Module UFMEQT-20-1

Stress and Dynamics

Dynamics

October 2010

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University of the West of England, Bristol

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1 Vectors and Co-ordinate Systems

While the basics of vectors are described in the revision notes, here we will discuss 2
some of the more complex aspects of vectors, particularly those that are applied to the subjects explored as part of this course.

1.1 Vectors

1.1.1 Scalars and Vectors

There are quantities in physics that are determined uniquely by just one number. These 3
are called scalars. These include:

- Mass
- Temperature
- Speed

There are others that need more than one number to be fully described: a magnitude 4
and a direction. For instance, in one dimensional motion, velocity has a certain magnitude, which is the speed, but you also have to know which direction it goes. Examples of vectors are:

- Displacement
- Velocity
- Acceleration
- Force

A vector has a length and a direction, and is normally represented graphically with an arrow ($\longrightarrow$). Its length indicates the magnitude. If you view a vector head on, it is drawn as a dot within a circle ($\odot$), whereas viewing a vector from behind, it is drawn as a cross within a circle ($\oplus$).

1.1.2 Vector Notation

Vector quantities are designated in **boldface** in text books (and in this text) and un- 5
derlined type or with an arrow over the letter in hand-written work. For example $\mathbf{a}$, $\underline{a}$ or $\overrightarrow{a}$ denotes an acceleration vector, with $\mathbf{v}$, $\underline{v}$ or $\overrightarrow{v}$ signifying a velocity vector.

1.1.3 Vector Addition

Vector addition is given in more detail in the revision notes, but in summary there are 6 essentially two methods: the triangle method of addition; and the parallelogram method of addition. Both are shown in Figure 1.1 with (a) showing the triangular method of addition, and (b) the parallelogram method. In both cases, $\mathbf{A}$ and $\mathbf{B}$ are displacement vectors, and when added, $\mathbf{A} + \mathbf{B} = \mathbf{R}$, the resultant vector $\mathbf{R}$ is the same.

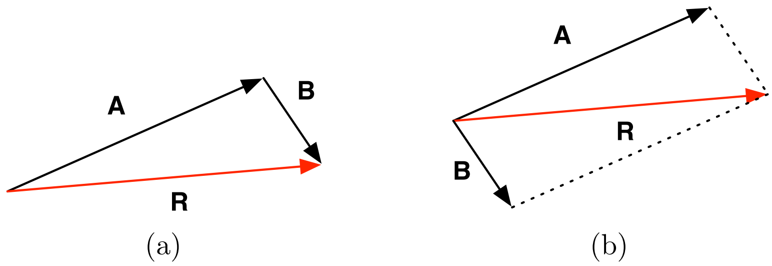


Figure 1.1: Vector addition — graphical method

7 A negative vector is a vector with the same length (the same magnitude) put pointing in the opposite direction. As such:

$$\mathbf{A} + (-\mathbf{A}) = 0$$

1.1.4 Vector Decomposition

We have seen that the sum of more than one vector can be described by one resultant vector. Similarly, we can *decompose* one vector in to the sum of others.

8 Consider vector $\mathbf{A}$ in three-dimensional space as shown in Figure 1.2, which connects the origin with point P.

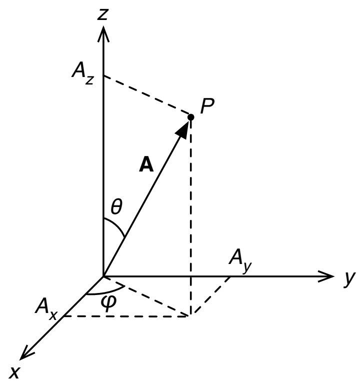


Figure 1.2: Vector $\mathbf{A}$ in three-dimensional space.

The projection of the vector on the x, y and z axes results in the numbers A_x, A_y and A_z. These values form the basis of the decomposition.

Unit Vectors

Unit vectors are special vectors that are always pointing in the direction of the three 9
axes and have a length of one. We denote the unit vectors as **i** for the x direction, **j** for the y direction and **k** for the z direction. (You may also see these denoted as $\hat{x}$, $\hat{y}$ and $\hat{z}$.) These unit vectors are shown in Figure 1.3.

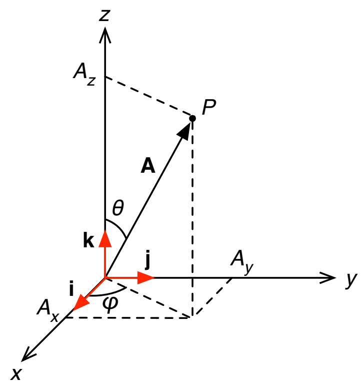


Figure 1.3: Vector **A** in three-dimensional space with unit vectors identified.

We can now rewrite vector **A** in terms of the three components that we have: 10

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

The three terms, $A_x\mathbf{i}$, $A_y\mathbf{j}$ and $A_z\mathbf{k}$ are vectors with magnitudes A_x, A_y and A_z lying along the three axes x, y and z respectively. Vector **A** is identical to the sum of these vectors.

Vector Magnitude and Direction

The magnitude of vector **A** is: 11

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

The magnitude of a vector, by definition, is always positive.
The angles θ and φ can be determined from trigonometry. From inspection:

$$\theta = \cos^{-1}\frac{A_z}{|\mathbf{A}|} \qquad \varphi = \tan^{-1}\frac{A_y}{A_x}$$

If we know the angles and the magnitude of **A**, then the values of A_x, A_y and A_z can 12
be determined as below (again, from basic trigonometry):

$$A_x = |\mathbf{A}|\sin\theta\cos\varphi \qquad A_y = |\mathbf{A}|\sin\theta\sin\varphi \qquad A_z = |\mathbf{A}|\cos\theta$$

Examples

1. Using $\mathbf{A}$ as defined earlier, we are given the following:

$$\mathbf{A} = 3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$$

Determine the magnitude, θ and φ of vector $\mathbf{A}$.

2. A small spacecraft is launched from a spaceship. The x, y and z-axes are defined relative to the spaceship, with the origin 0 at the spaceship. Radar emitted and received by the spaceship is used to track the small spacecraft. The parameters defined earlier (θ, φ and $\mathbf{A}$) are used in this example.
If the small spacecraft travels 3 km from the spaceship along $\theta = 70°$, $\varphi = 35°$ to point P, give the position of the small space craft $\mathbf{A}$ in decomposed form.

1.1.5 Relative and Absolute Displacement

13 Consider the positions of two points A and B relative to a third point, O where O is the origin of a co-ordinate system as shown in Figure 1.4.

The subscripts indicate the direction of the displacement:

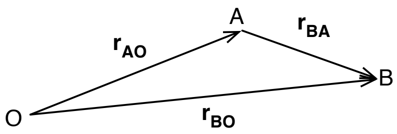


Figure 1.4: Displacement of A and B relative to the origin, O

- $\mathbf{r}_{\text{AO}}$ is the vector giving the displacement of A relative to O
- $\mathbf{r}_{\text{BO}}$ is the vector giving the displacement of B relative to O
- $\mathbf{r}_{\text{BA}}$ is the vector giving the displacement of B relative to A

As O is a fixed point (for example on the Earth's surface) $\mathbf{r}_{\text{AO}}$ and $\mathbf{r}_{\text{BO}}$ are **absolute** displacements, and the O subscript is often neglected resulting in $\mathbf{r}_{\text{A}}$ and $\mathbf{r}_{\text{B}}$. The vector $\mathbf{r}_{\text{BA}}$ is conversely a **relative** displacement.

Simple vector addition gives: 14

$$\mathbf{r}_{\text{B}} = \mathbf{r}_{\text{A}} + \mathbf{r}_{\text{AB}}$$

Rearranging $\mathbf{r}_{BA}$ can be found by rearranging:

$$\mathbf{r}_{\text{BA}} = \mathbf{r}_{\text{B}} - \mathbf{r}_{\text{A}}$$

Decomposing these equations in three dimensions results in the following relationship:

$$\begin{aligned}(r_{\text{BA}})_x\mathbf{i} + (r_{\text{BA}})_y\mathbf{j} + (r_{\text{BA}})_z\mathbf{k} = {} & [(r_{\text{B}})_x\mathbf{i} + (r_{\text{B}})_y\mathbf{j} + (r_{\text{B}})_z\mathbf{k}] - \\ & [(r_{\text{A}})_x\mathbf{i} + (r_{\text{A}})_y\mathbf{j} + (r_{\text{A}})_z\mathbf{k}]\end{aligned}$$

By grouping the $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ terms together, they can be treated individually:

$$\begin{aligned}(r_{\text{BA}})_x &= (r_{\text{B}})_x - (r_{\text{A}})_x \\ (r_{\text{BA}})_y &= (r_{\text{B}})_y - (r_{\text{A}})_y \\ (r_{\text{BA}})_z &= (r_{\text{B}})_z - (r_{\text{A}})_z\end{aligned}$$

1.2 Vector Multiplication

There are two ways to multiply vectors: 15

- Dot product (or scalar product)
- Cross product (or vector product)

1.2.1 Dot Product

The **Dot Product** is represented by a dot between the vectors. For example: 16

$$\mathbf{A} \cdot \mathbf{B}$$

There are two methods of calculating the dot product: an analytical method and a geometrical method.

Dot Product: Analytical Method

17 In three dimensional space, this is defined as:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$

The resultant is a scalar — it no longer has a direction. The $\mathbf{i}$ coefficients are multiplied together, the $\mathbf{j}$ coefficients are multiplied together and the $\mathbf{k}$ coefficients are multiplied together.

Dot Product: Geometrical Method

An alternative way to find a dot product may be more convenient depending on what you know. Let assume you know the length of $\mathbf{A}$ and $\mathbf{B}$, and you know the angle between the two, as shown in Figure 1.5(a).

18

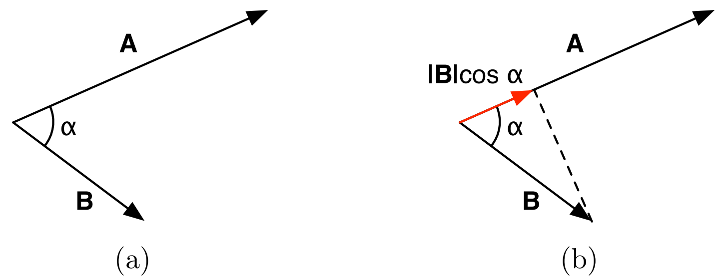


Figure 1.5: Vector multiplication — dot product

By projecting vector $\mathbf{B}$ on to vector $\mathbf{A}$, as shown in Figure 1.5(b), the length of the projection is the *magnitude of vector* $\boldsymbol{B}$ multiplied by $\cos\alpha$. The dot product is then defined as:

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\alpha$$

The two answers are completely identical. Depending on how the problem is presented, one method may be faster than the other.

Depending on the value for α, it can be seen that the dot product of two vectors can be either negative or positive, or indeed equal to zero. As stated earlier, the magnitudes of vectors $\mathbf{A}$ and $\mathbf{B}$ are always positive, so the sign of the dot product is determined by the the cosine of α. If the vectors are perpendicular to each other, then the dot product is zero. This will be revisited when we deal with the subject of work; we can have negative and positive work, and work and energy deal with dot products.

Dot Product: Examples

19 As an example, suppose we have $\mathbf{A}$ and $\mathbf{B}$. $\mathbf{A}$ and $\mathbf{B}$ are defined as:

$$\mathbf{A} = 3\mathbf{i} - 5\mathbf{j} + 6\mathbf{k} \qquad \mathbf{B} = 2\mathbf{j}$$

So, the dot product can be found by multiplying the **i** components together, the **j** components together and the **k** components together. Since the **i** and **k** components of **B** are zero, the dot product is simply:

$$\mathbf{A} \cdot \mathbf{B} = -5 \times 2 = -10$$

Another example: suppose we know that $\mathbf{A} = \mathbf{i}$ and $\mathbf{B} = \mathbf{k}$. These vectors are the unit vectors perpendicular to each other, and hence, the dot product of **A** and **B** is zero.

1.2.2 Cross Product

The cross product is represented by a cross between the two vectors: 20

$$\mathbf{A} \times \mathbf{B} = \mathbf{C}$$

Unlike the dot product, the resultant is a vector. Again, there are two methods to determine the cross product.

Cross Product: Analytical Method

The analytical method involves setting up a matrix and calculating its determinant: 21

$$\mathbf{A} \times \mathbf{B} = \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Using Sarrus' rule, we can see that the determinant is

$$\mathbf{C} = (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

So we can say: 22

$$\begin{aligned} A_yB_z - A_zB_y = C_x &\quad \rightarrow \quad \text{The } x \text{ component of } \mathbf{C} \\ A_zB_x - A_xB_z = C_y &\quad \rightarrow \quad \text{The } y \text{ component of } \mathbf{C} \\ A_xB_y - A_yB_x = C_z &\quad \rightarrow \quad \text{The } z \text{ component of } \mathbf{C} \end{aligned}$$

So we can also write:

$$\mathbf{C} = C_x\mathbf{i} + C_y\mathbf{j} + C_z\mathbf{k}$$

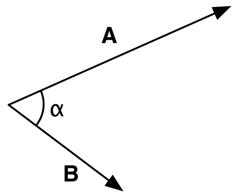


Figure 1.6: Vector multiplication — cross product

Cross Product: Geometrical Method

Again, imagine two vectors set up as shown in Figure 1.6.

23 The cross product is defined as:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin \alpha$$

24 It is clear to see that the cross product is zero if the angle is 0° or 180°. Again, $\sin \alpha$ can be greater or less than zero, making the magnitude of **C** positive or negative. As can be seen however, this equation only results in the magnitude.

To determine the direction of the cross product, we employ the **right hand rule** – see section 1.3.

Going back to Figure 1.6, you take **A**, because it is first mentioned, and you rotate **A** over the shortest possible angle to **B**. This is in the clockwise direction. Imagine you had a cork screw, and you rotated it clockwise, the cork screw goes into the cork. In this example, the direction of **C** is into the page, you will see the tail of the vector represented by $\oplus$.

25 The direction of a cross product is **always** perpendicular to **A** and **B**. It can either go out of the page or into the page.

Similarly, for $\mathbf{B} \times \mathbf{A}$, the shortest possible angle between the two vectors requires you to go in the anti-clockwise direction, so the direction of the resultant vector is out of the page, so we see the head of the vector, represented by $\odot$. (Remember, if you turned a corkscrew anti-clockwise, it would come out of the cork).

So we can say that:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

The cross product is therefore not commutative:

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$$

We will encounter cross products when we deal with torques and angular momentum.

1.3 Right-Handed Co-ordinate System

1.3.1 Special Features of Your Right Hand

To use the *right-hand rule*, hold up the thumb, index and middle finger. If the index 26
finger represents **A** and the middle finger, which is perpendicular to the index finger, represents **B**, then the direction of the cross product is the direction of the thumb. This is shown in Figure 1.7.

