Mechanics of Materials Summary

1. Stresses and Strains

1.1 Normal Stress

Let's consider a fixed rod. This rod has length L. Its cross-sectional shape is constant and has area A.

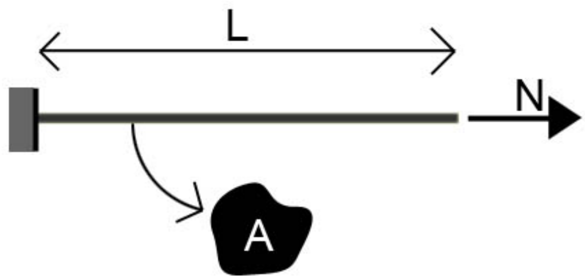

Figure 1.1: A rod with a normal force applied to it.

Now we will start pulling the rod. We apply a tensional load N to it, as shown in figure 1.1. At every point in the rod will then be a **normal force** N. The **normal stress** is now defined as

$$\sigma = \frac{N}{A}. \tag{1.1.1}$$

1.2 Normal Strain

Due to the normal force, the rod will elongate in its longitudinal direction. Its length will change by a **displacement** δ. The **strain** is now defined as

$$\varepsilon = \frac{\delta}{L}. \tag{1.2.1}$$

There is a relation between the normal stress and the strain. This relation is usually given by a **stress-strain diagram**. However, it is hard to do calculations if you only have a diagram. But as long as **yielding** does not occur, the relation between σ and ε is linear. The **E-modulus** (also called **Young's modulus** or **modulus of elasticity**) is then defined such that

$$E = \frac{\sigma}{\varepsilon}. \tag{1.2.2}$$

The normal force will not only cause the rod to elongate in its longitudinal direction. It will also contract in its axial direction. There is a relation between the longitudinal strain $\varepsilon_{longitudinal}$ and its axial strain ε_{axial}. This relation is

$$\nu = -\frac{\varepsilon_{axial}}{\varepsilon_{longitudinal}}. \tag{1.2.3}$$

Here ν is **Poisson's ratio**. Due to the minus sign, Poisson's ratio is always positive. Its value will usually be around 0.3.

1.3 Rod Elongation

Let's suppose we know the dimensions (A and L), properties (E) and loading conditions (N) of a rod. Can we find the elongation δ? In fact, we can. Using the definitions we just made, we find that

$$\delta = \frac{NL}{EA}. \tag{1.3.1}$$

There is a fundamental assumption behind this equation. We have assumed that the normal force N, the E-modulus E and the cross-sectional area A are constant throughout the length of the rod. If this is not the case, things will be a bit more difficult. Now we can find the displacement using

$$\delta = \int_0^L \frac{N}{EA}\,dx. \tag{1.3.2}$$

Note that for constant N, E and A this equation reduces back to equation (1.3.1).

1.4 Shear Stress

Now let's not pull the rod. Instead, we put a load on it as shown in figure 1.2. This will cause a **shear force** to be present in the rod.

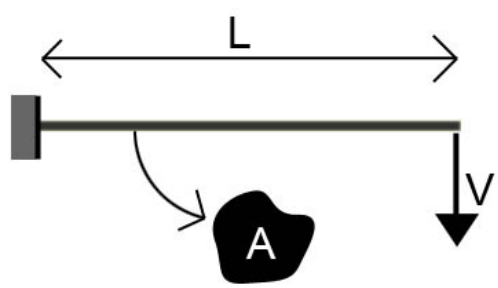

Figure 1.2: A rod with a shear force applied to it.

Just like a normal force results in a normal stress, so will a shear force result in a shear stress. The **shear stress** τ is

$$\tau = \frac{V}{A}. \tag{1.4.1}$$

This equation isn't entirely exact. This is because the shear stress isn't constant over the entire cross-section. We will examine shear stress in a later chapter.

1.5 Shear Strain

A shear force also causes some kind of deformation. This time we will have **shear strain**. Shear strain can be defined as the change of an angle that was previously 90°. So shear strain is an angle (with unit radians). Since the idea of the shear strain can be a little bit hard to grasp, figure 1.3 is present to clarify it a bit.

There is a relation between the shear stress τ and the shear force γ. This relation is given by

$$G = \frac{\tau}{\gamma}, \tag{1.5.1}$$

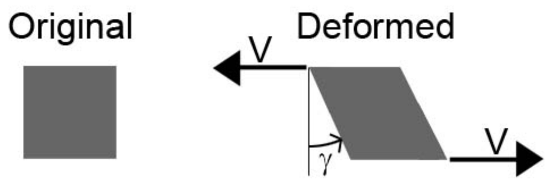

Figure 1.3: Clarification of the shear strain.

where G is the **shear modulus** (also called **shear modulus of elasticity** or **modulus of rigidity**). Just like the E-modulus, the shear modulus is a material property. There also is a relation between the E-modulus and the shear modulus. This relation involves Poisson's ratio ν. In fact, it is

$$\frac{E}{G} = 2\,(1+\nu)\,. \tag{1.5.2}$$

2. Thermal Effects and Prestress

2.1 Heating an Object

When an object is heated, it expands. An important parameter during this expansion if the **coefficient of thermal expansion** α. The strain due to thermal effects, the so-called **thermal strain** ε_T, can then be found using

$$\varepsilon_T = \alpha\,\Delta T \qquad \text{and also} \qquad \delta_T = L\alpha\,\Delta T, \tag{2.1.1}$$

where ΔT is the temperature difference and δ_T is the elongation of the bar due to this temperature difference.

