LINEAR ALGEBRA

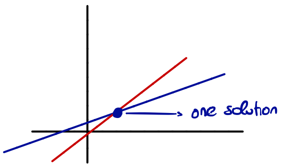
Linear Algebra

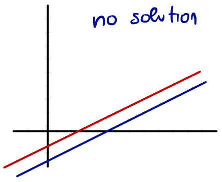
Lecture 1. Preparation Systems of linear equations

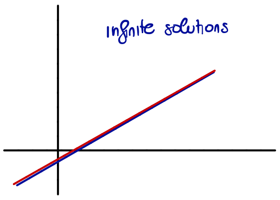
Linear equation

$$\begin{cases} 5x + y = 2 \longrightarrow 5x_1 + x_2 = 2 \\ 6x - 3y + 4z = 1 \longrightarrow 6x_1 - 3x_2 + 4x_3 = 1 \end{cases}$$ } the scalars are the coefficients of the equation

→ System of linear equations.







▹ Theorem: A linear system always has zero, one or infinitely many solutions.

- consistent: at least one solution
- inconsistent: no solutions

▹ Algorithm for solving a linear system.

Replace the system by an equivalent system easier to solve. Each being an operation of one of the following types:

1. One equation is replaced by the sum of itself and a multiple of another equation.
2. Two equations are interchanged
3. One equation is multiplied by a nonzero constant.

Pre-lecture HW

$x_1 + 5x_2 = 12$
$3x_1 + 8x_2 = 8$

$$\left[\begin{array}{cc|c} 1 & 5 & 12 \\ 3 & 8 & 8 \end{array}\right] -3 \sim \left[\begin{array}{cc|c} 1 & 5 & 12 \\ 0 & -7 & -28 \end{array}\right] \Rightarrow \quad \Rightarrow x_2 = 4$$

$X_1 + 5X^2 = 12$
$X_1 = -8$

Lecture 1

The augmented matrix:

$$\left.\begin{array}{l} x_1 + 5x_2 - 3x_3 = 1 \\ 2x_1 + x_2 - 15x_3 = 8 \\ -x_1 + x_2 + 3x_3 = 1 \end{array}\right\} \longrightarrow \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 2 & 1 & 15 & 8 \\ -1 & 1 & 3 & 1 \end{array}\right] -2 \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & -9 & 9 & 6 \\ -1 & 1 & 3 & 1 \end{array}\right] \times -1/3 \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & +3 & -3 & -2 \\ -1 & 1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ -1 & 1 & 3 & -2 \\ 0 & 3 & -3 & 1 \end{array}\right]$$

Row reduction: Example 1

Pivot ≠

focus on second and third block.

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 2 & 1 & 15 & 8 \\ -1 & 1 & 3 & 1 \end{array}\right] (-2)\ (1) \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & -9 & 9 & 6 \\ 0 & 6 & 6 & 2 \end{array}\right] (2/3) \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & -9 & 9 & 6 \\ 0 & 0 & 12 & 6 \end{array}\right].$$

Staircase form

no false equation
→ consistent

• no possibility of choice

• every column contains a pivot
→ unique solution

[0 0 0 | 2] it is not possible.

Example 2

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 2 & 1 & 15 & 8 \\ -1 & 1 & -9 & 1 \end{array}\right] (-2)\ (1) \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & -9 & 9 & 6 \\ 0 & 6 & -6 & 2 \end{array}\right] (2/3) \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & -9 & 9 & 6 \\ 0 & 0 & 0 & 6 \end{array}\right] \Big\}$$ No possible

Example 3

$$\left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 2 & 1 & 15 & 8 \\ -1 & 1 & -9 & -5 \end{array}\right] (-2)\ (1)\ (2/3) \sim \left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & -9 & 9 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ free variable

$x_3 = x_3$

Go back $\left[\begin{array}{ccc|c} 1 & 5 & 3 & 1 \\ 0 & -9 & 9 & 6 \\ 0 & 0 & 12 & 6 \end{array}\right]$ $(-1/4)\ (3/4)\ (1/12)$ Pivot $\sim \left[\begin{array}{ccc|c} 1 & 5 & 0 & -1/2 \\ 0 & -9 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$ $(5/9) \times \left(-\frac{1}{9}\right) \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -1/6 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$ $x_1 = 1/3$, $x_2 = -1/6$, $x_3 = 1/2$

→ continued $\left[\begin{array}{ccc|c} 1 & 0 & 8 & 1 \\ 0 & 1 & -1 & -2/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ solution $\left.\begin{cases} x_1 + 8x_3 = 1 \\ x_2 - x_3 = -2/3 \end{cases}\right\}$ x_3 is free $x_1 = 1 - 8x_3$, $x_2 = -2/3 + x_3$

Echelon Form

Definition: A rectangular matrix is in echelon form if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row (pivot) is in a colum to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros

Pivot, constat

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccccccccc|c} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

ceros

Solution Technique

1. Use elementary row operations to obtain echelon form of the augmented matrix
2. Determine if the system is consistent
3. Use backward substitution on the linear system of the echelon form to obtain the solutions

Definition:

A matrix in reduced echelon form if it has the following three properties:

1. It is in echelon form
2. The leading entry in each nonzero row is 1

Definition:

1. The basic variables of a linear system are the variables corresponding to pivot colums
2. The free variables of a linear system are the variables that have no pivot on the corresponding column.
3. The solution is found by expressing each basic variable as a function of the free variables.

Row equivalent: exists a sequence of row operations that transforms one matrix to another. Both have the same solution.

Pre lecture 2 Vectors and linear combinations

Vectors in $\mathbb{R}^n$

a vector $\underline{v}$ with $\underline{n}$ components is written as $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

- **Definition:**

  The set of all vectors with n components is called $\mathbb{R}^n$

- Equality and the zero vectors:

  Two vectors u and v in $\mathbb{R}^n$ are equal if all components are equal $v_1 = w_1$ $v_2 = u_2$ ...

  The zero vector 0 in $\mathbb{R}^n$ is a vector which has as components n zeros.