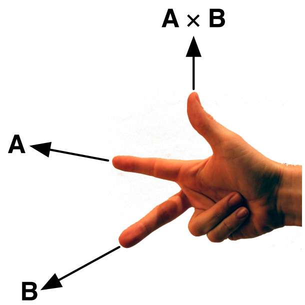


Figure 1.7: Right hand rule when using cross product

As an example, suppose we say that: 27

$$\mathbf{A} = \mathbf{i} \qquad \text{and} \qquad \mathbf{B} = \mathbf{j}$$

This means that:

$$\begin{array}{lll} A_x = 1 & A_y = 0 & A_z = 0 \\ B_x = 0 & B_y = 1 & B_z = 0 \end{array}$$

Clearly, these vectors are perpendicular to each other.
Determine:

$$\mathbf{A} \times \mathbf{B}$$

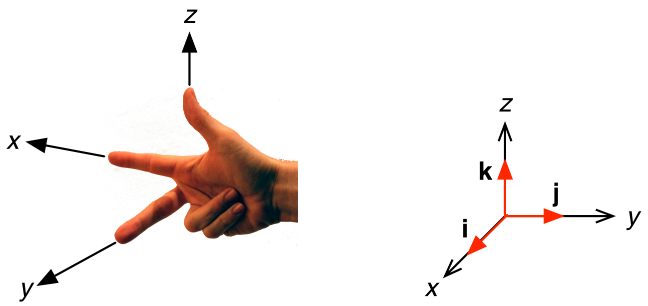
We could apply the recipe given above in the section on Cross Product, but it is much simpler to go to the co-ordinate axes we defined earlier (Figures 1.2 and 1.3), and shown in Figure 1.8.

Applying the right-hand-rule we can see that the direction of the resultant vector is 28
in the positive z direction. So:

$$\mathbf{A} \times \mathbf{B} = \mathbf{k}$$

It is no coincidence that the axes were set up in the way that they were. This arrangement of axes is called the **right-handed co-ordinate system**, and will be used throughout this course.

29



(a) (b)

Figure 1.8: Right-handed co-ordinate system

By definition, a right-handed co-ordinate system one in which:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

So, when you are working on problems involving torques and angular momentum, among others, it is ***essential*** that you draw yourself a set of right-handed co-ordinate system where $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. If you set up a system where $\mathbf{i} \times \mathbf{j} = -\mathbf{k}$, you will get incorrect answers, and the cross product calculations will not work.

1.3.2 Particle Motion in 3-Dimensions

30 Now we come to the section where this vector theory is applied to dynamics. Let us again revisit our co-ordinate system, with point P in three-dimensional space.

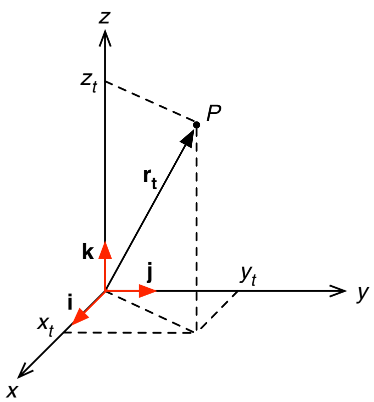


Figure 1.9: Point P in three-dimensional space

In Figure 1.9, the position of point P is defined as the vector $\mathbf{r}_t$. The subscript t indicates that it is a vector that varies with time. Its position on each axis can be found by decomposing the vector:

31

$$\mathbf{r}_t = x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k} \tag{1.1}$$

where x_t, y_t and z_t are the projections of the vector $\mathbf{r}_t$ on the three axes of the right-handed co-ordinate system.

We know that the velocity of a particle is the first derivative of its position, so we can 32
determine an equation for the velocity of vector $\mathbf{r}_t$:

$$\mathbf{v}_t = \frac{d\mathbf{r}_t}{dt} = \dot{x}_t\mathbf{i} + \dot{y}_t\mathbf{j} + \dot{z}_t\mathbf{k} \tag{1.2}$$

where we use the dot notation to represent the derivative with respect to time (see the revision notes).

Likewise, knowing that acceleration is the first derivative of its velocity (the second derivative of its position), the acceleration can be found:

$$\mathbf{a}_t = \frac{d\mathbf{v}_t}{dt} = \ddot{x}_t\mathbf{i} + \ddot{y}_t\mathbf{j} + \ddot{z}_t\mathbf{k} \tag{1.3}$$

Equations 1.1, 1.2 and 1.3 completely describe the motion of point P. The first terms in the equation describe the motion in the x direction, the second terms are the motion in the y direction and the third terms are the motion in the z direction. In other words, the three-dimensional motion has now been separated into three one-dimensional motions. Choosing just the first terms describes the behaviour *only* along the x axis.

Why does this help? Well, most motions that we will be dealing with in this level 1 course will involve motion in one or two dimensions. An example is plotting the trajectory of a projectile—it is only necessary to analyse these problems in two dimensions. The problems we face in dynamics can often be decomposed into x and y components, each of which can be analysed separately, but only together describe the motion of a particle.

Summary

Vectors

Vectors are used to describe many quantities in physics that need a *magnitude and* a 33
direction to be fully described. These are denoted with boldface text $\mathbf{A}$ in textbooks, or with an overhead arrow $\overrightarrow{A}$ in handwritten work.

Vector decomposition is required to describe one vector as the sum of others. We use 34
a co-ordinate system with unit vectors to do this:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

Magnitude and direction are given by: 35

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$
$$\theta = \cos^{-1}\frac{A_z}{|\mathbf{A}|}$$
$$\varphi = \tan^{-1}\frac{A_y}{A_x}$$

Multiplication

Dot Product

36 The Dot Product is denoted as $\mathbf{A} \cdot \mathbf{B}$. This is equal to:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z$$
$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\alpha$$

where α is the angle between vectors **A** and **B**. The resultant value is a scalar (it has no direction).

The dot product is used when dealing with work and energy.

Cross Product

37 The Cross product is denoted as $\mathbf{A} \times \mathbf{B}$. This is equal to:

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} + (A_z B_x - A_x B_z)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k}$$
$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\alpha$$

where α is the angle between vectors **A** and **B**. The resultant value is a vector, whose direction is perpendicular to **A** and **B**. If moving **A** to **B** requires a clockwise rotation, the direction is into the page, and follows the right-hand or corkscrew rule.

The cross product is used when dealing with torques and angular momentum.

Right-Handed Co-ordinate System

38 The Right-Handed Co-ordinate System is used throughout this course. This co-ordinate system is one in which:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

To help remember the direction of the axis, the right hand can be used.

39 When using the right-handed co-ordinate system, the motion of a particle in 3-dimensions can be described by three equations:

$$\mathbf{r}_t = x_t\mathbf{i} + y_t\mathbf{j} + z_t\mathbf{k}$$
$$\mathbf{v}_t = \frac{d\mathbf{r}_t}{dt} = \dot{x}_t\mathbf{i} + \dot{y}_t\mathbf{j} + \dot{z}_t\mathbf{k}$$
$$\mathbf{a}_t = \frac{d\mathbf{v}_t}{dt} = \ddot{x}_t\mathbf{i} + \ddot{y}_t\mathbf{j} + \ddot{z}_t\mathbf{k}$$

Each of these equations separate the 3-dimensional motion into three one-dimensional motions by vector decomposition. Most problems in this course will be based in only one or two dimensions, and only one set of terms will be required to solve motion in a single dimension.

Exercises

1. Two cars (A and B) are driven on a disused aerodrome. The x and y axes are defined to be aligned with the East direction and the North direction respectively and with the origin at the control tower. Note, this is a two dimension problem so the z axis (and the vector $\mathbf{k}$) is not needed. Use $\mathbf{r}$ as the position vector and the angle θ as the angle of $\mathbf{r}$ measuring anti-clockwise from the x axis. Remember vector $\mathbf{r} = r_x\mathbf{i} + r_y\mathbf{j}$
   Remember from the revision notes that in two dimensions the magnitude of a vector is:
   $$|\mathbf{r}| = \sqrt{r_x^2 + r_y^2}$$
   and the direction is:
   $$\theta = \tan^{-1}\frac{r_y}{r_x}$$
   Draw vector diagrams and answer the following questions
   a) i. Car A is driven 500 m due east from the control tower. Write $\mathbf{r}_\mathrm{A}$ in $r_x\mathbf{i}+r_y\mathbf{j}$ form.
      ii. Car B is driven 300 m due west from the control tower. Write $\mathbf{r}_\mathrm{B}$ in $r_x\mathbf{i} + r_y\mathbf{j}$ form. What is $\mathbf{r}_\mathrm{BA}$ (the vector describing the position of car B relative to car A).
   b) i. Car A is driven 500 m due east and the 300 m due north from the control tower. What is $\mathbf{r}_A$ and $r_x\mathbf{i} + r_y\mathbf{j}$ form? What is the magnitude and direction of $\mathbf{r}_A$?
      ii. If Car B is now driven 300 m due north and 400 m due west from the control tower, what is $\mathbf{r_B}$ in 4 form? What is the magnitude and direction of $\mathbf{r}_B$? What is $\mathbf{r}_{BA}$ in $r_x\mathbf{i} + r_y\mathbf{j}$ form?
2. A small spacecraft is launched from a spaceship. The x, y and z-axes are defined relative to the spaceship, with the origin 0 at the spaceship. Radar emitted and received by the spaceship is used to track the small spacecraft. The parameters defined earlier (θ, φ and $\mathbf{A}$) are used in this example.
   a) If the small spacecraft travels 3 km from the spaceship along $\theta = 70°$, $\varphi = 35°$ to point A, give the position of the small space craft $\mathbf{r}_\mathrm{A}$ in $\mathbf{r}_\mathrm{A}$ in $r_x\mathbf{i}+r_y\mathbf{j}+r_z\mathbf{k}$ form.
   b) If the small craft travels 5 km from point A along $\theta = 120°$ and $\varphi = 10°$ to point B:
      i. Give the position of B relative to A in $r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$ form.
      ii. Find the absolute position of point B (the displacement relative to the origin) of point B ($\mathbf{r}_\mathrm{B}$). Express your answer in $r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$ form.

3. Dot and Cross Products. If:
   - $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
   - $\mathbf{B} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$
   - $\mathbf{C} = \mathbf{i}$
   - $\mathbf{D} = \mathbf{j}$

   Calculate:

   a) $\mathbf{A} \cdot \mathbf{B}$
   b) $\mathbf{A} \cdot \mathbf{C}$
   c) $\mathbf{C} \cdot \mathbf{D}$
   d) $\mathbf{A} \times \mathbf{B}$
   e) $\mathbf{B} \times \mathbf{A}$
   f) $\mathbf{C} \times \mathbf{D}$
   g) $\mathbf{D} \times \mathbf{C}$

2 Newton’s Laws and Free-Body Diagrams

Newtonian mechanics forms the basis of this course. Here we will discuss Newton’s Laws, and the use of Free Body Diagrams to represent the forces acting on a system. 2

2.1 Newton’s Laws

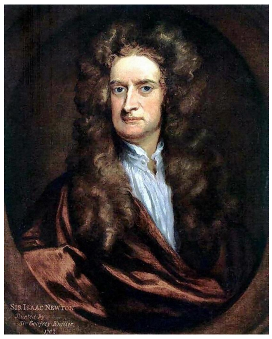


Figure 2.1: Newton

Sir Isaac Newton (Figure 2.1) is considered by many to one of the most influential people in human history, and his book *Principia* published in 1687 is one of the most significant scientific texts. In it, Newton describes universal gravitation and of particular interest to us the three laws of motion which define classical mechanics. This chapter we consider Newton’s laws of motion and a method of applying them to engineering problems involving dynamics. 3

It must be noted that Newtonian mechanics have now been superseded by quantum mechanics for very small scales of time and dimensions and by Einstein’s theory of relativity for relative speeds approaching the speed of light. Despite this, for a wide range of cases (and certainly for all cases dealt with in this course), Newtonian mechanics is the method of choice.

2.1.1 Newton’s First Law

The basis of Newton’s first law actually started earlier in the 17th century by Galileo 4
Galilei who also studied the motion of objects. He stated that:

> A body at rest remains at rest and a body in motion continues to move at a constant velocity in a straight line unless acted upon by an external force.

It is relatively simple to see that if something is at rest, and no external force is applied to it, it will continue to remain at rest. But what Galileo discovered was that **if an object is travelling at a certain velocity, there is no force necessary to keep it travelling at that velocity**. There is no obvious terrestrial example of this, so this was an important discovery. In the real world, drag or friction will eventually bring most things to rest.

Newton’s work continued from this and in his words, from *Principia* states: 5

> Every body perseveres in its state of rest or in uniform motion in a right line unless it is compelled to change that state by forces impressed upon it.

It is important that this fundamental law does not hold in all reference frames, for example in one that is being accelerated.

Imagine you are being accelerated (in a car, for example). Bystanders will see you being accelerated noticing your change in velocity, so according to the first law, there must be an external force being applied. You can of course feel this accelerative force. Now from *your* reference frame, you can see the bystanders coming towards you apparently being accelerated in the opposite direction. But if the first law works, the bystanders would feel an accelerative force, but of course, they do not. Therefore, the first law does not work for your frame of reference.

Another example is in plane that is accelerating down the runway just before taking off. If you left something on the floor of the plane, as the plane accelerates this object which was formally at rest, slides down the floor of the plane to the back row, so it seems that the object of rest is moving without a force acting on it, seemingly contradicting Newton's first law.

The first law only works in an *intertial frame of reference*, which is a reference frame in which there are no accelerations of any kind. This is an impossible situation—as the earth rotates about its axis, there is centripetal acceleration. The earth rotates about the sun which also results in a centripetal acceleration, and the sun rotates around the Milky Way which gives an additional centripetal acceleration and so on. So, for any point in the universe, there is no inertial reference frame.

These centripetal accelerations, which can be measured or calculated, are very small compared to gravitational acceleration. Therefore, in spite of these accelerations, we accept that a point on Earth is an approximate inertial reference frame.

- Can Newton's first law be proven? No, because it is impossible to be sure that your reference frame is without any accelerations.
- Do we believe in Newton's first law? Yes we do.
- Why? Because it is consistent within the uncertainties of the measurements of all experiments that have been done.

2.1.2 Newton's Second Law

6 Neglecting gravity for a moment, take the situation given in Figure 2.2.

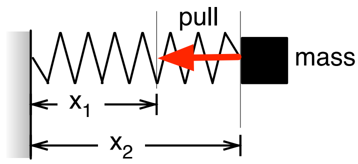


Figure 2.2: Spring accelerating a mass

The term x_1 is the relaxed length of the spring. If this spring is extended by a certain amount to x_2, we can imagine that there is a 'pull', indicated by the red arrow, applied at the end of the spring attempted to return the spring to its original length. If we attach mass m_1 to the end of the spring we can measure it's acceleration, $\mathbf{a}_1$ when it is released. If we then attach a different mass of m_2 to the end of the spring, we can measure this acceleration, $\mathbf{a}_2$. The 'pull' being applied by the spring is the same, and it is an experimental fact that the product of these can be equated:

$$m_1\mathbf{a}_1 = m_2\mathbf{a}_2$$

This product, $m\mathbf{a}$ is called a **force**. (We will discuss the specifics around forces being applied by a spring later in this course.)

Newton's second law states that: 7

> A force action on a body gives it an acceleration which is in the direction of the force and has a magnitude given by $m\mathbf{a}$.

So, mathematically:

$$\mathbf{F} = m\mathbf{a} \tag{2.1}$$

Equation 2.1 is the probably the most important law in all physics, and certainly one of the most important in this course. The units of force are [kg $\cdot$ m/s^2] which we call 1 Newton (N). Like the first law, the second law only holds in inertial reference frames.

- Can Newton's second law be proven? No, because it is impossible to be sure that your reference frame is without any accelerations.
- Do we believe in Newton's second law? Yes we do.
- Why? Because it is consistent within the uncertainties of the measurements of all experiments that have been done.

Gravitational Force

If dealing with the gravitational force of an object, the acceleration in Newton's second 8
law is the gravitational acceleration, $\mathbf{g} = 9.81\,\text{m/s}^2$. The gravitational force can therefore be written:

$$\mathbf{F}_g = m\mathbf{g} \tag{2.2}$$

which is always directed downwards.

