# AE1110ab – Aeronautics

## Equation of state

$$pV = nRT$$

$$\rho = \frac{m}{V} \qquad n = \frac{m}{M}$$

$$p\frac{m}{\rho} = \frac{m}{M}RT$$

$$p\frac{1}{\rho} = \frac{R}{M}T$$

$$p = \rho RT$$

## Archimedes' rule

*"The upward force on the balloon is equal to the weight of the air displaced by the balloon"*

$$F_L = mg = \rho V g$$

$$F_L = \rho_{atm}Vg - \rho_{gas}Vg \quad (-Fg_{payload})$$

$$F_L = \rho_{atm}Vg\left(1 - \frac{\rho_{gas}}{\rho_{atm}}\right)$$

$$p_1 = p_2 \qquad \rightarrow \qquad \rho_1 \frac{R}{M_1}T_1 = \rho_2 \frac{R}{M_2}T_2$$

$$\frac{\rho_1}{\rho_2} = \frac{T_2}{T_1} \cdot \frac{M_1}{M_2}$$

## Lift for gas balloons ($T_1 = T_2, M_1 \neq M_2$)

$$F_L = \rho_{atm}Vg\left(1 - \frac{\rho_{gas}}{\rho_{atm}}\right)$$

$$F_L = \rho_{atm}Vg\left(1 - \frac{M_{gas}}{M_{atm}}\right)$$

## Lift for hot-air balloons ($T_1 \neq T_2, M_1 = M_2$)

$$F_L = \rho_{atm}Vg\left(1 - \frac{\rho_{gas}}{\rho_{atm}}\right)$$

$$F_L = \rho_{atm}Vg\left(1 - \frac{T_{atm}}{T_{gas}}\right)$$

$$F_L = \rho_{atm}Vg\left(1 - \frac{T}{T + \Delta T}\right)$$

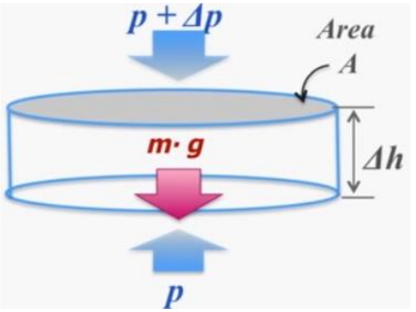
$$F_L = \rho_{atm}Vg\left(\frac{T + \Delta T}{T + \Delta T} - \frac{T}{T + \Delta T}\right)$$

$$F_L = \rho_{atm}Vg\left(\frac{\Delta T}{T + \Delta T}\right)$$

$$L_{hot\ air} \ll L_{helium}$$

$$n_{balloons} = \frac{W}{F_L} = \frac{mg}{\rho_{atm}Vg\left(1 - \frac{\rho_{gas}}{\rho_{atm}}\right)}$$

## Hydrostatic equation




$m = \rho V = \rho A \Delta h$

$$\sum F_{down} = \sum F_{up}$$

$$mg + (p + \Delta p)A = pA$$

$$(\rho A \Delta h)g + (pA + \Delta pA) = pA$$

$$\rho A \Delta h g = pA - pA - \Delta pA$$

$$\rho \Delta h g = -\Delta p$$

$$dp = -\rho g dh$$

## Pressure for ISA $(a \neq 0)$

$$dp = -\rho g dh$$

$$dp = -\frac{p}{RT} g dh$$

$$\frac{dp}{p} = -\frac{g}{RT} dh$$

Toussaint's formula: $T_1 = T_0 + a(h_1 - h_0)$ $\frac{dT}{dh} = a$

$dh = \frac{dT}{a}$ = lapse rate [K/m]

$$\frac{dp}{p} = -\frac{g}{RT}\frac{dT}{a}$$

$$\int_{p_0}^{p_1} \frac{1}{p} dp = -\frac{g}{aR} \int_{T_0}^{T_1} \frac{1}{T} dT$$

$$\ln p_1 - \ln p_0 = -\frac{g}{aR}(\ln T_1 - \ln T_0)$$

$$e^{\ln p_1 - \ln p_0} = e^{-\frac{g}{aR}(\ln T_1 - \ln T_0)}$$

$$\frac{e^{\ln p_1}}{e^{\ln p_0}} = e^{(\ln T_1 - \ln T_0)^{-\frac{g}{aR}}}$$

$$\frac{p_1}{p_0} = \left(\frac{T_1}{T_0}\right)^{-\frac{g}{aR}}$$

## Density for ISA $(a \neq 0)$

$$\frac{p_1}{p_0} = \left(\frac{T_1}{T_0}\right)^{-\frac{g}{aR}}$$

$$\frac{\rho_1 R T_1}{\rho_0 R T_0} = \left(\frac{T_1}{T_0}\right)^{-\frac{g}{aR}}$$

$$\frac{\rho_1}{\rho_0} = \left(\frac{T_1}{T_0}\right)^{-\frac{g}{aR}-1}$$

## Pressure for isothermal layer $(a = 0)$

$$dp = -\rho g dh$$

$$dp = -\frac{p}{RT} g dh$$

$$\frac{dp}{p} = -\frac{g}{RT} dh$$

$a = \frac{dT}{dh} = 0 \quad \rightarrow \quad dT = 0 \quad \therefore T_0 = T_1 = T$

$$\int_{p_0}^{p_1} \frac{1}{p} dP = -\frac{g}{RT} \int_{h_0}^{h_1} dh$$

$$\ln p_1 - \ln p_0 = -\frac{g}{RT}(h_1 - h_0)$$

$$e^{\ln p_1 - \ln p_0} = e^{-\frac{g}{RT}(h_1 - h_0)}$$

$$\frac{p_1}{p_0} = e^{-\frac{g}{RT}(h_1 - h_0)}$$

## Density for isothermal layer $(a = 0)$

$$\frac{p_1}{p_0} = e^{-\frac{g}{RT}(h_1 - h_0)}$$

$$\frac{\rho_1 R T_1}{\rho_0 R T_0} = e^{-\frac{g}{RT}(h_1 - h_0)}$$

$$\frac{\rho_1}{\rho_0} = e^{-\frac{g}{RT}(h_1 - h_0)}$$

| | **ISA** $(a \neq 0)$ | **Isothermal layer** $(a = 0)$ |
|---|---|---|
| **Pressure** | $p_1 = \left(\frac{T_1}{T_0}\right)^{-\frac{g}{aR}} \cdot p_0$ | $p_1 = e^{-\frac{g}{RT}(h_1 - h_0)} \cdot p_0$ |
| **Density** | $\rho_1 = \left(\frac{T_1}{T_0}\right)^{-\frac{g}{aR}-1} \cdot \rho_0$ | $\rho_1 = e^{-\frac{g}{RT}(h_1 - h_0)} \cdot \rho_0$ |