- **Vector Adition**

  - sum of u and v is equal to $u + v$

  $$u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \qquad u + v = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$$

- **Scalar Multiplication**

  - Given a vector "u" in $\mathbb{R}^n$ and a scalar c

  The scalar product of u by c is a new vector cu

  $$u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ and } c = 3 \qquad cu = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix}$$

- Linear combination

  vectors: $v_1, v_2, \dots v_p$

  scalars: $c_1, c_2 \dots c_p$ } vector y $\quad y = \underbrace{c_1}_{\text{weights}} \cdot v_1 + c_2 \cdot v_2 \dots\dots + c_p \cdot v_p$

- **Algebraic properties of $\mathbb{R}^n$** vectors u, v, w scalars: c, d

  a) $u + v = v + u$
  b) $(u+v) + w = u + (v + w)$
  c) $u + 0 = u$
  d) $u - u = 0$
  e) $c(u+v) = cu + cv$
  f) $(c+d)u = cu + du$
  g) $c(du) = (cd)u$
  h) $1u = u$

Example) Is $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

1. Scalar multiplication

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

2. Vector adition

$$\begin{bmatrix} x_1 + 2x_2 = 7 \\ -2x_1 + 5x_2 = 4 \\ -5x_1 + 6x_2 = -3 \end{bmatrix}$$

3. Augmented Matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{array}{l} x_1 = 3 \\ x_2 = 2 \end{array}}$$

Vector Equations

- A vector equation is an equation of the form
$$x_1 a_1 + x_2 a_2 + \dots x_n a_n = b \quad (1)$$
with $a_1, a_2 \dots a_n$ and b in $\mathbb{R}^n$ known.

- Theorem: A vector equation (1) has the same solutions as the linear system with augmented matrix
$$[a_1 \; a_2 \dots a_n \mid b]$$

Lecture 2 Spans and Matrix-Vector Products

Consistency: of a vector equation

- $x_1 a_1 + x_2 \cdot a_2 + \dots x_n \cdot a_n = b$
- b is a linear combination of the vectors $a_1, \dots, a_n$
- for which b is the linear system consistent.

Spans

- Definition: The subset of $\mathbb{R}^n$ spanned (or generated) by $a_1, a_2 \dots a_p$ in $\mathbb{R}^n$ is the set of all linear combinations of $a_1, a_2, \dots, a_p$
$$\text{Span}\{a_1, a_2, \dots, a_p\}$$
This set contains all vectors that can be written as
$$x_1 a_1 + x_2 a_2 + \dots + x_p \cdot a_p$$

Matrix-Vector Product

- Definition:

If A is an $m \times n$ matrix, with columns $a_1, \dots a_n$, and if x is in $\mathbb{R}^n$, then the product of A and x, denoted by Ax, is the linear combination of the columns of A using the corresponding entries in x as weight; that is.
$$Ax = [a_1 \; a_2 \; \dots \; a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 \dots a_n x_n$$

Matrix Equations

- Theorem:

If A is an $m \times n$ matrix, with columns $a_1 \dots a_n$, and if b is in $\mathbb{R}^m$, then the matrix equation $Ax = b$ has the same solution as the vector equation. $x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$ which has the same solution as $[a_1 \; a_2 \dots a_n \mid b]$

Pre Lecture 3 Solution Sets of Linear Equations

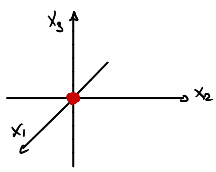
Homogeneous Equation:

- Systems for which the right hand sides consists of only zeros $\left.\begin{matrix} x_2 + x_3 = 0 \\ 2x_1 + 4x_2 + 4x_3 = 0 \end{matrix}\right\}$
- Trivial solution: to a homogeneous system is the solution with only zeros: $x_1 = 0 \; x_2 = 0 \; x_3 = 0$

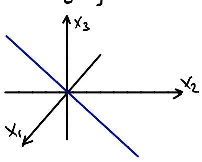
contains the origin.

- Non-trivial solutions: $x_1 = 0 \; x_2 = -x_3$ $\rightarrow \begin{cases} x_1 = 0 \\ x_2 = -t \\ x_3 = t \end{cases} \quad x = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
- If it has a trivial solution is consistent
- If there is a pivot in each column: one solution
- If there is no pivot in a colum: infinite solutions.

- $x = 0$

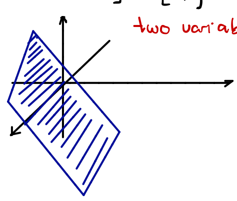


- $x = t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$



- $x = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

two variables.



Inhomogeneous Equations

$$\begin{cases} x_2 + x_3 = 2 \\ 2x_1 + 4x_2 + 4x_3 = 8 \end{cases} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{array}\right] \rightarrow \begin{cases} x_1 = 0 \\ x_2 = 2 - t \\ x_3 = 0 + t \end{cases} \rightarrow x = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}}_{\text{Particular solution.}} + t\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

- Translate of the homogeneous equation. from origin to vector tip

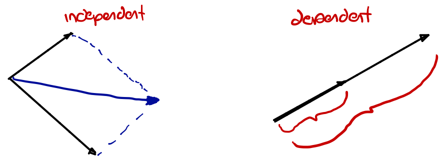
Solution Sets of Inhomogeneous Equations

- Theorem:
  - Consider the inhomogeneous equation $Ax=b$
  - The set of solutions is one of the:
    - empty
    - a translate of the solution set of the equation $Ax=0$
- Thus if $Ap=b$, then all solutions of $Ax=b$ are given in the form $x=p+v_h$ with $Av_h=0$

Lecture 3

Linearly Independent

A set of vectors $\{v_1, v_2 \dots v_p\}$ in $\mathbb{R}^n$ is linearly independent if the homogeneous equation $x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0$ only has the trivial solution.
Otherwise the set is called linearly dependent.



Linear Dependence

Theorem:
Any set $\{v_1, v_2, \dots v_p\}$ of p vectors in $\mathbb{R}^n$ is linearly dependent if $p>n$.

Theorem:
A set $\{v_1, v_2, \dots v_3\}$ is linearly dependent if and only if at least one of these vectors is a linear combination of the other vectors in the set.

Theorem:
$\{v_1, v_2\}$ with $v_2=0$ is linearly dependent

Solutions of a Homogeneous Linear Differential Equation

- Consider differential equation
  $y'' + 3y' + 2y = 0$
  - $y_1(t) = e^{-t}$
  - $y_2(t) = e^{-2t}$

Superposition tell us that linear combinations of solutions are solutions as well

$y(t) = c_1 e^{-t} + c_2 e^{-2t}$ $(c_1, c_2 \in \mathbb{R})$

- $c_1 e^{-t} + c_2 e^{-2t} = 0$ linearly independent?

  $\underbrace{c_2 = -c_1 e^{t}}$

  only works if c_2 and $c_1 = 0$ $y_1(t)$ and $y_2(t)$ are independent.

Pre-Lecture 4

- Consider $f(x)=b$
  - is there a solution?
  - How many solutions are there?
- Range:
  Set of all outcomes $f(x)$
- A function is onto if for each b in the codomain of f the equation $f(x)=b$ has at least one solution.
- A function is 1-1 if for each b in the codomain of f, the equation $f(x)=b$ has at most one solution.