We can see that the gravitation force due to the earth on a particular mass is linearly proportional with the mass. If the mass is 10 times larger, then the force due to gravity goes up by a factor of 10.

2.1.3 Newton's Third Law

9 Newton's third law states that:

> If one object exerts a force on another, the other exerts the same in opposite direction on the one.

This can be usefully summarised as:

$$\text{action} = -\text{reaction}$$

where the minus sign indicates that it opposes.

So, if you are sitting down, you are applying a force on the seat due to gravity, and at the same time, the seat is applying a force with the same magnitude on you. This opposing forces is often termed a **contact force**.

Let's look at a simple example. We have two masses, $m_1 = 5\,\text{kg}$ and $m_2 = 15\,\text{kg}$, as shown in Figure 2.3. There is a force, $\mathbf{F}$ which has a magnitude of 20 Newtons. Again, we neglect gravity.

10

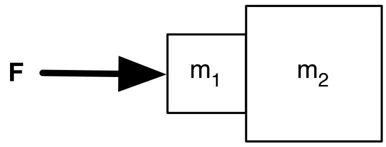


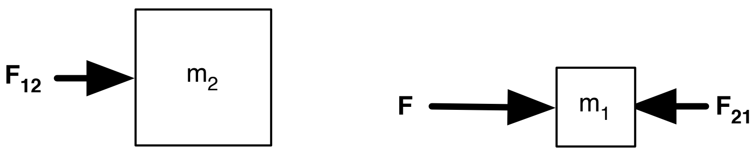
Figure 2.3: 3rd Law Example

Newton's second law states:

$$\mathbf{F} = m\mathbf{a} = (m_1 + m_2)\mathbf{a}$$
$$\mathbf{a} = \frac{m_1 + m_2}{\mathbf{F}} = \frac{5 + 15}{20} = 1\,\text{m/s}^2$$

While the system accelerates at $1\,\text{m/s}^2$, m_1 is applying a force to m_2, otherwise m_2 would not move. If you single out m_2 and draw the forces being applied on it (Figure 2.4(a)), we have force the m_1 exerts on m_2, which we call $\mathbf{F}_{12}$.

11



(a) (b)

Figure 2.4: Forces applied to m_2 (a) and m_1 (b)

For m_2, Newton's second law states that:

$$\mathbf{F}_{12} = m_2\mathbf{a} = (15)(1) = 15\,\mathrm{N}$$

Now, drawing the same sort of figure for m_1, which experiences the initial force $\mathbf{F}$ *and* it must experience a contact force from m_2. If m_1 is pushing on m_2, then m_2 must be pushing on m_1. We call this force $\mathbf{F}_{21}$. We know that m_1 is being accelerated.

For m_2 Newton's second law states that: 12

$$\mathbf{F} + \mathbf{F}_{21} = m_1\mathbf{a} \quad \rightarrow \quad \mathbf{F}_{21} = m_1\mathbf{a} - \mathbf{F} = (5)(1) - 20 = -15\,\mathrm{N}$$

which is equal and opposite to $\mathbf{F}_{12}$. The mass m_1 is pushing on m_2 with 15 N, and m_2 is pushing **back** on m_1 with 15 N. This is consistent with Newton's 3rd law.

Was this simple example a proof? No.

- Can Newton's third law be proven? No.
- Do we believe in Newton's third law? Yes we do.
- Why? Because it is consistent within the uncertainties of the measurements of all experiments that have been done.

2.2 Free-Body Diagrams

2.2.1 Particles and Rigid Bodies

In mechanics, real bodies (e.g. planets, cars, planes, tables, crates etc.) are represented 13
or *modelled* using certain idealisations which simplify application of the relevant theory. We refer to only two such models: particles and rigid bodies.

Particle

A **particle** has a mass but a size or shape that can be neglected. For example, the size of an aircraft is insignificant when compared to the size of the earth and therefore the aircraft can be modelled as a particle when studying its three-dimensional motion in space.

Rigid Body

A **rigid body** represents the next level of sophistication after the particle. A rigid body is a collection of particles which has a size or shape but this size or shape cannot change. In other words, when a body is modelled as a rigid body, we assume that any deformations (changes in shape) are relatively small and can be neglected. For example, the actual deformations occurring in most structures and machines are relatively small so that the rigid body assumption is suitable in most cases.

2.2.2 Equation of Motion for a Particle

14 When a system of forces acts on a particle, the Newton's second law may be written in the form:

$$\sum \mathbf{F} = m\mathbf{a} \tag{2.3}$$

where $\sum \mathbf{F}$ is the sum of all the external forces acting on the particle.

Successful application of equation 2.3 requires a complete specification of all the known and unknown external forces that act on the object. The best way to account for these is to draw the object's **Free-Body Diagram**: a sketch of the object free from its surroundings shown all the external forces that act on it.

In dynamics problems, since the resultant of these external forces produces the vector $m\mathbf{a}$, this can be indicated in the free-body diagram. Alternatively a separate **Kinetic Diagram** is often used to represent graphically the magnitude *and* direction of the vector $m\mathbf{a}$. In other words:

$$\text{Free-Body Diagram} = \text{Kinetic Diagram}$$

which is equivalent to equation 2.3.

2.2.3 Equations of Motion for a Rigid Body

15 Since rigid bodies have a definite size or shape, their motion is governed by both translational and rotational quantities. The translational equation of motion for the centre of mass of a rigid body is basically equation

$$\sum \mathbf{F} = m\mathbf{a}_G \tag{2.4}$$

where $m\mathbf{a}_G$ is the body's mass multiplied by the acceleration of its mass centre G.

The rotational equation of motion for a rigid body is given by:

$$\sum \mathbf{M}_G = I_G\alpha \tag{2.5}$$

which states that the sum of the applied couple moments and moments of all the external forces computed about a body's mass centre G is equal to the product of the moment of inertia of the body about an axis passing through G (often referred to as I_G) and the body's *angular* acceleration, α.

Alternatively, equation 2.5 can be re-written in the more general form as:

$$\sum \mathbf{M}_P = \sum (\mathbf{M}_k)_P \tag{2.6}$$

Here, $\sum \mathbf{M}_P$ represents the sum of the applied couple moments and the external moments taken about a general point P and $\sum(\mathbf{M}_k)_P$ represents the sum of the kinetic moments about P, in other words, the sum of $I_G\alpha$ and the moments generated by the components of the vectors $m\mathbf{a}_G$ about the point P.

When applying the equations of motion, one should always draw a free-body diagram in order to account for the terms involved in $\sum \mathbf{F}$, $\sum \mathbf{M}_G$ and $\sum \mathbf{M}_P$. The **kinetic diagram** is also useful in that it accounts graphically for the acceleration components of $m\mathbf{a}_G$ and the term $I_G\alpha$ and it is especially convenient when used to determine the components of $m\mathbf{a}_G$ and the moment terms in $\sum(\mathbf{M}_k)_P$.

2.2.4 Friction

The force of friction between an object and a surface is related to the application of 16
Newton's Third Law. For an object that is in contact with a surface, the object is subject to the force of gravity on the object following the equation:

$$\mathbf{F}_g = m\mathbf{g}$$

which is directed downwards. As Newton's 3rd law states, for the object to remain at rest, an equal but opposite force is applied by the surface to the object. This is known as the **normal** force or **normal** reaction force, often denoted as **N** or **R** and is *perpendicular* to the friction surface, even if the gravitational force is not acting perpendicular to the surface.

The friction force is defined as the friction coefficient, denoted by the Greek letter mu μ_k multiplied by this normal reaction:

$$\mathbf{F}_f = \mu_k \mathbf{N} \tag{2.7}$$

The coefficient of friction μ_k has no units.

The subscript k indicates this the coefficient of *kinetic* or *dynamic* friction, as this is the coefficient relating to the object actually moving relative to the surface.

The other coefficient of friction μ_s is known as the *static* friction coefficient, which is the friction associated with acceleration from rest to motion. The equation is the same as equation 2.7 with μ_k replaced by μ_s

There are few points to remember about friction:

- Friction force always **opposes** motion
- Friction is independent of contact area
- Dynamic friction is less than static friction

2.2.5 Drawing a Free Body Diagram

The following steps should be followed for drawing a free-body diagram. 17

1. Select a coordinate system
2. Identify the object you wish to analyse and draw its outline shape
3. Draw all external forces and couple moments* acting on the object and label them. These will include:
   a) Applied loadings
   b) Reactions occurring at points of contact with other bodies

   c) The weight of the body (applied at the body's centre of gravity, G)
   d) Frictional forces
   e) Indicate the necessary dimensions in order to calculate the moments*
4. The direction of force having a unknown magnitude can be assumed. (Remember, the magnitude of a vector is always positive, so if the solution yields a negative result, the minus sign indicates that the vector's sense is opposite to that which was originally assumed.)
5. The direction of the acceleration of the body's mass centre, $\mathbf{a}_G$ should be established and either identified on a kinetic diagram or in the coordinate axes.

The items marked with an asterisk are not necessary if dealing with an object modelled as a particle.

2.2.6 Free-Body Diagram Example: Particle

The 50 kg crate shown in Figure 2.5 rests on a horizontal plane for which the coefficient of friction is $\mu_k = 0.3$. The crate is subjected to a towing fore of magnitude 400 N and moves to the right without tipping over. Draw the free-body and kinetic diagrams of the crate.

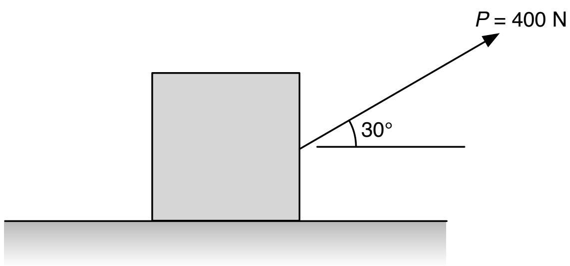


Figure 2.5: Particle Example

So, following the steps above, the following is the free-body and kinetic diagram of the crate.

We will study how to apply the laws of motion defined above to determine the normal force **N** and the acceleration of the crate **a** in the next lecture.

2.2.7 Free-Body Diagram Example: Rigid Body

The equations of motion 2.4, 2.5 (or 2.6) are used to determine the unknown forces, moments and acceleration components acting on an object (modelled as a rigid body) subjected to an unbalanced system of forces and moments. The first step in doing this again to draw the free-body diagram of the object to identify *all* of the external forces and moments acting on it. The procedure for drawing a free-body diagram for a rigid body is much the same as that for a particle with the main difference being that now, because the object a size or shape, it can support also eternal couple moments and moments of external forces.

This example also involves a 50 kg crate. A force of $P = 600\,\mathrm{N}$ is applied to the crate as shown in Figure 2.6. The coefficient of kinetic friction, $\mu_k = 0.2$.

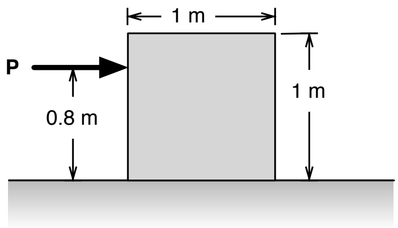


Figure 2.6: Rigid Body Example

So, following the steps above, the following is the free-body and kinetic diagram of the crate.

Again, we will study how to apply the laws of motion defined above to determine the normal force **N** and the acceleration of the crate **a** in the next lecture.

Summary

Newton's Laws

Newton's First Law

18 Newton's First Law:

> Every body perseveres in its state of rest or in uniform motion in a right line unless it is compelled to change that sate by forces impressed upon it.

This is only valid for inertial frames of reference. Earth is an approximate inertial frame of reference. Despite being unproven, we believe it because it is consistent within the uncertainties of any measurements we take.

Newton's Second Law

19 Newton's Second Law:

> A force action on a body gives it an acceleration which is in the direction of the force and has a magnitude given by $m\mathbf{a}$.

So, mathematically:

$$\mathbf{F} = m\mathbf{a}$$

This is only valid for inertial frames of reference. Earth is an approximate inertial frame of reference. Despite being unproven, we believe it because it is consistent within the uncertainties of any measurements we take.

If dealing with gravity, we replace $\mathbf{a}$ with $\mathbf{g}$. The gravitational force can be written:

$$\mathbf{F}_g = m\mathbf{g}$$

which is *always* directed downwards.

Newton's Third Law

20 Newton's Third Law:

> If one object exerts a force on another, the other exerts the same in opposite direction on the one.

This can be usefully summarised as:

$$\text{action} = -\text{reaction}$$

where the minus sign indicates that it opposes.

Despite being unproven, we believe it because it is consistent within the uncertainties of any measurements we take.

Free-Body Diagrams

Particles and Rigid Bodies

- **A Particle** has a mass but no size or shape. Particles cannot have couple moments applied to them. We only use a rectangular coordinate system with particles. 21
- **A Rigid Body** has mass *and* a size or shape. Rigid Bodies can experience couple moments applied to them by forces that are not directed at their mass centres, G. We can use a rectangular co-ordinate system or if motion is rotational, then we can use a curvilinear co-ordinate system.

Equations of Motion for a Particle

Newton's second law is: 22

$$\sum \mathbf{F} = m\mathbf{a}$$

The left hand side of the equation is drawn as the *free-body diagram*, with the right hand side being the *kinetic diagram*.

Equations of Motion for a Rigid Body

For rigid bodies, we have Newton's second law being applied to the mass centre: 23

$$\sum \mathbf{F} = m\mathbf{a}_G$$

where $\mathbf{a}_G$ is the acceleration of its mass centre, G.

The rotational equation of motion is given by:

$$\sum \mathbf{M}_G = I_G\alpha$$

where I_G is the moment of inertia about an axis passing through the mass centre, G, and α is the bodies angular acceleration.

An alternative form can be written:

$$\sum \mathbf{M}_P = \sum (\mathbf{M}_k)_P$$

where $\sum \mathbf{M}_P$ is the sum of the applied couple moments and the external moment take about a general point P and $\sum(\mathbf{M}_k)_P$ represents the sum of the kinetic moments about P.

Friction

The kinetic friction coefficient is represented by μ_k and the static friction coefficient is 24
represented by μ_s. In general, $\mu_s > \mu_k$.

The force required to overcome friction is given by:

$$\mathbf{F}_f = \mu_k \mathbf{N}$$

where $\mathbf{N}$ is the normal force, which is generally the opposing reaction force against gravity, $m\mathbf{g}$.

Drawing a Free-Body Diagram

25 Essentially, the main rules are:

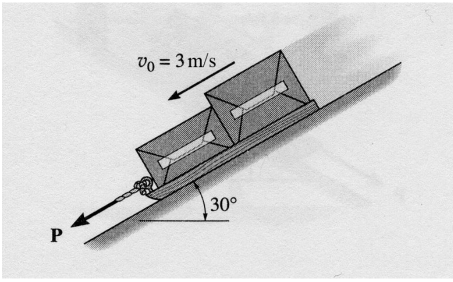
1. Establish an appropriate coordinate system
2. Isolate the object or body of interest
3. Draw all the forces being applied to the object
4. If the object is a rigid body (the forces are not applied to the mass centre), remember to identify the dimensions involved
5. Establish the direction of the object acceleration $\mathbf{a}$.

Exercises

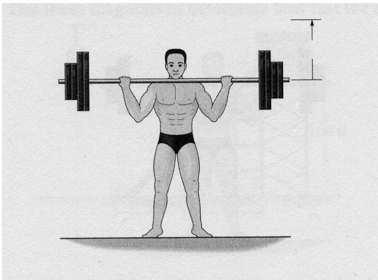
Draw the Free-Body diagram and kinetic diagram for the following bodies. Remember to separate the body from its surroundings and include all the forces applied by the outside world to the body. Include coordinates.

