Chapter 1

Fundamentals

1.1 Equations of motion for a rocket

As a start the definition of the impulse of a system and the relation with Newton's second law is given (for constant mass):

$$p = \int_M \boldsymbol{V} dM \qquad ; \qquad \frac{dp}{dt} = M\boldsymbol{a} = \boldsymbol{F}_u \tag{1.1}$$

The **Thrust** can be derived by determining the change in impulse:

$$dp = p_{t+dt} - p_t = (M + dM)(V + dV) - udM - MV \tag{1.2}$$

This results in the equation of motion for a rocket, with external forces F_u and an exhaust velocity c.

$$M\frac{dV}{dt} = mc + \sum F_u \quad with \quad m = -\frac{dM}{dt} \quad and \quad c = V - u \tag{1.3}$$

Introducing the thrust, drag and the gravity loss gives:

$$M\frac{dV}{dt} = T - D - Mg \quad with \quad T = mc + A_e(p_e - p_0) = mc_{eff} \tag{1.4}$$

And introducing the parameter I_{sp}:

$$I_{sp} = \frac{T}{mg_0} = \frac{c_{eff}}{g_0} \tag{1.5}$$

1.2 Velocity increase: Tsiolkovsky's equation

To derive the ΔV the equation of motion is integrated.

$$M\frac{dV}{dt} = mc_{eff} = -c_{eff}\frac{dM}{dt} \quad \rightarrow \quad dV = -c_{eff}\frac{dM}{M} \tag{1.6}$$

Integration yields the following equation, with $\Lambda = M_0/M_e$:

$$(V_e)_{id} = c_{eff}\ \ln(M_0/M_e) = c_{eff}\ \ln\Lambda \qquad with\ \ V_0 = 0 \tag{1.7}$$

This equation gives the **Ideal end velocity** V_e, which is derived for a gravity-free space and vacuum. The equation shows that the velocity increase is independent of burn-time and burn-program $m = m(t)$.

1.3 Travelled distance

The travelled distance is derived for a gravity free space and in vacuum. For the burn-program with a **Constant Mass Flow** the rocket also has **constant thrust**, because c_{eff} is constant. The mass varies linearly and the acceleration will increase with time. The burn will be:

$$t_b = \frac{M_0 - M_e}{m} = \frac{I_{sp}}{\Psi_0}\left(1 - \frac{1}{\Lambda}\right) \quad with \quad \Psi_0 = \frac{T}{M_0 g_0} \tag{1.8}$$

Using Tsiolkovsky's equation the following equation for the travelled distance can be derived:

$$S_e = \int_0^{t_b} V dt = \frac{c_{eff}}{m} \int_{M_e}^{M_0} (\ln M_0 - lnM)\, dM \tag{1.9}$$

Evaluating the integral results in the following equation:

$$S_e = \frac{c_{eff}^2}{\Psi_0 g_0}\left(1 - \frac{1}{\Lambda}(1 + \ln \Lambda)\right) \tag{1.10}$$

For the burn-program with a **Constant Thrust Load** $\Psi = \frac{T}{M g_0}$, the acceleration is constant.

$$\frac{dV}{dt} = \frac{T}{M} = \Psi_0 g_0 \tag{1.11}$$

Substituting the equation of the thrust $T = m c_{eff}$ and integration results in the following relationship between the thrust load and the momentary mass:

$$\Psi = \frac{T}{M g_0} = -\frac{c_{eff}\frac{dM}{dt}}{M g_0} \quad \rightarrow \quad M = M_0 e^{-\frac{\Psi g_0}{c_{eff}} t} \tag{1.12}$$

The burn-time is then:

$$t_b = \frac{c_{eff}}{\Psi g_0} \ln \Lambda = \frac{I_{sp}}{\Psi} \ln \Lambda \tag{1.13}$$

The travelled distance is obtained by substituting the time and acceleration into the equation for the distance.

$$S_e = \frac{1}{2} a t_b^2 = \frac{1}{2} \frac{c_{eff}^2}{\Psi g_0} \ln^2 \Lambda \tag{1.14}$$

1.4 Comparison of burn programs

The burn programs are compared for identical values of c_{eff}, I_{sp} and Λ. The following relationship is found for the velocities of the different burn-programs. The relationship shows that the (ideal) end velocity is independent of the burn-program.

$$\frac{(V_e)_{T=constant}}{(V_e)_{\Psi=constant}} = 1 \tag{1.15}$$

The burn time of the burn-program with constant thrust is found to be smaller than the burn time of the burn-program with constant thrust load.

$$\frac{(t_b)_{T=constant}}{(t_b)_{\Psi=constant}} = \frac{1 - \frac{1}{\Lambda}}{ln\Lambda} < 1 \quad for \;\; \Psi = \Psi_0 \tag{1.16}$$

The travelled distance is also found to be smaller with a constant thrust.

$$\frac{(S_e)_{T=constant}}{(S_e)_{\Psi=constant}} = 2\frac{1 - \frac{1}{\Lambda}(1 + ln\Lambda)}{ln^2\Lambda} < 1 \quad for \;\; \Psi = \Psi_0 \tag{1.17}$$

Chapter 2

Launch Trajectories

The launch Trajectory can be devided in three parts. The first part is while the Rocket is burning fuel and accelerates until the point of extinction. After the point of extinction the Rocket will pursue a ballistic trajectory without thrust. The Rocket continues to increase altitude until the culmination point where the vertical velocity is zero. After that the Rocket starts descending.