Definition:

Matrix transformation: function T of the form $T(x) = Ax$ for some matrix A.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix} \quad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+3y+5z \\ 2y \quad 4z \end{bmatrix}$$

Domain $T: \mathbb{R}^3$
Codomain $T: \mathbb{R}^2$

$\boxed{T \text{ onto or 1-1?}}$ every row has a pivot
So $Ax=b$ have a solution for all b

Lecture 4 Slides

- Linear transformations if:
  - $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T
  - $T(cu) = cT(u)$ for all scalars c and u in the domain of T
  - Types: Rotation, Reflection, Shear, Contraction/Expansion, Projection

- Theorem: For any matrix A, the matrix transformation $T(x) = Ax$ is a linear transformation.

- Nonlinear transformations
  - $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 + x_1 \\ 3 - x_1 + x_2 \end{bmatrix}$
  - $S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_1 - x_2 \end{bmatrix}$

- Matrix of a linear transformations
  - Theorem:
    - Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique $m \times n$ matrix A such that $T(x) = Ax$ for all x in $\mathbb{R}^n$.
    - The columns of the matrix A are the images under T of the standard unit vectors: $A = [T(e_1) \dots T(e_n)]$. A is called the standard matrix of T.

- Properties of linear transformation
  - Theorem

    Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, with standard matrix A. Then:
    - T is one-to-one if and only if the following statements hold:
      - $T(x) = 0$ has only the trivial solution.
      - The columns of A are linearly independent.
      - Every column of A contains a pivot.
    - T is onto if and only if the following equivalent statements hold:
      - The columns of A span $\mathbb{R}^m$
      - Every row of A contains a pivot.

Lecture 5 | Matrix Operations

Algebraic properties:

Theorem: Let A, B, C be matrices of the same size and let r and s be scalars.

a. $A + B = B + A$

b. $(A+B)+C = A+(B+C)$

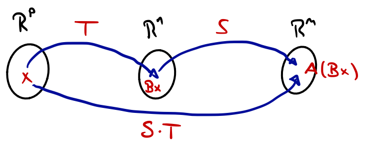
c. $A + 0 = A$

d. $r(A+B) = rA + rB$

e. $(r+s)A = rA + sA$

f. $r(sA) = (rs)A$

Compositions of linear transformations



Calculating AB in other ways

- column rule: column j of AB equals AB
- row-column rule: entry (i,j) of AB equals $row_i(A)b_j$
- row rule: row i of AB equals $row_i(A)B$

The identity matrix

Definition:

The identity matrix I_n is an $n \times n$ matrix with ones on the diagonal and zeros everywhere else.

$$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ALGEBRAIC PROPERTIES

Theorem: Let A be an $m\times n$ matrix, and let B and C be matrices with sizes for which the indicated sums and products are defined. Then:

a. $A(BC)=(AB)C$ b. $A(B+C)=AB+AC$ c. $(B+C)A=BA+CA$ d. $r(AB)=(rA)B=A(rB)$ for any scalar r e. $I_m\cdot A=A=A\,I_n$

WARNING

- If AB and BA are both well-defined then in general they are not equal.
- If $AB=AC$, then in general it is not true that $B=C$
- If $AB=0$, then in general you can not conclude that $A=0$ or $B=0$

PITCH, ROLL, YAW

- FOLLOWING ROTATION MATRICES

Roll: $R_x(\theta)=\begin{bmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{bmatrix}$ Pitch: $R_y(\theta)=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}$ Yaw: $R_z(\theta)=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$

- Multiple rotations can be obtained by multiplying rotation matrices.
- Are not commutative: final position may differ depending on the order.

POWERS OF A SQUARE MATRIX:

If A is an $n\times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A: $A^k=\underbrace{A\cdot A\dots A}_{k\text{ factors}}$

THE TRANSPOSE OF A MATRIX

DEFINITION: For each $m\times n$ matrix A the transpose of A, denoted by A^T is the matrix of size $n\times m$ which is obtained from A by interchanging the rows and the columns of A.

THEOREM: Let A and B be matrices with sizes such that the following operations are allowed.

a. $(A^T)^T=A$ b. $(A+B)^T=A^T+B^T$ c. $(rA)^T=rA^T$ for any scalar r d. $(AB)^T=A^T\cdot B^T$

LECTURE 6

INVERTIBLE MATRIX

DEFINITION: A square $n\times m$ matrix A is invertible if there is an $n\times n$ matrix C such that $CA=I_n$ and $AC=I_n$

If C exists, it is unique and it is called the inverse of A. It is denoted by A^{-1}. If C does not exist, we call A singular.

THEOREM:

Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $ad-bc\neq$, then A is invertible and $A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ if $ad-bc=0$, then A is singular.

USING AN INVERSE MATRIX

THEOREM: If A is an invertible $n\times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $Ax=b$ has the unique solution $x=A^{-1}\cdot b$

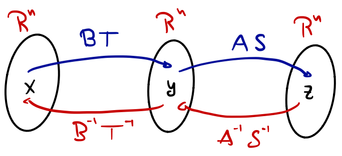
ALGEBRAIC PROPERTIES

THEOREM:

a: If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1}=A$

b: If A and B are $n\times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. that is $(AB)^{-1}=B^{-1}\cdot A^{-1}$

c: If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is, $(A^T)^{-1}=(A^{-1})^T$



ELEMENTARY MATRICES

Definition: An $n \times n$ elementary matrix is a matrix obtained by performing a single elementary row operation on I_n.

$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$E_3 \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \cdot 5 = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 5 & 10 \end{bmatrix}$$

INVERTIBLE MATRIX

Theorem: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}

Algorithm for finding A^{-1}:

Row reduce the augmented matrix $[A | I_n]$. If A is row equivalent to I_n, then $[A | I_n]$ is row equivalent to $[I_n | A^{-1}]$. Otherwise, A does not have an inverse.

THE INVERTIBLE MATRIX THEOREM

Theorem: Let A be a square $n \times n$ matrix. Then the following statements are logically equivalent:

a. A is an invertible matrix
b. A is row equivalent to I_n
c. A has n pivot positions.
d. The equation $Ax = 0$ has only the trivial solution
e. The columns of A form a linearly independent set
f. The linear transformation $x \to Ax$ is one-to-one
g. The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.

---

a. The columns of A span $\mathbb{R}^n$
b. The linear transformation $x \to Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
c. There is an $n \times n$ matrix C such that $CA = I$
d. There is an $n \times n$ matrix D such that $AD = I$
e. A^T is an invertible matrix.