Particles

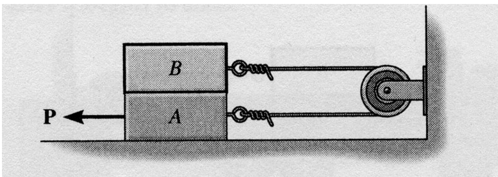
1. Sled being pulled down slope. Neglect air resistance but there is friction between surface and sled.



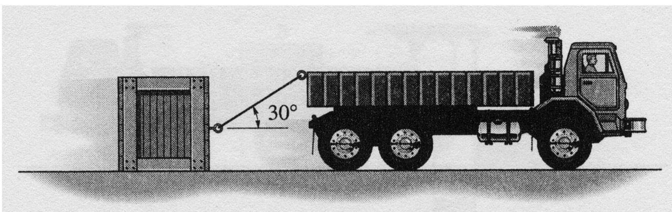
2. Man lifting bar bells:



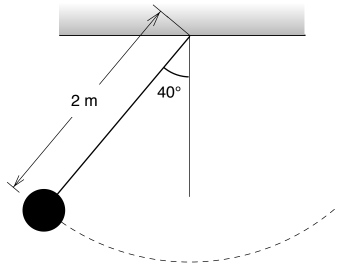
3. Two blocks: Pulling force applied to block A. There is friction between the blocks.



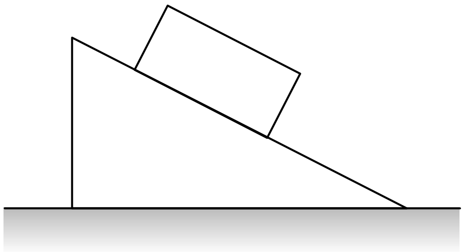
4. Truck pulling crate:



5. Pendulum falling due to gravity. Neglect air resistance.

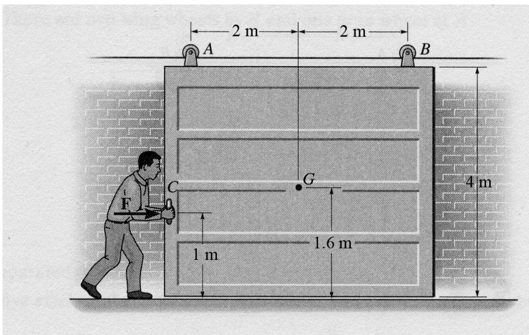


6. Draw free-body and kinetic diagrams for *both* block and wedge. There is friction between the wedge and the ground and between the wedge and the block.

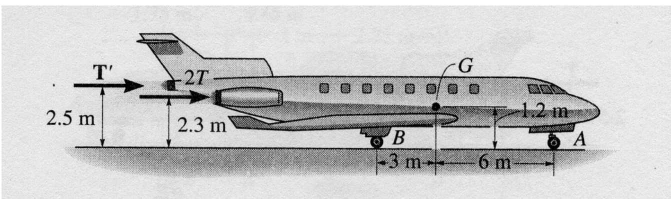


2.2.8 Rigid Bodies

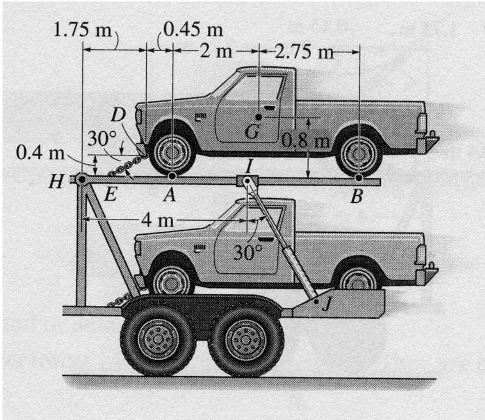
7. Draw the free-body diagram and kinetic diagrams for the garage door with centre of gravity, G, if man pushes on it at C with a horizontal force with magnitude F. There are rollers at A and B.



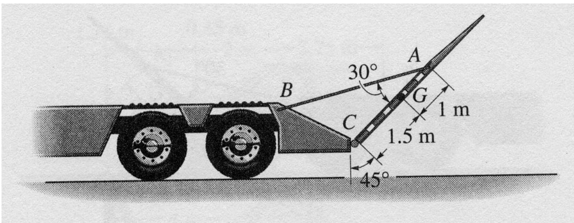
8. The jet has a centre of gravity at G. Initially at take off the engines provide a total thrust force of $2T + T\prime$. Neglect the mass of the wheels, and due to low velocity, neglect any lift caused by the wings. There are two wing wheels at B and one nose wheel at A.



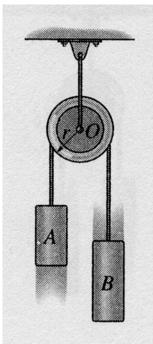
9. The top truck has a centre of gravity G. It is tied to the transporter using a chain DE. The transporter is accelerating. Draw Free-Body and Kinetic diagrams of the *top* truck.



10. The drop gate at the end of the trailer has a centre of gravity G, and is supported by the cable AB and hinge at C. The truck begins to accelerate. Draw the free-body and kinetic diagrams of the drop gate.



11. The two blocks A and B have masses such that $m_B > m_A$. The pulley has a mass. Regaring the pulley and two block as a single system, draw the free-body and kinetic diagrams of this system. Neglect the mass of the cord joining the masses, and no slipping occurs on the pulley.



3 Applying Newton's Laws

Following on from Chapter 2 on Newton's Laws and Free-Body Diagrams, we now focus 2
on applying these concepts to analyse and solve dynamics problems. We will use the examples explored in Chapter 2 to illustrate the process involved in solving these types of problems.

3.1 Solving Problems: Particle Motion

3.1.1 Free-Body Diagram

From last week's lecture, we now know the process to build up a free-body diagram for 3
a problem where the body is modelled as a particle. The equation of motion that we apply for particle motion is:

$$\sum \mathbf{F} = m\mathbf{a}$$

which states the *sum* of the external forces acting on the system is equal to the total mass m of the particles multiplied by the acceleration $\mathbf{a}$ of the particle.

You will also remember that the first step of producing a free-body diagram was to 4
identify an appropriate coordinate system. Often, for these problems, a rectangular coordinate system with x and y axes is suitable. These axes could also be identified using the unit vectors discussed in chapter 1, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. The two sets of axes shown in Figure 3.1 are equivalent: one uses the standard cartesian coordinates x and y, while the other is a right-handed coordinate system identified with the unit vectors. We will demonstrate how to solve these problems using the unit vector coordinate system.

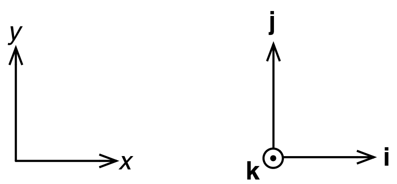


Figure 3.1: Suitable coordinate axes for particle motion problems

3.1.2 Equations of Motion: Component Form

5 Once the Free-Body diagram has been determined, we apply the equations of motion in their **scalar component form**. The equation of motion :

$$\sum \mathbf{F} = m\mathbf{a} \tag{3.1}$$

can be broken into two equations: one representing the **i** axis, and one representing the **j** axis:

$$\sum F_x \mathbf{i} = m a_x \mathbf{i} \qquad \text{and} \qquad \sum F_y \mathbf{j} = m a_y \mathbf{j} \tag{3.2}$$

In some reference texts, you may see these equations written without the unit vectors **i** and **j** as such:

$$\sum F_x = m a_x \qquad \text{and} \qquad \sum F_y = m a_y \tag{3.3}$$

Either set of equations 3.2 or 3.3 will yield the same results, and your choice will depend on the problem you are given. The main point is that we **deal with horizontal and vertical motion separately.**

We make the assumption that the components are **positive** if they are directed along a positive axis and negative if they are directed along a negative axis.

If there is friction involved, remember that a **friction force always opposes motion** relative to the surface it contacts.

Perhaps this is best illustrated with an example.

3.1.3 Particle Motion: Example

We shall return to the crate example first shown in the last lecture:

6 The 50 kg crate shown in Figure 3.2 rests on a horizontal plane for which the coefficient of friction is $\mu_k = 0.3$. The crate is subjected to a towing fore of magnitude 400 N and moves to the right without tipping over. Draw the free-body and kinetic diagrams of the crate.

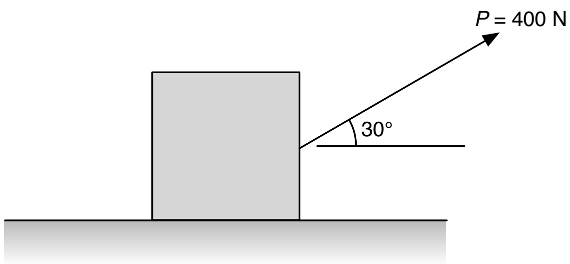


Figure 3.2: Particle Example

7 By applying the process, we resulted in a free-body diagram and kinetic diagram that
8 looked something like that shown in Figure 3.3:

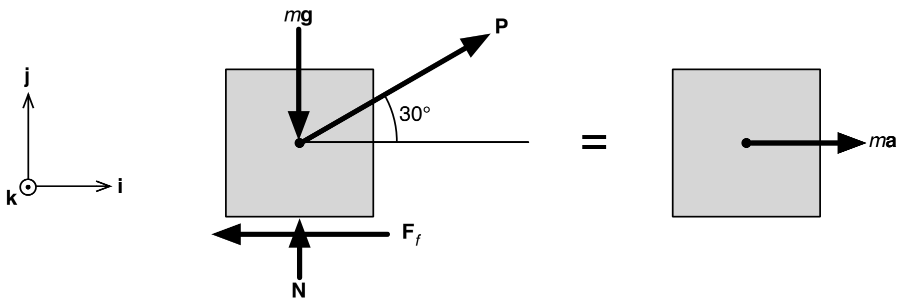


Figure 3.3: Particle Example — Free-Body Diagram

Equating the free-body diagram and its corresponding diagram (applying equation 3.1):

$$\sum \mathbf{F} = m\mathbf{a} = \mathbf{P} + m\mathbf{g} + \mathbf{N} + \mathbf{F}_f \tag{3.4}$$

We can see that every force in the free-body diagram lies either along the **i** or **j** axes 9
except the applied force, **P**. As such, in order to apply the equations in 3.2, we need to decompose **P** into its component parts, P_x and P_y:

$$\begin{aligned}
\mathbf{P} &= P_x\mathbf{i} + P_y\mathbf{j} \\
P_x &= |\mathbf{P}|\cos 30^\circ = 400\cos 30^\circ = 346.4\,\mathrm{N} \\
P_y &= |\mathbf{P}|\sin 30^\circ = 400\sin 30^\circ = 200\,\mathrm{N} \\
\therefore \mathbf{P} &= 346.4\mathbf{i} + 200\mathbf{j}\,\mathrm{N}
\end{aligned}$$

Decomposing 3.4 into component forms (equation 3.2) we have: 10

$$\sum F_x\mathbf{i} = ma_x\mathbf{i} = P_x\mathbf{i} - F_{f_x}\mathbf{i} \tag{3.5}$$

$$\sum F_y\mathbf{j} = ma_y\mathbf{j} = P_y\mathbf{j} - mg\mathbf{j} + N_y\mathbf{j} \tag{3.6}$$

Note the signs in equations 3.5 and 3.6 — the terms that are in the same direction as the axes are positive, and those that are in the opposite direction (for example, mg and F_{f_x}) are negative.

Note that the vectors $m\mathbf{g}$ and $\mathbf{N}$ have no horizontal component, therefore do not feature in equation 3.5, and correspondingly, the friction force $\mathbf{F}_f$ has no vertical component, therefore does not feature in equation 3.6.

The sharp-eyed among you will have noticed that there are three unknowns but only two equations. In equation 3.5, we know P_x but we do not know F_{f_x} nor a_x, and in equation 3.6, we know P_y, but not N_y. We, do however, have another equation relating two of our unknowns: 11

$$F_{f_x} = \mu_k N_y \tag{3.7}$$

12 Since the vertical motion of the crate is zero, we can say that $a_y = 0$, so we can solve equation 3.6:

$$\begin{aligned} 0 &= P_y\mathbf{j} - mg\mathbf{j} + N_y\mathbf{j} \\ N_y\mathbf{j} &= mg\mathbf{j} - P_y\mathbf{j} = [(50)(9.81) - 200]\,\mathbf{j} = 290.5\mathbf{j}\,\mathrm{N} \end{aligned}$$

13 Applying equation 3.7:

$$F_{f_x} = \mu_k N_y = (0.3)(290.5) = 87.15\,\mathrm{N}$$

Plugging these into equation 3.5, we can determine the acceleration of the crate:

$$\begin{aligned} ma_x\mathbf{i} &= P_x\mathbf{i} - F_{f_x}\mathbf{i} \\ a_x\mathbf{i} &= \left[\frac{P_x - F_{f_x}}{m}\right]\mathbf{i} = \left[\frac{346.4 - 87.15}{50}\right] = 5.19\mathbf{i}\,\mathrm{m/s^2} \end{aligned}$$

So, the acceleration of the crate is **5.19 m/s²** to the right (along the **i** axis).

3.1.4 Another Example: VTOL Aircraft

The mass of a VTOL (vertical take off and landing) aircraft is 4000 kg. Its engines exert 14
a force $\mathbf{F}_F$ and the force due to air resistance is $\mathbf{F}_R$. Calculate the acceleration for $\mathbf{F}_F = 11000\mathbf{i} + 51000\mathbf{j}\,\mathrm{N}$; $\mathbf{F}_R = -1000\mathbf{i} - 1000\mathbf{j}\,\mathrm{N}$

3.2 Solving Problems: Rigid Body Motion

15 For rigid bodies, there were two equations of motion that we need to consider: one relating to translational motion (equation 2.4) and one relating to rotation (equation 2.5 or 2.6). These are reproduced below:

$$\sum \mathbf{F} = m\mathbf{a}_G \qquad \sum \mathbf{M}_G = I_G\alpha$$

These equations are used to determine unknown forces, moments and acceleration components acting on an object modelled as a rigid body subjected to an unbalanced system of forces and moments.

3.2.1 Rigid Body: Example

We again return to the example given last week.

16 This example also involves a 50 kg crate. A force of $P = 600$ N is applied to the crateas shown in Figure 3.4. The coefficient of kinetic friction, $\mu_k = 0.2$.

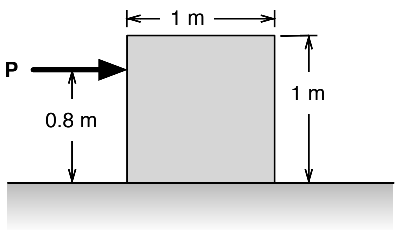


Figure 3.4: Rigid Body Example

The first step in doing this again to draw the free-body diagram of the object to identify all of the external forces and moments acting on it. This was done as part of an example in the last lecture, and should have resulted in the following drawing shown

17 in Figure 3.5.