In this case a homogeneous gravity field and vacuum are assumed. It follows that the equations of motion in x and z direction are:

$$M\frac{dV_x}{dt} = T\cos\theta \tag{2.1}$$

$$M\frac{dV_z}{dt} = T\sin\theta - Mg_0 \tag{2.2}$$

2.1 Constant pitch angle

2.1.1 General expressions

The equations of motion with a constant pitch angle are:

$$\frac{a_x}{g_0} = \frac{T}{Mg_0}\cos\theta_0 \quad ; \quad \frac{a_z}{g_0} = \frac{T}{Mg_0}\sin\theta_0 - 1 \tag{2.3}$$

Integrating the equations of motion, yields the velocity profile.

$$V_x = c_{eff}\ln\left(\frac{M_0}{M}\right)\cos\theta_0 \quad ; \quad V_z = c_{eff}\ln\left(\frac{M_0}{M}\right)\sin\theta_0 - g_0 t \tag{2.4}$$

From this the flight path angle can be found:

$$\tan\gamma = \frac{V_z}{V_x} = \tan\theta_0 - \frac{g_0 t}{c_{eff}\ln\left(\frac{M_0}{M}\right)\cos\theta_0} \tag{2.5}$$

The intial flight path angle is determined by taking the limit, using l'Hopital:

$$\lim_{t\to 0}\tan\gamma = \tan\gamma_0 = \tan\theta_0 - \frac{g_0}{c_{eff}\frac{m}{M_0}\cos\theta_0} = \tan\theta_0 - \frac{1}{\psi_0\cos\theta_0} \tag{2.6}$$

And using the following derivative:

$$\frac{d}{dt}\left(\ln\left(\frac{M_0}{M}\right)\right) = \frac{d}{dM}\left(\ln\left(\frac{M_0}{M}\right)\right)\frac{dM}{dt} = \frac{m}{M} \tag{2.7}$$

The conditions in the **Extinction point** or burnout point follow from the equations of motion, since the pitch angle is constant :

$$V_{x_e} = c_{eff} \ln \Lambda \cos \theta_0 = V_{e_{id}} \cos \theta_0 \tag{2.8}$$

$$V_{z_e} = c_{eff} \ln \Lambda \sin \theta_0 - g_0 t_b = V_{e_{id}} \sin \theta_0 - g_0 t_b \tag{2.9}$$

$$\frac{V_e}{V_{e_{id}}} = \sqrt{1 - \frac{2 g_0 t_b \sin \theta_0}{V_{e_{id}}} + \left(\frac{g_0 t_b}{V_{e_{id}}} \right)^2} \tag{2.10}$$

In these expressions, for a pre-determined burn time, the condition of motion in the burnout point is independent of the thrust as a function of time.

2.1.2 Flight with constant Thrust

The equations of motion for the Flight with constant Thrust and constant pitch angle in the burnout point are:

$$\frac{(a_x)_e}{g_0} = \frac{T}{M_e g_0} \frac{M_0}{M_0} \cos \theta_0 = \Psi_0 \Lambda \cos \theta_0 \quad ; \quad \frac{(a_z)_e}{g_0} = \frac{T}{M_e g_0} \frac{M_0}{M_0} \sin \theta_0 - 1 = \Psi_0 \Lambda \sin \theta_0 - 1 \tag{2.11}$$

The burn time is found by using the definition of the thrust load and assuming constant thrust.

$$t_b = \frac{c_{eff}}{g_0 \Psi_0} \left(1 - \frac{1}{\Lambda} \right) \tag{2.12}$$

Substituting the burn time in the equation for the velocity and integrating this equation yields the end velocity and the coordinates of the extinction point:

$$X_e = \int_0^{t_b} V_x dt = \frac{c_{eff}^2}{\Psi_0 g_0} \left(1 - \frac{1}{\Lambda} (1 + ln\Lambda) \right) \cos \theta_0 \tag{2.13}$$

$$Z_e = \int_0^{t_b} V_z dt = \frac{c_{eff}^2}{\Psi_0 g_0} \left[\left(1 - \frac{1}{\Lambda} (1 + ln\Lambda) \right) \sin \theta_0 - \frac{1}{2\Psi_0} \left(1 - \frac{1}{\Lambda} \right)^2 \right] \tag{2.14}$$

The flight path angle can be determined by substituting the burn time. After the extinction point, the rocket will pursue a ballistic trajectory with $T = 0$.

$$\frac{dV_x}{dt} = 0 \quad ; \quad V_x = (V_x)_e \quad ; \quad \frac{dV_z}{dt} = -g_0 \quad ; \quad V_z = (V_z)_e - g_0(t - t_b) \tag{2.15}$$

The **Culmination point** can be derived by using its property $V_z = 0$. Resulting in:

$$t_c = t_b + \frac{(V_z)_e}{g_0} = \frac{(V_e)_{id}}{g_0} \sin \theta_0 \tag{2.16}$$

The coordinates can be determined by substituting the culmination time. The point of impact can be determined by using the property $Z = 0$.

For a launch by means of an **impulsive shot**, the properties below are valid. These are the expressions for the **"bullet trajectories"** over a flat Earth (parabolic trajectories).

$$\Psi_0 = \infty \quad t_b = 0 \tag{2.17}$$

2.1.3 Flight with constant Thrust-load

For a constant Thrust load, the accelerations are constant. Therefore, the velocities increase linearly and the flight path angle γ and angle of attack remain constant. The trajectory is thus rectilinear.

$$\frac{a_x}{g_0} = \Psi \cos\theta_0 \quad ; \quad \frac{a_z}{g_0} = \Psi \sin\theta_0 - 1 \quad ; \quad \tan\gamma = \tan\theta_0 - \frac{1}{\Psi\cos\theta_0} = const. \tag{2.18}$$

The extinction point, culmination point and impact point can be determined by substituting the burn time.

$$t_b = \frac{c_{eff}}{\Psi g_0} \ln\Lambda = \frac{(V_e)_{id}}{\Psi g_0} \tag{2.19}$$

2.1.4 Maximum shooting range

The maximum shooting range is derived with the use of the variable K:

$$K = \frac{\frac{1}{\Lambda} + \ln\Lambda - 1}{\Psi_0 \ln^2\Lambda} \tag{2.20}$$

For a burn program with **constant thrust**, the total flight time becomes:

$$\frac{t_i g_0}{(V_e)_{id}} = \sin\theta_0 + \sqrt{(\sin\theta_0 - 2K)\sin\theta_0} \tag{2.21}$$

The shooting range becomes:

$$\frac{X_i g_0}{(V_e)_{id}} = \cos\theta_0 \left[(\sin\theta_0 - K) + \sqrt{(\sin\theta_0 - 2K)\sin\theta_0}\right] \tag{2.22}$$

For the impuslive shot launch $K = 0$ and $\Psi_0 = \infty$ it holds that:

$$\frac{X_i g_0}{(V_e)_{id}} = 2\sin\theta_0\cos\theta_0 = \sin 2\theta_0 \tag{2.23}$$