The value of ΔT is not always equal for different points on the rod. If the heating is performed unevenly, it is more difficult to find δ_T. This time we have to use

$$\delta_T = \int_0^L \varepsilon_T\,dx = \int_0^L \alpha\,\Delta T\,dx. \tag{2.1.2}$$

2.2 Heating a Blocked Rod

Now suppose we have a rod, clamped on both sides. Let's heat this rod. It tries to expand, but the walls won't make way for this expansion. Instead, the walls exert a compressive force N on the rod to keep it at its original length. The entire situation is shown in figure 2.1.

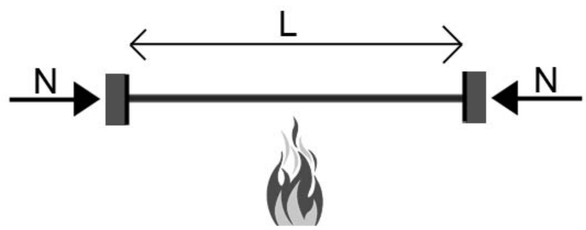

Figure 2.1: The heating of a blocked rod.

The bar retains its original length. So the total strain is 0. This strain consists of a thermal part and a part due to the normal force. So we find that

$$\varepsilon_T + \varepsilon_N = 0 \qquad \Rightarrow \qquad \alpha\,\Delta T - \frac{N}{EA} = 0. \tag{2.2.1}$$

Note that the minus sign is present because the force N is compressive. The force N in the original equation (from the previous chapter) was tensional. The above equation is called a **compatibility equation**. It is the extra equation necessary to solve the problem.

Now the force exerted by the walls can be found. Also the stress due to this thermal effect can be found. They are

$$N = EA\alpha\,\Delta T \qquad \text{and} \qquad \sigma_T = -\frac{N}{A} = -E\alpha\,\Delta T = -E\varepsilon_T. \tag{2.2.2}$$

Once more the minus sign is present, for the same reason as in the last equation.

2.3 Prestress

In the previous chapter we saw that there was a constant force acting on the rod. It was as if the rod was too small to fit between the walls. This gives us an idea.

Let's take a rod that is only slightly too long to fit between two walls. When we want to install the rod, we first need to compress it (to shorten it). Let's suppose a (compressive) **prestress force** Q is necessary for this. After the rod has been placed, there will always be a certain stress in the rod. This **prestress** σ_P is then equal to

$$\sigma_P = -\frac{Q}{A}. \tag{2.3.1}$$

The elongation of the bar due to this prestress then is

$$\delta_P = \int_0^L \frac{\sigma_P}{E} dx = -\int_0^L \frac{Q}{AE} dx. \tag{2.3.2}$$

There are minus signs again because Q is a compressive force. Note that this is logical, as we have compressed the bar, so δ_P must be negative as well.

2.4 The Turnbuckle

Now we don't want to place a rod between two walls, but a cable. Installed in this cable is a **turnbuckle**. This is a device with which you can shorten the cable slightly. It does this by (sort of) removing a piece of the cable.

If you turn a turnbuckle once, it removes a length p from one side of the cable, where p is the **turnbuckle pitch**. However, a turnbuckle has a cable on both of its sides (both left and right). So one turn will cause a shortening of the rope of $2p$. In general, we can now say that the displacement of the rope due to the turnbuckle is

$$\delta_P = -2np, \tag{2.4.1}$$

where n is the amount of turns you have made. The minus sign is present because the turnbuckle decreases the length of the rope.

Figure 2.2: Situation sketch of the cable with the turnbuckle.

Let's now install a cable between two walls, as shown in figure 2.2. Initially this cable is simply hanging horizontally, without any stress in it. Then we start turning the turnbuckle. Since the cable is fixed to the walls, it needs to retain its original length. This causes the walls to exert a force on the cables. This force can be found using

$$\delta_P + \delta_N = 0 \qquad \Rightarrow \qquad -2np + \frac{NL}{EA} = 0, \tag{2.4.2}$$

where N is the tensional force exerted by the walls on the cable. It follows that

$$N = \frac{2np}{L} EA \qquad \text{and} \qquad \sigma_P = \frac{N}{A} = \frac{2np}{L} E. \tag{2.4.3}$$

3. Mechanics of Materials Theory

3.1 Thin-Walled Structures

Before we venture further into the depths of mechanics of materials, we first need to discuss some basic ideas. The first idea we will discuss is that of thin-walled structures. Let's consider the I-beam cross-section shown in figure 3.1. Such a cross-section is often used in constructions for reasons we will discover in later chapters.

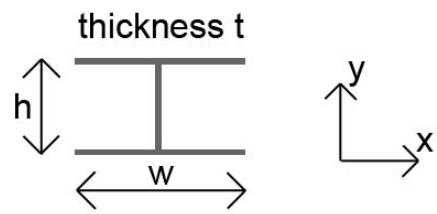

Figure 3.1: Another example of a cross-sectional shape: an I-beam.

We can try to calculate the area of this I-beam. We see a horizontal part of the beam with width w, a vertical part with height h and another horizontal part with height w. So we might initially think that

$$A = wt + ht + wt = 2wt + ht. \tag{3.1.1}$$

However, now we've counted the overlapping parts twice. If we keep those into account, we will get

$$A = wt + (h - 2t)t + wt = 2wt + ht - 2t^2. \tag{3.1.2}$$

The second way of calculating things is more exact. However, as cross-sectional shapes get more complicated, this second way will become rather difficult.