INVERTIBLE LINEAR TRANSFORMATIONS

Definition: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$

$S(T(x)) = x \qquad T(S(x)) = x$

If S exists, S is the inverse of T and we write T^{-1}

Theorem:

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case $T^{-1}(x) = A^{-1}x$

LECTURE 7 LINEAR SUBSPACES

- **Superposition**

$\begin{bmatrix} 2 & 4 & 3 \\ 4 & 8 & 6 \end{bmatrix} x = 0$

- $x = 0$ is a solution
- $\underbrace{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ is a solution so is } \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}}_{\text{multiplication}}$
- $\underbrace{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \text{ are solutions, thus so is } \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{\text{Sum}}$

- **Linear Subspace**

- **Definition:**

A linear subspace of $\mathbb{R}^n$ is a set H satisfying the properties:

- 0 is in H
- If v and w are in H then so is $v + w$
- If v is in H and c in $\mathbb{R}$ then cv is in H

▹ Nullspaces

Definition:

The null space $Nul(A)$ of a matrix A is the set of solutions of the homogeneous equation $Ax=0$

$A=\begin{bmatrix}2&4&3\\4&8&6\end{bmatrix}$ $Ax=0:\left[\begin{array}{ccc|c}2&4&3&0\\4&8&6&0\end{array}\right] \overset{-2}{\sim} \begin{bmatrix}2&4&3&0\\0&0&0&0\end{bmatrix}$ $x_1=-2x_2-3/_2\,x_3$ x_2, x_3 free variables $x=t\begin{bmatrix}-2\\1\\0\end{bmatrix}+s\begin{bmatrix}-3\\0\\2\end{bmatrix}$

Definition:

The span of a set of vectors $\{v_1, v_2, \ldots v_n\}$ is the set of all linear combinations $c_1v_1+c_2v_2+\ldots+c_nv_n$

$Ax=0 \rightarrow x=t\begin{bmatrix}-2\\1\\0\end{bmatrix}+s\begin{bmatrix}-3\\0\\2\end{bmatrix}$

$Nul(A)=Span\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-3\\0\\2\end{bmatrix}\right\}$

Column Space

▹ **Definition:**

The column space $Col(A)$ of a matrix A is the set of all vectors of the form Ax

$A=\begin{bmatrix}2&4&3\\4&8&6\end{bmatrix}$ $Ax=x_1\begin{bmatrix}2\\4\end{bmatrix}+x_2\begin{bmatrix}4\\8\end{bmatrix}+x_3\begin{bmatrix}3\\6\end{bmatrix}$

▹ **Theorem**

The column space $Col(A)$ is the span of the column vectors of A

Lecture 7

- A set H is vector space if for every $\underline{u}, \underline{v} \in H$ and $c, d \in \mathbb{R}$ $c\underline{u}+d\underline{v} \in H$
- $Nul\,A=\{v \mid A\underline{v}=\underline{0}\}$ ▹ $Col\,B=Span\{\underbrace{\underline{b}_1, \underline{b}_2}_{\text{columns of }B}\}$ $\left[\begin{array}{cc|c}1&2&1\\4&1&1\\-1&0&1\end{array}\right]$ inconsistent

$v=\begin{bmatrix}1\\1\\1\end{bmatrix}$ $A=\begin{bmatrix}1&-4&3\\2&-3&1\end{bmatrix}$ $v=Nul(A)$

TWO IMPORTANT EXAMPLES OF SUBSPACES.

Theorem: the set of solutions to a homogeneous system of m equations with n unknowns is a linear subspace of $\mathbb{R}^n$.

Theorem: The span of a set of vectors $\{v_1 \ldots v_n\}$ in $\mathbb{R}^m$ is a linear subspace of $\mathbb{R}^m$ is a linear subspace of $\mathbb{R}^n$.

- $Nul(A)$ is a linear subspace of $\mathbb{R}^n$.
- **Proof**

let $\underline{u}, v \in Nul(A)$ $\rightarrow A\underline{u}=A\underline{v}=0$; let $c, d \in \mathbb{R}$ $\Big\}$ then $A(c\underline{u}+d\underline{v})=cA\underline{u}+dA\underline{v}=c\underline{0}+d\underline{0}=0$ $\rightarrow c\underline{u}+d\underline{v} \in Nul(A)$

- $Span\{v_1, \ldots, v_n\}=H$ (vector in $\mathbb{R}^n$) is linear subspace
- **Proof**

let $x, y \in H$ $\rightarrow$ then, there exists $c_1 \ldots c_n \in \mathbb{R}$, $d_1 \ldots d_n$ $\Big\}$ such that $x=c_1\cdot v_1 \ldots + c_n\cdot v_n$, $y=d_1\cdot v_1 \ldots + d_n\cdot v_n$; let $c, d \in \mathbb{R}$ $\Big\}$ now $cx+dy=c(c_1\cdot v_1 \ldots + c_n\cdot v_n)+d(d_1\cdot v_1+\ldots+d_n\cdot v_n)$

Null space and column space

Definition:

The column space of a matrix A is the set $Col(A)$ of all linear combinations of the columns of A

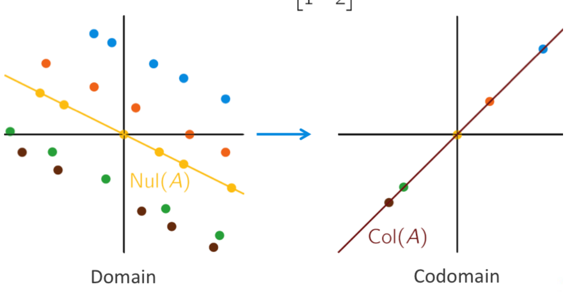
Remark: The column space $Col(A)$ of an $m\times n$ matrix A is a subspace of $\mathbb{R}^m$.

Definition:

The null space of a matrix A is the set $Nul(A)$ of all solutions of the homogeneous equation $Ax=0$

Remark: The null space $Nul(A)$ of an $m\times n$ matrix A is a subspace of $\mathbb{R}^n$.

$$T(\mathbf{x})=\begin{bmatrix}1&2\\1&2\end{bmatrix}\mathbf{x}$$



Basis

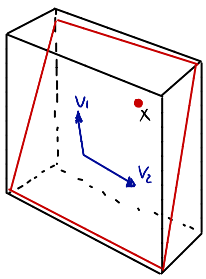
Definition:

- A basis for a subspace H of $\mathbb{R}^n$ is a set of vector which:
  - is linearly independent and
  - spans H

Theorem:

- The pivot columns of a matrix A form a basis for the column space of A.