The maximum shooting range is achieved in this case when $\theta_0 = 45°$.
For a finite value of Ψ_0 the optimal pitch angle can be determined by differentiating the shooting range with respect to θ_0, yielding:

$$K = \frac{2\sin^2\theta_0 - 1}{2\sin^3\theta_0} \quad ; \quad \frac{X_i g_0}{(V_e)_{id}} = \frac{\cos\theta_0}{2\sin^3\theta_0} \quad ; \quad \frac{t_i g_0}{(V_e)_{id}} = \frac{1}{\sin\theta_0} \tag{2.24}$$

For the flight with a **constant thrust load** the shooting range is given below, which is the same as for the constant thrust provided one read $1/2\Psi$ for K.

$$\frac{X_i g_0}{(V_e)_{id}} = \cos\theta_0 \left[(\sin\theta_0 - \frac{1}{2\Psi}) + \sqrt{(\sin\theta_0 - \frac{1}{\Psi})\sin\theta_0}\right] \tag{2.25}$$

The optimal pitch angle is found by differentiating, yielding a similar result:

$$\Psi = \frac{2\sin^2\theta_0 - 1}{2\sin^3\theta_0} \tag{2.26}$$

2.1.5 Optimal steering program

The steering program $\theta_0 = \textit{constant}$ yields the a maximal velocity V_e in the burn out point at a pre-determined path angle γ_e. This is determined by using the relationship between V_x and V_z and determining the maximum velocity. This result is independent of the burn program. The following relationship between the final flight path angle and the pitch angle is known:

$$\tan\gamma_e = \tan\theta_0 - \frac{g_0 t_b}{(V_e)_{id}\cos\theta_0} \tag{2.27}$$

The following constant pitch angle is derived to be optimal:

$$\cos\theta_0 = \left(-\frac{g_0 t_b}{(V_e)_{id}}\sin\gamma_e + \sqrt{1 - \left(\frac{g_0 t_b}{(V_e)_{id}}\cos\gamma_e\right)^2}\right) \tag{2.28}$$

2.2 Ascent trajectories without angle of attack - "Gravity-turn")

The ascent trajectories without an angle of attack are important for rocket flight through the more dense layers of the atmosphere, in order to keep the aerodynamic loads on the rocket as low as possible (no lift acting as transverse force on the rocket). For simplification of the analysis, the motion is said to take place in vacuum.

The equations of motion become:

$$M\frac{dV_x}{dt} = T\frac{V_x}{V} \quad ; \quad M\frac{dV_z}{dt} = T\frac{V_z}{V} - Mg_0 \quad ; \quad M\frac{dV}{dt} = T - Mg_0\sin\gamma \tag{2.29}$$

If the equation of motion in the X-direction is multiplied with V_z and the equation of motion in the Z-direction is multiplied with V_x, substraction yields:

$$M\left(V_x\frac{dV_z}{dt} - V_z\frac{dV_x}{dt}\right) = -Mg_0V_x \quad \rightarrow \quad \frac{1}{V_x}\frac{dV_z}{dt} - \frac{V_z}{V_x^2}\frac{dV_x}{dt} = \frac{d\left(V_z/V_x\right)}{dt} = -\frac{g_0}{V_x} \tag{2.30}$$

Realizing that $\tan\gamma = V_z/V_x$ and $V_x = V\cos\gamma$, the following equation of motion can be derived:

$$\frac{d(\tan\gamma)}{dt} = \frac{1}{\cos^2\gamma}\frac{d\gamma}{dt} = -\frac{g_0}{V\cos\gamma} \quad \rightarrow \quad V\frac{d\gamma}{dt} = -g_0\cos\gamma \quad \rightarrow \quad V^2\frac{d\gamma}{ds} = \frac{V^2}{R} = -g_0\cos\gamma \tag{2.31}$$

In this equation R is the radius of curvature of the path. This indicates that the centrifugal force perpendicular to the trajectory is in equilibrium with the weight component in this direction. The described trajectory is commonly named a gravity turn for this reason.

For the lift-off of launch rockets, the rocket is accelerated vertically from the launch pad to a certain V_0 and the rocket receives a **kick-angle** δ, then with V_0 and $\gamma_0 = \pi/2 - \delta$ the initial conditions have been set for the gravity turn. For burn program with constant thrust the e.o.m. can only be solved numerically. For burn program with constant specific thrust the e.o.m. can be solved analytically. The pitch-over maneuver for the gravity turn should occur when the vertical velocity is not that too large so soon after launch but also the atmosphere should not be too dense; basically it is a trade-off to avoid large aerodynamic loads. On average, but depending on the size and construction of the rocket, the rocket parameters and the intended orbit a gravity turn takes place at around $10km$ height.

In conclusion, the gravity turn is a form of trajectory optimization that uses the gravity to steer the rocket such that enough horizontal velocity is created (heading into the right orbit) and to keep the angle of attack as low as possible to minimize aerodynamic stress.

2.3 Vertical Ascent Trajectories

This type of trajectories is amongst others of importance to sounding rockets. In reality these rockets are usually launched under an angle, which deviates somewhat from 90°, for example 80°.

2.3.1 Vertical flight with aerodynamic drag

In the vertical flight through the atmosphere, the equation of motion is:

$$M\frac{dV}{dt} = T - D - Mg_0 \qquad with \;\; D = C_D\frac{1}{2}\rho V^2 S \tag{2.32}$$

Numerical integration can be used to determine the trajectory of the rocket and the drag losses. During the vertical powered flight the velocity V increases, while the density ρ decreases with increasing altitude. In the beginning of the flight the velocity dominates, yet at greater altitudes the density becomes so small that the term approaches zero. At a certain altitude the drag will be maximum after which it starts to decrease. Evidently the dynamic pressure q has a maximum at an altitude of roughly 10 km. For altitudes greater than 60 km, the dynamic pressure q is so low that, despite the high velocity, the aerodynamic resistance can be neglected.

The velocity losses ΔV_g and ΔV_D are heavily dependent on the thrust load $\Psi_0 = T_0/M_0g_0$. With increasing Ψ_0, the burn time t_b decreases and thus ΔV_g decreases. On the other hand, with increasing Ψ_0, the velocity is greater in the more dense layers of the atmosphere and therefore ΔV_D increases. Unlike the vertical flight in vacuum, the culmination altitude wil not increase monotonously with Ψ_0, but will nevertheless display a maximum for a certain value of Ψ_0.