## Geopotential altitude

$$h = \frac{R_e}{R_e + h_g} h_g$$

where $h_g$= distance to the surface of the Earth and $R_e = 6378\ m$

## Straight, horizontal and steady flight

$$L = W \qquad D = T$$

## Glide angle and ratio

$$\theta = \tan^{-1}\left(\frac{h}{b}\right) \qquad ratio = \frac{h}{b}$$

## Lift

$$L = C_L \frac{1}{2} \rho V^2 S \qquad (= W)$$

## Aspect ratio

$$A = \frac{b^2}{S} = \frac{b}{c}$$

## Bernoulli's principle

*"Along a streamline, sum of static and dynamic pressure is constant"*

$$p_{tot} = p_s + q = p_s + \frac{1}{2}\rho V^2 = constant$$

## Airspeed

$$V = \sqrt{\frac{2(p_{tot} - p_s)}{\rho}}$$

## Drag

$$D = C_D \frac{1}{2}\rho V^2 S \qquad (= T)$$

## Drag coefficient

$$C_D = C_{D_0} + C_{D_i} = C_{D_0} + \frac{{C_L}^2}{\pi A e} = C_{D_0} + k_1 C_L + k_2 C_L^2$$

## Glide ratio

$$ratio = \left(\frac{C_L}{C_D}\right)_{max}$$

## Mach number

$$M = \frac{V}{a} = \frac{flow\ velocity}{speed\ of\ sound} = \frac{V}{\sqrt{\gamma R T}}$$

## True air speed (TAS) and equivalent air speed (EAS)

$$\frac{1}{2}\rho_0 {V_{EAS}}^2 = \frac{1}{2}\rho_1 {V_{TAS}}^2$$

$$V_{TAS} = \sqrt{\frac{\rho_0}{\rho_1}}\, V_{EAS}$$

$$IAS = CAS = EAS$$

## Stall speed

$$V_{stall} = \sqrt{\frac{2W}{\rho S C_{L_{max}}}}$$

## Magnetic heading

$$magnetic\ heading = compass\ heading + compass\ deviation$$

## True heading

$$true\ heading = magnetic\ heading + magnetic\ variation$$

## Load factor

$$n = \frac{L}{W}$$

## Ultimate load (UL)

$$UL = limit\ load \times safety\ factor$$

## Hoop (circumferential) stress

$$\sigma_{circ} = \frac{\Delta pR}{t}$$

## Longitudinal stress

$$\sigma_{long} = \frac{\Delta pR}{2t}$$

## Pitch angle ($\theta$)

$$\theta = \alpha + \gamma = angle\ of\ attack + flight\ path\ angle$$

## Course angle ($\chi$)

$$\chi = \psi + \beta = heading + sideslip$$

## Longitudinal static stability

Reaction to disturbance is in opposite direction to initial disturbance

$$\Delta C_m < 0 \qquad\qquad \Delta\alpha > 0$$

$$C_{m_\alpha} = \frac{dC_m}{d\alpha} < 0$$

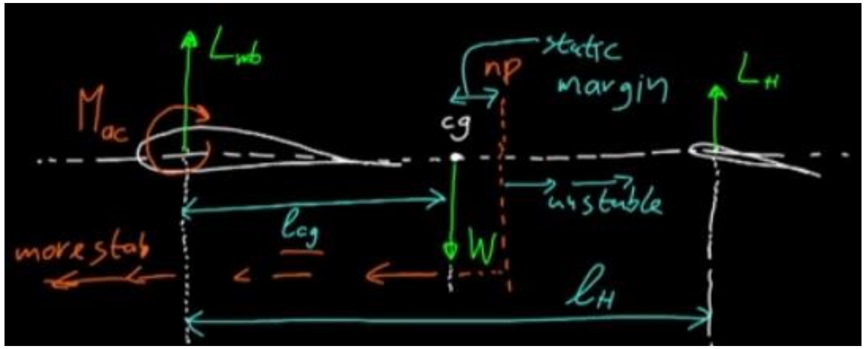
## Angle of attack of horizontal tail surface ($\alpha_H$)

$$\alpha_H = \alpha + i_H - \varepsilon$$

where $i_H$ = tail surface setting and $\varepsilon$ = downwash

$$\frac{d\alpha_H}{d\alpha} = \frac{d}{d\alpha}(\alpha + i_H - \varepsilon) = 1 - \frac{d\varepsilon}{d\alpha}$$

## Moment around the centre of gravity




$$W = L = L_{wb} + L_H$$

$$M_{cg} = M_{ac} + L_{wb} \cdot l_{cg} - L_H(l_H - l_{cg})$$

$$M_{cg} = M_{ac} + L_{wb} \cdot l_{cg} - L_H \cdot l_H + L_H \cdot l_{cg}$$

$$M_{cg} = M_{ac} + (L_{wb} + L_H) \cdot l_{cg} - L_H \cdot l_H$$

$$M_{cg} = M_{ac} + L \cdot l_{cg} - L_H \cdot l_H$$

## Moment coefficient

$$M = C_m \frac{1}{2}\rho V^2 S\bar{c}$$

$$C_m = \frac{M}{\frac{1}{2}\rho V^2 S\bar{c}}$$

$$C_m = \frac{M_{ac} + L \cdot l_{cg} - L_H \cdot l_H}{\frac{1}{2}\rho V^2 S\bar{c}}$$

$$C_m = \frac{M_{ac}}{\frac{1}{2}\rho V^2 S\bar{c}} + \frac{L \cdot l_{cg}}{\frac{1}{2}\rho V^2 S\bar{c}} - \frac{L_H \cdot l_H}{\frac{1}{2}\rho V^2 S\bar{c}}$$