Lecture 8 Dimensions Prelecture



$x_1 - x_2 + 2x_3 = 0$

$v_1 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

Consistency

$$\left[\begin{array}{cc|c} -1 & 2 & x_1 \\ 3 & 2 & x_2 \\ 2 & 0 & x_3 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 2 & x_1 \\ 0 & 8 & x_2 + 3x_1 \\ 0 & 0 & x_1 - x_2 + 2x_3 \end{array}\right]$$

$\mathcal{B} = \{v_1, v_2\}$

$$\boxed{x = \begin{bmatrix} 1 \\ 13 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}} \rightarrow \left[\begin{array}{cc|c} -1 & 2 & 1 \\ 3 & 2 & 13 \\ 2 & 0 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

No free variables
unique solutions

$\left.\begin{array}{l} c_1 = 3 \\ c_2 = 2 \end{array}\right\} \quad x_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Another Basis

$w_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad w_2 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \qquad \mathcal{C} = \{w_1, w_2\} \qquad x = \begin{bmatrix} 1 \\ 13 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 4 & 13 \\ 0 & 2 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} c_1 = 1 \\ c_2 = 3 \end{array}\right\} \quad x_{\mathcal{C}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Lecture 8 Dimension

Dimensions and Rank

Definition:

The dimension of a nonzero subspace H, denoted by $\dim(H)$, is the number of vectors in any basis for H.

Definition:

The rank of a matrix A, denoted by $\operatorname{rank}(A)$, is the dimension of the column space of A.

Rank Theorem

If a matrix A has n columns, then $\operatorname{rank}(A) + \dim(\operatorname{Nul}(A)) = n$

Theorem:

We can control the system if the matrix C has a rank n where

$C = [B \;\; AB \;\; \ldots \;\; A^{n-1}B]$

The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$.

- Any linearly independent set of exactly p elements in H is a basis for H
- Any set of p elements of H that spans H is a basis for H

Extension to Invertible Matrix Theorem

Suppose A is an $m \times n$ matrix. The following are equivalent:
- $\operatorname{Col}(A) = \mathbb{R}^m$
- $\operatorname{rank}(A) = m$
- Every row A has a pivot

Suppose A is an $m \times n$ matrix. The following are equivalent:
- $\operatorname{Nul}(A) = \{0\}$
- $\dim(\operatorname{Nul}(A)) = 0$
- Every column of A has a pivot

Pre-Lecture 9: Determinants

Theorem: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ invertible if $ad - bc \neq 0$ $\qquad A^{-1} = \overbrace{\frac{1}{ad-bc}}^{\text{determinants}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Submatrices

Definition: A_{ij} is a submatrix obtained from a matrix A with row i and column j removed.

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \qquad A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$$

Definition: The (i,j)-cofactor of a matrix A is C_{ij} and is given by $\boxed{C_{ij} = (-1)^{i+j} \det(A_{ij})}$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} \qquad C_{23} = (-1)^5 \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = 2$$

Determinants

Definition: The determinant of an $n \times n$ matrix A, with $n \geq 2$ is given by $\det(A) = a_{11} C_{11} + a_{12} + C_{12} + \dots + a_{1n} \cdot C_{1n}$ if $n = 1$ $\det(A) = A$

Lecture 9 Determinants

Theorem:

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column.

The cofactor expansion across row i is given by

$$\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \dots + a_{in} \cdot C_{in}$$

The cofactor expansion down column j is given by

$$\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} \cdot C_{nj}$$

Determinants of triangular matrices

Theorem:

If A is an $n \times n$ triangular matrix, then $\det(A)$ is the product of the entries of the main diagonal of A: $\det(A) = a_{11} \cdot a_{22} \cdots a_{nn}$

Row operations and determinants

Elementary row operations:

1. One row is replaced by the sum of itself and a multiple of another row.
2. Two rows are interchanged.
3. One row is multiplied by a nonzero constant k.

Theorem:

det A be a square matrix and $A \sim B$ using one row operation.

a. If row operation 1 was used, then $\det B = \det A$

b. If row operation 2 was used then $\det B = -\det A$

c. If row operation 3 was used then $\det B = k \cdot \det A$

INVERTIBILITY AND DETERMINANTS

Theorem:

A square matrix A is invertible if and only if $\det A \neq 0$

Theorem:

$\det A = 0$ if and only if the columns of A are linearly dependent.

PROPERTIES OF DETERMINANTS

Theorem:

If A is an $n \times n$ matrix then $\det A^T = \det A$

Column operations are handled in the same manner as row operations

Theorem:

If A and B are $n \times n$ matrices, then $\det(AB) = \det(A) \cdot \det(B)$

Corollary:

If A is an invertible matrix then $\det(A^{-1}) = \dfrac{1}{\det(A)}$

PRE LECTURE 10 CRAMER'S RULE

SYSTEM OF N EQUATIONS IN N UNKNOWNS

- $Ax = b$ for an $(n \times n)$-matrix A and vector b in $\mathbb{R}^n$
- System has unique solution only if ($\det A \neq 0$)
- In that case $x = A^{-1} b$

FORMULAS FOR INDIVIDUAL ENTRIES

- Solution $x = A^{-1} b$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \longrightarrow \begin{matrix} x_1 = ? \\ x_2 = ? \\ \vdots \\ x_n = ? \end{matrix}$$

(2x2) MATRIX

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad b = \begin{bmatrix} p \\ q \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \times \begin{bmatrix} p \\ q \end{bmatrix}$$

$$x = \frac{1}{ad - bc} \begin{bmatrix} dp - bq \\ -cp + aq \end{bmatrix}$$

$$\left\{ \begin{aligned} x_1 &= \frac{dp - bq}{ad - bc} \\ x_2 &= \frac{-cp + aq}{ad - bc} \end{aligned} \right.$$

CRAMER'S RULE IN (2x2)-CASE

$$A_1(b) = \begin{bmatrix} p & b \\ q & d \end{bmatrix} \qquad A_2(b) = \begin{bmatrix} a & p \\ c & q \end{bmatrix}$$

$$x_1 = \frac{\det A_1(b)}{\det A} \qquad x_2 = \frac{\det A_2(b)}{\det A}$$

CRAMER'S RULE GENERAL CASE

System $Ax = b$ with invertible $(n \times n)$-matrix A and b in $\mathbb{R}^n$ has a unique solution $x = [x_1, x_2, \dots, x_n]^T$

Cramer's rule: $x_i = \dfrac{\det(A_i(b))}{\det(A)}$

$A_i(b) = [a_1 \dots a_{i-1} \; b \; a_{i+1} \dots a_n]$

LECTURE 10 APPLICATION OF DETERMINANTS

Determinants as area or volume:

Theorem: If A is a 2x2 matrix, the area of the parallelogram determined by the columns of A is $|\det(A)|$.

If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det(A)|$.

