ENGINEERING MECHANICS

Volume 1

Equilibrium

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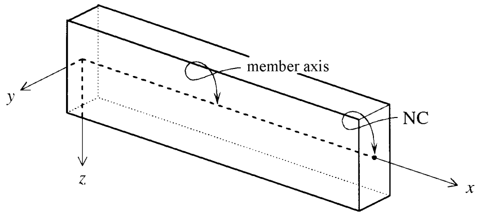
Springer

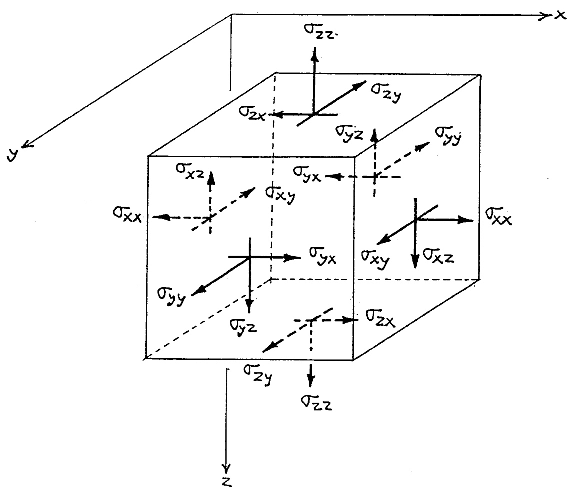
Coordinate system in a member (bar, beam, column, etc.)

The x axis is selected along the *member axis*, through the *normal (force) centre* NC of the consecutive cross-sections. The y and z axis are chosen parallel to the plane of a cross-section.



Normal and shear forces

σ_{ij} is the stress on a plane with the normal in the i direction ($i = x, y, z$), and acting in the j direction. σ_{ij} is a normal stress when $i = j$ and a shear stress when $i \neq j$. The positive directions are shown in the figure.



Relationship between section forces and stresses in the cross-section

Normal force

$$N = \int_A \sigma_{xx}\,\mathrm{d}A$$

Shear forces

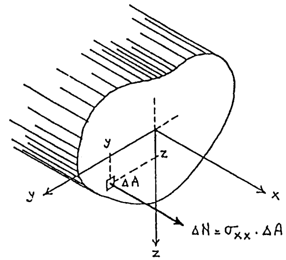
$$V_y = \int_A \sigma_{xy}\,\mathrm{d}A$$

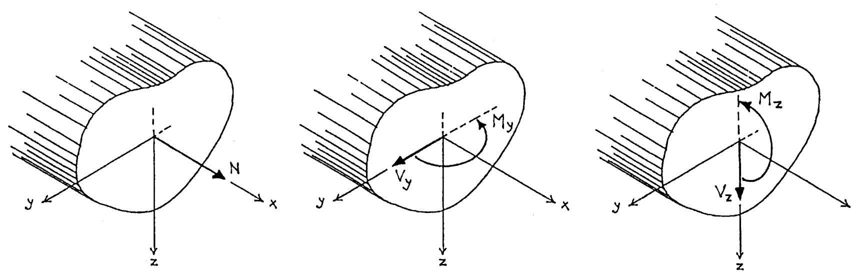
$$V_z = \int_A \sigma_{xz}\,\mathrm{d}A$$

Bending moments

$$M_y = \int_A y\sigma_{xx}\,\mathrm{d}A$$

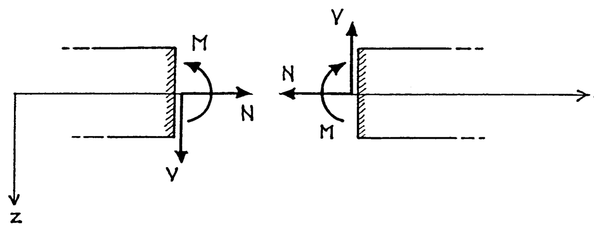
$$M_z = \int_A z\sigma_{xx}\,\mathrm{d}A$$





Positive directions of N, V_z and M_z

The figure below shows the positive directions of the normal force N, shear force $V_z = V$ and bending moment in the xz plane $M_z = M$.



Relationship between section forces and load

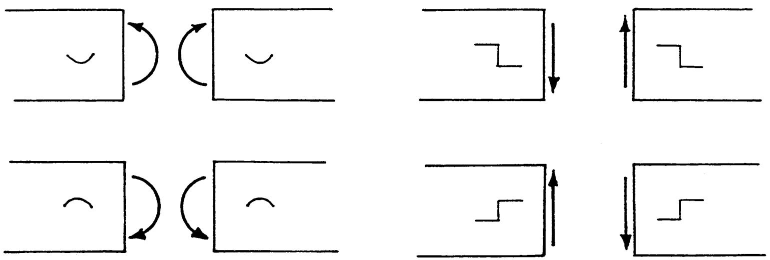
$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0, \qquad N = -\int q_x \,\mathrm{d}x,$$

$$\frac{\mathrm{d}V_z}{\mathrm{d}x} + q_z = 0, \qquad V_z = -\int q_z \,\mathrm{d}x,$$

$$\frac{\mathrm{d}M_z}{\mathrm{d}x} - V_z = 0, \qquad M_z = \int V_z \,\mathrm{d}x = -\iint q_z \,\mathrm{d}x\,\mathrm{d}x.$$

Deformation symbols

The deformation symbols for bending are given in the left-hand figure, those for the shear forces are shown in the right-hand figure.

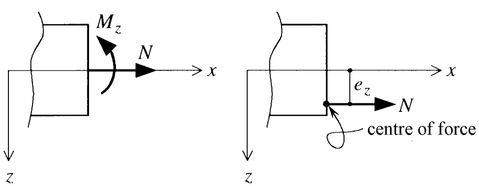


Cable equation

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -q_z.$$

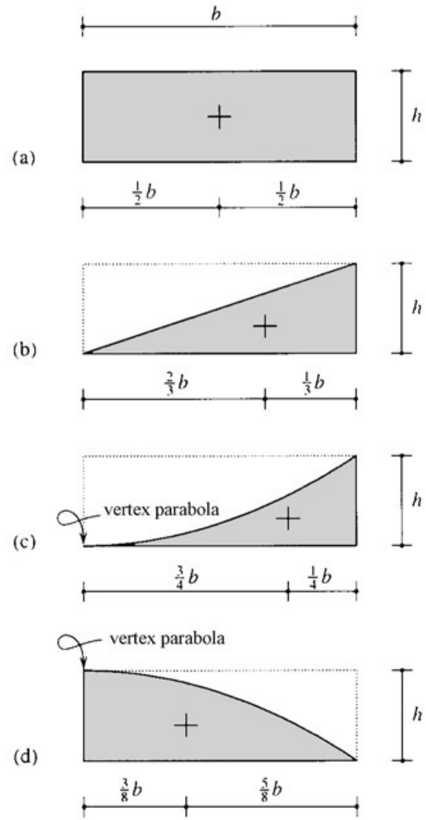
Centre of force

$$e_z = \frac{M_z}{N}, \quad \text{see figure below.}$$



Area A and centroid (+) of a number of simple shapes

Shape	Area A	Figure
Rectangle	bh	a
Right-angles triangle	$\frac{1}{2}bh$	b
Parabola (concave)	$\frac{1}{3}bh$	c
Parabola (convex)	$\frac{2}{3}bh$	d



ENGINEERING MECHANICS

Engineering Mechanics

Volume 1: Equilibrium

by

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Preface

This Volume is the first of a series of two:

- Volume 1 : Equilibrium
- Volume 2 : Stresses, deformations and displacements

Volume 1 introduces the fundamentals of structural and continuum mechanics in a comprehensive and consistent way. All theoretical developments are presented in the text and by means of an extensive set of figures. Numerous examples support the theory and provide a link to engineering practice. Combined with an extensive set of problems in each chapter, students are given ample opportunities to exercise.

The book consists of distinct modules, each divided into sections which are conveniently sized to be used as lectures. Both formal and intuitive (engineering) arguments are used in parallel to derive the important principles, for instance in bending moment diagrams and shear force diagrams. An important feature of the book is the straightforward and consistent sign convention, based on the stress definitions of continuum mechanics which will be used in Volume 2.

The modular content of the book shows a clear order of topics, starting with the introduction of forces and equilibrium of a particle followed by the extension to moments and the equilibrium of rigid bodies. An important aspect that is used throughout the series is the interaction between rigid bodies and the forces that act upon rigid bodies. These forces play an important role in Chapter 4, where structural elements and support conditions are introduced, followed by Chapter 5, which deals with the interaction forces and support reactions. A comprehensive chapter on loads gives an overview not only of the origin of loads, but also provides an introduction how to treat loads in engineering codes and in structural calculations. Examples of specific loads from gases, from liquids and from soils can be found in Chapters 7 and 8. These chapters can be regarded as an introduction in soil and fluid mechanics, and can be omitted when treating only structural mechanics.

After the basic theory of equilibrium of rigid bodies, boundary conditions and the method of calculating the reactions, the focus shifts to the section forces (internal forces) in trusses (Chapter 9), and beam and frame structures (Chapters 10 to 13). The formal treatment of the beam theory of Chapter 11 uses as little mathematics as possible and shows the fundamental relations between bending moments, shear forces and distributed loads. This fundamental approach is supported with an extensive intuitive approach based on the visual use of bending moment diagrams and shear force diagrams. Chapters 12 and 13 are therefore the most important chapters, and use all previously introduced definitions and sign conventions.

The last part of Volume 1 consists of some special topics like cables (Chapter 14), virtual work and influence lines (Chapters 15 and 16). Virtual work is introduced as an alternative to the ordinary equilibrium conditions as used in the first part of this book. Using the principle of virtual work proves to be a fast method to calculate sectional forces and reactions in statically determinate structures. The theory of virtual work is also needed to obtain influence lines. Chapter 16 can therefore only be used in combination with Chapter 15.

Although the books introduce the fundamentals of engineering mechanics, not much mathematical knowledge is required. Examples in which use is made of integral calculus or differential equations can be omitted, although they contribute to the mathematical explanation of the relations between bending moments, shear forces and distributed loads. The educational value is therefore not only fundamental knowledge. It is also a demonstration how to translate physical problems into abstract models, which can be solved with mathematical tools.

Finding the right balance between the abstract fundamentals and practical application should be the challenge for the lecturer.

Coenraad Hartsuijker
Hans Welleman
Delft, The Netherlands
July 2006

Foreword

Structural or Engineering Mechanics is one of the core courses for new students in engineering studies. At Delft University of Technology a joint educational program for Statics and Strength of Materials has been developed by the Koiter Institute, and has subsequently been incorporated in the curricula of faculties like Civil Engineering, Aeronautical Engineering, Architectural Engineering, Mechanical Engineering, Maritime Engineering and Industrial Design.

In order for foreign students also to be able to benefit from this program an English version of the Dutch textbook series written by Coenraad Hartsuijker, which were already used in most faculties, appeared to be necessary. It is fortunate that in good cooperation between the writers, Springer and the Koiter Institute Delft, an English version of two text books could be realized, and it is believed that this series of books will greatly help the student to find his or her way into Engineering or Structural Mechanics.

Indeed, the volumes of this series offer some advantages not found elsewhere, at least not to this extent. Both formal and intuitive approaches are used, which is more important than ever. The books are modular and can also be used for self-study. Therefore, they can be used in a flexible manner and will fit almost any educational system. And finally, the SI system is used consistently. For these reasons it is believed that the books form a very valuable addition to the literature.

René de Borst
Scientific Director, Koiter Institute Delft

1 Introduction

This chapter provides a number of definitions and describes various concepts. Following a brief description of the field of mechanics in Section 1.1, Section 1.2 addresses the character of a number of important quantities in mechanics, and the units in which they are expressed. Quantities of a magnitude and direction that meet the conditions of the so-called parallelogram rule are called vectors, which are covered in Section 1.3.

Newton's three Laws of Motion and his Law of Gravitation were an important step forward in the development of mechanics. We look at these laws at the end of the chapter in Section 1.4.

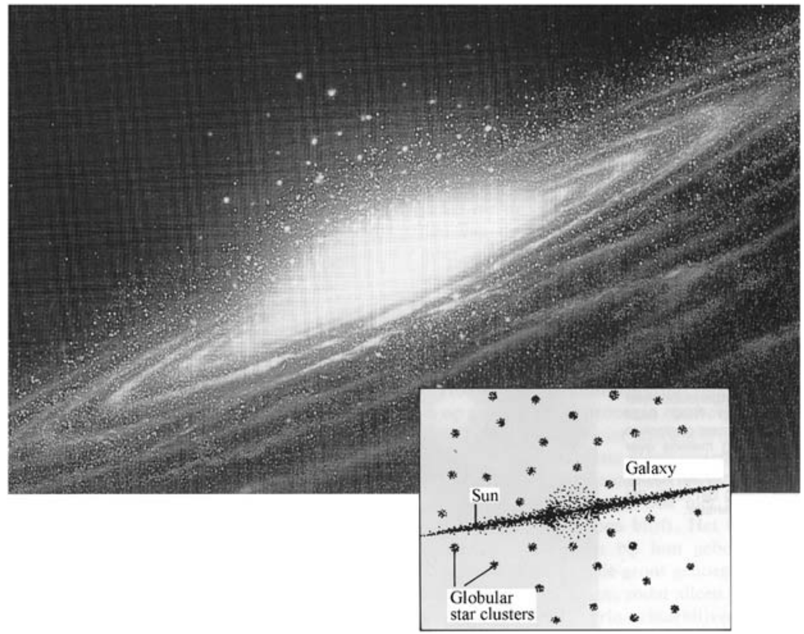


Figure 1.1 Model of our Galaxy and its globular star clusters. Source: *Natuur en Techniek* 89/10, p. 757.