$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$ $\quad L_H = C_{L_H}\frac{1}{2}\rho V^2 S_H$ $\quad V_H = \frac{S_H \cdot l_H}{S\bar{c}}$

$$C_m = C_{M_{ac}} + C_L \cdot \frac{l_{cg}}{\bar{c}} - \frac{\left(C_{L_H}\frac{1}{2}\rho V^2 S_H\right) \cdot l_H}{\frac{1}{2}\rho V^2 S\bar{c}}$$

$$C_m = C_{M_{ac}} + C_L \cdot \frac{l_{cg}}{\bar{c}} - C_{L_H} V_H$$

$$\frac{dC_m}{d\alpha} = \frac{dC_{M_{ac}}}{d\alpha} + \frac{dC_L}{d\alpha}\frac{l_{cg}}{\bar{c}} - \frac{dC_{L_H}}{d\alpha}V_H$$

$a = \frac{dC_L}{d\alpha} = gradient\ 'C_L\ vs\ \alpha'$ $\quad \frac{dC_{M_{ac}}}{d\alpha} = 0$

$a_t = \frac{dC_{L_H}}{d\alpha_H}$ $\quad \rightarrow \quad$ $\frac{dC_{L_H}}{d\alpha} \cdot \frac{d\alpha_H}{d\alpha_H} = \frac{dC_{L_H}}{d\alpha_H} \cdot \frac{d\alpha_H}{d\alpha} = at \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right)$

$$C_{m_\alpha} = a \cdot \frac{l_{cg}}{\bar{c}} - at \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right) \cdot V_H < 0$$

$$a \cdot \frac{l_{cg}}{\bar{c}} < at \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right) \cdot V_H$$

$$\frac{l_{cg}}{\bar{c}} < \frac{at}{a} \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right) \cdot V_H$$

where $V_H$ is the tail volume (design parameter)

$$l_{cg} < l_{np}$$

## Neutral static stability

$$C_{m_\alpha} = 0$$

$$\frac{l_{cg}}{\bar{c}} = \frac{l_{np}}{\bar{c}} = \frac{at}{a} \cdot \left(1 - \frac{d\varepsilon}{d\alpha}\right) \cdot V_H$$

## Thrust ($T$) and mass flow ($\dot{m}$)

$$F = \frac{dI}{dt} = \frac{d(\dot{m}V)}{dt} = \dot{m}\frac{dV}{dt} + \frac{d\dot{m}}{dt}V = \dot{m}\Delta V$$

$$T = \dot{m} \cdot \left(V_j - V_{TAS}\right)$$

$$\dot{m} = \rho_{atm} \cdot A_{inlet} \cdot V_{TAS}$$

## Rotational velocity of a propeller

$$V_{rot} = \omega r$$

where $\omega$ is the angular speed [rad/s] and *r* is the radius

## Work performed by and available power of a propeller

$$W = T\Delta x$$

$$P_a = \frac{T\Delta x}{\Delta t} = TV_0$$

$$Q = \dot{m}_f H$$

where H is the amount of energy per unit of fuel

$$\eta_{th} = \frac{P_a}{Q}$$

## Propulsive efficiency

$$\eta = \frac{P_a}{P_{br}} = \frac{TV_0}{\frac{1}{2}\dot{m}\left({V_j}^2 - {V_0}^2\right)}$$

## Bypass ratio

$$B = \frac{\dot{m}_c}{\dot{m}_h} = \frac{bypass\ airflow\ (cold)}{core\ airflow\ (hot)}$$

## Jet efficiency

$$P_j = \frac{1}{2}\dot{m}{V_j}^2 - \frac{1}{2}\dot{m}{V_0}^2 = \frac{1}{2}\dot{m}\left({V_j}^2 - {V_0}^2\right)$$

$$T = \dot{m}\cdot\left(V_j - V_0\right) \qquad P_a = TV_0$$

$$\eta_j = \frac{P_a}{P_{br}} = \frac{TV_0}{P_j}$$

$$\eta_j = \frac{\dot{m}\left(V_j - V_0\right)V_0}{\frac{1}{2}\dot{m}{V_j}^2 - \frac{1}{2}\dot{m}{V_0}^2}$$

$$\eta_j = \frac{\dot{m}\left(V_j - V_0\right)V_0}{\frac{1}{2}\dot{m}\left({V_j}^2 - {V_0}^2\right)}$$

$$\eta_j = \frac{\left(V_j - V_0\right)V_0}{\frac{1}{2}\left(V_j - V_0\right)\left(V_j + V_0\right)}$$

$$\eta_j = \frac{2V_0}{\left(V_j + V_0\right)}$$

$$\eta_j = \frac{2}{1 + \frac{V_j}{V_0}}$$

## Stress ($\sigma$) and strain of a material ($\varepsilon$)

$$\sigma = \frac{F}{A}$$

$$\varepsilon = \frac{\Delta L}{L}$$

## Specific property

$$specific\ property = \frac{property}{density}$$

## Mach angle ($\mu$) of shockwaves ($M > 1$)

$$\mu = \sin^{-1}\left(\frac{1}{M}\right)$$

$$\mu = \sin^{-1}\left(\frac{a}{V}\right)$$

## Rotor torque ($Q$)

$$Q = T_{tr}\cdot l_{tr}$$

where $T_{tr}$ is the tail rotor thrust and $l_{tr}$ is the tail rotor moment arm

# AE1110e – Aerodynamics

## Specific volume

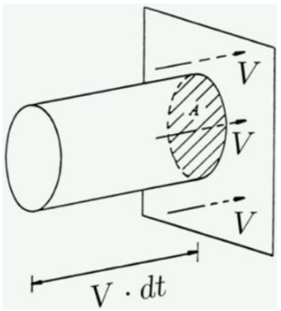
$$SV = \frac{1}{\rho}$$

## Kinetic energy

$$KE = \frac{3}{2}kT$$

where $k = 1.38 \cdot 10^{-23} \frac{J}{K}$ is the Stefan-Boltzmann constant

## Continuity equation for steady flow




$$m = \rho V \qquad\qquad V = VdtA$$

$$\dot{m} = \frac{\rho VdtA}{dt}$$

$$\dot{m} = \rho VA = constant$$

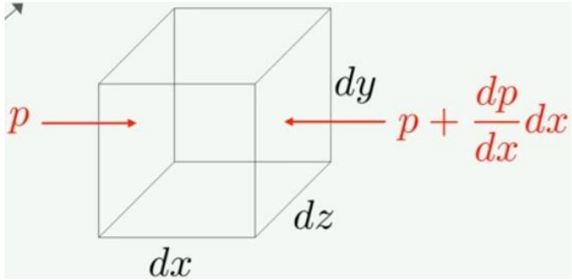
$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

- where $\rho$ and $V$ are mean values over the area
- for incompressible flow, $VA = constant$