Linear transformations and areas or volumes

Theorem:

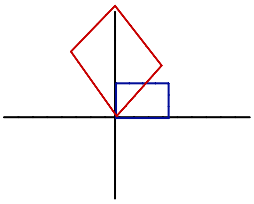
Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix A. If S is a finite region in $\mathbb{R}^2$, then:

$$\text{Area of } T(S) = |\det(A)| \cdot (\text{area of } S)$$

Similarly, if $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with standard matrix A and S is a finite region in $\mathbb{R}^3$, then:

$$\text{Volume of } T(S) = |\det(A)| \cdot (\text{volume of } S)$$

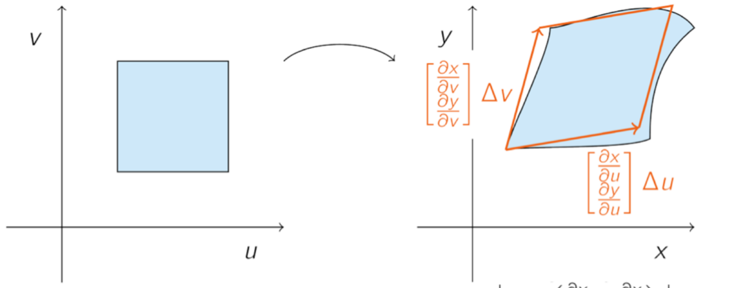
$A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ Determinant = 3



The linear transformation maps the rectangle with vertices $(0,0), (3,0), (3,2)$ and $(0,0)$ to the parallelogram with vertices $(0,0), (-2,4), (1,7)$ and $(3,3)$

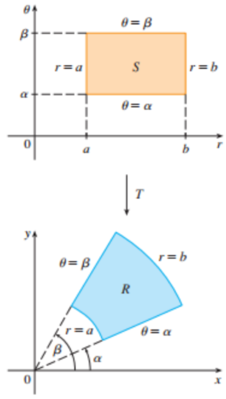
Integration and change of variables

A unit rectangle is mapped from (u,v)-coordinates to a region in (x,y)-coordinates that is approximated by a parallelogram.



Area: $\Delta u \Delta v$

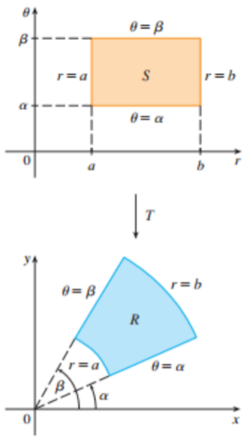
Area: $\left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial u} \end{pmatrix}\right| \Delta u \Delta v$



When you change variables during integration, you need the Jacobian. This is a determinant that described the scaling factor after a change of variables.

For example for polar coordinates we have

$$\iint_S f(x,y)\,dA$$
$$= \iint_R f(r\cos(\theta), r\sin(\theta))\,|\det(J)|\,dr\,d\theta$$



$$\iint_S f(x,y)\,dA$$
$$= \iint_R f(r\cos(\theta), r\sin(\theta))\det(J)\,dr\,d\theta$$

Here, the polar coordinates are

$$x = r\cos(\theta) \qquad y = r\sin(\theta),$$

so the Jacobian equals

$$\det(J) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$
$$= r\cos^2(\theta) + r\sin^2(\theta) = r$$

Prelecture 11 Inner Product and Orthogonality

Inner product in two dimensions

Definition:

ALGEBRAIC

For two vectors $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ The inner product: $a \cdot b = a_1 \cdot b_1 + a_2 \cdot b_2$

GEOMETRIC

$a \cdot b = \|a\| \cdot \|b\| \cdot \cos\theta$

length of a: $\sqrt{a_1^2 + a_2^2} = \sqrt{a \cdot a}$

Distance between a and b: $\text{dist}(a,b) = \|a - b\| = \sqrt{(a-b)(a-b)}$

Inner Product in $\mathbb{R}^n$

Definition:

For two vectors $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ $\quad a \cdot b = a_1 \cdot b_1 + a_2 \cdot b_2 + \dots + a_n \cdot b_n = a^T b$

Elementary properties:

Theorem:

- $a \cdot b = b \cdot a$ (symmetry)
- $a \cdot (b + c) = a \cdot b + a \cdot c$ (linearity)
- $a \cdot (k \cdot b) = k \cdot (a \cdot b) = (k \cdot a) \cdot b$ (linearity)
- $a \cdot a \geq 0$ and $a \cdot a = 0$ only if $a = 0$ (positivity)

$a \cdot a = a_1^2 + a_2^2 + \dots + a_n^2$

Geometry in $\mathbb{R}^n$

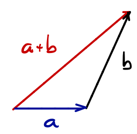
Definitions: for two vectors a and b in $\mathbb{R}^n$:

- The norm of a: $\|a\| = \sqrt{a \cdot a}$
- The distance between a and b: $dist(a,b) = \|a - b\|$
- orthogonal if $[a \cdot b = 0]$

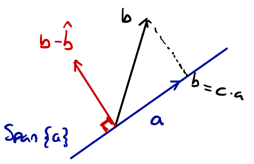
(A few) properties of norm and distance

$\|r a\| = |r| \, \|a\|$ $\qquad$ $\|a + b\| \leq \|a\| + \|b\|$ (triangle inequality)

$dist(a,b) = dist(b,a)$



Orthogonal Projection onto a line



$(b - \hat{b}) = (b - ca) \perp a \rightarrow (b - ca) \cdot a = 0$

$\boxed{c = \frac{b \cdot a}{a \cdot a}}$ $\qquad$ $\boxed{\hat{b} = \frac{b \cdot a}{a \cdot a} \cdot a}$

Lecture 11 Inner Product and Orthogonality

Orthogonal Complement

Definition: A vector x is orthogonal to a subspace S of $\mathbb{R}^n$ if $x \perp s$ for each s in S notation: $x \perp S$

The orthogonal complement of S, denoted $S^\perp$, is the set of all vectors x that are orthogonal to S, and is always a subspace of $\mathbb{R}^n$

Orthogonal Set

Definition: A set S of vectors $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is called an orthogonal set if $v_i \cdot v_j = 0$ for each pair

Theorem: An orthogonal set $S = \{v_1 \dots v_p\}$ of nonzero vectors is a linearly independent set.

Coordinates with respect to an orthogonal basis

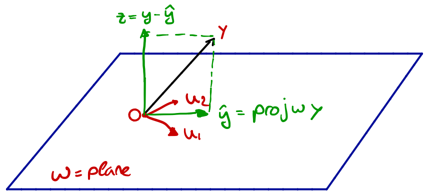
Definition: An orthogonal basis for a subspace W is a basis for W that is also an orthogonal set.