2.3.2 The influence of wind

In the first phase of the flight through the atmosphere, the motion of the rocket will also be subjected to the influence of horizontal wind velocities.

The rocket can be **Pitch stabilised** which means that the pitch angle remains fixed. A sideways motion with respect to the ground and the windspeed yield an angle of attack.

$$\tan\alpha = \frac{V_w - V_x}{V_z} \tag{2.33}$$

As a result of the angle of attack a normal force arises perpendicular to the longitudinal body axis, so that the equation of motion in the X-direction becomes:

$$M\frac{dV_x}{dt} = N = C_{N_\alpha}\alpha\frac{1}{2}\rho V^2 S \tag{2.34}$$

During the larger part of the flight $V >> V_w$, thus $V_z \approx V$ and $\tan\alpha \approx \alpha$. Also assuming that the wind velocity V_w is independent of the altitude Z, results:

$$MV\frac{dV_x}{dZ} = C_{N_\alpha}\frac{V_w - V_x}{V_z}\frac{1}{2}\rho V^2 S \quad ; \quad \frac{d(V_w - V_x)}{V_w - V_x} = -n_\alpha dZ \quad ; \quad n_\alpha = \frac{C_{N_\alpha}\frac{1}{2}\rho S}{M} \tag{2.35}$$

Integration leads to the horizontal wind profile. This can be approximated by assuming the value N_α to be constant.

$$\frac{V_x}{V_w} = 1 - e^{-\int_0^z n_\alpha dZ} \approx 1 - e^{-n_\alpha dZ} \tag{2.36}$$

It has been shown that sounding rockets obtain a horizontal velocity during powered flight that approaches the (horizontal) wind velocity. In the free flight the horizontal velocity does not change if the wind velocity does not change with altitude. Thus, as a global estimate the wind velocity can be assumed to be equal for the whole flight. Resulting in the following "wind displacement".

$$\Delta X_i = V_w t_f \tag{2.37}$$

An unguided, **Statically stable rocket** will direct itself towards the relative velocity of the wind with respect to the rocket. As a result the thrust will get a horizontal component and the rocket will thus move in a direction against the direction of the wind. If the rocket has a large static stability, the momentary angle of attack will repeatedly be zero. With respect to a reference frame attached to the wind, the rocket will thus still describe a gravity turn. At the end of the powered flight, the rocket has acquired a horizontal velocity with respect to the wind, which is maintained during the free flight that follows. When the wind velocity is independent of altitude, the reference frame attached to the wind has a uniform motion with respect to the inertial reference frame, so that the equations of motion remain unchanged. The resulting wind displacement will be:

$$\Delta X_i = V_w t_f \tag{2.38}$$

The trajectory displacement resulting from the gravity turn is considerably greater than that due to the wind displacement.

Chapter 3

The Multi-Stage Rocket

A Multi-stage rocket is given in the figure below.

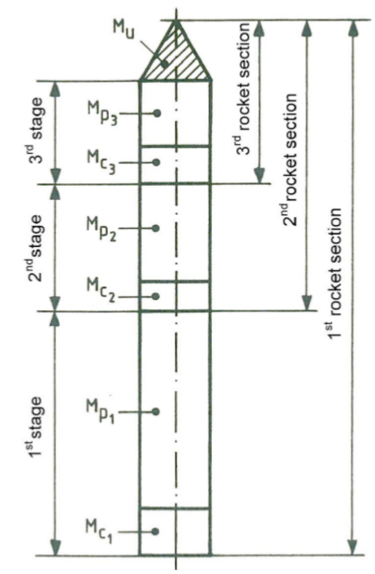

Figure 3.1: Nomenclature of a Multi-Stage rocket.

3.1 Rocket Characteristic Quantities and Ratios

The derivations of the equations for a Multi-Stage Rocket repeatedly make use of Ratios of Characteristic Quantities.

The total rocket mass or start/initial mass M_0 is made up of the construction mass M_c, the propellant mass M_p and the payload (useful load) M_u.

$$\frac{M_0}{M_0} = \frac{M_c}{M_0} + \frac{M_p}{M_0} + \frac{M_u}{M_0} = 1 \tag{3.1}$$

Herein the **Payload Ratio** λ is:

$$\lambda = \frac{M_u}{M_0} \tag{3.2}$$

Since the rocket can be carrying various payloads, which are not directly of any influence on the construction mass M_c, the **Construction Mass Ratio** ϵ is defined by:

$$\epsilon = \frac{M_c}{M_c + M_p} \tag{3.3}$$

The **Propellant Mass Ratio** φ can be expressed in terms of payload ratio and the construction mass ratio:

$$\varphi = \frac{M_p}{M_0} = \frac{M_p}{M_c + M_p}\frac{M_c + M_p}{M_0} = (1-\epsilon)(1-\lambda) \tag{3.4}$$

The **Mass ratio** Λ is defined by:

$$\Lambda = \frac{M_0}{M_e} = \frac{M_0}{M_0 - M_p} = \frac{1}{1 - M_p/M_0} = \frac{1}{1-\varphi} \quad \rightarrow \quad \frac{1}{\Lambda} = 1 - \varphi = \lambda(1-\epsilon) + \epsilon \tag{3.5}$$

Using the Mass ratio ,the **ideal end velocity** can be rewritten as:

$$(V_e)_{id} = c_{eff}\ \ln\Lambda = -c_{eff}\ \ln\left(\lambda(1-\epsilon)+\epsilon\right) \tag{3.6}$$

From which, the payload ratio and the proppelant mass ratio can be derived.

$$\lambda = \frac{e^{-\frac{(V_e)_{id}}{c_{eff}}} - \epsilon}{1-\epsilon} \qquad and \qquad \varphi = 1 - \frac{1}{\Lambda} = 1 - e^{-\frac{(V_e)_{id}}{c_{eff}}} \tag{3.7}$$

In the limit case, the maximum $\frac{(V_e)_{id}}{c_{eff}}$ is reached when the payload ratio is equal to zero.