Now let's look at the differences. The difference between the two methods is in this case $2t^2$. Usually the thickness t is small compared to the width w and height h. The quantity t^2 is then very small. If this is the case, we say we have a **thin-walled structure**. We then may neglect any factors with t^2 and such.

So the key idea is as follows: In thin-walled structures you do not have to consider the very small parts (with high powers of t). There will be no terms like $(h - 2t)$ and such. These terms all simplify to just h. By using this assumption, the analysis of many cross-sections is simplified drastically.

3.2 Center of Gravity - Method 1

In the next chapters we will be analyzing forces in beams. The way in which the beams behave, strongly depends on the shape of its cross-section. So the coming couple of paragraphs we will examine the cross-sections of beams.

A first thing which we need to find is the position of the **center of gravity** of a cross-section. This position has coordinates $(\overline{x}, \overline{y})$. The values of $\overline{x}$ and $\overline{y}$ can be found using

$$\overline{x} = \frac{1}{A}\int_A x\,dA \qquad \text{and} \qquad \overline{y} = \frac{1}{A}\int_A y\,dA. \tag{3.2.1}$$

To explain how to use these equations, we use an example. Let's consider the rectangle shown in figure 3.2.

Figure 3.2: An example of a cross-sectional shape: a square.

The value of $\overline{x}$ can be found using

$$\overline{x} = \frac{1}{A}\int_A x\,dA = \frac{1}{hw}\int_0^w h\,x\,dx = \frac{1}{hw}\frac{1}{2}hw^2 = \frac{1}{2}w. \tag{3.2.2}$$

Identically we can find that $\overline{y} = \frac{1}{2}h$. So the center of gravity is exactly in the middle of the rectangle, as was expected.

3.3 Center of Gravity - Method 2

The method of the previous paragraph has a few downsides. For complicated cross-sections it is often very hard to evaluate the integral. Cross-sectional shapes often consist of an amount of sub-shapes, of which you already know the position of the center of gravity. It would be much easier to use that fact to find the position of the center of gravity.

Now we will do exactly that. Let's define A_{tot} as the total area of the cross-section. We can now find $\overline{x}$ and $\overline{y}$ using

$$\overline{x} = \frac{1}{A_{tot}}\sum x_i A_i \qquad \text{and} \qquad \overline{y} = \frac{1}{A_{tot}}\sum y_i A_i, \tag{3.3.1}$$

where x_i, y_i is the position of the center of gravity of every subpart i and A_i the corresponding area. For the shape of figure 3.1 (the I-beam) we will thus get

$$\overline{y} = \frac{1}{A_{tot}}\sum y_i A_i = \frac{1}{2wt+ht}\left((0)(wt) + \left(\frac{1}{2}h\right)(ht) + (h)(wt)\right) = \frac{\frac{1}{2}h^2t + wht}{2wt+ht} = \frac{1}{2}h. \tag{3.3.2}$$

The center of gravity is exactly in the middle. We could have expected that, since the structure is symmetric. Instead of having to integrate things, we only need to add up things with this method. That's why this method is mostly used to find the center of gravity of a cross-section.

3.3.1 Moment of Inertia - Method 1

A quantity that often occurs when analyzing bending moments is

$$I_x = \int_A (y - y_r)^2\,dA \qquad \text{and} \qquad I_y = \int_A (x - x_r)^2\,dA. \tag{3.3.3}$$

Here the point x_r, y_r is a certain reference point. The parameters I_x and I_y are called the **second moment of inertia** about the x-axis/y-axis. The second moment of inertia is also called the **area moment of inertia**, or shortened it is just **moment of inertia**. The moment of inertia is always minimal if you calculate it with respect to the center of gravity (so if $x_r = \overline{x}$ and $y_r = \overline{y}$).

To see how these equations work, we will find the value I_x for a rectangle, as shown in figure 3.2. We do this with respect to the center of gravity. We thus get

$$I_x = \int_A (y - \overline{y})^2 \, dA = \int_0^h \left(y - \frac{1}{2}h \right)^2 w \, dx = \left[\frac{1}{3} w \left(y - \frac{1}{2}h \right)^3 \right]_0^h = \frac{1}{12} wh^3. \tag{3.3.4}$$

Identically, we find that the moment of inertia about the y-axis is $I_y = \frac{1}{12} hw^3$.

Next to I_x and I_y, there is also the **polar moment of inertia** I_p, defined as

$$I_p = \int_A \left(x^2 + y^2 \right) \, dA = I_x + I_y. \tag{3.3.5}$$

This quantity is also often denoted as J.

3.4 Moment of Inertia - Method 2

Evaluating an integral every time we need to know a moment of inertia is rather annoying. There must be a more simple method. It would be great if we can evaluate a cross-section part by part. Well, why can't we do this?

There is a good reason for that. A moment of inertia is always with respect to a certain reference point. We can't add up the moment of inertias of parts with different reference points. So what we need to do, is make the reference points for every part equal. We need to move them! More specifically, we need to move them to the center of gravity of the entire cross-section. How do we do this?