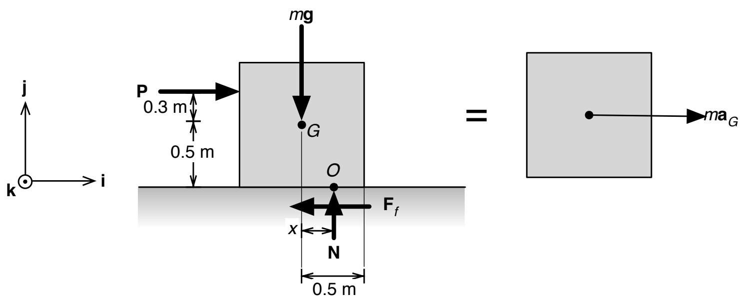


Figure 3.5: Rigid Body Example — Free-Body Diagram

18

Here, since the force $\mathbf{P}$ can cause the crate to either slide or top tip over, we model the crate as a rigid body. This model allows us to account for the effects of moments arising from $\mathbf{P}$ and any other external forces. We begin by *assuming that the crate slides.* Note that the normal force $\mathbf{N}$ acts at point O, a distance x (where $0 < x \leq 0.5\,\text{m}$) from the crate's centre line. The reason for this is that the line of action does not necessarily pass through the mass centre G ($x = 0$) since $\mathbf{N}$ must counteract the tendency for tipping caused by $\mathbf{P}$ about the mass centre, G.

To find the acceleration of the crate, we can apply the equation for translational motion and decompose in to the $\mathbf{i}$ and $\mathbf{j}$ components: 19

$$\begin{aligned}
\sum \mathbf{F} &= m\mathbf{a}_G = \mathbf{P} + m\mathbf{g} + \mathbf{N} + \mathbf{F}_f \\
\sum F_x\mathbf{i} &= ma_x\mathbf{i} = P_x\mathbf{i} - F_{f_x}\mathbf{i} \quad \rightarrow \quad ma_x\mathbf{i} = P_x\mathbf{i} - \mu_k N_y\mathbf{i} && (3.8) \\
\sum F_y\mathbf{j} &= ma_y\mathbf{j} = -mg\mathbf{j} + N_y\mathbf{j} && (3.9)
\end{aligned}$$

Again, there is no motion in the $\mathbf{j}$ direction, so $a_y = 0$, so we can calculate N_y from 20
equation 3.9:

$$0 = -mg\mathbf{j} + N_y\mathbf{j} \quad \rightarrow \quad N_y\mathbf{j} = mg\mathbf{j} = (50)(9.81)\mathbf{j} = 490.5\mathbf{j}\,\text{N}$$

Plugging this into equation 3.8, we can calculate the acceleration: 21

$$\begin{aligned}
ma_x\mathbf{i} &= P_x\mathbf{i} - \mu_k N_y\mathbf{i} \\
a_x\mathbf{i} &= \left[\frac{P_x - \mu_k N_y}{m}\right]\mathbf{i} = \left[\frac{600 - (0.2)(490.5)}{50}\right]\mathbf{i} = 10\mathbf{i}\,\text{m/s}^2
\end{aligned}$$

So the acceleration of the block is $\mathbf{10\,m/s^2}$.

The question is, are we correct to assume that the crate does not tip? To check this, 22
we should calculate x and if this is within the range $0 < x \leq 5\,\text{m}$, then this assumption is correct. To determine x, we should use the moment equation. In terms of the signs, we take all clock-wise moments are positive (you can use the opposite, as the answers will come out the same, but be consistent!). For a non-tipping crate, $\alpha = 0$. 23

$$\begin{aligned}
\circlearrowright \sum \mathbf{M}_G &= I_g\alpha = P_x(0.3\,\text{m}) + F_{f_x}(0.5\,\text{m}) - N_y(x) = 0 \\
N_y(x) &= P_x(0.3\,\text{m}) + \mu_k N_y(0.5\,\text{m}) \\
(490.5)x &= 600(0.3) + (0.2)(490.5)(0.5) \\
x &= \frac{600(0.3) + (0.2)(490.5)(0.5)}{490.5} = 0.467\,\text{m}
\end{aligned}$$

Since $0.467 < 0.5\,\text{m}$, we were correct to assume the crate does not tip.

3.2.2 Another Example: Bottle on a Conveyor Belt

24 A bottle is on a supermarket conveyor belt. The bottle has a mass of 1 kg and its radius is 5 cm. If the belt has an initial acceleration of $1\,\mathrm{m/s^2}$, does the bottle topple when the belt is turned on?

Summary

The main points in this section involve applying the equations of motion defined in section 2 to the free-body and kinetic diagrams also defined in section 2. The points to remember are:

- Remember to decompose the equations of translational motion (i.e. Newton's second law) in to **i** and **j** directions (or x and y directions). 25
- Make sure your signs make sense with your coordinate system.
- A friction force *opposes* motion.
- When computing moments, remember to identify your positive direction, and be consistent with it.
- Lastly, if you are asked to find velocities and position, use the standard constant acceleration equations (see the revision notes).

Exercises

1. A crane is lifting a crate of mass 20 kg with an upward acceleration of 6 m/s^2 by means of a single cable. What is the tension in the cable? *(Ans: 316 N)*
2. A caravan of mass 750 kg is towed along a level road by a car of mass 950 kg, the two bar being rigid. The car engine exerts a force, i.e. a tractive effort, of 2.5 kN, and there is no resistance to motion. Calculate the resulting acceleration of the car and caravan and the tensile force in the tow bar. *(Ans: 1.471 m/s^2; 1103 N)*
3. A man of mass 68 kg stands on the floor of a lift. If $\mathbf{j}$ is upwards calculate, stating any assumptions, the reaction force exerted by the floor on the man when:
   a) The lift accelerating upwards at a rate of 1.6 m/s^2.
   b) The lift is moving at a constant velocity of 2 m/s.
   c) The lift's upward velocity is decelerating at a rate of 1.8 m/s^2.
   d) The lift is accelerating downwards at 1.6 m/s^2.

   (Ans: 775.9$\boldsymbol{j}$ N; 667.1$\boldsymbol{j}$ N; 544.7$\boldsymbol{j}$ N; 558.3$\boldsymbol{j}$ N)
4. The mass of a VTOL aircraft is 4000 kg. Its engines exert a force $\mathbf{F}_F$ for t seconds. The force due to air resistance is $\mathbf{F}_R$. Calculate the change in velocity of the aircraft for:
   a) $\mathbf{F}_F = 62000\mathbf{j}\,\mathrm{N}; \mathbf{F}_R = -2000\mathbf{j}\,\mathrm{N}; \mathbf{t} = \mathbf{2}\,\mathrm{s}.$
   b) $\mathbf{F}_F = 11000\mathbf{i} + 51000\mathbf{j}\,\mathrm{N}; \mathbf{F}_R = -1000\mathbf{i} - 1000\mathbf{j}\,\mathrm{N}; \mathbf{t} = \mathbf{3}\,\mathrm{s}.$
   c) $\mathbf{F}_F = 7500\mathbf{i} + 36000\mathbf{j}\,\mathrm{N}; \mathbf{F}_R = -500\mathbf{i} - 1000\mathbf{j}\,\mathrm{N}; \mathbf{t} = \mathbf{2}\,\mathrm{s}.$

   (Assume right-handed coordinates, with $\mathbf{i}$ horizontal and $\mathbf{j}$ upwards.) *(Ans: 10.38$\boldsymbol{j}$ m/s; 7.5$\boldsymbol{i}$ + 8.1$\boldsymbol{j}$ m/s; 3.5$\boldsymbol{i}$ + 2.1$\boldsymbol{j}$ m/s)*
5. An aircraft has a mass of 5000 kg and experiences the following forces whilst in flight:

Horizontal thrust $\mathbf{F}_T$:	6×10^3 N	Vertical lift $\mathbf{F}_L$:	54×10^3 N
Horizontal drag $\mathbf{F}_{DH}$:	1.5×10^3 N	Vertical drag $\mathbf{F}_{DV}$:	0.5×10^3 N

   Assuming right-handed coordinates, with $\mathbf{i}$ horizontal and $\mathbf{j}$ upwards, calculate for the aircraft:
   a) Its acceleration in **i-j** component form.
   b) Its acceleration as a magnitude and angle from the horizontal.
   c) Its change in velocity after 2 seconds in **i-j** component form.

   (Ans: 0.9$\boldsymbol{i}$ + 0.89$\boldsymbol{j}$ m/s^2; 1.266 m/s^2 at 44.7°; 1.8$\boldsymbol{i}$ + 1.78$\boldsymbol{j}$ m/s)
6. An aircraft has a mass of 3000 kg and experiences the following forces whilst in flight:

Horizontal thrust $\mathbf{F}_T$:	4×10^3 N	Vertical lift $\mathbf{F}_L$:	30×10^3 N
Horizontal drag $\mathbf{F}_{DH}$:	1.25×10^3 N	Vertical drag $\mathbf{F}_{DV}$:	0.4×10^3 N

   Assuming right-handed coordinates, with $\mathbf{i}$ horizontal and $\mathbf{j}$ upwards, calculate for the aircraft:
   a) Its acceleration in **i-j** component form.
   b) Its acceleration as a magnitude and angle from the horizontal.
   c) Its change in velocity after 2 seconds in **i-j** component form.

*(Ans: 0.917**i** + 0.0567**j** m/s²; 0.9188 m/s² at 3.54°; 1.834**i** + 0.1134**j** m/s)*

7. An aircraft has a mass of 8800 kg and experiences the following forces whilst in flight:

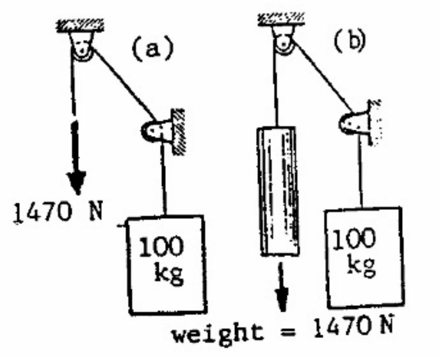
Horizontal thrust $\mathbf{F}_T$:	9.9×10^3 N	Vertical lift $\mathbf{F}_L$:	89.1×10^3 N
Horizontal drag $\mathbf{F}_{DH}$:	2.475×10^3 N	Vertical drag $\mathbf{F}_{DV}$:	1.2×10^3 N

Assuming right-handed coordinates, with **i** horizontal and **j** upwards, calculate for the aircraft:

a) Its acceleration in **i-j** component form.
b) Its acceleration as a magnitude and angle from the horizontal.
c) Its change in velocity after 2 seconds in **i-j** component form.

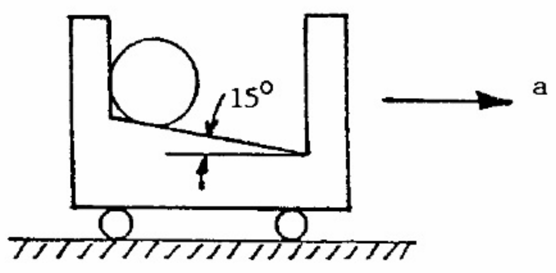
*(Ans: 0.844**i** + 0.178**j** m/s²; 0.863 m/s² at 11.9°; 8.44**i** + 1.78**j** m/s)*

8. Determine the magnitude of the acceleration of the 100 kg mass for each of the cases illustrated. The mass and friction of the pulleys are negligible. Hint: Take care to draw accurate free-body diagrams.



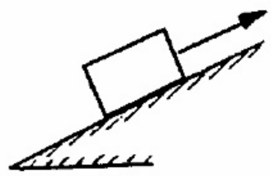
(Ans: 4.89 m/s²; 1.96 m/s²)

9. The frame shown is given a steady horizontal acceleration $a = 2g$. Determine the magnitude of the reaction force between the sphere, which weighs 10 N, and the vertical surface.



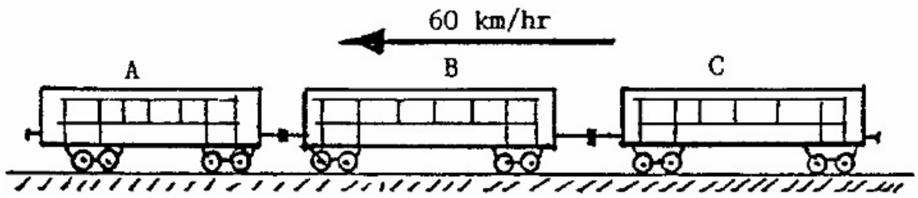
(Ans: 17.32 N)

10. The crate shown has a mass of 15 kg and is being pulled up a plane at 30° by a cord parallel to the plane. Calculate the magnitude of the tension necessary to give the crate an acceleration parallel to the plane of $5\,\mathrm{m/s^2}$. The coefficient of friction is 0.3.



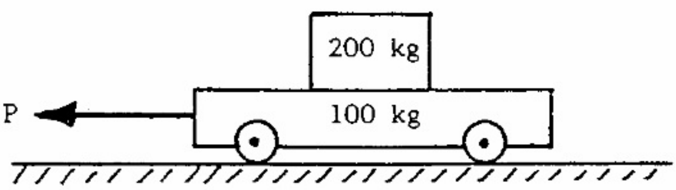
(Ans: 186.7 N)

11. The electric train shown consists of three coaches A, B and C of mass 35 t, 45 t and 35 t respectively. It is travelling at 60 km/h when the brakes are applied to coaches A and B, giving a braking force on each of these coaches of 25 kN, but no braking force on C. Calculate the magnitude of the force in each coupling.



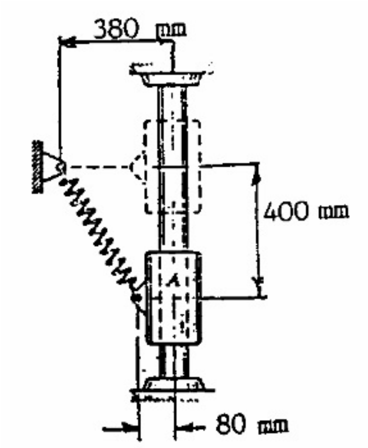
(Ans: 9.78 kN and 15.21 kN compression)

12. A 200 kg crate rests on a 100 kg cart; the coefficient of friction between the crate and cart is 0.25. If the crate is not to slip with respect to the cart, calculate:
    a) The maximum allowable magnitude of the force P which may be applied to the cart.
    b) The corresponding acceleration of the cart.



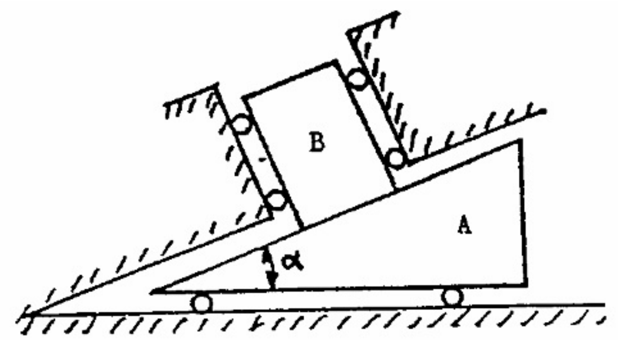
(Ans: 735 N, 2.45 m/s^2)

13. The collar A has a mass of 10 kg and slides on a vertical shaft. The spring is uncompressed when the collar is in the dotted position. Determine the initial acceleration magnitude of the collar when it is released from rest in the position illustrated. The coefficient of friction between the collar and the shaft is 0.2, and the stiffness of the spring is 2500 N/m. Note: the pull force provided by the spring is $F = kx$, where x is the extension from the spring's natural length and k is the stiffness of the spring.