$$\lim_{\lambda\to 0}\frac{(V_e)_{id}}{c_{eff}} = -\ln\epsilon \tag{3.8}$$

The **Total Payload Ratio** can be rewritten as the multiplication of the payload ratio of all rocket sections:

$$\lambda_{tot} = \frac{(M_u)_N}{(M_0)_1} = \frac{(M_u)_1}{(M_0)_1}\cdots\frac{(M_u)_N}{(M_0)_N} = \lambda_1\cdots\lambda_N = \prod_{i=1}^{N}\lambda_i \tag{3.9}$$

The ideal end velocity can then be rewritten as:

$$(V_e)_N = \sum_{i=1}^{N}\left(c_{eff}\right)_i \ln\Lambda_i = -\sum_{i=1}^{N}\left(c_{eff}\right)_i \ln\left(\lambda_i(1-\epsilon_i)+\epsilon_i\right) = f(\lambda_i) \tag{3.10}$$

To obtain the **Maximum end velocity** for a given λ_{tot} the distribution of the values λ_i can be determined. This done by using the above equation $f(\lambda_i)$ and adding a subsidiary requirement (adding a zero to get a solution).

$$F(\lambda_i) = f(\lambda_i) + \mu g(\lambda_i) \quad with \quad g(\lambda_i) = \sum_{i=1}^{N}\ln(\lambda_i) - \ln(\lambda_{tot}) = 0 \tag{3.11}$$

Differentiating the equation $F(\lambda_i)$ with respect to $\lambda_)$ results in $N+1$ equations and N unknown λ_i and the unknown μ. Yielding the optimal values for λ_i.

$$\lambda_i = \frac{\mu\epsilon_i}{\left((c_{eff})_i - \mu\right)(1-\epsilon_i)} \quad and \quad \prod_{i=1}^{N}\lambda_i = \lambda_{tot} = \sum_{i=1}^{N}\frac{\mu\epsilon_i}{\left((c_{eff})_i - \mu\right)(1-\epsilon_i)} \tag{3.12}$$

3.2 Identical stages

A special case is the multi-stage rocket with identical stages for which both the effective exhaust velocity c_{eff} and the construction mass ratio are the same for each stage.

$$(c_{eff})_1 = (c_{eff})_2 \cdots \text{ and } \epsilon_1 = \epsilon_2 \cdots \text{ and } \varphi_1 = \varphi_2 \cdots$$

Since μ is a constant, it follows that the optimal multi-stage rocket also has identical rocket sections. With $\lambda_1 = \lambda_2 = \ldots = \lambda_N$. The ideal end velocity can be rewritten as:

$$(V_e)_N = -N c_{eff} \ln(\lambda(1-\epsilon)+\epsilon) \quad and \quad (V_e)_N = -N c_{eff} \ln(\lambda_{tot}^{1/N}(1-\epsilon)+\epsilon) \tag{3.13}$$

In the limit case, the maximum $\frac{(V_e)_{id}}{c_{eff}}$ is reached when the total payload ratio is equal to zero.

$$\lim_{\lambda_{tot}\to 0} \frac{(V_e)_{id}}{c_{eff}} = -N \ \ln \epsilon \tag{3.14}$$

The dimensionless end velocity $(V_e)_N / c_{eff}$ increases as the number of stages increase, reaching a maximum for $N = \infty$. Using 'l Hopital's rule it follows:

$$\frac{(V_e)_N}{c_{eff}} = \frac{-c_{eff} \ln(\lambda_{tot}^{1/N}(1-\epsilon)+\epsilon)}{1/N} \qquad \lim_{N\to\infty} \frac{(V_e)_N}{c_{eff}} = -(1-\epsilon)\ln \lambda_{tot} \tag{3.15}$$

With the use of:

$$a = \lambda^{1/N} \ ; \quad \ln \ a = \frac{1}{N} \ln \ \lambda \ ; \quad \frac{d\ (\ln \ a)}{dN} = \frac{1}{a}\frac{da}{dN} = -\frac{1}{N^2} \ln \ \lambda \quad \rightarrow \quad \frac{da}{dN} = -\frac{\lambda^{1/N}}{N^2} \ln\ (\lambda) \tag{3.16}$$

3.3 Coasting

In the previous sections of this chapter it was assumed that the $(i+1)^{th}$ rocket stage is ignited directly after ejection of the previous (i^{th}) stage. In some missions, however, a free (unpowered) flight is executed before igniting the next rocket stage. This is known as 'coasting'.

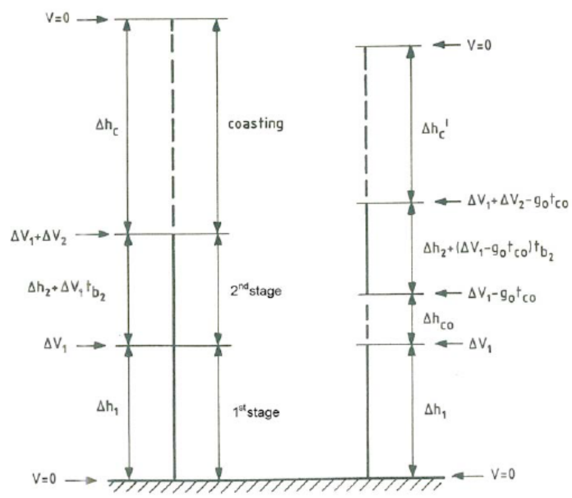

Figure 3.2: Vertical flight with and without coasting

The culmination altitude without coasting follows as:

$$h_c = \Delta h_1 + \Delta h_2 + \Delta V_1 t_{b_2} + \frac{(\Delta V_1 + \Delta V_2)^2}{2g_0} \tag{3.17}$$

The travelled distance during the coasting phase and the starting velocity of the second stage become:

$$\Delta h_{co} = \Delta V_1 t_{co} - \frac{1}{2} g_0 t_{co}^2 \quad ; \quad (V_2)_0 = \Delta V_1 - g_0 t_{co} \tag{3.18}$$

Substituting above equations gives the culmination altitude with coasting:

$$h_c' = \Delta h_1 + \left(\Delta h_2 - \frac{1}{2} g_0 t_{co}^2 \right) + (\Delta V_1 - g_0 t_{co})\, t_{b_2} + \frac{(\Delta V_1 + \Delta V_2)^2 - g_0 t_{co}}{2g_0} \tag{3.19}$$

In this expression it holds that $\Delta V_2 + g_0 (t_b)_2 = (\Delta V_2)_{id}$. Thus the difference in culmination altitude will become:

$$\Delta h_c' = h_c' - h_c = -c_{eff,2} \ln \Lambda_2 t_{co} \tag{3.20}$$

From analyzing the flight time, it can be derived that the culmination flight time is independent of the coast time.