There is a rule, called **Steiner's rule** (also called the **parallel axis theorem**), with which you can move the reference point. Suppose we have the moment of inertia with respect to the center of gravity and want to know it with respect to another point x_r, y_r. The equations we can then use are

$$I_{x_r} = I_{x_{cog}} + A \left(y_r - y_{cog} \right)^2 \qquad \text{and} \qquad I_{y_r} = I_{y_{cog}} + A \left(x_r - x_{cog} \right)^2. \tag{3.4.1}$$

So if we move the reference point away from the center of gravity by a distance d in a certain direction, then the moment of inertia for the corresponding axis increases by Ad^2. This term Ad^2 is called **Steiner's term**.

So, taking into account the Steiner's term, we can derive another expression for the moment of inertia. It will become

$$I_{x_{tot}} = \sum \left(I_{x_i} + A_i \left(y_{cog_{tot}} - y_{cog_i} \right)^2 \right) \qquad \text{and} \qquad I_{y_{tot}} = \sum \left(I_{y_i} + A_i \left(x_{cog_{tot}} - x_{cog_i} \right)^2 \right). \tag{3.4.2}$$

If we apply this to the I-shaped beam of figure 3.1, we find

$$I_x = 2 \left(\frac{1}{12} wt^3 + (wt) \left(\frac{1}{2}h \right)^2 \right) + \frac{1}{12} th^3 = \frac{1}{2} wth^2 + \frac{1}{12} th^3. \tag{3.4.3}$$

We don't have a Steiner's term for the vertical part of the beam (the so-called **web**). This is because the center of gravity of this part coincides with the center of gravity of the whole cross-section. Also note that we have ignored the term involving t^3 at the horizontal parts (the **flanges**). This is because we assumed we are dealing with a thin-walled structure.

3.5 The Moment of Inertia for Common Cross-Sectional Shapes

To apply the method of the previous paragraph, we need to know the moment of inertia for a couple of shapes. Some of them are given here. The following moment of inertias are with respect to the center of gravity of that cross-section. For that reason, the position of the center of gravity is also given.

- **A rectangle**, with width w and height h. The center of gravity is in its center (at height $\frac{1}{2}h$ and width $\frac{1}{2}w$). The moment of inertias are

$$I_x = \frac{1}{12} w h^3 \qquad \text{and} \qquad I_y = \frac{1}{12} h w^3. \tag{3.5.1}$$

- **An isosceles triangle**, with its base down and tip up. (The tip then lies centered above the base.) The base has width w, and the triangle has height h. The center of gravity now lies on height $\frac{1}{3}h$ and on width $\frac{1}{2}w$. The moment of inertias are

$$I_x = \frac{1}{36} w h^3 \qquad \text{and} \qquad I_y = \frac{1}{48} h w^3. \tag{3.5.2}$$

- **A circle**, with radius R. Its center of gravity lies at the center of the circle. The moment of inertias are

$$I_x = I_y = \frac{1}{4}\pi R^4 \tag{3.5.3}$$

- **A tube**, with inner radius R_1 and outer radius R_2. Its center of gravity lies at the center of the tube. The moment of inertias are

$$I_x = I_y = \frac{1}{4}\pi \left(R_2^4 - R_1^4\right) \tag{3.5.4}$$

4. Normal Forces and Bending Moments

4.1 Introduction to Bending

Let's consider a beam. We can bend this beam with only a bending moment M. This form of bending (bending without any normal forces) is called **pure bending**. But what will happen to the beam under pure bending? To find that out, we have to look at a small part dx of the beam, as is done in figure 4.1.

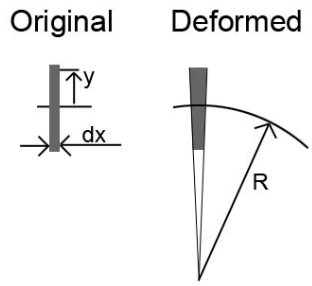

Figure 4.1: Deformation of a small part of beam under bending.

We see that one part of the beam (in this case the top part) elongates, while the other part is being contracted. So part of the beam has tensile stress, while the other has compressive stress. Also, there is some part in the beam without any stresses. This part is called the **neutral axis** (also called **neutral line**).

Let's see if we can find an expression for the stress. For that, we first ought to consider the strain. This strain depends on the **bend radius** R and the position in the beam y. Here y is the distance from the neutral axis. We will find that the strain due to a bending moment is

$$\varepsilon_M(y) = \frac{L_{new} - L_{old}}{L_{old}} = \frac{\frac{R+y}{R}dx - dx}{dx} = \frac{y}{R}. \tag{4.1.1}$$

For some reason engineers don't like to work with a bend radius. Instead, the **curvature** is defined as $\kappa = 1/R$. Let's now assume that no yielding (no permanent deformation) occurs. Then we have as stress

$$\sigma_M = \varepsilon_M E = \frac{Ey}{R} = Ey\kappa. \tag{4.1.2}$$

So the stress varies linearly with the distance y. That's nice to know! It's often relatively easy to work with linear relations. However, our job of analyzing bending is not finished yet.

4.2 Stress as a Function of the Bending Moment

One thing we would still like to know, is where the neutral axis will be. To find it, we have two pieces of data: The equations of the previous paragraph and the fact that there is no normal force (since we are dealing with pure bending). So let's see if we can find the normal force. This force is

$$0 = N = \int_A dF = \int_A \sigma_M dA = \int_A Ey\kappa\, dA = E\kappa \int_A y\, dA. \tag{4.2.1}$$

E and κ are constant for this cross-section. They are also nonzero, so the integral must be zero. It can be shown that this is only the case if y is the distance from the center of gravity of the cross-section. So the neutral line is the center of gravity of the cross-section!