(Ans: 24.2 m/s^2)

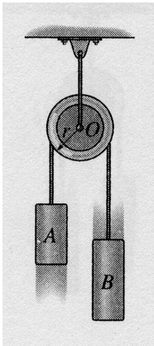
14. The wedge A is free slide without vertical friction on the fixed horizontal surface, and friction may also be neglected between the plunger B and its guides, which constrain the plunger to slide at right angles to the face of the wedge. Between the plunger and the wedge the coefficient of friction is 0.3. The mass of the wedge is 40 kg and that of the plunger 60 kg. The system is released from rest. Determine the magnitude of the acceleration of the wedge if the wedge angle is 20 degrees. Hint: the plunger is accelerating as well as the wedge.



(Ans: 2.59 m/s^2)

15. The two blocks A and B have mass m_A and m_B respectively, where $m_B > m_A$. The pulley can be treated as a disk of mass M. Regarding the pulley and two blocks as a single system, formulate the appropriate equation of motion which, which when solved, will lead to the acceleration of block A. Neglect the mass of the cord and any slipping on the pulley. (Hint: we want an equation made up the variables given, not a numerical answer). Note that the moment of inertia of a solid pulley is:

$$I_G = \frac{1}{2}Mr^2$$



(Ans: $a = \dfrac{\alpha}{r} = \dfrac{g(m_B - m_A)}{\left(\frac{1}{2}M + m_B + m_A\right)}$), direction $\uparrow$.

4 Non-Uniform Acceleration

So far, we have only dealt with constant acceleration problems. What about when accel- 2
eration varies? A simple example would be a car accelerating. The aerodynamic resistive 3
forces created by an object moving through air is given by the following equation:

$$\mathbf{F}_{aero} = \frac{1}{2}\rho C_d A V^2$$

where:

- ρ is the density of air (a constant)
- C_d the drag coefficient (a constant)
- A the frontal area of the car (a constant)
- V the vehicle's velocity (not a constant)

As you can see, the aerodynamic force is proportional with the *square* of the speed, 4
so as the speed rises, the force resisting motion increases with the *square* of velocity. With a constant tractive force, $\mathbf{F}_t$ from the engine, the acceleration will therefore be non-constant, and Newton's second law will have the form (in the horizontal direction):

$$ma_x = F_{t_x} - F_{aero,x} = F_{t_x} - BV_x^2$$

where $B = \frac{1}{2}\rho C_d A$. The acceleration a is clearly non-constant.

Note: For the following analysis, we make the assumption that the motion of a particle is along a straight line. This means that the motion is one-dimensional, so for these sections, we deal with the scalar quantities of displacement, velocity and acceleration.

4.1 Expressions For Velocity and Acceleration

4.1.1 Instantaneous Velocity

If we use x as the symbol for displacement, then the *instantaneous velocity* is the rate 5
of change of displacement with respect to time:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}$$

using dot notation, as reviewed in the revision notes.

Now that we have this relationship, we can determine that the *change in displacement* is:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t} \quad \rightarrow \quad \int \mathrm{d}x = \int v \,\mathrm{d}t \quad \rightarrow \quad x_2 - x_1 = \int v \,\mathrm{d}t$$

4.1.2 Instantaneous Acceleration

6 Similarly, the*instantaneous acceleration* is the rate of change of velocity with respect to time:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \dot{v}$$

This can be continued:

$$a = \dot{v} = \frac{\mathrm{d}}{\mathrm{d}t}v = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}$$

7 Similarly, this relationship reveals the *change in velocity*:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \rightarrow \quad \int \mathrm{d}v = \int a\,\mathrm{d}t \quad \rightarrow \quad v_2 - v_1 = \int a\,\mathrm{d}t$$

The other relationship we can determine to do with acceleration andvelocity is the following, requiring a little bit of algebraic manipulation:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

4.2 Graphical and Numerical Methods

4.2.1 Plotting Velocity against Time

8 Figure 4.1 shows a velocity-time profile of a particle undergoing non-uniform acceleration in a straight line.

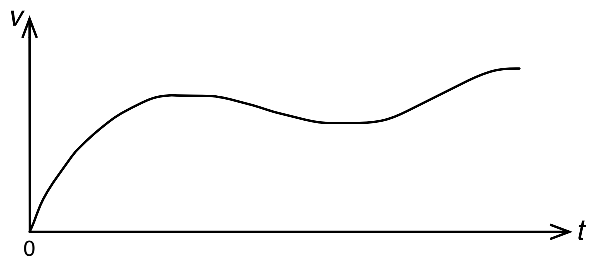


Figure 4.1: Particle undergoing non-uniform acceleration

As can be seen, the slope of the curve ($\frac{\mathrm{d}v}{\mathrm{d}t}$) varies with time, indicating that the acceleration is non-constant. To determine the instantaneous acceleration at any point along this line would involve calculating the derivative of the function that represents this line. However, for such a case, it would be difficult or impossible to fit a single analytical function to the curve shown in Figure 4.1.

Mathematical differentiation of such a velocity-time graph (to determine acceleration) is defined as:

$$\lim_{\delta t \to 0} \frac{\text{change in } v}{\text{change in } t} = \lim_{\delta t \to 0} \frac{\delta v}{\delta t}$$

Thus, differentiation involves consideration of changes that take place in an time interval that approaches zero.

Since we do not have a function, an *approximation* of the derivative can be made 9
by a simple graphical or numerical method that involves splitting the curve in to small sections of time (say 1 second, for example), and assuming that the line between each time step is a straight, representingconstant acceleration. The curve shown in Figure 4.1 is converted into the Figure 4.2, with each section during one time step a straight line. (This is the method computers and calculators often use to evaluate functions, although the splitting is often a much greater resolution, providing a more accurate answer. Your calculator may have a button indicated [$\int$ dx] which can perform this function.)

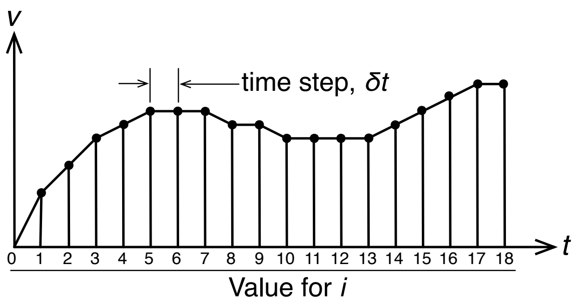


Figure 4.2: Particle undergoing non-uniform acceleration split into discrete time steps

The rate of acceleration is then assumed to remain constant at its average value during 10
the time step δt, where

$$\delta t = t_{i-1} - t_i$$

Hence the constant acceleration formulae may be applied within each time step δt.

4.2.2 Acceleration

Since the acceleration is constant for each time step, δt, the *approximate* value for 11
acceleration between t_{i-1} and t_i is the change in velocity, $v_i - v_{i-1}$ divided by the time step:

$$a_i = \frac{\text{change in velocity}}{\text{time step}} = \frac{v_i - v_{i-1}}{\delta t}$$

Note that the average velocity or acceleration values are the actual values occurring half-way through the time interval when linear variation of velocity or acceleration with time are being considered.

4.2.3 Displacement

The approximate *additional* displacement δx_i occurring between t_{i-1} and t_i is given by the average velocity between t_{i-1} and t_i multiplied by the time step, δt. (This will result in the area of each of the segments, and the sum of the area will be the total displacement.)

$$\delta x_i = \text{average velocity} \times \text{timestep} = \frac{v_i + v_{i-1}}{2}\delta t$$

The approximate overall displacement is simply the sum of the additional displacement for each time step. Hence:

$$\text{Approximate overall displacement} x = \sum_{i=1}^{n} \delta x_i$$

Remember, since the integral of acceleration is velocity, the area for each segment under a acceleration time graph will be the velocity value for that segment. Similarly, the area under an the complete velocity-time graph will be the total displacement.

4.2.4 Example

As an example, let us assume that we wish to find the acceleration, a, the average velocity v and the additional displacement $x_2 - x_1$ for the second time step, where $i = 2$.

Time time step $\delta t = t_2 - t_1$

The rate of acceleration is assumed to remain constant at its average value during the time step.

13 The average acceleration

$$a = \frac{\text{change in velocity}}{\text{change in time}} = \frac{v_2 - v_1}{t_2 - t_1}$$

The average velocity:

$$v_{av} = \frac{v_1 + v_2}{2} \left[= \frac{\text{displacement}}{\text{time interval}} = \frac{x_2 - x_1}{t_2 - t_1} \right]$$

The additional displacement:

$$(x_2 - x_1) = \text{average velocity} \times \text{time} = \frac{1}{2}(v_1 + v_2)(t_2 - t_1)$$

4.3 Solution by Integration

4.3.1 Differentiation and Integration

Conversely to the above section, if you have knowledge of the function of acceleration 14
versus time or velocity versus time, we can use differentiation and integrationto determine the necessary information about the motion of a particle.

Remember that:

$$\begin{aligned} x &= \text{displacement} = \int v \, \mathrm{d}t \\ v &= \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \qquad = \int a \, \mathrm{d}t \\ a &= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \ddot{x} \end{aligned}$$

Differentiation and integration tables can be found in the revision notes.

4.3.2 Functions of Acceleration

This section uses the equations detailed above to develop the various forms in which problems could be presented. Here, we say $x = x_0$, $v = v_0$ and $t = 0$ designated at the beginning of the interval.

In each of the following cases when the acceleration varies according to some functional relationship, the ability to solve the equations by direct mathematical integration will depend on the form of the function. In cases where the integration is excessively awkward or difficult, integration by graphical or numerical methods may be used. Alternatively, if the problem does not prohibit it, a computer based method may be used to evaluate the function.

Acceleration as a function of time

Here:

$$a = f(t)$$

Therefore, we can say:

$$a = f(t) = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \rightarrow \quad \int_0^t f(t) \, \mathrm{d}t = \int_{v_0}^{v} \mathrm{d}v$$

Integrating results in:

$$v - v_0 = \int_0^t f(t) \, \mathrm{d}t$$

which gives us an equation of v as a function of time, t.

We can then do the following to get position:

$$v = \frac{dx}{dt} \quad \rightarrow \quad \int_0^t v \, \mathrm{d}t = \int_{x_0}^x \mathrm{d}x$$

Integrating:

$$x - x_0 = \int_0^t v \, \mathrm{d}t$$

Acceleration as a function of velocity

Here:

$$a = f(v)$$

Therefore, we can say:

$$f(v) = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \rightarrow \quad \int_{v_0}^v \frac{1}{f(v)} \, \mathrm{d}v = \int_0^t \mathrm{d}t = t$$

which gives t as a function of v. Rearranging to solve for v as a function of t, we can then use:

$$v = \frac{\mathrm{d}x}{\mathrm{d}t}$$

to determine x.

Alternatively, we know that:

$$a = v\frac{\mathrm{d}v}{\mathrm{d}x} = f(v) \quad \rightarrow \quad \int_{v_0}^v \frac{v}{f(v)} \, \mathrm{d}v = \int_{x_0}^x \mathrm{d}x$$

Integrating:

$$x - x_0 = \int_{v_0}^v \frac{v}{f(v)} \, \mathrm{d}v$$

which is also a function of x in terms of v, without explicit reference to time, t.

Acceleration as a function of displacement

Here:

$$a = f(x)$$

We know that:

$$a = v\frac{\mathrm{d}v}{\mathrm{d}x} = f(t) \quad \rightarrow \quad \int_{x_0}^x f(x) \, \mathrm{d}x = \int_{v_0}^v v \, \mathrm{d}v$$

Integrating:

$$v^2 - v_0^2 = 2\int_{x_0}^x f(x) \, \mathrm{d}x$$

which gives us v as a function of x. If we call this function $g(x)$. We can then use the definition of velocity to gain an equation for time, t:

$$v = \frac{dx}{dt} = g(x) \quad \rightarrow \quad \int_{x_0}^x \frac{1}{g(x)} \, \mathrm{d}x = \int_0^t \mathrm{d}t = t$$

4.3.3 Example

The linear acceleration a of an electric vehicle starting from rest is given by $a = 2 - 0.1t$, 15
for $t < 5\,\mathrm{s}$. Find the time taken for the vehicle to reach $5\,\mathrm{m/s}$ and the distance travelled in that time.

The acceleration is not constant so the constant acceleration formulae cannot be used.

Summary

Solution by Graphical and Numerical Methods

- Split continuous graph in small sections 16
- Acceleration = constant for each section
- Apply standard formulae for each section

Remember, since the integral of acceleration is velocity, the area for each segment under a acceleration time graph will be the velocity value for that segment. Similarly, the area under an the complete velocity-time graph will be the total displacement.

Solution by Integration

17

- Suitable if you have knowledge of function
- If you have a as a function of t: integrate to get v. Integrate again to get x.
- If you have a as a function of v: integrate to get t as a function of v. Rearrange to get v as a function of t, then integrate to get x.
- If you have a as a function of x: Integrate to gain v as a function of x, then use the definition of velocity to gain t.
- If you have v as a function of t: differentiate to get a.

Exercises

Graphical and Numerical Methods

1. The following table gives the speed of a train at intervals of 1 minute.

t (min)	0	1	2	3	4	5	6	7	8	9	10
v (m/s)	0	5	15	20	30	33	33	27	20	7	0

   Sketch the graph of velocity against time and hence determine the distance travelled and the average speed.
   (Ans: 11400 m, 19 m/s)
2. A car starts from rest with an acceleration that increases uniformly from zero to $3\,\mathrm{m/s^2}$ over a time period of 10 s. The acceleration of the car remains constant at $3\,\mathrm{m/s^2}$ for 7 s. The acceleration then reduces to zero instantaneously, after which it decreases uniformly from zero to $-6\,\mathrm{m/s^2}$ in a time period of 12 s.
   Draw the graph of acceleration against time, and hence construct the graphs of velocity against time and displacement against time. From the grpahs, determine:
   a) The maximum velocity.
   b) The total distance travelled.
   (Ans: 36 m/s; 515.5 m)
3. A motorcycle accelerates in a straight line from rest for a time period of 18 s, the acceleration varying with time as shown in the table, which gives values of its acceleration every 2 s.
   Using numerical integration, complete the table, and hence determine for the motorcycle at the end of the period of acceleration:
   a) its approximate velocity.
   b) its approximate total displacement from its starting point.

Time (s)	Acceleration ($\mathrm{m/s^2}$)	Velocity (m/s)	Displacement (m)
0	5.4	0	0
2	4.4		
4	3.5		
6	2.7		
8	2.0		
10	1.4		
12	0.9		
14	0.5		
16	0.2		
18	0		

   (Ans: 36.6 m/s; 473.4 m)

4. A car is raced from a standing start in a straight line. Its velocity is recorded every second. Using *numerical* differentiation and integration, complete the table and hence determine its displacement at $t = 10\,\mathrm{s}$ and its acceleration at $t = 9.5\,\mathrm{m/s}$.

Time (s)	Displacement (m)	Velocity (m/s)	Acceleration (m/s^2)
0	0	0	—
	—		4.5
1		4.5	—
	—		
2		8.75	—
	—		
3		12.75	—
	—		
4		16.50	—
	—		
5		20.00	—
	—		
6		23.25	—
	—		
7		26.25	—
	—		
8		29.00	—
	—		
9		31.50	—
	—		
10		33.75	—

(Ans: $189.375\,\mathrm{m}$; $2.25\,\mathrm{m/s^2}$)

5. The displacement of a moving body was recorded on each frame of a motion picture film, using a cine-camera with a shutter speed of 20 frames/s. The displacements from the initial body position were determined from each successive frame and are given in the table below.
   Using numerical differentiation, complete the table and hence determine:
   a) The approximate velocity when the displacement is 2.925 m.
   b) The approximate acceleration when the displacement is 2.425 m.

