of a parallel coordinate system $x'y'z'$ with its origin at the plate's center of mass.

(b) If the plate is rotating with angular velocity $\boldsymbol{\omega} = 20\mathbf{i} - 12\mathbf{j} + 16\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

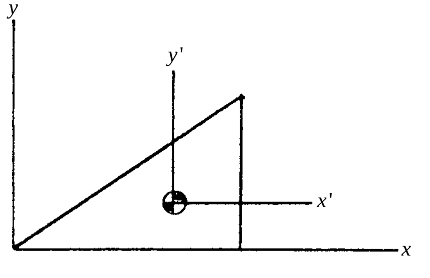
Solution: See the solution of Problem 20.84

(a) The coordinates of the plate's center of mass are

$$d_x = \frac{2}{3}(1.83) = 1.22 \text{ m},$$

$$d_y = \frac{1}{3}(1.22) = 0.41 \text{ m},$$

$$d_z = 0.$$



From the parallel-axis theorems (Eq. 20.42), we obtain

$$I_{x'x'} = 3.26 - (0.41)^2(13.6) = 1.12 \text{ kg-m}^2,$$

$$I_{y'y'} = 22.73 - (1.22)^2(13.6) = 2.53 \text{ kg-m}^2,$$

$$I_{z'z'} = 26.1 - [(0.41)^2 + (1.22)^2](13.6) = 3.65 \text{ kg-m}^2,$$

$$I_{x'y'} = 7.58 - (0.41)(1.22)(13.6) = 0.842 \text{ kg-m}^2,$$

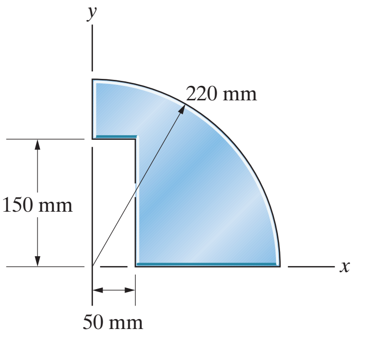
$$I_{y'z'} = I_{z'x'} = 0$$

(b) $$[I]\omega = \begin{bmatrix} 1.12 & -0.842 & 0 \\ -0.842 & 2.53 & 0 \\ 0 & 0 & 3.65 \end{bmatrix} \begin{bmatrix} 20 \\ -12 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} 32.5 \\ -47.2 \\ 58.4 \end{bmatrix}$$

$$\mathbf{H} = 32.5\mathbf{i} - 47.2\mathbf{j} + 58.4\mathbf{k} \text{ (kg-m}^2\text{)}.$$

Problem 20.86 Determine the inertia matrix of the 2.4-kg steel plate in terms of the coordinate system shown.



Solution: Equation (20.39) gives the plate's moments and products of inertia in terms of the moments and product of inertia of its area. Treating the area as a quarter-circle using Appendix B, the moments and products of inertia of the area are

$$I_x = \frac{1}{16}\pi(0.22)^4 - \frac{1}{3}(0.05)(0.15)^3 = 0.000404 \text{ m}^4$$

$$I_y = \frac{1}{16}\pi(0.22)^4 - \frac{1}{3}(0.15)(0.05)^3 = 0.000454 \text{ m}^4$$

$$I_{xy}^A = \frac{1}{8}(0.22)^4 - \frac{1}{4}(0.05)^2(0.15)^2 = 0.000279 \text{ m}^4.$$

The area is

$$A = \tfrac{1}{4}\pi(0.22)^2 - (0.05)(0.15) = 0.0305 \text{ m}^2.$$

The moments of inertia of the plate are

$$I_{xx} = \frac{m}{A}I_x = 0.0318 \text{ kg-m}^2,$$

$$I_{yy} = \frac{m}{A}I_y = 0.0357 \text{ kg-m}^2,$$

$$I_{zz} = I_{xx} + I_{yy} = 0.0674 \text{ kg-m}^2,$$

$$I_{xy} = \frac{m}{A}I_{xy}^A = 0.0219 \text{ kg-m}^2, \text{ and } I_{yz} = I_{zx} = 0.$$

Problem 20.87 The mass of the steel plate is 2.4 kg.

(a) Determine its moments and products of inertia in terms of a parallel coordinate system $x'y'z'$ with its origin at the plate's center of mass.
(b) If the plate is rotating with angular velocity $\omega = 20\mathbf{i} + 10\mathbf{j} - 10\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

Solution:

(a) The x and y coordinates of the center of mass coincide with the centroid of the area:

$$A_1 = \frac{1}{4}\pi(0.22)^2 = 0.0380 \text{ m}^2,$$

$$A_2 = (0.05)(0.15) = 0.0075 \text{ m}^2$$

$$\mathbf{x} = \frac{\dfrac{4(0.22)}{3\pi}A_1 - (0.025)A_2}{A_1 - A_2} = 0.1102 \text{ m},$$

$$\mathbf{y} = \frac{\dfrac{4(0.22)}{3\pi}A_1 - (0.075)A_2}{A_1 - A_2} = 0.0979 \text{ m}.$$

Using the results of the solution of Problem 20.86 and the parallel axis theorems,

$$I_{x'x'} = I_{xx} - m\mathbf{y}^2 = 0.00876 \text{ kg-m}^2:$$

$$I_{y'y'} = I_{yy} - m\mathbf{x}^2 = 0.00655 \text{ kg-m}^2:$$

$$I_{z'z'} = I_{x'x'} + I_{y'y'} = 0.01531 \text{ kg-m}^2$$

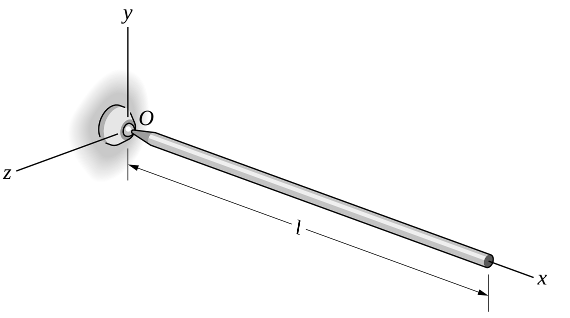
$$I_{x'y'} = I_{xy} - m\mathbf{xy} = -0.00396 \text{ kg-m}^2, \text{ and } I_{y'z'} = I_{z'x'} = 0.$$

The angular momentum is

$$\begin{bmatrix} H_{x'} \\ H_{y'} \\ H_{z'} \end{bmatrix} = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & 0 \\ -I_{x'y'} & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ -10 \end{bmatrix}$$

$$= \begin{bmatrix} 0.215 \\ 0.145 \\ -0.153 \end{bmatrix} (\text{kg-m}^2/\text{s}).$$

Problem 20.88 The slender bar of mass m rotates about the fixed point O with angular velocity $\boldsymbol{\omega} = \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Determine its angular momentum (a) about its center of mass and (b) about O.



Solution:

(a) From Appendix C and by inspection, the moments and products of inertia about the center of mass of the bar are:

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = \frac{mL^2}{12},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The angular momentum about its center of mass is

$$[H]_G = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix}$$

Alternatively, in terms of the **i**, **j**, **k**,

$$\boxed{\mathbf{H}_G = \frac{mL^2}{12}(\omega_y\mathbf{j} + \omega_z\mathbf{k})}.$$

(b) From Appendix C and by inspection, the moments and products of inertia about O are

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = \frac{mL^2}{3},$$

$$I_{xy} = I_{xz} = I_{yz} = 0$$

The angular momentum about O is

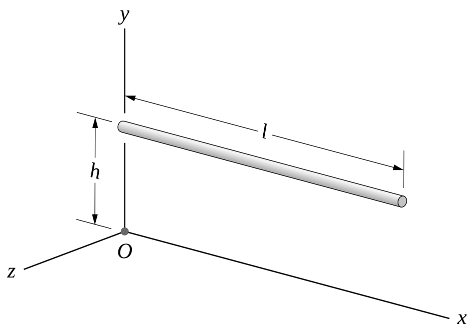
$$[H]_O = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{3} & 0 \\ 0 & 0 & \frac{mL^2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{mL^2\omega_y}{3} \\ \frac{mL^2\omega_z}{3} \end{bmatrix},$$

or, alternatively, in terms of the unit vectors **i**, **j**, **k**,

$$\boxed{\mathbf{H}_O = \frac{mL^2}{3}(\omega_y\mathbf{j} + \omega_z\mathbf{k})}$$

Problem 20.89 The slender bar of mass m is parallel to the x axis. If the coordinate system is body fixed and its angular velocity about the fixed point O is $\boldsymbol{\omega} = \omega_y \mathbf{j}$, what is the bar's angular momentum about O?



Solution: From Appendix C and by inspection, the moments and products of inertia about the center of mass of the bar are:

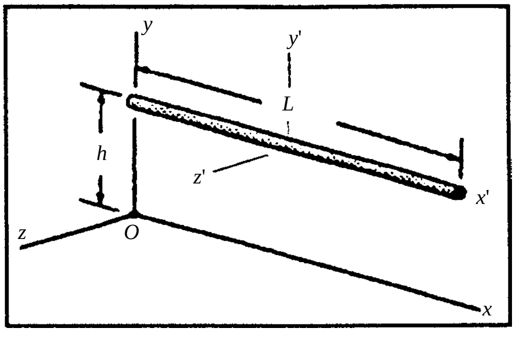
$$I_{x'x'} = 0,$$

$$I_{y'y'} = I_{z'z'} = \frac{mL^2}{12},$$

$$I_{x'y'} = I_{x'z'} = I_{y'z'} = 0.$$

The coordinates of the center of mass are

$$(d_x, d_y, d_z) = \left(\frac{L}{2}, h, 0\right).$$



From Eq. (20.42),

$$I_{xx} = I_{x'x'} + (d_y^2 + d_z^2)m = \mathrm{mh}^2$$

$$I_{yy} = I_{y'y'} + (d_x^2 + d_z^2)m = \frac{mL^2}{12} + \frac{mL^2}{4} = \frac{mL^2}{3}$$

$$I_{zz} = I_{z'z'} + (d_x^2 + d_y^2)m = m\left(h^2 + \frac{L^2}{3}\right),$$

$$I_{xy} = I_{x'y'} + d_x d_y m = 0 + \frac{mLh}{2},$$

$$I_{xz} = I_{x'z'} + d_x d_z m = 0$$

$$I_{yz} = I_{y'z'} + d_y d_z m = 0$$

The angular momentum about O is

$$[H]_O = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ 0 \end{bmatrix}$$

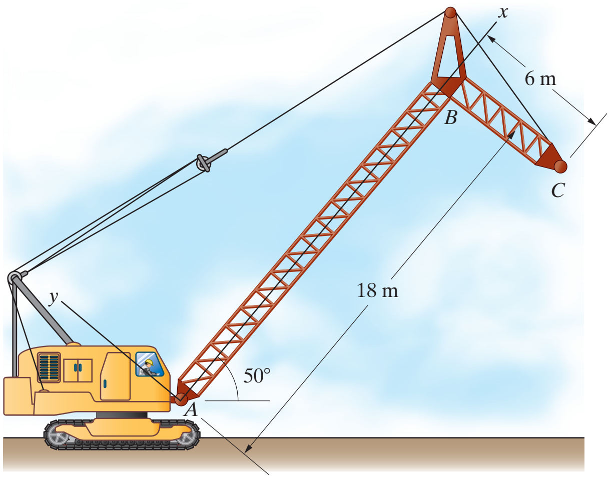
$$= \begin{bmatrix} mh^2 & -\dfrac{mLh}{2} & 0 \\ -\dfrac{mLh}{2} & \dfrac{mL^2}{3} & 0 \\ 0 & 0 & m\left(h^2 + \dfrac{L^2}{3}\right) \end{bmatrix} \begin{bmatrix} 0 \\ \omega_y \\ 0 \end{bmatrix}$$

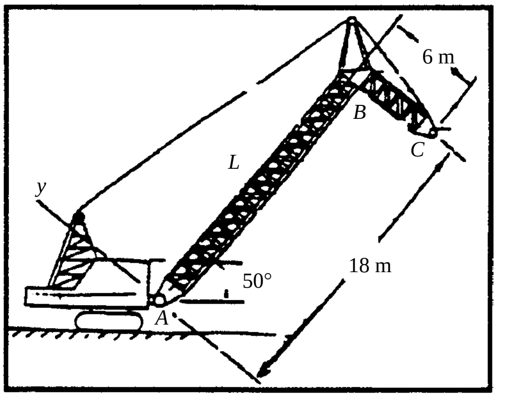
$$= \begin{bmatrix} -\dfrac{mLh\omega_y}{2} \\ \dfrac{mL^2\omega_y}{3} \\ 0 \end{bmatrix}.$$

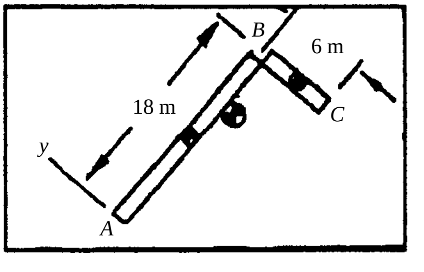
Alternatively,

$$\boxed{\mathbf{H}_O = -\frac{mLh}{2}\omega_y \mathbf{i} + \frac{mL^2}{3}\omega_y \mathbf{j}}$$

Problem 20.90 In Example 20.8, the moments and products of inertia of the object consisting of the booms AB and BC were determined in terms of the coordinate system shown in Fig. 20.34. Determine the moments and products of inertia of the object in terms of a parallel coordinate system $x'y'z'$ with its origin at the center of mass of the object.







Solution: From Example 20.8, the inertia matrix for the two booms in the x, y, z system is

$$[I]_O = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 19{,}200 & 86{,}400 & 0 \\ 86{,}400 & 1{,}036{,}800 & 0 \\ 0 & 0 & 1{,}056{,}000 \end{bmatrix} \text{ kg-m}^2.$$

The mass of the boom AB is $m_{AB} = 4800$ kg. The mass of the boom BC is $m_{BC} = 1600$ kg. The coordinates of the center of mass of the two booms are

$$x' = \frac{m_{AB}\dfrac{L}{2} + m_{BC}L}{m_{AB} + m_{BC}} = 11.25 \text{ m}.$$

$$y' = \frac{m_{AB}(0) + m_{BC}(-3)}{m_{AB} + m_{BC}} = -0.75 \text{ m}.$$

$z' = 0$, from which $(d_x, d_y, d_z) = (11.25, -0.75, 0)$ m. Algebraically rearrange Eq. (20.42) to obtain the moments and products of inertia about the parallel axis passing through the center of mass of the two booms when the moments and products of inertia in the x, y, z system are known:

$$\boxed{I^{(G)}_{x'x'} = I^{(0)}_{xx} - (d_y^2 + d_x^2)m = 15600 \text{ kg-m}^2}.$$

$$\boxed{I^{(G)}_{y'y'} = I^{(o)}_{yy} - (d_x^2 + d_z^2)m = 226800 \text{ kg-m}^2}.$$

$$\boxed{I^{(G)}_{z'z'} = I^{(o)}_{zz} - (d_x^2 + d_y^2)m = 242400 \text{ kg-m}^2}$$

$$\boxed{I^{(G)}_{x'y'} = I^{(0)}_{xy} - d_x d_y m = -32400 \text{ kg-m}^2},$$

$$\boxed{I^{(G)}_{x'z'} = 0}, \quad \boxed{I^{(G)}_{y'z'} = 0}.$$

The inertia matrix for the x', y', z' system is

$$[I]_G = \begin{bmatrix} 15{,}600 & 32{,}400 & 0 \\ 32{,}400 & 226{,}800 & 0 \\ 0 & 0 & 242{,}400 \end{bmatrix} \text{ kg-m}^2.$$

Problem 20.91 Suppose that the crane described in Example 20.8 undergoes a rigid-body rotation about the vertical axis at 0.1 rad/s in the counterclockwise direction when viewed from above.

(a) What is the crane's angular velocity vector $\boldsymbol{\omega}$ in terms of the body-fixed coordinate system shown in Fig. 20.34?

(b) What is the angular momentum of the object consisting of the booms AB and BC *about its center of mass*?

Solution: The unit vector parallel to vertical axis in the x', y', z' system is

$$\mathbf{e} = \mathbf{i}\sin 50^\circ + \mathbf{j}\cos 50^\circ = 0.7660\mathbf{i} + 0.6428\mathbf{j}.$$

The angular velocity vector is

$$\boxed{\boldsymbol{\omega} = (0.1)\mathbf{e} = 0.07660\mathbf{i} + 0.06428\mathbf{j}}$$

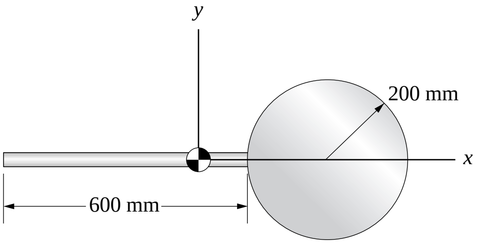
From the inertia matrix given in the solution of Problem 20.90, the angular moment about the center of mass is

$$[H]_G = \begin{bmatrix} 15{,}600 & 32{,}400 & 0 \\ 32{,}400 & 226{,}800 & 0 \\ 0 & 0 & 242{,}400 \end{bmatrix} \begin{bmatrix} 0.07660 \\ 0.06428 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3277.7 \\ 17060.4 \\ 0 \end{bmatrix} \text{ kg-m}^2/\text{s},$$

or, $\boxed{\mathbf{H}_G = 3277.7\mathbf{i} + 17060.4\mathbf{j}}$ (kg-m^2/s).

Problem 20.92 A 3-kg slender bar is rigidly attached to a 2-kg thin circular disk. In terms of the body-fixed coordinate system shown, the angular velocity of the composite object is $\boldsymbol{\omega} = 100\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ (rad/s). What is the object's angular momentum about its center of mass?



Solution: Choose a coordinate system x, y, z originating at the left end of the bar. From Appendix C and by inspection, the inertia matrix for the bar about its left end is

$$[I]_B = \begin{bmatrix} 0 & 0 & \\ 0 & \dfrac{m_B L^2}{3} & 0 \\ 0 & 0 & \dfrac{m_B L^2}{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.36 & 0 \\ 0 & 0 & 0.36 \end{bmatrix} \text{ kg-m}^2.$$

From Eq. (20.42) the inertia matrix of the disk about the left end of the bar is

$$[I]_D = \begin{bmatrix} \dfrac{m_D R^2}{4} & 0 & 0 \\ 0 & \dfrac{m_D R^2}{4} & 0 \\ 0 & 0 & \dfrac{m_D R^2}{2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & m_D(L+R)^2 & 0 \\ 0 & 0 & m_D(L+R)^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 1.32 \end{bmatrix} \text{ kg-m}^2.$$

The inertia matrix of the composite is

$$[I]_{\text{left\_end}} = [I]_B + [I]_D = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 1.66 & 0 \\ 0 & 0 & 1.68 \end{bmatrix}.$$

The coordinates of the center of mass of the composite in the x, y, z system are

$$x' = \frac{m_B(L/2) + m_D(R+L)}{m_B + m_D} = 0.5 \text{ m}.$$

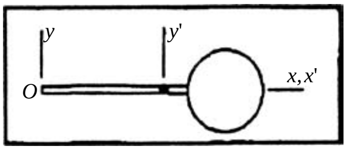
$$y' = 0, \quad z' = 0,$$

from which $(d_x, d_y, d_z) = (0.5, 0, 0)$ m. Rearrange Eq. (20.42) to yield the inertia matrix in the x', y', z' system when the inertia matrix in the x, y, z system is known:

$$[I]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 1.66 & 0 \\ 0 & 0 & 1.68 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & 0 \\ 0 & d_x^2(m_B + m_D) & 0 \\ 0 & 0 & d_x^2(m_B + m_D) \end{bmatrix}$$

$$= \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \text{ kg-m}^2.$$

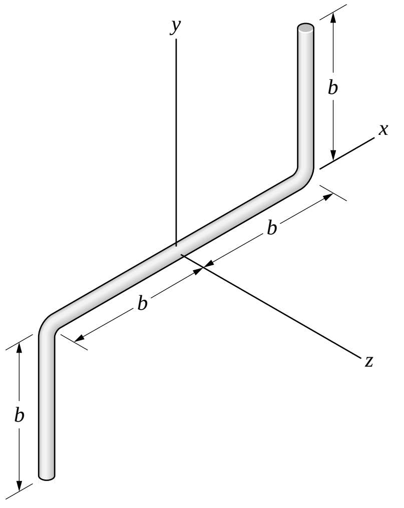


The angular momentum is

$$[H]_G = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.41 & 0 \\ 0 & 0 & 0.43 \end{bmatrix} \begin{bmatrix} 100 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -1.64 \\ 2.58 \end{bmatrix},$$

$$\boxed{\mathbf{H} = 2\mathbf{i} - 1.64\mathbf{j} + 2.58\mathbf{k} (\text{kg-m}^2/\text{s})}$$

Problem 20.93* The mass of the homogeneous slender bar is m. If the bar rotates with angular velocity $\boldsymbol{\omega} = \omega_0(24\mathbf{i} + 12\mathbf{j} - 6\mathbf{k})$, what is its angular momentum about its center of mass?



Solution: The strategy is to transfer the moments of inertia of the ends about their attached ends to the center of mass, and sum the resulting moments and products of inertia. The mass of the central element is $m_C = \frac{m}{2}$, and the mass of each end element is $m_E = \frac{m}{4}$. For the central element about its center of mass:

$$I_{xx} = 0,$$

$$I_{yy} = I_{zz} = \frac{m_C(2b)^2}{12} = \frac{mb^2}{6},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

For each end element about its center of mass:

$$I_{xx} = \frac{m_E b^2}{12} = \frac{mb^2}{48},$$

$$I_{yy} = 0,$$

$$I_{zz} = \frac{m_E b^2}{12} = \frac{mb^2}{48},$$

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

The coordinates of the center of mass (at the origin) relative to the center of mass of the left end is $(d_x, d_y, d_z) = \left(-b, -\frac{b}{2}, 0\right)$, and from the right end $(d_x, d_y, d_z) = \left(b, \frac{b}{2}, 0\right)$. From Eq. (20.42), the moments of inertia of each the end pieces about the center of mass (at the origin) are

$$I^{(G)}_{x'x'} = I^{(L)}_{xx} + (d_y^2 + d_z^2)m_E = \frac{mb^2}{48} + \frac{m_E b^2}{4} = \frac{mb^2}{12}.$$

$$I^{(G)}_{y'y'} I^{(L)}_{yy} + (d_x^2 + d_z^2)m_E = \frac{mb^2}{4}.$$

$$I^{(G)}_{z'z'} = I^{(L)}_{zz} + (d_x^2 + d_y^2)m_E = \frac{mb^2}{48} + m_E\left(b^2 + \frac{b^2}{4}\right) = \frac{mb^2}{3}$$

$$I^{(G)}_{x'y'} = I^{(L)}_{xy} + d_x d_y m_E = mb^2/8,\ I^{(G)}_{x'z'} = 0,\ I^{(G)}_{y'z'} = 0.$$

The sum of the matrices:

$$[I]_G = [I]_{GC} + 2[I]_{\text{end}}$$

$$= mb^2 \begin{bmatrix} 0.1667 & -0.25 & 0 \\ -0.25 & 0.6667 & 0 \\ 0 & 0 & 0.8333 \end{bmatrix}.$$

The angular momentum about the center of mass is

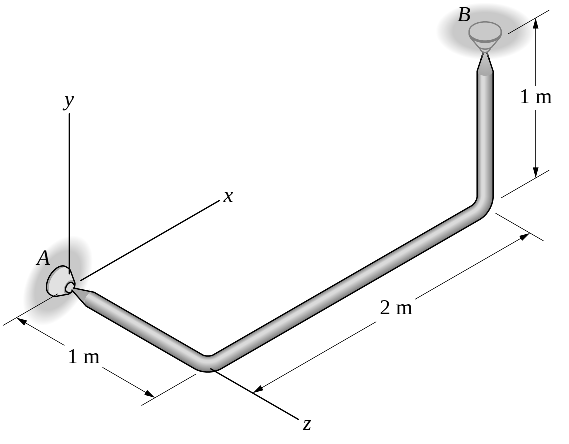
$$[H]_G = \omega_0 mb^2 \begin{bmatrix} 0.1667 & -0.25 & 0 \\ -0.25 & 0.6667 & 0 \\ 0 & 0 & 0.8333 \end{bmatrix} \begin{bmatrix} 24 \\ 12 \\ -6 \end{bmatrix}$$

$$= \omega_0 mb^2 \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}.$$

$$\boxed{\mathbf{H} = \omega_0 mb^2(\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})}$$

Problem 20.94* The 8-kg homogeneous slender bar has ball-and-socket supports at A and B.

(a) What is the bar's moment of inertia about the axis AB?
(b) If the bar rotates about the axis AB at 4 rad/s, what is the magnitude of its angular momentum about its axis of rotation?



Solution: Divide the bar into three elements: the central element, and the two end element. The strategy is to find the moments and products of inertia in the x, y, z system shown, and then to use Eq. (20.43) to find the moment of inertia about the axis AB. Denote the total mass of the bar by $m = 8$ kg, the mass of each end element by $m_E = \frac{m}{4} = 2$ kg, and the mass of the central element by $m_C = \frac{m}{2} = 4$ kg.

The left end element: The moments and products of inertia about point A are:

$$I_{xx}^{(LA)} = \frac{m_E(1^2)}{3} = \frac{m}{12} = 0.6667 \text{ kg-m}^2,$$

$$I_{yy}^{(LA)} = \frac{m_E(1^2)}{3} = \frac{m}{12} = 0.6667 \text{ kg-m}^2,$$

$$I_{zz}^{(LA)} = 0,$$

$$I_{xy}^{(LA)} = I_{xz}^{(LA)} = I_{yz}^{(LA)} = 0.$$

The right end element: The moments and products of inertia about its center of mass are

$$I_{xx}^{(RG)} = \frac{m_E(1^2)}{12} = \frac{m}{48} = 0.1667 \text{ kg-m}^2,$$

$$I_{yy}^{(RG)} = 0,$$

$$I_{zz}^{(RG)} = \frac{m_E(1^2)}{12} = \frac{m}{48} = 0.1667 \text{ kg-m}^2,$$

$$I_{xx}^{(RG)} = I_{xz}^{(RG)} = I_{yz}^{(RG)} = 0.$$

The coordinates of the center of mass of the right end element are $(d_x, d_y, d_z) = (2, 0.5, 1)$. From Eq. (20.42), the moments and products of inertia in the x, y, z system are

$$I_{xx}^{(RA)} = I_{xx}^{(RG)} + (d_y^2 + d_z^2)\frac{m}{4} = 2.667 \text{ kg-m}^2,$$

$$I_{yy}^{(RA)} = I_{yy}^{(RG)} + (d_x^2 + d_z^2)\frac{m}{4} = 10.0 \text{ kg-m}^2$$

$$I_{zz}^{(RA)} = I_{zz}^{(RG)} + (d_x^2 + d_y^2)\frac{m}{4} = 8.667 \text{ kg-m}^2,$$

$$I_{xy}^{(RA)} = I_{xy}^{(RG)} + d_x d_y \frac{m}{4} = 2.0 \text{ kg-m}^2$$

$$I_{xz}^{(RA)} = I_{xz}^{(RG)} + d_x d_z \frac{m}{4} = 4.0 \text{ kg-m}^2,$$

$$I_{yz}^{(RA)} = I_{yz}^{(RG)} + d_y d_z \frac{m}{4} = 1.0 \text{ kg-m}^2$$

Sum the two inertia matrices:

$$[I]_{RLA} = [I]_{RA} + [I]_{LA} = \begin{bmatrix} 3.333 & -2 & -4 \\ -2 & 10.67 & -1 \\ -4 & -1 & 8.667 \end{bmatrix} \text{ kg-m}^2,$$

where the negative signs are a consequence of the definition of the inertia matrix.

The central element: The moments and products of inertia of the central element about its center of mass are:

$$I_{xx}^{(CG)} = 0, I_{yy}^{(CG)} = \frac{m_C(2^2)}{12} = \frac{m}{6} = 1.333 \text{ kg-m}^2,$$

$$I_{zz}^{(CG)} = \frac{m_C(2^2)}{12} = \frac{m}{6} = 1.333 \text{ kg-m}^2.$$

$$I_{xy}^{(CG)} = I_{xz}^{(CG)} = I_{yz}^{(CG)} = 0.$$

The coordinates of the center of mass of the central element are $(d_x, d_y, d_z) = (1, 0, 1)$. From Eq. (20.42) the moments and products of inertia in the x, y, z system are:

$$I_{xx}^{(CA)} = I_{xx}^{(CG)} + (d_y^2 + d_z^2)\frac{m}{2} = 4 \text{ kg-m}^2,$$

$$I_{yy}^{(CA)} = I_{yy}^{(CG)} + (d_x^2 + d_z^2)\frac{m}{2} = 9.333 \text{ kg-m}^2$$

$$I_{zz}^{(CA)} = I_{zz}^{(CG)} + (d_x^2 + d_y^2)\frac{m}{2} = 5.333 \text{ kg-m}^2,$$

$$I_{xy}^{(CA)} = I_{xy}^{(CG)} + d_x d_y \frac{m}{2} = 0,$$

$$I_{xz}^{(CA)} = I_{xz}^{(CG)} + d_x d_z \frac{m}{2} = 4 \text{ kg-m}^2,$$

$$I_{yz}^{(CA)} = I_{yz}^{(CG)} + d_y d_z \frac{m}{2} = 0$$

Sum the inertia matrices:

$$[I]_A = [I]_{RLA} + [I]_{CA} = \begin{bmatrix} 7.333 & -2 & -8 \\ -2 & 20.00 & -1 \\ -8 & -1 & 14.00 \end{bmatrix} \text{ kg-m}^2.$$

(a) *The moment of inertia about the axis AB*: The distance AB is parallel to the vector $\mathbf{r}_{AB} = 2\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$ (m). The unit vector parallel to $\mathbf{r}_{AB}$ is

$$\mathbf{e}_{AB} = \frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|} = 0.8165\mathbf{i} + 0.4082\mathbf{j} + 4082\mathbf{k}.$$

From Eq. (20.43), the moment of inertia about the AB axis is

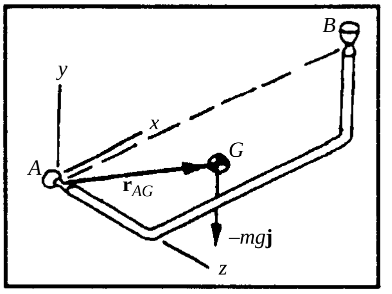
$$I_{AB} = I_{xx}^{(A)} e_x^2 + I_{yy}^{(A)} e_y^2 + I_{zz}^{(A)} e_z^2 - 2I_{xy}^{(A)} e_x e_y - 2I_{xz}^{(A)} e_x e_z$$

$$- 2I_{yz}^{(A)} e_y e_z, \boxed{I_{AB} = 3.56 \text{ kg-m}^2}.$$

(b) The angular momentum about the AB axis is

$$\boxed{H_{AB} = I_{AB}\omega = 3.56(4) = 14.22 \text{ kg-m}^2/\text{s}}$$

Problem 20.95* The 8-kg homogeneous slender bar in Problem 20.94 is released from rest in the position shown. (The x-z plane is horizontal.) What is the magnitude of the bar's angular acceleration about the axis AB at the instant of release?



Solution: The center of mass of the bar has the coordinates:

$$x_G = \frac{1}{m}\left(\left(\frac{m}{4}\right)0 + \left(\frac{m}{2}\right)(1) + \left(\frac{m}{4}\right)(2)\right) = 1 \text{ m}.$$

$$y_G = \frac{1}{m}\left(\left(\frac{m}{4}\right)(0) + \left(\frac{m}{2}\right)(0) + \left(\frac{m}{4}\right)(0.5)\right) = 0.125 \text{ m}.$$

$$z_G = \frac{1}{m}\left(\left(\frac{m}{4}\right)(0.5) + \left(\frac{m}{2}\right)(1) + \left(\frac{m}{4}\right)(1)\right) = 0.875 \text{ m}.$$

The line from A to the center of mass is parallel to the vector $\mathbf{r}_{AG} = \mathbf{i} + 0.125\mathbf{j} + 0.875\mathbf{k}$ (m). From the solution to Problem 20.94 the unit vector parallel to the line AB is $\mathbf{e}_{AB} = 0.8165\mathbf{i} + 0.4082\mathbf{j} + 0.4082\mathbf{k}$. The magnitude of the moment about line AB due to the weight is

$$e[\mathbf{r}_{AB} \times (-mg\mathbf{j})] = \begin{vmatrix} 0.8165 & 0.4082 & 0.4082 \\ 1.000 & 0.125 & 0.875 \\ 0 & -78.48 & 0 \end{vmatrix} = 24.03 \text{ N-m}.$$

From the solution to Problem 20.94, $I_{AB} = 3.556$ kg-m^2. From the equation of angular motion about axis AB, $M_{AB} = I_{AB}\alpha$, from which

$$\boxed{\alpha = \frac{24.03}{3.556} = 6.75 \text{ rad/s}^2}.$$

Problem 20.96 In terms of a coordinate system $x'y'z'$ with its origin at the center of mass, the inertia matrix of a rigid body is

$$[I'] = \begin{bmatrix} 20 & 10 & -10 \\ 10 & 60 & 0 \\ -10 & 0 & 80 \end{bmatrix} \text{ kg-m}^2.$$

Determine the principal moments of inertia and unit vectors parallel to the corresponding principal axes.

Solution: *Principal Moments of Inertia*: The moments and products:

$I_{x'x'} = 20 \text{ kg-m}^2$,

$I_{y'y'} = 60 \text{ kg-m}^2$,

$I_{z'z'} = 80 \text{ kg-m}^2$,

$I_{x'y'} = -10 \text{ kg-m}^2$,

$I_{x'z'} = 10 \text{ kg-m}^2$,

$I_{y'z'} = 0$.

From Eq. (20.45), the principal values are the roots of the cubic equation. $AI^3 + BI^2 + CI + D = 0$, where

$A = +1$,

$B = -(I_{x'x'} + I_{y'y'} + I_{z'z'}) = -160$,

$C = (I_{x'x'}I_{y'y'} + I_{y'y'}I_{z'z'} + I_{z'z'}I_{x'x'} - I_{x'y'}^2 - I_{x'z'}^2 - I_{y'z'}^2)$

$= 7400$,

$D = -(I_{x'x'}I_{y'y'}I_{z'z'} - I_{x'x'}I_{y'z'}^2 - I_{y'y'}I_{x'z'}^2$

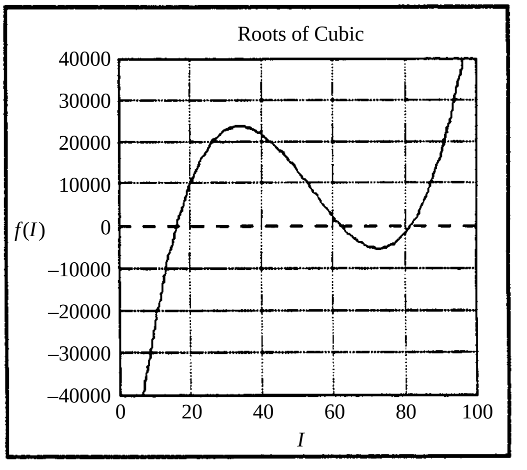
$- I_{z'z'}I_{x'y'}^2 - 2I_{x'y'}I_{y'z'}I_{x'z'})$

$= -82000$.

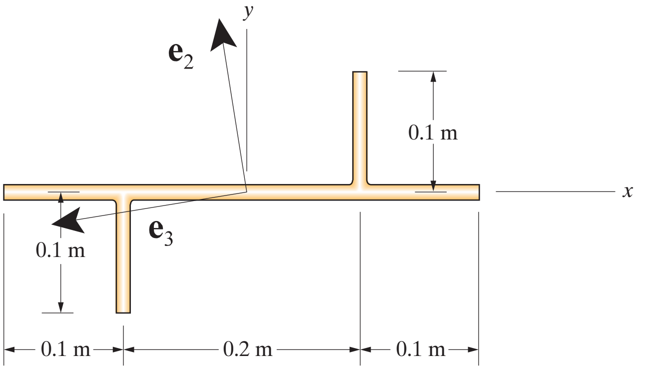
The function $f(I) = AI^3 + BI^2 + CI + D$ is graphed to find the zero crossings, and these values are refined by iteration. The graph is shown. The principal moments of inertia are:

$\boxed{I_1 = 16.15 \text{ kg-m}^2}$,

$\boxed{I_2 = 62.10 \text{ kg-m}^2}$, $\boxed{I_3 = 81.75 \text{ kg-m}^2}$.



Problem 20.97 For the object in Problem 20.81, determine the principal moments of inertia and unit vectors parallel to the corresponding principal axes. Draw a sketch of the object showing the principal axes.



Solution: In Problem 20.81, we found the inertia matrix to be

$$[I] = \begin{bmatrix} 0.00667 & -0.01 & 0 \\ -0.01 & 0.0733 & 0 \\ 0 & 0 & 0.08 \end{bmatrix} \text{ kg-m}^2$$

To find the principal inertia we expand the determinant to produce the characteristic equation

$$\det \begin{vmatrix} 0.00667 - I & -0.01 & 0 \\ -0.01 & 0.0733 - I & 0 \\ 0 & 0 & 0.08 - I \end{vmatrix}$$

$$= (0.08 - I)[(0.00667 - I)(0.0733 - I) - (0.01)^2] = 0$$

Solving this equation, we have

$$\boxed{I_1 = 0.08 \text{ kg-m}^2, \; I_2 = 0.0748 \text{ kg-m}^2, \; I_3 = 0.0052 \text{ kg-m}^2.}$$

Substituting these principal inertias into Eqs. (20.46) and dividing the resulting vector **V** by its magnitude, we find a unit vector parallel to the corresponding principal value.

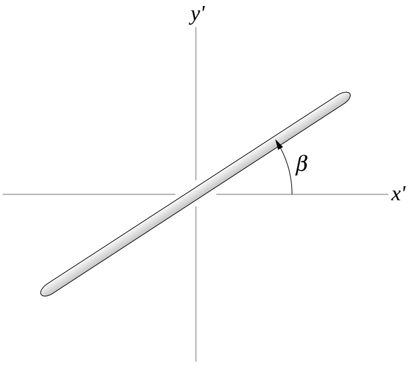
$$\boxed{\mathbf{e}_1 = 0.989\mathbf{i} + 0.145\mathbf{j}, \; \mathbf{e}_2 = -0.145\mathbf{i} + 0.989\mathbf{j}, \mathbf{e}_3 = \mathbf{k}.}$$

The principal axes are shown on the figure at the top of the page with $\mathbf{e}_1$ pointing out of the paper. The axes are rotated counterclockwise through the angle $\theta = \cos^{-1}(0.989) = 8.51°$.

Problem 20.98 The 1-kg, 1-m long slender bar lies in the x'-y' plane. Its moment of inertia matrix is

$$[I'] = \begin{bmatrix} \frac{1}{12}\sin^2\beta & -\frac{1}{12}\sin\beta\cos\beta & 0 \\ -\frac{1}{12}\sin\beta\cos\beta & \frac{1}{12}\cos^2\beta & 0 \\ 0 & 0 & \frac{1}{12} \end{bmatrix}.$$

Use Eqs. (20.45) and (20.46) to determine the principal moments of inertia and unit vectors parallel to the corresponding principal axes.



Solution: [*Preliminary Discussion:* The moment of inertia about an axis coinciding with the slender rod is zero; it follows that one principal value will be zero, and the associated principal axis will coincide with the slender bar. Since the moments of inertia about the axes normal to the slender bar will be equal, there will be two equal principal values, and Eq. (20.46) will fail to yield unique solutions for the associated characteristic vectors. However the problem can be solved by inspection: the unit vector parallel to the axis of the slender rod will be $\mathbf{e}_1 = \mathbf{i}\cos\beta + \mathbf{j}\sin\beta$. A unit vector orthogonal to $\mathbf{e}_1$ is $\mathbf{e}_2 = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta$. A third unit vector orthogonal to these two is $\mathbf{e}_3 = \mathbf{k}$. The solution based on Eq. (20.46) must agree with these preliminary results.]