3.4 Boosters

Up until now only successive rocket stages have been discussed, those that are mounted on top of each other and are ignited one after the other. It is also possible to place the rocket stages side-byside and allowing them to operate simultaneously. In the last case one speaks of 'parallel staging', whereas the previous is often referred to as 'tandem staging'.

Because the effective exhaust velocity of the boosters can differ from the core stage, an average effective exhaust velocity can be determined by:

$$\bar{c}_{eff} = \frac{T_1 + T_b}{m_1 + m_b} = \frac{m_1 c_{eff,1} + m_b c_{eff,b}}{m_1 + m_b} \tag{3.21}$$

add ratios etc

Chapter 4

The Ballistic Flight over the Earth

The ballistic flight is exclusively determined by the gravitational attraction acting on the rocket. The Kepler equation yields,

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\theta} = \frac{p}{1+e\cos\theta} \tag{4.1}$$

for which $r = p$ when $\theta = \pi/2$ and $\theta = 3\pi/2$

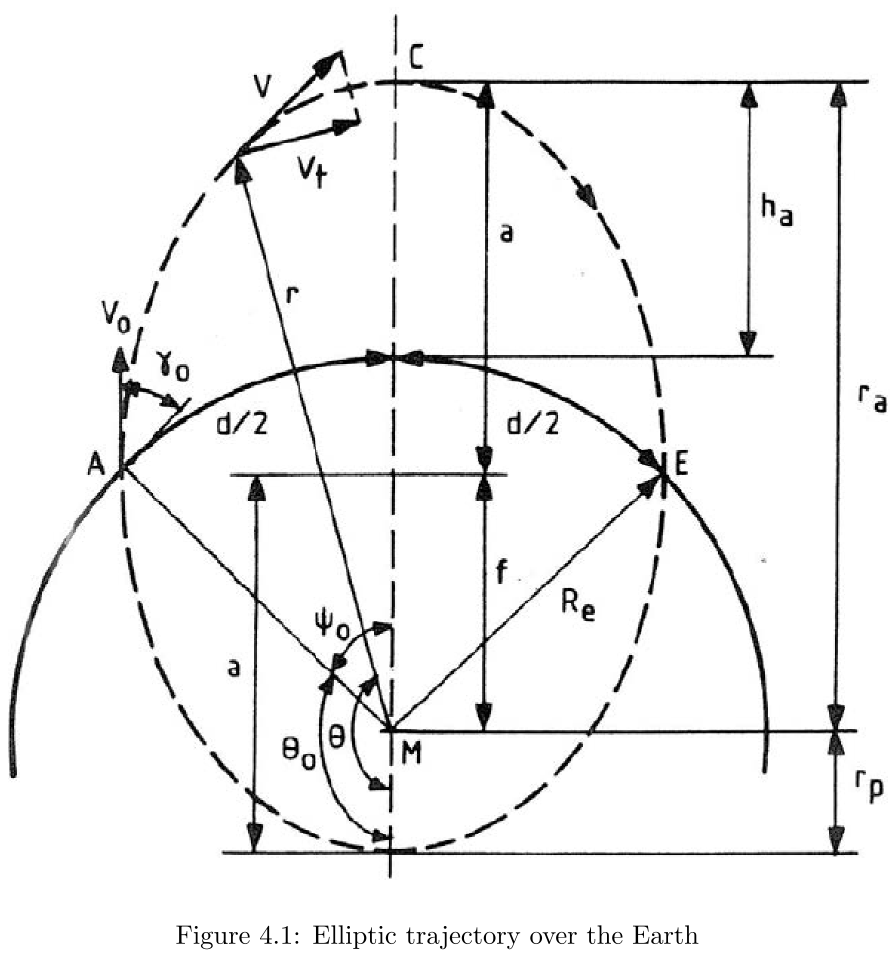

Figure 4.1: Elliptic trajectory over the Earth

The semi-latus rectum and the angular momentum are related by the following equation. The relationship between the angular momentum and the Launch conditions is given in the second equation below.

$$p = \frac{H^2}{\mu} = a\left(1 - e^2\right) \quad ; \quad H = rV_t = R_E V_0 cos\gamma_0 \tag{4.2}$$

The law of energy yields:

$$\frac{V^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \quad with \quad V_c^2 = \frac{\mu}{R_E} \quad and \quad S_0 = \frac{V_0}{V_{c,0}} \tag{4.3}$$

From which the **semi-major axis** can be derived as a function of the launch conditions.

$$\frac{a}{R_E} = \frac{1}{2 - \frac{V_0^2}{\mu / R_E}} = \frac{1}{2 - S_0^2} \tag{4.4}$$

Using the definition of S_0 and the definition of H yields the equation for the **semi-latus rectum**

$$\frac{p}{R_E} = \frac{H^2}{\mu R_E} = \frac{V_0^2 \cos^2\gamma_0 R_E}{\mu} = S_0^2 \cos^2\gamma_0 \tag{4.5}$$

and after substitution in the equation for the semi-latus rectum the **eccentricity** becomes:

$$e = \sqrt{1 - S_0^2 \cos^2\gamma_0 \left(2 - S_0^2\right)} \tag{4.6}$$

Substituting this in the **Kepler equation** yields:

$$\frac{r}{R_E} = \frac{S_0^2 \cos^2\gamma_0}{1 + \sqrt{1 - S_0^2 \cos^2\gamma_0 \left(2 - S_0^2\right)} cos\theta} \tag{4.7}$$

The **Semi-shooting range** Ψ_0 is determined by using the above equation and realizing that $r/R_e = 1$ yielding:

$$1 = \frac{S_0^2 \cos^2\gamma_0}{1 + \sqrt{1 - S_0^2 \cos^2\gamma_0 \left(2 - S_0^2\right)} \cos\theta_0} \quad \Psi_0 = \pi - \theta_0 = \frac{d/2}{R_e} \tag{4.8}$$

And the final result, using $\cos\Psi_0 = -\cos\theta_0$:

$$\Psi_0 = \frac{d/2}{R_e} = \arccos\left(\frac{1 - S_0^2 cos^2\gamma_0}{\sqrt{1 - S_0^2 cos^2\gamma_0 (2 - S_0^2)}}\right) \tag{4.9}$$

flat earth approx...