That's very nice to know, but we still can't calculate much. Although we know the stress as a function of the curvature κ, this curvature is usually unknown. Can we express the stress as a function of the bending moment M? In fact, we can. This bending moment can be found using

$$M = -\int_A y \, dF = -E\kappa \int_A y^2 \, dA = -E\kappa I, \tag{4.2.2}$$

where I is the moment of inertia about the horizontal axis of the cross-section. The minus sign in the equation is present due to the sign convention of the bending moment. A couple of engineers decided that to bend the beam as shown in figure 4.1, a negative bending moment was required.

So now we have expressed the curvature κ as a function of the bending moment. Combining this with the equations of the previous paragraph, we find the equation for the stress under pure bending (the so-called **flexure formula**). It is

$$\sigma_M = -\frac{My}{I}. \tag{4.2.3}$$

We can derive another important fact from this equation. We can find where maximum stress occurs. M and I are constant for a cross-section. So maximum stress occurs if y is maximal. This is thus at the top/bottom of the cross-section.

4.3 Adding a Normal Force

So now we know the normal stress in a beam when it is subject to only a bending moment. What happens when we subject a beam to a normal force N? Well, if there is only a normal force, then the normal stress is easy to find. It is

$$\sigma_N = \frac{N}{A}. \tag{4.3.1}$$

But what do we do if we have a normal force and a bending moment simultaneously? The answer to that question is quite simple. We add the stress up. So the total stress σ_T then becomes

$$\sigma_T = \sigma_N + \sigma_M = \frac{N}{A} - \frac{My}{I}. \tag{4.3.2}$$

The principle used in this relation is the principle of **superposition**.

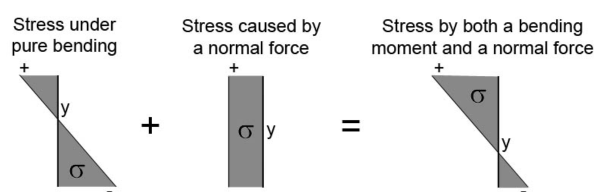

Figure 4.2: Stress diagram for different loading cases.

Now let's examine the stress in a beam. We do this using the stress diagrams of figure 4.2. When there was only a bending moment, the neutral axis was at the center of gravity of the cross-section. Applying a normal force shifts the neutral line by a distance d. This distance can be found using

$$\frac{N}{A} - \frac{Md}{I} = 0 \qquad \Rightarrow \qquad d = \frac{NI}{MA}. \tag{4.3.3}$$

When the normal force gets sufficiently big, the neutral line may move outside of the cross-section. Then there isn't any part in the beam anymore without stress.

5. Shear Stress and Shear Flow

5.1 The Shear Stress Equation

Now we have expressions for the normal stress caused by bending moments and normal forces. It would be nice if we can also find an expression for the shear stress due to a shear force. So let's do that.

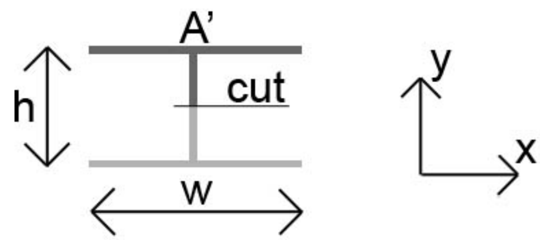

Figure 5.1: An example cross-section, where a cut has been made.

We want to know the shear stress at a certain point. For that, we need to look at the cross-section of the beam. One such cross-section is shown in figure 5.1. To find the shear stress at a given point, we make a cut. We call the thickness of the cut t. We take the area on one side of the cut (it doesn't matter which side) and call this area A'. Now it can be shown that

$$\tau = \frac{V}{It} \int_{A'} y \, dA. \tag{5.1.1}$$

Here y is the vertical distance from the center of gravity of the entire cross-section. (Not just the part A'!) Let's evaluate this integral for the cross-section. We find that

$$\tau = \frac{V}{It} \int_{A'} y \, dA = \frac{V}{It} \left(\int_0^{h/2} yt \, dy + \int_{h/2}^{h/2+t} yw \, dy \right) = \frac{V}{It} \left(\frac{1}{8} h^2 t + \frac{1}{2} wht \right). \tag{5.1.2}$$

Note that we have used the thin-walled structure principle in the last step.

So now we have found the shear stress. Note that this is the shear stress at the position of the cut! At different places in the cross-section, different shear stresses are present.

One thing we might ask ourselves now is: Where does maximum shear stress occur? Well, it can be shown that this always occurs in the center of gravity of the cross-section. So if you want to calculate the maximum shear stress, make a cut through the center of gravity of the cross-section. (An exception may occur if torsion is involved, but we will discuss that in a later chapter.)

5.2 A Slight Simplification

Evaluating the integral of (5.1.1) can be a bit difficult in some cases. To simplify things, let's define Q as

$$Q = \int_{A'} y \, dA, \tag{5.2.1}$$

implying that

$$\tau = \frac{VQ}{It}. \tag{5.2.2}$$

We have seen this quantity Q before. It was when we were calculating the position of the center of gravity. And we had a nice trick back then to simplify calculations. We split A' up in parts. Now we have

$$Q = \sum y_i A_i', \tag{5.2.3}$$

with A_i' the area of a certain part and y_i the position of its center of gravity. We can apply this to the cross-section of figure 5.1. We would then get

$$\tau = \frac{V}{It}\left(\left(\frac{1}{4}h\right)\left(\frac{1}{2}ht\right) + \left(\frac{1}{2}h\right)(wt)\right). \tag{5.2.4}$$

Note that this is exactly the same as what we previously found (as it should be).