Principal Moments of Inertia: The moments and products:

$$I_{x'x'} = \frac{\sin^2\beta}{12},\ I_{y'y'} = \frac{\cos^2\beta}{12},\ I_{z'z'} = \frac{1}{12},$$

$$I_{x'y'} = +\frac{\sin\beta\cos\beta}{12},\ I_{x'z'} = 0,\ I_{y'z'} = 0.$$

From Eq. (20.45), the principal values are the roots of the cubic, $AI^3 + BI^2 + CI + D = 0$. The coefficients are:

$$A = +1,$$

$$B = -(I_{x'x'} + I_{y'y'} + I_{z'z'}) = -\left(\frac{\sin^2\beta}{12} + \frac{\cos^2\beta}{12} + \frac{1}{12}\right) = -\frac{1}{6},$$

$$C = (I_{x'x'}I_{y'y'} + I_{y'y'}I_{z'z'} + I_{z'z'}I_{x'x'} - I_{x'y'}^2 - I_{x'z'}^2 - I_{y'z'}^2),$$

$$C = \frac{\sin^2\beta\cos^2\beta}{144} + \frac{\cos^2\beta}{144} + \frac{\sin^2\beta}{144} - \frac{\sin^2\beta\cos^2\beta}{144} = \frac{1}{144}.$$

$$D = -(I_{x'x'}I_{y'y'}I_{z'z'} - I_{x'x'}I_{y'z'}^2 - I_{y'y'}I_{x'z'}^2 - I_{z'z'}I_{x'y'}^2$$

$$- 2I_{x'y'}I_{y'z'}I_{x'z'}),$$

$$D = -\left(\frac{\sin^2\beta\cos^2\beta}{12^3} - \frac{\sin^2\beta\cos^2\beta}{12^3}\right) = 0.$$

The cubic equation reduces to

$$I^3 - \left(\frac{1}{6}\right)I^2 + \left(\frac{1}{144}\right)I = \left(I^2 - \left(\frac{1}{6}\right)I + \left(\frac{1}{144}\right)\right)I = 0.$$

By inspection, the least root is $\boxed{I_1 = 0}$. The other two roots are the solution of the quadratic $I^2 + 2bI + c = 0$ where $b = -\frac{1}{12}$, $c = \frac{1}{144}$, from which $I_{1,2} = -b \pm \sqrt{b^2 - c} = \frac{1}{12}$, from which $\boxed{I_2 = \frac{1}{12}}$, $\boxed{I_3 = \frac{1}{12}}$

Principal axes: The characteristic vectors parallel to the principal axes are obtained from Eq. (20.46),

$$V_{x'}^{(j)} = (I_{y'y'} - I_j)(I_{z'z'} - I_j) - I_{y'z'}^2$$

$$V_{y}^{(j)} = I_{x'y'}(I_{z'z'} - I_j) + I_{x'z'}I_{y'z'}$$

$$V_{z'}^{(j)} = I_{x'z'}(I_{y'y'} - I_j) + I_{x'y'}I_{y'z'}.$$

For the first root,

$$I_1 = 0,\ \mathbf{V}^{(1)} = \frac{\cos^2\beta}{144}\mathbf{i} + \frac{\cos\beta\sin\beta}{144}\mathbf{j} = \frac{\cos\beta}{144}(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}).$$

The magnitude:

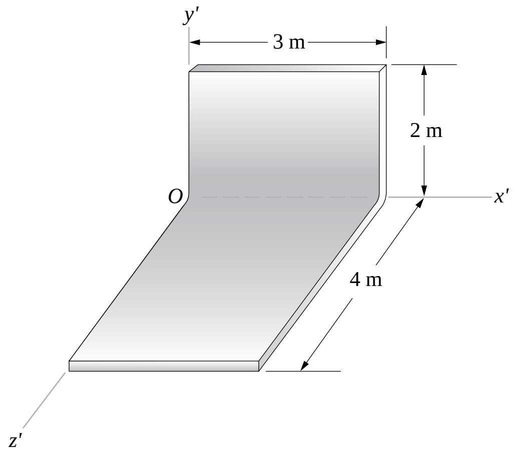
$$|\mathbf{V}^{(1)}| = \frac{|\cos\beta|}{144}\sqrt{\cos^2\beta + \sin^2\beta} = \frac{|\cos\beta|}{144},$$

and the unit vector is $\mathbf{e}_1 = \mathrm{sgn}(\cos\beta)(\cos\beta\mathbf{i} + \sin\beta\mathbf{j})$, where $\mathrm{sgn}(\cos\beta)$ is equal to the sign of $\cos\beta$. Without loss of generality, β can be restricted to lie in the first or fourth quadrants, from which

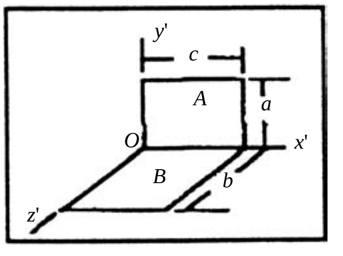
$$\boxed{\mathbf{e}_1 = (\cos\beta\mathbf{i} + \sin\beta\mathbf{j})}$$

For $I_2 = I_3 = \left(\frac{1}{12}\right)$, $\mathbf{V}^{(2)} = 0$, and $\mathbf{V}^{(3)} = 0$, from which *the equation* Eq. (20.46) *fails for the repeated principal values, and the characteristic vectors are to determined from the condition of orthogonality with* $\mathbf{e}_1$. From the preliminary discussion, $\boxed{\mathbf{e}_2 = -\mathbf{i}\sin\beta + \mathbf{j}\cos\beta}$ and $\boxed{\mathbf{e}_3 = 1\mathbf{k}}$.

Problem 20.99* The mass of the homogeneous thin plate is 3 kg. For a coordinate system with its origin at O, determine the plate's principal moments of inertia and unit vectors parallel to the corresponding principal axes.



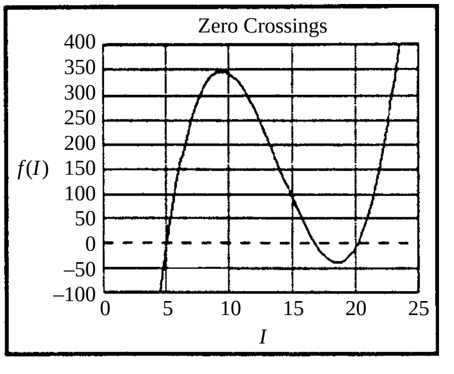
Solution: Divide the plate into A and B sheets, as shown. Denote $m = 3$ kg, $a = 2$ m, $b = 4$ m, and $c = 3$ m. The mass of plate A is $m_A = \dfrac{6m}{18} = \dfrac{m}{3} = 1$ kg. The mass of plate B is $m_B = \dfrac{12m}{18} =$ 2 kg. The coordinates of the center of mass of A are



$$(d_x^{(A)}, d_y^{(A)}, d_z^{(A)}) = \left(\frac{c}{2}, \frac{a}{2}, 0\right) = (1.5, 1, 0) \text{ m}.$$

The coordinates of the center of mass of B are

$$(d_x^{(B)}, d_y^{(B)}, d_z^{(B)}) = \left(\frac{c}{2}, 0, \frac{b}{2}\right) = (1.5, 0, 2) \text{ m}.$$



Principal Values: From Appendix C, the moments and products of inertia for plate A are

$$I_{x'x'}^{(A)} = \frac{m_A a^2}{12} + (d_y^2 + d_z^2)m_A = 1.333 \text{ kg-m}^2,$$

$$I_{y'y'}^{(A)} = \frac{m_A c^2}{12} + (d_x^2 + d_z^2)m_A = 3 \text{ kg-m}^2,$$

$$I_{z'z'}^{(A)} = \frac{m_A(c^2 + a^2)}{12} + (d_x^2 + d_y^2)m_A = 4.333 \text{ kg-m}^2,$$

$$I_{x'y'}^{(A)} = d_x d_y m_A = 1.5 \text{ kg-m}^2,$$

$$I_{x'z'}^{(A)} = 0,\ I_{y'z'}^{(A)} = 0.$$

The moments and products of inertia for plate B are

$$I_{x'x'}^{(B)} = \frac{m_B b^2}{12} + (d_y^2 + d_z^2)m_B = 10.67 \text{ kg-m}^2,$$

$$I_{y'y'}^{(B)} = \frac{m_B(c^2 + b^2)}{12} + (d_x^2 + d_z^2)m_B = 16.67 \text{ kg-m}^2,$$

$$I_{z'z'}^{(B)} = \frac{m_B c^2}{12} + (d_x^2 + d_y^2)m_B = 6 \text{ kg-m}^2,\ I_{x'y'}^{(B)} = 0,$$

$$I_{x'z'}^{(B)} = d_x d_z m_B = 6 \text{ kg-m}^2,\ I_{y'z'}^{(B)} = 0.$$

The inertia matrix is the sum of the two matrices:

$$[I'] = \begin{bmatrix} I_{x'x'} & -I_{x'y'} & -I_{x'z'} \\ -I_{y'x'} & I_{y'y'} & -I_{y'z'} \\ -I_{z'x'} & -I_{z'y'} & I_{z'z'} \end{bmatrix} = \begin{bmatrix} 12 & -15 & -6 \\ -1.5 & 19.67 & 0 \\ -6 & 0 & 10.33 \end{bmatrix}.$$

From Eq. (20.45), the principal values are the roots of the cubic equation $AI^3 + BI^2 + CI + D = 0$, where

$$A = +1,$$

$$B = -(I_{x'x'} + I_{y'y'} + I_{z'z'}) = -42,$$

$$C = (I_{x'x'}I_{y'y'} + I_{y'y'}I_{z'z'} + I_{z'z'}I_{x'x'} - I_{x'y'}^2 - I_{x'z'}^2 - I_{y'z'}^2)$$

$$= 524.97,$$

$$D = -(I_{x'x'}I_{y'y'}I_{z'z'} - I_{x'x'}I_{y'z'}^2 - I_{y'y'}I_{x'z'}^2 - I_{z'z'}I_{x'y'}^2$$

$$- 2I_{x'y'}I_{y'z'}I_{x'z'}) = -1707.4.$$

The function $f(I) = AI^3 + BI^2 + CI + D$ is graphed to determine the zero crossings, and the values refined by iteration. The graph is shown. The principal values are

$\boxed{I_1 = 5.042 \text{ kg-m}^2}$, $\boxed{I_2 = 16.79 \text{ kg-m}^2}$,

$\boxed{I_3 = 20.17 \text{ kg-m}^2}$.

Principal axes: The characteristic vectors parallel to the principal axes are obtained from Eq. (9.20),

$$V_{x'}^{(j)} = (I_{y'y'} - I_j)(I_{z'z'} - I_j) - I_{y'z'}^2$$

$$V_{y}^{(j)} = I_{x'y'}(I_{z'z'} - I_j) + I_{x'z'}I_{y'z'}$$

$$V_{z'}^{(j)} = I_{x'z'}(I_{y'y'} - I_j) + I_{x'y'}I_{y'z'}.$$

For $I_1 = 5.042$, $\mathbf{V}^{(1)} = 77.38\mathbf{i} + 7.937\mathbf{j} + 87.75\mathbf{k}$,

and $\boxed{\mathbf{e}_1 = 0.6599\mathbf{i} + 0.06768\mathbf{j} + 0.7483\mathbf{k}}$.

For $I_2 = 16.79$, $\mathbf{V}^{(2)} = -18.57\mathbf{i} - 9.687\mathbf{j} - 17.25\mathbf{k}$,

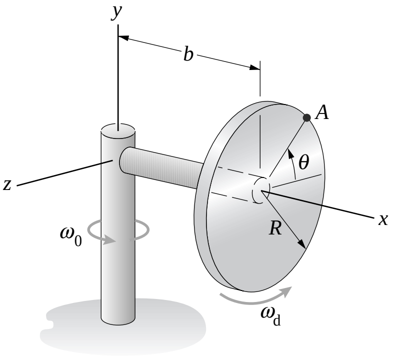
and $\boxed{\mathbf{e}_2 = -0.6843\mathbf{i} - 0.3570\mathbf{j} + 0.6358\mathbf{k}}$.

For $I_3 = 20.17$, $\mathbf{V}^{(3)} = 4.911\mathbf{i} - 14.75\mathbf{j} - 2.997\mathbf{k}$,

and $\mathbf{e}_3 = +0.3102\mathbf{i} - 0.9316\mathbf{j} - 0.1893\mathbf{k}$.

Problem 20.100 The disk is pinned to the horizontal shaft and rotates relative to it with angular velocity ω_0. Relative to an earth-fixed reference frame, the vertical shaft rotates with angular velocity ω_0.

(a) Determine the disk's angular velocity vector $\boldsymbol{\omega}$ relative to the earth-fixed reference frame.
(b) What is the velocity of point A of the disk relative to the earth-fixed reference frame?



Solution: From Problem 20.43, the inertia matrix is

$$[I] = \begin{bmatrix} 4.167 \times 10^5 & 2.5 \times 10^5 & 0 \\ 2.5 \times 10^5 & 4.167 \times 10^5 & 0 \\ 0 & 0 & 8.333 \times 10^5 \end{bmatrix} \text{ kg-m}^2.$$

For rotation at a constant rate, the angular acceleration is zero, $\boldsymbol{\alpha} = 0$. The body-fixed coordinate system rotates with angular velocity Eq. (20.19) reduces to:

$$\begin{bmatrix} \sum M_{Ox} \\ \sum M_{Oy} \\ \sum M_{Oz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.01047 \\ 0 & 0.01047 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 4.167 \times 10^5 & 2.5 \times 10^5 & 0 \\ 2.5 \times 10^5 & 4.167 \times 10^5 & 0 \\ 0 & 0 & 8.333 \times 10^5 \end{bmatrix}$$

$$\times \begin{bmatrix} 0.01047 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -8726.7 \\ 2618.0 & 4363.3 & 0 \end{bmatrix} \begin{bmatrix} 0.01047 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 27.42 \end{bmatrix} \text{ N-m}, \quad |\mathbf{M}_0| = 27.4 \text{ (N-m)}$$

Problem 20.101 The disk is pinned to the horizontal shaft and rotates relative to it with constant angular velocity ω_0. Relative to an earth-fixed reference frame, the vertical shaft rotates with constant angular velocity ω_0. What is the acceleration of point A of the disk relative to the earth-fixed reference frame?

Solution: (See Figure in solution to Problem 20.100.) The angular acceleration of the disk is given by

$$\frac{d\boldsymbol{\omega}}{dt} = \frac{d}{dt}(\omega_d \mathbf{i} + \omega_O \mathbf{j}) + \boldsymbol{\omega}_O \times \boldsymbol{\omega}_d = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \omega_O & 0 \\ \omega_d & 0 & 0 \end{bmatrix}$$

$$= -\omega_O \omega_d \mathbf{k}.$$

The velocity of point A relative to O is

$$\mathbf{a}_{A/O} = \boldsymbol{\alpha} \times \mathbf{r}_{A/O} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O})$$

$$= (-\omega_O \omega_d)(\mathbf{k} \times \mathbf{r}_{A/O}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}).$$

Term by term:

$$-\omega_O \omega_d (\mathbf{k} \times \mathbf{r}_{A/O}) = -\omega_O \omega_d \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \omega_O \omega_d R \sin\theta \mathbf{i} - \omega_O \omega_d b \mathbf{j},$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/O}) = \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ b & R\sin\theta & -R\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_d & \omega_O & 0 \\ -R\omega_O \cos\theta & R\omega_d \cos\theta & R\omega_d \sin\theta - b\omega_O \end{bmatrix}$$

$$= (R\omega_d \sin\theta - b\omega_O)(\omega_O \mathbf{i} - \omega_d \mathbf{j}) + (R\cos\theta)(\omega_d^2 + \omega_O^2)\mathbf{k}.$$

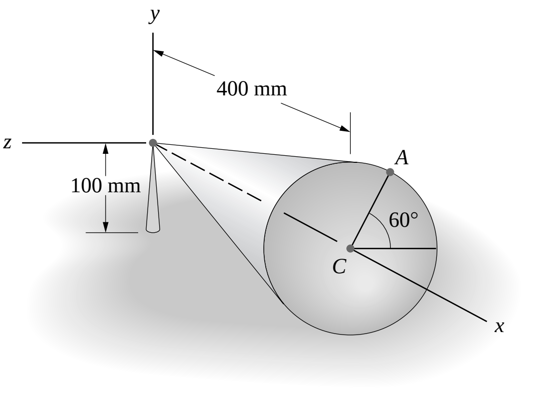
Collecting terms:

$$\mathbf{a}_{A/O} = (2R\omega_O\omega_d \sin\theta - b\omega_O^2)\mathbf{i} - (R\omega_d^2 \sin\theta)\mathbf{j}$$

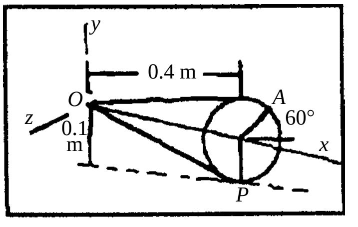
$$+ (R\omega_d^2 \cos\theta + R\omega_O^2 \cos\theta)\mathbf{k}.$$

Problem 20.102 The cone is connected by a ball-and-socket joint at its vertex to a 100-mm post. The radius of its base is 100 mm, and the base rolls on the floor. The velocity of the center of the base is $\mathbf{v}_C = 2\mathbf{k}$ (m/s).

(a) What is the cone's angular velocity vector $\boldsymbol{\omega}$?
(b) What is the velocity of point A?



Solution: Denote $\theta = 60^\circ$, $h = 0.1$ m, $L = 0.4$ m, $R = 0.1$ m.



(a) The strategy is to express the velocity of the center of the base of the cone, point C, in terms of the known (zero) velocities of O and P to formulate simultaneous equations for the angular velocity vector components.

The line OC is parallel to the vector $\mathbf{r}_{C/O} = \mathbf{i}L$ (m). The line PC is parallel to the vector $\mathbf{r}_{C/P} = R\mathbf{j}$ (m). The velocity of the center of the cone is given by the two expressions: $\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{C/O}$, and $\mathbf{v}_C = \mathbf{v}_P + \boldsymbol{\omega} \times \mathbf{r}_{C/P}$, where $\mathbf{v}_C = 2\mathbf{k}$ (m/s), and $\mathbf{v}_O = \mathbf{v}_P = 0$.

Expanding:

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ L & 0 & 0 \end{bmatrix} = \omega_z L\mathbf{j} - \omega_y L\mathbf{k}.$$

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_{C/P} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 0 & R & 0 \end{bmatrix} = -R\omega_z\mathbf{i} + R\omega_x\mathbf{k}.$$

Solve by inspection:

$$\omega_x = \frac{v_C}{R} = 20 \text{ rad/s}, \quad \omega_y = -\frac{v_C}{L} = -5 \text{ m/s},$$

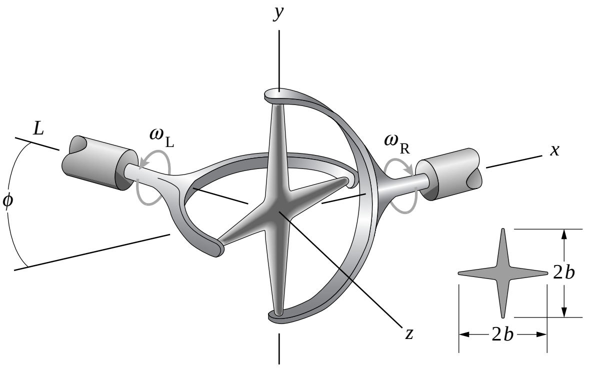
$$\omega_z = 0. \quad \boxed{\boldsymbol{\omega} = 20\mathbf{i} - 5\mathbf{j} \text{ (rad/s)}}$$

(b) The line OA is parallel to the vector $\mathbf{r}_{A/O} = \mathbf{i}L + \mathbf{j}R\sin\theta - \mathbf{k}R\cos\theta$. The velocity of point A is given by: $\mathbf{v}_A = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{r}_{A/O}$, where $\mathbf{v}_O = 0$.

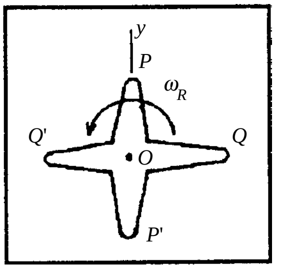
$$\mathbf{v}_A = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & -5 & 0 \\ L & R\sin\theta & -R\cos\theta \end{bmatrix},$$

$$\boxed{\mathbf{v}_A = 0.25\mathbf{i} + 1\mathbf{j} + 3.73\mathbf{k} \text{ (m/s)}}$$

Problem 20.103 The mechanism shown is a type of universal joint called a yoke and spider. The axis L lies in the x-z plane. Determine the angular velocity ω_L and the angular velocity vector $\boldsymbol{\omega}_S$ of the cross-shaped "spider" in terms of the angular velocity ω_R at the instant shown.



Solution: Denote the center of mass of the spider by point O, and denote the line coinciding with the vertical arms of the spider (the y axis) by $P'P$, and the line coinciding with the horizontal arms by $Q'Q$. The line $P'P$ is parallel to the vector $\mathbf{r}_{P/O} = b\mathbf{j}$. The angular velocity of the right hand yoke is positive along the x axis, $\boldsymbol{\omega}_R = \omega_R\mathbf{i}$, from which the angular velocity $\boldsymbol{\omega}_L$ is positive toward the right, so that $\boldsymbol{\omega}_L = \omega_L(\mathbf{i}\cos\phi + \mathbf{k}\sin\phi)$. The velocity of the point P on the extremities of the line $P'P$ is $\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}_R \times \mathbf{r}_{P/O}$, where $\mathbf{v}_O = 0$, from which



$$\mathbf{v}_P = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_R & 0 & 0 \\ 0 & b & 0 \end{bmatrix} = b\omega_R\mathbf{k}.$$

The velocity v_P is also given by

$$\mathbf{v}_P = \mathbf{v}_O + \boldsymbol{\omega}_S \times \mathbf{r}_{P/O} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_{Sx} & \omega_{Sy} & \omega_{Sz} \\ 0 & b & 0 \end{bmatrix}$$

$$= -\mathbf{i}b\omega_{Sz} + \mathbf{k}b\omega_{Sx},$$

from which $\omega_{Sz} = 0$, $\omega_{Sx} = \omega_R$. The line $Q'Q$ is parallel to the vector $\mathbf{r}_{Q/O} = \mathbf{i}b\sin\phi - \mathbf{k}b\cos\phi$. The velocity of the point Q is

$$\mathbf{v}_Q = \mathbf{v}_O + \boldsymbol{\omega}_S \times \mathbf{r}_{Q/O} = 0 + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_{Sx} & \omega_{Sy} & 0 \\ b\sin\phi & 0 & -b\cos\phi \end{bmatrix},$$

$$\mathbf{v}_Q = -\mathbf{i}(\omega_{Sy}b\cos\phi) + \mathbf{j}(\omega_{Sx}b\cos\phi) - \mathbf{k}(\omega_{Sy}b\sin\phi).$$

The velocity $\mathbf{v}_Q$ is also given by $\mathbf{v}_Q = \mathbf{v}_O + \boldsymbol{\omega}_L \times \mathbf{r}_{Q/O}$, from which

$$\mathbf{v}_Q = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_L\cos\phi & 0 & \omega_L\sin\phi \\ b\sin\phi & 0 & -b\cos\phi \end{bmatrix} = \mathbf{j}b\omega_L(\cos^2\phi + \sin^2\phi)$$

$$= \mathbf{j}b\omega_L,$$

from which $\omega_{Sy} = 0$, from which

$$\omega_S = \omega_R,$$

$\boxed{\boldsymbol{\omega}_S = \omega_R\mathbf{i}}$, $\boxed{\omega_L = \omega_{Sx}\cos\phi = \omega_R\cos\phi}$.

Problem 20.104 The inertia matrix of a rigid body in terms of a body-fixed coordinate system with its origin at the center of mass is

$$[I] = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \text{ kg-m}^2.$$

If the rigid body's angular velocity is $\boldsymbol{\omega} = 10\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$ (rad/s), what is its angular momentum about its center of mass?

Solution: The angular momentum is

$$\begin{bmatrix} H_{Ox} \\ H_{Oy} \\ H_{Oz} \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix} = \begin{bmatrix} 25 \\ 0 \\ 50 \end{bmatrix} \text{ kg-m}^2/\text{s}$$

In terms of the unit vectors **i**, **j**, **k**, $\boxed{\mathbf{H} = 25\mathbf{i} + 50\mathbf{k} \text{ kg-m}^2/\text{s}}$

Problem 20.105 What is the moment of inertia of the rigid body in Problem 20.104 about the axis that passes through the origin and the point (4, −4, 7) m?

Strategy: Determine the components of a unit vector parallel to the axis, and use Eq. (20.43).

Solution: The unit vector parallel to the line passing through (0, 0, 0) and (4, −4, 7) is

$$\mathbf{e} = \frac{4\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}}{\sqrt{4^2 + 4^2 + 7^2}} = 0.4444\mathbf{i} - 0.4444\mathbf{j} + 0.7778\mathbf{k}.$$

The inertia matrix in Problem 20.104 is

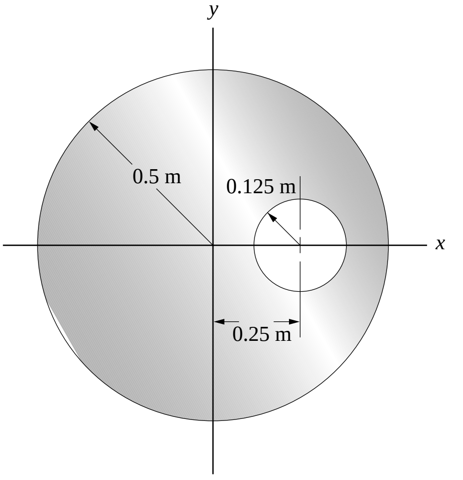
$$[I] = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix},$$

where advantage is taken of the symmetric property of the inertia matrix. From Eq. (20.43), the new moment of inertia about the line through (0, 0, 0) and (4, −4, 7) is

$$I_O = 4e_x^2 + 2e_y^2 + 6e_z^2 + 2(1)(e_x e_y) + 2(-1)(e_x e_z) + 2(0)(e_y e_z),$$

$$\boxed{I_O = 3.728 \text{ kg-m}^2}$$

Problem 20.106 Determine the inertia matrix of the 0.6 kg thin plate in terms of the coordinate system shown.



Solution: The strategy is to determine the moments and products for a solid thin plate about the origin, and then subtract the moments and products of the cut-out. The mass density is

$$\rho = \frac{0.6}{\left(\pi(0.5^2) - \pi\left(\frac{3}{24}\right)^2\right)T} = \frac{0.8149}{T}\ \text{kg/m}^3,$$

from which $\rho T = 0.8149$ kg/m^2, where T is the (unknown) thickness of the plate. From Appendix C and by inspection, the moments and products of inertia for a thin plate of radius R are:

$$I_{xx} = I_{yy} = \frac{mR^2}{4},\ I_{zz} = \frac{mR^2}{2},\ I_{xy} = I_{xz} = I_{yz} = 0.$$

For a 6 *in.* radius *solid thin plate*, $m = \rho T\pi(0.5^2) = 0.64$ kg. $I_{xx} = I_{yy} = 0.04$ kg-m^2, $I_{zz} = 0.08$ kg-m^2, $I_{xy} = I_{xz} = I_{yz} = 0$. The coordinates of the 0.125 m radius cut-out are $(d_x, d_y, d_x) = (3, 0, 0)$. The mass removed by the cut-out is $m_C = \rho T\pi R_C^2 = 0.04$ kg. The moments and products of inertia of the cut-out are

$$I_{xx}^C = \frac{m_C R_C^2}{4} = 1.563 \times 10^{-4}\ \text{kg-m}^2,$$

$$I_{yy}^C = \frac{m_C R_C^2}{4} + \left(\frac{3}{12}\right)^2 m_C = 2.656 \times 10^{-3}\ \text{kg-m}^2,$$

$$I_{zz}^C = \frac{m_C R_C^2}{2} + d_x^2 m_C = 2.813 \times 10^{-3}\ \text{kg-m}^2,$$

$$I_{xy}^C = 0,\ I_{xz}^C = 0,\ I_{yz}^C = 0.$$

The inertia matrix of the plate with the cut-out is

$$[I]_0 = \begin{bmatrix} 0.04 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.08 \end{bmatrix}$$

$$- \begin{bmatrix} 1.563 \times 10^{-4} & 0 & 0 \\ 0 & 2.656 \times 10^{-3} & 0 \\ 0 & 0 & 2.813 \times 10^{-3} \end{bmatrix},$$

$$\boxed{[I]_0 = \begin{bmatrix} 0.0398 & 0 & 0 \\ 0 & 0.0373 & 0 \\ 0 & 0 & 0.0772 \end{bmatrix} \text{kg-m}^2}$$

Problem 20.107 At $t = 0$, the plate in Problem 20.106 has angular velocity $\boldsymbol{\omega} = 10\mathbf{i} + 10\mathbf{j}$ (rad/s) and is subjected to the force $\mathbf{F} = -10\mathbf{k}$ (N) acting at the point $(0, 0.5, 0)$ m. No other forces or couples act on the plate. What are the components of its angular acceleration at that instant?

Solution: The coordinates of the center of mass are $(-0.01667, 0, 0)$ m. The vector from the center of mass to the point of application of the force is $\mathbf{r}_{F/G} = 0.01667\mathbf{i} + 0.5\mathbf{j}$ (m). The moment about the center of mass of the plate is

$$\mathbf{M}_G = \mathbf{r}_{F/G} \times \mathbf{F} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.01667 & 0.5 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

$$= -5\mathbf{i} + 0.1667\mathbf{j} \text{ (N-m)}.$$

Eq. (20.19) reduces to

$$\begin{bmatrix} -5 \\ 0.1667 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.03984 & 0 & -0 \\ 0 & 0.03718 & 0 \\ 0 & 0 & 0.07702 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -0.267 \end{bmatrix}.$$

Carry out the matrix multiplication to obtain the three equations:

$0.03984\alpha_x = -5$, $0.03718\alpha_y = 0.1667$, $0.07702\alpha_z - 0.267 = 0$.

Solve: $\boxed{\boldsymbol{\alpha} = -125.5\mathbf{i} + 4.484\mathbf{j} + 3.467\mathbf{k} \text{ rad/s}^2}$

Problem 20.108 The inertia matrix of a rigid body in terms of a body-fixed coordinate system with its origin at the center of mass is

$$[I] = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \text{ kg-m}^2.$$

If the rigid body's angular velocity is $\boldsymbol{\omega} = 10\mathbf{i} - 5\mathbf{j} + 10\mathbf{k}$ (rad/s) and its angular acceleration is zero, what are the components of the total moment about its center of mass?

Solution: Use general motion, Eq. (20.19),

$$\begin{bmatrix} \sum M_{\phi x} \\ \sum M_{\phi y} \\ \sum M_{\phi z} \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

with $\boldsymbol{\alpha} = \dfrac{d\boldsymbol{\omega}}{dt} = 0$. The coordinate system is rotating with angular velocity $\boldsymbol{\omega}$, from which $\boldsymbol{\omega} = \boldsymbol{\omega}$. Eq. (20.19) reduces to

$$\begin{bmatrix} \sum M_{\phi x} \\ \sum M_{\phi y} \\ \sum M_{\phi z} \end{bmatrix} = \begin{bmatrix} 0 & -10 & -5 \\ 10 & 0 & -10 \\ 5 & 10 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -20 & -30 \\ 50 & 10 & -70 \\ 30 & 25 & -5 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix} \text{ N-m},$$

$$\boxed{\mathbf{M} = -250\mathbf{i} - 250\mathbf{j} + 125\mathbf{k} \text{ (N-m)}}$$

Problem 20.109 If the total moment about the center of mass of the rigid body described in Problem 20.108 is zero, what are the components of its angular acceleration?

Solution: Use general motion, Eq. (20.19), with the moment components equated to zero,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} + \begin{bmatrix} 0 & -10 & -5 \\ 10 & 0 & -10 \\ 5 & 10 & 0 \end{bmatrix}$$

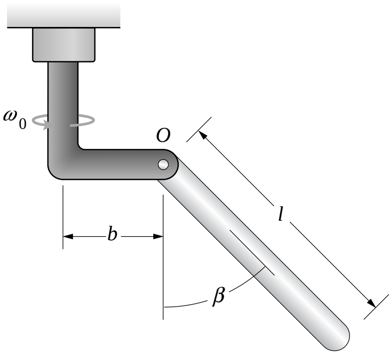
$$\times \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 10 \end{bmatrix},$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} + \begin{bmatrix} -250 \\ -250 \\ 125 \end{bmatrix}.$$

Carry out the matrix multiplication to obtain the three simultaneous equations in the unknowns: $4\alpha_x + \alpha_y - \alpha_z = 250$, $\alpha_x + 2\alpha_y + 0 = 250$, $-\alpha_x + 0 + 6\alpha_z = -125$.

Solve: $\boxed{\boldsymbol{\alpha} = 31.25\mathbf{i} + 109.4\mathbf{j} - 15.63\mathbf{k} \text{ (rad/s}^2)}$

Problem 20.110 The slender bar of length l and mass m is pinned to the L-shaped bar at O. The L-shaped bar rotates about the vertical axis with a constant angular velocity ω_0. Determine the value of ω_0 necessary for the bar to remain at a constant angle β relative to the vertical.



Solution: Since the point O is not fixed, this is general motion, in which Eq. (20.19) applies. Choose a coordinate system with the origin at O and the x axis parallel to the slender bar.

The moment exerted by the bar. Since axis of rotation is fixed, the acceleration must be taken into account in determining the moment. The vector distance from the axis of rotation in the coordinates system shown is

$$\mathbf{r}_0 = b(\mathbf{i}\cos(90^\circ - \beta) + \mathbf{j}\sin(90^\circ - \beta)) = b(\mathbf{i}\sin\beta + \mathbf{j}\cos\beta).$$

The vector distance to the center of mass of the slender bar is $\mathbf{r}_{G/O} = \mathbf{i}\left(\frac{L}{2}\right)$. The angular velocity is a constant and the coordinate system is rotating with an angular velocity $\boldsymbol{\omega} = \omega_0(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta)$. The acceleration of the center of mass relative to O is

$$\mathbf{a}_G = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times (\mathbf{r}_0 + \mathbf{r}_{G/O}))$$

$$= \boldsymbol{\omega} \times \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega_0\cos\beta & \omega_0\sin\beta & 0 \\ b\sin\beta + \dfrac{L}{2} & b\cos\beta & 0 \end{bmatrix},$$

$$\mathbf{a}_G = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\omega_0\cos\beta & \omega_0\sin\beta & 0 \\ 0 & 0 & -\omega_0 b - \dfrac{\omega_0 L\sin\beta}{2} \end{bmatrix}$$

$$= a_{Gx}\mathbf{i} + a_{Gy}\mathbf{j}$$

$$\mathbf{a}_G = -\omega_0^2\left(+b\sin\beta + \frac{L}{2}\sin^2\beta\right)\mathbf{i}$$

$$-\omega_0^2\left(b\cos\beta + \frac{L}{2}\sin\beta\cos\beta\right)\mathbf{j}.$$

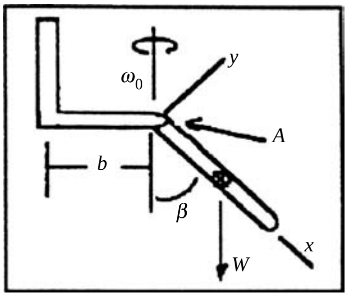
From Newton's second law, $m\mathbf{a}_G = \mathbf{A} + \mathbf{W}$, from which $\mathbf{A} = m\mathbf{a}_G - \mathbf{W}$. The weight is $\mathbf{W} = mg(\mathbf{i}\cos\beta - \mathbf{j}\sin\beta)$. The moment about the center of mass is

$$\mathbf{M}_G = \mathbf{r}_{O/G} \times \mathbf{A} = \mathbf{r}_{O/G} \times (m\mathbf{a}_G - \mathbf{W})$$

$$= \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\dfrac{L}{2} & 0 & 0 \\ ma_{Gx} - mg\cos\beta & ma_{Gy} + mg\sin\beta & 0 \end{bmatrix}.$$

$$\mathbf{M}_G = +\left(\frac{m\omega_o^2 bL}{2}\cos\beta - \frac{mgL}{2}\sin\beta + \frac{m\omega_0^2 L^2}{4}\sin\beta\cos\beta\right)\mathbf{k}$$

$$= M_z\mathbf{k}$$



The Euler Equations: The moments of inertia of the bar about the center of mass are $I_{xx} = 0$, $I_{yy} = I_{zz} = \dfrac{mL^2}{12}$, $I_{xy} = I_{xz} = I_{yz} = 0$. Eq. (20.19) becomes:

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \omega_o^2 \begin{bmatrix} 0 & 0 & +\sin\beta \\ 0 & 0 & \cos\beta \\ -\sin\beta & -\cos\beta & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \dfrac{mL^2}{12} & 0 \\ 0 & 0 & \dfrac{mL^2}{12} \end{bmatrix}$$

$$\times \begin{bmatrix} -\cos\beta \\ \sin\beta \\ 0 \end{bmatrix}.$$

Carry out the matrix multiplication:

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{\omega_0^2 mL^2}{12}\cos\beta\sin\beta \end{bmatrix}.$$

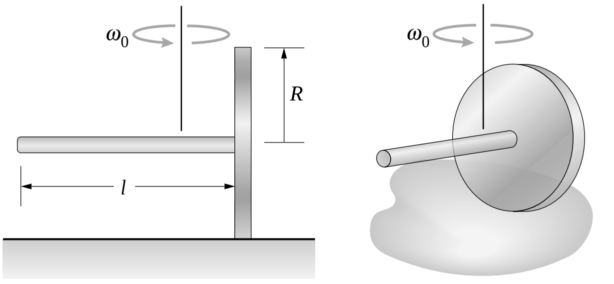
Substitute:

$$\frac{m\omega_0^2 bL}{2}\cos\beta - \frac{mgL}{2}\sin\beta + \frac{m\omega_0^2 L^2}{4}\sin\beta\cos\beta$$

$$= -\omega_0^2\frac{mL^2}{12}\sin\beta\cos\beta.$$

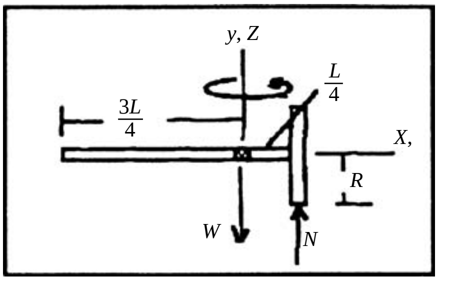
Solve : $\boxed{\omega_0 = \sqrt{\dfrac{g\sin\beta}{\left(\dfrac{2}{3}\right)L\sin\beta\cos\beta + b\cos\beta}}}$

Problem 20.111 A slender bar of length l and mass m is rigidly attached to the center of a thin circular disk of radius R and mass m. The composite object undergoes a motion in which the bar rotates in the horizontal plane with constant angular velocity ω_0 about the center of mass of the composite object and the disk rolls on the floor. Show that $\omega_0 = 2\sqrt{g/R}$.



Solution: Measuring from the left end of the slender bar, the distance to the center of mass is

$$d_G = \frac{\left(\frac{L}{2}\right)m + mL}{2m} = \frac{3L}{4}.$$



Choose an X, Y, Z coordinate system with the origin at the center of mass, the Z axis parallel to the vertical axis of rotation and the X axis parallel to the slender bar. Choose an x, y, z coordinate system with the origin at the center of mass, the z axis parallel to the slender bar, and the y axis parallel to the Z axis. By definition, the nutation angle is the angle between Z and z, $\theta = 90^\circ$. The precession rate is the rotation about the Z axis, $\dot{\psi} = \omega_0$ rad/s. The velocity of the center of mass of the disk is $v_G = (L/4)\dot{\psi}$, from which the spin rate is $\dot{\phi} = \frac{v}{R} = \left(\frac{L}{4R}\right)\dot{\psi}$. From Eq. (20.29), the moment about the x-axis is

$$M_x = (I_{zz} - I_{xx})\dot{\psi}^2 \sin\theta\cos\theta + I_{zz}\dot{\psi}\dot{\phi}\sin\theta = I_{zz}\dot{\phi}\dot{\psi} = \frac{\omega_0^2 L}{4R} I_{zz}.$$

The moment of inertia is $I_{zz} = \frac{mR^2}{2}$, from which

$$M_x = \left(\frac{mRL}{8}\right)\omega_0^2.$$

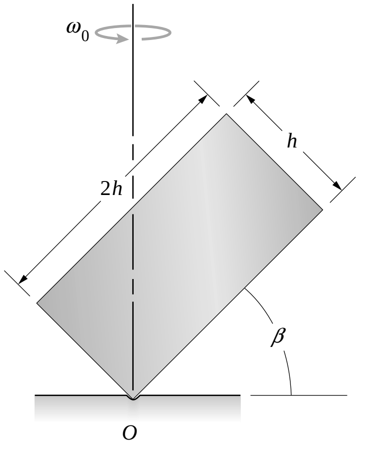
The normal force acting at the point of contact is $N = 2mg$. The moment exerted about the center of mass

$$M_G = N\frac{L}{4} = mg\left(\frac{L}{2}\right).$$

Equate the moments: $mg\left(\frac{L}{2}\right) = m\left(\frac{RL}{8}\right)\omega_0^2$, from which

$$\boxed{\omega_0 = 2\sqrt{\frac{g}{R}}}.$$

Problem 20.112* The thin plate of mass m spins about a vertical axis with the plane of the plate perpendicular to the floor. The corner of the plate at O rests in an indentation, so that it remains at the same point on the floor. The plate rotates with constant angular velocity ω_0 and the angle β is constant.