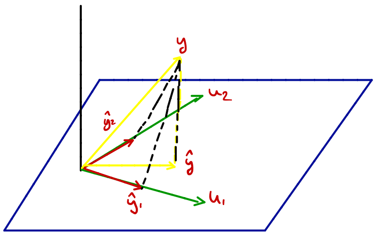
Theorem: If $S = \{v_1 \dots v_p\}$ is an orthogonal basis for a subspace W in $\mathbb{R}^n$ we can write any vector $y \in W$ in the following way:

$$y = \frac{y \cdot v_1}{v_1 \cdot v_1} \cdot v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 + \dots + \frac{y \cdot v_p}{v_p \cdot v_p} \cdot v_p$$

Pre lecture 12 Orthogonal projections

Orthogonal projection onto a plane





$\hat{y}_1 = proj_{u_1} y = \dfrac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1$

$\hat{y}_2\ proj_{u_2} y = \dfrac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2$

$\hat{y} = \hat{y}_1 + \hat{y}_2$

$\|y - \hat{y}\|$ = distance y to W : shortest distance of $\|y - \hat{y}\|$ is distance from y to W

Formula of orthogonal projection of Y onto W

Theorem:
The orthogonal projection $\hat{y}$ of y onto a plane W with orthogonal basis $\{u_1, u_2\}$ is:

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2$$

Lecture 12 Orthogonal Projections

Orthonormal sets; Orthogonal matrices

Definition:
An orthonormal set S is an orthogonal set $\{u_1, \dots, u_p\}$ of unit vectors

So, $u_i \cdot u_j = 0$ if $i \neq j$, and $u_i \cdot u_i = 1$

Theorem:
An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$

Definition:
An orthogonal matrix is a square invertible matrix U such that $U^{-1} = U^T$

Orthogonal Projections: The general case

Definition:

- Let W be a subspace of $\mathbb{R}^n$. Then each y in $\mathbb{R}^n$ can be written in the form $y = \hat{y} + z$.
- Where $\hat{y}$ is in W and z is in the orthogonal complement $W^\perp$.
- $\hat{y}$ is called the orthogonal projection of y.

Theorem:
The decomposition $y = \hat{y} + z$ is unique. If $\{u_1, \dots, u_p\}$ is an orthogonal basis of W then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \dots \frac{y \cdot u_p}{u_p \cdot u_p} \cdot u_p, \quad \text{and} \quad z = y - \hat{y}$$

Orthogonal Projection

- You can determine the matrix of the projection onto the (x,z)-plane. This is achieved by the mapping $T(x,y,z) = (x,0,z)$
- The matrix is thus given by $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Best Approximation

Theorem:

Let W be a subspace of $\mathbb{R}^n$, y a vector in $\mathbb{R}^n$ and let $\hat{y}$ be the orthogonal projection y onto W. Then

$\|y-\hat{y}\| \le \|y-v\|$ for all vectors v in W

Pre Lecture 13 The Gram-Schmidt Process

How to find an orthogonal basis?

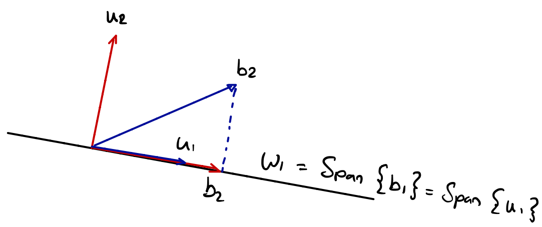
Suppose $H = \text{span}\{b_1, b_2, \dots, b_k\}$ is a subspace in $\mathbb{R}^n$.

1. Is there an orthogonal basis $\{u_1, u_2, \dots, u_k\}$ for H?
2. How to find $\{u_1, u_2, \dots, u_k\}$?

Numerical Example

$b_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ -8 \\ -7 \\ 6 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 6 \\ 1 \\ -2 \\ -7 \end{bmatrix}$

Find an orthogonal basis for $H = \text{span}\{b_1, b_2, b_3\}$



1. $u_1 = b_1$
2. $u_2 = b_2 - \hat{b}_2 = b_2 - \dfrac{b_2 \cdot b_1}{b_1 \cdot b_1} \cdot b_1 = b_2 - \dfrac{b_2 \cdot u_1}{u_1 \cdot u_1} u_1$
3. $u_3 = b_3 - \hat{b}_3 = b_3 - \dfrac{b_3 \cdot u_1}{u_1 \cdot u_1} \cdot u_1 - \dfrac{b_3 \cdot u_2}{u_2 \cdot u_2} \cdot u_2$

General Step

$$\boxed{u_{i+1} = b_{i+1} - \frac{(b_{i+1}) \cdot u_1}{u_1 \cdot u_1} \cdot u_1 - \frac{b_{i+1} \cdot u_2}{u_2 \cdot u_2} \cdot u_2 - \dots - \frac{b_{i+1} \cdot u_i}{u_i \cdot u_i} \cdot u_i}$$

What if the vectors are dependent?

Suppose $H = \text{span}\{b_1, b_2, \dots, b_k\}$ is a subspace in $\mathbb{R}^n$. If $\{b_1, b_2, \dots, b_k\}$ is a dependent set, can Gram-Schmidt still help to find an orthogonal basis for H?

Lecture 13 The Gram-Schmidt Process

- Construction of projection matrix from orthonormal basis

Corollary:

If $\{u_1, \dots, u_p\}$ is an orthonormal basis of W and $U = [u_1\ u_2 \cdots u_p]$, then

$$\text{proj}_W y = UU^T y$$

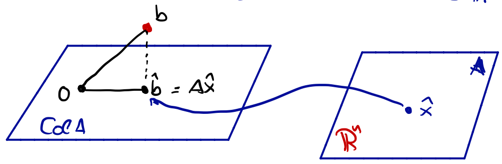
Pre-Lecture 14 Least-Squares Problems

Least Squares Solution

Definition:

- A is $m \times n$ matrix, $b \in \mathbb{R}^m$ and $Ax = b$ (in)consistent

A least-squares solution of $Ax = b$ is an $\hat{x} \in \mathbb{R}^n$ such that $\|b - A\hat{x}\| \le \|b - Ax\|$, for all $x \in \mathbb{R}^n$



$\hat{x}$ is least-square solution of $Ax = b$ if and only if $A\hat{x} = \hat{b}$.