## Euler equation (momentum) for steady & compressible flow

$$F = ma$$

(for this derivation we neglect fiction (viscosity) & gravitational force)




$$F = pdydz - \left(p + \frac{dp}{dx}dx\right)dydz$$

$$F = -\frac{dp}{dx}dxdydz$$

$$m = \rho V = \rho dxdydz$$

$$a = \frac{dV}{dt} = \frac{\frac{dV}{dx}dx}{dt} = \frac{dV}{dx}V$$

$$-\frac{dp}{dx}dxdydz = (\rho dxdydz)\frac{dV}{dx}V$$

$$dp = -\rho VdV$$

## Bernoulli's principle (along a streamline)

$$dp = -\rho V dV$$

$$dp + \rho V dV = 0$$

$$\int_{p_1}^{p_2} dp + \rho \int_{V_1}^{V_2} V dV = 0$$

$$(p_2 - p_1) + \rho\left(\frac{1}{2}V_2^2 - \frac{1}{2}V_1^2\right) = 0$$

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2$$

$$p + \frac{1}{2}\rho V^2 = constant\ along\ a\ streamline$$

$$p + \frac{1}{2}\rho V^2 = p_s + q = p_{tot}$$

## Flight speed

$$V = \sqrt{\frac{2q}{\rho}}$$

## Internal energy of a system ($e$): 1st law of thermodynamics

$$de = dq + dW$$

## Work done on a system

$$W = force \times distance = Fs = (pdA)s$$

$$dW = \int pdAs = p\int sdA$$

where $\int sdA$ is the change of specific volume ($dV$)

$$dW = -pdV$$

$$de = dq - pdV$$

## Enthalpy ($h$)

$$h = e + pV$$

$$dh = de + d(pV)$$

$$dh = de + Vdp + pdV$$

$$dh = (dq - pdV) + Vdp + pdV$$

$$dh = dq + Vdp$$

## Specific heat

$$c = \frac{dq}{dT}$$

$$dq = cdT$$

1. For constant volume ($dV = 0$):

$$de = dq$$

$$de = c_v dT$$

$$e = c_v T$$

2. For constant pressure ($dp = 0$):

$$dh = dq$$

$$dh = c_p dT$$

$$h = c_p T$$

## Isentropic relations for compressible flow

$$de = dq - pdV$$

$$dq = 0$$

$$de = -pdV = c_v dT$$

$$dh = dq + Vdp$$

$$dq = 0$$

$$dh = Vdp = c_p dT$$

dividing $de$ by $dh$:

$$\frac{de}{dh} = \frac{-pdV}{Vdp} = \frac{c_v dT}{c_p dT} = \frac{c_v}{c_p}$$

$$\frac{Vdp}{pdV} = -\frac{c_p}{c_v}$$

$$\frac{dp}{p} = -\frac{c_p}{c_v}\frac{dV}{V}$$

$$\frac{dp}{p} = -\gamma\frac{dV}{V}$$

where $\gamma = 1.4$ is the ratio of specific heats

$$\int_1^2 \frac{1}{p}dp = -\gamma\int_1^2 \frac{1}{V}dV$$

$$\ln p_2 - \ln p_1 = -\gamma(\ln V_2 - \ln V_1)$$

$$\ln\frac{p_2}{p_1} = -\gamma\ln\frac{V_2}{V_1}$$

$$V_1 = \frac{1}{\rho_1} \qquad V_2 = \frac{1}{\rho_2}$$

$$\frac{p_2}{p_1} = \left(\frac{\rho_1}{\rho_2}\right)^{-\gamma}$$

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

from equation of state:

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

## Energy equation for compressible flow

Using Euler's equation, adiabatic process and specific volume:

$$dp = -\rho V dV \qquad dq = 0 \qquad V = \frac{1}{\rho}$$

$$dh = dq + Vdp$$

$$dh - Vdp = 0$$

$$dh + V\rho VdV = 0$$

$$dh + \frac{1}{\rho}\rho VdV = 0$$

$$dh + VdV = 0$$

$$h + \frac{1}{2}V^2 = constant$$

$$h = c_p T$$

$$c_p T + \frac{1}{2}V^2 = constant$$

$$c_p T_1 + \frac{1}{2}{V_1}^2 = c_p T_2 + \frac{1}{2}{V_2}^2$$

| **For steady, isentropic, compressible flows** | |
|---|---|
| **Continuity equation** | $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$ |
| **Isentropic relations** | $\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$ |
| **Energy equation** | $c_p T_1 + \frac{1}{2}{V_1}^2 = c_p T_2 + \frac{1}{2}{V_2}^2$ |
| **Equation of state** | $p_1 = \rho_1 R T_1$ $p_2 = \rho_2 R T_2$ |

## Speed of sound ($a$)

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

$$\rho a A_1 = (p + dp) A_2 (a + da)$$

1 dimensional flow, so $A_1 = A_2 = A = constant$

$$\rho a = (\rho + d\rho)(a + da)$$

$$\rho a = \rho a + a d\rho + \rho da + d\rho da$$

$$a = -\rho \frac{da}{d\rho}$$

$$dp = -\rho V dV = -\rho a da \qquad da = -\frac{1}{\rho a} dp$$

$$a = -\rho \frac{da}{d\rho} = \rho \frac{1}{\rho a} \frac{dp}{d\rho}$$

$$a^2 = \frac{dp}{d\rho}$$

$$a = \sqrt{\frac{dp}{d\rho}}_{isentropic}$$

$$a = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma \frac{\rho R T}{\rho}}$$

$$a = \sqrt{\gamma R T}$$

## Mach number ($M$)

$$M = \frac{V}{a} = \frac{V}{\sqrt{\gamma RT}}$$

## Second form of isentropic relations

$$c_pT_0 + \frac{1}{2}{V_0}^2 = c_pT_1 + \frac{1}{2}{V_1}^2$$

at stagnation point, ($V = 0$)

$$c_pT_0 = c_pT_1 + \frac{1}{2}{V_1}^2$$

$$\frac{T_0}{T_1} = 1 + \frac{\gamma - 1}{2}\frac{V_1^2}{\gamma RT_1}$$

$$\frac{T_0}{T_1} = 1 + \frac{\gamma - 1}{2}\frac{V_1^2}{a_1^2}$$

$$\frac{T_0}{T_1} = 1 + \frac{\gamma - 1}{2}{M_1}^2$$

bringing flow isentropically to rest, isentropic relation can be used:

$$\frac{p_0}{p_1} = \left(1 + \frac{\gamma - 1}{2}{M_1}^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho_1} = \left(1 + \frac{\gamma - 1}{2}{M_1}^2\right)^{\frac{1}{\gamma-1}}$$

where $T_0, p_0$ $and$ $\rho_0$ are TOTAL temperature, pressure and density

## Area-velocity relation

$$\rho VA = constant$$

$$\ln\rho + \ln V + \ln A = \ln(constant)$$

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

$$dp = -\rho VdV \qquad \rho = -\frac{dp}{VdV}$$

$$\frac{-d\rho VdV}{dp} + \frac{dV}{V} + \frac{dA}{A} = 0$$

$$a^2 = \frac{dp}{d\rho} \qquad \frac{d\rho}{dp} = \frac{1}{a^2}$$

$$-\frac{VdV}{a^2} + \frac{dV}{V} + \frac{dA}{A} = 0$$

$$\frac{dA}{A} = \frac{VdV}{a^2} - \frac{dV}{V}$$

$$\frac{dA}{A} = (M^2 - 1)\frac{dV}{V}$$

When $M < 1$ (subsonic), $dV > 0$ then $dA < 0$

When $M > 1$ (supersonic), $dV > 0$ then $dA > 0$

When $M = 1$ (sonic), then $\frac{dA}{A} = 0$

$$\frac{dV}{V} = \frac{1}{M^2 - 1}\frac{dA}{A} = \frac{1}{1^2 - 1}\frac{dA}{A} = \frac{1}{0}\frac{dA}{A} = \infty = impossible$$

## Expansion ratio

$$\rho_{ts}V_{ts}A_{ts} = \rho_* V_* A_*$$

$$\frac{A_{ts}}{A_*} = \frac{\rho_* V_*}{\rho_{ts}V_{ts}}$$

## d'Alembert's paradox

inviscid flow = no drag (no friction)

viscous flow = finite drag (friction)

## Reynold's number

$$R_{e_x} = \frac{\rho_\infty V_\infty x}{\mu} = \frac{V_\infty x}{\nu_\infty}$$

where $x$ = chord, $\mu$ = dynamic viscosity and $\nu_\infty$= kinematic viscosity

## Shear stress ($\tau_w$)

$$\tau_w = \mu\left(\frac{du}{dy}\right)_{y=0}$$

where $\mu$ is the dynamic (absolute) viscosity coefficient

## Local Reynold's number

$$R_{e_L} = \frac{Vx}{\nu} = \frac{Vx}{\frac{\mu}{\rho}}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity

| | **Laminar bound. Lay.** | **Turbulent bound. lay.** |
|---|---|---|
| **Thickness** | $d = \frac{5.2x}{\sqrt{R_{e_x}}}$ | $d = \frac{0.37x}{(R_{e_x})^{0.2}}$ |
| **Skin friction coefficient** | $c_f = \frac{1.328}{\sqrt{R_{e_L}}}$ | $c_f = \frac{0.074}{(R_{e_x})^{0.2}}$ |
| **Local skin friction coefficient** | $c_{f_x} = \frac{\tau_w}{q_\infty} = \frac{0.664}{\sqrt{R_{e_x}}}$ | – |
| **Total skin friction drag coefficient** | $c_{f_x} = \frac{1.328qL}{\sqrt{R_{e_L}}}$ | – |

## Critical Reynold's number

$$R_{e_{cr}} = \frac{V_\infty x_{cr}}{\nu_\infty}$$

## Pressure coefficient for incompressible flow

$$c_p = \frac{p_1 - p_0}{q_0} = \frac{p_1 - p_0}{\frac{1}{2}\rho V_0^2}$$

$$p_1 + \frac{1}{2}\rho V_1^2 = p_0 + \frac{1}{2}\rho V_0^2$$

$$p_1 - p_0 = \frac{1}{2}\rho V_0^2$$

$$c_p = 1\ (at\ stagnation\ point)$$

$$c_p = 1 - \frac{V_1^2}{V_0^2}$$

## Viscous (profile) drag

$$c_d = c_{d_{pressure}} + c_{d_{friction}}$$

## Airfoil characteristics

$$c_l = \frac{L}{\frac{1}{2}\rho V^2 c}$$

$$c_d = \frac{D}{\frac{1}{2}\rho V^2 c}$$

$$c_n = \int_0^1 (c_{p_l} - c_{p_u})\, d(\frac{x}{c})$$

## Compressibility correction for pressure & lift coefficient

$$c_p = \frac{c_{p,0}}{\sqrt{1 - M_\infty^2}}$$

$$c_l = \frac{c_{l,0}}{\sqrt{1 - M_\infty^2}}$$

where $c_{p,0}$ and $c_{l,0}$ are the incompressible pressure & lift coefficients

## Critical pressure coefficient

$$c_{p_{cr}} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{2 + (\gamma - 1)M_\infty^2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

## Critical Mach number for swept wings

$$M_{cr}\ for\ swept\ wings = \frac{M_{cr}}{\cos\Lambda}$$

where $\Lambda$ is the sweep angle

## Induced drag

$$D_i = L \cdot \alpha_i$$

## Drag coefficient and induced drag coefficient

$$\alpha_i = \frac{C_L}{\pi A} \qquad A = \frac{b^2}{S}$$

$$C_{D_i} = \frac{C_L^2}{\pi A e_1}$$

- When $e_1 = 1$, minimum induced drag
- When $e_1 < 1$, higher induced drag

$$C_D = C_{d_0} + C_{d_i}$$

$$C_D = C_{d_0} + \frac{C_L^2}{\pi A e_1}$$

## Lift curve slope

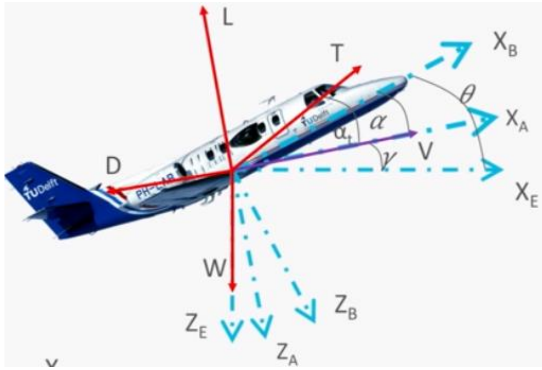
$$\alpha = \frac{dC_L}{d\alpha} = \frac{\alpha_0}{1 + \frac{\alpha_0}{\pi A e_1}} = \frac{\alpha_0}{1 + \frac{75.3\alpha_0}{\pi A e_1}}$$