(a) Show that the angular velocity ω_0 is related to the angle β by

$$\frac{h\omega_0^2}{g} = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\sin\beta\cos\beta - \cos^2\beta}$$

(b) The equation you obtained in (a) indicates that $\omega_0 = 0$ when $2\cos\beta - \sin\beta = 0$. What is the interpretation of this result?

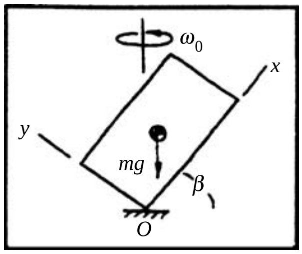
Solution:

Choose a body-fixed coordinate system with its origin at the fixed point O and the axes aligned with the plate's edges. Using the moments of intertia for a rectangular area,

$$I_{xx} = \frac{mh^2}{3},\; I_{yy} = \frac{4mh^2}{3},\; I_{zz} = I_{xx} + I_{yy} = \frac{5mh^2}{3},$$

$$I_{xy} = \frac{mh^2}{2},\; I_{xz} = I_{yz} = 0.$$

The plate's angular velocity is $\boldsymbol{\omega} = \omega_0 \sin\beta\mathbf{i} + \omega_0\cos\beta\mathbf{j}$, and the moment about O due to the plate's weight is $\sum M_x = 0$, $\sum M_y = 0$, $\sum M_z = \frac{h}{2}mg\sin\beta - hmg\cos\beta$.



Choose a coordinate system with the x axis parallel to the right lower edge of the plate and the y axis parallel to the left lower edge of the plate, as shown. The body fixed coordinate system rotates with angular velocity $\boldsymbol{\omega} = \omega_0(\mathbf{j}\sin\beta + \mathbf{k}\cos\beta)$. From Eq. (20.13),

$$\begin{bmatrix} 0 \\ 0 \\ M_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0\cos\beta \\ 0 & 0 & -\omega_0\sin\beta \\ -\omega_0\cos\beta & \omega_0\sin\beta & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{mh^2}{3} & -\frac{mh^2}{2} & 0 \\ -\frac{mh^2}{2} & \frac{4mh^2}{3} & 0 \\ 0 & 0 & \frac{5m^2}{3} \end{bmatrix} \begin{bmatrix} \omega_0\cos\beta \\ \omega_0\cos\beta \\ 0 \end{bmatrix}.$$

Expand, $$M_z = hmg\left(\frac{\sin\beta}{2} - \cos\beta\right)$$

$$= mh^2\omega_a^2\left(-\frac{\sin\beta\cos\beta}{3} + \frac{\cos^2\beta}{2} - \frac{\sin^2\beta}{2}\right.$$

$$\left. + \frac{4\sin\beta\cos\beta}{3}\right)$$

Solve: $$\boxed{\frac{\omega_0^2 h}{g} = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\sin\beta\cos\beta - \cos^2\beta}}$$

(b) The perpendicular distance from the axis of rotation to the center of mass of the plate is

$$d = \left|\mathbf{r}_{O/G} \times \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right| = \left\| \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\frac{h}{2} & -h \\ 0 & \cos\beta & \sin\beta \end{bmatrix} \right\|$$

$$= h\left(\cos\beta - \frac{\sin\beta}{2}\right).$$

If this distance is zero, $\beta = \tan^{-1}(2) = 63.43^\circ$, the accelerations of the center of mass and the external moments are zero (see equations above, where for convenience the term $\cos\beta - \frac{\sin\beta}{2}$ has been kept as a factor) and the plate is balanced.

The *angular velocity of rotation is zero* (the plate is stationary) *if* $\beta = \tan^{-1}(2) = 63.435^\circ$, since the numerator of the right hand term in the boxed expression vanishes (the balance at this point would be very unstable, since an infinitesimally small change in β would induce a destabilizing moment.).

Problem 20.113* In Problem 20.112, determine the range of values of the angle β for which the plate will remain in the steady motion described.

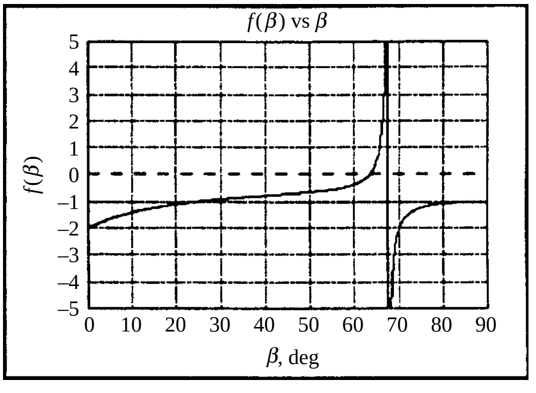
Solution: From the solution to Problem 20.112

$$\frac{\omega_0^2 h}{g} = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\sin\beta\cos\beta - \cos^2\beta}.$$

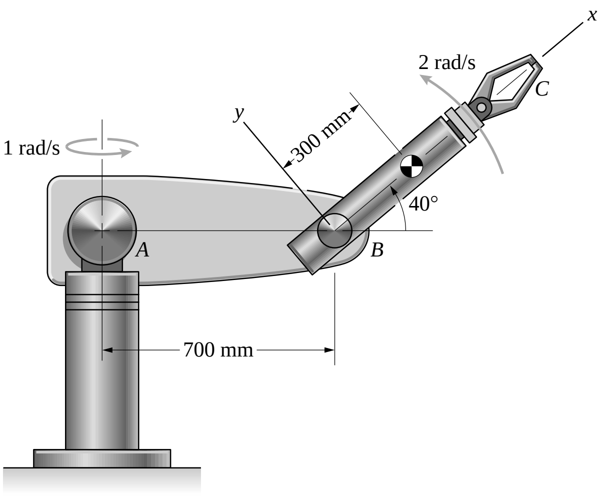
The angular velocity is a real number, from which $\omega_0^2 \geq 0$, from which

$$f(\beta) = \frac{2\cos\beta - \sin\beta}{\sin^2\beta - 2\cos\beta\sin\beta - \cos^2\beta} \geq 0.$$

A graph of $f(\beta)$ for values of $0 \leq \beta \leq 90^\circ$ is shown. The function is positive over the half-open interval $63.4348^\circ \leq \beta < 67.50^\circ$. The angular velocity is zero at the lower end of the interval, and "blows up" (becomes infinite) when the denominator vanishes, which occurs at *exactly* $\beta = \dfrac{3\pi}{8} = 67.50^\circ$.



Problem 20.114* Arm BC has a mass of 12 kg, and its moments and products of inertia, in terms of the coordinate system shown, are $I_{xx} = 0.03$ kg-m^2, $I_{yy} = I_{zz} = 4$ kg-m^2, and $I_{xy} = I_{yz} = I_{xz} = 0$. At the instant shown, arm AB is rotating in the horizontal plane with a constant angular velocity of 1 rad/s in the counterclockwise direction viewed from above. Relative to arm AB, arm BC is rotating about the z axis with a constant angular velocity of 2 rad/s. Determine the force and couple exerted on arm BC at B.



Solution: In terms of the body-fixed coordinate system shown, the moments and products of inertia are

$I_{xx} = 0.03$ kg-m^2,

$I_{yy} = I_{zz} = 4 - (0.3)^2(12) = 2.92$ kg-m^2,

$I_{xy} = I_{yz} = I_{zx} = 0.$

In terms of the angle θ, the angular velocity of the coordinate system is

$$\boldsymbol{\omega} = (1)\sin\theta\,\mathbf{i} + (1)\cos\theta\,\mathbf{j} + (2)\mathbf{k} \text{ (rad/s)},$$

so its angular acceleration is Eq. (20.4)

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} = \cos\theta\frac{d\theta}{dt}\mathbf{i} - \sin\theta\frac{d\theta}{dt}\mathbf{j}.$$

Setting $\theta = 40^\circ$ and $d\theta/dt = 2$ rad/s, we obtain

$$\boldsymbol{\omega} = 0.643\mathbf{i} + 0.766\mathbf{j} + 2\mathbf{k} \text{ (rad/s)}, \quad \textbf{(1)}$$

$$\boldsymbol{\alpha} = 1.532\mathbf{i} - 1.286\mathbf{j} \text{ (rad/s}^2). \quad \textbf{(2)}$$

Equation (20.19) is

$$\begin{bmatrix} M_{Bx} \\ M_{By} + 0.3F_{Bz} \\ M_{Bz} - 0.3F_{By} \end{bmatrix} = \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 2.92 & 0 \\ 0 & 0 & 2.92 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 2.92 & 0 \\ 0 & 0 & 2.92 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}.$$

Using Eqs. (1) and (2), this gives the equations

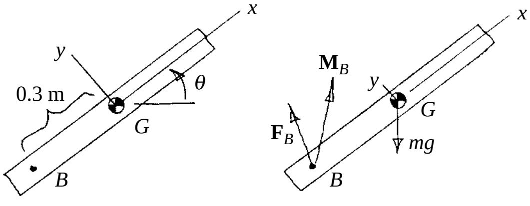
$$M_{Bx} = 0.046,$$

$$M_{By} + 0.3F_{Bz} = -7.469,$$

$$M_{Bz} - 0.3F_{By} = 1.423,$$

From which

$$\underline{\mathbf{M}_B = 0.046\mathbf{i} - 10.25\mathbf{j} + 30.63\mathbf{k} \text{ (N-m)}}.$$



The acceleration of B toward A is $(1 \text{ rad/s})^2(0.7 \text{ m}) = 0.7$ m/s^2, so

$$\mathbf{a}_B = -0.7\cos 40^\circ\mathbf{i} + 0.7\sin 40^\circ\mathbf{j}$$

$$= -0.536\mathbf{i} + 0.450\mathbf{j} \text{ (m/s}^2).$$

The acceleration of the center of mass is

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{G/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{G/B})$$

$$= -1.912\mathbf{i} + 0.598\mathbf{j} + 0.771\mathbf{k} \text{ (m/s}^2).$$

From Newton's second law (see free-body diagram),

$$F_{Bx} - mg\sin 40^\circ = m(-1.912),$$

$$F_{By} - mg\cos 40^\circ = m(0.598),$$

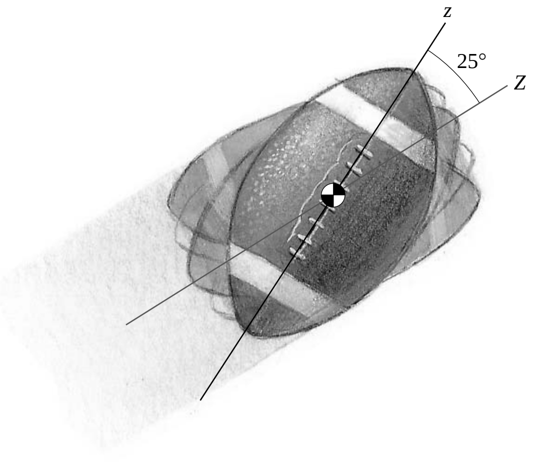
$$F_{Bz} = m(0.771).$$

Solving, $\underline{\mathbf{F}_B = 52.72\mathbf{i} + 97.35\mathbf{j} + 9.26\mathbf{k} \text{ (N)}}$. The moment about the center of mass is

$$\sum \mathbf{M} = \mathbf{M}_B + (-0.3\mathbf{i}) \times \mathbf{F}_B$$

$$= \mathbf{M}_B + 0.3F_{Bz}\mathbf{j} - 0.3F_{By}\mathbf{k}.$$

Problem 20.115 Suppose that you throw a football in a wobbly spiral with a nutation angle of 25°. The football's moments of inertia are $I_{xx} = I_{yy} = 0.0041$ kg-m^2 and $I_{zz} = 0.00136$ kg-m^2. If the spin rate is $\dot{\phi} = 4$ revolutions per second, what is the magnitude of the precession rate (the rate at which the football wobbles)?



Solution: This is modeled as moment-free, steady precession of an axisymmetric object. From Eq. (20.33), $(I_{zz} - I_{xx})\dot{\psi}\cos\theta + I_{zz}\dot{\phi} = 0$, from which

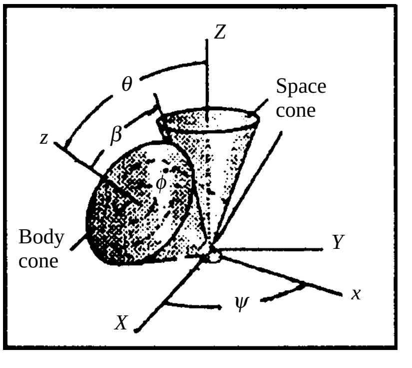
$$\dot{\psi} = -\frac{I_{zz}\dot{\phi}}{(I_{zz} - I_{xx})\cos\theta}.$$

Substitute $\dot{\psi} = -\dfrac{(0.00136 \text{ kg-m}^2)(4)}{(0.00136 \text{ kg-m}^2 - 0.0041 \text{ kg-m}^2)\cos 25^\circ} = 2.21$ rev/s.

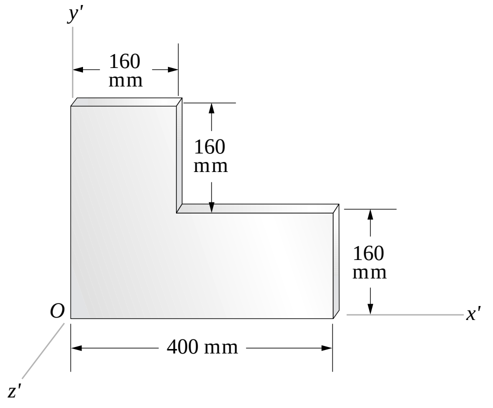
$$\boxed{|\dot{\psi}| = 2.21 \text{ rev/s}}$$

Problem 20.116 Sketch the body and space cones for the motion of the football in Problem 20.115.

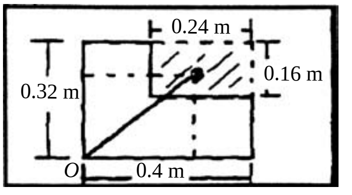
Solution: The angle β is given by $\tan\beta = \left(\dfrac{I_{zz}}{I_{xx}}\right)\tan\theta$, from which $\beta = 8.8^\circ$. $\beta < \theta$, and the body cone revolves outside the space cone. The sketch is shown. The angle $\theta = 25^\circ$.



Problem 20.117 The mass of the homogeneous thin plate is 1 kg. For a coordinate system with its origin at O, determine the plate's principal moments of inertia and the directions of unit vectors parallel to the corresponding principal axes.



Solution: The moment of inertia is determined by the strategy of determining the moments and products of a larger plate, and then subtracting the moments and products of inertia of a cutout, as shown in the sketch. Denote $h = 0.32$ m, $b = 0.4$ m, $c = 0.16$ m, $d = b - c = 0.24$ m. The area of the plate is $A = hb - cd = 0.0896$ m^2. Denote the mass density by ρ kg/m^3. The mass is $\rho AT = 1$ kg, from which $\rho T = \frac{1}{A} = 11.16$ kg/m^2, where T is the (unknown) thickness. *The moments and products of inertia of the large plate*: The moments and products of inertia about O are (See Appendix C)



$$I_{xx}^{(p)} = \frac{\rho T b h^3}{3},\ I_{yy}^{(p)} = \frac{\rho T h b^3}{3},$$

$$I_{zz}^{(p)} = I_{yy}^{(p)} + I_{xx}^{(p)} = \frac{\rho T b h (h^2 + b^2)}{3},$$

$$I_{xy}^{(p)} = \frac{\rho T h^2 b^2}{4},\ I_{xz}^{(p)} = I_{yz}^{(p)} = 0.$$

The moments and products of inertia for the cutout: The moments and products of inertia about the center of mass of the cutout are:

$$I_{xx}^{(c)} = \frac{\rho T d c^3}{12},\ I_{yy}^{(c)} = \frac{\rho T c d^3}{12},\ I_{zz}^{(c)} = \frac{\rho T c d (c^2 + d^2)}{12},$$

$$I_{xy}^{(c)} = I_{xz}^{(c)} = I_{yz}^{(c)} = 0.$$

The distance from O to the center of mass of the cutout is $(d_x, d_y, 0) = (0.28, 0.24, 0)$ m. The moments and products of inertia about the point O are

$$I_{xx}^{(0)} = I_{xx}^{(c)} + \rho T c d (d_y^2),\ I_{yy}^{(0)} = I_{yy}^{(c)} + \rho T c d (d_x^2),$$

$$I_{zz}^{(0)} = I_{zz}^{(c)} + \rho T c d (d_x^2 + d_y^2),$$

$$I_{xy}^{(0)} = I_{xy}^{(c)} + \rho T c d (d_x d_y),\ I_{xz}^{(0)} = I_{yz}^{(0)} = 0.$$

The moments and products of inertia of the object: The moments and products of inertia about O are

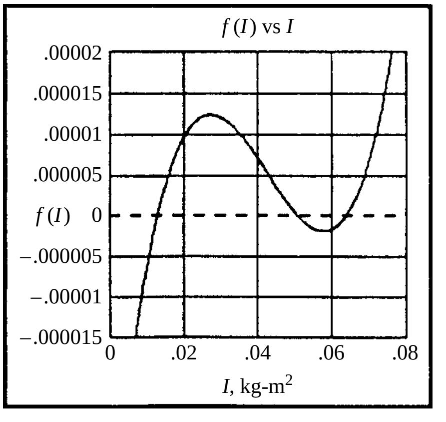
$$I_{xx} = I_{xx}^{(p)} - I_{xx}^{(0)} = 0.02316 \text{ kg-m}^2,$$

$$I_{yy} = I_{yy}^{(p)} - I_{yy}^{(0)} = 0.04053 \text{ kg-m}^2,$$

$$I_{zz} = I_{zz}^{(p)} - I_{zz}^{(0)} = 0.06370 \text{ kg-m}^2,$$

$$I_{xy} = I_{xy}^{(p)} - I_{xy}^{(0)} = 0.01691 \text{ kg-m}^2,$$

$$I_{xz} = I_{yz} = 0.$$



The principal moments of inertia. The principal values are given by the roots of the cubic equation $AI^3 + BI^2 + CI + D = 0$, where

$$A = 1,\ B = -(I_{xx} + I_{yy} + I_{zz}) = -0.1274,$$

$$C = I_{xx}I_{yy} + I_{yy}I_{zz} + I_{xx}I_{zz} - I_{xy}^2 - I_{xz}^2 - I_{yz}^2 = 4.71 \times 10^{-3},$$

$$D = -(I_{xx}I_{yy}I_{zz} - I_{xx}I_{yz}^2 - I_{yy}I_{xz}^2 - I_{zz}I_{xy}^2 - 2I_{xy}I_{yz}I_{xz})$$

$$= -4.158 \times 10^{-5}.$$

The function $f(I) = AI^3 + BI^2 + CI + D$ is graphed to get an estimate of the roots, and these estimates are refined by iteration. The graph is shown. The refined values of the roots are $I_1 = 0.01283$ kg-m^2, $I_2 = 0.05086$ kg-m^2, $I_3 = 0.06370$ kg-m^2.

The principal axes. The principal axes are obtained from a solution of the equations

$$V_x = (I_{yy} - I)(I_{zz} - I) - I_{yz}^2$$

$$V_y = I_{xy}(I_{zz} - I) + I_{xz}I_{yz}$$

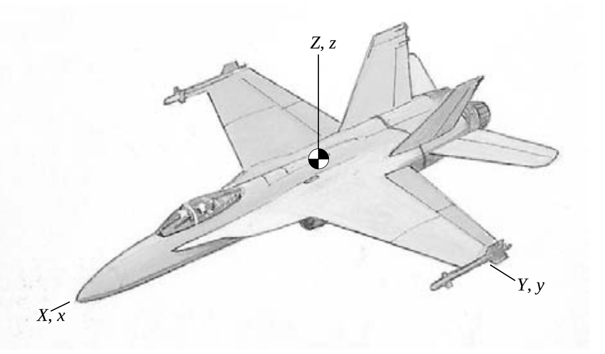
$$V_z = I_{xz}(I_{yy} - I) + I_{xz}I_{yz}.$$

Since $I_{xz} = I_{yz} = 0$, $V_z = 0$, and the solution fails for this axis, and the vector is to be determined from the orthogonality condition. Solving for V_x, V_y, the unit vectors are: for $I = I_1$, $V_x^{(1)} = 0.00141$, $V_y^{(1)} = 8.6 \times 10^{-4}$, from which the unit vectors are $\boxed{\mathbf{e}_1 = 0.8535\mathbf{i} + 0.5212\mathbf{j}}$. For $I = I_2$, $V_x^{(2)} = -1.325 \times 10^{-4}$, $V_y^{(2)} = 2.170 \times 10^{-4}$, from which $\boxed{\mathbf{e}_2 = -0.5212\mathbf{i} + 0.8535\mathbf{j}}$. The third unit vector is determined from orthogonality conditions: $\boxed{\mathbf{e}_3 = \mathbf{k}}$.

Problem 20.118 The airplane's principal moments of inertia, in kg-m^2, are $I_{xx} = 10844$, $I_{yy} = 65062$, and $I_{zz} = 67773$.

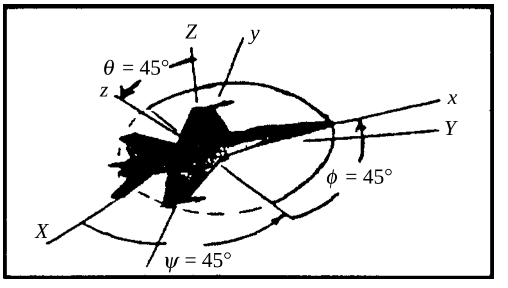
(a) The airplane begins in the reference position shown and maneuvers into the orientation $\psi = \theta = \phi = 45^\circ$. Draw a sketch showing the plane's orientation relative to the XYZ system.

(b) If the airplane is in the orientation described in (a), the rates of change of the Euler angles are $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.2$ rad/s, and the second derivatives of the angles with respect to time are zero, what are the components of the total moment about the airplane's center of mass?



Solution:

(a)



(b) The Eqs. (20.36) apply.

$$M_x = I_{xx}(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi$$

$$+ \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi) - (I_{yy} - I_{zz})$$

$$\times (\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)(\dot{\psi}\cos\theta + \phi),$$

$$M_y = I_{yy}(\ddot{\psi}\sin\theta\cos\phi - \ddot{\theta}\sin\phi + \dot{\psi}\dot{\theta}\cos\theta\cos\phi$$

$$- \dot{\psi}\dot{\phi}\sin\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi) - (I_{zz} - I_{xx})$$

$$\times (\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)(\dot{\psi}\cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta) - (I_{xx} - I_{yy})(\dot{\psi}\sin\theta\sin\phi$$

$$+ \dot{\theta}\cos\phi)(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi).$$

Substitute $I_{xx} = 10844$, $I_{yy} = 65062$, and $I_{zz} = 67773$, in kg-m^2, and $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.2$ rad/s, and $\ddot{\psi} = \ddot{\theta} = \ddot{\phi} = 0$. The moments:

$\boxed{M_x = -383.5 \text{ N-m}}$, $\boxed{M_y = -3451.2 \text{ N-m}}$,

$\boxed{M_z = -1084.6 \text{ N-m}}$

Problem 20.119 What are the x, y, and z components of the angular acceleration of the airplane described in Problem 20.118?

Solution: The angular accelerations are given by Eq. (20.35):

$$\frac{d\omega_x}{dt} = \ddot{\psi} \sin\theta \sin\phi + \dot{\psi}\dot{\theta} \cos\theta \sin\phi + \dot{\psi}\dot{\phi} \sin\theta \cos\phi$$

$$+ \ddot{\theta} \cos\phi - \dot{\theta}\dot{\phi} \sin\phi,$$

$$\frac{d\omega_y}{dt} = \ddot{\psi} \sin\theta \cos\phi + \dot{\psi}\dot{\theta} \cos\theta \cos\phi - \dot{\psi}\dot{\phi} \sin\theta \sin\phi$$

$$- \ddot{\theta} \sin\phi - \dot{\theta}\dot{\phi} \cos\phi,$$

$$\frac{d\omega_z}{dt} = \ddot{\psi} \cos\theta - \dot{\psi}\dot{\theta} \sin\theta + \ddot{\phi}.$$

Substitute $I_{xx} = 10844$, $I_{yy} = 65062$, and $I_{zz} = 67773$, in kg-m^2, and $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.2$ rad/s, and $\ddot{\psi} = \ddot{\theta} = \ddot{\phi} = 0$, to obtain

$\boxed{\alpha_x = -0.0283 \text{ rad/s}^2}$, $\boxed{\alpha_y = -0.0283 \text{ rad/s}}$, $\boxed{\alpha_z = 0}$

Problem 20.120 If the orientation of the airplane in Problem 20.118 is $\psi = 45^\circ$, $\theta = 60^\circ$, and $\phi = 45^\circ$, the rates of change of the Euler angles are $\dot{\psi} = 0$, $\dot{\theta} = 0.2$ rad/s, and $\dot{\phi} = 0.1$ rad/s, and the components of the total moment about the center of mass of the plane are $\Sigma M_x = 542$ N-m, $\Sigma M_y = 1627$ N-m, and $\Sigma M_z = 0$, what are the x, y, and z components of the airplane's angular acceleration?

Solution: The strategy is to solve Eqs. (20.36) for $\ddot{\theta}$, $\ddot{\phi}$, and $\ddot{\psi}$, and then to use Eqs. (20.35) to determine the angular accelerations.

$$M_x = I_{xx}(\ddot{\psi} \sin\theta \sin\phi + \ddot{\theta} \cos\phi + \dot{\psi}\dot{\theta} \cos\theta \sin\phi$$

$$+ \dot{\psi}\dot{\phi} \sin\theta \cos\phi - \dot{\theta}\dot{\phi} \sin\phi) - (I_{yy} - I_{zz})$$

$$\times (\dot{\psi} \sin\theta \cos\phi - \dot{\theta} \sin\phi)(\dot{\psi} \cos\theta + \dot{\phi}),$$

$$M_y = I_{yy}(\ddot{\psi} \sin\theta \cos\phi - \ddot{\theta} \sin\phi + \dot{\psi}\dot{\theta} \cos\theta \cos\phi$$

$$- \dot{\psi}\dot{\phi} \sin\theta \sin\phi - \dot{\theta}\dot{\phi} \cos\phi) - (I_{zz} - I_{xx})$$

$$\times (\dot{\psi} \sin\theta \sin\phi + \dot{\theta} \cos\phi)(\dot{\psi} \cos\theta + \dot{\phi}),$$

$$M_z = I_{zz}(\ddot{\psi} \cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta} \sin\theta) - (I_{xx} - I_{yy})(\dot{\psi} \sin\theta \sin\phi$$

$$+ \dot{\theta} \cos\phi)(\dot{\psi} \sin\theta \cos\phi - \theta \sin\phi).$$

Substitute numerical values and solve to obtain $\ddot{\phi} = -0.03266$ rad/s^2, $\ddot{\theta} = 0.01143$ rad/s^2, $\ddot{\psi} = 0.09732$ rad/s^2. These are to be used (with the other data) in Eqs. (20.35),

$$\frac{d\omega_x}{dt} = \ddot{\psi} \sin\theta \sin\phi + \dot{\psi}\dot{\theta} \cos\theta \sin\phi + \dot{\psi}\dot{\phi} \sin\theta \cos\phi$$

$$+ \ddot{\theta} \cos\phi - \dot{\theta}\dot{\phi} \sin\phi,$$

$$\frac{d\omega_y}{dt} = \ddot{\psi} \sin\theta \cos\phi + \dot{\psi}\dot{\theta} \cos\theta \cos\phi - \dot{\psi}\dot{\phi} \sin\theta \sin\phi$$

$$- \ddot{\theta} \sin\phi - \dot{\theta}\dot{\phi} \cos\phi,$$

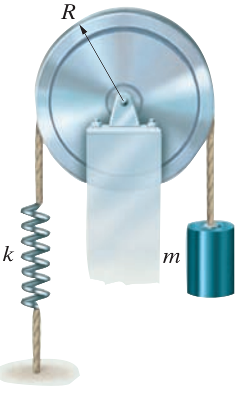
$$\frac{d\omega_z}{dt} = \ddot{\psi} \cos\theta - \dot{\psi}\dot{\theta} \sin\theta + \ddot{\phi}.$$

Substitute, to obtain:

$\boxed{\alpha_x = 0.05354 \text{ rad/s}^2}$, $\boxed{\alpha_y = 0.03737 \text{ rad/s}^2}$,

$\boxed{\alpha_z = 0.016 \text{ rad/s}^2}$

Problem 21.1 In Active Example 21.1, suppose that the pulley has radius $R = 100$ mm and its moment of inertia is $I = 0.005$ kg-m^2. The mass $m = 2$ kg, and the spring constant is $k = 200$ N/m. If the mass is displaced downward from its equilibrium position and released, what are the period and frequency of the resulting vibration?



Solution: From Active Example 21.1 we have

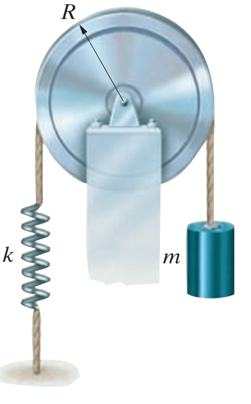
$$\omega = \sqrt{\frac{k}{m + \dfrac{I}{R^2}}} = \sqrt{\frac{(200\ \text{N/m})}{(2\ \text{kg}) + \dfrac{(0.005\ \text{kg-m}^2)}{(0.1\ \text{m})^2}}} = 8.94\ \text{rad/s}$$

Thus

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{8.94\ \text{rad/s}} = 0.702\ \text{s}, \quad f = \frac{1}{\tau} = 1.42\ \text{Hz}.$$

$$\boxed{\tau = 0.702\ \text{s}, \quad f = 1.42\ \text{Hz}.}$$

Problem 21.2 In Active Example 21.1, suppose that the pulley has radius $R = 4$ cm and its moment of inertia is $I = 0.06$ kg-m^2. The suspended object weighs 5 N, and the spring constant is $k = 10$ N/m. The system is initially at rest in its equilibrium position. At $t = 0$, the suspended object is given a downward velocity of 1 m/s. Determine the position of the suspended object relative to its equilibrium position as a function of time.



Solution: From Active Example 21.1 we have

$$\omega = \sqrt{\frac{k}{m + \dfrac{I}{R^2}}} = \sqrt{\frac{(10\ \text{N/m})}{\left(\dfrac{5\ \text{N}}{9.81\ \text{m/s}^2}\right) + \dfrac{(0.06\ \text{kg-m}^2)}{(0.04\ \text{m})^2}}} = 0.51\ \text{rad/s}$$

The general solution is

$$x = A \sin \omega t + B \cos \omega t, \quad v = A\omega \cos \omega t - B\omega \sin \omega t.$$

Putting in the initial conditions, we have

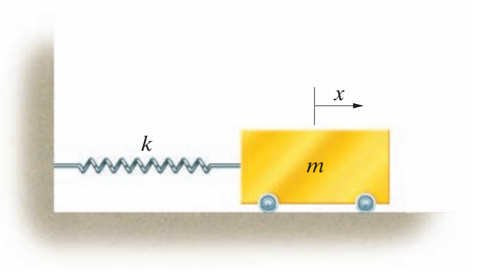
$$x(t = 0) = B = 0 \Rightarrow B = 0,$$

$$v(t = 0) = A\omega = (1\ \text{ft/s}) \Rightarrow A = \frac{1\ \text{m/s}}{0.51\ \text{rad/s}} = 1.96\ \text{m}.$$

Thus the equation is

$$\boxed{x = 1.96 \sin 0.51t\ (\text{m}).}$$

Problem 21.3 The mass $m = 4$ kg. The spring is unstretched when $x = 0$. The period of vibration of the mass is measured and determined to be 0.5 s. The mass is displaced to the position $x = 0.1$ m and released from rest at $t = 0$. Determine its position at $t = 0.4$ s.



Solution: Knowing the period, we can find the natural frequency

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.5\ \text{s}} = 12.6\ \text{rad/s}.$$

The general solution is

$$x = A \sin \omega t + B \cos \omega t, \quad v = A\omega \cos \omega t - B\omega \sin \omega t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B = 0.1\ \text{m} \Rightarrow B = 0.1\ \text{m},$$

$$v(t = 0) = A\omega = 0 \Rightarrow A = 0.$$

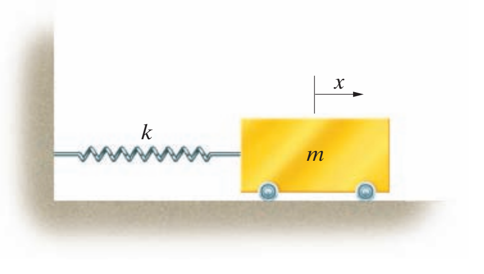
Thus the equation is

$$x = (0.1\ \text{m}) \cos(12.6\ \text{rad/s}\ t)$$

At the time $t = 0.4$ s, we find

$$\boxed{x = 0.0309\ \text{m}.}$$

Problem 21.4 The mass $m = 4$ kg. The spring is unstretched when $x = 0$. The frequency of vibration of the mass is measured and determined to be 6 Hz. The mass is displaced to the position $x = 0.1$ m and given a velocity $dx/dt = 5$ m/s at $t = 0$. Determine the amplitude of the resulting vibration.



Solution: Knowing the frequency, we can find the natural frequency

$$\omega = 2\pi f = 2\pi(6\ \text{Hz}) = 37.7\ \text{rad/s}.$$

The general solution is

$$x = A \sin \omega t + B \cos \omega t, \quad v = A\omega \cos \omega t - B\omega \sin \omega t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B = 0.1\ \text{m} \Rightarrow B = 0.1\ \text{m},$$

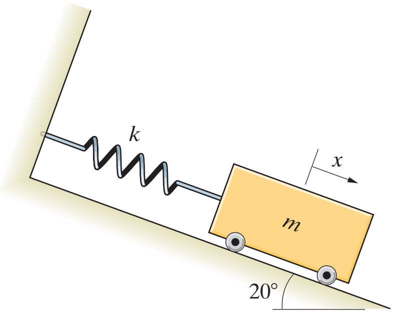
$$v(t = 0) = A\omega = 5\ \text{m/s} \Rightarrow A = \frac{5\ \text{m/s}}{37.7\ \text{rad/s}} = 0.133\ \text{m}.$$

The amplitude of the motion is given by

$$\sqrt{A^2 + B^2} = \sqrt{(0.1\ \text{m})^2 + (0.133\ \text{m})^2} = 0.166\ \text{m}.$$

$$\boxed{\text{Amplitude} = 0.166\ \text{m}.}$$

Problem 21.5 The mass $m = 4$ kg and the spring constant is $k = 64$ N/m. For vibration of the spring-mass oscillator relative to its equilibrium position, determine (a) the frequency in Hz and (b) the period.



Solution: Since the vibration is around the equilibrium position, we have

(a) $$\boxed{\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{64 \text{ N/m}}{4 \text{ kg}}} = 4 \text{ rad/s}\left(\frac{1 \text{ cycle}}{2\pi \text{ rad}}\right) = 0.637 \text{ Hz}}$$

(b) $$\boxed{\tau = \frac{2\pi}{\omega} = \frac{2\pi}{4 \text{ rad/s}} = 1.57 \text{ s}}$$

Problem 21.6 The mass $m = 4$ kg and the spring constant is $k = 64$ N/m. The spring is unstretched when $x = 0$. At $t = 0$, $x = 0$ and the mass has a velocity of 2 m/s down the inclined surface. What is the value of x at $t = 0.8$ s?

Solution: The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = mg\sin 20^\circ \Rightarrow \frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = g\sin 20^\circ$$

Putting in the numbers we have

$$\frac{d^2x}{dt^2} + (16\text{s}^{-2})x = 3.36 \text{ m/s}^2$$

The solution to this nonhomogeneous equation is

$$x = A\sin([4 \text{ s}^{-1}]t) + B\cos([4 \text{ s}^{-1}]t) + 0.210 \text{ m}$$

Using the initial conditions we have

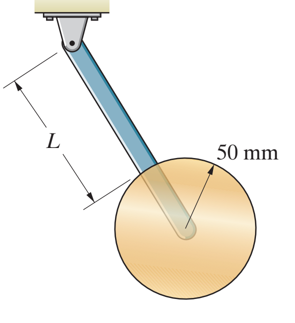
$$\left.\begin{aligned} 0 &= B + 0.210 \text{ m} \\ 2 \text{ m/s} &= A(4 \text{ s}^{-1}) \end{aligned}\right\} \Rightarrow A = 0.5 \text{ m}, \ B = -0.210 \text{ m}$$

The motion is

$$x = (0.5 \text{ m})\sin([4 \text{ s}^{-1}]t) - (0.210 \text{ m})\cos([4 \text{ s}^{-1}]t) + 0.210 \text{ m}$$

At $t = 0.8$ s we have $\boxed{x = 0.390 \text{ m}}$

Problem 21.7 Suppose that in a mechanical design course you are asked to design a pendulum clock, and you begin with the pendulum. The mass of the disk is 2 kg. Determine the length L of the bar so that the period of small oscillations of the pendulum is 1 s. For this preliminary estimate, neglect the mass of the bar.



Solution: Given $m = 2$ kg, $r = 0.05$ m

For small angles the equation of motion is

$$\left(\frac{1}{2}mr^2 + m[r+L]^2\right)\frac{d^2\theta}{dt^2} + mg(L+r)\theta = 0$$

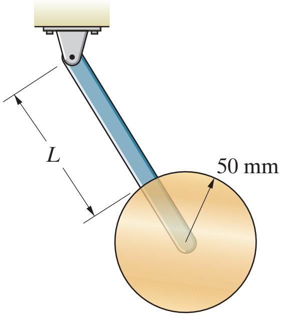
$$\Rightarrow \frac{d^2\theta}{dt^2} + \left(\frac{2g[L+r]}{r^2 + [r+L]^2}\right)\theta = 0$$

The period is $\tau = 2\pi\sqrt{\dfrac{r^2 + 2(r+L)^2}{2g(L+r)}}$

Set $\tau = 1$ s and solve to find $\boxed{L = 0.193 \text{ m}}$

Problem 21.8 The mass of the disk is 2 kg and the mass of the slender bar is 0.4 kg. Determine the length L of the bar so that the period of small oscillations of the pendulum is 1 s.

Strategy: Draw a graph of the value of the period for a range of lengths L to estimate the value of L corresponding to a period of 1 s.



Solution: We have

$$m_d = 2 \text{ kg}, \; m_b = 0.4 \text{ kg}, \; r = 0.05 \text{ m}, \; \tau = 1 \text{ s}.$$

The moment of inertia of the system about the pivot point is

$$I = \frac{1}{3}m_b L^2 + \frac{1}{2}m_d r^2 + m_d(L+r)^2.$$

The equation of motion for small amplitudes is

$$\left(\frac{1}{3}m_b L^2 + \frac{1}{2}m_d r^2 + m_d(L+r)^2\right)\ddot{\theta} + \left(m_b\frac{L}{2} + m_d[L+r]\right)g\theta = 0$$

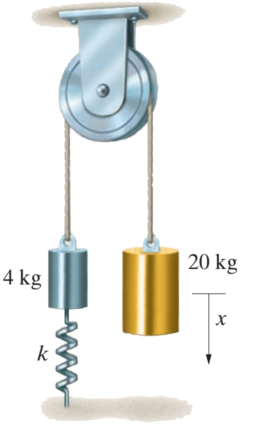
Thus, the period is given by

$$\tau = 2\pi\sqrt{\frac{\frac{1}{3}m_b L^2 + \frac{1}{2}m_d r^2 + m_d(L+r)^2}{\left(m_b + \frac{L}{2} + m_d[L+r]\right)g}}$$

This is a complicated equation to solve. You can draw the graph and get an approximate solution, or you can use a root solver in your calculator or on your computer.