1.1 Mechanics

1.1.1 Examples from the field of mechanics

Mechanics is the subdivision of physics which addresses equilibrium and the motion of matter.

Mechanics therefore includes for example:

- The description of the movement of natural and artificial heavenly bodies. Figure 1.1 is a schematic representation of our Galaxy. The vast majority of all stars are in a flat disk. Above and below this disk, there are some 200 globular star clusters that revolve in ellipsoidal orbits

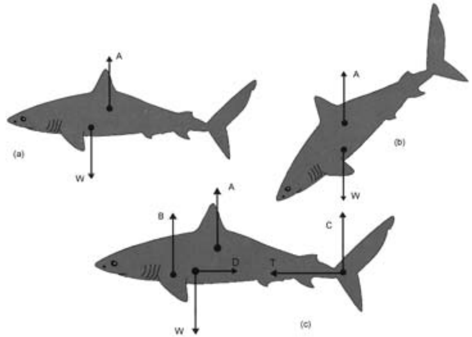


Figure 1.2 The forces exerted on a swimming shark. Source: *Natuur en Techniek* 90/02, p. 136.

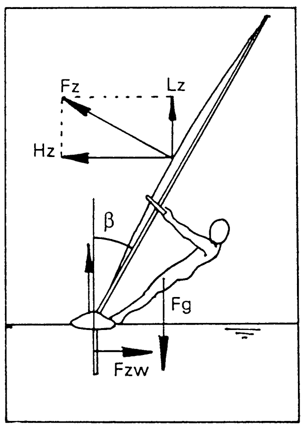


Figure 1.3 A surfer balancing on his board. Source: *Leidraad voor surfers*, Vereniging Zeilscholen Nederland.

around the centre of the Galaxy.

- A calculation of the forces exerted on a swimming shark. In Figure 1.2a, in which the shark is at rest, the shark is subject to forces resultant from its weight W and the upward force A caused by the water pressure. As a result, the animal tips over (see Figure 1.2b). If the shark's tail generates a thrust T, vertical forces are generated that keep the shark in vertical equilibrium (see Figure 1.2c).
- Balancing on a surfboard (see Figure 1.3).
- A calculation as to the deformation of an oil platform at sea subject to wave action. Figure 1.4 shows a concrete platform designed for the Norwegian Troll field with a water depth of 340 metres. The seabed consists of extremely weak clay. The sea conditions are extremely rough with waves over 10 metres in height. The mass of the deck is 60,000 tons (60×10^6 kg).
- The description of water currents in a river, estuary, or sea. Figure 1.5 represents a current model for the North Sea. The arrows indicate the direction and strength of the current for certain areas. This type of model can be used to investigate the distribution of toxic materials.
- The investigation of stresses in prostheses, such as an artificial hand, hip joint or knee joint. As shown in Figure 1.6, the attachment of the prosthesis in a knee joint is extremely important. Figure 1.7a shows the magnitude of the forces calculated by using an arithmic model. Major tensile stress occurs at the end of the prosthesis (black area). This stress can lead to fractures in the cement (adhesive) as shown in the X-ray in Figure 1.7b.
- Finding the right shape for a high tower by effectively transferring the loading by wind and its dead weight onto the foundation. Figure 1.8 shows the 300-metre Eiffel Tower, completed in 1889 and the first 1000-foot tower, built for the 1889 World Exhibition in Paris. The tower is constructed of wrought iron.

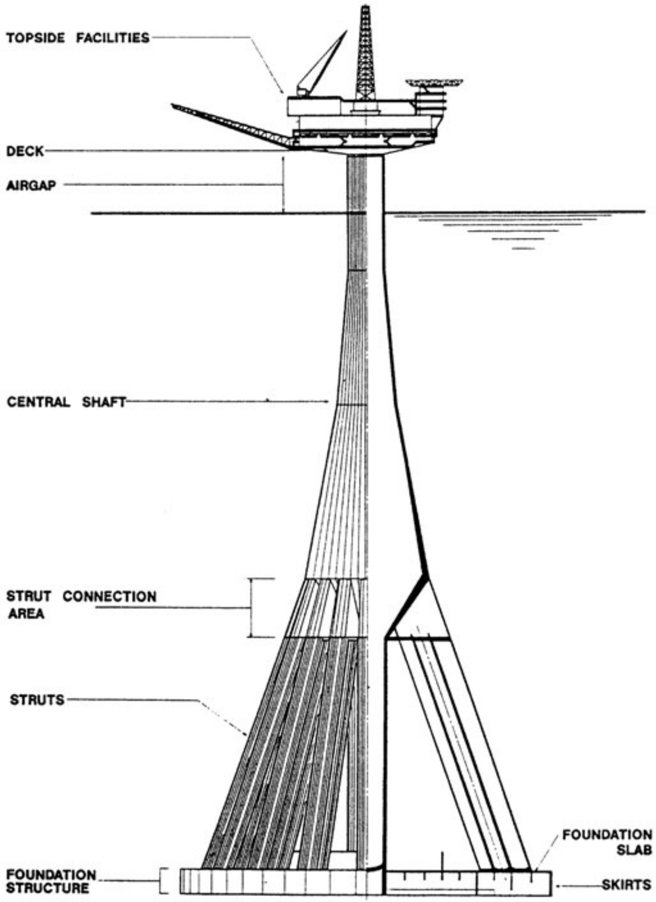


Figure 1.4 Design of a concrete platform in 340-metre deep water. Source: *Heron* 1986, no. 1, p. 86.

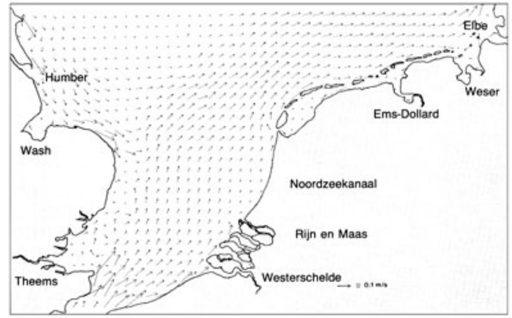


Figure 1.5 A current model for the North Sea. Source: *Natuur en Techniek* 90/04, p. 292.

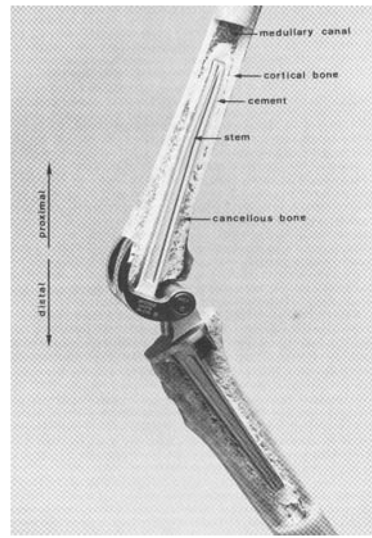


Figure 1.6 Open knee prosthesis. Source: *Heron* 1986, no. 1, p. 100.

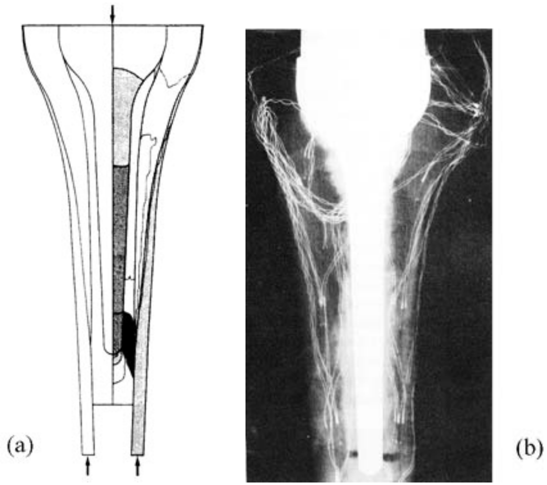


Figure 1.7 In the model of the knee prosthesis (a) the grey shades indicate the size of the stress according to an arithmetic model. The largest tensile force occurs in the black area near the end of the prosthesis. The X-ray (b) shows a fracture in the cement at that point. Source: *Heron* 1986, no. 1, p. 105.

- Water flow through a dam. In Figure 1.9, you can see stream lines and equipotential lines for a dam on an impermeable subsoil. They are perpendicular to one another and form a so-called flow net. The EF section of the slope is known as the seepage surface. Here the water leaves the dam and flows down along the slope.
- Closing the Maeslant barrier (see Figure 1.10).
  The Maeslant barrier, the storm barrier in the Nieuwe Waterweg in the west of the Netherlands, consists of two 22-metre high sector doors shaped like an arc with an arc length of 214 metres. The doors are turned towards each other afloat from docks. When closed, the doors are sunk onto a threshold by the inlet of water. The water pressure on the doors is diverted to foundation blocks by means of two 260-metre truss arms. The truss arm and the foundation block are joined by means of a ball-and-socket joint with a 10-metre diameter.

1.1.2 Subdivisions within mechanics

Mechanics' extensive field of operation can be subdivided in various ways.

A subdivision addressed in the given description of mechanics is based on the perspective of rest and movement:
- *Statics*, or the study of material at rest.
- *Dynamics*, or the study of moving material.

A subcomponent of dynamics is *kinematics*, the study that describes the displacement of bodies, without addressing the cause of the movement.

Another subdivision of mechanics is that which describes the degree of deformability of matter:
- *Theoretical mechanics*, the mechanics of particles and rigid (non-deformable) bodies.
- *Solid mechanics*, the mechanics of solid deformable bodies.
- *Fluid mechanics*.
- *Gas mechanics*.

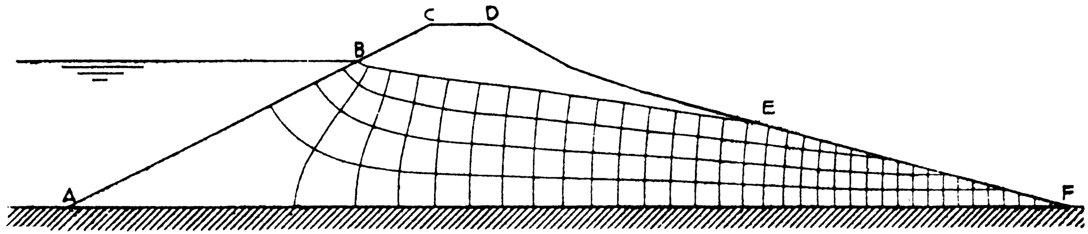


Figure 1.9 Stream lines and equipotential lines in a dam on an impermeable subsoil.

Figure 1.8 The Eiffel tower (1889) was the world's first 1000-foot tower. Photograph: Hans Welleman.

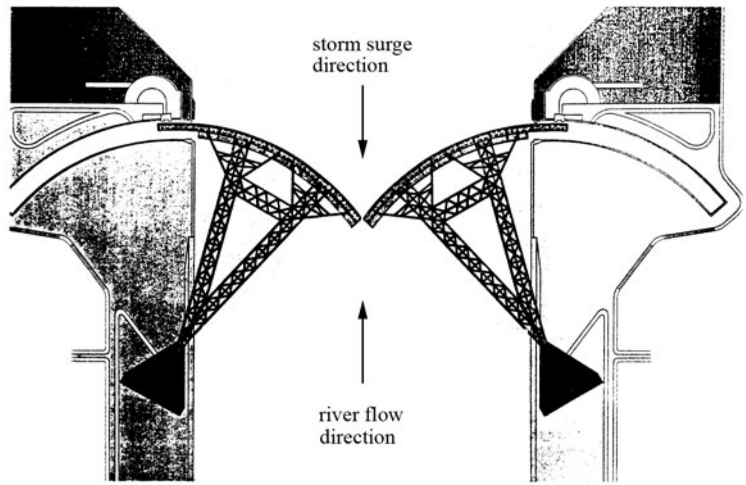


Figure 1.10 The Maeslant barrier – a storm barrier in the Nieuwe Waterweg near Rotterdam in the Netherlands.

1.1.3 Applied mechanics

In principle, the *mechanics of structures* addresses both the statics and dynamics of structures. This book solely covers the statics of structures.

Mechanics allows us to investigate to what degree a structure, both in its entirety and with respect to its individual components, is effective and reliable regarding strength, stiffness, and stability.[1]

For structures made of solid, deformable materials (concrete, wood, synthetics, or metals such as steel or aluminium), the field of mechanics is also known as *applied mechanics*.

The part of applied mechanics which focusses on calculating the forces in a structure is known as *structural mechanics*. The part in which the focus is on stress and deformation (strength and stiffness) is known as *mechanics of materials*. The division between structural mechanics and the mechanics of materials is only effective for so-called *statically-determinate structures*,[2] or structures in which the force flow can be determined directly from the equilibrium. For calculations relating to structures other than those that are statically-determinate (so-called *statically-indeterminate structures*) one has to use elements from both structural mechanics and the mechanics of materials.