where $\alpha_0 = 0.1097° = 2\pi$ for a 2D thin airfoil

# AE1110f – Flight mechanics

## Derivations of the equations of motion

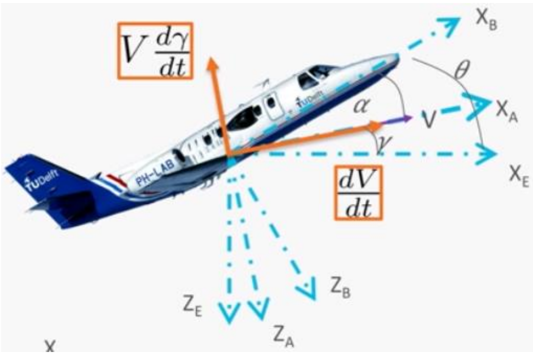
1. Free body diagram

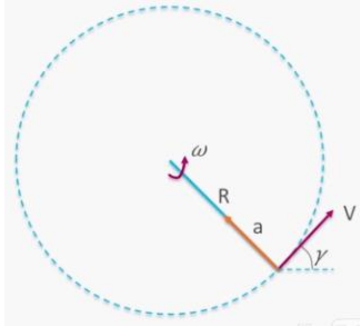



$$\sum\nolimits_{\parallel V} F = T\cos(\alpha_t) - D - W\sin(\gamma)$$

$$\sum\nolimits_{\perp V} F = T\sin(\alpha_t) + L - W\cos(\gamma)$$

2. Kinematic diagram




$$\sum\nolimits_{\parallel V} F = ma = \frac{W}{g}\frac{dV}{dt}$$

$$\sum\nolimits_{\perp V} F = ma = m\frac{V^2}{R} = m\frac{(\omega R)V}{R} = m\frac{\left(\frac{d\gamma}{dt}R\right)V}{R} = \frac{W}{g}\frac{d\gamma}{dt}V$$

## Equations of motion

$$\sum\nolimits_{\parallel V} F = \frac{W}{g}\frac{dV}{dt} = T\cos(\alpha_t) - D - W\sin(\gamma)$$

$$\sum\nolimits_{\perp V} F = \frac{W}{g}\frac{d\gamma}{dt}V = T\sin(\alpha_t) + L - W\cos(\gamma)$$

## Flight conditions

Straight flight: $\frac{d\gamma}{dt} = 0$

Steady flight: $\frac{d\gamma}{dt} = \frac{dV}{dt} = 0$

Horizontal flight: $\gamma = 0$

Symmetric flight: $\beta$ (angle of sideslip) $= 0$ and plane $\perp$ to Earth

## Lift

$$L = W$$

$$C_L \frac{1}{2}\rho V^2 S = W$$

$$C_L = \frac{W}{S}\frac{2}{\rho}\frac{1}{V^2}$$

## Drag

$$T = D$$

$$T = C_D \frac{1}{2}\rho V^2 S = D$$

$$T = \left(C_{D_0} + C_{D_i}\right)\frac{1}{2}\rho V^2 S = D$$

$$T = \left(C_{D_0} + \frac{C_L^2}{\pi A e}\right)\frac{1}{2}\rho V^2 S = D$$

$$T = \left(C_{D_0} + k_1 C_L + k_2 C_L^2\right)\frac{1}{2}\rho V^2 S = D$$

$$T = \left[C_{D_0} + k_1\left(\frac{W}{S}\frac{2}{\rho}\frac{1}{V^2}\right) + k_2\left(\frac{W}{S}\frac{2}{\rho}\frac{1}{V^2}\right)^2\right]\frac{1}{2}\rho V^2 S = D$$

$$T = C_{D_0}\left(\frac{1}{2}\rho V^2 S\right) + k_1\frac{W}{S}\frac{2}{\rho}\frac{1}{V^2}\left(\frac{1}{2}\rho V^2 S\right) + k_2\left(\frac{W}{S}\frac{2}{\rho}\frac{1}{V^2}\right)^2\left(\frac{1}{2}\rho V^2 S\right)$$

$$T = C_{D_0}\left(\frac{1}{2}\rho V^2 S\right) + k_1 W + k_2\frac{W^2}{S}\frac{2}{\rho}\frac{1}{V^2} = D$$

## Propulsion

$$T = \left(\dot{m} + \dot{m}_f\right)V_j - \dot{m}V_0 = \dot{m}\left(V_j - V_0\right)$$

## Total efficiency ($\eta_t$)

$W = T\Delta x$ $\quad$ $P_a = TV_0$ $\quad$ $Q = \dot{m}_f H$

where $Q$ = thermal power and $H$ = amount of energy per unit of fuel

$$\eta_t = \frac{P_a}{Q} = \frac{P_a}{Q}\cdot\frac{P_j}{P_j} = \frac{P_a}{P_j}\cdot\frac{P_j}{Q} = \eta_j \cdot \eta_{th}$$

## Jet efficiency ($\eta_j$)

$$\eta_j = \frac{P_a}{P_j}$$

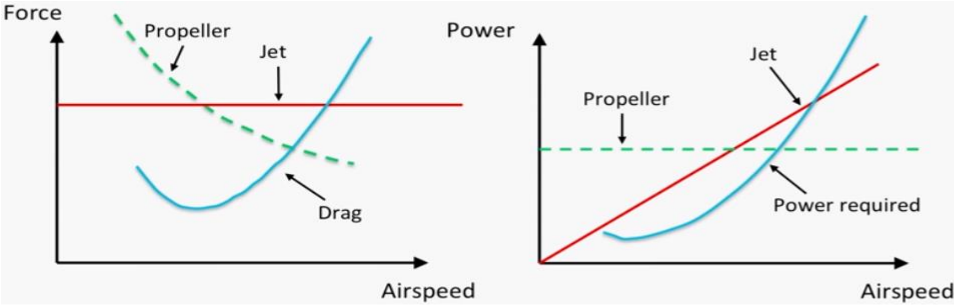
$$\eta_j = \frac{TV_0}{\frac{1}{2}\dot{m}V_j^2 - \frac{1}{2}\dot{m}V_0^2}$$

$$\eta_j = \frac{\left(\dot{m}\left(V_j - V_0\right)\right)V_0}{\frac{1}{2}\dot{m}(V_j^2 - V_0^2)}$$