Using a root solver, we find $\boxed{L = 0.203 \text{ m.}}$

Problem 21.9 The spring constant is $k = 785$ N/m. The spring is unstretched when $x = 0$. Neglect the mass of the pulley, that is, assume that the tension in the rope is the same on both sides of the pulley. The system is released from rest with $x = 0$. Determine x as a function of time.



Solution: We have the equations

$$T - (4 \text{ kg})(9.81 \text{ m/s}^2) - (785 \text{ N/m})x = (4 \text{ kg})\ddot{x}$$

$$T - (20 \text{ kg})(9.81 \text{ m/s}^2) = -(20 \text{ kg})\ddot{x}$$

If we eliminate the tension T from these equations, we find

$$(24 \text{ kg})\ddot{x} + (785 \text{ N/m})x = (16 \text{ kg})(9.81 \text{ m/s}^2)$$

$$\ddot{x} + (5.72 \text{ rad/s})^2 x = 6.54 \text{ m/s}^2.$$

The solution of this equation is

$$x = A\sin\omega t + B\cos\omega t + (0.200 \text{ m}), \quad v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Using the initial conditions, we have

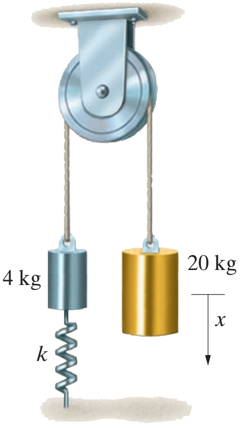
$$x(t=0) = B + (0.200 \text{ m}) \Rightarrow B = -0.200 \text{ m},$$

$$v(t=0) = A\omega = 0 \Rightarrow A = 0.$$

Thus the equation is

$$\boxed{x = (0.200 \text{ m})(1 - \cos[5.72 \text{ rad/s } t]).}$$

Problem 21.10 The spring constant is $k = 785$ N/m. The spring is unstretched with $x = 0$. The radius of the pulley is 125 mm, and moment of inertia about its axis is $I = 0.05$ kg-m^2. The system is released from rest with $x = 0$. Determine x as a function of time.



Solution: Let T_1 be the tension in the rope on the left side, and T_2 be the tension in the rope on the right side. We have the equations

$$T_1 - (4\text{ kg})(9.81\text{ m/s}^2) - (785\text{ N/m})x = (4\text{ kg})\ddot{x}$$

$$T_2 - (20\text{ kg})(9.81\text{ m/s}^2) = -(20\text{ kg})\ddot{x}$$

$$(T_2 - T_1)(0.125\text{ m}) = (0.05\text{ kg-m}^2)\frac{\ddot{x}}{(0.125\text{ m})}$$

If we eliminate the tensions T_1 and T_2 from these equations, we find

$$(27.2\text{ kg})\ddot{x} + (785\text{ N/m})x = (16\text{ kg})(9.81\text{ m/s}^2)$$

$$\ddot{x} + (5.37\text{ rad/s})^2 x = 5.77\text{ m/s}^2.$$

The solution of this equation is

$$x = A\sin\omega t + B\cos\omega t + (0.200\text{ m}),\ \ v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Using the initial conditions, we have

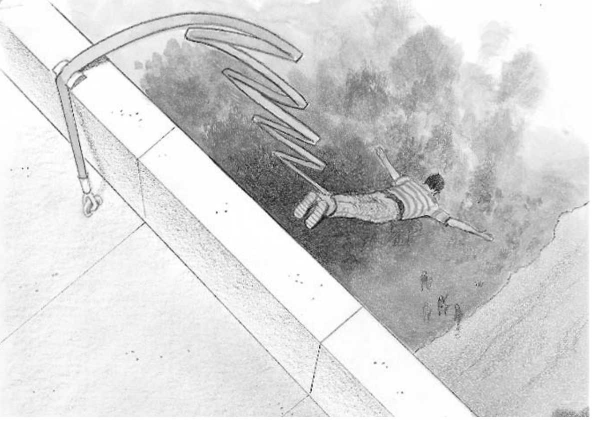
$$x(t = 0) = B + (0.200\text{ m}) \Rightarrow B = -0.200\text{ m},$$

$$v(t = 0) = A\omega = 0 \Rightarrow A = 0.$$

Thus the equation is

$$\boxed{x = (0.200\text{ m})(1 - \cos[5.37\text{ rad/s } t]).}$$

Problem 21.11 A "bungee jumper" who weighs 711.7 N leaps from a bridge above a river. The bungee cord has an unstretched length of 18.3 m, and it stretches an additional 12.2 m before the jumper rebounds. Model the cord as a linear spring. When his motion has nearly stopped, what are the period and frequency of his vertical oscillations? (You can't model the cord as a linear spring during the early part of his motion. Why not?)



Solution: Use energy to find the spring constant

$$T_1 = 0,\ V_1 = 0,\ T_2 = 0,\ V_2 = -(711.7\text{ N})(30.5\text{ m}) + \frac{1}{2}k(12.2\text{ m})^2$$

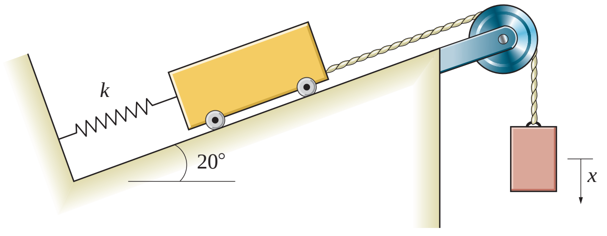
$$T_1 + V_1 = T_2 + V_2 \ \Rightarrow\ k = 291.9\text{ N/m}$$

$$\boxed{\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{291.9\text{ N/m}}{(711.7\text{ N})/(9.81\text{ m/s}^2)}} = 0.319\text{ Hz}, \\ \tau &= \frac{1}{f} = 3.13\text{ s} \end{aligned}}$$

The cord cannot be modeled as a linear spring during the early motion because it is slack and does not support a compressive load.

Problem 21.12 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The mass of each object is 30 kg. The inclined surface is smooth. Neglect the mass of the pulley. The system is released from rest with $x = 0$.

(a) Determine the frequency and period of the resulting vibration.
(b) What is the value of x at $t = 4$ s?



Solution: The equations of motion are

$$T - mg\sin\theta - kx = m\ddot{x}, \quad T - mg = -m\ddot{x}.$$

If we eliminate T, we find

$$2m\ddot{x} + kx = mg(1 - \sin\theta), \quad \ddot{x} + \frac{k}{2m}x = \frac{1}{2}g(1 - \sin\theta),$$

$$\ddot{x} + \frac{1}{2}\left(\frac{800 \text{ N/m}}{30 \text{ kg}}\right)x = \frac{1}{2}(9.81 \text{ m/s}^2)(1 - \sin 20^\circ),$$

$$\ddot{x} + (3.65 \text{ rad/s})^2 x = 3.23 \text{ m/s}^2.$$

(a) The natural frequency, frequency, and period are

$$\omega = 3.65 \text{ rad/s}, \quad f = \frac{\omega}{2\pi} = 0.581 \text{ Hz}, \quad \tau = \frac{1}{f} = 1.72 \text{ s}.$$

$$\boxed{\begin{aligned} f &= 0.581 \text{ s}, \\ \tau &= 1.72 \text{ s}. \end{aligned}}$$

(b) The solution to the differential equation is

$$x = A\sin\omega t + B\cos\omega t + 0.242 \text{ m}, \quad v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.242 \text{ m} = 0 \Rightarrow B = -0.242 \text{ m},$$

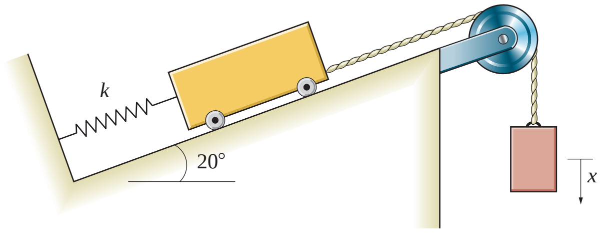
$$v(t = 0) = A\omega = 0 \Rightarrow A = 0.$$

Thus the equation is $x = (0.242 \text{ m})(1 - \cos[3.65 \text{ rad/s } t])$

At $t = 4$ s we have $\boxed{x = 0.351 \text{ m.}}$

Problem 21.13 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The mass of each object is 30 kg. The inclined surface is smooth. The radius of the pulley is 120 mm and its moment of inertia is $I = 0.03$ kg-m^2. At $t = 0$, $x = 0$ and $dx/dt = 1$ m/s.

(a) Determine the frequency and period of the resulting vibration.
(b) What is the value of x at $t = 4$ s?



Solution: Let T_1 be the tension in the rope on the left of the pulley, and T_2 be the tension in the rope on the right of the pulley. The equations of motion are

$$T_1 - mg\sin\theta - kx = m\ddot{x}, \; T_2 - mg = -m\ddot{x}, \; (T_2 - T_1)r = I\frac{\ddot{x}}{r}.$$

If we eliminate T_1 and T_2, we find

$$\left(2m + \frac{I}{r^2}\right)\ddot{x} + kx = mg(1 - \sin\theta),$$

$$\ddot{x} + \frac{kr^2}{2mr^2 + I}x = \frac{mgr^2}{2mr^2 + I}(1 - \sin\theta), \quad \ddot{x} + (3.59 \text{ rad/s})^2 x = 3.12 \text{ m/s}^2.$$

(a) The natural frequency, frequency, and period are

$$\omega = 3.59 \text{ rad/s}, \quad f = \frac{\omega}{2\pi} = 0.571 \text{ Hz}, \quad \tau = \frac{1}{f} = 1.75 \text{ s}.$$

$$\boxed{\begin{aligned} f &= 0.571 \text{ s}, \\ \tau &= 1.75 \text{ s}. \end{aligned}}$$

(b) The solution to the differential equation is

$$x = A\sin\omega t + B\cos\omega t + 0.242 \text{ m}, \quad v = A\omega\cos\omega t - B\omega\sin\omega t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.242 \text{ m} = 0 \Rightarrow B = -0.242 \text{ m},$$

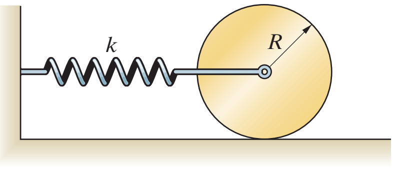
$$v(t = 0) = A\omega = (1 \text{ m/s}) \Rightarrow A = \frac{1 \text{ m/s}}{3.59 \text{ rad/s}} = 0.279 \text{ m}.$$

Thus the equation is

$$x = (0.242 \text{ m})(1 - \cos[359 \text{ rad/s} t]) + (0.279 \text{ m})\sin[359 \text{ rad/s } t]$$

At $t = 4$ s we have $\boxed{x = 0.567 \text{ m.}}$

Problem 21.14 The 89 N disk rolls on the horizontal surface. Its radius is $R = 152.4$ mm. Determine the spring constant k so that the frequency of vibration of the system relative to its equilibrium position is $f = 1$ Hz.



Solution: Given $m = \dfrac{89\text{ N}}{9.81\text{ m/s}^2}$, $R = 0.152$ m

The equations of motion

$$\left.\begin{aligned} -kx + F &= m\frac{d^2x}{dt^2} \\ FR &= -\left(\frac{1}{2}mR^2\right)\frac{1}{R}\frac{d^2x}{dt^2} \end{aligned}\right\} \Rightarrow \frac{d^2x}{dt^2} + \left(\frac{2k}{3m}\right)x = 0$$

We require that $f = \dfrac{1}{2\pi}\sqrt{\dfrac{2k}{3m}} = 1\text{ Hz} \Rightarrow \boxed{k = 537\text{ N/m}}$

Problem 21.15 The 89 N disk rolls on the horizontal surface. Its radius is $R = 152.4$ mm. The spring constant is $k = 218.9$ N/m. At $t = 0$, the spring is unstretched and the disk has a clockwise angular velocity of 2 rad/s. What is the amplitude of the resulting vibrations of the center of the disk?

Solution: See the solution to 21.14

$$\frac{d^2x}{dt^2} + \left(\frac{2k}{3m}\right)x = 0 \Rightarrow \frac{d^2x}{dt^2} + (16.09\text{ rad/s})^2 x = 0$$

The solution is

$$x = A\cos([16.09\text{ rad/s}]t) + B\sin([16.09\text{ rad/s}]t)$$

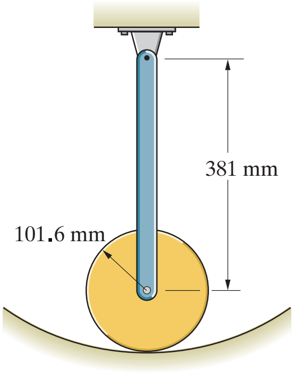
Using the initial conditions

$$\left.\begin{aligned} 0 &= A \\ (2\text{ rad/s})(0.152\text{ m}) &= B(16.09\text{ rad/s}) \end{aligned}\right\} \Rightarrow A = 0,\ B = 0.076\text{ m}$$

Thus $x = (0.076\text{ m})\sin([16.09\text{ rad/s}]t)$

The amplitude is $\boxed{B = 0.076\text{ m}}$

Problem 21.16 The 8.9 N bar is pinned to the 22.2 N disk. The disk rolls on the circular surface. What is the frequency of small vibrations of the system relative to its vertical equilibrium position?



Solution: Use energy methods.

$$T = \frac{1}{2}\left(\frac{1}{3}\left[\frac{8.9\text{ N}}{9.81\text{ m/s}^2}\right][0.381\text{ m}]^2\right)\omega_{\text{bar}}^2 + \frac{1}{2}\left(\frac{22.2\text{ N}}{9.81\text{ m/s}^2}\right)(0.381\text{ m})^2\,\omega_{\text{bar}}^2$$

$$+\frac{1}{2}\left(\frac{1}{2}\left[\frac{22.2\text{ N}}{9.81\text{ m/s}^2}\right][0.102\text{ m}]^2\right)\left(\frac{0.381}{0.102}\right)^2\omega_{\text{bar}}^2 = (0.268\text{ N-m-s}^2)\omega_{\text{bar}}^2$$

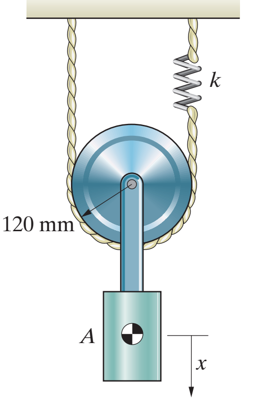
$$V = -(8.9\text{ N})(0.191\text{ m})\cos\theta - (22.2\text{ N})(0.381\text{ m})\cos\theta = -(10.17\text{ N-m})\cos\theta$$

Differentiating and linearizing we find

$$(0.537\text{ N-m-s}^2)\frac{d^2\theta}{dt^2} + (10.17\text{ N-m})\theta = 0 \;\Rightarrow\; \frac{d^2\theta}{dt^2} + (4.35\text{ rad/s})^2\theta = 0$$

$$\boxed{f = \frac{4.35\text{ rad/s}}{2\pi\text{ rad}} = 0.692\text{ Hz}}$$

Problem 21.17 The mass of the suspended object A is 4 kg. The mass of the pulley is 2 kg and its moment of inertia is 0.018 N-m^2. For vibration of the system relative to its equilibrium position, determine (a) the frequency in Hz and (b) the period.



Solution: Use energy methods

$$T = \frac{1}{2}(6\text{ kg})v^2 + \frac{1}{2}(0.018\text{ kg-m}^2)\left(\frac{v}{0.12\text{ m}}\right)^2 = (3.625\text{ kg})v^2$$

$$V = -(6\text{ kg})(9.81\text{ m/s}^2)x + \frac{1}{2}(150\text{ N/m})(2x)^2$$

$$= (300\text{ N/m})x^2 - (58.9\text{ N})x$$

Differentiating we have

$$(7.25\text{ kg})\frac{d^2x}{dt^2} + (600\text{ N/m})x = 58.9\text{ N}$$

$$\Rightarrow \frac{d^2x}{dt^2} + (9.10\text{ rad/s})^2x = 8.12\text{ m/s}^2$$

(a) $\boxed{f = \frac{9.10\text{ rad/s}}{2\pi\text{ rad}} = 1.45\text{ Hz}}$

(b) $\boxed{\tau = \frac{1}{f} = 0.691\text{ s}}$

Problem 21.18 The mass of the suspended object A is 4 kg. The mass of the pulley is 2 kg and its moment of inertia is 0.018 N-m^2. The spring is unstretched when $x = 0$. At $t = 0$, the system is released from rest with $x = 0$. What is the velocity of the object A at $t = 1$ s?

Solution: See the solution to 21.17

The motion is given by

$x = A\cos([9.10 \text{ rad/s}]t) + B\sin([9.10 \text{ rad/s}]t) + 0.0981 \text{ m}$

$v = \dfrac{dx}{dt} = (9.10 \text{ rad/s})\{-A\sin([9.10 \text{ rad/s}]t) + B\cos([9.10 \text{ rad/s}]t)\}$

Use the initial conditions

$$\left.\begin{aligned} A + 0.0981 \text{ m} &= 0 \\ B(0.910 \text{ rad/s}) &= 0 \end{aligned}\right\} \Rightarrow A = -0.0981 \text{ m}, \ B = 0$$

Thus we have

$x = (0.0981 \text{ m})(1 - \cos([9.10 \text{ rad/s}]t))$

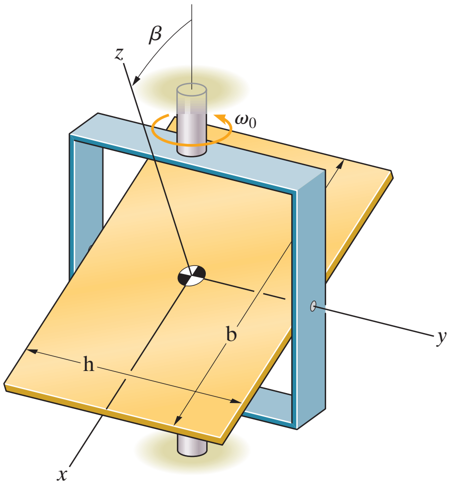
$v = \dfrac{dx}{dt} = -(0.892 \text{ m/s})\sin([9.10 \text{ rad/s}]t)$

At $t = 1$ s, $\boxed{v = 0.287 \text{ m/s}}$

Problem 21.19 The thin rectangular plate is attached to the rectangular frame by pins. The frame rotates with constant angular velocity $\omega_0 = 6$ rad/s.The angle β between the z axis of the body-fixed coordinate system and the vertical is governed by the equation

$$\frac{d^2\beta}{dt^2} = -\omega_0^2 \sin\beta\cos\beta.$$

Determine the frequency of small vibrations of the plate relative to its horizontal position.



Strategy: By writing $\sin\beta$ and $\cos\beta$ in terms of their Taylor series and assuming that β is small, show that the equation governing β can be expressed in the form of Eq. (21.5).

Solution:

$$\frac{d^2\beta}{dt^2} + \omega_0^2\sin\beta\cos\beta = 0 \ \Rightarrow \ \frac{d^2\beta}{dt^2} + \frac{1}{2}\omega_0^2\sin 2\beta = 0$$

Linearizing we have

$$\frac{d^2\beta}{dt^2} + \frac{1}{2}\omega_0^2 2\beta = 0 \ \Rightarrow \ \frac{d^2\beta}{dt^2} + \omega_0^2\beta = 0$$

$$\omega = \omega_0 = 6 \text{ rad/s} \ \Rightarrow \ \boxed{f = \frac{6 \text{ rad/s}}{2\pi \text{ rad}} = 0.954 \text{ Hz}}$$

Problem 21.20 Consider the system described in Problem 21.19. At $t = 0$, the angle $\beta = 0.01$ rad and $d\beta/dt = 0$. Determine β as a function of time.

Solution: See 21.19

The equation of motion is

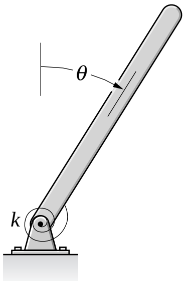
$$\frac{d^2\beta}{dt^2} + (6 \text{ rad/s})^2\beta = 0$$

The solution is

$$\boxed{\beta = (0.01 \text{ rad}) \cos([6.00 \text{ rad/s}]t)}$$

Problem 21.21 A slender bar of mass m and length l is pinned to a fixed support as shown. A torsional spring of constant k attached to the bar at the support is unstretched when the bar is vertical. Show that the equation governing small vibrations of the bar from its vertical equilibrium position is

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \omega^2 = \frac{(k - \frac{1}{2}mgl)}{\frac{1}{3}ml^2}.$$



Solution: The system is conservative. The pivot is a fixed point. The moment of inertia about the fixed point is $I = mL^2/3$. The kinetic energy of the motion of the bar is

$$T = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2.$$

The potential energy is the sum of the energy in the spring and the gravitational energy associated with the change in height of the center of mass of the bar,

$$V = \frac{1}{2}k\theta^2 - \frac{mgL}{2}(1 - \cos\theta).$$

For a conservative system,

$$T + V = \text{const.} = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}k\theta^2 - \frac{mgL}{2}(1 - \cos\theta).$$

Take the time derivative and reduce:

$$\left(\frac{d\theta}{dt}\right)\left[\frac{mL^2}{3}\left(\frac{d^2\theta}{dt^2}\right) + k\theta - \frac{mgL}{2}\sin\theta\right] = 0.$$

Ignore the possible solution $\dfrac{d\theta}{dt} = 0$, from which

$$\frac{mL^2}{3}\left(\frac{d^2\theta}{dt^2}\right) + k\theta - \frac{mgL}{2}\sin\theta = 0.$$

For small amplitude vibrations $\sin\theta \to \theta$, and the canonical form (see Eq. (21.4)) of the equation of motion is

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \boxed{\omega^2 = \frac{k - \dfrac{mgL}{2}}{\dfrac{1}{3}mL^2}}$$

Problem 21.22 The initial conditions of the slender bar in Problem 21.21 are

$$t = 0 \begin{cases} \theta = 0 \\ \dfrac{d\theta}{dt} = \dot{\theta}_0. \end{cases}$$

(a) If $k > \frac{1}{2}mgl$, show that θ is given as a function of time by

$$\theta = \frac{\dot{\theta}_0}{\omega}\sin\omega t, \quad \text{where } \omega^2 = \frac{(k - \frac{1}{2}mgl)}{\frac{1}{3}ml^2}.$$

(b) If $k < \frac{1}{2}mgl$, show that θ is given as a function of time by

$$\theta = \frac{\dot{\theta}_0}{2h}(e^{ht} - e^{-ht}), \quad \text{where } h^2 = \frac{(\frac{1}{2}mgl - k)}{\frac{1}{3}ml^2}.$$

Strategy: To do part (b), seek a solution of the equation of motion of the form $x = Ce^{\lambda t}$, where C and λ are constants.

Solution: Write the equation of motion in the form

$$\frac{d^2\theta}{dt^2} + p^2\theta = 0, \quad \text{where } p = \sqrt{\frac{k - \dfrac{mgL}{2}}{\dfrac{1}{3}mL^2}}.$$

Define $\omega = \sqrt{\dfrac{\left(k - \dfrac{mgL}{2}\right)}{\left(\dfrac{1}{3}mL^2\right)}}$ if $k > \dfrac{mgL}{2}$,

and $h = \sqrt{\dfrac{\left(\dfrac{1}{2}mgL - k\right)}{\left(\dfrac{1}{3}mL^2\right)}}$,

if $k < \dfrac{mgL}{2}$, from which $p = \omega$ if $k > \dfrac{mgL}{2}$, and $p = ih$, if $k < \dfrac{mgL}{2}$, where $i = \sqrt{-1}$. Assume a solution of the form $\theta = A\sin pt + B\cos pt$. The time derivative is $\dfrac{d\theta}{dt} = pA\cos pt - pB\sin pt$. Apply the initial conditions at $t = 0$, to obtain $B = 0$, and $A = \dfrac{\dot{\theta}_0}{p}$, from which the solution is $\theta = \dfrac{\dot{\theta}_0}{p}\sin pt$.

(a) Substitute: if $k > \dfrac{mgL}{2}$, $\boxed{\theta = \dfrac{\dot{\theta}_0}{\omega}\sin\omega t}$.

(b) If $k < \dfrac{mgL}{2}$, $\theta = \dfrac{\dot{\theta}_0}{ih}\sin(iht)$. From the definition of the hyperbolic sine,

$$\frac{\sinh(ht)}{h} = \frac{\sin(iht)}{ih} = \frac{1}{2h}\left(e^{ht} - e^{-ht}\right),$$

from which the solution is $\boxed{\theta = \dfrac{\dot{\theta}_0}{2h}(e^{ht} - e^{-ht})}$. [*Check*: An alternate solution for part (b) based on the suggested strategy is:

For $k < \dfrac{mgL}{2}$

write the equation of motion in the form

$$\frac{d^2\theta}{dt^2} - h^2\theta = 0,$$

and assume a general solution of the form

$$\theta = Ce^{\lambda t} + De^{-\lambda t}.$$

Substitute into the equation of motion to obtain $(\lambda^2 - h^2)\theta = 0$, from which $\lambda = \pm h$, and the solution is $\theta = Ce^{ht} + De^{-ht}$, where the positive sign is taken without loss of generality. The time derivative is

$$\frac{d\theta}{dt} = hCe^{ht} - hDe^{-ht}.$$

Apply the initial conditions at $t = 0$ to obtain the two equations: $0 = C + D$, and $\dot{\theta}_0 = hC - hD$. Solve:

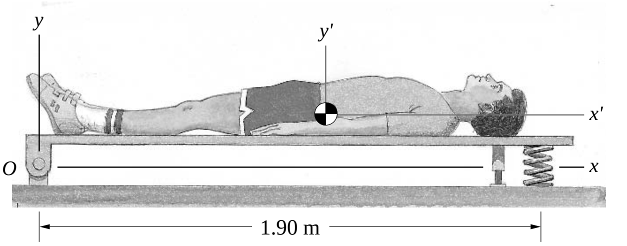
$$C = \frac{\dot{\theta}_0}{2h},$$

$$D = -\frac{\dot{\theta}_0}{2h},$$

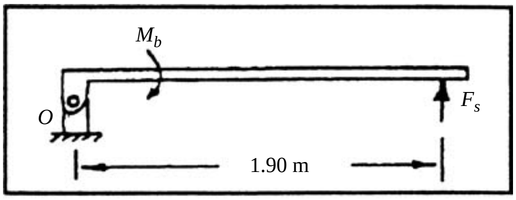
from which the solution is

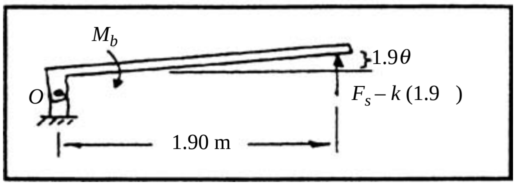
$$\theta = \frac{\dot{\theta}_0}{2h}(e^{ht} - e^{-ht}). \quad \textit{check}]$$

Problem 21.23 Engineers use the device shown to measure an astronaut's moment of inertia. The horizontal board is pinned at O and supported by the linear spring with constant $k = 12$ kN/m. When the astronaut is not present, the frequency of small vibrations of the board about O is measured and determined to be 6.0 Hz. When the astronaut is lying on the board as shown, the frequency of small vibrations of the board about O is 2.8 Hz. What is the astronaut's moment of inertia about the z axis?



Solution: *When the astronaut is not present*: Let F_s be the spring force and M_b be the moment about 0 due to the board's weight when the system is in equilibrium. The moment about 0 equals zero, $\sum M_{(\text{pt }0)} = (1.9)F_s - M_b = 0(1)$. When the system is in motion and displaced by a small counterclockwise angle θ, the spring force decreases to $F_s - k(1.9\theta)$: The equation of angular motion about 0 is





$$\sum M_{(\text{pt}0)} = (1.9)[F_s - k(1.9\theta)] - M_b = I_b \frac{d^2\theta}{dt^2},$$

where I_b is the moment of inertial of the board about the z axis. Using Equation (1), we can write the equation of angular motion as

$$\frac{d^2\theta}{dt^2} + \omega_1^2\theta = 0, \quad \text{where } \omega_1^2 = \frac{k(1.9)^2}{I_b} = \frac{(12{,}000)(1.9)^2}{I_b}.$$

We know that $f_1 = \omega_1/2\pi = 6$ Hz, so $\omega_1 = 12\pi = 37.7$ rad/s and we can solve for I_b: $I_b = 30.48$ kg-m^2.

When the astronaut is present: Let F_s be the spring force and M_{ba} be the moment about 0 due to the weight of the board and astronaut when the system is in equilibrium. The moment about 0 equals zero, $\sum M_{(pt0)} = (1.9)F_s - M_{ba} = 0(2)$. When the system is in motion and displaced by a small counterclockwise angle θ, the spring force decreases to $F_s - k(1.9\theta)$: The equation of angular motion about 0 is

$$\sum M_{(\text{pt}0)} = (1.9)[F_s - k(1.9\theta)] - M_{ba} = (I_b + I_a)\frac{d^2\theta}{dt}^2,$$

where I_a is the moment of inertia of the astronaut about the z axis. Using Equation (2), we can write the equation of angular motion as

$$\frac{d^2\theta}{dt^2} + \omega_2^2\theta = 0, \quad \text{where } \omega_2^2 = \frac{k(1.9)^2}{I_b + I_a} = \frac{(12{,}000)(1.9)^2}{I_b + I_a}.$$

In this case $f_2 = \omega_2/2\pi = 2.8$ Hz, so $\omega_2 = 2.8(2\pi) = 17.59$ rad/s. Since we know I_b, we can determine I_a, obtaining $I_a = 109.48$ kg-m^2.

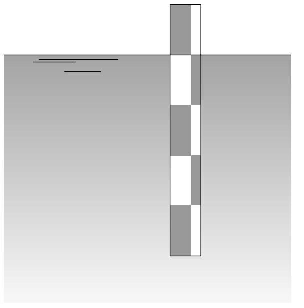
Problem 21.24 In Problem 21.23, the astronaut's center of mass is at $x = 1.01$ m, $y = 0.16$ m, and his mass is 81.6 kg. What is his moment of inertia about the z' axis through his center of mass?

Solution: From the solution of Problem 21.23, his moment of inertial about the z axis is $I_z = 109.48$ kg-m^2. From the parallel-axis theorem,

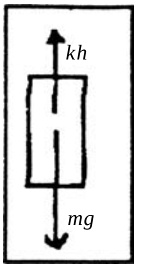
$$I_{z'} = I_z - (d_x^2 + d_y^2)m = 109.48 - [(1.01)^2 + (0.16)^2](81.6)$$

$$= 24.2 \text{ kg-m}^2.$$

Problem 21.25* A floating sonobuoy (sound-measuring device) is in equilibrium in the vertical position shown. (Its center of mass is low enough that it is stable in this position.) The device is a 10-kg cylinder 1 m in length and 125 mm in diameter. The water density is 1025 kg/m^3, and the buoyancy force supporting the buoy equals the weight of the water that would occupy the volume of the part of the cylinder below the surface. If you push the sonobuoy slightly deeper and release it, what is the frequency of the resulting vertical vibrations?



Solution: Choose a coordinate system with y positive downward. Denote the volume beneath the surface by $V = \pi R^2 d$, where $R = 0.0625$ m. The density of the water is $\rho = 1025$ kg/m^3. The weight of the displaced water is $W = \rho V g$, from which the buoyancy force is $F = \rho V g = \pi \rho R^2 g d$. By definition, the spring constant is



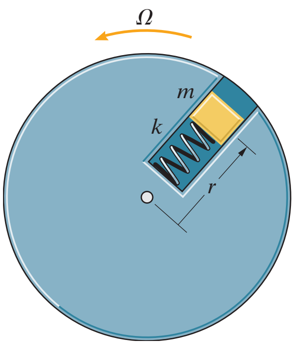
$$k = \frac{\partial F}{\partial d} = \pi \rho R^2 g = 123.4 \text{ N/m}.$$

If h is a positive change in the immersion depth from equilibrium, the force on the sonobuoy is $\sum F_y = -kh + mg$, where the negative sign is taken because the "spring force" kh opposes the positive motion h. From Newton's second law, $m\dfrac{d^2h}{dt^2} = mg - kh$. The canonical form (see Eq. (21.4)) is $\dfrac{d^2h}{dt^2} + \omega^2 h = g$, where $\omega = \sqrt{\dfrac{k}{m}} = 3.513$ rad/s. The frequency is $\boxed{f = \dfrac{\omega}{2\pi} = 0.5591 \text{ Hz}}$.

Problem 21.26 The disk rotates *in the horizontal plane* with constant angular velocity $\Omega = 12$ rad/s. The mass $m = 2$ kg slides in a smooth slot in the disk and is attached to a spring with constant $k = 860$ N/m. The radial position of the mass when the spring is unstretched is $r = 0.2$ m.

(a) Determine the "equilibrium" position of the mass, the value of r at which it will remain stationary relative to the center of the disk.
(b) What is the frequency of vibration of the mass relative to its equilibrium position?

Strategy: Apply Newton's second law to the mass in terms of polar coordinates.



Solution: Using polar coordinates, Newton's second law in the r direction is

$$\Sigma F_r : -k(r - r_0) = m(\ddot{r} - r\Omega^2) \Rightarrow \ddot{r} + \left(\frac{k}{m} - \Omega^2\right) r = \frac{k}{m} r_0$$

(a) The "equilibrium" position occurs when $\ddot{r} = 0$

$$r_{eq} = \frac{\frac{k}{m} r_0}{\frac{k}{m} - \Omega^2} = \frac{k r_0}{k - m\Omega^2} = \frac{(860 \text{ N/m})(0.2 \text{ m})}{(860 \text{ N/m}) - (2 \text{ kg})(12 \text{ rad/s})^2} = 0.301 \text{ m}.$$

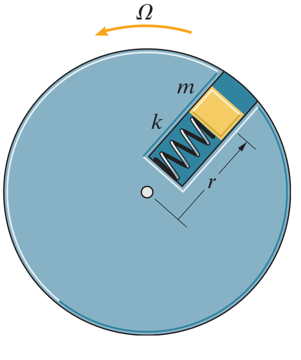
$$\boxed{r_{eq} = 0.301 \text{ m}.}$$

(b) The frequency of vibration is found

$$\omega = \sqrt{\frac{k}{m} - \Omega^2} = \sqrt{\frac{860 \text{ N/m}}{2 \text{ kg}} - (12 \text{ rad/s})^2} = 16.9 \text{ rad/s}, \quad f = \frac{\omega}{2\pi} = 2.69 \text{ Hz}.$$

$$\boxed{f = 2.69 \text{ Hz}.}$$

Problem 21.27 The disk rotates *in the horizontal plane* with constant angular velocity $\Omega = 12$ rad/s. The mass $m = 2$ kg slides in a smooth slot in the disk and is attached to a spring with constant $k = 860$ N/m. The radial position of the mass when the spring is unstretched is $r = 0.2$ m. At $t = 0$, the mass is in the position $r = 0.4$ m and $dr/dt = 0$. Determine the position r as a function of time.



Solution: Using polar coordinates, Newton's second law in the r direction is

$$\Sigma F_r : -k(r - r_0) = m(\ddot{r} - r\Omega^2),$$

$$\ddot{r} + \left(\frac{k}{m} - \Omega^2\right) r = \frac{k}{m} r_0$$

$$\ddot{r} + (16.9 \text{ rad/s})^2 r = 86 \text{ m/s}^2.$$

The solution is

$$r = A \sin\omega t + B \cos\omega t + 0.301 \text{ m}, \quad \frac{dr}{dt} = A\omega \cos\omega t - B\omega \sin\omega t.$$

Putting in the initial conditions, we have

$$r(t = 0) = B + 0.301 \text{ m} = 0.4 \text{ m} \Rightarrow B = 0.0993 \text{ m},$$

$$\frac{dr}{dt}(t = 0) = A\omega = 0 \Rightarrow A = 0.$$

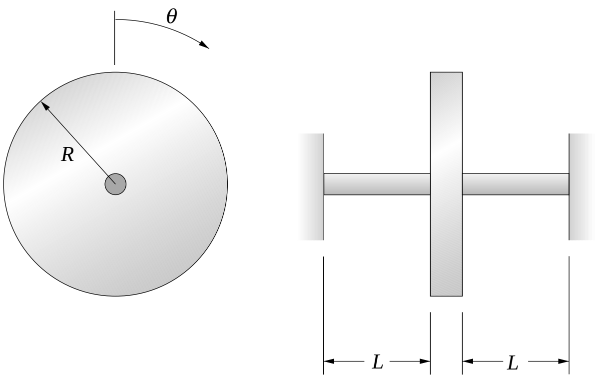
Thus the equation is

$$\boxed{r = (0.0993 \text{ m}) \cos[16.9 \text{ rad/s } t] + (0.301 \text{ m}).}$$

Problem 21.28 A homogeneous 44.5 N disk with radius $R = 0.31$ m is attached to two identical cylindrical steel bars of length $L = 0.31$m. The relation between the moment M exerted on the disk by one of the bars and the angle of rotation, θ, of the disk is

$$M = \frac{GJ}{L}\theta,$$

where J is the polar moment of inertia of the cross section of the bar and $G = 8.14 \times 10^{10}$ N/m^2 is the shear modulus of the steel. Determine the required radius of the bars if the frequency of rotational vibrations of the disk is to be 10 Hz.



Solution: The moment exerted by two bars on the disk is $M = k\theta$, where the spring constant is

$$k = \frac{\partial M}{\partial \theta} = \frac{2GJ}{L}.$$

The polar moment of the cross section of a bar is

$$J = \frac{\pi r^4}{2},$$

from which $k = \dfrac{\pi r^4 G}{L}$.

From the equation of angular motion,

$$I\frac{d^2\theta}{dt^2} = -k\theta.$$

The moment of inertia of the disk is

$$I = \frac{W}{2g}R_d^2,$$

from which

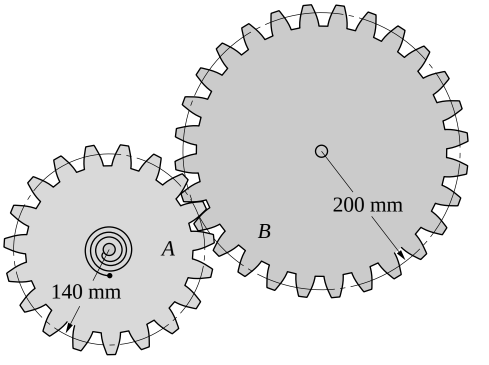
$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \omega = \sqrt{\frac{2\pi G g r^4}{WLR_d^2}} = \left(\frac{r^2}{R_d}\right)\sqrt{\frac{2\pi G g}{WL}}.$$

Solve: $r = \sqrt{R_d\sqrt{\dfrac{WL}{2\pi G g}}\,\omega} = \sqrt{R_d\sqrt{\dfrac{2\pi WL}{Gg}}\,f}.$

Substitute numerical values: $R_d = 0.31$ m, $L = 0.31$ m, $W = 445$ N, $G = 8.14 \times 10^{10}$ N/m^2, $g = 9.81$ m/s^2, from which

$\boxed{r = 0.01 \text{ m} = 10 \text{ mm.}}$

Problem 21.29 The moments of inertia of gears A and B are $I_A = 0.025$ kg-m^2 and $I_B = 0.100$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 10$ N-m/rad. What is the frequency of small angular vibrations of the gears?