The behaviour of a structure must be investigated "*beyond the base*". For example, it is important that the slides in the Oosterschelde barrier in Figure 1.11 are sufficiently strong, but it is equally important that the structure can be properly carried by the subsoil. Since the behaviour of soil clearly

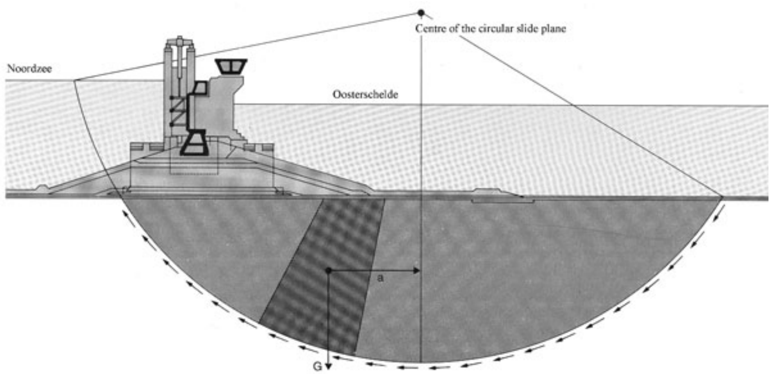


Figure 1.11 The Oosterschelde barrier; the subsoil has to be able to bear the structure.

[1] *Stability* is defined as the *reliability of the equilibrium*. Since the stability of the equilibrium depends on the stiffness of the structure, the stability demand can also be interpreted as a stiffness demand.

[2] The concepts *statically-determinate* and *statically-indeterminate* are covered in more detail in Chapter 4.

differs from that of regular solid material, the investigation into the forces and deformations in soil is part of a separate field of expertise known as *soil mechanics*.

1.1.4 Theory and experiment

Mechanics, as a part of physics, is a science that addresses the determination of laws and patterns that can be used to describe natural phenomena and, more importantly, that can be used to predict them. As such, mechanics is an *empirical science*: it aims to formulate the phenomena investigated and their mutual relationship as accurately as possible. In doing so, it is not the results from the calculations that are decisive, but rather their agreement with what we learn from observation and experimentation. After all, we want to be able to predict with a certain degree of accuracy whether a satellite we launch will end up in its orbit, or whether a bridge is sufficiently strong and rigid.

Reality is however far too complex to be described fully. For this reason, one always has to work with a *model*, a simplified representation of reality, and one which addresses only a limited number of factors concurrently. Which aspects are addressed and which *schematisations* (simplifications) are used, depends on the objective in question.

1.1.5 Schematisation

Schematisation is an abstraction that at the same time includes all the relevant issues. When investigating the movement of the earth around the sun, the dimensions of the earth are of subsidiary importance, and we schematize the earth as a particle. If, however, one is looking to investigate the rotation of the earth around its axis, we do have to take the dimensions of the earth into consideration.

When calculating the force flow in a framework, it is common practice to schematize the columns and beams as so-called line elements, and to

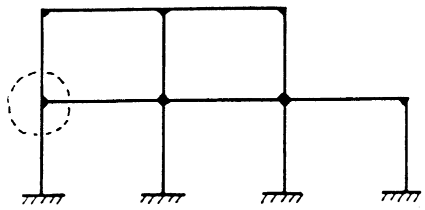


Figure 1.12 When performing calculations for a building, the columns and beams are generally modelled as so-called line elements; the column-beam joints are thereby reduced to particles, with negligible dimensions.

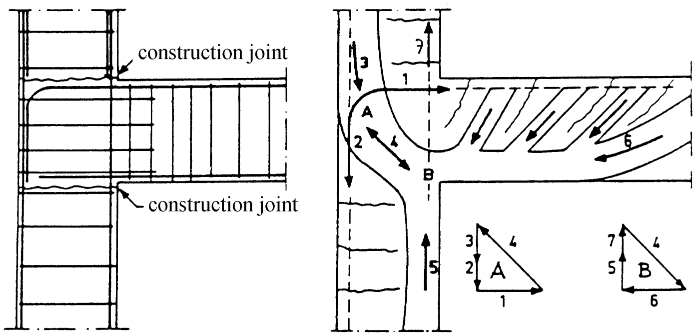


Figure 1.13 When detailing the column-beam joints, like here in reinforced concrete, the dimensions are no longer negligible.

represent the joints between the columns and beams as particles (see Figure 1.12). If we want to find out more about the interaction of forces in a column-beam joint, for example in the case of a concrete structure to be able to determine where the reinforcement has to be placed, the dimensions of the joint can no longer be ignored, and one has to use another schematisation for the joint (see Figure 1.13).

Mechanics uses a range of concepts that offer a schematisation of reality. For example, dimensions are ignored for *particles*, while for *rigid bodies* deformation is ignored. Another concept is that of *stress*, in which a continuous structure of matter is assumed, while in reality (on a micro level) this is discrete, with molecules and atoms. You should be aware of these schematisations.

Much knowledge within mechanics is set down using mathematical formulas, based on certain schematisations and modelling. Mathematics subsequently offers a language that enables us to formulate and solve the problems, and interpret the solution unambiguously. We can then use the findings to make predictions relating to the behaviour of a structure. It is from this predictive capacity that the science of mechanics derives its practical use.

Within a given schematisation, mathematical models are used, and the results are exact; for this reason, mechanics is often called an *exact science*.

1.2 Quantities, units, dimensions

1.2.1 Quantities and their units

Mechanics involves measurable, physical quantities. A *quantity* X is generally characterised by a *numerical value* $\{X\}$ and a *unit* $[X]$. This can be symbolically described as

$$\begin{aligned} \text{quantity} &= \text{numerical value} \times \text{unit} \\ X &= \{X\} \times [X]. \end{aligned}$$

The unit $[X]$ is the degree to which the quantity X is measured.

In mechanics, one uses the International Units System (Système International d'Unité), abbreviated in all languages to SI. The SI includes

- seven basic units (Table 1.1);
- two supplementary units (Table 1.2);
- and a large number of derived units.

Basic units, critical in the structural mechanics, are *length*, *mass* and *time*.

Length (ℓ)
A measure for measuring distances in space. Space is defined as the geometric area in which people live and work and in which they build their structures. The basic unit of length is the metre [m].

Mass (m)
A measure for the characteristic of a body that it resists a change in its movement. This characteristic is known as the *inertia* of the body. The basic unit for mass is the kilogram [kg] (not grams!).

Time (t)
A measure for the sequence of events. The fundamental unit for time is the second [s].

SI derivative units are obtained from the definitions of the derived basic quantities as products and quotients of powers of basic units. A number

Table 1.1 Basic quantities and basic units.

Basic quantity		**Basic unit**	
Name	**Symbol**	**Name**	**Symbol**
length	ℓ	metre	m
mass	M	kilogram	kg
time	T	second	s
electric current	I	amp	A
thermodynamic temperature	T	Kelvin	K
amount of material	N	mol	mol
luminosity	I	candela	cd

Table 1.2 Supplementary quantities and units.

Supplementary quantity		**Supplementary unit**	
Name	**Symbol**	**Name**	**Symbol**
(plane) angle	α	radian	rad
solid angle	Ω	steradian	sr

of derived units have their own names and their own symbols. You will find a number of these units in Table 1.3.

Table 1.3 Derived units with their own name and symbol.

Derived quantity	**Derived unit**	
	Name	**Symbol**
area	square metre	m^2
volume, content	cubic metre	m^3
frequency	Hertz	$\mathrm{Hz} = \mathrm{s}^{-1}$
force	Newton	$\mathrm{N} = \mathrm{kgm/s}^2$
pressure, tension	Pascal	$\mathrm{Pa} = \mathrm{N/m}^2$
work, energy, amount of warmth	Joule	$\mathrm{J} = \mathrm{Nm}$
capacity, energy flow	Watt	$\mathrm{W} = \mathrm{J/s}$

1.2.2 Prefixes

If numbers are either very large or very small, you can use a prefix for the unit. Frequently used prefixes are shown in Table 1.4.

Example:

$$1\ \mathrm{MPa} = 10^6\ \mathrm{Pa} = 10^6\ \mathrm{N/m}^2 = 1\ \mathrm{N/mm}^2.$$

For derived units as a product of a number of units, we can join up the symbol group, unless this gives rise to confusion. In the latter case, place a multiplication point between the units. In this vein, Nms could either be Newton-metre-second or Newton-milliseconds. Depending on what one is trying to say, you should therefore write Nm·s or N·ms.

Table 1.4 Common SI prefixes.

Prefix	**Symbol**	**Factor**
giga	G	10^9
mega	M	10^6
kilo	k	10^3
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}

1.2.3 Dimensions

Besides the unit $[X]$ in which a quantity X is expressed and the associated numerical value $\{X\}$, a quantity also has a dimension $\dim(X)$. The *dimension* indicates the type of quantity without saying anything about the choice of unit or the magnitude of the numerical value.

The dimensions of the basic quantities are called the basic dimensions. For the basic dimensions of length (ℓ), mass (m), and time (t) one writes

$$\begin{aligned}\dim(\ell) &= L,\\ \dim(m) &= M,\\ \dim(t) &= T.\end{aligned}$$

You will find a number of examples of derived quantities and their dimensions in Table 1.5.

Dimension formulas can be used to determine whether inaccuracies have occurred in deriving a physical relationship. This is known as a *dimension check*. They can be used to check that the expressions to the left and the right of the equals sign have the same dimensions. The same can be achieved by checking whether the products of all the units, expressed in terms of the basic units, are the same on both sides.

The radian and solid angle are considered dimensionless quantities. When performing a dimension check, we must assign the symbols for rad and sr the dimension 1.

Table 1.5 Examples of derived quantities and their dimensions.

Type of quantity	**Definition**	**Dimension formula**	**SI unit**
velocity	$v = \mathrm{d}u/\mathrm{d}t$	LT^{-1}	m/s
force	$F = m \cdot a$	LMT^{-2}	$\mathrm{N} = \mathrm{kgm/s}^2$
energy, work	$E = A = F \cdot \ell$	$\mathrm{L}^2\mathrm{MT}^{-2}$	$\mathrm{J} = \mathrm{kgm}^2/\mathrm{s}^2$

1.3 Vectors

1.3.1 Scalars and vectors

Certain physical quantities are fully determined by a numerical value with the associated unit. These include length, mass, time, temperature, work and energy, and are referred to as scalar quantities, or *scalars*. Other physical quantities can be fully described only if, in addition to the magnitude (determined by a number and a unit), one also defines in which direction in space the quantity is oriented. If these quantities with a magnitude and direction meet the conditions of the so-called *parallelogram rule* (see Section 1.3.4), they are known as *vectors*. Vectors include motion, velocity, impulse, acceleration, and force.

In order to distinguish vectors from scalars, the symbols for a vector are printed in bold (**a**) or we place an arrow over the symbol ($\vec{a}$).

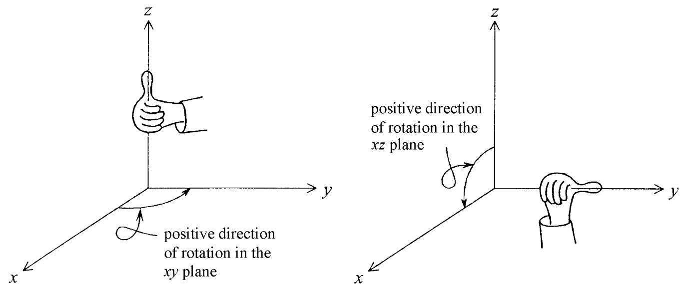


Figure 1.14 A right orthogonal coordinate system.

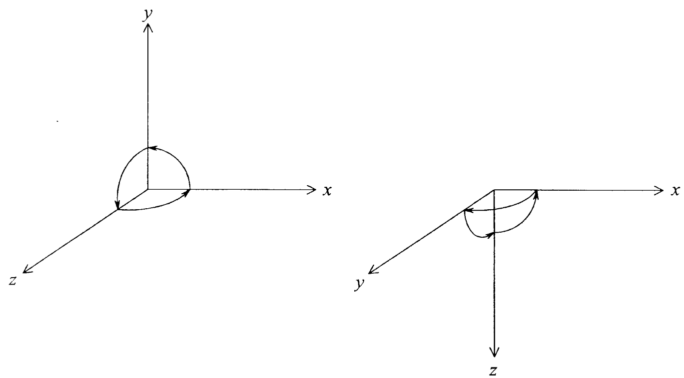


Figure 1.15 A right orthogonal coordinate system in two other positions, with the positive direction of rotation in the coordinate planes.

Other physical quantities besides scalars and vectors are *tensors*[1] (of the second order and above). Tensors are not covered in this book.

1.3.2 Coordinate system

Vectors are quantities with a direction in space. Space is seen as three dimensional and *Euclidian* (after Euclid[2]). When describing phenomena in space, one uses a *right orthogonal coordinate system*. This is a system of three mutually perpendicular axes x, y and z, that are oriented in such a way that they meet the conditions of the so-called *right-hand rule*: if you make a fist with the fingers of your right hand, as shown in Figure 1.14, and you point the free thumb in the z direction, the bent fingers in your fist have to point in the direction of a rotation with the smaller angle of the x axis to the y axis. This direction of the rotation with the smaller angle of the x axis to the y axis is called the positive *direction of rotation* in the xy plane (about the z axis). In this description, x, y and z can be exchanged cyclically (see Figure 1.14).