5.3 Shear Flow

Let's define the **shear flow** q as

$$q = \tau t = \frac{VQ}{I}. \tag{5.3.1}$$

Now why would we do this? To figure that out, we take a look at the shear stress distribution and the shear flow distribution. They are both plotted in figure 5.2.

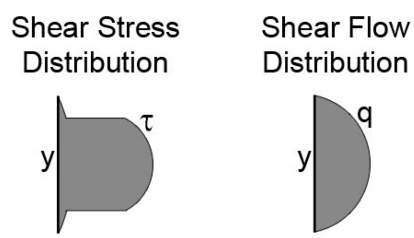

Figure 5.2: Distribution of the shear stress and the shear flow over the structure.

We see that the shear stress suddenly increases if the thickness decreases. This doesn't occur for the shear flow. The shear flow is independent of the thickness. You could say that, no matter what the thickness is, the shear flow flowing through a certain part of the cross-section stays the same.

5.4 Bolts

Suppose we have two beams, connected by a number of n bolts. In the example picture 5.3, we have $n = 2$. These bolts are placed at intervals of s meters in the longitudinal direction, with s being the **spacing** of the bolts.

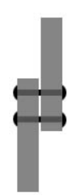

Figure 5.3: Example of a cross-section with bolts.

Suppose we can measure the shear force V_b in every bolt. Let's assume that these shear forces are equal for all bolts. (In asymmetrical situations things will be a bit more complicated, but we won't go into detail on that.) The shear flow in all the bolts together will then be

$$q_b = \frac{nV_b}{s}. \tag{5.4.1}$$

Now we can reverse the situation. We can calculate the shear flow in all bolts together using the methods from the previous paragraph. In our example figure 5.3, we would have to make a vertical cut through both bolts. Now, with the above equation, the shear force per bolt can be calculated.

5.5 The Value of Q for Common Cross-Sectional Shapes

It would be nice to know the maximum values of Q for some common cross-sectional shapes. This could save us some calculations. If you want to know Q_{max} for a rather common cross-section, just look it up in the list below.

- **A rectangle**, with width w and height h.

$$Q_{max} = \frac{wh^2}{8} \qquad \text{and} \qquad \tau_{max} = \frac{3}{2}\frac{V}{A} = \frac{3}{2}\frac{V}{wh}. \tag{5.5.1}$$

- **A circle** with radius R.

$$Q_{max} = \frac{2}{3}R^3 \qquad \text{and} \qquad \tau_{max} = \frac{4}{3}\frac{V}{A} = \frac{4}{3}\frac{V}{\pi R^2}. \tag{5.5.2}$$

- **A tube** with inner radius R_1 and outer radius R_2.

$$Q_{max} = \frac{2}{3}\left(R_2^3 - R_1^3\right) \qquad \text{and} \qquad \tau_{max} = \frac{4}{3}\frac{V}{\pi\left(R_2^2 - R_1^2\right)}\frac{R_2^2 + R_1R_2 + R_1^2}{R_2^2 + R_1^2}. \tag{5.5.3}$$

- **A thin-walled tube** with radius R and thickness t.

$$Q_{max} = 2R^2t \qquad \text{and} \qquad \tau_{max} = 2\frac{V}{A} = \frac{V}{\pi Rt}. \tag{5.5.4}$$

6. Torsion

6.1 Introduction to Torsion

We have now dealt with normal forces, shear forces and bending moments. Only torsion is left.

Torsion is caused by a **torque** T. This torque causes the bar to twist by an angle ϕ; the so-called **angle of twist**. A visualization of this is given in figure 6.1.

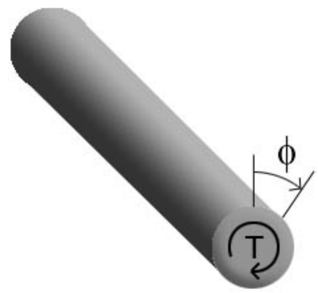

Figure 6.1: A bar with a torque T applied to it.

Another important angle is the **rotation per meter** θ, defined as

$$\theta = \frac{\phi}{L}. \tag{6.1.1}$$

Here L is the length of the bar.

In this chapter, we will mainly consider **pure torsion**. This is when the torque acts on the center of gravity of the cross-section. At the end we will combine this with a shear force.

6.2 The Torsion Formula

Let's consider a beam with a circular cross-section. What happens when we put a torque T on it? The torque causes a shear stress τ in the bar. And where there is shear stress, there is shear strain. This shear strain γ is given by

$$\gamma = \rho\theta. \tag{6.2.1}$$

The variable ρ is the distance with respect to the center of gravity of the bar. The shear stress can now be found using

$$\tau = G\gamma = G\rho\theta. \tag{6.2.2}$$

However, usually θ isn't known. But we usually do know the torque T that acts on a bar. The torque can also be found using

$$T = \int_A \rho\tau \, dA = G\theta \int_A \rho^2 \, dA = G\theta I_p. \tag{6.2.3}$$

Combining the above equations, we find an expression for the shear stress. Namely,

$$\tau = \frac{T\rho}{I_p}. \tag{6.2.4}$$

This equation is known as the **torsion formula**. Let's take a close look at this equation. It turns out that the shear stress τ increases as we go further from the center of gravity of the cross-section. So maximum shear stress occurs at the edges of the cross-section.