Solution: The system is conservative. Denote the rotation velocities of A and B by $\dot{\theta}_A$, and $\dot{\theta}_B$ respectively. The kinetic energy of the gears is $T = \frac{1}{2}I_A\dot{\theta}_A^2 + \frac{1}{2}I_B\dot{\theta}_B^2$. The potential energy of the torsional spring is $V = \frac{1}{2}k\theta_A^2$. $T + V = \text{const.} = \frac{1}{2}I_A\dot{\theta}_A^2 + \frac{1}{2}I_B\dot{\theta}_B^2 + \frac{1}{2}k\theta_A^2$. From kinematics, $\dot{\theta}_B = -\left(\frac{R_A}{R_B}\right)\dot{\theta}_A$. Substitute, define

$$M = \left(I_A + \left(\frac{R_A}{R_B}\right)^2 I_B\right) = 0.074 \text{ kg-m}^2,$$

and take the time derivative:

$$\left(\frac{d\theta_A}{dt}\right)\left(M\left(\frac{d^2\theta_A}{dt^2}\right) + k\theta_A\right) = 0.$$

Ignore the possible solution $\left(\frac{d\theta_A}{dt}\right) = 0$, from which

$$\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = 0, \quad \text{where } \omega = \sqrt{\frac{k}{M}} = 11.62 \text{ rad/s.}$$

The frequency is $\boxed{f = \frac{\omega}{2\pi} = 1.850 \text{ Hz}}$.

Problem 21.30 At $t = 0$, the torsional spring in Problem 21.29 is unstretched and gear B has a counterclockwise angular velocity of 2 rad/s. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.

Solution: It is convenient to express the motion in terms of gear A, since the equation of motion of gear A is given in the solution to Problem 21.29: $\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = 0$, where

$$M = \left(I_A + \left(\frac{R_A}{R_B}\right)^2 I_B\right) = 0.074 \text{ kg-m}^2,$$

$$\omega = \sqrt{\frac{k}{M}} = 11.62 \text{ rad/s.}$$

Assume a solution of the form $\theta_A = A\sin\omega t + B\cos\omega t$. Apply the initial conditions,

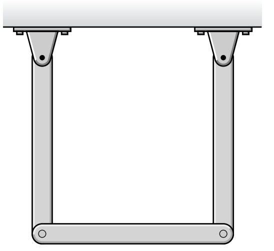
$$\theta_A = 0, \theta_A = -\left(\frac{R_B}{R_A}\right)\theta_B = -2.857 \text{ rad/s,}$$

from which $B = 0$, $A = \frac{\theta_A}{\omega} = -0.2458$, and

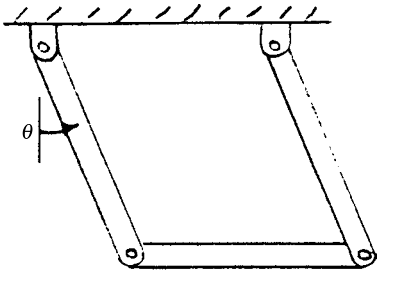
$$\theta_A = -0.2458\sin(11.62t) \text{ rad,}$$

and $\boxed{\theta_B = -\left(\frac{R_A}{R_B}\right)\theta_A = 0.172\sin(11.6t) \text{ rad}}$

Problem 21.31 Each 2-kg slender bar is 1 m in length. What are the period and frequency of small vibrations of the system?



Solution: The total energy is



$$T + V = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\left(\frac{1}{3}ml^2\right)\left(\frac{d\theta}{dt}\right)^2$$

$$+ \frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2 - mg\frac{L}{2}\cos\theta - mg\frac{L}{2}\cos\theta - mgL\cos\theta$$

$$= \frac{5}{6}mL^2\left(\frac{d\theta}{dt}\right)^2 - 2mgL\cos\theta.$$

$$\frac{d}{dt}(T+V) = \frac{5}{3}mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + 2mgL\sin\theta\frac{d\theta}{dt} = 0,$$

so the (linearized) equation of motion is

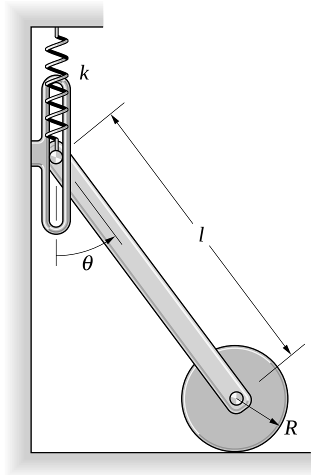
$$\frac{d^2\theta}{dt^2} + \frac{6g}{5L}\theta = 0.$$

Therefore

$$\omega = \sqrt{\frac{6g}{5L}} = \sqrt{\frac{6(9.81)}{5(1)}} = 3.43 \text{ rad/s},$$

so $\tau = \dfrac{2\pi}{\omega} = 1.83$ s, $f = 0.546$ Hz.

Problem 21.32* The masses of the slender bar and the homogeneous disk are m and m_d, respectively. The spring is unstretched when $\theta = 0$. Assume that the disk rolls on the horizontal surface.



(a) Show that the motion of the system is governed by the equation

$$\left(\frac{1}{3} + \frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2} - \frac{3m_d}{2m}\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$-\frac{g}{2l}\sin\theta + \frac{k}{m}(1-\cos\theta)\sin\theta = 0.$$

(b) If the system is in equilibrium at the angle $\theta = \theta_e$ and $\tilde{\theta} = \theta - \theta_e$ show that the equation governing small vibrations relative to the equilibrium position is

$$\left(\frac{1}{3} + \frac{3m_d}{2m}\cos^2\theta_e\right)\frac{d^2\tilde{\theta}}{dt^2} + \left[\frac{k}{m}(\cos\theta_e - \cos^2\theta_e\right.$$

$$\left. + \sin^2\theta_e) - \frac{g}{2l}\cos\theta_e\right]\tilde{\theta} = 0.$$

Solution: (See Example 21.2.) The system is conservative.

(a) The kinetic energy is

$$T = \tfrac{1}{2}I\dot{\theta}_2 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}I_d\dot{\theta}_d^2 + \tfrac{1}{2}m_d v_d^2,$$

where $I = \dfrac{mL^2}{12}$ is the moment of inertia of the bar about its center of mass, v is the velocity of the center of mass of the bar, $I_d = \dfrac{mR^2}{2}$ is the polar moment of inertia of the disk, and v_d is the velocity of the center of mass of the disk. The height of the center of mass of the bar is $h = \dfrac{L\cos\theta}{2}$, and the stretch of the spring is $S = L(1-\cos\theta)$, from which the potential energy is

$$V = \frac{mgL}{2}\cos\theta + \frac{1}{2}kL^2(1-\cos\theta)^2.$$

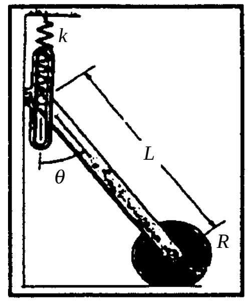
The system is conservative,

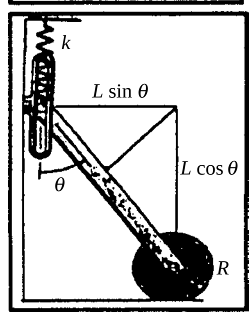
$$T + V = \text{const.} \quad \dot{\theta}l_d = \frac{v}{R} = \frac{L\cos\theta}{R}\dot{\theta}.$$

Choose a coordinate system with the origin at the pivot point and the x axis parallel to the lower surface. The instantaneous center of rotation of the bar is located at $(L\sin\theta, L\cos\theta)$. The center of mass of the bar is located at $\left(\frac{L}{2}\sin\theta, \frac{L}{2}\cos\theta\right)$. The distance from the center of mass to the center of rotation is

$$r = \sqrt{\left(L - \frac{L}{2}\right)^2\sin^2\theta + \left(L - \frac{L}{2}\right)^2\cos^2\theta} = \frac{L}{2},$$

from which $v = \dfrac{L}{2}\dot{\theta}$. The velocity of the center of mass of the disk is $v_d = (L\cos\theta)\dot{\theta}$. The angular velocity of the disk is $\dot{\theta}l_d = \dfrac{v}{R} = \dfrac{L\cos\theta}{R}\dot{\theta}$.





Substitute and reduce:

$$\frac{1}{2}\left(\frac{1}{3} + \frac{3m_d}{2m}\cos^2\right)\dot{\theta}^2 + \frac{g}{2L}\cos\theta + \frac{k}{2m}(1-\cos\theta)^2 = \text{const.}$$

Take the time derivative:

$$\dot{\theta}\left[\left(\frac{1}{3} + \frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2} - \frac{3m_d}{2m}\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2\right.$$

$$\left. - \frac{g}{2L}\sin\theta + \frac{k}{m}(1-\cos\theta)\sin\theta\right] = 0,$$

from which $$\left(\frac{1}{3} + \frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2} - \frac{3m_d}{2m}\sin\theta\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$-\frac{g}{2L}\sin\theta + \frac{k}{m}(1-\cos\theta)\sin\theta = 0$$

(b) The non-homogenous term is $\frac{g}{2L}$, as can be seen by dividing the equation of motion by $\sin\theta \neq 0$,

$$\frac{\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta\right)}{\sin\theta}\frac{d^2\theta}{dt^2}-\frac{3m_d}{2m}\cos\theta\left(\frac{d\theta}{dt}\right)^2$$

$$+\frac{k}{m}(1-\cos\theta)=\frac{g}{2L}.$$

Since the non-homogenous term is independent of time and angle, the equilibrium point can be found by setting the acceleration and the velocity terms to zero, $\cos\theta_e = 1-\frac{mg}{2Lk}$. [*Check*: This is identical to Eq. (21.15) in Example 21.2, as expected. *check.*]. Denote $\tilde{\theta}=\theta-\theta_e$. For small angles:

$$\cos\theta = \cos\tilde{\theta}\cos\theta_e - \sin\tilde{\theta}\sin\theta_e \to \cos\theta_e - \tilde{\theta}\sin\theta_e.$$

$$\cos^2\theta \to \cos^2\theta_e - 2\tilde{\theta}\sin\theta_e\cos\theta_e.$$

$$\sin\theta = \sin\tilde{\theta}\cos\theta_e + \cos\tilde{\theta}\sin\theta_e \to \tilde{\theta}\cos\theta_e + \sin^2\theta_e.$$

$$(1-\cos\theta)\sin\theta \to (1-\cos\theta_e)\sin\theta_e + \tilde{\theta}(\cos\theta_e - \cos\theta_e$$

$$+\sin^2\theta_e).$$

$$\sin\theta\cos\theta \to \tilde{\theta}(\cos^2\theta_e - \sin^2\theta_e).$$

Substitute and reduce:

$$(1)\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta\right)\frac{d^2\theta}{dt^2} \to \left(\frac{1}{3}+\frac{3m_d}{m}\tilde{\theta}\sin\theta_e\cos\theta_e\right)$$

$$\times\frac{d^2\tilde{\theta}}{dt^2} \to \left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta_e\right)\frac{d^2\tilde{\theta}}{dt^2}.$$

$$(2) = -\frac{3m_d}{2m}\cos\theta\sin\theta\left(\frac{d\theta}{dt}\right)^2 \to -\frac{3m_d}{2m}\tilde{\theta}\sin\theta_e\cos\theta_e$$

$$\times\left(\frac{d\tilde{\theta}}{dt}\right)^2 \to 0.$$

$$(3)\quad -\frac{g}{2L}\sin\theta \qquad \to -\frac{g}{2L}\tilde{\theta}\cos\theta_e - \frac{g}{2L}\sin\theta_e.$$

$$(4)\quad \frac{k}{m}(1-\cos\theta)\sin\theta \to \frac{k}{m}(1-\cos\theta_e)\sin\theta_e$$

$$+\frac{k}{m}\tilde{\theta}(\cos\theta_e - \cos^2\theta_e + \sin^2\theta_e),$$

where the terms $\tilde{\theta}\frac{d^2\tilde{\theta}}{dt^2} \to 0$, $\tilde{\theta}\left(\frac{d\tilde{\theta}}{dt}\right)^2 \to 0$, and terms in $\tilde{\theta}^2$ have been dropped.

Collect terms in (1) to (4) and substitute into the equations of motion:

$$\frac{\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta_e\right)}{\sin\theta_e}\frac{d^2\tilde{\theta}}{dt^2}+\left[\frac{k}{m}\frac{(\cos\theta_e-\cos^2\theta_e+\sin^2\theta_e)}{\sin\theta_e}\right.$$

$$\left.-\frac{g}{2L}\frac{\cos\theta_e}{\sin\theta_e}\right]=\left[\frac{g}{2L}-\frac{k}{m}(1-\cos\theta_e)\right].$$

The term on the right $\frac{g}{2L}-\frac{k}{m}(1-\cos\theta_e)=0$, as shown by substituting the value $\cos\theta_e = 1-\frac{mg}{2Lk}$, from which

$$\left(\frac{1}{3}+\frac{3m_d}{2m}\cos^2\theta_e\right)\frac{d^2\tilde{\theta}}{dt^2}$$

$$+\left[\frac{k}{m}(\cos\theta_e-\cos^2\theta_e+\sin^2\theta_e)-\frac{g}{2L}\cos\theta_e\right]\tilde{\theta}=0$$

is the equation of motion for small amplitude oscillations about the equilibrium point.

Problem 21.33* The masses of the bar and disk in Problem 21.32 are $m = 2$ kg and $m_d = 4$ kg, respectively. The dimensions $l = 1$ m and $R = 0.28$ m, and the spring constant is $k = 70$ N/m.

(a) Determine the angle θ_e at which the system is in equilibrium.

(b) The system is at rest in the equilibrium position, and the disk is given a clockwise angular velocity of 0.1 rad/s. Determine θ as a function of time.

Solution:

(a) From the solution to Problem 21.32, the static equilibrium angle is

$$\theta_e = \cos^{-1}\left(1 - \frac{mg}{2kL}\right) = 30.7^\circ = 0.5358 \text{ rad.}$$

(b) The canonical form (see Eq. (21.4)) of the equation of motion is $\dfrac{d^2\tilde{\theta}}{dt^2} + \omega^2\tilde{\theta} = 0$, where

$$\omega = \sqrt{\frac{\dfrac{k}{m}(\cos\theta_e - \cos^2\theta_e + \sin^2\theta_e) - \dfrac{g}{2L}\cos\theta_e}{\left(\dfrac{1}{3} + \dfrac{3m_d}{2m}\cos^2\theta_e\right)}}$$

$$= 1.891 \text{ rad/s.}$$

Assume a solution of the form

$$\tilde{\theta} = \theta - \theta_e = A\sin\omega t + B\cos\omega t,$$

from which

$$\theta = \theta_e + A\sin\omega t + B\cos\omega t.$$

From the solution to Problem 21.32, the angular velocity of the disk is

$$\dot{\theta}_d = \frac{v_d}{R} = \frac{(L\cos\theta_e)}{R}\dot{\theta}.$$

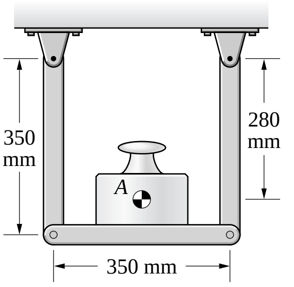
The initial conditions are

$$t = 0, \theta = \theta_e, \dot{\theta} = \frac{R\dot{\theta}_d}{L\cos\theta_e} = \frac{(0.1)(0.28)}{0.86} = 0.03256 \text{ rad/s,}$$

from which $B = 0$, $A = 0.03256/\omega = 0.01722$, from which the solution is

$$\theta = 0.5358 + 0.01722\sin(1.891t).$$

Problem 21.34 The mass of each slender bar is 1 kg. If the frequency of small vibrations of the system is 0.935 Hz, what is the mass of the object A?



Solution: The system is conservative. Denote $L = 0.350$ m, $L_A = 0.280$ m, $m = 1$ kg, and M the mass of A. The moments of inertia about the fixed point is the same for the two vertical bars. The kinetic energy is

$$T = \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}I\dot{\theta}^2 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}Mv_A^2,$$

where v is the velocity of the center of mass of the lower bar and v_A is the velocity of the center of mass of A, from which $T = \frac{1}{2}(2I\dot{\theta}^2 + mv^2 + Mv_A^2)$. The potential energy is

$$V = \frac{mgL}{2}(1-\cos\theta) + \frac{mgL}{2}(1-\cos\theta) + mgL(1-\cos\theta)$$

$$+ MgL_A(1-\cos\theta),$$

$$V = (MgL_A + 2mgL)(1-\cos\theta).$$

$$T + V = \text{const.} = \tfrac{1}{2}(2I\dot{\theta}^2 + mv^2 + Mv_A^2) + (MgL_A + 2mgL)$$

$$\times (1-\cos\theta).$$

From kinematics, $v = L\cos\theta(\dot{\theta})$, and $v_A = L_A\cos\theta(\dot{\theta})$. Substitute:

$$\tfrac{1}{2}(2I + (mL^2 + ML_A^2)\cos^2\theta)\dot{\theta}^2 + (MgL_A + 2mgL)(1-\cos\theta)$$

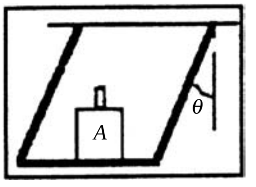
$$= \text{const.}$$

For small angles: $\cos^2\theta \to 1$, $(1-\cos\theta) \to \dfrac{\theta^2}{2}$, from which

$$\frac{1}{2}(2I + mL^2 + ML_A^2)\dot{\theta}^2 + \left(\frac{MgL_A}{2} + mgL\right)\theta^2 = \text{const.}$$

Take the time derivative:

$$\dot{\theta}\left[(2I + mL^2 + ML_A^2)\frac{d^2\theta}{dt^2} + (MgL_A + 2mgL)\theta\right] = 0.$$



Ignore the possible solution $\dot{\theta} = 0$, from which

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \omega = \sqrt{\frac{2mgL + MgL_A}{2I + mL^2 + ML_A^2}}.$$

The moment of inertia of a slender bar about one end (the fixed point) is $I = \dfrac{mL^2}{3}$, from which

$$\omega = \sqrt{\frac{g(2+\eta)}{\frac{5}{3}L + \eta L_A}}, \quad \text{where } \eta = \frac{ML_A}{mL}.$$

The frequency is

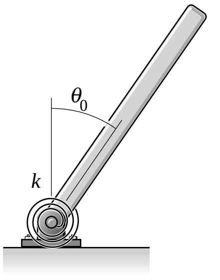
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3g(2+\eta)}{5L + 3\eta L_A}} = 0.935 \text{ Hz}.$$

Solve:

$$\eta = \frac{5L\omega^2 - 6g}{3(g - L_A\omega^2)} = 3.502,$$

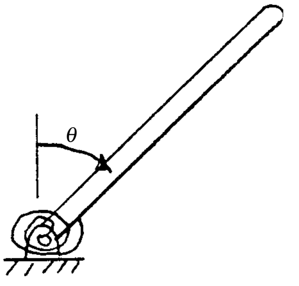
from which $\boxed{M = \dfrac{mL}{L_A}(3.502) = 4.38 \text{ kg}}$

Problem 21.35* The 4-kg slender bar is 2 m in length. It is held in equilibrium in the position $\theta_0 = 35^\circ$ by a torsional spring with constant k. The spring is unstretched when the bar is vertical. Determine the period and frequency of small vibrations of the bar relative to the equilibrium position shown.



Solution: The total energy is

$$T + v = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}k\theta^2 + mg\frac{L}{2}\cos\theta.$$



$$\frac{d}{dt}(T+v) = \frac{1}{3}mL^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + k\theta\frac{d\theta}{dt} - mg\frac{L}{2}\sin\theta\frac{d\theta}{dt} = 0,$$

so the equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{3k}{mL^2}\theta - \frac{3g}{2L}\sin\theta = 0. \quad \textbf{(1)}$$

Let $\theta_0 = 35^\circ$ be the equilibrium position:

$$\frac{3k}{mL^2}\theta_0 - \frac{3g}{2L}\sin\theta_0 = 0. \quad \textbf{(2)}$$

Solving for k,

$$k = \frac{mgL}{2}\frac{\sin\theta_0}{\theta_0} = \frac{(4)(9.81)(2)}{2}\frac{\sin 35^\circ}{(35\pi/180)} = 36.8 \text{ N-m.}$$

Let $\tilde{\theta} = \theta - \theta_0$. Then

$$\sin\theta = \sin(\theta_0 + \tilde{\theta}) = \sin\theta_0 + (\cos\theta_0)\tilde{\theta} + \cdots$$

Using this expression and Eq. (2), Eq. (1) (linearized) is

$$\frac{d^2\tilde{\theta}}{dt^2} + \left(\frac{3k}{mL^2} - \frac{3g}{2L}\cos\theta_0\right)\tilde{\theta} = 0.$$

Therefore $\omega = \sqrt{\dfrac{3k}{mL^2} - \dfrac{3g}{2L}\cos\theta_0}$

$$= \sqrt{\frac{(3)(36.8)}{(4)(2)^2} - \frac{(3)(9.81)}{(2)(2)}\cos 35^\circ}$$

$$= 0.939 \text{ rad/s,}$$

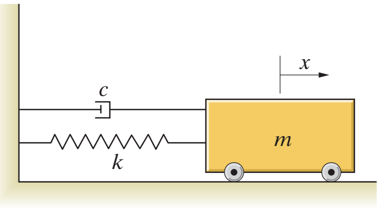
so $\tau = \dfrac{2\pi}{\omega} = 6.69$ s,

$$f = \frac{1}{\tau} = 0.149 \text{ Hz.}$$

Problem 21.36 The mass $m = 2$ kg, the spring constant is $k = 72$ N/m, and the damping constant is $c = 8$ N-s/m. The spring is unstretched when $x = 0$. The mass is displaced to the position $x = 1$ m and released from rest.

(a) If the damping is subcritical, what is the frequency of the resulting damped vibrations?
(b) What is the value of x at $t = 1$ s?

(See Active Example 21.3.)



Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0,\quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0,\quad \ddot{x} + \frac{8}{2}\dot{x} + \frac{72}{2}x = 0,$$

$$\ddot{x} + 2(2)\dot{x} + (6)^2 x = 0.$$

We recognize $\omega = 6$, $d = 2$, $d < \omega \Rightarrow$ yes, the motion is subcritical.

(a) $\omega_d = \sqrt{\omega^2 - d^2} = \sqrt{6^2 - 2^2} = 5.66$ rad/s, $f = \dfrac{\omega}{2\pi}$, $\boxed{f = 0.900 \text{ Hz.}}$

(b) The solution to the differential equation is

$$x = e^{-dt}(A \sin \omega_d t + B \cos \omega_d t)$$

$$\frac{dx}{dt} = e^{-dt}([A\omega_d - Bd] \cos \omega_d t - [B\omega_d + Ad] \sin \omega_d t)$$

Putting in the initial conditions we have

$$x(t = 0) = B = 1 \text{ m},$$

$$\frac{dx}{dt}(t = 0) = (A\omega_d - Bd) = 0 \Rightarrow A = \frac{d}{\omega_d}B = \frac{2}{5.66}(1 \text{ m}) = 0.354 \text{ m}.$$

The equation of motion is now $x = e^{-2t}(0.354 \sin[5.66t] + \cos[5.66t])$.

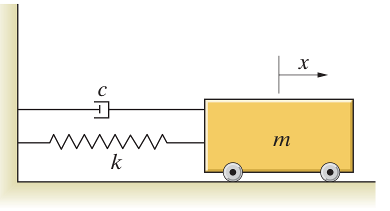
At $t = 1$ s, we have $\boxed{x = 0.0816 \text{ m.}}$

Problem 21.37 The mass $m = 2$ kg, the spring constant is $k = 72$ N/m, and the damping constant is $c = 32$ N-s/m. The spring is unstretched with $x = 0$. The mass is displaced to the position $x = 1$ m and released from rest.

(a) If the damping is subcritical, what is the frequency of the resulting damped vibrations?

(b) What is the value of x at $t = 1$ s?

(See Active Example 21.3.)



Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0,\ \ \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0,\ \ \ddot{x} + \frac{32}{2}\dot{x} + \frac{72}{2}x = 0,$$

$$\ddot{x} + 2(8)\dot{x} + (6)^2 x = 0.$$

(a) We recognize $\omega = 6,\ d = 8,\ d > \omega \Rightarrow$ $\boxed{\text{Damping is supercritical.}}$

(b) We have $h = \sqrt{d^2 - \omega^2} = \sqrt{8^2 - 6^2} = 5.29$ rad/s.

The solution to the differential equation is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t},\ \ \frac{dx}{dt} = -(d-h)Ce^{-(d-h)t} - (d+h)De^{-(d+h)t}.$$

Putting in the initial conditions, we have

$$\left.\begin{aligned} x(t=0) &= C + D = 1 \text{ m}, \\ \frac{dx}{dt}(t=0) &= -(d-h)C - (d+h)D = 0 \end{aligned}\right\} \Rightarrow C = \frac{d+h}{2h},\ \ D = \frac{h-d}{2h}$$

$C = 1.26$ ft, $D = -0.256$ m.

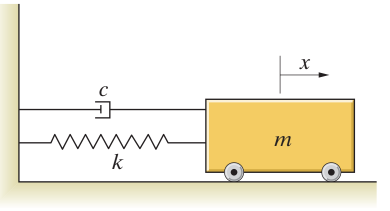
The general solution is then $x = (1.26)e^{-2.71t} - (0.256)e^{-13.3t}$.

At $t = 1$ s we have $\boxed{x = 0.0837 \text{ m.}}$

Problem 21.38 The mass $m = 4$ kg, the spring constant is $k = 72$ N/m. The spring is unstretched when $x = 0$.

(a) What value of the damping constant c causes the system to be critically damped?
(b) Suppose that c has the value determined in part (a). At $t = 0$, $x = 1$ m and $dx/dt = 4$ m/s. What is the value of x at $t = 1$ s?

(See Active Example 21.3.)



Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0,\ \ \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0,\ \ \ddot{x} + \frac{c}{4}\dot{x} + \frac{72}{4}x = 0,$$

$$\ddot{x} + 2\left(\frac{c}{8}\right)\dot{x} + (4.24)^2 x = 0.$$

(a) The system is critically damped when $d = \dfrac{c}{8} = \omega = 4.24 \Rightarrow \boxed{c = 33.9 \text{ N-s/m.}}$

(b) The solution to the differential equation is

$$x = Ce^{-dt} + Dte^{-dt},\ \ \frac{dx}{dt} = (D - Cd - Ddt)e^{-dt}.$$

Putting in the initial conditions, we have

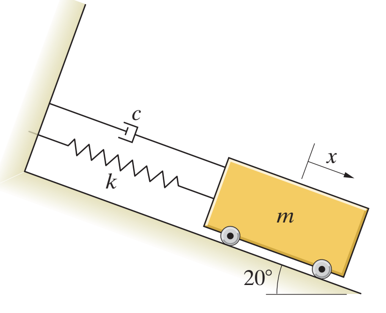
$$x(t = 0) = C = 1 \text{ m},$$

$$\frac{dx}{dt}(t = 0) = D - Cd = 4 \text{ m/s} \Rightarrow D = (1)(4.24) + 4 = 8.24 \text{ m}.$$

The general solution is then $x = (1 \text{ m})e^{-4.24t} + (8.24 \text{ m})te^{-4.24t}$.

At $t = 1$ s we have $\boxed{x = 0.133 \text{ m.}}$

Problem 21.39 The mass $m = 2$ kg, the spring constant is $k = 8$ N/m, and the damping coefficient is $c = 12$ N-s/m. The spring is unstretched when $x = 0$. At $t = 0$, the mass is released from rest with $x = 0$. Determine the value of x at $t = 2$ s.



Solution: The equation of motion is

$$(2 \text{ kg})\frac{d^2x}{dt^2} + (12 \text{ N-s/m})\frac{dx}{dt} + (8 \text{ N/m})x = (2 \text{ kg})(9.81 \text{ m/s}^2)\sin 20^\circ$$

$$\frac{d^2x}{dt^2} + 2(3 \text{ rad/s})\frac{dx}{dt} + (2 \text{ rad/s})^2 x = 3.36 \text{ m/s}^2$$

We identify $\omega = 2$ rad/s, $d = 3$ rad/s, $h = \sqrt{d^2 - \omega^2} = 2.24$ rad/s.

Since $d > \omega$, we have a supercritical case. The solution is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t} + 0.839 \text{ m}$$

$$v = \frac{dx}{dt} = -C(d-h)e^{-(d-h)t} - D(d+h)e^{-(d+h)t}$$

Using the initial conditions we have

$$\left.\begin{array}{l} 0 = C + D + 0.839 \text{ m} \\ 0 = -C(d-h) - D(d+h) \end{array}\right\} \Rightarrow C = -0.982 \text{ m},\ D = 0.143 \text{ m}$$

The motion is

$$x = -(0.982 \text{ m})e^{-(0.764 \text{ rad/s})t} + (0.143 \text{ m})e^{-(5.24 \text{ rad/s})t} + 0.839 \text{ m}$$

At $t = 2$ s $\boxed{x = 0.626 \text{ m}}$

Problem 21.40 The mass $m = 2.19$ kg, the spring constant is $k = 7.3$ N/m, and the damping coefficient is $c = 11.67$ N-s/m. The spring is unstretched when $x = 0$. At $t = 0$, the mass is released from rest with $x = 0$. Determine the value of x at $t = 2$ s.

Solution: The equation of motion is

$$(2.19\text{ kg})\frac{d^2x}{dt^2} + (11.67\text{ N-s/m})\frac{dx}{dt} + (7.3\text{ N/m})x$$

$$= (2.19\text{ kg})(9.81\text{ m/s}^2)\sin 20^\circ$$

$$\frac{d^2x}{dt^2} + 2(2.67\text{ rad/s})\frac{dx}{dt} + (1.826\text{ rad/s})^2 x = 3.36\text{ m/s}^2$$

We identify

$$\omega = 1.83\text{ rad/s},\ d = 2.67\text{ rad/s},\ h = \sqrt{d^2 - \omega^2} = 1.94\text{ rad/s}$$

Since $d > \omega$, we have the supercritical case. The solution is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t} + 1.01\text{ m}$$

$$v = \frac{dx}{dt} = -C(d-h)e^{-(d-h)t} - D(d+h)e^{-(d+h)t}$$

Using the initial conditions we find

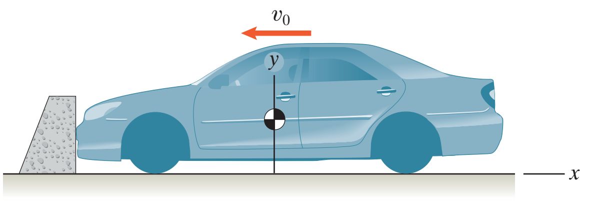
$$\left.\begin{aligned} 0 &= C + D + 1.01\text{ m} \\ 0 &= -C(d-h) - D(d+h) \end{aligned}\right\} \Rightarrow C = -1.19\text{ m},\ D = 0.187\text{ m}$$

Thus the solution is

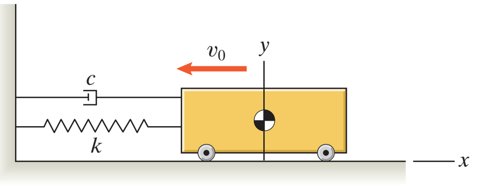
$$x = -(1.19\text{ m})e^{-(0.723\text{ rad/s})t} + (0.187\text{ m})e^{-(4.61\text{ rad/s})t} + 1.01\text{ m}$$

At $t = 2$ s, $\boxed{x = 0.725\text{ m}}$

Problem 21.41 A 79.8 kg test car moving with velocity $v_0 = 7.33$ m/s collides with a rigid barrier at $t = 0$. As a result of the behavior of its energy-absorbing bumper, the response of the car to the collision can be simulated by the damped spring-mass oscillator shown with $k = 8000$ N/m and $c = 3000$ N-s/m. Assume that the mass is moving to the left with velocity $v_0 = 7.33$ m/s and the spring is unstretched at $t = 0$. Determine the car's position (a) at $t = 0.04$ s and (b) at $t = 0.08$ s.



Car colliding with a rigid barrier



Simulation model

Solution: The equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0,$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \frac{3000}{79.8}\dot{x} + \frac{8000}{79.8}x = 0, \quad \ddot{x} + 2(18.8)\dot{x} + (10.0)^2 x = 0$$

We recognize that $\omega = 10$, $d = 18.8$ $d > \omega \Rightarrow$ Supercritical damping.

We have $h = \sqrt{d^2 - \omega^2} = 15.9$ rad/s, $v_0 = 7.33$ m/s.

The solution to the differential equation is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t}, \quad \frac{dx}{dt} = -(d-h)Ce^{-(d-h)t} - (d+h)De^{-(d+h)t}.$$

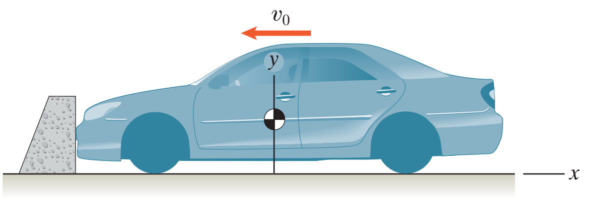
Putting in the initial conditions, we have

$$\left.\begin{aligned} &x(t=0) = C + D = 0 \\ &\frac{dx}{dt}(t=0) = -(d-h)C - (d+h)D = -v_0 \end{aligned}\right\} \Rightarrow C = -D = -\frac{v_0}{2h} = -0.231 \text{ m}.$$

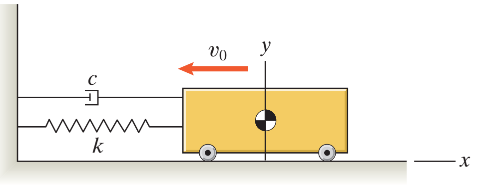
The general solution is then $x = (-0.231)(e^{-2.89t} - e^{-34.7t})$

At the 2 specified times we have $\boxed{\text{(a) } x = -0.148 \text{ m}, \text{ (b) } x = -0.169 \text{ m}}$

Problem 21.42 A 79.8 kg test car moving with velocity $v_0 = 7.33$ m/s collides with a rigid barrier at $t = 0$. As a result of the behavior of its energy-absorbing bumper, the response of the car to the collision can be simulated by the damped spring-mass oscillator shown with $k = 8000$ N/m and $c = 3000$ N-s/m. Assume that the mass is moving to the left with velocity $v_0 = 7.33$ m/s and the spring is unstretched at $t = 0$. Determine the car's deceleration (a) immediately after it contacts the barrier; (b) at $t = 0.04$ s and (c) at $t = 0.08$ s.



Car colliding with a rigid barrier



Simulation model

Solution: From Problem 21.41 we know that the motion is given by

$$x = (-0.231)(e^{-2.89t} - e^{-34.7t}),$$

$$\frac{dx}{dt} = (-0.231)([-2.89]e^{-2.89t} - [-34.7]e^{-34.7t}),$$

$$\frac{d^2x}{dt^2} = (-0.231)([-2.89]^2 e^{-2.89t} - [-34.7]^2 e^{-34.7t})$$

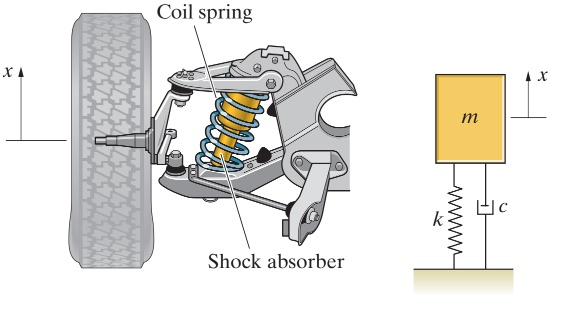
Thus the deceleration a is given by

$$a = -\frac{d^2x}{dt^2} = -(0.587)e^{-2.89t} + (84.5)e^{-34.7t}$$

Putting in the required times, we find

$$\boxed{\begin{array}{l}\text{(a) } a = 276 \text{ m/s}^2, \\ \text{(b) } a = 67.6 \text{ m/s}^2, \\ \text{(c) } a = 15.8 \text{ m/s}^2.\end{array}}$$

Problem 21.43 The motion of the car's suspension shown in Problem 21.42 can be modeled by the damped spring–mass oscillator in Fig. 21.9 with $m = 36$ kg, $k = 22$ kN/m, and $c = 2.2$ kN-s/m. Assume that no external forces act on the tire and wheel. At $t = 0$, the spring is unstretched and the tire and wheel are given a velocity $dx/dt = 10$ m/s. Determine the position x as a function of time.



Solution: Calculating ω and d, we obtain

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{22{,}000}{36}} = 24.72 \text{ rad/s}$$

and $\quad d = \dfrac{c}{2m} = \dfrac{2200}{2(36)} = 30.56$ rad/s.

Since $d > \omega$, the motion is supercritically damped and Equation (21.24) is the solution, where

$$h = \sqrt{d^2 - \omega^2} = 17.96 \text{ rad/s}.$$

Equation (21.24) is

$$x = Ce^{-(d-h)t} + De^{-(d+h)t},$$

or $\quad x = Ce^{-12.6t} + De^{-48.5t}$.

The time derivative is

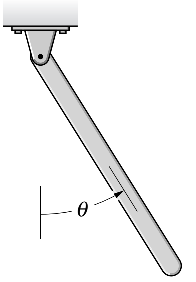
$$\frac{dx}{dt} = -12.6Ce^{-12.6t} - 48.5De^{-48.5t}.$$

At $t = 0$, $x = 0$ and $dx/dt = 10$ m/s: Hence, $0 = C + D$ and $10 = -12.6C - 48.5D$. Solving for C and D, we obtain $C = 0.278$ m, $D = -0.278$ m. The solution is

$$x = 0.278(e^{-12.6t} - e^{-48.5t}) \text{ m}.$$

Problem 21.44 The 4-kg slender bar is 2 m in length. Aerodynamic drag on the bar and friction at the support exert a resisting moment about the pin support of magnitude $1.4(d\theta/dt)$ N-m, where $d\theta/dt$ is the angular velocity in rad/s.

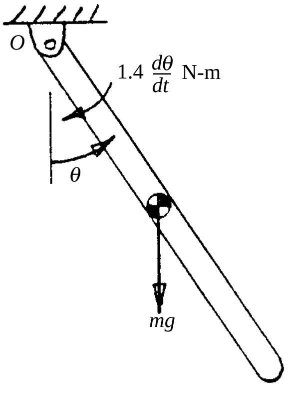
(a) What are the period and frequency of small vibrations of the bar?
(b) How long does it take for the amplitude of vibration to decrease to one-half of its initial value?



Solution:

(a) $\sum M_0 = I_0\alpha$:

$$-1.4\frac{d\theta}{dt} - mg\frac{L}{2}\sin\theta = \frac{1}{3}mL^2\alpha.$$



The (linearized) equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{4.2}{mL^2}\frac{d\theta}{dt} + \frac{3g}{2L}\theta = 0.$$

This is of the form of Eq. (21.16) with

$$d = \frac{4.2}{2mL^2} = \frac{4.2}{2(4)(2)^2} = 0.131 \text{ rad/s},$$

$$\omega = \sqrt{\frac{3g}{2L}} = \sqrt{\frac{3(9.81)}{2(2)}} = 2.71 \text{ rad/s}.$$

From Eq. (21.18), $\omega_d = \sqrt{\omega^2 - d^2} = 2.71$ rad/s, and from Eqs. (21.20),

$$\tau_d = \frac{2\pi}{\omega_d} = 2.32 \text{ s},$$

$$f_d = \frac{1}{\tau_d} = 0.431 \text{ Hz}.$$

(b) Setting $e^{-dt} = e^{-0.131t} = 0.5$, we obtain $t = 5.28$ s.

Problem 21.45 The bar described in Problem 21.44 is given a displacement $\theta = 2^\circ$ and released from rest at $t = 0$. What is the value of θ (in degrees) at $t = 2$ s?

Solution: From the solution of Problem 21.44, the damping is subcritical with $d = 0.131$ rad/s, $\omega = 2.71$ rad/s. From Eq. (21.19),

$$\theta = e^{-0.131t}(A\sin 2.71t + B\cos 2.71t),$$

so $$\frac{d\theta}{dt} = -0.131e^{-0.131t}(A\sin 2.71t + B\cos 2.71t)$$

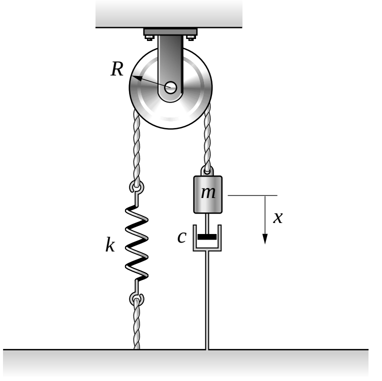
$$+ e^{-0.131t}(2.71A\cos 2.71t - 2.71B\sin 2.71t).$$

At $t = 0$, $\theta = 2^\circ$ and $d\theta/dt = 0$. Substituting these conditions, $2^\circ = B$, $0 = -0.131B + 2.71A$, we see that $B = 2^\circ$, $A = 0.0969^\circ$, so

$$\theta = e^{-0.131t}(0.0969^\circ \sin 2.71t + 2^\circ\cos 2.71t).$$

At $t = 2$ s, we obtain $\theta = 0.942^\circ$.