Frame	Time (s)	Displacement (m)	Velocity (m/s)	Acceleration (m/s^2)
1	0	0	—	—
				—
2		0.125	—	
				—
3		0.375	—	
				—
4		0.925	—	
				—
5		1.575	—	
				—
6		2.425	—	
				—
7		3.425	—	—

   (Ans: 20 m/s; 60 m/s^2)

Integration Method

6. A vehicle starts from rest with a linear acceleration given by $a = 4 - 0.5t$, where a is in m/s^2 and t is the time elapsed from the start in seconds. Calculate, using integration for parts (c) and (d):
   a) the initial acceleration.
   b) the time taken for the acceleration to reduce to zero.
   c) the distance travelled up to the instant when the acceleration is zero.
   d) the time which elapses before the vehicle comes to rest again.
   (Ans: (a) 4 m/s^2; (b) 8 s; (c) 85.3 m; (d) 16 s)
7. The linear acceleration of a body is given by $a = 10 - 3(t)^{\frac{1}{2}}$, where a is in m/s^2 and t is the time elapsed from the start in seconds. If the body is initially at rest, calculate by integration:
   a) the time which elapses before the body comes to rest again.
   b) the distance travelled in this time.
   (Ans: (a) 25 s; (b) 625 m)
8. A motor car starts from rest with a linear acceleration given by $a = 4 - 0.04x$, where a is in m/s^2 and x is the distance travelled from the start in metres. Calculate, using integration for part (c):

a) the initial acceleration.
b) the distance travelled by the car from the start to the point at which its acceleration is zero.
c) the velocity at the instant when the acceleration is zero. (Use $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$)

(Ans: (a) 4 m/s^2; (b) 100 m; (c) 20 m/s)

9. An aircraft lands on a straight runway with a touchdown speed of 50m/s. The brakes are then applied causing a deceleration which is proportional to the velocity of the plane, given by a = -0.03v, where a is in m/s2 and v is in m/s.
Calculate by integration:
   a) the time required for the plane to reduce its velocity to 5m/s; Use $a = dv/dt$, and note that $\int \frac{1}{v}\,\mathrm{d}v = \ln v$. [76.75s]
   b) the distance travelled during that time period. (Use $a = v\frac{\mathrm{d}v}{\mathrm{d}x}$) [1500m]

   (Ans: (a) 76.75 s; (b) 1500 m)
10. It is proposed that an electric vehicle is to have a regenerative braking system which when applied gives the vehicle a velocity of:

$$v = ue^{\frac{-t}{\tau}}$$

where u is the velocity before braking starts, t is the time and τ is a time constant for the system. Obtain an expression for the displacement x of the vehicle as a function of the time, and sketch the graph of displacement against time. Briefly discuss the practical implications of such a braking system.
Note! $\int e^{kt}\,\mathrm{d}t = \frac{e^{kt}}{k} + C$
(Ans: $x = u\tau(1 - e^{\frac{-1}{\tau}})$)

5 Momentum and Force Impulse

5.1 Momentum 2

5.1.1 Definition

Momentum is a measure of the quantity of motion possessed by a body, and is defined 3
as $\mathbf{p}$:

$$\mathbf{p} = m\mathbf{v} \tag{5.1}$$

Momentum is a vector, having both magnitude and direction, and its units are [kgm/s].

There is a relationship between Newton's Second Law and momentum. Newton's 4
Second Law states that:

$$\mathbf{F} = m\mathbf{a}$$

With some substitution, we can see that **the force is the rate of change of momentum**:

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}m\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{5.2}$$

This means that:

- if particle changes its momentum, a force must have acted upon it
- if a force acts on a particle, it will change its momentum

From this relationship, we can see that:

$$\int \mathbf{F}\,\mathrm{d}t = \int \mathrm{d}\mathbf{p} \quad \rightarrow \quad \mathbf{F}t = \mathbf{p}$$

which means that the units of momentum could also be the units of force (N) times the units of time (s), so you may see the units of momentum expressed as [Ns]. [Ns] is equivalent to [kgm/s].

5.1.2 Internal and External Forces

Imagine we have large number of particles which are interacting with each other. The 5
interaction could be gravitational, or electrical, it doesn't matter. Such a system could be a star cluster. These are shown in Figure 5.1.

We pick two of these arbitrarily, and call them m_i and m_j. These are exposed to the 6
external forces acting on the system, $\mathbf{F}_{i,ext}$ and $\mathbf{F}_{j,ext}$, as shown in Figure 5.2(a).

7

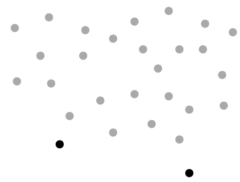
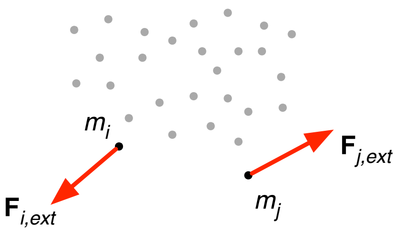


Figure 5.1: A cluster of particles



(a) (b)

Figure 5.2: A cluster of particles — External (a) and Internal (b) forces

But as we stated, they are interacting with each other, either attracting or repelling, so in addition to these external forces, there are internal forcesbetween the two particles, $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$, where

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

as they are equal and opposite, following Newton's Third Law. These are shown in Figure 5.2(b).

8 If these were the only two forces, then the net forces on i would be $\mathbf{F}_{i,net}$ and the net forces on j will be $\mathbf{F}_{j,net}$, as shown in Figure 5.3. Obviously, since there are many particles, and there are interactions between them all, these net forces will be different that those indicated in the diagram, but they would still be the sum of the external forces and all the internal forces.

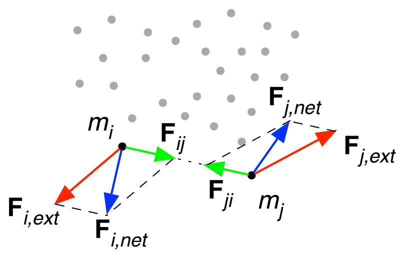


Figure 5.3: A cluster of particles — Net forces

9

What is the total momentum of these particles? This is the sum of the individual momenta.

$$\mathbf{p}_{tot} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_i + \cdots$$

Taking the derivative of this:

$$\frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = \mathbf{F}_{1,net} + \mathbf{F}_{2,net} + \cdots + \mathbf{F}_{i,net} + \cdots = \mathbf{F}_{tot}$$

where $\mathbf{F}_{tot}$ is the total force on the entire system.

Now comes the special part: due to Newton's third law, all these internal forces cancel 10
each other out: for example F_{ji} cancels out F_{ij} if you look at the *system as a whole*, and the same goes for all the other interactions that are happening between all the individual particles.

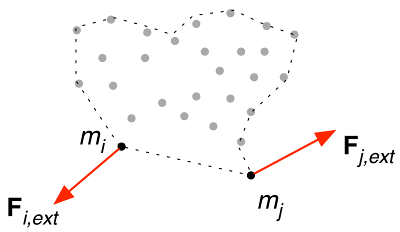


Figure 5.4: A cluster of particles — Total forces

So the total force on the system is simply the total external force.

$$\frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = \mathbf{F}_{tot} = \mathbf{F}_{tot,ext}$$

This results in a key conclusion: if the total external forces of the system as a whole 11
are zero, then *the momentum of the system cannot change*, and is therefore *conserved*. This is known as the ***conservation of momentum.*** If $\mathbf{F}_{tot,ext} = 0$

$$\frac{\mathrm{d}\mathbf{p}_{tot}}{\mathrm{d}t} = 0$$

It doesn't matter how many particles you have, and they could collide with each other, or explode, but it does not matter — these are all internal forces, and if they are not affected by external forces, the momentum of the entire system does not change. **If the net sum of the total external forces is zero, momentum is conserved.**

The principle of conservation of momentum is usefully demonstrated when studying collisions.

5.2 Conservation of Momentum

12 We study three types of collisions in dynamics: inelastic collisions (where colliding objects stick together), elastic collisions (where colliding objects bounce apart), and superelastic collisions. The difference between these different types of collisions is explained by an understanding of the change in kinetic energy in a system, so this will be discussed first.

5.2.1 Kinetic Energy

13 Taking Newton's second law in one dimension results in:

$$\mathbf{F} = m\mathbf{a} \quad \rightarrow \quad F_x = ma_x$$

The discussion of instantaneous acceleration in chapter 4 resulted in the following equation for acceleration (again in one dimension)

$$a_x = v\frac{\mathrm{d}v_x}{\mathrm{d}x}$$

Substituting this into Newton's second law, rearranging and integrating results in the following:

$$\begin{aligned} F_x &= mv\frac{\mathrm{d}v}{\mathrm{d}x} \\ \int F_x \,\mathrm{d}x &= m\int v \,\mathrm{d}v = \frac{1}{2}mv_{x1}^2 - \frac{1}{2}mv_{x0}^2 \end{aligned}$$

14 Similar expressions can be derived for the other two dimensions, and in the general vector form:

$$\text{Change in kinetic energy} = \frac{1}{2}m\mathbf{v}_1^2 - \frac{1}{2}m\mathbf{v}_0^2 \tag{5.3}$$

Energy is often expressed with the term U, and kinetic energy has the subscript k. Hence, kinetic energy is U_k. The units of energy are [kgm^2/s^2], which are more commonly known as Joules (J).

15 Energy will be discussed in more detail in the next lecture, but we accept for now that there are many forms of energy; kinetic energy (U_k) is one form, gravitational potential energy (U_p) is another, heat (or internal energy) yet another, and all energy follows the law of conservation of energy, which, neglecting any work given or taken from a system follows the energy balance equation:

$$\text{energy in state 1} = \text{energy in state 2} \tag{5.4}$$

If we put all the non-kinetic energies together, we can rewrite the energy balance equation:

$$U_{k1} + Q = U_{k2} \tag{5.5}$$

where Q represents the change in the other forms of energy and determines the type of collision undertaken:

- If $Q < 0$, kinetic energy has been lost in the system (normally as heat) and results in a completely **inelastic collision** 16
- If $Q = 0$, kinetic energy is conserved, and results in a completely **elastic collision**
- If $Q > 0$, kinetic energy has been gained in the system (as a result of an explosion, or a spring, for example), and results in a **superelastic collision**

We will discuss each of these individually.

5.2.2 Collisions

Inelastic Collisions

Consider two bodies having masses m_1 and m_2 as shown in Figure 5.5, with m_1 travelling at initial velocity u_1, and m_2 travelling at initial velocity u_2. 17

Figure 5.5: Two masses

We are dealing with an inelastic collision so we imagine that these masses are covered with glue, such that when colliding, they stick together, resulting in an inelastic collision as shown in Figure 5.6. 18

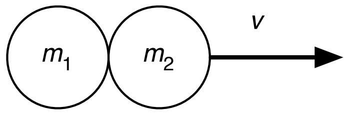


Figure 5.6: Two masses — Inelastic collision

Just before the impact, the masses had the momentum (dealing in one dimension, we only need to deal with the scalar values of velocity and momentum): 19

$$p_{\text{before}} = m_1 u_1 + m_2 u_2$$

Just after the impact, the momentum has changed to:

$$p_{\text{after}} = (m_1 + m_2)v$$

Conservation of momentum indicates that, in the absence of external forces: 20

$$\begin{aligned} p_{\text{before}} &= p_{\text{after}} \\ m_1 v_1 + m_2 v_2 &= (m_1 + m_2) v \end{aligned} \tag{5.6}$$

So, if we know the initial condition and the masses, then v can be found from equation 5.6:

$$m_1 u_1 + m_2 u_2 = (m_1 + m_2) v \quad \rightarrow \quad v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \tag{5.7}$$

21 Taking the following values:

- $m_1 = 1\,\text{kg}$, $u_1 = 5\,\text{m/s}$
- $m_2 = 2\,\text{kg}$, $u_2 = 3\,\text{m/s}$

we can calculate v from equation 5.7:

$$v = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} = \frac{(1)(5) + (2)(3)}{(1) + (2)} = 3.67\,\text{m/s}$$

Now let us compare the kinetic energy at the beginning and the kinetic energy at the end:

22

$$\begin{aligned} U_{k,\text{before}} &= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2}(1)(5)^2 + \frac{1}{2}(2)(3)^2 = 21.5\,\text{J} \\ U_{k,\text{after}} &= \frac{1}{2}(m_1 + m_2) v^2 = \frac{1}{2}(3)(3.67)^2 = 20.2\,\text{J} \end{aligned}$$

As you can see, some energy was lost in the collision. Although the actual amount of energy lost was small, it is possible to set up an inelastic collision in which all kinetic energy is lost: imagine two equal masses, travelling towards each other at equal but opposite velocities. The total momentum of the system before the collision is zero (remember, momentum is a vector, so the opposing velocities cancel each other out), but the system does have kinetic energy. After the collision, the masses stick together, but have zero velocity, meaning zero kinetic energy.

So **energy is always lost in an inelastic collision**.

Elastic Collisions

23 In a completely elastic collision, the masses do not stick together, but ‘bounce’ off each other. A completely perfect elastic collision *does* conserve kinetic energy as well as momentum.

Again, we take masses m_1 and m_2 as shown in Figure 5.7 travelling at initial velocities u_1 and u_2.

24 This time, when the masses collide, they do not stick together, and leave collision with velocities of v_1 and v_2 respectively as shown in Figure 5.8.

25 Just before the impact, the masses had the momentum:

$$p_{\text{before}} = m_1 u_1 + m_2 u_2$$

Figure 5.7: Two masses before an elastic collision

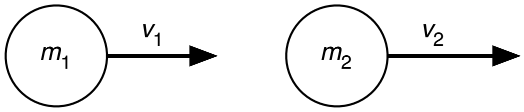


Figure 5.8: Two masses after an elastic collision

Just after the impact, the momentum is now:

$$p_{\text{after}} = m_1 v_1 + m_2 v_2$$

Conservation of momentum means that these two equations can be equated: 26

$$\begin{aligned} p_{\text{before}} &= p_{\text{after}} \\ m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \end{aligned} \tag{5.8}$$

Notice that we have two unknowns, v_1 and v_2, so if we are to solve this, we need another equation. As we know that kinetic energy is conserved in a perfectly elastic collision, we can say that:

$$\begin{aligned} U_{k,\text{before}} &= U_{k,\text{after}} \\ \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \end{aligned} \tag{5.9}$$

Now we have two equations (equation 5.8 and 5.9) to solve the two unknowns, v_1 and 27
v_2. Determining v_1 and v_2 requires quite a bit of longwinded and error-prone algebraic manipulation, which is not shown here, but to save you from calculating them yourselves, the results come out to be:

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_2 \tag{5.10}$$

$$v_2 = \frac{m_2 - m_1}{m_1 + m_2} u_2 + \frac{2m_1}{m_1 + m_2} u_1 \tag{5.11}$$

Taking the same initial conditions: 28

- $m_1 = 1\,\text{kg}$, $u_1 = 5\,\text{m/s}$
- $m_2 = 2\,\text{kg}$, $u_2 = 3\,\text{m/s}$

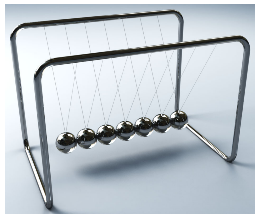
and using equations 5.10 and 5.11 v_1 and v_2 can be determined:

$$
\begin{aligned}
v_1 &= \frac{m_1 - m_2}{m_1 + m_2} u_1 + \frac{2m_2}{m_1 + m_2} u_2 \\
v_1 &= \frac{1-2}{3} 5 + \frac{4}{3} 3 = 2.33\,\mathrm{m/s} \\
v_2 &= \frac{m_2 - m_1}{m_1 + m_2} u_2 + \frac{2m_1}{m_1 + m_2} u_1 \\
v_2 &= \frac{2-1}{3} 3 + \frac{2}{3} 5 = 4.33\,\mathrm{m/s}
\end{aligned}
$$

Inserting these results into equation 5.8 confirms the conservation of momentum.