One final thing we would like to know is the angle of twist. We can find it using

$$\phi = \int_0^L \theta \, dx = \int_0^L \frac{\tau}{G\rho} dx = \int_0^L \frac{T}{GI_p} dx = \frac{TL}{GI_p}. \tag{6.2.5}$$

6.3 Closed Thin-Walled Cross-Sections

Previously we considered a circular cross-section. Now we will look at **closed thin-walled cross-sections**. A cross-section is **closed** if it consists of an uninterrupted curve. Let's define L_m as the length of this curve. Also, A_m is the **mean enclosed area** (the area which the curve encloses). It can now be shown that the shear flow is given by

$$q = \frac{T}{2A_m}. \tag{6.3.1}$$

The shear stress at a given point in the cross-section can now be found using

$$\tau = \frac{T}{2tA_m}, \tag{6.3.2}$$

where t is the thickness at that point of the cross-section.

To find the angle of twist, we can still use the familiar equation

$$\phi = \frac{TL}{GI_p}. \tag{6.3.3}$$

There is, however, one slight problem. It is usually rather difficult to find the polar moment of inertia for these cross-sections in the conventional way. Luckily, there is another equation which we can use. It is

$$I_p = \frac{4A_m^2}{\oint_0^{L_m} \frac{1}{t} ds}. \tag{6.3.4}$$

The sign $\oint$ means that the integration must be performed along the entire boundary. If the thickness t varies only in steps (which it usually does), then you can also use

$$I_p = \frac{4A_m^2}{\sum (L_{m_i}/t_i)}. \tag{6.3.5}$$

Here t_i is the thickness at a certain part of the cross-section and L_{m_i} is the length of that part.

6.4 Combining Shear Forces and Torsion

What do we do if we have both a shear force and a torque acting on a beam? In that case, we can use the principle of superposition. First find the shear stress distribution for the beam when only the shear force is present. Then find the shear stress distribution for the beam when only the torque is present. (Keep in mind the direction of the shear stress!) Then simply add it all up to find the real shear stress distribution. Sounds easy, doesn't it?

7. Bending Deflection

7.1 The Differential Equations

In the last chapter we saw the effect of a torque: A twist angle. What is the effect of bending a beam? There are two things that are of interest: The beam **rotation** θ and the **deflection** $u = \delta$.

To be able to analyze this problem without having awfully difficult equations, we have to assume that the beam rotations and deflections are small. If we make that assumption, we can derive five equations. These are

$$q(x) = EIu'''' = EI\frac{d^4u}{dx^4} \tag{7.1.1}$$

$$V(x) = EIu''' = EI\frac{d^3u}{dx^3} \tag{7.1.2}$$

$$M(x) = EIu'' = EI\frac{d^2u}{dx^2} = EI\kappa \tag{7.1.3}$$

$$\theta(x) = u' = \frac{du}{dx} \tag{7.1.4}$$

$$\delta(x) = u \tag{7.1.5}$$

Let's discuss the sign convention. The distributed load q is assumed to act downward, and so is the vertical force V. The moment M is assumed to be clockwise. The rotation θ is assumed to be clockwise as well. Finally, the displacement is positive when the beam is deflected downward.

So, if we have a distributed load q acting on a beam, all we have to do is integrate four times and we have the deflection. It sounds easy, but actually, it isn't easy at all. Not only is it annoying to integrate four times. But during every integration, a new constant pops up. That'll be a lot of constants! You need to solve for these constants using boundary conditions.

So, it's a lot of work. And it's very easy to make an error. I can imagine you've got better things to do. So let's look for an easier method to find the rotation/deflection of a beam.

7.2 Vergeet-Me-Nietjes/Forget-Me-Nots

The second (and mostly used) method to determine deflections is by considering standard load types. Sounds vague so far? Well, the idea is as follows. We have a clamped beam. On this beam is a certain load type working. The load types we will be considering are shown in figure 7.1.

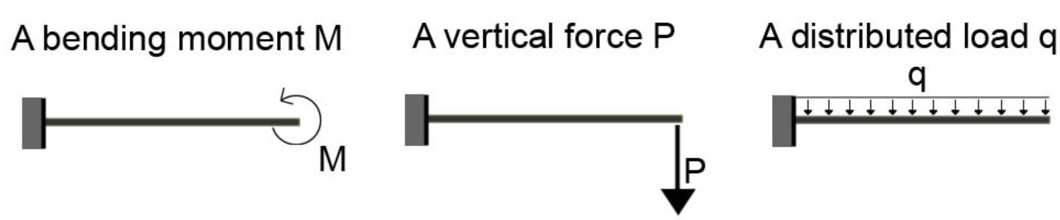

Figure 7.1: The three load types we will be considering.

These are the three standard load types. The rotation and displacement of the tip has been calculated for these examples. The results are shown in table 7.1. These equations are called the **vergeet-me-nietjes** (in Dutch) or the **forget-me-nots** (in English), named after the myosotis flower. They are therefore sometimes also called the **myosotis equations**.

Load type:	θ:	δ:
A distributed load q, pointing downward	$\frac{qL^3}{6EI}$	$\frac{qL^4}{8EI}$
A single force P, pointing downward	$\frac{PL^2}{2EI}$	$\frac{PL^3}{3EI}$
A single moment M, applied CCW	$\frac{ML}{EI}$	$\frac{ML^2}{2EI}$

Table 7.1: The vergeet-me-nietjes/forget-me-nots.