Figure 1.15 shows two more examples of such coordinate systems, with the positive directions of rotation in the various coordinate planes.

An orthogonal coordinate system is called *Cartesian* (after Descartes[3]) if equal units are chosen along the coordinate axes.

---

1 Scalars and vectors can be seen as members of the family of tensors. Vectors are also known as tensors of the first order. Scalars are tensors of the zero order. Second order tensors in mechanics include strain tensor, stress tensor, and bending stiffness tensor. Tensors can be recognised by the transformation rules for their components when rotating the coordinate system.

2 Euclid (approx. 300 BC), Greek mathematician in Alexandria.

3 René Descartes (Cartesius) (1596–1650), French mathematician and philosopher. Main work: “Discours de la méthode” (1637).

If equal unit vectors are chosen along the axes, this is referred to as an *orthonormal* coordinate system (contraction of orthogonal and normalised).

1.3.3 Types of vectors

In a diagram, a vector in space can be represented by an arrow. The direction of the arrow represents the direction of the vector. The length of the arrow (in a particular scale) can be drawn to represent the magnitude of the vector.

There are three types of vector:

- *Fixed vectors*
  Fixed vectors, in addition to their magnitude and direction, also have a *point of application* (see Figure 1.16).
  Example: a force on a deformable body.
- *Sliding vectors*
  Sometimes the location of the point of application is of no importance and may be moved in the direction of the vector. This is called a sliding vector. Sliding vectors do not have a fixed point of action, but have only a *line of action* (see Figure 1.17).
  Example: the force on a rigid body.
- *Free vectors*
  When the place of the line of action of a vector is not important either, one refers to a free vector.
  Example: The translation of a rigid body. All points of the body are subject to the same displacement. The free vector stands for the entire collection of displacement vectors (see Figure 1.18).

Comment:
If we want to investigate the equilibrium (or the motion) of a body as a whole, the body can often be considered a rigid (non-deformable) body, with the forces as sliding vectors. After all, it does not make a difference for rigid bodies whether it is kept in equilibrium by a force from above or

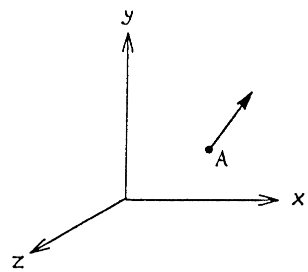


Figure 1.16 Fixed vector with point of application A.

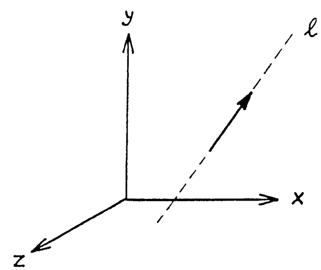


Figure 1.17 Sliding vector with line of action ℓ.

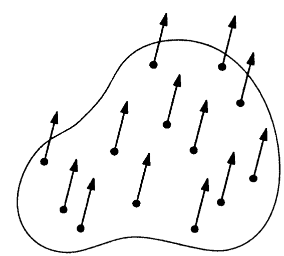


Figure 1.18 Free vector.

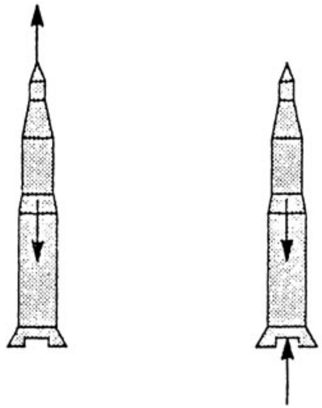


Figure 1.19 For the equilibrium of a rigid body, it is not relevant whether a force is moved along its line of action. It certainly makes a difference with respect to internal phenomena.

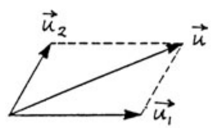


Figure 1.20 Vector addition using the parallelogram rule.

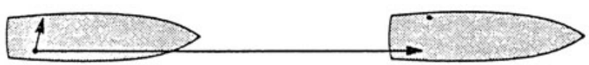


Figure 1.21 The vector addition illustrated using a sailor walking on a moving ship.

below (see Figure 1.19). On the other hand, if one is looking to investigate deformations or internal phenomena within the body, the points of application of the forces do play a role and the forces must be considered fixed vectors. For phenomena inside bodies (such as human bodies), it certainly makes a difference whether the body is hung from above or is supported from below!

1.3.4 Parallelogram rule

We can add two vectors with the same point of application into a single vector using the so-called *parallelogram rule* in Figure 1.20. The parallelogram rule is easy to understand if one imagines, as in Figure 1.21, the movement of a sailor walking on a moving ship. The displacement $\vec{u}$ of the sailor with respect to the earth consists of the sum of the displacement $\vec{u}_1$ of the ship with respect to the earth and his own displacement $\vec{u}_2$ with respect to the ship. In the same way, one can also add up velocity vectors and forces.

For the vector addition, as shown in Figure 1.20, one writes

$$\vec{u} = \vec{u}_1 + \vec{u}_2.$$

In reverse, we say that $\vec{u}_2$ is the difference between $\vec{u}$ and $\vec{u}_1$, or

$$\vec{u}_2 = \vec{u} - \vec{u}_1.$$

1.3.5 Vector components and scalar components

We often describe a vector by means of its so-called *components*. If $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are the unit vectors along respectively the x, y and z axis (vectors directed along the axes and with a length equal to 1), the vector can also be defined as the vector sum of its three components (see Figure 1.22):

$$\vec{a} = \vec{a}_x + \vec{a}_y + \vec{a}_z = a_x\vec{e}_x + a_y\vec{e}_y + a_z\vec{e}_z.$$

The vector quantities $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ are known as the *vector components* of vector $\vec{a}$. The scalar quantities a_x, a_y and a_z are the *scalar components*[1] of vector $\vec{a}$.

In this book we will usually take the word *component* to mean *scalar component*.

The *magnitude* or *norm*[2] of the vector $\vec{a}$ is:

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \qquad (a \geq 0).$$

To add two vectors given by their components we add the respective components. The sum of two vectors $\vec{a}$ and $\vec{b}$ with components a_x, a_y and a_z, respectively b_x, b_y and b_z, is:

$$\vec{a} + \vec{b} = (a_x + b_x)\vec{e}_x + (a_y + b_y)\vec{e}_y + (a_z + b_z)\vec{e}_z.$$

This is illustrated in Figure 1.23 for two vectors in the xy plane (with $a_z = b_z = 0$).

1.3.6 Formal and visual notation of a vector

So far in the figures, the arrow for a vector included the vector symbol (letter with an arrow above). In addition to this *formal notation* there is also a *visual notation*. Both notations are shown in Figure 1.24.

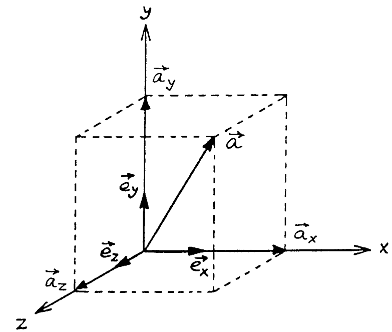


Figure 1.22 The vector components $\vec{a}_x$, $\vec{a}_y$, $\vec{a}_z$ of vector $\vec{a}$.

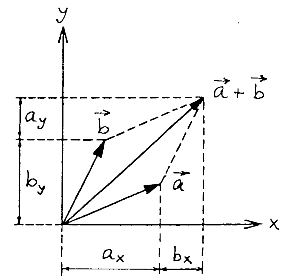


Figure 1.23 It is possible to add two vectors by adding their associated scalar components.

Figure 1.24 Vector notation (a) and visual notation (b).

---

1 The scalar components a_x; a_y; a_z of vector $\vec{a}$ are not scalars: they depend on the coordinate system that is used.

2 The magnitude or norm a of vector $\vec{a}$ is a scalar: it is independent of the coordinate system that is used.

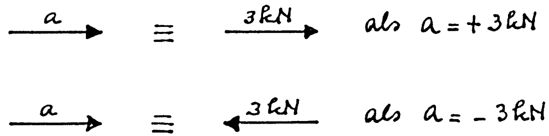


Figure 1.25 In visual notation, the arrow depicted should be seen as a unit vector that has to be multiplied by the depicted value.

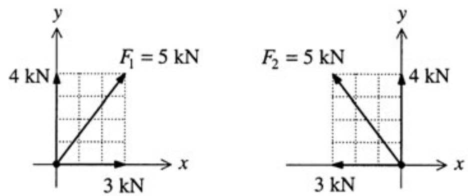


Figure 1.26 The forces $\vec{F}_1$ and $\vec{F}_2$ resolved into their components.

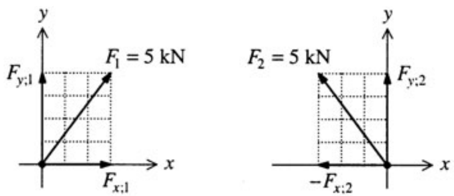


Figure 1.27 If we want to define the components of force $\vec{F}_2$ as $F_{x;2}$ and $F_{y;2}$ in a visual model, we have to place a minus sign next to $F_{x;2}$, the x component of $\vec{F}_2$.

In the visual notation, each arrow shown reflects a unit vector, which has to be multiplied by the value shown with the arrow. If this value is negative, the vector works in the direction opposite to the one shown (see Figure 1.25). Since in visual notation the emphasis lies on "*seeing what is happening*", it is preferred to not include a negative value alongside a vector arrow.

The visual notation is frequently used in mechanics for manual calculations. When setting up manual calculations, the visual aspect plays an important role as one generally links the calculation to a "*picture*" on the basis of which one can better imagine what is happening.

In Figure 1.26, forces $\vec{F}_1$ and $\vec{F}_2$ have been resolved into components along the x and y axis. All the forces have been drawn in the directions in which they operate and include their magnitude.

If one wants to name the components in the xy coordinate system shown, one has to imagine that F_x and F_y relate to the (not shown) unit vectors in the coordinate system, respectively $\vec{e}_x$ and $\vec{e}_y$.

Therefore[1]

$$F_{x;1} = +3 \text{ kN}; \qquad F_{y;1} = +4 \text{ kN};$$
$$F_{x;2} = -3 \text{ kN}; \qquad F_{y;2} = +4 \text{ kN}.$$

The x component of force $\vec{F}_2$ opposes the x direction (is opposite to the direction of the unit vector $\vec{e}_x$) and is therefore negative. If one wants to denote the components of $\vec{F}_1$ and $\vec{F}_2$ by F_x and F_y in a visual representation, as in Figure 1.27, one must place a minus sign next to $F_{x;2}$, the x

[1] The *direction indices* x and y always precede the other indices. It is common practice to separate the indices by a semicolon. Sometimes the *separator* is omitted.

component of force $\vec{F}_2$.

Forces are vectors. In the formal notation they are indicated with an arrow over the symbol: $\vec{F}$. In structural mechanics the visual notation is generally used. In that case it is usual to indicate a force by its magnitude $F = |\vec{F}|$. In this book we will principally use the visual notation for a force.

1.3.7 Vector properties

Quantities can be imagined as vectors if they meet the calculation rules for vectors. These rules include the *commutative* property for addition:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

and the *associative* property for addition:

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}).$$

These properties indicate that the vector sum is independent of the order in which the vectors are added.

Not every quantity that is defined by a magnitude and a direction is a vector. For example, the rotation of a body, a quantity with a magnitude and a direction, is not a vector, as the quantity does not meet the commutative and associative properties of the addition. This can be checked for a book in the xz plane by first rotating it through 90° about the y axis and then through 90° about the z axis. As shown in Figure 1.28, the final position changes if the rotation is performed in a different order.

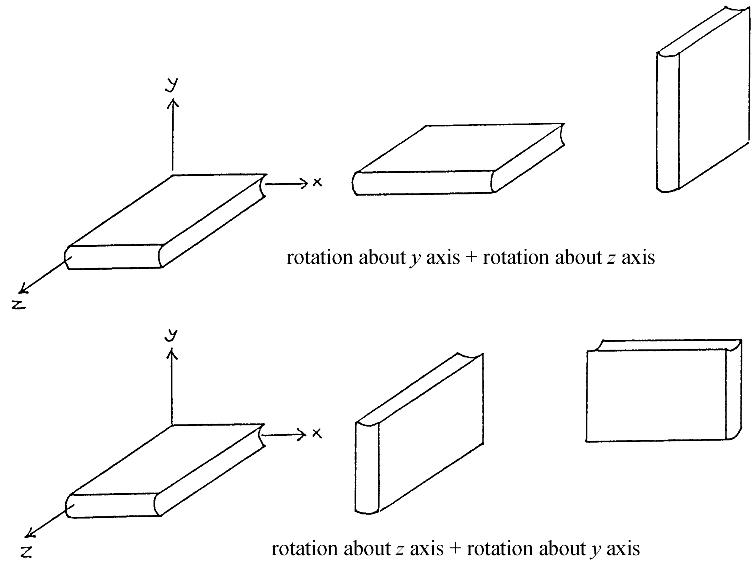


Figure 1.28 If the order changes when adding (finite) rotations, the end result also changes.