Problem 21.46 The radius of the pulley is $R = 100$ mm and its moment of inertia is $I = 0.1$ kg-m^2. The mass $m = 5$ kg, and the spring constant is $k = 135$ N/m. The cable does not slip relative to the pulley. The coordinate x measures the displacement of the mass relative to the position in which the spring is unstretched. Determine x as a function of time if $c = 60$ N-s/m and the system is released from rest with $x = 0$.



Solution: Denote the angular rotation of the pulley by θ. The moment on the pulley is $\sum M = R(kx) - RF$, where F is the force acting on the right side of the pulley. From the equation of angular motion for the pulley,

$$I\frac{d^2\theta}{dt^2} = Rkx - RF,$$

from which $F = -\frac{I}{R}\frac{d^2\theta}{dt^2} + kx$.

The force on the mass is $-F + f + mg$, where the friction force $f = -c\frac{dx}{dt}$ acts in opposition to the velocity of the mass. From Newton's second law for the mass,

$$m\frac{d^2x}{dt^2} = -F - c\frac{dx}{dt} + mg = \frac{I}{R}\frac{d^2\theta}{dt^2} - kx - e\frac{dx}{dt} + mg.$$

From kinematics, $\theta = -\frac{x}{R}$, from which the equation of motion for the mass is

$$\left(\frac{I}{R^2} + m\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = mg.$$

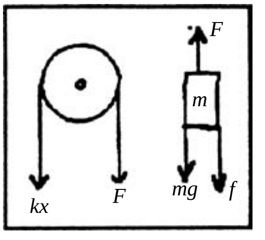
The canonical form (see Eq. (21.16)) of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = \frac{R^2 mg}{I + R^2 m},$$

where $d = \frac{cR^2}{2(I + R^2 m)} = 2$ rad/s, $\omega^2 = \frac{kR^2}{(I + R^2 m)} = 9$ (rad/s)^2.

The damping is sub-critical, since $d^2 < \omega^2$. The solution is the sum of the solution to the homogenous equation of motion, of the form $x_c = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} = 2.236$ rad/s, and the solution to the non-homogenous equation, of the form

$$x_p = \frac{mgR^2}{(I + R^2 m)\omega^2} = \frac{mg}{k} = 0.3633.$$



(The particular solution x_p is obtained by setting the acceleration and velocity to zero and solving, since the non-homogenous term mg is not a function of time or position.) The solution is

$$x = x_c + x_p = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t) + \frac{mg}{k}.$$

Apply the initial conditions: at $t = 0$, $x = 0$, $\frac{dx}{dt} = 0$, from which $0 = B + \frac{mg}{k}$, and $0 = -d[x_c]_{t=0} + \omega_d A = -dB + \omega_d A$, from which $B = -0.3633$, $A = \frac{dB}{\omega_d} = -0.3250$, and

$$x(t) = e^{-dt}\left(-\frac{dmg}{k\omega_d}\sin\omega_d - \left(\frac{mg}{k}\right)\cos\omega_d t\right) + \left(\frac{mg}{k}\right)$$

$$\boxed{x(t) = e^{-2t}(-0.325\sin(2.236\,t) - 0.363\cos(2.236\,t)) + 0.3633 \text{ (m)}}$$

Problem 21.47 For the system described in Problem 21.46, determine x as a function of time if $c =$ 120 N-s/m and the system is released from rest with $x = 0$.

Solution: From the solution to Problem 21.46 the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = \frac{R^2mg}{I + R^2m},$$

where $d = \dfrac{cR^2}{2(I + R^2m)} = 4$ rad/s,

and $\omega^2 = \dfrac{kR^2}{(I + R^2m)} = 9$ (rad/s)^2.

The system is supercritically damped, since $d^2 > \omega^2$. The homogenous solution is of the form (see Eq. (21.24)) $x_c = Ce^{-(d-h)t} + De^{-(d+h)t}$, where

$h = \sqrt{d^2 - \omega^2} = 2.646$ rad/s, $(d - h) = 1.354$, $(d + h) = 6.646$.

The particular solution is $x_p = \dfrac{mg}{k} = 0.3633$. The solution is

$$x(t) = x_c + x_{p_c} = Ce^{-(d-h)t} + De^{-(d+h)t} + \frac{mg}{k}.$$

Apply the initial conditions, at $t = 0$, $x = 0$, $\dfrac{dx}{dt} = 0$, from which $0 = C + D + \dfrac{mg}{k}$, and $0 = -(d - h)C - (d + h)D$. Solve:

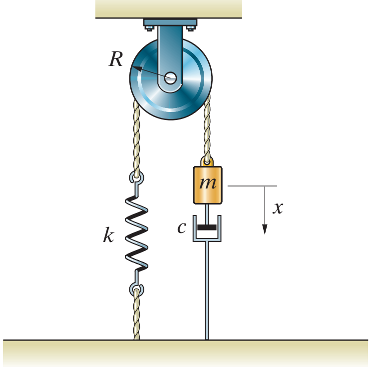
$$C = -\frac{(d+h)}{2h}\left(\frac{mg}{k}\right),$$

and $D = \dfrac{(d-h)}{2h}\left(\dfrac{mg}{k}\right)$,

from which

$$\boxed{\begin{aligned} x(t) &= \frac{mg}{k}\left(1 - \frac{(d+h)}{2h}e^{-(d-h)t} + \frac{(d-h)}{2h}e^{-(d+h)t}\right). \\ &= 0.3633(1 - 1.256e^{-1.354t} + 0.2559e^{-6.646t})\ \text{(m)} \end{aligned}}$$

Problem 21.48 For the system described in Problem 21.46, choose the value of c so that the system is critically damped, and determine x as a function of time if the system is released from rest with $x = 0$.



Solution: From the solution to Problem 21.46, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = \frac{R^2mg}{I + R^2m},$$

where $d = \dfrac{cR^2}{2(I + R^2m)}$, and $\omega^2 = \dfrac{kR^2}{(I + R^2m)} = 9$ (rad/s)^2.

For critical damping, $d^2 = \omega^2$, from which $d = 3$ rad/s. The homogenous solution is (see Eq. (21.25)) $x_c = Ce^{-dt} + Dte^{-dt}$ and the particular solution is $x_p = \dfrac{mg}{k} = 0.3633$ m. The solution:

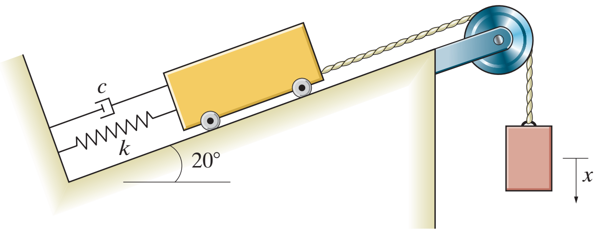
$$x(t) = x_c + x_p = Ce^{-dt} + Dte^{-dt} + \frac{mg}{k}.$$

Apply the initial conditions at $t = 0$, $x = 0$, $\dfrac{dx}{dt} = 0$, from which $C + \dfrac{mg}{k} = 0$, and $-dC + D = 0$. Solve: $C = -\dfrac{mg}{k} = -0.3633$, $D = dC = -1.09$. The solution is

$$x(t) = \frac{mg}{k}(1 - e^{-dt}(1 + dt)),$$

$$\boxed{x(t) = 0.363(1 - e^{-3t}(1 + 3t))\ \text{m}}$$

Problem 21.49 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The mass of each object is 30 kg. The inclined surface is smooth. The radius of the pulley is 120 mm and it moment of inertia is $I = 0.03$ kg-m^2. Determine the frequency and period of vibration of the system relative to its equilibrium position if (a) $c = 0$, (b) $c = 250$ N-s/m.



Solution: Let T_1 be the tension in the rope on the left of the pulley, and T_2 be the tension in the rope on the right of the pulley. The equations of motion are

$$T_1 - mg\sin\theta - c\dot{x} - kx = m\ddot{x},\ T_2 - mg = -m\ddot{x},\ (T_2 - T_1)\,r = I\frac{\ddot{x}}{r}.$$

If we eliminate T_1 and T_2, we find

$$\left(2m + \frac{I}{r^2}\right)\ddot{x} + c\dot{x} + kx = mg(1 - \sin\theta),$$

$$\ddot{x} + \frac{cr^2}{2mr^2 + I}\dot{x} + \frac{kr^2}{2mr^2 + I}x = \frac{mgr^2}{2mr^2 + I}(1 - \sin\theta),$$

$$\ddot{x} + (0.016c)\dot{x} + (3.59 \text{ rad/s})^2 x = 3.12 \text{ m/s}^2.$$

(a) If we set $c = 0$, the natural frequency, frequency, and period are

$$\omega = 3.59 \text{ rad/s},\ f = \frac{\omega}{2\pi} = 0.571 \text{ Hz},\ \tau = \frac{1}{f} = 1.75 \text{ s}. \quad \boxed{\begin{array}{l} f = 0.571 \text{ s}, \\ \tau = 1.75 \text{ s}. \end{array}}$$

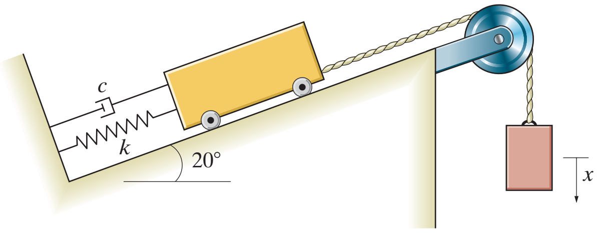
(b) If we set $c = 250$ N/m, then

$$\ddot{x} + 2(2.01)\dot{x} + (3.59 \text{ rad/s})^2 x = 3.12 \text{ m/s}^2.$$

We recognize $\omega = 3.59,\ d = 2.01,\ \omega_d = \sqrt{\omega^2 - d^2} = 2.97$ rad/s.

$$f = \frac{\omega_d}{2\pi}, \quad \tau = \frac{1}{f}, \quad \boxed{f = 0.473 \text{ Hz},\ \tau = 2.11 \text{ s}.}$$

Problem 21.50 The spring constant is $k = 800$ N/m, and the spring is unstretched when $x = 0$. The damping constant is $c = 250$ N-s/m. The mass of each object is 30 kg. The inclined surface is smooth. The radius of the pulley is 120 mm and it moment of inertia is $I = 0.03$ kg-m^2. At $t = 0$, $x = 0$ and $dx/dt = 1$ m/s. What is the value of x at $t = 2$ s?



Solution: From Problem 21.49 we know that the damping is sub-critical and the key parameters are

$$\omega = 3.59 \text{ rad/s},\ d = 2.01 \text{ rad/s},\ \omega_{\text{d}} = 2.97 \text{ rad/s}.$$

The equation of motion is $\ddot{x} + 2(2.01)\dot{x} + (3.59 \text{ rad/s})^2 x = 3.12 \text{ m/s}^2$.

The solution is

$$x = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t) + 0.242$$

$$\frac{dx}{dt} = e^{-dt}([A\omega_d - Bd]\cos\omega_d t - [Ad + B\omega_d]\sin\omega_d t)$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.242 = 0 \Rightarrow B = -0.242,$$

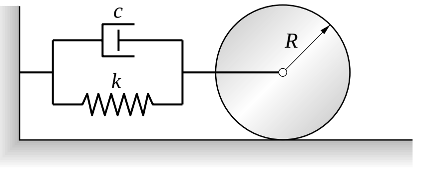
$$\frac{dx}{dt}(t = 0) = A\omega_d - Bd = 1 \Rightarrow A = 0.172$$

Thus the motion is governed by

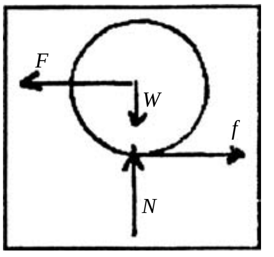
$$x = e^{-2.01t}(0.172\sin[2.97t] - 0.242\cos[2.97t]) + 0.242$$

At time $t = 2$ s we have $\boxed{x = 0.237 \text{ m.}}$

Problem 21.51 The homogeneous disk weighs 445 N and its radius is $R = 0.31$ m. It rolls on the plane surface. The spring constant is $k = 1459.3$ N/m and the damping constant is $c = 43.8$ N-s/m. Determine the frequency of small vibrations of the disk relative to its equilibrium position.



Solution: Choose a coordinate system with the origin at the center of the disk and the positive x axis parallel to the floor. Denote the angle of rotation by θ. The *horizontal* forces acting on the disk are



$$\sum F = -kx - c\frac{dx}{dt} + f.$$

From Newton's second law,

$$m\frac{d^2x}{dt^2} = \sum F = -kx - c\frac{dx}{dt} + f.$$

The moment about the center of mass of the disk is $\sum M = Rf$. From the equation of angular motion, $I\frac{d^2\theta}{dt^2} = Rf$, from which $f = \frac{I}{R}\frac{d^2\theta}{dt^2}$, where the moment of inertia is $I = \frac{mR^2}{2} = 0.642$ kg-m^2. Substitute:

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + \frac{I}{R}\frac{d^2\theta}{dt^2}.$$

From kinematics, $\theta = -\frac{x}{R}$, from which the equation of motion is

$$\left(m + \frac{I}{R^2}\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

The canonical form (see Eq. (21.16)) is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where $d = \frac{cR^2}{2(mR^2 + I)} = \frac{c}{3m} = 0.3217$ rad/s

and $\omega^2 = \frac{kR^2}{(I + R^2m)} = \frac{2k}{3m} = 21.45$ (rad/s)^2.

The damping is sub-critical, since $d^2 < \omega^2$. The frequency is

$$\boxed{f_d = \frac{1}{2\pi}\sqrt{\omega^2 - d^2} = 0.7353 \text{ Hz}}$$

Problem 21.52 In Problem 21.51, the spring is unstretched at $t = 0$ and the disk has a clockwise angular velocity of 2 rad/s. What is the angular velocity of the disk when $t = 3$ s?

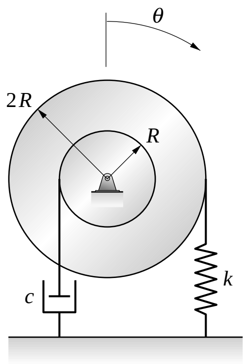
Solution: From the solution to Problem 21.51, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

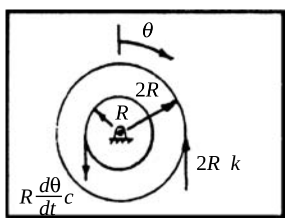
where $d = \dfrac{c}{3m} = 0.322$ rad/s and $\omega^2 = \dfrac{2k}{3m} = 21.47$ (rad/s)^2.

The system is sub-critically damped, so that the solution is of the form $x = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} =$ 4.622 rad/s. Apply the initial conditions: $x_0 = 0$, and from kinematics, $\dot{\theta}_0 = \dot{x}_0/R = -2$ rad/s, from which $\dot{x}_0 = 0.61$ m/s, from which $B = 0$ and $A = \dot{x}_0/\omega_d = 0.132$. The solution is $x(t) = e^{-dt}\left(\dfrac{\dot{x}_0}{\omega_d}\right)\sin\omega_d t$, and $\dot{x}(t) = -dx + \dot{x}_0 e^{-dt}\cos\omega_d t$. At $t = 3$ s, $x = 0.048$ m and $\dot{x} = 0.0466$ m/s. From kinematics, $\theta(t) = -\dfrac{\dot{x}(t)}{R}$.
At $t = 3$ s, $\boxed{\dot{\theta} = -0.153 \text{ rad/s}}$ clockwise.

Problem 21.53 The moment of inertia of the stepped disk is I. Let θ be the angular displacement of the disk relative to its position when the spring is unstretched. Show that the equation governing θ is identical in form to Eq. (21.16), where $d = \dfrac{R^2c}{2I}$ and $\omega^2 = \dfrac{4R^2k}{I}$.



Solution: The sum of the moments about the center of the stepped disk is



$$\sum M = -R\left(Rc\frac{d\theta}{dt}\right) - 2R(2Rk\theta).$$

From the equation of angular motion

$$\sum M = I\frac{d^2\theta}{dt^2}.$$

The equation of motion is

$$I\frac{d^2\theta}{dt^2} + R^2c\frac{d\theta}{dt} + 4R^2k\theta = 0.$$

The canonical form is

$$\boxed{\frac{d^2\theta}{dt^2} + 2d\frac{d\theta}{dt} + \omega^2\theta = 0}, \text{ where } d = \frac{R^2c}{2I}, \omega^2 = \frac{4R^2k}{I}.$$

Problem 21.54 In Problem 21.53, the radius $R = 250$ mm, $k = 150$ N/m, and the moment of inertia of the disk is $I = 2$ kg-m^2.

(a) At what value of c will the system be critically damped?

(b) At $t = 0$, the spring is unstretched and the clockwise angular velocity of the disk is 10 rad/s. Determine θ as a function of time if the system is critically damped.

(c) Using the result of (b), determine the maximum resulting angular displacement of the disk and the time at which it occurs.

Solution: From the solution to Problem 21.53, the canonical form of the equation of motion is

$$\frac{d^2\theta}{dt^2} + 2d\frac{d\theta}{dt} + \omega^2\theta = 0,$$

where $d = \dfrac{R^2 c}{2I}$ and $\omega^2 = \dfrac{4R^2 k}{I}$.

(a) For critical damping, $d^2 = \omega^2$, from which

$\boxed{c = \dfrac{4}{R}\sqrt{kI} = 277 \text{ N-s/m}}$, and $d = 4.330$ rad/s.

(b) The solution is of the form (see Eq. (21.25)) $\theta = Ce^{-dt} + Dte^{-dt}$. Apply the initial conditions: $\theta_0 = 0$, $\dot{\theta}_0 = -10$ rad/s, from which $C = 0$, and $D = -10$. The solution is

$\boxed{\theta(t) = \dot{\theta}_0 te^{-dt} = -10te^{-4.330t} \text{ rad/s}}$.

(c) The maximum (or minimum) value of the angular displacement is obtained from

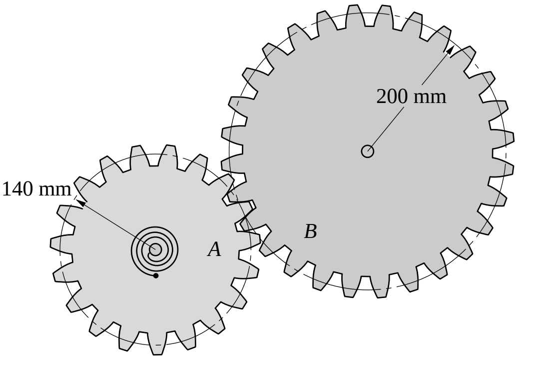
$$\frac{d\theta}{dt} = 0 = -10e^{-4.330t}(1 - 4.33t) = 0,$$

from which the maximum/minimum occurs at

$\boxed{t_{\max} = \dfrac{1}{4.330} = 0.231 \text{ s}}$.

The angle is $[\theta]_{t=t_{\max}} = -0.850$ rad *clockwise*.

Problem 21.55 The moments of inertia of gears A and B are $I_A = 0.025$ kg-m^2 and $I_B = 0.100$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 10$ N-m/rad. The bearing supporting gear B incorporates a damping element that exerts a resisting moment on gear B of magnitude $2(d\theta_B/dt)$ N-m, where $d\theta_B/dt$ is the angular velocity of gear B in rad/s. What is the frequency of small angular vibrations of the gears?



Solution: The sum of the moments on gear A is $\sum M = -k\theta_A + R_A F$, where the moment exerted by the spring opposes the angular displacement θ_A. From the equation of angular motion,

$$I_A \frac{d^2\theta_A}{dt^2} = \sum M = -k\theta_A + R_A F,$$

from which $F = \left(\frac{I_A}{R_A}\right)\frac{d^2\theta_A}{dt^2} + \left(\frac{k}{R_A}\right)\theta_A.$

The sum of the moments acting on gear B is

$$\sum M = -2\frac{d\theta_B}{dt} + R_B F,$$

where the moment exerted by the damping element opposes the angular velocity of B. From the equation of angular motion applied to B,

$$I_B \frac{d^2\theta_B}{dt^2} = \sum M = -2\frac{d\theta_B}{dt} + R_B F.$$

Substitute the expression for F,

$$I_B \frac{d^2\theta_B}{dt^2} + 2\frac{d\theta_B}{dt} - \left(\frac{R_B}{R_A}\right)\left(I_A \frac{d^2\theta_A}{dt^2} + k\theta_A\right) = 0.$$

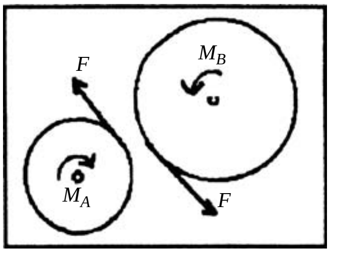
From kinematics,

$$\theta_A = -\left(\frac{R_B}{R_A}\right)\theta_B,$$

from which the equation of motion for gear B is

$$\left(I_B + \left(\frac{R_B}{R_A}\right)^2 I_A\right)\frac{d^2\theta_B}{dt^2} + 2\frac{d\theta_B}{dt} + \left(\frac{R_B}{R_A}\right)^2 k\theta_B = 0.$$

Define $M = I_B + \left(\frac{R_B}{R_A}\right)^2 I_A = 0.1510$ kg-m^2.



The canonical form of the equation of motion is

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = 0,$$

where $d = \frac{1}{M} = 6.622$ rad/s

and $\omega^2 = \left(\frac{R_B}{R_A}\right)^2 \frac{k}{M} = 135.1$ (rad/s)^2.

The system is sub critically damped, since $d^2 < \omega^2$, from which $\omega_d = \sqrt{\omega^2 - d^2} = 9.555$ rad/s, from which the frequency of small vibrations is

$$\boxed{f_d = \frac{\omega_d}{2\pi} = 1.521 \text{ Hz}}$$

Problem 21.56 At $t = 0$, the torsional spring in Problem 21.55 is unstretched and gear B has a counterclockwise angular velocity of 2 rad/s. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.

Solution: From the solution to Problem 21.55, the canonical form of the equation of motion for gear B is

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = 0,$$

where $M = I_B + \left(\frac{R_B}{R_A}\right)^2 I_A = 0.1510$ kg-m^2,

$$d = \frac{1}{M} = 6.622 \text{ rad/s},$$

and $\omega^2 = \left(\frac{R_B}{R_A}\right)^2 \frac{k}{M} = 135.1 \text{ (rad/s)}^2$.

The system is sub critically damped, since $d^2 < \omega^2$, from which $\omega_d = \sqrt{\omega^2 - d^2} = 9.555$ rad/s. The solution is of the form $\theta_B(t) = e^{-dt}(A \sin\omega_d t + B\cos\omega_d t)$. Apply the initial conditions, $[\theta_B]_{t=0} = 0$, $[\dot{\theta}_B]_{t=0} = 2$ rad/s, from which $B = 0$, and $A = \frac{2}{\omega_d} = 0.2093$, from which the solution is

$$\boxed{\theta_B(t) = e^{-6.62t}(0.209\sin(9.55t))}$$

Problem 21.57 For the case of critically damped motion, confirm that the expression $x = Ce^{-dt} + Dte^{-dt}$ is a solution of Eq. (21.16).

Solution: Eq. (21.16) is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0.$$

We show that the expression is a solution by substitution. The individual terms are:

(1) $x = e^{-dt}(C + Dt)$,

(2) $\frac{dx}{dt} = -dx + De^{-dt}$,

(3) $\frac{d^2x}{dt^2} = -d\frac{dx}{dt} - dDe^{-dt} = d^2x - 2dDe^{-dt}$.

Substitute:

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = (d^2x - 2De^{-dt}) + 2d(-dx + De^{-dt})$$

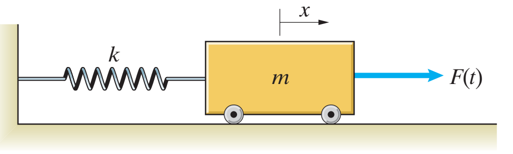
$$+ \omega^2 x = 0.$$

Reduce:

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = (-d^2 + \omega^2)x = 0.$$

This is true if $d^2 = \omega^2$, which is the definition of a critically damped system. *Note*: Substitution leading to an identity shows that $x = Ce^{-dt} + Dte^{-dt}$ is *a solution*. It *does not* prove that it is *the only solution*.

Problem 21.58 The mass $m = 2$ kg and the spring constant is $k = 72$ N/m. The spring is unstretched when $x = 0$. The mass is initially stationary with the spring unstretched, and at $t = 0$ the force $F(t) = 10 \sin 4t$ N is applied to the mass. What is the position of the mass at $t = 2$ s?



Solution: The equation of motion is

$$m\ddot{x} + kx = F(t) \Rightarrow \ddot{x} + \frac{k}{m}x = \frac{F(t)}{m} \Rightarrow \ddot{x} + \frac{72}{2}x = \frac{10}{2}\sin 4t$$

$$\ddot{x} + (6)^2 x = 5 \sin 4t$$

We recognize the following:

$\omega = 6$, $\omega_0 = 4$, $d = 0$, $a_0 = 5$, $b_0 = 0$.

$$A_p = \frac{(6^2 - 4^2)5}{(6^2 - 4^2)^2} = 0.25, \; B_p = 0$$

Therefore, the complete solution (homogeneous plus particular) is

$$x = A \sin 6t + B \cos 6t + 0.25 \sin 4t, \; \frac{dx}{dt} = 6A \cos 6t - 6B \sin 6t + \cos 4t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B = 0, \; \frac{dx}{dt}(t = 0) = 6A + 1 \Rightarrow A = -0.167.$$

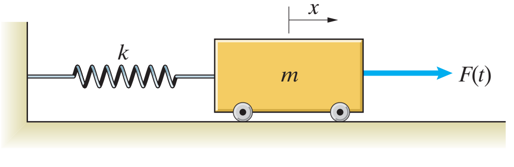
The complete solution is now

$$x = -0.167 \sin 6t + 0.25 \sin 4t$$

At $t = 6$ s we have

$$\boxed{x = 0.337 \text{ m.}}$$

Problem 21.59 The mass $m = 2$ kg and the spring constant is $k = 72$ N/m. The spring is unstretched when $x = 0$. At $t = 0$, $x = 1$ m, $dx/dt = 1$ m/s, and the force $F(t) = 10 \sin 4t + 10 \cos 4t$ N is applied to the mass. What is the position of the mass at $t = 2$ s?



Solution: The equation of motion is

$$m\ddot{x} + kx = F(t) \Rightarrow \ddot{x} + \frac{k}{m}x = \frac{F(t)}{m} \Rightarrow \ddot{x} + \frac{72}{2}x = \frac{10}{2}\sin 4t + \frac{10}{2}\cos 4t$$

$$\ddot{x} + (6)^2 x = 5 \sin 4t + 5 \cos 4t.$$

We recognize the following:

$\omega = 6$, $\omega_0 = 4$, $d = 0$, $a_0 = 5$, $b_0 = 5$.

$$A_p = \frac{(6^2 - 4^2)5}{(6^2 - 4^2)^2} = 0.25, \; B_p = \frac{(6^2 - 4^2)5}{(6^2 - 4^2)^2} = 0.25.$$

Therefore, the complete solution (homogeneous plus particular) is

$$x = A \sin 6t + B \cos 6t + 0.25 \sin 4t + 0.25 \cos 4t,$$

$$\frac{dx}{dt} = 6A \cos 6t - 6B \sin 6t + \cos 4t - \sin 4t.$$

Putting in the initial conditions, we have

$$x(t = 0) = B + 0.25 = 1 \Rightarrow B = 0.75$$

$$\frac{dx}{dt}(t = 0) = 6A + 1 = 1 \Rightarrow A = 0.$$

The complete solution is now

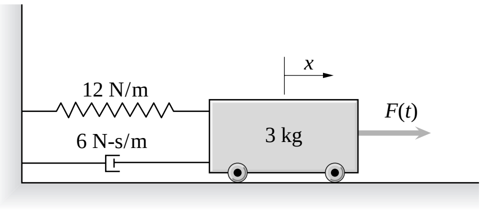
$$x = 0.75 \cos 6t + 0.25 \sin 4t + 0.25 \cos 4t.$$

At $t = 6$ s we have

$$\boxed{x = 0.844 \text{ m.}}$$

Problem 21.60 The damped spring–mass oscillator is initially stationary with the spring unstretched. At $t = 0$, a constant force $F(t) = 6$ N is applied to the mass.

(a) What is the steady-state (particular) solution?
(b) Determine the position of the mass as a function of time.



Solution: Writing Newton's second law for the mass, the equation of motion is

$$F(t) - c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2} \text{ or}$$

$$F(t) - 6\frac{dx}{dt} - 12x = 3\frac{d^2x}{dt^2},$$

which we can write as

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = \frac{F(t)}{3}. \quad \mathbf{(1)}$$

(a) If $F(t) = 6$ N, we seek a particular solution of the form $x_p = A_0$, a constant. Substituting it into Equation (1), we get $4x_p = \frac{F(t)}{3} = 2$ and obtain the particular solution: $x_p = 0.5$ m.

(b) Comparing equation (1) with Equation (21.26), we see that $d = 1$ rad/s and $\omega = 2$ rad/s. The system is subcritically damped and the homogeneous solution is given by Equation (21.19) with $\omega_d = \sqrt{\omega^2 - d^2} = 1.73$ rad/s. The general solution is

$$x = x_h + x_p = e^{-1}(A \sin 1.73t + B \cos 1.73t) + 0.5 \text{ m}.$$

The time derivative is

$$\frac{dx}{dt} = -e^{-1}(A \sin 1.73t + B \cos 1.73t)$$

$$+ e^{-t}(1.73A \cos 1.73t - 1.73t - 1.73B \sin 1.73t).$$

At $t = 0$, $x = 0$, and $dx/dt = 0 \quad 0 = B + 0.5$, and $0 = -B + 1.73A$. We see that $B = -0.5$ and $A = -0.289$ and the solution is

$$x = e^{-t}(-0.289 \sin 1.73t - 0.5 \cos 1.73t) + 0.5 \text{ m}.$$

Problem 21.61 The damped spring–mass oscillator shown in Problem 21.60 is initially stationary with the spring unstretched. At $t = 0$, a force $F(t) = 6 \cos 1.6t$ N is applied to the mass.

(a) What is the steady-state (particular) solution?
(b) Determine the position of the mass as a function of time.

Solution: Writing Newton's second law for the mass, the equation of motion can be written as

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = \frac{F(t)}{3} = 2 \cos 1.6t.$$

(a) Comparing this equation with Equation (21.26), we see that $d = 1$ rad/s, $\omega = 2$ rad/s, and the forcing function is $a(t) = 2 \cos 1.6t$. This forcing function is of the form of Equation (21.27) with $a_0 = 0$, $b_0 = 2$ and $\omega_0 = 1.6$. Substituting these values into Equation (21.30), the particular solution is $x_p = 0.520 \sin 1.6t + 0.234 \cos 1.6t$ (m).

(b) The system is subcritically damped so the homogeneous solution is given by Equation (21.19) with $\omega_d = \sqrt{\omega^2 - d^2} = 1.73$ rad/s. The general solution is

$$x = x_k + x_p = e^{-t}(A \sin 1.73t + B \cos 1.73t) + 0.520 \sin 1.6t$$

$$+ 0.234 \cos 1.6t.$$

The time derivative is

$$\frac{dx}{dt} = -e^{-t}(A \sin 1.73t + B \cos 1.73t)$$

$$+ e^{-t}(1.73A \cos 1.73t - 1.73B \sin 1.73t)$$

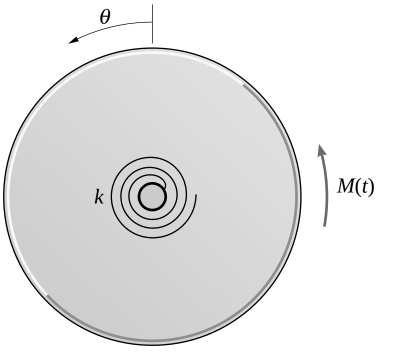
$$+ (1.6)(0.520) \cos 1.6t - (1.6)(0.234) \sin 1.6t$$

At $t = 0$, $x = 0$ and $dx/dt = 0 : 0 = B + 0.2340 = -B + 1.73A + (1.6)(0.520)$. Solving, we obtain $A = -0.615$ and $B = -0.234$, so the solution is

$$x = e^{-t}(-0.615 \sin 1.73t - 0.234 \cos 1.73t) + 0.520 \sin 1.6t$$

$$+ 0.234 \cos 1.6t \text{ m}.$$

Problem 21.62 The disk with moment of inertia $I = 3$ kg-m^2 rotates about a fixed shaft and is attached to a torsional spring with constant $k = 20$ N-m/rad. At $t = 0$, the angle $\theta = 0$, the angular velocity is $d\theta/dt = 4$ rad/s, and the disk is subjected to a couple $M(t) = 10\sin 2t$ N-m. Determine θ as a function of time.



Solution: The equation of angular motion for the disk is

$$M(t) - k\theta = 1\frac{d^2\theta}{dt^2}: \text{ or } 10\sin 2t - (20)\theta = 3\frac{d^2\theta}{dt^2},$$

or, rewriting in standard form, we have

$$\frac{d^2\theta}{dt^2} + \frac{20}{3}\theta = \frac{10}{3}\sin 2t$$

(a) Comparing this equation with equation (21.26), we see that $d = 0$, $\omega = \sqrt{20/3} = 2.58$ rad/s and the forcing function is $a(t) = \dfrac{10}{3}\sin 2t$. This forcing function is of the form of Equation (21.27), with $a_0 = 10/3$, $b_0 = 0$ and $\omega_0 = 2$. Substituting these values into Equation (21.30), the particular solution is $\theta_p = 1.25\sin 2t$.

(b) The general solution is

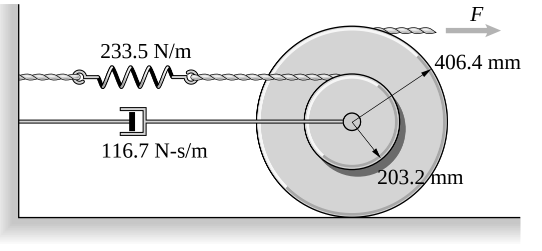
$$\theta = \theta_h + \theta_p = A\sin 2.58t + B\cos 2.58t + 1.25\sin 2t$$

The time derivative is

$$\frac{d\theta}{dt} = 2.58A\cos 2.58t - 2.58B\sin 2.58t + 2.50\cos 2t.$$

At $t = 0$, $\theta = 0$ and $d\theta/dt = 4$ rad/s, $0 = B$, and $4 = 2.58A + 2.50$. Solving, we obtain $A = 0.581$ and $B = 0$. The solution is $\theta = 0.581\sin 2.58t + 1.25\sin 2t$ rad.

Problem 21.63 The stepped disk weighs 89 N and its moment of inertia is $I = 0.81$ kg-m^2. It rolls on the horizontal surface. The disk is initially stationary with the spring unstretched, and at $t = 0$ a constant force $F = 44.5$ N is applied as shown. Determine the position of the center of the disk as a function of time.



Solution: The strategy is to apply the free body diagram to obtain equations for both θ and x, and then to eliminate one of these. An essential element in the strategy is the determination of the stretch of the spring. Denote $R = 0.203$ m, and the stretch of the spring by S. Choose a coordinate system with the positive x axis to the right. The sum of the moments about the center of the disk is $\sum M_C = RkS + 2Rf - 2RF$. From the equation of angular motion,

$$I\frac{d^2\theta}{dt^2} = \sum M_C = RkS + 2Rf - 2RF.$$

Solve for the reaction at the floor:

$$f = \frac{I}{2R}\frac{d^2\theta}{dt^2} - \frac{k}{2}S + F.$$

The sum of the horizontal forces:

$$\sum F_x = -kS - c\frac{dx}{dt} + F + f.$$

From Newton's second law:

$$m\frac{d^2x}{dt^2} = \sum F_x = -kS - c\frac{dx}{dt} + F + f.$$

Substitute for f and rearrange:

$$m\frac{d^2x}{dt^2} + \frac{I}{2R}\frac{d^2\theta}{dt^2} + c\frac{dx}{dt} + \frac{3}{2}kS = 2F.$$

From kinematics, the displacement of the center of the disk is $x = -2R\theta$. The stretch of the spring is the amount wrapped around the disk plus the translation of the disk, $S = -R\theta - 2R\theta = -3R\theta = \frac{3}{2}x$. Substitute:

$$\left(m + \frac{I}{(2R)^2}\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \left(\frac{3}{2}\right)^2 kx = 2F.$$

Define $M = m + \dfrac{I}{(2R)^2} = 14$ kg.

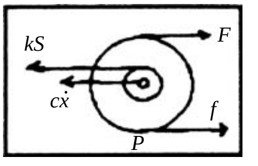
The canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a,$$

where $d = \dfrac{c}{2M} = 4.170$ rad/s,

$$\omega^2 = \left(\frac{3}{2}\right)^2\frac{k}{M} = 37.53 \text{ (rad/s)}^2,$$

and $a = \dfrac{2F}{M} = 6.35$ m/s^2.

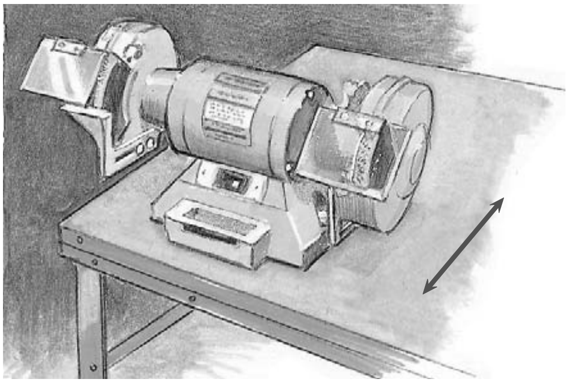


The particular solution is found by setting the acceleration and velocity to zero and solving: $x_p = \dfrac{a}{\omega^2} = \dfrac{8F}{9k} = 0.169$ m. Since $d^2 < \omega^2$, the system is sub-critically damped, so the homogenous solution is $x_h = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} = 4.488$ rad/s. The complete solution is $x = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t) + \dfrac{8F}{9k}$. Apply the initial conditions: at $t = 0$, $x_0 = 0$, $\dot{x}_0 = 0$, from which $B = -\dfrac{8F}{9k} = -0.169$, and $A = \dfrac{dB}{\omega_d} = -0.157$. Adopting $g = 9.81$ m/s^2 the solution is

$$x = \frac{8F}{9k}\left[1 - e^{dt}\left(\frac{d}{\omega_d}\sin\omega_d t + \cos\omega_d t\right)\right]$$

$$\boxed{= 0.169 - e^{-4.170t}(0.157\sin 4\,488t + 0.169\cos 4\,488t) \text{ m}},$$

Problem 21.64* An electric motor is bolted to a metal table. When the motor is on, it causes the tabletop to vibrate horizontally. Assume that the legs of the table behave like linear springs, and neglect damping. The total weight of the motor and the tabletop is 667 N. When the motor is not turned on, the frequency of horizontal vibration of the tabletop and motor is 5 Hz. When the motor is running at 600 rpm, the amplitude of the horizontal vibration is 0.25 mm. What is the magnitude of the oscillatory force exerted on the table by the motor at its 600-rpm running speed?



Solution: For $d = 0$, the canonical form (see Eq. (21.26)) of the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = a(t),$$

where $\omega^2 = (2\pi f)^2 = (10\pi)^2 = 986.96$ (rad/s)^2

and $a(t) = \dfrac{F(t)}{m} = \dfrac{gF(t)}{W}$.