7.3 Applying the Forget-Me-Nots

So, how do we apply the forget-me-nots? Just follow the following steps:

- Split the beam up in segments. Every segment should be one of the standard load types.
- For every segment, do the following:
  - Let's call the left side of the segment point A and the right side point B. First find the forces and moments acting on B. (Don't forget the internal forces in the beam!)
  - Now express the rotation and displacement in B as a function of the rotation and displacement in A. For that, use the equations

$$\theta_B = \theta_A + \frac{q_{AB}L_{AB}^3}{6EI} + \frac{P_B L_{AB}^2}{2EI} + \frac{M_B L_{AB}}{EI}, \tag{7.3.1}$$

$$\delta_B = \delta_A + \frac{q_{AB}L_{AB}^4}{8EI} + \frac{P_B L_{AB}^3}{3EI} + \frac{M_B L_{AB}^2}{2EI} + L_{AB}\theta_A. \tag{7.3.2}$$

$$\tag{7.3.3}$$

An important part is the term $L\theta_A$ at the end of the last equation. This term is present due to the so-called **wagging tail effect**. It is often forgotten, so pay special attention to it.

To demonstrate this method, we consider an example. Let's take a look at figure 7.2. Can we determine the displacement at point C?

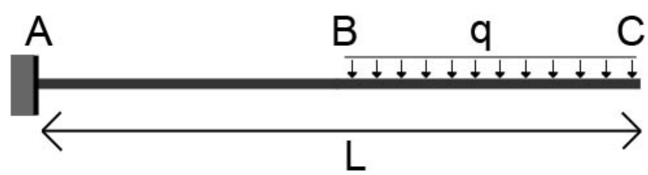

Figure 7.2: An example problem for the forget-me-nots.

It's quite clear that we need to split the beam up in two parts. First let's consider part BC. We have

$$\delta_C = \delta_B + \frac{q_{BC}L_{BC}^4}{8EI} + L_{BC}\theta_B = \delta_B + \frac{q\,(L/2)^4}{8EI} + (L/2)\theta_B = \delta_B + \frac{qL^4}{128EI} + (L/2)\theta_B. \tag{7.3.4}$$

Now let's look at part AB. Since the beam is clamped, we have $\theta_A = 0$ and $\delta_A = 0$, which is nice. We do need to take into account the internal forces in point B though. In B is a shear force $P_B = \frac{1}{2}qL$, pointing downward, and a bending moment $M_B = \frac{1}{8}qL^2$, directed clockwise. So we have

$$\theta_B = \frac{P_B L_{AB}^2}{2EI} + \frac{M_B L_{AB}}{EI} = \frac{qL^3}{16EI} + \frac{qL^3}{8EI}, \tag{7.3.5}$$

$$\delta_B = \frac{P_B L^3}{3EI} + \frac{M_B L^2}{2EI} = \frac{qL^4}{48EI} + \frac{qL^4}{64EI}. \tag{7.3.6}$$

Fill this in into equation (7.3.4) and you have your answer! That's a lot easier than difficult integrations.

7.4 Moment Area Method

Another way to find the rotation/deflection is by using the **moment area method**. For that, we first need to plot the moment diagram of the beam. Now let's consider a certain segment AB. Let A_M be the area under the bending moment diagram between points A and B. Now we have

$$\theta_B = \theta_A + \frac{A_M}{EI}. \tag{7.4.1}$$

Let's examine the moment diagram a bit closer. The diagram has a center of gravity. For example, if the diagram is a triangle (with $M = 0$ at A and $M =$ something at B), then the center of gravity of the moment diagram will be at $2/3^{rd}$ of length AB. Let's call x_{cog} the distance between the center of gravity of the moment diagram and point B. We then have

$$\delta_B = \delta_A + \frac{A_M}{EI}x_{cog} + L\theta_A. \tag{7.4.2}$$

Note that the wagging tail effect once more occurs in the moment area method. Although the moment area method is usually a bit more difficult to apply than the forget-me-nots, it can sometimes save time. You do ought to be familiar with the method though.

7.5 Statically Indeterminate Beams

Let's call R the amount of reaction forces acting on a beam. (A clamp has 3 reaction forces, a hinge has 2, etcetera.) Also, let's call J the amount of hinges in the system. If $R - 2J = 3$, the structure is statically determinate. You can use the above methods to solve reaction forces.

If, however, $R - 2J > 3$, the structure is statically indeterminate. We need compatibility equations to solve the problem. How do we approach such a problem? Simply follow the following steps.

- Remove constraints and replace them by reaction forces/moments, until the structure is statically determinant. Carefully choose the constraints to replace, since it might simplify your calculations.
- Express all the remaining reaction forces/moments in the reaction forces/moments you just added.
- Express the displacement of the removed constraints in the new reaction forces/moments.
- Assume that the displacement of the removed constraints are 0, and solve the equations.

7.6 Symmetry

Sometimes you are dealing with structures that are symmetric. You can make use of that fact. There are actually two kinds of symmetry. Let's consider both of them.

A structure is **symmetric**, if you can mirror it, and get back the original structure. If this is the case, then the shear force and the rotation at the center are zero. In an equation this becomes

$$V_{center} = 0 \qquad \text{and} \qquad \theta_{center} = 0. \tag{7.6.1}$$

A structure is **skew symmetric** if you can rotate it 180° about a certain point and get back the original structure. If this is the case, then

$$M_{center} = 0 \qquad \text{and} \qquad \delta_{center} = 0. \tag{7.6.2}$$