1.4 Newton's Laws

1.4.1 Basic laws

The basic laws for the displacement of a particle (a body with negligibly small dimensions but with some mass) were first formulated by Newton (1687).[1] Newton's three laws are as follows:

- First law or *law of inertia*.
  Every particle persists in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces imposed on it.
- Second law or *law of motion*.
  The rate of change in *momentum* of a particle (the product of mass and velocity) is equal to the force applied to it, and has the direction of that force.
- Third law or the *law of action and reaction*.
  If particle (1) exerts a force on particle (2), particle (2) will exert an equal and opposite force on particle (1).

Law of inertia

The first law states that a particle at rest will remain at rest if no force is exerted on it, and that a particle that is in motion in a straight line at a constant speed, will continue that movement at that same speed in the same straight line if no forces are exerted on it. The property with which a particle resists a change in its state of rest or movement is called its *inertia*. Newton's first law is therefore also known as the law of inertia.

---

[1] Sir Isaac Newton (1642–1727), an English mathematician and physicist, published his laws at the age of 44 in his book "*Philosophiae naturalis principia mathematica*", also known as "*Principia*". In his laws, Newton uses the word *body*. Later developments in mechanics showed that it must relate to a body without dimensions, here referred to as a *particle*. A body with finite dimensions can still perform rotations, which are not mentioned by Newton.

Law of motion

The second law is defined by the following formula:

$$\vec{F} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t}.$$

Here, $\vec{F}$ is the force on the particle, m its mass, and $\vec{v}$ its velocity.[1] The notation with vectors shows that the change in momentum has the same direction as the force.

If the mass of the particle does not change during the motion, the second law can also be written as

$$\vec{F} = m\vec{a},$$

in which $\vec{a}$ represents the acceleration of the particle:

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}.$$

It should be noted that the first law is actually a special case of the second law: if the force on the particle is zero, its acceleration is also zero.

By not including a proportionality constant in the mathematical formulation of the second law (the formulation in words only refers to proportionality between force and a change in momentum), we actually define the unit of force as the force that gives a mass of 1 kilogram an acceleration of 1 metre per second squared. This unit of force is the Newton (symbol N,

---

[1] One of the essential distinctions in mechanics is between *speed* and *velocity*: speed is a scalar and velocity is a vector. The speed v is the magnitude of the velocity $\vec{v}$: $v = |\vec{v}|$. If a particle traverses, say, a circle, with constant speed v, then its velocity $\vec{v}$ will change, because its direction is changing.

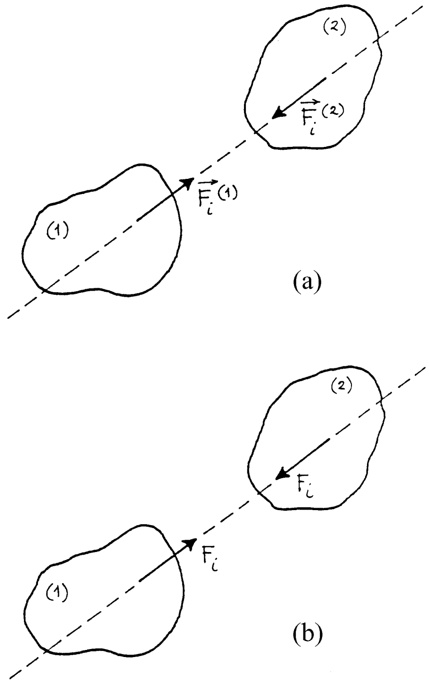


Figure 1.29 Newton's law of action and reaction in (a) vector notation ("*action* = –*reaction*") and (b) visual notation ("*action* = *reaction*").

see Section 1.2.2):

$$1\ \mathrm{N} = 1\ \mathrm{kg} \cdot \mathrm{m/s}^2.$$

Law of action and reaction

The third law is the so-called law of action and reaction. If one defines the force that body (1) exerts on body (2) by $\vec{F}_\mathrm{i}^{(2)}$ and the force which body (2) exerts on body (1) by $\vec{F}_\mathrm{i}^{(1)}$, the third law states that[1]

$$\vec{F}_\mathrm{i}^{(1)} = -\vec{F}_\mathrm{i}^{(2)}.$$

According to the law of action and reaction, forces always act in pairs of equal and opposite forces. The law of action and reaction is depicted in Figure 1.29 in both vector notation and visual notation.

In the vector notation in Figure 1.29a one would say

"*action* = −*reaction*".

In the visual notation in Figure 1.29b one would rather say

"*action* = *reaction*".

In both cases, the meaning is the same. In the visual notation it can clearly be seen that the *interaction* between both bodies occurs between the *pair of forces* F_i.

---

1 The upper index denotes the body on which the force is exerted, the lower index is the i of *interaction*.

1.4.2 Law of gravitation

Newton also formulated the law that describes the attraction between two bodies. This *Law of Universal Gravitation* states that the force between two particles with masses m_1 and m_2 at a distance r apart is an attraction that operates along the joining line of the two particles, with magnitude

$$F = G\frac{m_1 m_2}{r^2}.$$

Here, G is a universal constant, which is the same for all pairs of particles. The value of G, the *gravitation constant*, has been experimentally determined as

$$G = 66.71 \times 10^{-12}\ \mathrm{Nm^2/kg^2}.$$

In general, all attractive forces on earth between bodies are dominated by the attractive force of the Earth on those bodies, as the mass of the Earth is so much greater (5.975×10^{24} kg) than that of any other body.

On the basis of Newton's second law and the law of gravitation, it follows that in the event of a free fall near the surface of the earth, all masses (in the absence of friction) are subject to the same acceleration (denoted by g, the *gravitational acceleration*).

Assuming that one can imagine the mass of the earth as concentrated in its centre, this gives

$$g = \frac{GM}{R^2},$$

whereby M is the mass of the earth and R is the distance from the particle to the centre.

Since the Earth is flattened at the poles, the exact value of g depends on the location on earth. At the equator, g is approximately 9.790 m/s^2, at the poles it is approximately 9.832 m/s^2, while in the Netherlands (52° lattitude) it is 9.813 m/s^2.

For simplicity, in building practice, we assume

$$g = 10 \text{ m/s}^2.$$

The 2% error is minor if one considers all the uncertainties in, for example, the magnitudes and points of application of the loads, the dimensions of the structural elements, and the properties of the materials.

Since 1 N = 1 kg·m/s^2, we can also say:

$$g = 10 \text{ N/kg}.$$

In the gravitational field, a mass of 1 kg therefore weighs 10 N. The gravitational acceleration g is also known as the *gravitational field strength*.

2 Statics of a Particle

If several forces are exerted on a particle, they can be compounded as described in Sections 1.3.4 and 1.3.5 for vectors. The sum of all the forces is called the resultant force, or *resultant*. Since all the forces have the same point of application, namely the particle, the resultant also acts on that point. Section 2.1 addresses compounding and resolving forces on a particle in a plane. Section 2.2 looks at compounding and resolving forces on a particle in space. In Section 2.3 we show how to derive the equilibrium equation from the motion equation for a particle at rest.

2.1 Coplanar forces

We will first address compounding forces. Compounding is possible both analytically (Section 2.1.1) and graphically (Section 2.1.2). We will then show how to resolve a force into components with given directions. Here we can choose between an analytical approach (Section 2.1.3) and a graphical approach (Section 2.1.4).

2.1.1 Compounding forces analytically

We can compound forces analytically by adding together the respective components in each of the coordinate directions, see Section 1.3.5. This

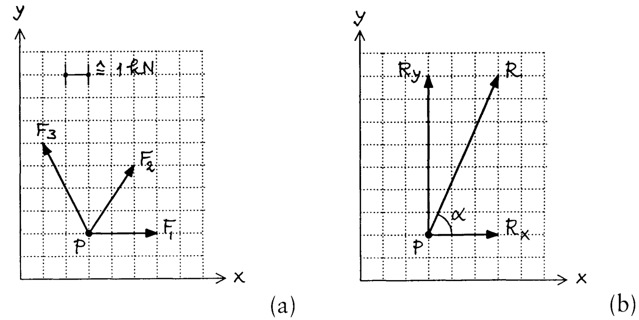


Figure 2.1 (a) Particle loaded by three forces and (b) the resultant R of the three forces.

is illustrated on the basis of an example.

Example

In Figure 2.1a, the three forces F_1, F_2 and F_3 located in the xy plane act on the particle P. Their magnitudes and directions can be derived using the squares from the figure.

Note: As you can see from the figure, here we use the *visual notation* for force vectors (see Section 1.3.6).

Question:
What is the magnitude and direction of the resultant R?

Solution: In the coordinate system shown, the components of the resultant R of the three forces are

$$R_x = F_{x;1} + F_{x;2} + F_{x;3} = (3\text{ kN}) + (2\text{ kN}) + (-2\text{ kN}) = 3\text{ kN},$$

$$R_y = F_{y;1} + F_{y;2} + F_{y;3} = (0\text{ kN}) + (3\text{ kN}) + (4\text{ kN}) = 7\text{ kN}.$$

Be careful with respect to the signs of the x and y components: they are related to the (not shown) unit vectors in the coordinate system!

The *magnitude* of the resultant is

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(3\text{ kN})^2 + (7\text{ kN})^2} = \sqrt{58}\text{ kN}.$$

From the angle α between the line of action of R and the x axis, it follows that

$$\tan\alpha = \frac{R_y}{R_x} = \frac{7\text{ kN}}{3\text{ kN}} = 2.33 \Rightarrow \alpha = 66.8^\circ + k \times 180^\circ.$$

The *direction* of R is determined by

$$\sin\alpha = \frac{R_y}{R} = 0.919 \quad \text{and} \quad \cos\alpha = \frac{R_x}{R} = 0.394 \Rightarrow \alpha = 66.8°.$$

The resultant R and its components are shown in Figure 2.1b.

2.1.2 Compounding forces graphically; force polygon

If all the forces are in the same plane, the vector addition can easily be performed graphically by repeatedly implementing the *parallelogram rule* (see Section 1.3.4) or by drawing a so-called *force polygon*. Two examples are given below.

Example 1

Figure 2.2 shows the result of the graphical approach for the example in the previous section. First, we determine the resultant of F_1 and F_2, after which we compound it with F_3. Using the square canvas the resultant R is

$$R = \sqrt{58}\ \text{kN}.$$

When compounding forces graphically, it is not necessary to draw all the parallelograms fully. A drawing in which all the forces with magnitudes and directions are drawn behind one another suffices, as shown in Figure 2.3. This type of drawing is called a *force polygon*.

If, as in the example, the starting point A (the tail of the arrow for the first force F_1) does *not* coincide with the end point B (the point of the arrow for the last force F_2), one refers to an *open force polygon*.

The starting point and the end point of the force polygon determine the direction and magnitude of the resultant R for all forces: the arrow for R runs from starting point A to end point B (see Figure 2.4). The magnitude and direction of R can be *measured* or *calculated* using the drawing. If you look closely, you will recognise the force polygon in Figure 2.2.

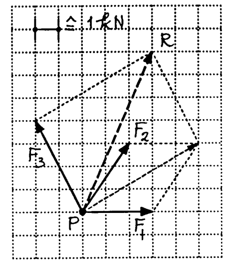


Figure 2.2 Graphical representation of the vector addition by repeatedly applying the parallelogram rule.

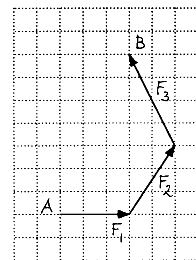


Figure 2.3 By placing the forces F_1 to F_3 head-tail behind each other you get an open force polygon.

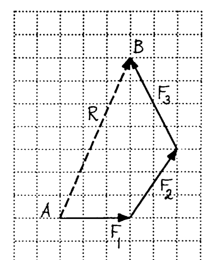


Figure 2.4 Graphical representation of the resultant using a force polygon.

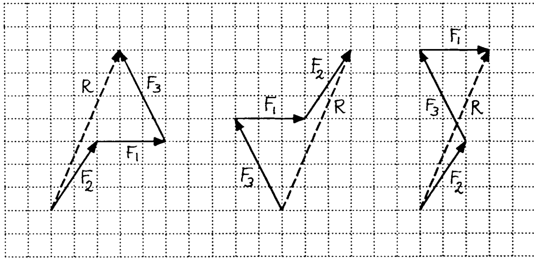


Figure 2.5 The end result is not influenced by the order in which the forces in a force polygon are exerted.

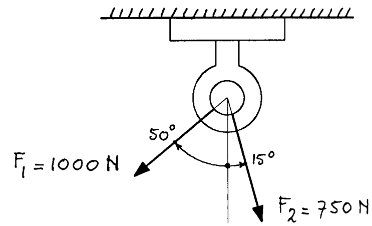


Figure 2.6 A ring subject to two forces.

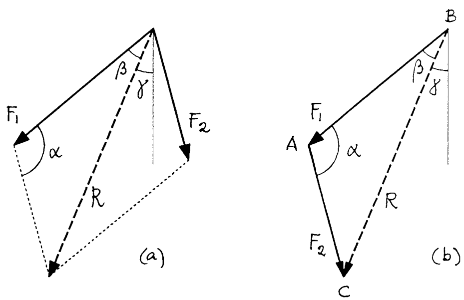


Figure 2.7 The forces drawn to scale in (a) a parallelogram and (b) force polygon.