An example of such an collision are snooker balls. These collisions are very close to a perfectly elastic collision (there is a very slight loss of energy which we can hear as sound when the two balls collide).

Another example of the principles of conservation of momentum and energy in action can be seen with a Newton's cradle.



Superelastic Collisions

29 A superelastic collision is where the kinetic energy after the 'collision' is greater than the kinetic energy to start with. An example of this could be two masses, m_1 and m_2 that are directly next to each other to start with, and have no velocity, as shown in Figure 5.9(a).

(a) (b)

Figure 5.9: Superelastic collision

30 An explosive charge between them is set off (Figure 5.9(b)), at which point the masses move apart, with speeds v_1 and v_2, as shown in Figure 5.10.

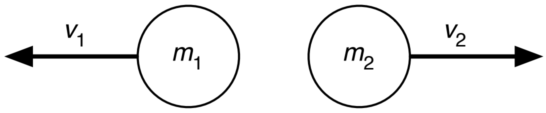


Figure 5.10: Superelastic collision

31

Again, the conservation of momentum applies:

$$\begin{aligned} p_{\text{before}} &= p_{\text{after}} \\ 0 &= m_1 v_1 - m_2 v_2 \quad \rightarrow \quad \frac{m_1}{m_2} = \frac{v_2}{v_1} \end{aligned} \tag{5.12}$$

(Notice the minus sign, as the masses are going in opposite directions.)

The only thing we can tell from equation 5.12 is the *ratio* of the velocities. The larger mass will have the lower speed.

The kinetic energy has clearly increased. Where has this kinetic energy come from? The answer is the potential chemical energy stored in the exlosive. Despite this, momentum has remained constant. This explosion is simply an *internal force*, and as we discussed earlier, momentum does not care about internal forces.

5.3 Force Impulse

As was commented on in Section 5.1.2, in the absence of any external forces on the system as whole, momentum is conserved. But what can be said about external forces that *are* applied to a system—what happens to the momentum and how can we quantify the change? For this, we use the term **Impulse**.

5.3.1 The Impulse of a Force

The impulse of a force is defined as the *product of the force and the time for which it acts*: 32

$$\text{Force Impulse} = \text{force} \times \text{time}$$

It is clear that small forces applied for a long time have the same *impulse* as large forces applied briefly.

It is quite common for the applied force to vary with respect to time, so force impulse is the integral of the force with respect to time:

$$\text{Force Impulse} = \int \mathbf{F}\, \mathrm{d}t = \int \mathrm{d}\mathbf{p} = \int \mathrm{d}m\mathbf{v} \tag{5.13}$$

which we can say because:

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

In certain cases, the mass can vary with respect to time; a particular example is rocket that uses up fuel as it flies. On the other hand, if we assume that the mass remains constant, then equation 5.13 becomes:

$$\text{Force Impulse} = \int \mathbf{F}\, \mathrm{d}t = m \int \mathrm{d}\mathbf{v} = m(\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{p}_2 - \mathbf{p}_1$$

So:

$$\text{Force Impulse} = \Delta \mathbf{p} = \text{change in momentum}$$

Note that taking the integral of force with respect to time is the same as calculating the area under the curve on a graph of forces against time, as shown in Figure 5.11.

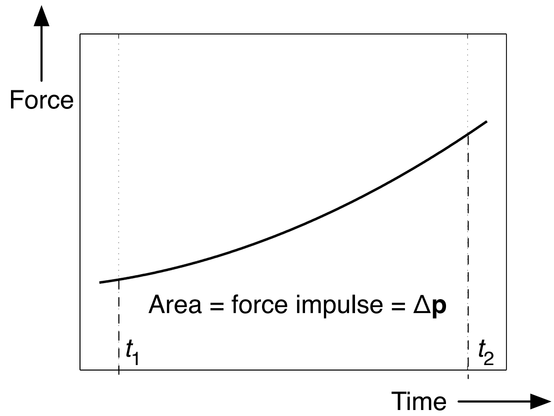


Figure 5.11: Graph of Force against time — Area = $\Delta \mathbf{p}$

5.3.2 Constant Force

34 The impulse of a constant force, equation 5.13 becomes:

$$F \int \mathrm{d}t = m(\mathbf{v}_2 - \mathbf{v}_1) \quad \rightarrow \quad \mathbf{F}(t_2 - t_1) = m(\mathbf{v}_2 - \mathbf{v}_1)$$

Therefore, for a constant force:

$$\text{Force impulse} = \mathbf{F}\Delta t = m\Delta \mathbf{v} = \Delta \mathbf{p}$$

where $\Delta t = t_2 - t_1$ and $\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$

5.3.3 Impulsive Force

An impulsive force is a force that acts for a very short time, as time approaches zero. This type of impulse is often idealised so that the change in momentum produces by the forces happens with no change in time, which can be represented as a *step change* in momentum, which is physically impossible.

You may reasonably think that the force impulse as Δt approaches zero to be zero, but there are notable exceptions, such as a hammer blow, when impulse forces generated are very large.

5.3.4 Example: Force Impulse

A golf ball of mass 95 gm is struck by a club and reaches a speed of 185 mph. If the 35
contact between the club and the ball lasts 0.46 ms, calculate the magnitude of the average fore exerted by the club on the ball, and also the magnitude of the acceleration experienced by the ball.

To solve this, we should first find the velocity of the golf ball on the tee, then in the air:

$$\mathbf{v}_1 = 0 \qquad \mathbf{v}_2 = 185\frac{\cancel{\text{miles}}}{\cancel{\text{hour}}} \times \frac{1609.33\,\text{m}}{1\,\cancel{\text{miles}}} \times \frac{\cancel{\text{hour}}}{3600\,\text{seconds}} = 82.7\,\text{m/s}$$

So the force impulse: 36

$$F\Delta t = m(\mathbf{v_2} - \mathbf{v_1}) \quad \rightarrow \quad F = \frac{m(\mathbf{v_2} - \mathbf{v_1})}{\Delta t}$$

$$F = \frac{(0.095)(82.7)}{0.46 \times 10^{-3}} = 17.08\,\text{kN}$$

To calculate the acceleration, we can either use Newton's second law:

$$\mathbf{F} = m\mathbf{a} \quad \rightarrow \quad \mathbf{a} = \frac{\mathbf{F}}{m} = \frac{17.08}{0.095} = 179800\,\text{m/s}^2$$

or use the constant acceleration equation from basic linear motion:

$$\mathbf{a} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{\Delta t} = \frac{82.7}{0.46 \times 10^{-3}} 179800\,\text{m/s}^2$$

Summary

Momentum

Momentum is defined as: 37

$$\mathbf{p} = m\mathbf{v}$$

which is a vector. Its units are [kgm/s] or alternatively [Ns].

Momentum and force are related by:

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$$

The total change in momentum on a system as a whole is the sum of the external 38
forces on the system as a whole. The *internal forces* are not considered by momentum. If there are no external forces, then there is no change in momentum, and this principles is known as:

The Conservation of Momentum

Collisions

39 In every collision, **momentum is conserved**. Energy may or may not be.

Three types of collisions are dealt with:

- *Inelastic collision* where kinetic energy is lost (the masses stick together)
- *Elastic collision* where kinetic energy is also conserved (the masses bounce off each other)
- *Superelastic collision* where kinetic energy is gained (a mass is separated by a potential energy, for example a spring or an explosion)

Inelastic Collision

For an inelastic collision, the final velocity is given by:

$$\mathbf{v} = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2}$$

where $\mathbf{v}$ is the final velocity, and $\mathbf{u}$ represents initial velocities.

Elastic Collision

For an elastic collision, the final velocities of the two masses, m_1 and m_2 are given by the following equations:

$$\mathbf{v}_1 = \frac{m_1 - m_2}{m_1 + m_2}\mathbf{u}_1 + \frac{2m_2}{m_1 + m_2}\mathbf{u}_2$$
$$\mathbf{v}_2 = \frac{m_2 - m_1}{m_1 + m_2}\mathbf{u}_2 + \frac{2m_1}{m_1 + m_2}\mathbf{u}_1$$

Superelastic Collision

In a superelastic 'collision', the final velocities cannot be determined analytically, as there are two unknowns but only one equation (unless the increase in kinetic energy is given). What can be determined is that the ratio of the masses is inversely proportional to the ratio of velocities — the higher the mass, the lower the velocity:

$$\frac{m_1}{m_2} = \frac{v_2}{v_1}$$

Force Impulse

40 The force impulse is defined as the force multiplied by the time for which it acts. It was shown that the force impulse is equivalent to the change in momentum:

$$\int \mathbf{F}\,\mathrm{d}t = \Delta\mathbf{p}$$

and is the area under a Force-Time graph.

If the force is constant, the following relationship applies:

$$\mathbf{F}\Delta t = m\Delta\mathbf{v} = \Delta\mathbf{p}$$

An *impulsive forces* is a type of impulse that occurs as the time approaches zero. This represents a step change in momentum.

Momentum and Force Impulse: Exercises

1. A golf ball of mass 100 gm is struck by a club and reaches a speed of 200 mph. The collision between the club and the ball lasts 0.5 ms. Given that 1 mile = 1609.33 m:

   a) Calculate the magnitude of the average force exerted by the club on the ball (and vice-versa) during the collision.
   b) Calculate the magnitude of the acceleration experienced by the ball

   (Ans: (a) 17.9 kN; (b) 179000 m/s^2)
2. A car (A) is stationary on level ground with its handbrake off. A second car (B) collides with the back of the car A at 5$\mathbf{i}$ mph. The mass of each car is 750 kg. Ignore rolling resistance, etc. Calculate the momentum $\mathbf{p}$ and U_k of each car before and after the collision and comment on whether $\mathbf{p}$ and U_k are conserved for the following cases, after collision.

   a) Car A 2.5$\mathbf{i}$ mph; Car B 2.5$\mathbf{i}$ mph
   b) Car A 4$\mathbf{i}$ mph; Car B 1$\mathbf{i}$ mph
   c) Car A 5$\mathbf{i}$ mph; Car B 0$\mathbf{i}$ mph

   (Ans: Before for all cases: A: 1676..4$\mathbf{i}$ kgm/s 1873.5 J; B: 0$\mathbf{i}$ kgm/s 0 J; (a) 838.2$\mathbf{i}$ kgm/s, 468.4 J; 838.2$\mathbf{i}$ kgm/s 468.4 J; (b) 1341.1$\mathbf{i}$ kgm/s, 1199 J; 335.28$\mathbf{i}$ kgm/s, 74.94 J; (c) 1676.4$\mathbf{i}$ kgm/s, 1874 J; 0$\mathbf{i}$ kgm/s, 0 J)
3. A golf ball of mass 120 gm is struck by a club which imparts a speed of 270 km/h to the ball, the collision between the club and the ball lasting 0.6 ms. Calculate for the ball during the collision:

   a) Its change in momentum.
   b) The impulse of the force acting.
   c) The average force acting, calculate from the force impulse.
   d) The acceleration.
   e) The average force acting, calculated directly from Newton's Second Law.

   (Ans: 9 kgm/s; 9 Ns; 15 kN; 125 m/s^2; 15 kN)
4. A car (A) is stationary on level ground with its handbrake off. A second car (B) collides with the back of the car A at 7.2 km/h. The mass of each car is 900 kg. Ignore rolling resistance, etc. Calculate the momentum $\mathbf{p}$ and U_k of each car before and after the collision and comment on whether $\mathbf{p}$ and U_k are conserved for the following cases, after collision.

   a) Car A 3.6 km/h; Car B 3.6 km/h
   b) Car A 5.4 km/h; Car B 1.8 km/h
   c) Car A 7.2 km/h; Car B at rest

Answers	*Before*		*After*		*Comments*
	A	*B*	*A*	*B*	
(a) Momentum	*0*	*1800*	*900*	*900*	*Conserved*
U_k	*0*	*1800*	*450*	*450*	*Loss = 900*
(b) Momentum	*0*	*1800*	*1350*	*450*	*Conserved*
U_k	*0*	*1800*	*1012.5*	*112.5*	*Loss = 675*
(c) Momentum	*0*	*1800*	*1800*	*0*	*Conserved*
U_k	*0*	*1800*	*1800*	*0*	*Conserved*

5. Two trucks, A and B, have masses of 30 and 50 tonnes respectively. Initially A is at rest and B is travelling at 24 km/h. Calculate the loss of kinetic energy if:
   a) The trucks lock together on impact
   b) The trucks travel on separately after impact with truck A having a velocity of 10 km/h

   (Ans: 414 kJ; 369 kJ)
6. A ballistic pendulum consists of a block of wood of mass 5 kg suspended on a wire 4 m long. A bullet of mass 40 gm is fired into the block and causes it to swing through an angle of 70°. Determine the velocity of the bullet just before it hits the block. *(Ans: 905.4 m/s)*
7. A pile driver of mass 275 kg falls 0.9 m on to a pile of mass 450 kg. Assuming the driver and pile remain in contact after impact, and that the pile moves 150 mm into the ground, calculate, allowing for gravity after impact:
   a) The velocity of the driver just before impact.
   b) The kinetic energy of the driver just before impact.
   c) The common velocity of the driver and pile just after impact.
   d) The kinetic energy of the driver plus pile just after impact.
   e) The loss in kinetic energy during impact.
   f) The average resisting force exerted by the ground on the pile as the pile penetrates the ground.

   (Ans: 4.2 m/s; 2430 J; 1.592 m/s; 920 J, 1510 J; 13250 J)
8. A pile-driver of mass 700 kg falling 0.2 m is used to drive a pile of mass 500 kg into the ground. Assuming there is no rebound, find the common velocity of the driver and pile at the end of the blow and the loss of kinetic energy that occurs during the impact. If the resistance of the ground is constant, find its value if the pile is driven 75 mmm into the ground. *(Ans: 1.155 m/s; 572 J; 22440 N)*
9. A stationary truck of total mass 9000 kg is set in motion by the action of a shunting locomotive which hits it with a force impulse of 30 kNs. The truck travels freely along a level track against a rolling resistance of 65 N/tonne, for a period of 15 s, when it collides with a second truck of mass 12000 kg which is moving at 0.6 m/s in the same direction as the first truck. The trucks lock together on impact, and the move on together.
   Determine their common speed immediately after impact, and the loss of kinetic energy at impact. *(Ans: 1.353 m/s; 7940 J)*
10. The spring buffers on a truck engage with similar buffers on a second truck which

the first catches up and collides with on a straight horizontal track. The first truck has a mass of 8 tonnes and its initial velocity is 3 m/s. The second truck has a mass of 12 tonnes and an initial velocity of 1 m/s in the same direction as the first.

Determine:

a) The common velocity of the trucks at the instant during the impact when the springs are just fully compressed and the trucks are thus moving together.
b) The maximum amount of strain energy stored in the springs during impact.
c) The velocity of each truck on separating if **only** one half of the energy stored in the springs during their compression in the initial part of the collision is returned to the trucks as the springs expand again. This part involves the solution of a pair of simulations equations and a quadratic equation.

(Ans: (a) 1.8 m/s; (b) 9600 J; (c) 0.95m/s; 2.366 m/s)