The forcing frequency is

$$f_0 = \left(\frac{600}{60}\right) = 10 \text{ Hz},$$

from which $\omega_0 = (2\pi)10 = 62.83$ rad/s. Assume that $F(t)$ can be written in the form $F(t) = F_0 \sin \omega_0 t$. From Eq. (21.31), the amplitude of the oscillation is

$$x_0 = \frac{a_0}{\omega^2 - \omega_0^2} = \frac{gF_0}{W(\omega^2 - \omega_0^2)}.$$

Solve for the magnitude:

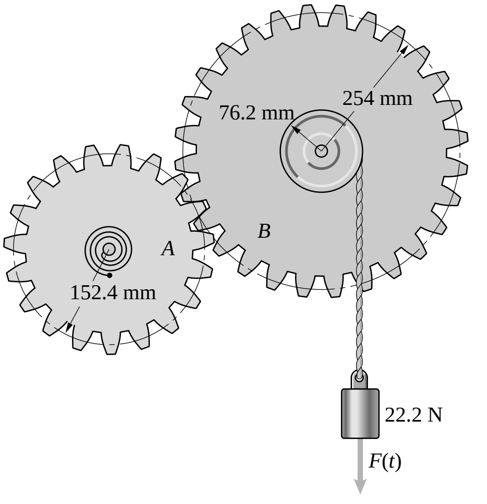
$$|F_0| = \frac{W}{g}|(\omega^2 - \omega_0^2)|x_0$$

Substitute numerical values:

$W = 667$ N, $g = 9.81$ m/s^2, $|(\omega^2 - \omega_0^2)| = 2960.9$ (rad/s)^2,

$x_0 = 0.25$ mm. $= 0.00025$ m, from which $\boxed{F_0 = 51.2 \text{ N}}$

Problem 21.65 The moments of inertia of gears A and B are $I_A = 0.019$ kg-m^2 and $I_B = 0.136$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 2.71$ N-m/rad. The system is in equilibrium at $t = 0$ when it is subjected to an oscillatory force $F(t) = 17.8 \sin 3t$ N. What is the downward displacement of the 22.2 N weight as a function of time?



Solution: Choose a coordinate system with the x axis positive upward. The sum of the moments on gear A is $\sum M = -k\theta_A + R_A F$. From Newton's second law,

$$I_A \frac{d^2\theta_A}{dt^2} = \sum M = -k\theta_A + R_A F,$$

from which $\quad F = \left(\frac{I_A}{R_A}\right)\frac{d^2\theta_A}{dt^2} + \left(\frac{k}{R_A}\right)\theta_A.$

The sum of the moments acting on gear B is $\sum M = R_B F - R_W F_d$. From the equation of angular motion applied to gear B,

$$I_B \frac{d^2\theta_B}{dt^2} = \sum M_B = R_B F - R_W F_d.$$

Substitute for F to obtain the equation of motion for gear B:

$$I_B \frac{d^2\theta_B}{dt^2} - \left(\frac{R_B}{R_A}\right) I_A \frac{d^2\theta_A}{dt^2} - \left(\frac{R_A}{R_B}\right) k\theta_A - R_W F_d = 0.$$

Solve:

$$F_d = \left(\frac{I_B}{R_W}\right)\frac{d^2\theta_B}{dt^2} - \left(\frac{R_B}{R_A R_W}\right)\frac{d^2\theta_A}{dt^2} - \left(\frac{R_B}{R_A R_W}\right) k\theta_A.$$

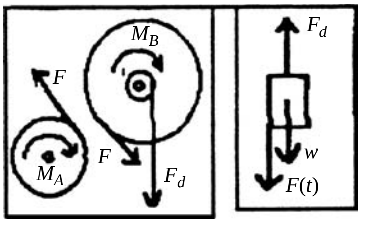
The sum of the forces on the weight are $\sum F = +F_d - W - F(t)$. From Newton's second law applied to the weight,

$$\left(\frac{W}{g}\right)\frac{d^2x}{dt^2} = F_d - W - F(t).$$

Substitute for F_d, and rearrange to obtain the equation of motion for the weight:

$$\frac{W}{g}\frac{d^2x}{dt^2} - \frac{I_B}{R_W}\frac{d^2\theta_B}{dt^2} + \frac{R_B I_A}{R_W R_A}\frac{d^2\theta_A}{dt^2} + \frac{R_B}{R_W R_A} k\theta_A$$

$$= -W - F(t).$$



From kinematics, $\theta_A = -\left(\frac{R_B}{R_A}\right)\theta_B$, and $x = -R_W\theta_B$, from which

$$\frac{d^2\theta_B}{dt^2} = -\frac{1}{R_W}\frac{d^2x}{dt^2}, \quad \frac{d^2\theta_A}{dt^2} = -\left(\frac{R_B}{R_A}\right)\frac{d^2\theta_B}{dt^2} = \frac{R_B}{R_W R_A}\frac{d^2x}{dt^2},$$

and $\quad \theta_A = \frac{R_B}{R_W R_A} x.$

Define $\quad \eta = \frac{R_B}{R_W R_A} = 21.87$ m^{-1},

and $\quad M = \frac{W}{g} + \frac{I_B}{(R_W)^2} + \left(\frac{R_B}{R_W R_A}\right)^2 I_A = 34.69$ kg,

from which the equation of motion for the weight *about the unstretched spring position* is:

$$M\frac{d^2x}{dt^2} + (k\eta^2)x = -W - F(t).$$

For $d = 0$, the canonical form (see Eq. (21.16)) of the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = a(t), \quad \text{where } \omega^2 = \frac{\eta^2 k}{M} = 37.39 \text{ (rad/s)}^2.$$

[*Check*: This agrees with the result in the solution to Problem 21.53, as it should, since nothing has changed except for the absence of a damping element. *check*.] The non-homogenous terms are $a(t) = \frac{W}{M} + \frac{F(t)}{M}$. Since $\frac{W}{M}$ is not a function of t,

$$x_{pw} = -\frac{W}{\omega^2 M} = -\frac{W}{\eta^2 k} = -0.017 \text{ m},$$

which is the equilibrium point. Make the transformation $\tilde{x}_p = x_p - x_{pw}$. The equation of motion about the equilibrium point is

$$\frac{d^2\mathbf{x}}{dt^2} + \omega^2 \tilde{x} = -\frac{17.8 \sin 3t}{M}.$$

Assume a solution of the form $\tilde{x}_p = A_p \sin 3t + B_p \cos 3t$. Substitute:

$$(\omega^2 - 3^2)(A_p \sin 3t + B_p \cos 3t) = -\frac{3 \sin 3t}{M},$$

from which $B_p = 0$, and

$$A_p = -\frac{17.8}{M(\omega^2 - 3^2)} = -0.0181 \text{ m}.$$

The particular solution is

$$\tilde{x}_p = -\frac{17.8}{M(\omega^2 - 3^2)} \sin 3t = -0.0181 \sin 3t \text{ m}.$$

The solution to the homogenous equation is

$$x_h = A_h \sin \omega t + B_h \cos \omega t,$$

and the complete solution is

$$\tilde{x}(t) = A_h \sin \omega t + B_h \cos \omega t - \frac{17.8}{M(\omega^2 - 3^2)} \sin 3t.$$

Apply the initial conditions: at $t = 0$, $x_0 = 0$, $\dot{x}_0 = 0$, from which $0 = B$,

$$0 = \omega A_h - \frac{53.4}{M(\omega^2 - 3^2)},$$

from which $A_h = \frac{53.4}{\omega M(\omega^2 - 3^2)} = 0.00887$.

The complete solution for vibration about the equilibrium point is:

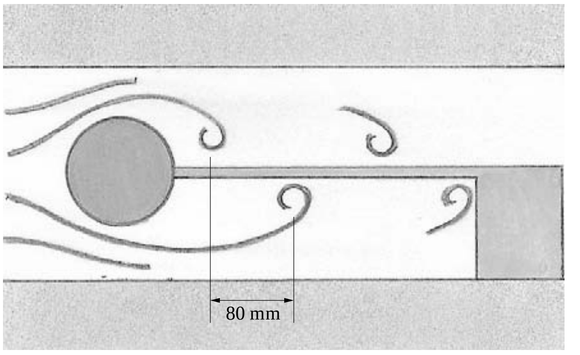
$$\tilde{x}(t) = \frac{17.8}{M(\omega^2 - \omega_0^2)} \left(\frac{3}{\omega} \sin \omega t - \sin 3t\right)$$

$$= 0.00887 \sin 6.114t - 0.0181 \sin 3t \text{ m}.$$

The downward travel is the negative of this:

$$\boxed{\tilde{x}_{\text{down}} = -0.00887 \sin 6.114t + 0.0181 \sin 3t \text{ m}}$$

Problem 21.66* A 1.5-kg cylinder is mounted on a sting in a wind tunnel with the cylinder axis transverse to the direction of flow. When there is no flow, a 10-N vertical force applied to the cylinder causes it to deflect 0.15 mm. When air flows in the wind tunnel, vortices subject the cylinder to alternating lateral forces. The velocity of the air is 5 m/s, the distance between vortices is 80 mm, and the magnitude of the lateral forces is 1 N. If you model the lateral forces by the oscillatory function $F(t) = (1.0)\sin\omega_0 t$ N, what is the amplitude of the steady-state lateral motion of the sphere?



Solution: The time interval between the appearance of an upper vortex and a lower vortex is $\delta_t = \dfrac{0.08}{5} = 0.016$ s, from which the period of a *sinusoidal-like* disturbance is $\tau = 2(\delta t) = 0.032$ s, from which $f_0 = \dfrac{1}{\tau} = 31.25$ Hz. [*Check*: Use the physical relationship between frequency, wavelength and velocity of propagation of a *small amplitude sinusoidal wave*, $\lambda f = v$. The wavelength of a traveling sinusoidal disturbance is the distance between two peaks or two troughs, or twice the distance between adjacent peaks and troughs, $\lambda = 2(0.08) = 0.16$ m, from which the frequency is $f_0 = \dfrac{v}{\lambda} = 31.25$ Hz. *check*] The circular frequency is $\omega_0 = 2\pi f_0 = 196.35$ rad/s. The spring constant of the sting is

$$k = \frac{F}{\delta} = \frac{10}{0.00015} = 66667 \text{ N/m}.$$

The natural frequency of the sting-cylinder system is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{66667}{1.5}} = 33.55 \text{ Hz},$$

from which $\omega = 2\pi f = 210.82$ rad/s. For $d = 0$, the canonical form (see Eq. 21.16)) of the equation of motion is $\dfrac{d^2x}{dt^2} + \omega^2 x = a(t)$, where

$$a(t) = \frac{F}{m} = \frac{1}{15}\sin\omega_0 t = 0.6667\sin 196.3t.$$

From Eq. (21.31) the amplitude is

$$\boxed{E = \frac{a_0}{\omega^2 - \omega_0^2} = \frac{0.6667}{5891} = 1.132 \times 10^{-4} \text{ m}}$$

[*Note*: This is a small deflection (113 *microns*) but the associated aerodynamic forces may be significant to the tests (e.g. $F_{amp} = 7.5$ N), since the sting is stiff. Vortices may cause undesirable noise in sensitive static aerodynamic loads test measurements.]

Problem 21.67 Show that the amplitude of the particular solution given by Eq. (21.31) is a maximum when the frequency of the oscillatory forcing function is $\omega_0 = \sqrt{\omega^2 - 2d^2}$.

Solution: Eq. (21.31) is

$$E_p = \frac{\sqrt{a_0^2 + b_0^2}}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4d^2\omega_0^2}}.$$

Since the numerator is a constant, rearrange:

$$\eta = \frac{E_p}{\sqrt{a_0^2 + b_0^2}} = [(\omega^2 - \omega_0^2)^2 + 4d^2\omega_0^2]^{-\frac{1}{2}}.$$

The maximum (or minimum) is found from

$$\frac{d\eta}{d\omega_0} = 0.$$

$$\frac{d\eta}{d\omega_0} = -\frac{1}{2}\frac{4[-(\omega^2 - \omega_0^2)(\omega_0) + 2d^2\omega_0]}{[(\omega^2 - \omega_0^2)^2 + 4d^2\omega_0^2]^{\frac{3}{2}}} = 0,$$

from which $-(\omega^2 - \omega_0^2)(\omega_0) + 2d^2\omega_0 = 0$. Rearrange, $(\omega_0^2 - \omega^2 + 2d^2)\omega_0 = 0$. Ignore the possible solution $\omega_0 = 0$, from which

$$\boxed{\omega_0 = \sqrt{\omega^2 - 2d^2}}.$$

Let $\tilde{\omega}_0 = \sqrt{\omega^2 - 2d^2}$ be the maximizing value. To show that η is indeed a maximum, take the second derivative:

$$\left[\frac{d^2\eta}{d\omega_0^2}\right]_{\omega_0} = \tilde{\omega}_0 = \frac{3}{4}\frac{[(4\omega_0)(\omega_0^2 - \omega^2 + 2d^2)]^2_{\omega_0=\tilde{\omega}_0}}{[(\omega^2 - \omega_0^2)^2 + 2d^2\omega_0^2]^{\frac{5}{2}}_{\omega_0=\tilde{\omega}_0}}$$

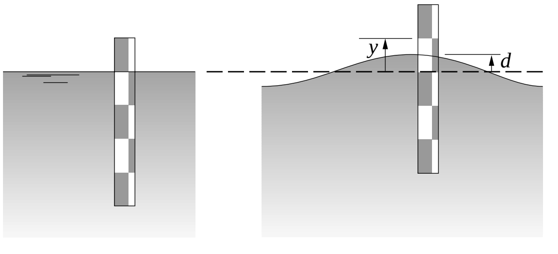
$$-\frac{1}{2}\frac{4[3\omega_0^2 - \omega^2 + 2d^2]_{\omega_0=\tilde{\omega}_0}}{[(\omega^2 - \omega_0^2)^2 + 2d^2\omega_0^2]^{\frac{3}{2}}_{\omega_0=\tilde{\omega}_0}},$$

from which $\left[\frac{d^2\eta}{dt^2}\right]_{\omega_0=\tilde{\omega}_0} = -\frac{4\omega_0^2}{[2\omega^2 d^2]^{\frac{3}{2}}} < 0$,

which demonstrates that it is a maximum.

Problem 21.68* A sonobuoy (sound-measuring device) floats in a standing-wave tank. The device is a cylinder of mass m and cross-sectional area A. The water density is ρ, and the buoyancy force supporting the buoy equals the weight of the water that would occupy the volume of the part of the cylinder below the surface. When the water in the tank is stationary, the buoy is in equilibrium in the vertical position shown at the left. Waves are then generated in the tank, causing the depth of the water at the sonobuoy's position *relative to its original depth* to be $d = d_0 \sin\omega_0 t$. Let y be the sonobuoy's vertical position relative to its original position. Show that the sonobuoy's vertical position is governed by the equation

$$\frac{d^2y}{dt^2} + \left(\frac{A\rho g}{m}\right)y = \left(\frac{A\rho g}{m}\right)d_0 \sin\omega_0 t.$$



Solution: The volume of the water displaced at equilibrium is $V = Ah$ where A is the cross-sectional area, and h is the equilibrium immersion depth. The weight of water displaced is $\rho V g = \rho g A h$, so that the buoyancy force is $F_b = \rho A g h$.

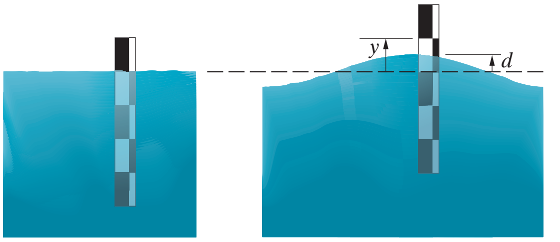
The sum of the vertical forces is $\sum F_y = \rho g A h - mg = 0$ at equilibrium, where m is the mass of the buoy. By definition, the spring constant is $\frac{\partial F_b}{\partial h} = k = \rho g A$. For any displacement δ of the immersion depth from the equilibrium depth, the net vertical force on the buoy is $\sum F_y = -\rho g A(h + \delta) + mg = -\rho A g \delta = -k\delta$, since h is the equilibrium immersion depth. As the waves are produced, $\delta = y - d$, where $d = d_0 \sin\omega_0 t$, from which $\sum F_y = -k(y - d)$. From Newton's second law,

$$m\frac{d^2y}{dt^2} = -k(y - d), \text{ from which } m\frac{d^2y}{dt^2} + ky = kd.$$

Substitute:

$$\boxed{\frac{d^2y}{dt^2} + \left(\frac{\rho g A}{m}\right)y = \left(\frac{\rho g A}{m}\right)d_0 \sin\omega_0 t}$$

Problem 21.69 Suppose that the mass of the sonobuoy in Problem 21.68 is $m = 10$ kg, its diameter is 125 mm, and the water density is $\rho = 1025$ kg/m^3. If $d = 0.1 \sin 2t$ m, what is the magnitude of the steady-state vertical vibrations of the sonobuoy?



Solution: From the solution to Problem 21.68,

$$\frac{d^2 y}{dt^2} + \left(\frac{\rho g A}{m}\right) y = \left(\frac{\rho g A}{m}\right) d_0 \sin \omega_0 t.$$

The canonical form is

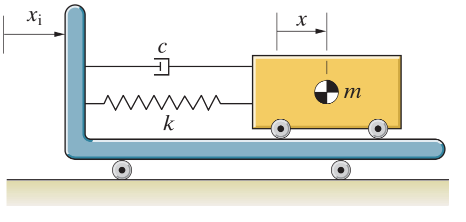
$$\frac{d^2 y}{dt}^2 + \omega^2 y = a(t),$$

where $\omega^2 = \dfrac{\rho \pi d^2 g}{4\ m} = \dfrac{1025\pi(0.125^2)(9.81)}{4(10)} = 12.34\ (\text{rad/s})^2,$

and $a(t) = \omega^2(0.1)\sin 2t$. From Eq. (21.31), the amplitude of the steady state vibrations is

$$\boxed{E_p = \frac{\omega^2(0.1)}{|(\omega^2 - 2^2)|} = 0.1480\ \text{m}}$$

Problem 21.70 The mass weighs 222 N. The spring constant is $k = 2919$ N/m, and $c = 146$ N-s/m. If the base is subjected to an oscillatory displacement x_i of amplitude 0.254 m and frequency $\omega_i = 15$ rad/s, what is the resulting steady-state amplitude of the displacement of the mass relative to the base?



Solution: The canonical form of the equation of motion is

$$\frac{d^2 x}{dt^2} + 2d\frac{dt}{dt} + \omega^2 x = a(t),$$

where $d = \dfrac{c}{2m} = \dfrac{gc}{2W} = 3.217$ rad/s,

$$\omega^2 = \frac{kg}{W} = 128.7\ (\text{rad/s})^2,$$

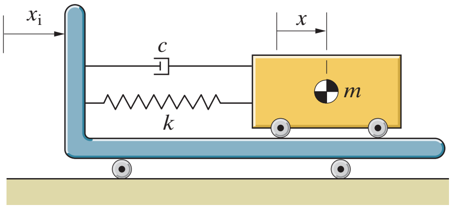
and (see Eq. (21.38),

$$a(t) = -\frac{d^2 x_i}{dt^2} = x_i \omega_i^2 \sin(\omega_i t - \phi).$$

The displacement of the mass relative to its base is

$$\boxed{\begin{aligned} E_p &= \frac{\omega_i^2 x_i}{\sqrt{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}} \\ &= \frac{(15^2)(0.254)}{\sqrt{(11.34^2 - 15^2)^2 + 4(3.217^2)(15^2)}} = 0.419\ \text{m}. \end{aligned}}$$

Problem 21.71 The mass in Fig. 21.21 is 100 kg. The spring constant is $k = 4$ N/m, and $c = 24$ N-s/m. The base is subjected to an oscillatory displacement of frequency $\omega_i = 0.2$ rad/s. The steady-state amplitude of the displacement of the mass relative to the base is measured and determined to be 200 mm. What is the amplitude of the displacement of the base? (See Example 21.7.)



Solution: From Example 21.7 and the solution to Problem 21.70, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a(t)$$

where $d = \dfrac{c}{2m} = 0.12$ rad/s, $\omega^2 = \dfrac{k}{m} = 0.04$ (rad/s)^2,

and (see Eq. (21.38))

$$a(t) = -\frac{d^2x_i}{dt^2} = x_i\omega_i^2 \sin(\omega_i t - \phi).$$

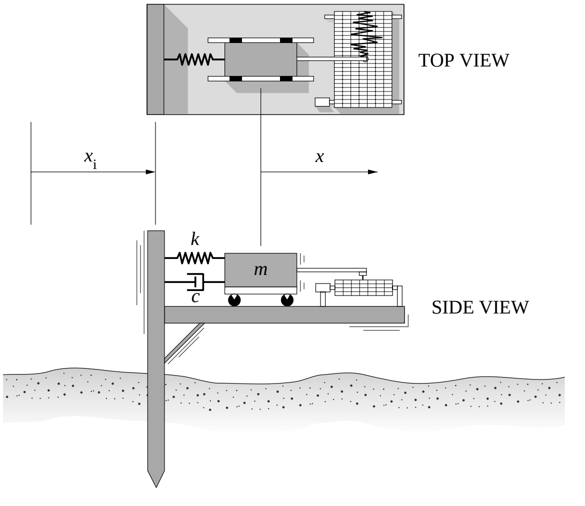
The displacement of the mass relative to its base is

$$E_p = 0.2 = \frac{\omega_i^2 x_i}{\sqrt{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}}$$

$$= \frac{(0.2^2)x_i}{\sqrt{(0.2^2 - 0.2^2)^2 + 4(0.12^2)(0.2^2)}} = 0.8333x_i \text{ m},$$

from which $\boxed{x_i = \dfrac{0.2}{0.8333} = 0.24 \text{ m}}$

Problem 21.72 A team of engineering students builds the simple seismograph shown. The coordinate x_i measures the local horizontal ground motion. The coordinate x measures the position of the mass relative to the frame of the seismograph. The spring is unstretched when $x = 0$. The mass $m = 1$ kg, $k = 10$ N/m, and $c = 2$ N-s/m. Suppose that the seismograph is initially stationary and that at $t = 0$ it is subjected to an oscillatory ground motion $x_i = 10 \sin 2t$ mm. What is the amplitude of the steady-state response of the mass? (See Example 21.7.)



Solution: From Example 21.7 and the solution to Problem 21.70, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a(t)$$

where $d = \dfrac{c}{2m} = 1$ rad/s, $\omega^2 = \dfrac{k}{m} = 10$ (rad/s)^2,

and (see Eq. (21.38))

$$a(t) = -\frac{d^2x_i}{dt^2} = x_i\omega_i^2 \sin\omega_i t \quad \text{where } x_i = 10 \text{ mm}, \omega_i = 2 \text{ rad/s}.$$

The amplitude of the steady state response of the mass relative to its base is

$$\boxed{E_p = \frac{\omega_i^2 x_i}{\sqrt{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}} = \frac{(2^2)(10)}{\sqrt{(3.162^2 - 2^2)^2 + 4(1^2)(2^2)}} = 5.55 \text{ mm}}$$

Problem 21.73 In Problem 21.72, determine the position x of the mass relative to the base as a function of time. (See Example 21.7.)

Solution: From Example 21.7 and the solution to Problem 21.70, the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = a(t)$$

where $d = \dfrac{c}{2m} = 1$ rad/s, $\omega^2 = \dfrac{k}{m} = 10$ (rad/s)^2,

and $a(t) = -\dfrac{d^2x_i}{dt^2} = x_i\omega_i^2 \sin\omega_i t$

where $x_i = 10$ mm, $\omega_i = 2$ rad/s. From a comparison with Eq. (21.27) and Eq. (21.30), the particular solution is $x_p = A_p \sin\omega_i t + B_p \cos\omega_i t$, where

$$A_p = \frac{(\omega^2 - \omega_i^2)\omega_i^2 x_i}{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2} = 4.615.$$

$$B_p = -\frac{2d\omega_i^3 x_i}{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2} = -3.077.$$

[*Check*: Assume a solution of the form $x_p = A_p \sin\omega_i t + B_p \cos\omega_i t$. Substitute into the equation of motion:

$$[(\omega^2 - \omega_i^2)A_p - 2d\omega_i B_p]\sin\omega_i t$$

$$+ [(\omega^2 - \omega_i^2)B_p + 2d\omega_i Ap]\cos\omega_i t = x_i\omega_i^2 \sin\omega_i t.$$

Equate like coefficients:

$$(\omega^2 - \omega_i^2)A_p - 2d\omega_i B_p = x_i\omega_i^2,$$

and $2d\omega_i A_p + (\omega^2 - \omega_i^2)B_p = 0.$

Solve:

$$A_p = \frac{(\omega^2 - \omega_i^2)\omega_i^2 x_i}{(\omega^2 - \omega_i^2) + 4d^2\omega_i^2},$$

$$B_p = \frac{-2d\omega_i^3 x_i}{(\omega^2 - \omega_i^2)^2 + 4d^2\omega_i^2}, \; check.]$$

Since $d^2 < \omega^2$, the system is sub-critically damped, and the homogenous solution is $x_h = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = \sqrt{\omega^2 - d^2} = 3$ rad/s. The complete solution is $x = x_h + x_p$. Apply the initial conditions: at $t = 0$, $x_0 = 0$, from which $0 = B + B_p$, and $0 = -dB + \omega_d A + \omega_i A_p$. Solve:

$$B = -B_p = 3.077,\; A = \frac{dB}{\omega_d} - \frac{\omega_i A_p}{\omega_d} = -2.051$$

The solution is

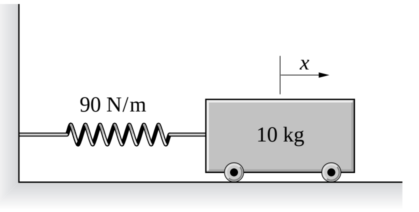
$$x = e^{-dt}(A\sin\omega t + B\cos\omega t) + A_p \sin\omega_i t + B_p \cos\omega_i t:$$

$$\boxed{x = -e^{-t}(2.051\sin 3t - 3.077\cos 3t) + 4.615\sin 2t - 3.077\cos 2t \text{ mm}}$$

Problem 21.74 The coordinate x measures the displacement of the mass relative to the position in which the spring is unstretched. The mass is given the initial conditions

$$t = 0 \begin{cases} x = 0.1 \text{ m}, \\ \dfrac{dx}{dt} = 0. \end{cases}$$

(a) Determine the position of the mass as a function of time.
(b) Draw graphs of the position and velocity of the mass as functions of time for the first 5 s of motion.



Solution: The canonical equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{90}{10}} = 3$ rad/s.

(a) The position is

$$x(t) = A \sin \omega t + B \cos \omega t,$$

and the velocity is

$$\frac{dx(t)}{dt} = \omega A \cos \omega t - \omega B \sin \omega t.$$

At $t = 0$, $x(0) = 0.1 = B$, and

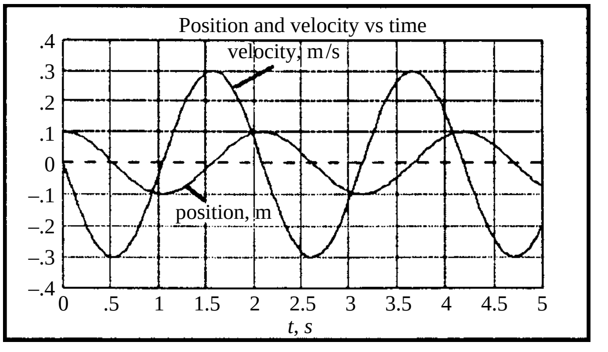
$$\frac{dx(0)}{dt} = 0 = \omega A,$$

from which $A = 0$, $B = 0.1$ m, and

$$\boxed{x(t) = 0.1 \cos 3t \text{ m}},$$

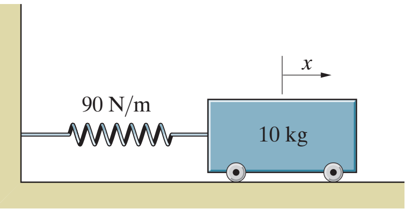
$$\frac{dx(t)}{dt} = -0.3 \sin 3t \text{ m/s}.$$

(b) The graphs are shown.



Problem 21.75 When $t = 0$, the mass in Problem 21.74 is in the position in which the spring is unstretched and has a velocity of 0.3 m/s to the right. Determine the position of the mass as functions of time and the amplitude of vibration

(a) by expressing the solution in the form given by Eq. (21.8) and
(b) by expressing the solution in the form given by Eq. (21.9)



Solution: From Eq. (21.5), the canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

where $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{90}{10}} = 3$ rad/s.

(a) From Eq. (21.6) the position is $x(t) = A\sin\omega t + B\cos\omega t$, and the velocity is

$$\frac{dx(t)}{dt} = \omega A\cos\omega t - \omega B\sin\omega t.$$

At $t = 0$, $x(0) = 0 = B$, and $\frac{dx(0)}{dt} = 0.3 = \omega A$ m/s, from which $B = 0$ and $A = \frac{0.3}{\omega} = 0.1$ m. The position is

$\boxed{x(t) = 0.1\sin 3t \text{ m}}$.

The amplitude is

$\boxed{|x(t)_{\max} = 0.1 \text{ m}}$

(b) From Eq. (21.7) the position is $x(t) = E\sin(\omega t - \phi)$, and the velocity is

$$\frac{dx(t)}{dt} = \omega E\cos(\omega t - \phi).$$

At $t = 0$, $x(0) = -E\sin\phi$, and the velocity is

$$\frac{dx(0)}{dt} = 0.3 = \omega E\cos\phi.$$

Solve: $\phi = 0, \quad E = \frac{0.3}{\omega} = 0.1,$

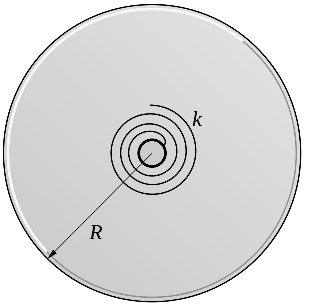
from which $\boxed{x(t) = 0.1\sin 3t}$.

The amplitude is

$\boxed{|x(t)| = E = 0.1 \text{ m}}$

Problem 21.76 A homogenous disk of mass m and radius R rotates about a fixed shaft and is attached to a torsional spring with constant k. (The torsional spring exerts a restoring moment of magnitude $k\theta$, where θ is the angle of rotation of the disk relative to its position in which the spring is unstretched.) Show that the period of rotational vibrations of the disk is

$$\tau = \pi R\sqrt{\frac{2m}{k}}.$$



Solution: From the equation of angular motion, the equation of motion is $I\alpha = M$, where $M = -k\theta$, from which

$$I\frac{d^2\theta}{dt^2} + k\theta = 0,$$

and the canonical form (see Eq. (21.4)) is

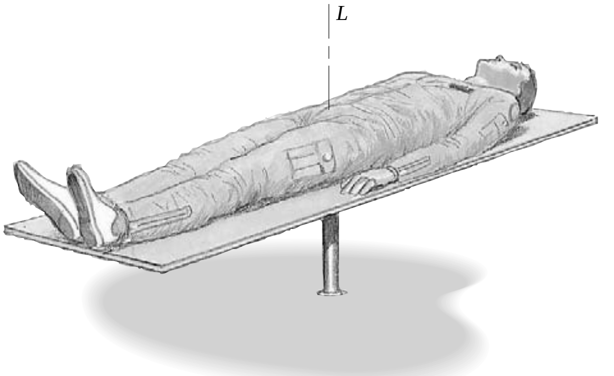
$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where} \quad \omega = \sqrt{\frac{k}{I}}.$$

For a homogenous disk the moment of inertia about the axis of rotation is

$$I = \frac{mR^2}{2}, \quad \text{from which} \quad \omega = \sqrt{\frac{2\,k}{mR^2}} = \frac{1}{R}\sqrt{\frac{2\,k}{m}}.$$

The period is $\boxed{\tau = \frac{2\pi}{\omega} = 2\pi R\sqrt{\frac{m}{2\,k}} = \pi R\sqrt{\frac{2\,m}{k}}}$

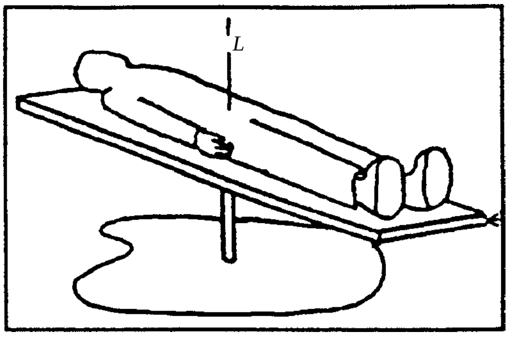
Problem 21.77 Assigned to determine the moments of inertia of astronaut candidates, an engineer attaches a horizontal platform to a vertical steel bar. The moment of inertia of the platform about L is 7.5 kg-m^2, and the frequency of torsional oscillations of the unloaded platform is 1 Hz. With an astronaut candidate in the position shown, the frequency of torsional oscillations is 0.520 Hz. What is the candidate's moment of inertia about L?



Solution: The natural frequency of the unloaded platform is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{I}} = 1 \text{ Hz},$$

from which $k = (2\pi f)^2 I = (2\pi)^2 7.5 = 296.1$ N-m/rad.



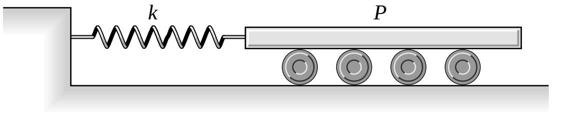
The natural frequency of the loaded platform is

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{I_1}} = 0.520 \text{ Hz},$$

from which $I_1 = \left(\frac{1}{2\pi f_1}\right)^2 k = 27.74$ kg-m^2,

from which $\boxed{I_A = I_1 - I = 20.24 \text{ kg-m}^2}$

Problem 21.78 The 22-kg platen P rests on four roller bearings that can be modeled as 1-kg homogenous cylinders with 30-mm radii. The spring constant is $k = 900$ N/m. What is the frequency of horizontal vibrations of the platen relative to its equilibrium position?



Solution: The kinetic energy is the sum of the kinetic energies of translation of the platen P and the roller bearings, and the kinetic energy of rotation of the roller bearings. Denote references to the platen by the subscript P and references to the ball bearings by the subscript B:

$$T = \frac{1}{2}m_P\left(\frac{dx_P}{dt}\right)^2 + \frac{1}{2}(4m_B)\left(\frac{dx_B}{dt}\right)^2 + \frac{1}{2}(4I_B)\left(\frac{d\theta}{dt}\right)^2.$$

The potential energy is the energy stored in the spring:

$$V = \frac{1}{2}kx_P^2.$$

From kinematics, $-R\theta = x_B$

and $x_B = \dfrac{x_P}{2}$.

Since the system is conservative, $T + V = \text{const.}$ Substitute the kinematic relations and reduce:

$$\left(\frac{1}{2}\right)\left(m_P + m_B + \frac{I_B}{R^2}\right)\left(\frac{dx_P}{dt}\right)^2 + \left(\frac{1}{2}\right)kx_P^2 = \text{const.}$$

Take the time derivative:

$$\left(\frac{dx}{dt}\right)\left[\left(m_P + m_B + \frac{I_B}{R^2}\right)\left(\frac{d^2x_P}{dt^2}\right) + kx\right] = 0.$$

This has two possible solutions,

$$\left(\frac{dx}{dt}\right) = 0$$

or $$\left(m_P + m_B + \frac{I_B}{R^2}\right)\left(\frac{d^2x_P}{dt^2}\right) + kx_P = 0.$$

The first can be ignored, from which the canonical form of the equation of motion is

$$\frac{d^2x_P}{dt^2} + \omega^2 x_P = 0,$$

where $\omega = R\sqrt{\dfrac{k}{R^2(m_P + m_B) + I_B}}$.

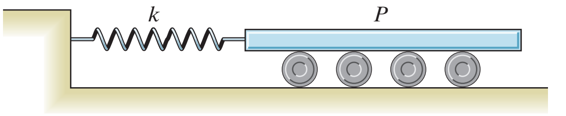
For a homogenous cylinder,

$$I_B = \frac{m_B R^2}{2},$$

from which $\omega = \sqrt{\dfrac{k}{m_P + \frac{3}{2}m_B}} = 6.189$ rad/s.

The frequency is $\boxed{f = \dfrac{\omega}{2\pi} = 0.985 \text{ Hz}}$.

Problem 21.79 At $t = 0$, the platen described in Problem 21.78 is 0.1 m to the left of its equilibrium position and is moving to the right at 2 m/s. What are the platen's position and velocity at $t = 4$ s?



Solution: The position is

$$x(t) = A \sin \omega t + B \cos \omega t,$$

and the velocity is

$$\frac{dx}{dt} = \omega A \cos \omega t - \omega B \sin \omega t,$$

where, from the solution to Problem 21.78, $\omega = 6.189$ rad/s. At $t = 0$, $x(0) = -0.1$ m, and

$$\left[\frac{dx}{dt}\right]_{t=0} = 2 \text{ m/s},$$

from which $B = -0.1$ m, $A = \dfrac{2}{\omega} = 0.3232$ m.

The position and velocity are

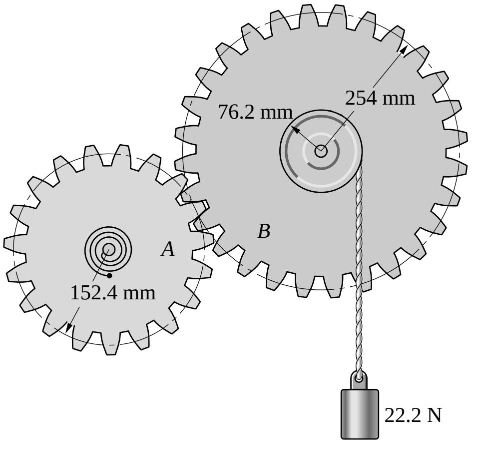
$$x(t) = 0.3232 \sin(6.189\ t) - 0.1 \cos(6.189\ t) \text{ (m)},$$

$$\frac{dx}{dt} = 2 \cos(6.189\ t) + 0.6189 \sin(6.189\ t) \text{ (m/s)}.$$

At $t = 4$ s, $\boxed{x = -0.2124 \text{ m}}$,

$$\boxed{\frac{dx}{dt} = 1.630 \text{ m/s}}.$$

Problem 21.80 The moments of inertia of gears A and B are $I_A = 0.019$ kg-m^2 and $I_B = 0.136$ kg-m^2. Gear A is attached to a torsional spring with constant $k = 2.71$ N-m/rad. What is the frequency of angular vibrations of the gears relative to their equilibrium position?