Figure 2.5 shows all the forces in changing order in a force polygon. The order clearly does not influence the end result (the vector addition is *associative* and *commutative*, see Section 1.3.7).

Example 2

Two forces F_1 and F_2 are acting on the ring in Figure 2.6. Their directions are shown in the figure. The forces are not shown to scale.

Question:

Find the magnitude and direction of the resultant force on the ring if

$$F_1 = 1000 \text{ N},$$

$$F_2 = 750 \text{ N}.$$

Solution:

In Figure 2.7, the forces have been drawn to scale, with 1 cm $\hat{=}$ 250 N. Using the parallelogram rule in Figure 2.7a or the force polygon in Figure 2.7b, we can *construct* the resultant R. Through *measuring* we find that R has a length of approximately 5.95 cm, so that

$$R \approx 5.95 \times 250 \text{ N} = 1488 \text{ N}.$$

With a protractor, we find that the line of action of R makes an angle γ of approximately 22.5° with the vertical.

Check:

The magnitude and direction of the resultant R can also be *calculated* from the force triangle ABC. In doing so, we use the *cosine rule* and the *sine rule*, as shown in Figure 2.8.

In the triangle ABC in Figure 2.7b

$$\alpha = 180° - (50° + 15°) = 115°.$$

Using the cosine rule we find that

$$\begin{aligned} R &= \sqrt{F_1^2 + F_2^2 - 2F_1F_2\cos\alpha} \\ &= \sqrt{(1000\ \mathrm{N})^2 + (750\ \mathrm{N})^2 - 2\times(1000\ \mathrm{N})(750\ \mathrm{N})\cos 115^\circ} \\ &= \sqrt{2.196\times 10^6\ \mathrm{N}^2} = 1482\ \mathrm{N}. \end{aligned}$$

The angle β in triangle ABC can be calculated using the sine rule:

$$\frac{R}{\sin\alpha} = \frac{F_2}{\sin\beta},$$

$$\sin\beta = \frac{F_2}{R}\sin\alpha = \frac{750\ \mathrm{kN}}{1482\ \mathrm{kN}}\sin 115^\circ = 0.459$$

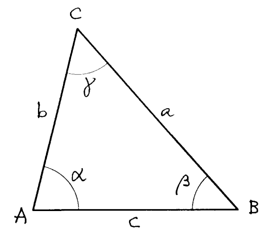
so that

$$\beta = 27.3^\circ.$$

The angle γ that the resultant R makes with the vertical is therefore

$$\gamma = 50^\circ - \beta = 22.7^\circ.$$

The values measured in Figure 2.7 correspond well with the values we have calculated.



cosine rule: $c = \sqrt{a^2 + b^2 - 2ab\cos\gamma}$

sine rule: $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

Figure 2.8 The cosine and sine rules.

2.1.3 Resolving a force in two given directions analytically

In a plane, we can resolve a force F into two components with given lines of action.

Example
The force $F = \sqrt{34}$ kN in Figure 2.9a has to be resolved into two forces F_a and F_b with the given lines of action a and b.

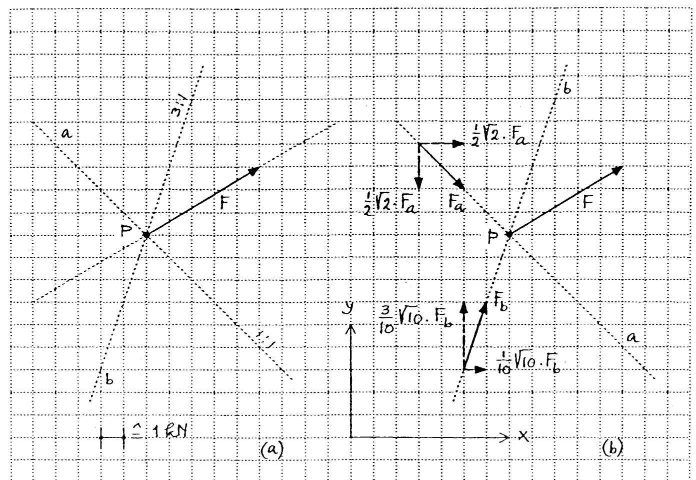


Figure 2.9 The force F in P has to be resolved into two components with given lines of action a and b.

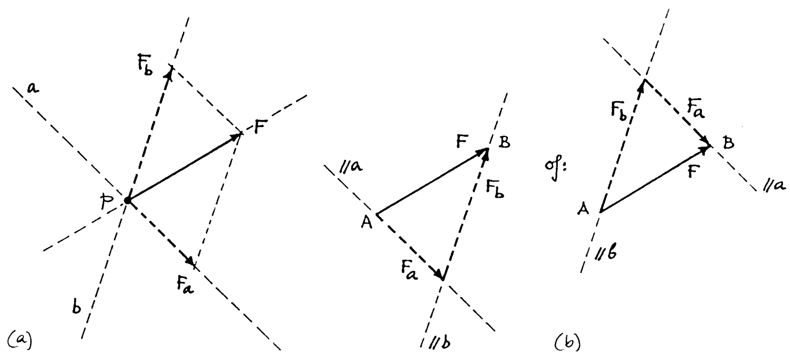


Figure 2.10 Resolving a force in two given directions graphically using (a) the parallelogram rule and (b) a force polygon.

Solution:
Figure 2.9b shows the forces F_a and F_b on the lines of action a and b. Along a and b the directions of the forces can be chosen freely. In the analytical approach, we calculate F_a and F_b on the basis of the condition that the sum of the components from F_a and F_b is equal to the corresponding component of F in each of the coordinate directions

$$\begin{aligned} F_{x;a} + F_{x;b} &= F_x, \\ F_{y;a} + F_{y;b} &= F_y. \end{aligned} \tag{a}$$

For the components of F in the coordinate system shown in Figure 2.9b

$$F_x = 5 \text{ kN}; \quad F_y = 3 \text{ kN}$$

and for the components of F_a and F_b respectively:

$$\begin{aligned} F_{x;a} &= \tfrac{1}{2}\sqrt{2} \times F_a; & F_{y;a} &= -\tfrac{1}{2}\sqrt{2} \times F_a, \\ F_{x;b} &= \tfrac{1}{10}\sqrt{10} \times F_b; & F_{y;b} &= +\tfrac{3}{10}\sqrt{10} \times F_b. \end{aligned}$$

Substitution in (a) gives two equations with F_a and F_b as unknowns:

$$\begin{aligned} \tfrac{1}{2}\sqrt{2} \times F_a + \tfrac{1}{10}\sqrt{10} \times F_b &= 5 \text{ kN}, \\ -\tfrac{1}{2}\sqrt{2} \times F_a + \tfrac{3}{10}\sqrt{10} \times F_b &= 3 \text{ kN}. \end{aligned}$$

The solution is

$$\begin{aligned} F_a &= 3\sqrt{2} \text{ kN}, \\ F_b &= 2\sqrt{10} \text{ kN}. \end{aligned}$$

The fact that F_a and F_b are both positive means that they act in the directions we chose in Figure 2.9b. If we had chosen the directions of F_a and F_b in the opposite sense, we would have found that F_a and F_b were negative.

2.1.4 Resolving a force in two given directions graphically

A force F can be resolved graphically into two components F_a and F_b, with given lines of action a en b using the *parallelogram rule* in Figure 2.10a or the *force polygon* in Figure 2.10b. The graphical approach has the advantage that you can at once see in which directions the components F_a and F_b are working.

Example
A force $F = 30$ kN acts on the trestle in Figure 2.11a. This has to be resolved into the components F_a and F_b with lines of action a and b.

Solution:
In Figure 2.12, the force has been resolved using a force polygon. The force scale is 1 cm $\hat{=}$ 5 kN. By *measuring*, you find (in the force polygon) F_a and F_b have lengths 5.7 cm and 6.3 cm respectively, so that

$$F_a \approx 5.7 \times 5 \text{ kN} = 28.5 \text{ kN},$$

$$F_b \approx 6.3 \times 5 \text{ kN} = 31.5 \text{ kN}.$$

In Figure 2.11b forces F_a and F_b are shown as they act on the trestle. It is clear that force F_a is pulling on the beam while force F_b is pushing against it. Together, they exert the same load as the single force F. The forces F_a and F_b are *statically equivalent* (equal in an equilibrium consideration) to their resultant F.

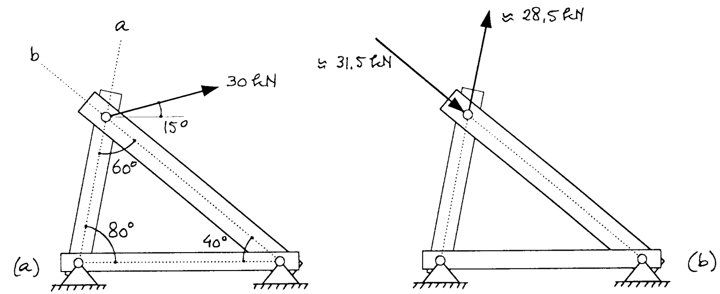


Figure 2.11 A trestle is loaded by a (tensile) force of 30 kN that has to be resolved into components along the lines a and b.

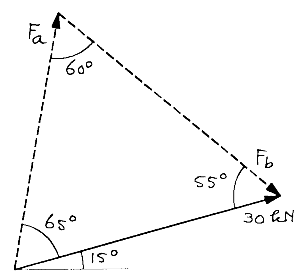


Figure 2.12 The forces drawn to scale in a force polygon.

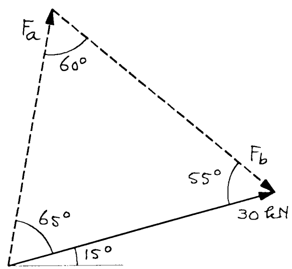


Figure 2.12 The forces drawn to scale in a force polygon.

Check: The magnitude of F_a and F_b can also be *calculated* from the force triangle by using the *sine rule* (see Figure 2.12):

$$\frac{F_a}{\sin 55^\circ} = \frac{F_b}{\sin 65^\circ} = \frac{F}{\sin 60^\circ}$$

so that:

$$F_a = F \cdot \frac{\sin 55^\circ}{\sin 60^\circ} = 28.4 \text{ kN},$$

$$F_b = F \cdot \frac{\sin 65^\circ}{\sin 60^\circ} = 31.4 \text{ kN}.$$

The values *measured* in Figure 2.12 correspond closely to the *calculated* values.

2.2 Forces in space

If not all the forces are in the same plane, the analytical approach is generally simpler than the graphical approach. For example, for the components R_x, R_y and R_z of the resultant $\vec{R}$:

$$R_x = \sum F_x,$$

$$R_y = \sum F_y,$$

$$R_z = \sum F_z.$$

The summation symbol means that all the forces exerted on the particle have to be added together.

For the magnitude R of $\vec{R}$ therefore

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}.$$

The direction of $\vec{R}$ is determined by the angles α_x, α_y and α_z which $\vec{R}$ makes with respectively the x, y and z axis, see Figure 2.13:

$$\cos\alpha_x = \frac{R_x}{R}; \quad \cos\alpha_y = \frac{R_y}{R}; \quad \cos\alpha_z = \frac{R_z}{R}.$$

The quantities $\cos\alpha_x$, $\cos\alpha_y$ and $\cos\alpha_z$ are called the *direction cosines*. Regardless of the direction of $\vec{R}$, the angles α_x, α_y and α_z are always between 0° and 180°.

When defining a vector, such as the force $\vec{R}$, we need three numbers (and a unit). The three numbers could be the values of the three components R_x, R_y and R_z, or, for example, the magnitude of $\vec{R}$ and two of the three direction cosines. In the latter case, the third direction cosine is given by the other two as

$$\cos^2\alpha_x + \cos^2\alpha_y + \cos^2\alpha_z = 1.$$

Working with forces in space is illustrated using two examples. The first example relates to resolving a force into its components (Section 2.2.1). The second example relates to compounding forces (Section 2.2.2).

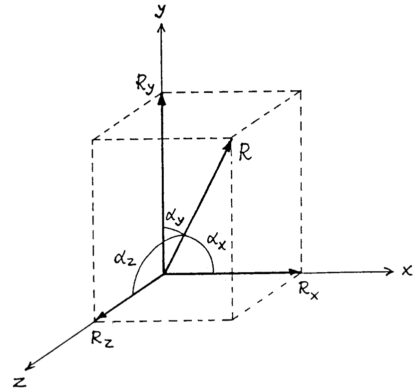


Figure 2.13 The resultant R, with its components R_x, R_y, R_z and the angles α_x, α_y, α_z that R makes with the coordinate axes.

2.2.1 Resolving a force into its components

In order to be able to resolve a force into its x, y and z component, we first have to calculate the direction cosines. This is illustrated by means of an example.

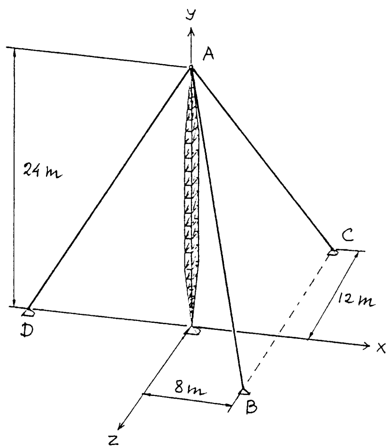


Figure 2.14 A secured mast.