$$\eta_j = \frac{\left(V_j - V_0\right)V_0}{\frac{1}{2}\left(V_j - V_0\right)\left(V_j + V_0\right)}$$

$$\eta_j = \frac{2V_0}{V_j + V_0} = \frac{2}{1 + \frac{V_j}{V_0}}$$

## Performance diagrams




## Minimum airspeed (@ $C_{L_{max}}$)

$$L = W = C_L \frac{1}{2} \rho V^2 S$$

$$V_{min} = \sqrt{\frac{W}{S} \frac{2}{\rho} \frac{1}{C_{L_{max}}}}$$

## Maximum airspeed (@ $P_{a_{max}} = P_r$)

$$P_{a_{max}} = DV$$

$$P_{a_{max}} = D \frac{L}{L} \cdot V = \frac{D}{L} W \cdot V$$

$$P_{a_{max}} = \frac{C_D \frac{1}{2} \rho V^2 S}{C_L \frac{1}{2} \rho V^2 S} W \cdot V$$

$$P_{a_{max}} = \frac{C_D}{C_L} W \cdot \sqrt{\frac{W}{S} \frac{2}{\rho} \frac{1}{C_L}}$$

$$P_{a_{max}} = \sqrt{\frac{W^3}{S} \frac{2}{\rho} \frac{C_D^2}{C_L^3}}$$

$$P_{a_{max}} = \sqrt{\frac{W^3}{S} \frac{2}{\rho} \frac{\left(C_{D_0} + C_{D_i}\right)^2}{C_L^3}}$$

$$P_{a_{max}} = \sqrt{\frac{W^3}{S} \frac{2}{\rho} \frac{\left(C_{D_0} + \frac{C_L^2}{\pi A e}\right)^2}{C_L^3}}$$

$$P_{a_{max}} = \sqrt{\frac{W^3}{S} \frac{2}{\rho} \frac{\left(C_{D_0} + k_1 C_L + k_2 C_L^2\right)^2}{C_L^3}}$$

$$P_{a_{max}}^2 = \frac{W^3}{S} \frac{2}{\rho} \frac{\left(C_{D_0} + k_1 C_L + k_2 C_L^2\right)^2}{C_L^3}$$

$$\frac{\left(C_{D_0} + k_1 C_L + k_2 C_L^2\right)^2}{C_L^3} = P_{a_{max}}^2 \frac{s}{W^3} \frac{\rho}{2}$$

$$\left(C_{D_0} + k_1 C_L + k_2 C_L^2\right)^2 = \left(P_{a_{max}}^2 \frac{s}{W^3} \frac{\rho}{2}\right) C_L^3$$

solve for $C_L$ if $k_1 = 0$

$$\left(C_{D_0}\right)^2 + 2 C_{D_0} k_2 C_L^2 + (k_2 C_L^2)^4 = \left(P_{a_{max}}^2 \frac{s}{W^3} \frac{\rho}{2}\right) C_L^3$$

$$k_2^2 C_L^4 - \left(P_{a_{max}}^2 \frac{s}{W^3} \frac{\rho}{2}\right) C_L^3 + (2 C_{D_0} k_2) C_L^2 + C_{D_0}^2 = 0$$

Maximum range (how far an aircraft can fly)

$$range_{max} = \left(\frac{V}{F}\right)_{max}$$

$$range_{max} = \frac{V}{c_p P_{br}}$$

$$range_{max} = \frac{V}{c_p \frac{P_a}{\eta_j}}$$

$$range_{max} = \frac{V}{c_p \frac{DV}{\eta_j}}$$

$$range_{max} = \frac{\eta_j}{c_p D}$$

Optimal lift coefficient for maximum range (jet & prop)

$$C_{L_{opt}} = \frac{-k_1 \pm \sqrt{k_1^2 + 12k_2 C_{D_0}}}{6k_2} = \sqrt{\frac{C_{D_0}}{k_2}}$$

Maximum endurance ($P_{r_{min}}: minimum\ fuel\ flow, F$)

$$E = \frac{W_f}{F}$$

$$F = c_p P_{br} = c_p \frac{P_a}{\eta}$$

$$F = \frac{c_p}{\eta} P_{r_{min}}$$

$$F = \frac{c_p}{\eta} (DV)_{min}$$

$$F = \frac{c_p}{\eta} \left(\frac{C_D}{C_L} W \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}}\right)_{min}$$

$$F = \frac{c_p}{\eta} \left(\sqrt{\frac{W^3}{S}\frac{2}{\rho}\frac{C_D^2}{C_L^3}}\right)_{min}$$

$$F = \frac{c_p}{\eta} \left(\sqrt{\frac{W^3}{S}\frac{2}{\rho}\left(\frac{C_D^2}{C_L^3}\right)_{min}}\right)_{min}$$

$$F = \frac{c_p}{\eta} \left(\sqrt{\frac{W^3}{S}\frac{2}{\rho}\left(\frac{C_L^3}{C_D^2}\right)_{max}}\right)_{min}$$

$$F = \frac{c_p}{\eta} \left(\sqrt{\frac{W^3}{S}\frac{2}{\rho}\left(\frac{C_L^3}{\left(C_{D_0} + k_1 C_L + k_2 C_L^2\right)^2}\right)_{max}}\right)_{min}$$

Optimal lift coefficient for maximum endurance (jet & pro)

$$C_{L_{opt}} = \sqrt{\frac{C_{D_0}}{k_2}} = \frac{k_1 \pm \sqrt{k_1^2 + 12k_2 C_{D_0}}}{2k_2}$$

## Optimal speed for maximum range & maximum endurance

$$V_{opt} = \sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_{L_{opt}}}}$$

| | $C_L$ **for max range** | $C_L$ **for max endurance** |
|---|---|---|
| **With respect to** | Distance | Time |
| **Factors** | $\left(\frac{V}{F}\right)_{max}$ | $P_{r_{min}} = (DV)_{min} = \frac{dC_D}{dC_L} = 0 \rightarrow (fuel\ flow)_{min}$ |
| **Jet** | $\frac{-k_1 \pm \sqrt{k_1^2 + 12k_2C_{D_0}}}{6k_2}$ | $\sqrt{\frac{C_{D_0}}{k_2}}$ |
| **Propeller** | $\sqrt{\frac{C_{D_0}}{k_2}}$ | $\frac{k_1 \pm \sqrt{k_1^2 + 12k_2C_{D_0}}}{2k_2}$ |