4.0.1 Launch trajectories

The possible lanch trajectories are derived by using the following equations:

$$\tan\Psi_0 \frac{S_0^2 \sin 2\gamma_0}{2\left(1 - S_0^2 \cos^2\gamma_0\right)} \qquad \frac{\sin\Psi_0}{\cos\Psi_0} = \frac{S_0^2 \sin 2\gamma_0}{2 - S_0^2\left(1 + \cos 2\gamma_0\right)} \tag{4.10}$$

This leads to the following equation. For $S_0 < \sqrt{2}$, the results of both sides should always be positive.

$$\sin\left(2\gamma_0 + \Psi_0\right) = \left(\frac{2}{S_0^2} - 1\right)\sin\Psi_0 \quad for\ S_0 < \sqrt{2} \quad \sin\left(2(\gamma_0)_1 + \Psi_0\right) = \sin\left(\pi - 2(\gamma_0)_2 - \Psi_0\right) \tag{4.11}$$

Two solutions exist for two different launch angles. The following equation can be found:

$$2(\gamma_0)_1 + \Psi_0 = \pi - 2(\gamma_0)_2 - \Psi_0 \quad \rightarrow \quad (\gamma_0)_2 = \frac{\pi}{2}\left((\gamma_0)_1 + \Psi_0\right) \tag{4.12}$$

These equations yield several solutions. For $S_0 < 1$, a high and a low trajectory yield the same shooting range. For $S_0 = 1$ a high trajectory and a trajectory over the surface of the Earth is found. For $1 < S_0 < \sqrt{2}$ a short and a long trajectory is found. The short trajectory is a high trajetory to the impact point. The long trajectory is a low trajectory in the other direction around the Earth, thus yielding a shooting range of $\Psi_0 > \pi/2$.

4.1 Maximum shooting range

4.2 Flight time

The flight time is determined by using the definiton for the Eccentric anomaly:

$$t_f = t_{\theta_i} - t_{\theta_0} = \sqrt{\frac{a^3}{\mu}}\left(E_i - E_0 - e(\sin E_i - \sin E_0)\right) \tag{4.13}$$

Resulting in the following equation for the flight time.

$$t_f = 2\sqrt{\frac{a^3}{\mu}}(\pi - E_0 + e\sin E_0)) \quad \textit{with} \quad E_i = 2\pi - E_0 \;\; \textit{and} \;\; \sin E_i = -\sin E_0 \tag{4.14}$$

To determine the flight time as a function of the launch conditions, information is needed about the semi-major axis and the Eccentric anomaly. The following expression is used to derived a function of the Eccentric anomaly:

$$r\cos\theta = a\cos E - ae \tag{4.15}$$

Substituting the equation for the radius r and the equation for the semi-latus rectum p as a function of a and e into the above equation yields:

$$\cos E_0 = \frac{e + \cos\theta_0}{1 + e\cos\theta_0} = \frac{e - \cos\Psi_0}{1 - e\cos\Psi_0} \quad \textit{with} \quad \theta_0 = \pi - \Psi_0 \tag{4.16}$$

The cosine of E_0 is determined by substituting equations 4.6 and 4.9 into the above equation. Realizing that for $S_0 < 1$, $\cos E_0$ is negative and thus $E_0 > 90°$.

$$\cos E_0 = -\frac{1 - S_0^2}{\sqrt{1 - S_0^2\cos^2\gamma_0\left(2 - S_0^2\right)}} \quad \rightarrow \quad E_0 = \frac{\pi}{2} + \arcsin\left(\frac{1 - S_0^2}{\sqrt{1 - S_0^2\cos^2\gamma_0\left(2 - S_0^2\right)}}\right) \tag{4.17}$$

It is also found that:

$$\sin E_0 = \sqrt{1 - \cos^2 E_0} = \frac{S_0\sin\gamma_0\sqrt{2 - S_0^2}}{\sqrt{1 - S_0^2\cos^2\gamma_0\left(2 - S_0^2\right)}} \quad \rightarrow \quad e\sin E_0 = S_0\sin\gamma_0\sqrt{2 - S_0^2} \tag{4.18}$$

Using the equation relating the semi-major axis and the Earth's radius, the following equation can be rewritten:

$$\sqrt{\frac{a^3}{\mu}} = \sqrt{\frac{R_e^3}{\mu}}\left(2 - S_0^2\right)^{-3/2} \tag{4.19}$$

Substituting the above expressions the flight time becomes:

$$t_f = \sqrt{\frac{R_e^3}{\mu}}\left(2 - S_0^2\right)^{-3/2}\left[\frac{\pi}{2} - \arcsin\left(\frac{1 - S_0^2}{\sqrt{1 - S_0^2\cos^2\gamma_0\left(2 - S_0^2\right)}}\right) + S_0\sin\gamma_0\sqrt{2 - S_0^2}\right] \tag{4.20}$$

4.3 Influence of launch errors

Launch errors have an influence on the distance travelled. This difference can be approximated by:

$$\Delta d = 2R_e \Delta\Psi_0 = 2R_e \left[\frac{\partial \Psi_0}{\partial V_0} \Delta V_0 + \frac{\partial \Psi_0}{\partial \gamma_0} \Delta \gamma_0 \right] \tag{4.21}$$

4.4 Three-dimensional ballistic flight across the Earth

To determine the Launch and Impact point on the Earth, the two-dimensional trajectory has to be mapped on the three-dimensional surface. This is done by using the following figure:

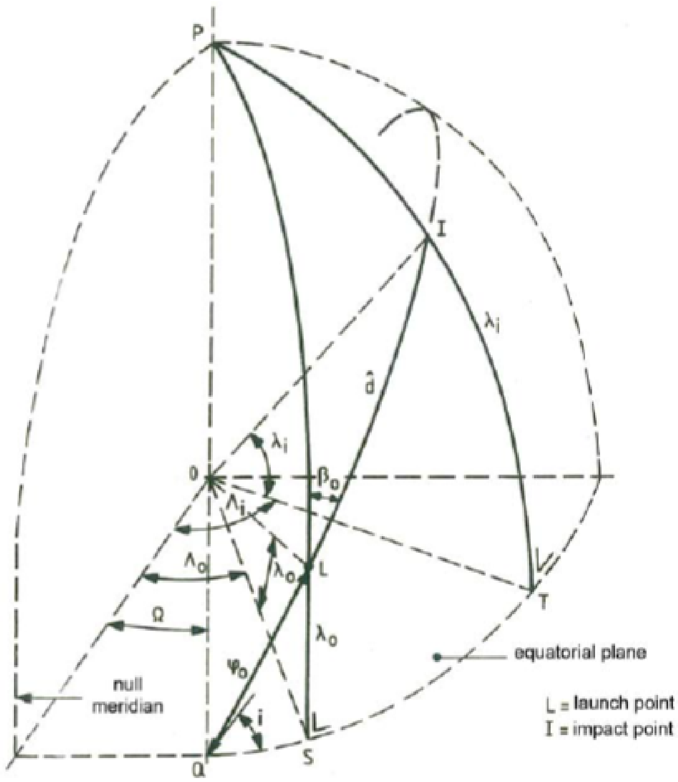

Figure 4.2: Location of point of launch and point of impact

The following equations can be derived from the figure above:

$$\cos i = \sin \beta_0 \cos \lambda_0 \quad ; \quad \tan \lambda_0 = \tan \varphi_0 \cos \beta_0 \quad ; \quad \tan(\Lambda_0 - \Omega) = \tan \varphi_0 \cos i = \sin \lambda_0 \tan \beta_0 \tag{4.22}$$

The **latitude** of the point of impact λ_i is determined by:

$$\sin \lambda_i = \sin \lambda_0 \cos \hat{d} + \cos \lambda_0 \sin \hat{d} \cos \beta_0 \tag{4.23}$$

The **longitude** of the point of impact Λ_i is determined by:

$$\tan(\Lambda_i - \Omega) = \tan\left(\varphi_0 + \hat{d}\right) \cos i \tag{4.24}$$

Spherical triogonometry
The impact point is derived using spherical trigonometry. The following equations hold:

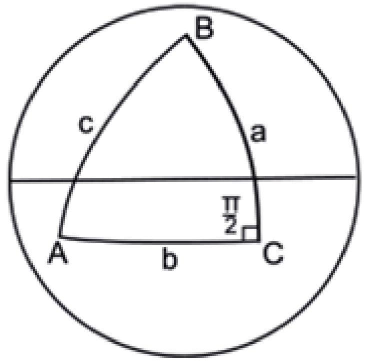

Figure 4.3: Spherical geometry

$\cos c = \cos a \cos b$
$\sin a = \sin A \sin c$
$\sin b = \sin B \sin c$

$$\tan a = \tan A \sin b$$

$\tan b = \tan B \sin a$
$\tan b = \cos A \tan c$
$\tan a = \cos B \tan c$

$$\cos A = \sin B \cos a$$

$\cos B = \sin A \cos b$
$\cos c = \cot A \cot B$

Rotating Earth

For a rotating Earth, the local velocities and angles have to be transformed to inertial parameters. The following figure illustrates the transformation. The transformation is dependent on the lattitute λ_0 and the heading angle β_0.

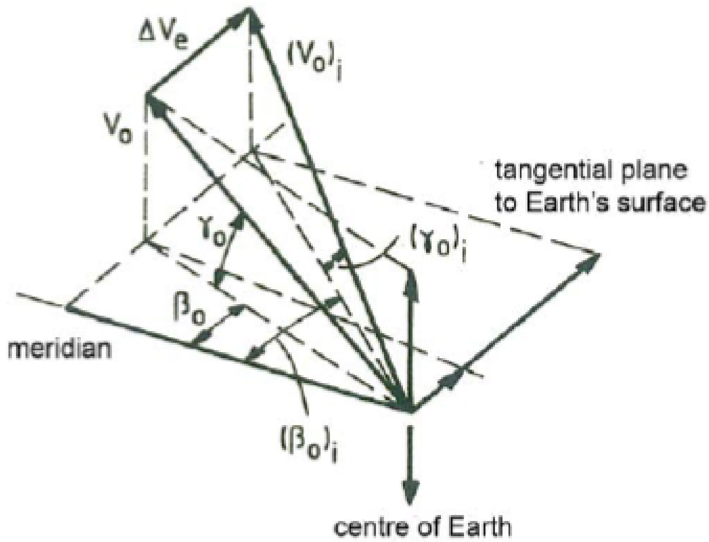

Figure 4.4: Transformation of local velocity and angles to inertial velocity and angles

The velocity is increased by the Earth's surface velocity in the longitudinal direction:

$$\Delta V_e = \omega_e R_e \cos\lambda_0 \tag{4.25}$$

The inertial velocities can be determined by the following equations:

$V_{0,i,z} = V_0 \sin\gamma_0$ (perpendicular to the Earth's surface)
$V_{0,i,y} = V_0 \cos\gamma_0 \cos\beta_0$ (direction of the meridian)
$V_{0,i,x} = V_0 \cos\gamma_0 \sin\beta_0 + \Delta V_e$ (longitudinal direction)

The inertial launch velocity is then given by:

$$V_{0,i} = \sqrt{V_{0,i,x}^2 + V_{0,i,y}^2 + V_{0,i,z}^2} = V_0\sqrt{1 + 2\cos\gamma_0 \sin\beta_0 \frac{\Delta V_e}{V_0} + \left(\frac{\Delta V_e}{V_0}\right)^2} \tag{4.26}$$

The inertial launch angle and azimuth-angle are then given by:

$$\sin\gamma_{0,i} = \frac{V_0 \sin\gamma_0}{V_{0,i}} \quad ; \quad \sin\beta_{0,i} = \frac{V_0\cos\gamma_0\sin\beta_0 + \Delta V_e}{V_{0,i}\cos\gamma_{0,i}} \tag{4.27}$$

During the flight the Earth rotates under the Rocket, therefore reducing or increasing the longitude. The true impact point Λ_i' is found by using the following equation.

$$\Lambda_i' = \Lambda_i + \omega_e t_f \tag{4.28}$$