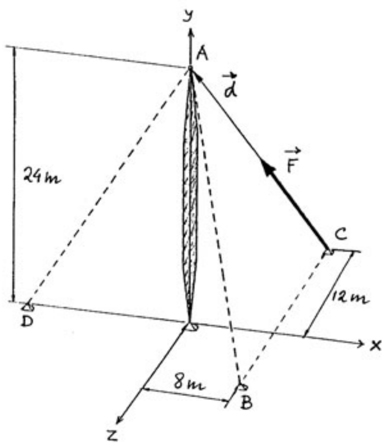


Figure 2.15 The (tensile) force $\vec{F}$ that the rope AC exerts on the foundation block in C.

Example

In Figure 2.14, a mast is being kept upright by three ropes. Rope AC is subject to a tensile force of 35 kN.

Question:

Find the x, y and z component of force $\vec{F}$ that the rope exerts on the foundation block in C (see Figure 2.15).

Note that here we use the *formal vector notation*.

Solution:

The force $\vec{F}$ that is working on the foundation block has the same direction as the vector **CA** (directed from C to A). This vector, which indicates the direction of $\vec{F}$ is hereby referred to as $\vec{d}$,[1] see Figure 2.15. $\vec{F}$ and $\vec{d}$ have the same direction cosines, so that

$$\cos\alpha_x = \frac{F_x}{F} = \frac{d_x}{d},$$

$$\cos\alpha_y = \frac{F_y}{F} = \frac{d_y}{d},$$

$$\cos\alpha_z = \frac{F_z}{F} = \frac{d_z}{d}.$$

Figure 2.15 shows that

$$d_x = -8 \text{ m},$$

$$d_y = +24 \text{ m},$$

$$d_z = +12 \text{ m}.$$

The x component of $\vec{d}$ is negative as it is pointing in the negative x

---

1 Remember the *d* of *direction*.

direction. The magnitude (length) d of $\vec{d}$ is

$$d = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{(-8\text{ m})^2 + (24\text{ m})^2 + (12\text{ m})^2} = 28\text{ m}.$$

Using this information, we can calculate the components of $\vec{F}$:

$$F_x = F\frac{d_x}{d} = (35\text{ kN}) \times \frac{-8\text{ m}}{28\text{ m}} = -10\text{ kN},$$

$$F_y = F\frac{d_y}{d} = (35\text{ kN}) \times \frac{24\text{ m}}{28\text{ m}} = +30\text{ kN},$$

$$F_z = F\frac{d_z}{d} = (35\text{ kN}) \times \frac{12\text{ m}}{28\text{ m}} = +15\text{ kN}.$$

Figure 2.16 shows the components of $\vec{F}$ as they are working on the foundation block.

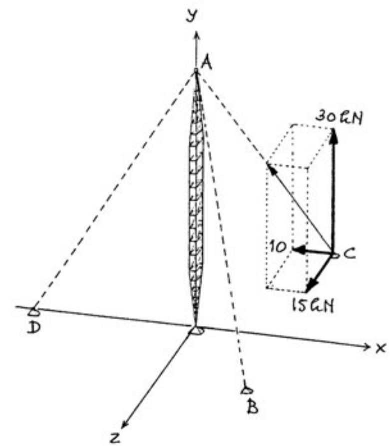


Figure 2.16 The components of $\vec{F}$.

2.2.2 Compounding forces

In order to determine the resultant of the forces on a particle in space, we first resolve all the forces into their x, y and z component, and then add all the associated components together. This is illustrated in an example.

Example

Figure 2.17 shows the schematised situation in a salvage operation. A shows the wreckage of a crashed lorry on a slope. People are trying to salvage the wreckage using cables AB and AC and winches in B and C. Cable AB is pulling on the wreckage with a force of magnitude $F_1 = 7.5$ kN; cable AC is pulling on the wreckage with force of magnitude $F_2 = 10$ kN.

Question:
Find the resultant force being exerted by the cables on the wreckage.

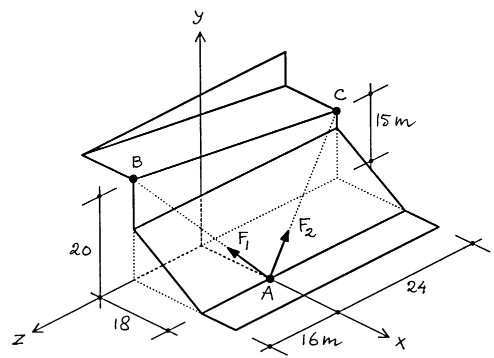


Figure 2.17 Schematisation of the situation surrounding a salvage operation. The wreckage of a crashed lorry is located on a slope at A, and is being salvaged with the cables AB and AC and winches in B and C.

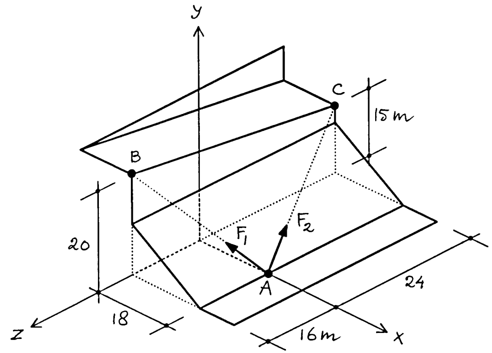


Figure 2.17 Schematisation of the situation surrounding a salvage operation. The wreckage of a crashed lorry is located on a slope at A, and is being salvaged with the cables AB and AC and winches in B and C.

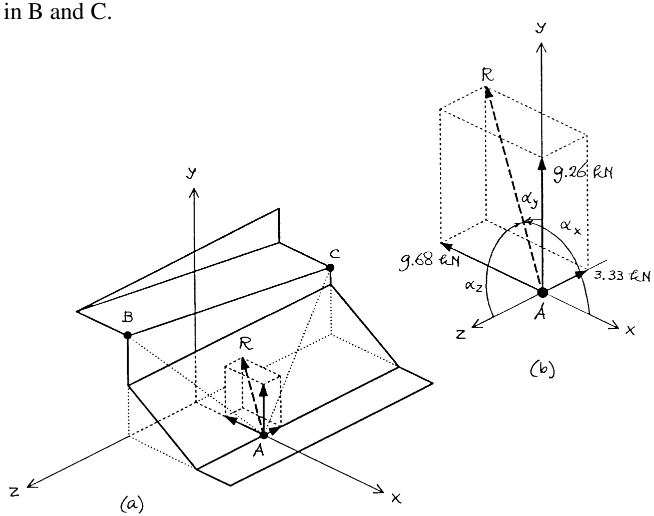


Figure 2.18 (a) The components of the resultant $\vec{R}$ that the cables exert on the wreckage and (b) the angles α_x, α_y, α_z that this force makes with the coordinate axes.

Table 2.1 Calculation of the components of the resultant $\vec{R}$.

	d_x	d_y	d_z	d	F	F_x	F_y	F_z
$\vec{d}_1 = \mathbf{AB}, \vec{F}_1$	–18	+20	+16	31.30	7.5	–4.31	+4.79	+3.83
$\vec{d}_2 = \mathbf{AC}, \vec{F}_2$	–18	+15	–24	33.54	10	–5.37	+4.47	–7.16
					$\sum =$	–9.68	+9.26	–3.33

Solution:
First, we have to resolve the forces $\vec{F}_1$ and $\vec{F}_2$ into their components in the same way as described in the previous section: $F_x = F d_x/d$, etc. The calculation is shown in Table 2.1 (d in m and F in kN).

The components of the resultant force $\vec{R}$ on the wreckage are therefore

$$R_x = \sum F_x = -9.68 \text{ kN},$$
$$R_y = \sum F_y = +9.26 \text{ kN},$$
$$R_z = \sum F_z = -3.33 \text{ kN},$$

so that

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$
$$= \sqrt{(-9.68 \text{ kN})^2 + (9.26 \text{ kN})^2 + (-3.33 \text{ kN})^2} = 13.80 \text{ kN}.$$

Figure 2.18a shows the components of the resultant $\vec{R}$ as they act on the wreckage. Figure 2.18b shows the angles that the resultant makes with the coordinate axes. The angles are calculated as follows:

$$\cos\alpha_x = \frac{R_x}{R} = \frac{-9.68 \text{ kN}}{13.80 \text{ kN}} = -0.701 \Rightarrow \alpha_x = 134.5°,$$

$$\cos\alpha_y = \frac{R_y}{R} = \frac{+9.26\text{ kN}}{13.80\text{ kN}} = +0.671 \Rightarrow \alpha_y = 47.8°,$$

$$\cos\alpha_z = \frac{R_z}{R} = \frac{-3.33\text{ kN}}{13.80\text{ kN}} = -0.241 \Rightarrow \alpha_z = 104.0°.$$

2.3 Equilibrium of a particle

According to Newton's first law, if the resultant of all the forces on a particle is zero, it will remain at rest if it was at rest originally. This means that the particle is in *equilibrium*.

If the particle is to be in equilibrium, then

$$\sum \vec{F} = \vec{0}.$$

This (vector) equation is called the *equilibrium condition for the particle*. The summation symbol means that all the forces acting on the particle have to be added together.

2.3.1 Graphical: closed force polygon

In a force polygon, the equilibrium condition means that all the forces acting on the particle have to form a *closed force polygon*: the resulting force is then zero.

An example of this is given in Figure 2.19. The four coplanar forces acting on the particle form a closed force polygon; the particle is therefore in equilibrium.

In Figure 2.20a, the particle is subject to the three coplanar forces F_1, F_2 and F_3. Together they form an open force polygon. The particle is not

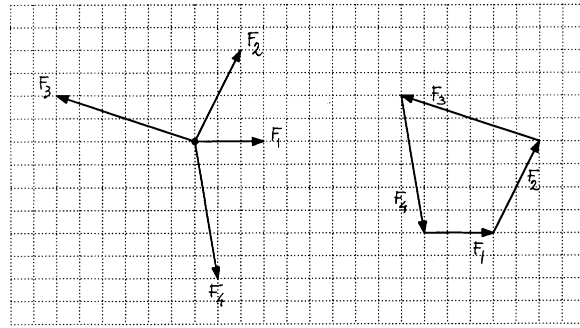


Figure 2.19 If the forces on a particle form a closed force polygon then the particle is in equilibrium.

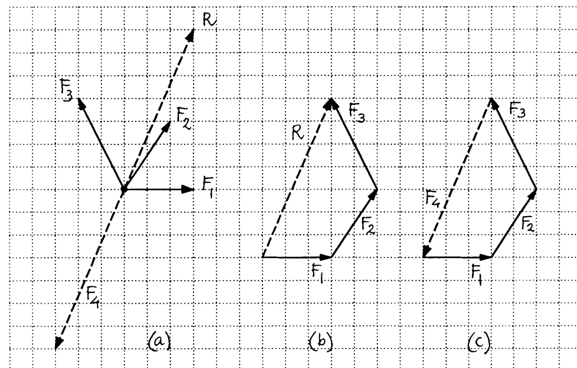


Figure 2.20 The force F_4 that closes the force polygon from F_1 to F_3 – and ensures equilibrium – is equal and opposite to the resultant R from F_1 to F_3.

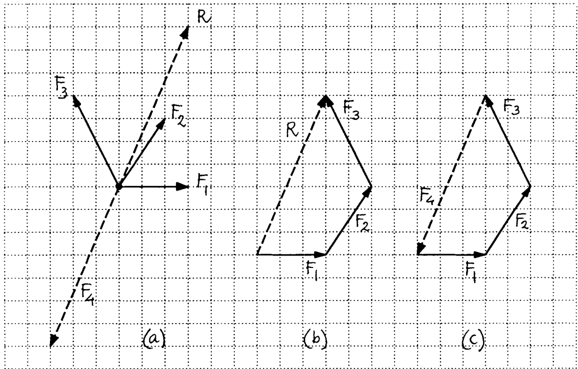


Figure 2.20 The force F_4 that closes the force polygon from F_1 to F_3 – and ensures equilibrium – is equal and opposite to the resultant R from F_1 to F_3.

in equilibrium. Due to the resultant force R (Figure 2.20b), a change in momentum occurs (Newton's second law): the resultant R on the particle will cause its velocity to change.

The force F_4 that closes the open force polygon (Figure 2.20c) is the force that brings the given forces into equilibrium. F_4 has the same magnitude, but opposite direction to the resultant R of the forces F_1, F_2 and F_3.

2.3.2 Analytical: equilibrium equations

A particle is and stays at rest if the resultant of all the forces acting on the particle is zero.

The *vector equation* for the force equilibrium

$$\sum \vec{F} = \vec{0}$$

resolves into three *algebraic equations* in space:

$$\sum F_x = 0,$$
$$\sum F_y = 0,$$
$$\sum F_z = 0.$$

These are referred to as the three equations for the *force equilibrium* of the particle in the x, y, and z direction respectively. If a particle is to be in equilibrium, each of the three components of the resultant must be zero.

If all the forces are coplanar, the number of equations for the force equilibrium is reduced to two. This is illustrated by two examples.

Example 1
Investigate whether the particle P in Figure 2.21, subject to the forces F_1 to F_4 in the xy plane is in equilibrium.