## Climbing and descending flight

$$T - D - W\sin\gamma = 0$$

$$L - W\cos\gamma \ = 0$$

$$L = W \qquad \sin\gamma = \frac{T-D}{W}$$

## Rate of climb (ROC)

$$ROC = V\sin\gamma = V\frac{T-D}{W} = \frac{TV-DV}{W} = \frac{P_a - P_r}{W}$$

## Maximum climb angle ($\gamma$) ($T_{max}$ & $D_{min}$: $\frac{dC_D}{dC_L} = 0$)

$$\sin(\gamma_{max}) = \frac{(T-D)_{max}}{W}$$

## Minimum descent angle ($\bar{\gamma}_{min} = \left(\frac{C_D}{C_L}\right)_{min} = \left(\frac{C_L}{C_D}\right)_{max}, T = 0$)

$$\sin\gamma = -\frac{D}{W} = -\frac{D}{L} = -\frac{C_D}{C_L} \qquad \bar{\gamma} = -\gamma \qquad C_L = \sqrt{\frac{C_{D_0}}{k_2}}$$

$$\sin\bar{\gamma} = \frac{D}{L}$$

$$\sin\bar{\gamma} = \frac{C_D}{C_L}$$

$$\sin\bar{\gamma} = \frac{C_{D_0} + k_1\sqrt{\frac{C_{D_0}}{k_2}} + k_2\frac{C_{D_0}}{k_2}}{\sqrt{\frac{C_{D_0}}{k_2}}}$$

$$\sin\bar{\gamma} = \frac{C_{D_0} + k_1\sqrt{\frac{C_{D_0}}{k_2}} + C_{D_0}}{\sqrt{\frac{C_{D_0}}{k_2}}}$$

$$\sin\bar{\gamma} = \frac{2C_{D_0}}{\sqrt{C_{D_0}/k_2}} + k_1$$

## Time to climb

$$t = \frac{h}{ROC}$$

## Maximum rate of climb in steady flight

$$\frac{dV}{dt} = 0$$

$$ROC_{max} = \frac{(P_a - P_r)_{max}}{W}$$

$$ROC_{max} = \frac{(TV - DV)_{max}}{W}$$

$$ROC_{max} = (DV)_{max}$$

$$ROC_{max} = \left(D\frac{L}{L}V\right)_{max}$$

$$ROC_{max} = \left(\frac{D}{L}WV\right)_{max}$$

$$ROC_{max} = \left(\frac{C_D}{C_L}W\sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}}\right)_{max}$$

$$ROC_{max} = \left(\sqrt{\frac{W^3}{S}\frac{2}{\rho}\frac{C_L^2}{C_L^3}}\right)_{max}$$

$$ROC_{max} = \left(\sqrt{\frac{W^3}{S}\frac{2}{\rho}\left(\frac{C_L^2}{C_L^3}\right)_{max}}\right)_{max}$$

## Minimum rate of descent in gliding flight

$$ROD_{min} = \frac{(P_a - P_r)_{min}}{W}$$

$$ROD_{min} = -\frac{(P_r)_{min}}{W} = -\frac{(DV)_{min}}{W}$$

$$ROD_{min} = -\frac{\left(D\frac{L}{L}V\right)_{min}}{W}$$

$$ROD_{min} = -\frac{\left(\frac{D}{L}W\sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_L}}\right)_{min}}{W}$$

$$ROD_{min} = -\sqrt{\frac{W}{S}\frac{2}{\rho}\left(\frac{C_D^2}{C_L^3}\right)_{min}}$$

$$ROD_{min} = -\sqrt{\frac{W}{S}\frac{2}{\rho}\left(\frac{C_L^3}{C_D^2}\right)_{max}}$$

## Load factor

$$n = \frac{L}{W}$$

$$n = \frac{C_L}{\frac{W}{S}\frac{2}{\rho}\frac{1}{V^2}}$$

## Minimum airspeed with load factor

$$V_{min} = \sqrt{n}\sqrt{\frac{W}{S}\frac{2}{\rho}\frac{1}{C_{L_{max}}}}$$

## Change of load factor

$$\Delta n = \frac{\Delta L}{W}$$

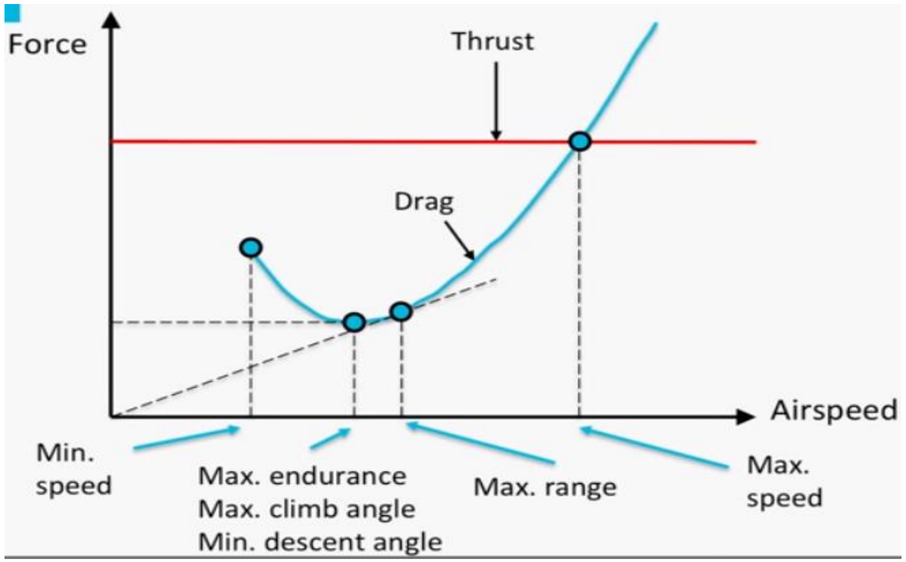
$$\Delta n = \frac{\Delta C_L \frac{1}{2}\rho(V^2 + U^2)S}{W}$$

$$\Delta n = \frac{\frac{dC_L}{d\alpha}\Delta\alpha\frac{1}{2}\rho V^2 S}{W}$$

$$\Delta n = \frac{\frac{dC_L}{d\alpha}\frac{U}{V}\frac{1}{2}\rho V^2 S}{W}$$

$$\Delta n = \frac{dC_L}{d\alpha}\frac{\rho}{2}\frac{S}{W}UV$$

## Performance diagram of an ideal jet




## Performance diagram of an ideal propeller