$Ax = b$ (inconsistent) $\longrightarrow$ $Ax = \hat{b}$ (always consistent)

Solutions of $Ax = \hat{b}$ are called least-squares solutions and denoted by $\hat{x}$

Lecture 14 Least-Squares Problems

1. $\hat{x}$ is a least-square solution of $Ax = b$ if and only if $A\hat{x} = \hat{b} = \text{proj}_{\text{col}(A)}(b)$
2. If the columns of A are linearly independent, then the least squares solution of the system $Ax = b$ is unique

   If the columns of A are linearly dependent, then $Ax = b$ has infinitely many least-squares solutions.

Normal Equations

Theorem:

- The set of least-squares solutions of $Ax = b$ coincides with the nonempty set of solutions of the system.

  $A^TAx = A^Tb$ (normal equations)

- If A^TA is invertible, then the system $Ax = b$ has a unique least-squares solution $\hat{x}$, which is given by $\hat{x} = (A^TA)^{-1}A^Tb$

Pre-Lecture 15 Eigenvectors and Eigenvalues

- An eigenvector of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ
- A scalar λ is an eigenvalue of an $n \times n$ matrix A if the equation $Ax = \lambda x$ has a non-trivial solution.

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $Au = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4u$

$v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ $Av = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda v$

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\lambda_1 = -4$ ¿$\lambda_2 = 1$?

$\rightarrow Ax = x \longrightarrow Ax - x = 0 \longrightarrow Ax - 1x = 0$

$\rightarrow (A - I)x = 0 \longrightarrow \left[\begin{array}{cc|c} 0 & 6 & 0 \\ 5 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ NO NON-TRIVIAL SOLUTIONS

Lecture 15 Eigenvectors and Eigenvalues

Definition:

- A real number λ is an eigenvalue of a matrix A if there exists a nonzero vector x for which $Ax = \lambda x$. In that case the vector x is called an eigenvector corresponding to λ.

- The eigenspace of an $n \times n$ matrix A corresponding to the eigenvalue λ consists of the zero vector and all the eigenvectors with eigenvalue λ, i.e. it is the set of all solutions of the equation $(A - \lambda I)x = 0$

Theorem:

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem:

If $v_1, v_2 \ldots v_r$ are eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ respectively of an $n \times n$ matrix A, then the set $\{v_1, v_2, \ldots, v_r\}$ is linearly independent.

Characteristic Polynomial

- Of an $n \times n$ matrix A is given by the determinant $|A - \lambda I|$

Theorem:

A scalar λ_i is an eigenvalue of the $n \times n$ matrix A if and only if λ_i is a solution of the characteristic polynomial of A.

Multiplicity of Eigenvalues

- Algebraic multiplicity: α_i of an eigenvalue λ_i is the number of factors $(\lambda - \lambda_i)$ in the characteristic polynomial.
- Geometric multiplicity: of an eigenvalue λ_i is defined as the dimension of the eigenspace E_{λ_i}. number of independent eigenvectors for λ_i.

Dimension of the Eigenspace

Theorem:

For each eigenvalue λ the geometric multiplicity is at most equal to the algebraic multiplicity:

$$1 \leq \dim E_\lambda \leq \alpha_\lambda$$

Invertible Matrix Theorem

Theorem:

If A is an $n \times n$ matrix, then the following statements are logically equivalent

a. Invertible matrix

s. Number 0 is not an eigenvalue

t. The determinant of A is not zero.

PreLecture 16 Diagonisable Matrices

- To diagonize a matrix

$P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

- A matrix A is called diagonisable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix

Lecture 16 Diagonalization

Similarity:

Square matrices A and B are similar if there is an invertible matrix P such that

$A = PBP^{-1}$

Theorem:

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities)

Diagonizable matrices:

- A matrix A is diagonizable if and only if A similar to a diagonal matrix D.
- This means that A is equal to PDP^{-1} for some invertible matrix P and diagonal matrix D.

Diagonalization Theorem:

Theorem:

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Theorem:

An $n \times n$ matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces is n. That is:

$$\sum_{\lambda} \dim E_\lambda = n$$

Diagonalization matrices

1. Find the eigenvalues of A.
2. Find a basis for each eigenspace
3. Construct P
4. Construct D

$n \times n$ matrices with fewer than n distinct eigenvalues

Theorem: det A be an $n \times n$ matrix all n eigenvalues are real. A is diagonizable if and only if for each eigenvalue λ, the dimension of the eigenspace E_λ is equal to the algebraic multiplicity of λ.

Pre-Lecture 17 Complex Eigenvalues and Eigenvectors

Complex Plane

The complex plane C^2 is the vector space that consists of all vectors with two coordinates z_1 and z_2, where both z_1 and z_2 are complex numbers.

Complex Vectors

Real part: of a complex vector $z = x + iy$ with x and y in $\mathbb{R}^n$ is: $z = x$

Imaginary part: of a complex vector $z = x + iy$ with x and y in $\mathbb{R}^n$ is: $z = y$

Conjugate is: $\bar{z} = x - iy$

Real-Valued Matrices

Theorem:

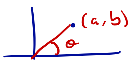
If A is an $m \times n$ matrix with real entries and λ is a (complex) eigenvalue of A with eigenvector v, then

- $\bar{\lambda}$ is also an eigenvalue of A
- $\bar{v}$ is the corresponding eigenvector.

Structure of a Matrix with Complex Eigenvalues

Theorem:

- Suppose a and b are real and not both zero and the matrix A is equal to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$
- The eigenvalues of A are $a + bi$ and $a - bi$.
- if $r = \sqrt{a^2 + b^2}$ $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$



Coordinate Vectors

Definition:

Assume $B = \{b_1, \dots, b_p\}$ is a basis for a subspace W of $\mathbb{R}^n$. The coordinate vector of x in W relative to B is $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$ with $c_1, \dots, c_p$ such that

$$x = c_1 \cdot b_1 + \dots + c_p \cdot b_p$$

Coordinate Transformations

Theorem:

Assume $B = \{b_1, \dots, b_n\}$ is a basis for $\mathbb{R}^n$ and $P = [b_1 \dots b_n]$. Then for x in $\mathbb{R}^n$

$$P[x]_B = x \leftrightarrow [x]_B = P^{-1}x$$

$$A = P \cdot C \cdot P^{-1}$$

$$\begin{tikzcd}
x \arrow[r, "\text{mult by } A"] \arrow[d, "\text{mult by } P^{-1}"'] & Ax \arrow[d, "\text{mult by } P"] \\
y = [x]_B \arrow[r, "\text{mult by } C"'] & {[Ax]_B = Cy}
\end{tikzcd}$$

Real Matrices with Complex Eigenvalues

Theorem:

Let A be a real 2×2 matrix with complex eigenvalue $a - bi$ and an associated eigenvector v in C^2. Then $A = PCP^{-1}$, where

$$P = [\operatorname{Re} v \cdot \operatorname{Im} v] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$