Solution: The system is conservative. The strategy is to determine the equilibrium position from the equation of motion about the unstretched spring position. Choose a coordinate system with the y axis positive upward. Denote $R_A = 0.152$ m, $R_B = 0.254$ m, and $R_M = 0.762$ m, and $W = 22.2$ N. The kinetic energy of the system is

$$T = \frac{1}{2} I_A \dot{\theta}_A^2 + \frac{1}{2} I_B \dot{\theta}_B^2 + \frac{1}{2}\frac{W}{g} v^2,$$

where $\dot{\theta}_A$, $\dot{\theta}_B$ are the angular velocities of gears A and B respectively, and v is the velocity of the 22.2 N weight. The potential energy is the sum of the energy stored in the spring plus the energy gain due to the increase in the height of the 22.2 N weight:

$$V = \frac{1}{2} k\theta_A^2 + W_y.$$

From kinematics,

$$v = R_M \dot{\theta}_B,$$

$$\dot{\theta}_B = -\left(\frac{R_A}{R_B}\right)\dot{\theta}_A,$$

$$y = R_M \theta_B = -R_M \left(\frac{R_A}{R_B}\right)\theta_A.$$

Substitute, $T + V =$ const.

$$= \left(\frac{1}{2}\right)\left[I_A + \left(\frac{R_A}{R_B}\right)^2 I_B + \left(\frac{W}{g}\right)(R_M^2)\left(\frac{R_A}{R_B}\right)^2\right]\dot{\theta}_A^2 + \left(\frac{1}{2}\right) k\theta_A^2 - (R_M)\left(\frac{R_A}{R_B}\right)\theta_A W.$$

Define $M = I_A + \left(\frac{R_A}{R_B}\right)^2 I_B + \frac{W}{g}(R_M^2)\left(\frac{R_A}{R_B}\right)^2$

$$= 0.0725 \text{ kg-m}^2,$$

and take the time derivative:

$$\dot{\theta}_A\left[M\left(\frac{d^2\theta_A}{dt^2}\right) + k\theta_A - W(R_M)\left(\frac{R_A}{R_B}\right)\right] = 0.$$

Ignore the possible solution $\dot{\theta}_A = 0$, to obtain

$$\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = F,$$

where $F = \frac{W R_M}{M}\left(\frac{R_A}{R_B}\right)$,

and $\omega = \sqrt{\frac{k}{M}} = 6.114$ rad/s,

is the equation of motion *about the unstretched spring position*. Note that

$$\frac{F}{\omega^2} = \frac{W}{k} R_M \left(\frac{R_A}{R_B}\right) = 0.375 \text{ rad}$$

is the equilibrium position of θ_A, obtained by setting the acceleration to zero (since the non-homogenous term F is a constant). Make the change of variable:

$$\tilde{\theta} = \theta_A - \frac{F}{\omega^2},$$

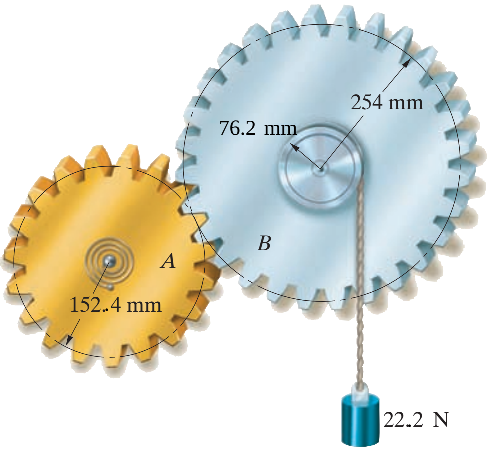
from which the canonical form (see Eq. (21.4)) of the equation of motion *about the equilibrium point* is

$$\frac{d^2\tilde{\theta}}{dt^2} + \omega^2\tilde{\theta} = 0,$$

and the natural frequency is

$$\boxed{f = \frac{\omega}{2\pi} = 0.9732 \text{ Hz.}}$$

Problem 21.81 The 22.2 N weight in Problem 21.80 is raised 12.7 mm from its equilibrium position and released from rest at $t = 0$. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 21.80, the equation of motion for gear A is

$$\frac{d^2\theta_A}{dt^2} + \omega^2\theta_A = F,$$

where $M = I_A + \left(\frac{R_A}{R_B}\right)^2 I_B + \frac{W}{g}(R_M^2)\left(\frac{R_A}{R_B}\right)^2$

$$= 0.0725 \text{ kg-m}^2,$$

$$\omega = \sqrt{\frac{k}{M}} = 6.114 \text{ rad/s},$$

and $F = \dfrac{W R_M \left(\dfrac{R_A}{R_B}\right)}{M} = 14.02 \text{ rad/s}^2.$

As in the solution to Problem 21.80, the *equilibrium angular position* θ_A associated with the equilibrium position of the weight is

$$[\theta_A]_{eq} = \frac{F}{\omega^2} = 0.375 \text{ rad}.$$

Make the change of variable:

$$\tilde{\theta}_A = \theta_A - [\theta_A]_{eq},$$

from which the canonical form of the equation of motion *about the equilibrium point* is

$$\frac{d^2\tilde{\theta}_A}{dt^2} + \omega^2\tilde{\theta}_A = 0.$$

Assume a solution of the form

$$\tilde{\theta}_A = A\sin\omega t + B\cos\omega t.$$

The displacement from the equilibrium position is, from kinematics,

$$\tilde{\theta}_A(t=0) = \left(\frac{R_B}{R_A}\right)\theta_A$$

$$= -\frac{1}{R_M}\left(\frac{R_B}{R_A}\right)y(t=0)$$

$$= -0.2778 \text{ rad},$$

from which the initial conditions are

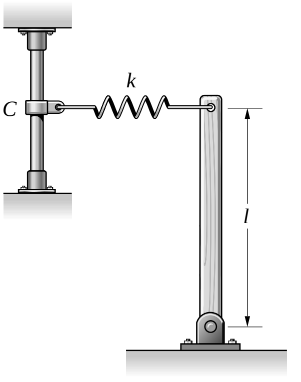
$$\tilde{\theta}_A(t=0) = -0.2778 \text{ rad} \quad \text{and} \quad \left[\frac{d\tilde{\theta}_A}{dt}\right]_{t=0} = 0,$$

from which $B = -0.2778$, $A = 0$. The angular position of gear A is $\theta_a = -0.2778\cos(6.114t)$ rad, from which the angular position of gear B is

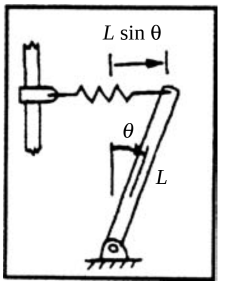
$$\boxed{\tilde{\theta}_B = -\left(\frac{R_A}{R_B}\right)\tilde{\theta}_A = 0.1667\cos(6.114t) \text{ rad}}$$

about the equilibrium position.

Problem 21.82 The mass of the slender bar is m. The spring is unstretched when the bar is vertical. The light collar C slides on the smooth vertical bar so that the spring remains horizontal. Determine the frequency of small vibrations of the bar.



Solution: The system is conservative. Denote the angle between the bar and the vertical by θ. The base of the bar is a fixed point. The kinetic energy of the bar is



$$T = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2.$$

Denote the datum by $\theta = 0$. The potential energy is the result of the change in the height of the center of mass of the bar from the datum and the stretch of the spring,

$$V = -\frac{mgL}{2}(1 - \cos\theta) + \frac{1}{2}k(L\sin\theta)^2.$$

The system is conservative,

$$T + V = \text{const} = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 + \frac{kL^2}{2}\sin^2\theta - \frac{mgL}{2}(1 - \cos\theta).$$

Take the time derivative:

$$\left(\frac{d\theta}{dt}\right)\left[I\frac{d^2\theta}{dt^2} + kL^2 \sin\theta\cos\theta - \frac{mgL}{2}\sin\theta\right] = 0.$$

From which

$$I\frac{d^2\theta}{dt^2} + kL^2 \sin\theta\cos\theta - \frac{mgL}{2}\sin\theta = 0.$$

For small angles, $\sin\theta \to \theta$, $\cos\theta \to 1$. The moment of inertia about the fixed point is

$$I = \frac{mL^2}{3},$$

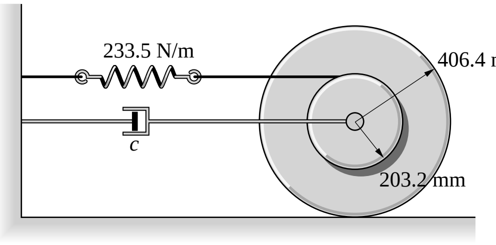
from which $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0,$

where $\omega = \sqrt{\dfrac{3k}{m} - \dfrac{3g}{2L}}.$

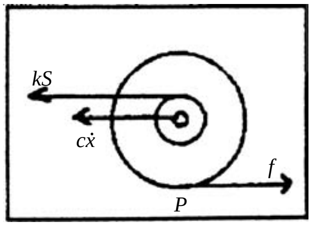
The frequency is

$$\boxed{f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3k}{m} - \frac{3g}{2L}}}$$

Problem 21.86 The stepped disk weighs 89 N, and its moment of inertia is $I = 0.81$ kg-m^2. It rolls on the horizontal surface. If $c = 116.7$ N-s/m, what is the frequency of vibration of the disk?



Solution: The strategy is to apply the free body diagram to obtain equations for both θ and x, and then to eliminate one of these. An essential element in the strategy is the determination of the stretch of the spring. Denote $R = 0.203$ m, and the stretch of the spring by S. Choose a coordinate system with the positive x axis to the right. The sum of the moments about the center of the disk is



$$\sum M_C = RkS + 2Rf.$$

From the equation of angular motion,

$$I\frac{d^2\theta}{dt^2} = \sum M_C = RkS + 2Rf.$$

Solve for the reaction at the floor:

$$f = \frac{I}{2R}\frac{d^2\theta}{dt^2} - \frac{k}{2}S.$$

The sum of the horizontal forces:

$$\sum F_x = -kS - c\frac{dx}{dt} + f.$$

From Newton's second law:

$$m\frac{d^2x}{dt^2} = \sum F_x = -kS - c\frac{dx}{dt} + f.$$

Substitute for f and rearrange:

$$m\frac{d^2x}{dt^2} + \frac{I}{2R}\frac{d^2\theta}{dt^2} + c\frac{dx}{dt} + \frac{3}{2}kS = 0$$

From kinematics, the displacement of the center of the center of the disk is $x = -2R\theta$. The stretch of the spring is the amount wrapped around the disk plus the translation of the disk,

$$S = -R\theta - 2R\theta = -3R\theta = \frac{3}{2}x.$$

Substitute: $$\left(m + \frac{I}{(2R)^2}\right)\frac{d^2x}{dt^2} + c\frac{dx}{dt} + \left(\frac{3}{2}\right)^2 kx = 0.$$

Define $M = m + \dfrac{I}{(2R)^2} = 14$ kg.

The canonical form of the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

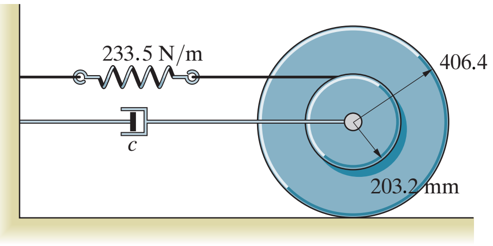
where $d = \dfrac{c}{2M} = 4.170$ rad/s,

$$\omega^2 = \left(\frac{3}{2}\right)^2 \frac{k}{M} = 37.53 \text{ (rad/s)}^2.$$

Therefore $\omega_d = \sqrt{\omega^2 - d^2} = 4.488$ rad/s, and the frequency is

$$\boxed{f_d = \frac{\omega_d}{2\pi} = 0.714 \text{ Hz}}$$

Problem 21.87 The stepped disk described in Problem 21.86 is initially in equilibrium, and at $t = 0$ it is given a clockwise angular velocity of 1 rad/s. Determine the position of the center of the disk relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 21.86, the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where $d = 4.170$ rad/s, $\omega^2 = 37.53$ (rad/s)^2. Therefore

$$\omega_d = \sqrt{\omega^2 - d^2} = 4.488 \text{ rad/s}.$$

Since $d^2 < \omega^2$, the system is sub-critically damped. The solution is of the form

$$x = e^{-dt}(A \sin \omega_d t + B \cos \omega_d t).$$

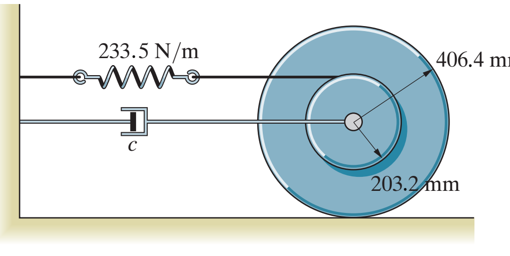
Apply the initial conditions: at $t = 0$, $\theta_0 = 0$, and $\dot{\theta}_0 = -1$ rad/s. From kinematics, $\dot{x}_0 = -2R\dot{\theta}_0 = 2R$ m. Substitute, to obtain

$$B = 0 \quad \text{and} \quad A = \frac{\dot{x}_0}{\omega_d} = 0.0906,$$

and the position of the center of the disk is

$$\boxed{x = 0.0906e^{-4.170t} \sin 4.488t}$$

Problem 21.88 The stepped disk described in Problem 21.86 is initially in equilibrium, and at $t = 0$ it is given a clockwise angular velocity of 1 rad/s. Determine the position of the center of the disk relative to its equilibrium position as a function of time if $c = 233.5$ N-s/m.



Solution: From the solution to Problem 21.86, the equation of motion is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

$$\text{where} \quad d = \frac{c}{2M} = 8.340 \text{ rad/s},$$

$$\omega^2 = \left(\frac{3}{2}\right)^2 \frac{k}{M} = 37.53 \text{ (rad/s)}^2.$$

Since $d^2 > \omega^2$,

the system is supercritically damped. The solution is of the form (see Eq. (21.24)) $x = e^{-dt}(Ce^{ht} + De^{-ht})$, where $h = \sqrt{d^2 - \omega^2} = 5.659$ rad/s. Apply the initial conditions: at $t = 0$, $\theta_0 = 0$, and $\dot{\theta}_0 = -1$ rad/s. From kinematics, $\dot{x}_0 = -2R\dot{\theta}_0 = 2R$ m. Substitute, to obtain $0 = C + D$ and $\dot{x}_0 = -(d - h)C - (d + h)D$. Solve:

$$C = \frac{x_0}{2h} = 0.036,$$

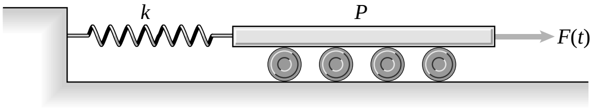
$$D = -\frac{x_0}{2h} = -0.036,$$

from which the position of the center of the disk is

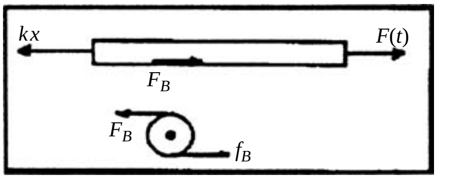
$$x = 0.036e^{-8.340t}(e^{5.659t} - e^{-5.659t})$$

$$\boxed{= 0.036(e^{-2.680t} - e^{-14.00t}) \text{ m}}$$

Problem 21.89 The 22-kg platen P rests on four roller bearings that can be modeled as 1-kg homogeneous cylinders with 30-mm radii. The spring constant is $k = 900$ N/m. The platen is subjected to a force $F(t) = 100 \sin 3t$ N. What is the magnitude of the platen's steady-state horizontal vibration?



Solution: Choose a coordinate system with the origin at the wall and the x axis parallel to the plane surface. Denote the roller bearings by the subscript B and the platen by the subscript P.



The roller bearings: The sum of the moments about the mass center of a roller bearing is

$$\sum M_{B\text{-cm}} = +RF_B + Rf_B.$$

From Newton's second law:

$$I_B \frac{d^2\theta}{dt^2} = RF_B + Rf_B.$$

Solve for the reaction at the floor:

$$f_B = \frac{I_B}{R}\frac{d^2\theta}{dt^2} - F_B.$$

The sum of the *horizontal* forces on each roller bearing:

$$\sum F_x = -F_B + f_P.$$

From Newton's second law

$$m_B \frac{d^2x_B}{dt^2} = -F_B + f_B,$$

where x_B is the translation of the center of mass of the roller bearing. Substitute

$$f_p, \quad m_B\frac{d^2x_B}{dt^2} = \frac{I_B}{R}\frac{d^2\theta}{dt^2} - 2F_B.$$

From kinematics, $\theta_B = -\dfrac{x_B}{R}$,

from which $\left(m_B + \dfrac{I_B}{R^2}\right)\dfrac{d^2x_B}{dt^2} = -2F_B.$

The platen: The sum of the forces on the platen are

$$\sum F_P = -kx + 4F_B + F(t).$$

From Newton's second law,

$$m_P\frac{d^2x_P}{dt^2} = -kx_p + 4F_B + F(t).$$

Substitute for F_B and rearrange:

$$m_p\frac{d^2x_p}{dt^2} + kx_P + 2\left(m_B + \frac{I_B}{R^2}\right)\frac{d^2x_B}{dt^2} = F(t).$$

From kinematics, $x_B = \dfrac{x_P}{2}$,

from which $\left(m_P + m_B + \dfrac{I_B}{R^2}\right)\dfrac{d^2x_P}{dt^2} + kx_P = F(t).$

For a homogenous cylinder,

$$I_B = \frac{m_B R^2}{2},$$

from which we define

$$M = m_p + \frac{3}{2}m_B = 23.5 \text{ kg}.$$

For $d = 0$, the canonical form of the equation of motion (see Eq. (21.26)) is

$$\frac{d^2x_p}{dt^2} + \omega^2 x_p = a(t),$$

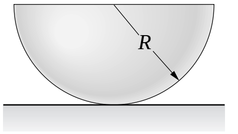
where $\omega^2 = 38.30\ (\text{rad/s})^2$,

and $a(t) = \dfrac{F(t)}{M} = 4.255 \sin 3t\ (\text{m/s}^2).$

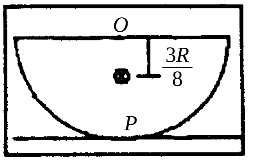
The amplitude of the steady state motion is given by Eq. (21.31):

$$\boxed{E_p = \frac{4.255}{(\omega^2 - 3^2)} = 0.1452 \text{ m}}$$

Problem 21.83 A homogeneous hemisphere of radius R and mass m rests on a level surface. If you rotate the hemisphere slightly from its equilibrium position and release it, what is the frequency of its vibrations?



Solution: The system is conservative. The distance from the center of mass to point O is $h = 3R/8$. Denote the angle of rotation about P by θ. Rotation about P causes the center of mass to rotate relative to the radius center OP, suggesting the analogy with a pendulum suspended from O. The kinetic energy is $T = (1/2)I_p\dot{\theta}^2$. The potential energy is $V = mgh(1 - \cos\theta)$, where $h(1 - \cos\theta)$ is the increase in height of the center of mass. $T + V = \text{const} = I_p\dot{\theta}^2 + mgh(1 - \cos\theta)$. Take the time derivative:



$$\dot{\theta}\left[I_p\frac{d^2\theta}{dt^2} + mgh\sin\theta\right] = 0,$$

from which $\dfrac{d^2\theta}{dt^2} + \dfrac{mgh}{I_p}\sin\theta = 0.$

For small angles $\sin\theta \to \theta$, and the moment of inertia about P is

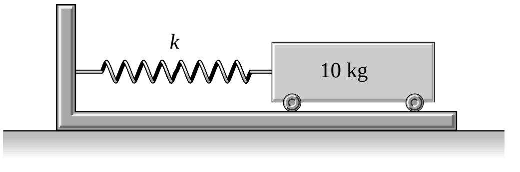
$$I_p = I_{CM} + m(R-h)^2 = \frac{83}{320}mR^2 + \left(\frac{5}{8}\right)^2 mR^2 = \frac{13}{20}mR^2,$$

from which $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = 0,$

where $\omega = \sqrt{\dfrac{20(3)g}{13(8)R}} = \sqrt{\dfrac{15g}{26R}}.$

The frequency is $\boxed{f = \dfrac{1}{2\pi}\sqrt{\dfrac{15g}{26R}}}$

Problem 21.84 The frequency of the spring-mass oscillator is measured and determined to be 4.00 Hz. The oscillator is then placed in a barrel of oil, and its frequency is determined to be 3.80 Hz. What is the logarithmic decrement of vibrations of the mass when the oscillator is immersed in oil?



Solution: The undamped and damped frequencies are $f = 4$ Hz and $f_d = 3.8$ Hz, so

$$\tau_d = \frac{1}{f_d} = 0.263 \text{ s},$$

$$\omega = 2\pi f = 25.13 \text{ rad/s},$$

$$\omega_d = 2\pi f_d = 23.88 \text{ rad/s}.$$

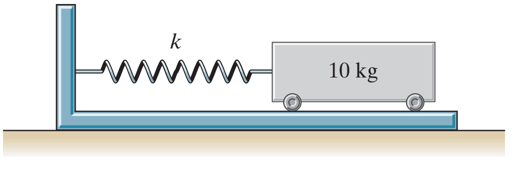
From the relation

$$\omega_d = \sqrt{\omega^2 - d^2},$$

we obtain $d = 7.85$ rad/s, so the logarithmic decrement is $\delta = d\tau_d = (7.85)(0.263) = 2.07$.

$\underline{\delta = 2.07}$.

Problem 21.85 Consider the oscillator immersed in oil described in Problem 21.84. If the mass is displaced 0.1 m to the right of its equilibrium position and released from rest, what is its position relative to the equilibrium position as a function of time?



Solution: The mass and spring constant are unknown. The canonical form of the equation of motion (see Eq. (21.16)) is

$$\frac{d^2x}{dt^2} + 2d\frac{dx}{dt} + \omega^2 x = 0,$$

where, from the solution to Problem 21.84, $d = 7.848$ rad/s, and $\omega = 2\pi(4) = 25.13$ rad/s. The solution is of the form (see Eq. (21.19)) $x = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$, where $\omega_d = 2\pi(3.8) = 23.88$ rad/s. Apply the initial conditions: at $t = 0$, $x_0 = 0.1$ m, and $\dot{x}_0 = 0$, from which $B = x_0$, and $0 = -dB + \omega_d A$,

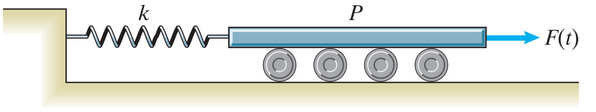
from which $A = \dfrac{dx_0}{\omega_d}$,

The solution is

$$x = x_0 e^{-dt}\left(\frac{d}{\omega_d}\sin\omega_d + \cos\omega_d\right)$$

$$\boxed{= 0.1e^{-7.848t}(0.3287\sin 23.88t + \cos 23.88t)}$$

Problem 21.90 At $t = 0$, the platen described in Problem 21.89 is 0.1 m to the right of its equilibrium position and is moving to the right at 2 m/s. Determine the platen's position relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 21.89, the equation of motion is

$$\frac{d^2x_p}{dt^2} + \omega^2 x_p = a(t),$$

where $\omega^2 = 38.30\ (\text{rad/s})^2$, and

$$a(t) = \frac{F(t)}{M} = 4.255 \sin 3t\ \text{m/s}^2.$$

The solution is in the form $x = x_h + x_p$, where the homogenous solution is of the form $x_h = A \sin \omega t + B \cos \omega t$ and the particular solution x_p is given by Eq. (21.30), with $d = 0$ and $b_0 = 0$. The result:

$$x = A \sin \omega t + B \cos \omega t + \frac{a_0}{(\omega^2 - \omega_0^2)} \sin \omega_0 t,$$

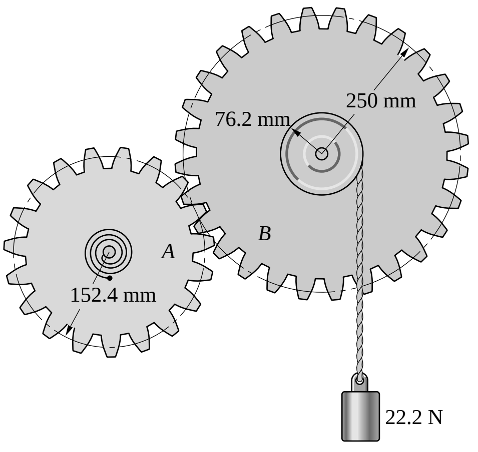
where $a_0 = 4.255$ m, $\omega = 6.189$ rad/s, and $\omega_0 = 3$ rad/s. Apply the initial conditions: at $t = 0$, $x_0 = 0.1$ m, and $\dot{x}_0 = 2$ m/s, from which $B = 0.1$, and

$$A = \frac{2}{\omega} - \left(\frac{\omega_0}{\omega}\right) \frac{a_0}{(\omega^2 - \omega_0^2)} = 0.2528,$$

from which

$$\boxed{x = 0.253 \sin 6.19t + 0.1 \cos 6.19t + 0.145 \sin 3t\ \text{m}}$$

Problem 21.91 The moments of inertia of gears A and B are $I_A = 0.019$ kg-m^2 and $I_B = 0.136$ kg-m^2. Gear A is connected to a torsional spring with constant $k = 2.71$ N-m/rad. The bearing supporting gear B incorporates a damping element that exerts a resisting moment on gear B of magnitude $2.03(d\theta_B/dt)$ N-m, where $d\theta_B/dt$ is the angular velocity of gear B in rad/s. What is the frequency of angular vibration of the gears?



Solution: Choose a coordinate system with the x axis positive downward. The sum of the moments on gear A is $\sum M = -k\theta_A + R_A F$, where the moment exerted by the spring opposes the angular displacement θ_A. From the equation of angular motion,

$$I_A \frac{d^2\theta_A}{dt^2} = \sum M = -k\theta_A + R_A F,$$

from which $F = \left(\frac{I_A}{R_A}\right)\frac{d^2\theta_A}{dt^2} + \left(\frac{k}{R_A}\right)\theta_A.$

The sum of the moments acting on gear B is

$$\sum M = -2.03\frac{d\theta_B}{dt} + R_B F - R_W F_W,$$

where $W = 22.2$ N, and the moment exerted by the damping element opposes the angular velocity of B. From the equation of angular motion applied to B,

$$I_B \frac{d^2\theta_B}{dt^2} = \sum M = -2.03\frac{d\theta_B}{dt} + R_B F - R_W F_W.$$

The sum of the forces on the weight are $\sum F = +F_W - W$. From Newton's second law applied to the weight,

$$\left(\frac{W}{g}\right)\frac{d^2x}{dt^2} = F_W - W,$$

from which $F_W = \left(\frac{W}{g}\right)\frac{d^2x}{dt^2} + W.$

Substitute for F and F_W to obtain the equation of motion for gear B:

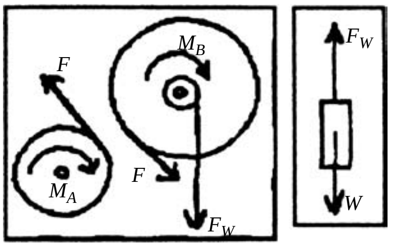
$$I_B \frac{d^2\theta_B}{dt^2} + 2.03\frac{d\theta_B}{dt} - \left(\frac{R_B}{R_A}\right)\left(I_A\frac{d^2\theta_A}{dt^2} + k\theta_A\right)$$

$$- R_W\left(\left(\frac{W}{g}\right)\frac{d^2x}{dt^2} + W\right) = 0.$$

From kinematics, $\theta_A = -\left(\frac{R_B}{RA}\right)\theta_B$, and $x = -R_W\theta_B$, from which

$$M\frac{d^2\theta_B}{dt^2} + 2.03\frac{d\theta_B}{dt} + \left(\frac{R_B}{R_A}\right)^2 k\theta_B = R_W W,$$

where $M = I_A + \left(\frac{R_B}{R_A}\right)^2 I_A + R_W^2\left(\frac{W}{g}\right) = 0.201$ kg-m^2



The canonical form of the equation of motion is

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = P,$$

where $d = \frac{2.03}{2M} = 5.047$ rad/s,

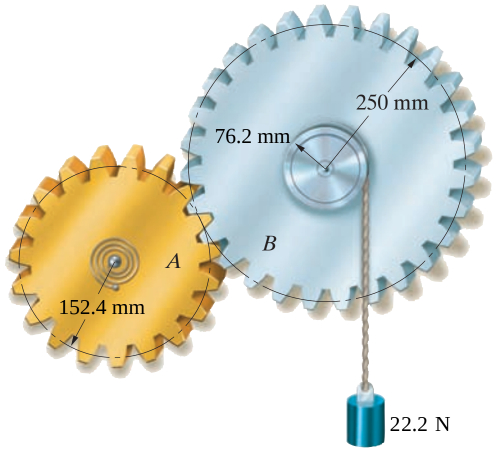
$$\omega^2 = \frac{\left(\frac{R_B}{R_A}\right)^2 k}{M} = 37.39 \text{ (rad/s)}^2,$$

and $P = \frac{R_W W}{M} = 8.412 \text{ (rad/s)}^2.$

The system is sub critically damped, since $d^2 < \omega^2$, from which $\omega_d = \sqrt{\omega^2 - d^2} = 3.452$ rad/s, and the frequency is

$$\boxed{f_d = \frac{\omega_d}{2\pi} = 0.5493 \text{ Hz}}$$

Problem 21.92 The 22.2 N weight in Problem 21.91 is raised 12.7 mm. from its equilibrium position and released from rest at $t = 0$. Determine the counterclockwise angular position of gear B relative to its equilibrium position as a function of time.



Solution: From the solution to Problem 81.91,

$$\frac{d^2\theta_B}{dt^2} + 2d\frac{d\theta_B}{dt} + \omega^2\theta_B = P,$$

where $d = 5.047$ rad/s, $\omega^2 = 37.39$ (rad/s)^2, and $P = 8.412$ (rad/s)^2. Since the non homogenous term P is independent of time and angle, the equilibrium position is found by setting the acceleration and velocity to zero in the equation of motion and solving: $\theta_{eq} = \frac{P}{\omega^2}$. Make the transformation $\tilde{\theta}_B = \theta_B - \theta_{eq}$, from which, by substitution,

$$\frac{d^2\tilde{\theta}_B}{dt^2} + 2d\frac{d\tilde{\theta}_B}{dt} + \omega^2\tilde{\theta} = 0$$

is the equation of motion about the equilibrium point. Since $d^2 < \omega^2$, the system is sub critically damped, from which the solution is $\tilde{\theta}_B = e^{-dt}(A\sin\omega_d t + B\cos\omega_d t)$. Apply the initial conditions: $x_0 = -0.0127$ m, from which

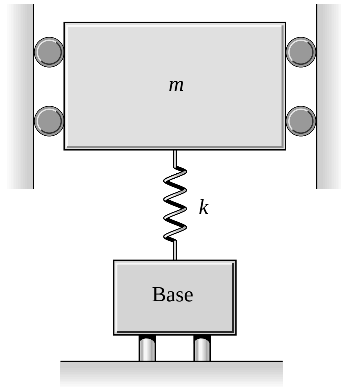
$$[\tilde{\theta}_B]_{t=0} = -\frac{x_0}{R_w} = 0.1667 \text{ rad}, \quad \frac{d\tilde{\theta}_B}{dt} = 0,$$

from which $B = 0.1667$, $A = \dfrac{d(0.1667)}{\omega_d} = 0.2437$.

The solution is

$$\boxed{\tilde{\theta}(t) = e^{-5.047t}(0.244\sin(3.45t) + 0.167\cos(3.45t))}$$

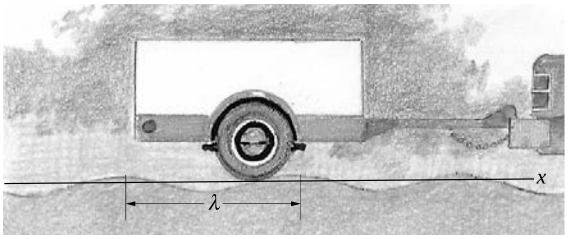
Problem 21.93 The base and mass m are initially stationary. The base is then subjected to a vertical displacement $h\sin\omega_i t$ relative to its original position. What is the magnitude of the resulting steady-state vibration of the mass m *relative to the base*?



Solution: From Eq. (21.26), for $d = 0$, $\dfrac{d^2x}{dt^2} + \omega^2 x = a(t)$, where $\omega^2 = \dfrac{k}{m}$, and $a(t) = \omega_i^2 h\sin\omega_i t$. From Eq. (21.31), the steady state amplitude is

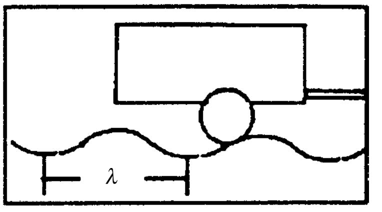
$$\boxed{E_p = \frac{\omega_i^2 h}{(\omega^2 - \omega_i^2)} = \frac{\omega_i^2 h}{\left(\dfrac{k}{m} - \omega_i^2\right)}}$$

Problem 21.94* The mass of the trailer, not including its wheels and axle, is m, and the spring constant of its suspension is k. To analyze the suspension's behavior, an engineer assumes that the height of the road surface relative to its mean height is $h\sin(2\pi/\lambda)$. Assume that the trailer's wheels remain on the road and its horizontal component of velocity is v. Neglect the damping due to the suspension's shock absorbers.



(a) Determine the magnitude of the trailer's vertical steady-state vibration *relative to the road surface*.
(b) At what velocity v does resonance occur?

Solution: Since the wheels and axle act as a base that moves with the disturbance, this is analogous to the transducer problem (Example 21.7). For a *constant velocity* the distance $x = \int_0^t v\,dt = vt$, from which the movement of the axle-wheel assembly as a function of time is $h_f(t) = h\sin(\omega_0 t)$, where $\omega_0 = \frac{2\pi v}{\lambda}$ rad/s. [*Check*: The velocity of the disturbing "waves" in the road relative to the trailer is v. Use the physical relationship between frequency, wavelength and velocity of propagation $\lambda f = v$. The wavelength of the road disturbance is λ, from which the forcing function frequency is $f_0 = \frac{v}{\lambda}$, and the circular frequency is $\omega_0 = 2\pi f_0 = \frac{2\pi v}{\lambda}$. *check*.] The forcing function on the spring-mass oscillator (that is, the trailer body and spring) is (see Example 21.7)



$$F(t) = -m\frac{d^2 h_f(t)}{dt^2} = m\left(\frac{2\pi v}{\lambda}\right)^2 h\sin\left(\frac{2\pi v}{\lambda}t\right).$$

For $d = 0$, the canonical form of the equation of motion is $\frac{d^2y}{dt^2} + \omega^2 y = a(t)$, where $\omega = \sqrt{\frac{k}{m}} = 8.787$ rad/s, and

$$a(t) = \frac{F(t)}{m} = \left(\frac{2\pi v}{\lambda}\right)^2 h\sin\left(\frac{2\pi v}{\lambda}t\right)$$

$$= a_0\left(\frac{2\pi v}{\lambda}\right)^2 \sin\left(\frac{2\pi v}{\lambda}t\right),$$

where $a_0 = h$.

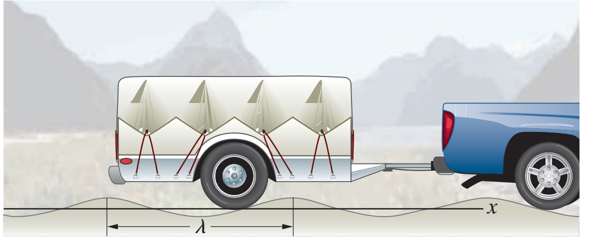
(a) The magnitude of the steady state amplitude of the motion relative to the wheel-axle assembly is given by Eq. (21.31) and the equations in Example 21.7 for $d = 0$ and $b_0 = 0$,

$$\boxed{E_p = \frac{\left(\frac{2\pi v}{\lambda}\right)^2 h}{\left(\frac{k}{m} - \left(\frac{2\pi v}{\lambda}\right)^2\right)}}$$

(b) Resonance, by definition, occurs when the denominator vanishes, from which

$$\boxed{v = \frac{\lambda}{2\pi}\sqrt{\frac{k}{m}}}$$

Problem 21.95* The trailer in Problem 21.94, not including its wheels and axle, weighs 4448 N. The spring constant of its suspension is $k = 35024$ N/m, and the damping coefficient due to its shock absorbers is $c = 2919$ N-s/m. The road surface parameters are $h = 5.1$ mm and $\lambda = 2.44$ m. The trailer's horizontal velocity is $v = 9.65$ km/h. Determine the magnitude of the trailer's vertical steady-state vibration relative to the road surface,



(a) neglecting the damping due to the shock absorbers and
(b) not neglecting the damping.

Solution:

(a) From the solution to Problem 21.94, for zero damping, the steady state amplitude relative to the road surface is

$$E_p = \frac{\left(\frac{2\pi v}{\lambda}\right)^2 h}{\left(\frac{k}{m} - \left(\frac{2\pi v}{\lambda}\right)^2\right)}.$$

Substitute numerical values:

$v = 9.65$ km/h $= 2.68$ m/s, $\lambda = 2.44$ m, $k = 35024$ N/m,

$$m = \frac{W}{g} = 453.5 \text{ kg},$$

to obtain $\boxed{E_p = 0.082 \text{ m} = 82 \text{ mm.}}$

(b) From Example 21.7, and the solution to Problem 21.94, the canonical form of the equation of motion is

$$\frac{d^2y}{dt^2} + 2d\frac{dy}{dt} + \omega^2 y = a(t),$$

where $d = \dfrac{c}{2m} = \dfrac{200}{2(31.08)} = 3.217$ rad/s, $\omega = 8.787$ rad/s,

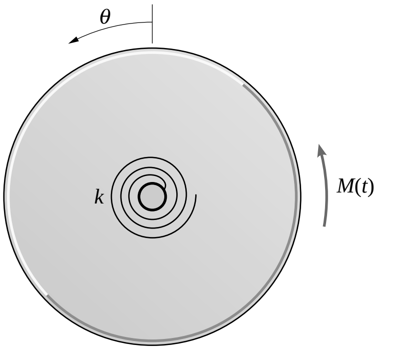
and $a(t) = a_0\left(\dfrac{2\pi v}{\lambda}\right)^2 \sin\left(\dfrac{2\pi v}{\lambda}t\right)$,

where $a_0 = h$. The system is sub-critically damped, from which $\omega_d = \sqrt{\omega^2 - d^2} = 8.177$ rad/s. From Example 21.7, the magnitude of the steady state response is

$$\boxed{E_p = \frac{\left(\frac{2\pi v}{\lambda}\right)^2 h}{\sqrt{\left(\omega^2 - \left(\frac{2\pi v}{\lambda}\right)^2\right)^2 + 4d^2\left(\frac{2\pi v}{\lambda}\right)^2}} = 0.045 \text{ m} = 45 \text{ mm.}}$$

Problem 21.96* A disk with moment of inertia I rotates about a fixed shaft and is attached to a torsional spring with constant k. The angle θ measures the angular position of the disk relative to its position when the spring is unstretched. The disk is initially stationary with the spring unstretched. At $t = 0$, a time-dependent moment $M(t) = M_0(1 - e^{-t})$ is applied to the disk, where M_0 is a constant. Show that the angular position of the disk as a function of time is

$$\theta = \frac{M_0}{I}\left[-\frac{1}{\omega(1+\omega^2)}\sin\omega t - \frac{1}{\omega^2(1+\omega^2)}\cos\omega t + \frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}e^{-t}\right].$$



Strategy: To determine the particular solution, seek a solution of the form

$$\theta_p = A_p + B_p e^{-t},$$

where A_p and B_p are constants that you must determine.

Solution: The sum of the moments on the disk are $\sum M = -k\theta + M(t)$. From the equation of angular motion, $I\dfrac{d^2\theta}{dt^2} = -k\theta + M(t)$. For $d = 0$ the canonical form is $\dfrac{d^2\theta}{dt^2} + \omega^2\theta = a(t)$, where $\omega = \sqrt{\dfrac{k}{I}}$, and $a(t) = \dfrac{M_0}{I}(1 - e^{-t})$. (a) For the particular solution, assume a solution of the form $\theta_p = A_p + B_p e^{-t}$. Substitute into the equation of motion,

$$\frac{d^2\theta_p}{dt^2} + \omega^2\theta_p = B_p e^{-t} + \omega^2(A_p + B_p e^{-t}) = \frac{M_0}{I}(1 - e^{-t}).$$

Rearrange:

$$\omega^2 A_p + B_p(1+\omega^2)e^{-t} = \frac{M_0}{I} - \frac{M_0}{I}e^{-t}.$$

Equate like coefficients:

$$\omega^2 A_p = \frac{M_0}{I},\ B_p(1+\omega^2) = -\frac{M_0}{I},$$

from which $A_p = \dfrac{M_0}{\omega^2 I}$, and $B_p = -\dfrac{M_0}{(1+\omega^2)I}$.

The particular solution is

$$\theta_p = \frac{M_0}{I}\left[\frac{1}{\omega^2} - \frac{e^{-t}}{(1+\omega^2)}\right].$$

The system is undamped. The solution to the homogenous equation has the form $\theta_h = A\sin\omega t + B\cos\omega t$. The trial solution is:

$$\theta = \theta_h + \theta_p = A\sin\omega t + B\cos\omega t + \frac{M_0}{I}\left(\frac{1}{\omega^2} - \frac{e^{-t}}{(1+\omega^2)}\right).$$

Apply the initial conditions: at $t = 0$, $\theta_0 = 0$ and $\dot\theta_0 = 0$:

$$\theta_0 = 0 = B + \frac{M_0}{I}\left(\frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}\right),$$

$$\dot\theta_0 = 0 = \omega A + \frac{M_0}{I}\left(\frac{1}{(1+\omega^2)}\right),$$

from which

$$A = -\frac{M_0}{I}\left(\frac{1}{\omega(1+\omega^2)}\right),$$

$$B = -\frac{M_0}{I}\left(\frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}\right) = -\frac{M_0}{I}\left(\frac{1}{\omega^2(1+\omega^2)}\right),$$

and the complete solution is

$$\boxed{\theta = \frac{M_0}{I}\left[-\frac{1}{\omega(1+\omega^2)}\sin\omega t - \frac{1}{\omega^2(1+\omega^2)}\cos\omega t + \frac{1}{\omega^2} - \frac{1}{(1+\omega^2)}e^{-t}\right].}$$