Solution (forces in kN):

$$R_x = \sum F_x = F_{x;1} + F_{x;2} + F_{x;3} + F_{x;4} = +6 - 2 - 1 - 3 = 0,$$

$$R_y = \sum F_y = F_{y;1} + F_{y;2} + F_{y;3} + F_{y;4} = -1 - 4 + 6 - 1 = 0.$$

The particle is in equilibrium since the resultant is zero: the forces F_1 to F_4 therefore together form an *equilibrium system*.[1]

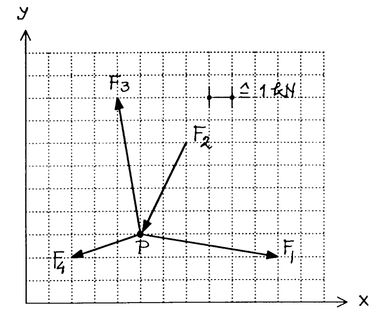


Figure 2.21 Is particle P in equilibrium?

Example 2
In Figure 2.22, a container with mass 880 kg is being unloaded. Forces F_1, F_2 and F_3, act on joint A. Here, F_3 stands for the weight of the container. In the figure, the system is in equilibrium. The gravitational field strength is $g = 10$ N/kg.

Question:
How large are F_1 and F_2?

Solution:
The weight of the container is

$$F_3 = mg = (880\ \text{kg})(10\ \text{N/kg}) = 8800\ \text{N}.$$

The unknown forces F_1 and F_2 are obtained from the two equations for the force equilibrium in the x and y directions:

$$\sum F_x = F_1 \cdot \sin 3° - F_2 \cdot \cos 20° = 0,$$

$$\sum F_y = F_1 \cdot \cos 3° - F_2 \cdot \sin 20° - (8800\ \text{N}) = 0.$$

Here we have two equations with two unknowns, namely the magnitude of

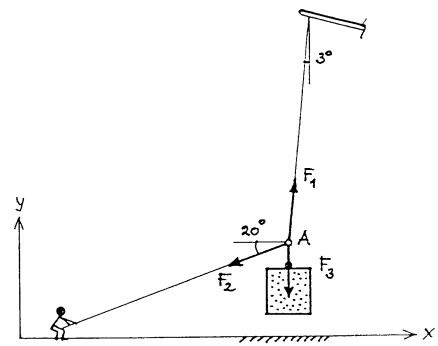


Figure 2.22 The forces acting on joint A when unloading a container.

[1] It is incorrect to say that "the forces neutralise one another" as the forces continue to exist.

the forces F_1 and F_2. We can write the equations as

$$0.0523 \times F_1 - 0.9397 \times F_2 = 0,$$

$$0.9986 \times F_1 - 0.3420 \times F_2 = 8800 \text{ N},$$

so that:

$$F_1 = 8984 \text{ N},$$

$$F_2 = 500 \text{ N}.$$

Alternative solution:
We also can calculate the forces on the basis of the closed force polygon for the equilibrium of junction A. A rough sketch of the force polygon, such as that in Figure 2.23, suffices.

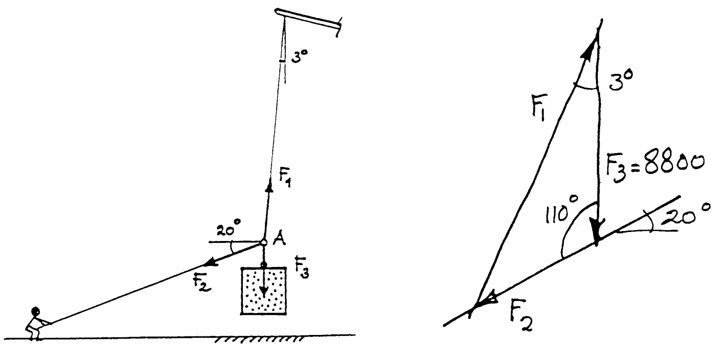


Figure 2.23 A rough sketch of the closed force polygon for the force equilibrium of joint A.

According to the sine rule this gives

$$\frac{F_1}{\sin 110^\circ} = \frac{F_2}{\sin 3^\circ} = \frac{F_3}{\sin(180^\circ - 110^\circ - 3^\circ)} = \frac{F_3}{\sin 67^\circ}.$$

With $F_3 = 8800$ N this means that

$$F_1 = F_3 \cdot \frac{\sin 110^\circ}{\sin 67^\circ} = 8983 \text{ N},$$

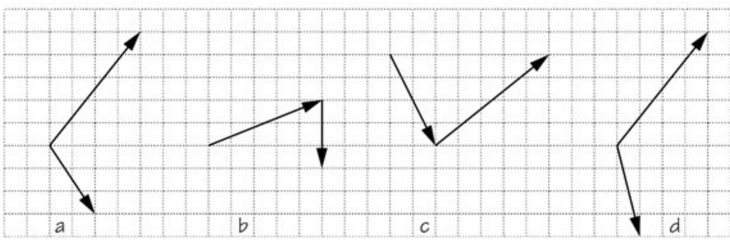
$$F_2 = F_3 \cdot \frac{\sin 3^\circ}{\sin 67^\circ} = 500 \text{ N}.$$

The fact that F_1 is 1 N less than before is the result of rounding off the goniometric function values (to four decimal places) in the previous solution.

2.4 Problems

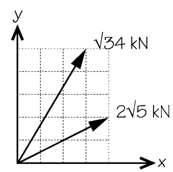
Compounding coplanar forces (Sections 2.1.1 and 2.1.1)

2.1 Which combination of forces has the smallest resultant?



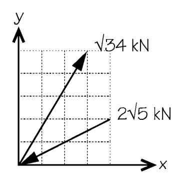
2.2 Two forces are acting on a particle, of which the magnitude and direction are shown in the figure.

Question:
Determine the magnitude and direction of the resultant for both forces:
a. graphically (choose a scale of 5 mm ≡ 1 kN);
b. analytically.



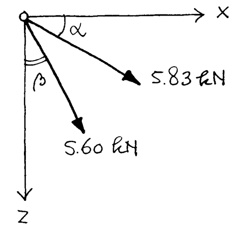
2.3 Two forces are acting on a particle, of which the magnitude and direction are shown in the figure. The resultant is R.

Question:
Calculate the components R_x and R_y:
a. analytically;
b. graphically (choose a scale of 5 mm ≡ 1 kN).



2.4 Two forces are acting on a particle. The values are included in kN. The directions are: $\tan\alpha = 3/5$ and $\tan\beta = 1/2$.

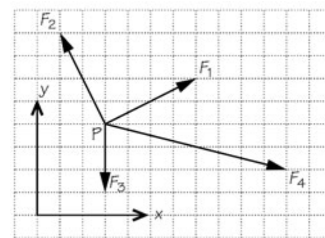
Question:
a. Calculate the resultant of the two forces graphically.
b. Check the result analytically.



2.5 Four forces are acting on particle P. The directions of the forces are shown in the figure. The forces are not drawn to scale.

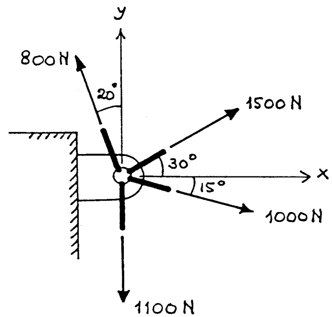
$$F_1 = 15\sqrt{5}\text{ kN},\ F_2 = 10\sqrt{5}\text{ kN},\ F_3 = 30\text{ kN and } F_4 = 5\sqrt{17}\text{ kN}.$$

Question:
Determine the magnitude and direction of the resultant of these four forces:
a. graphically (choose a scale of 1 mm ≡ 1 kN);
b. analytically.



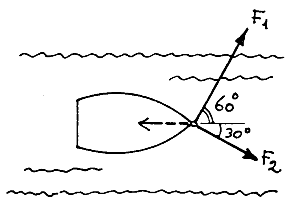
2.6 A number of coplanar cables are attached to a console. They exert the forces as shown in the figure.

Question:
Determine the magnitude and direction of the resultant force on the console.



2.7 A boat is kept in the middle of a river by means of two cables. The direction in which the boat pulls on both cables is shown by the dashed arrow.

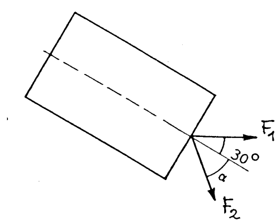
Question:
Calculate F_2 if $F_1 = 1000$ N.



2.8 A vessel is being pulled in the direction of its longitudinal axis by two tugs with a force of 20 kN. The directions of the forces F_1 and F_2 as exerted by the tow lines on the vessel are shown in the figure.

Question:

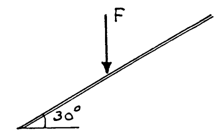
a. Determine F_1 and F_2 if $\alpha = 45°$.
b. The value of α whereby F_2 is minimal. How large in this case are F_1 and F_2?



Resolving coplanar forces (Sections 2.1.3 and 2.1.4)

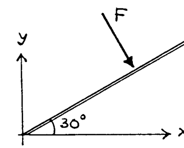
2.9 A vertical force F on a sloping roof has to be resolved into components perpendicular to and parallel to the surface of the roof.

Question:
Determine these components.



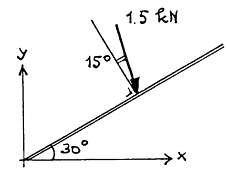
2.10 A force F perpendicular to a sloping roof has to be resolved into the components F_x and F_y, parallel to the x and y axis respectively.

Question:
Determine these components.



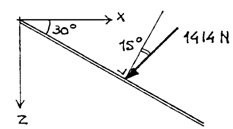
2.11 Resolve the force of 1.5 kN on the surface of the roof into:

a. components perpendicular to and parallel to the surface of the roof;
b. components in the x and y direction.



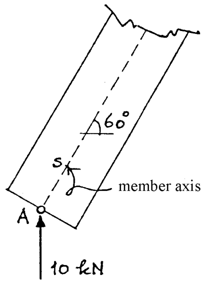
2.12 Resolve the force of 1414 N on the surface of the roof into:

a. components perpendicular to and parallel to the surface of the roof;
b. components in the x and z direction.



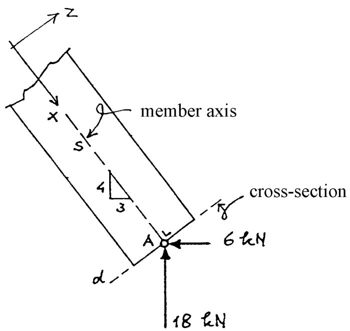
2.13 Resolve the force of 10 kN shown at A into components perpendicular to and parallel to the bar axis s:

a. graphically (mention the scale used for the forces);
b. analytically.

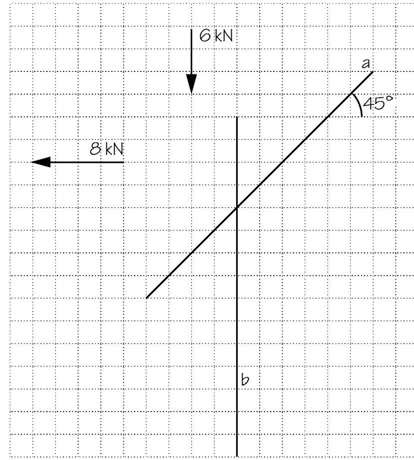


2.14 At the end A of the bar, the forces of 6 and 18 kN as shown in the diagram are exerted on the cross-section d.

Question:
Resolve the resulting force at A into:
a. components perpendicular to and parallel to cross-section d;
b. the components F_x and F_y.



2.15 The resultant R of the two forces shown is resolved into the components R_a and R_b, parallel to the directions a and b.

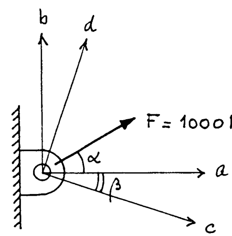


Question:
a. Draw the lines of action of R_a and R_b.
b. Determine R_a and R_b.
c. Draw R_a and R_b on their lines of action (in the directions in which they are acting) and record their values next to them.

2.16 A console is subject to a force $F = 1000$ N.

In addition: $\alpha = \beta = 30°$.

Question:
Resolve these forces into components in respectively:
a. the a and b direction;
b. the c and d direction;
c. the a and c direction;
d. the b and c direction.

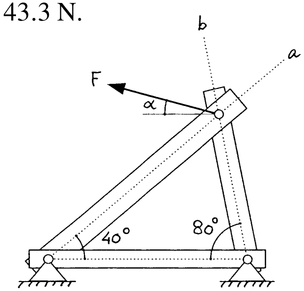


2.17 As 2.16 but now with $\alpha = 20°$ and $\beta = 30°$.

2.18 A trestle is subject to a tensile force F with an angle α.

Question:
Resolve this force into the components F_a and F_b along the lines of action a and b, for:

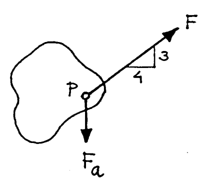
a. $\alpha = 20°$ and $F = 38.0$ N;
b. $\alpha = 35°$ and $F = 36.8$ N;
c. $\alpha = 50°$ and $F = 43.3$ N.



2.19 A force F at P is resolved into the components F_a and F_b.

Question:
Determine the magnitude and direction of F_b if $F = 5$ kN and $F_a = 1$ kN:

a. graphically (choose a scale of 5 mm ≡ 1 kN);
b. analytically.

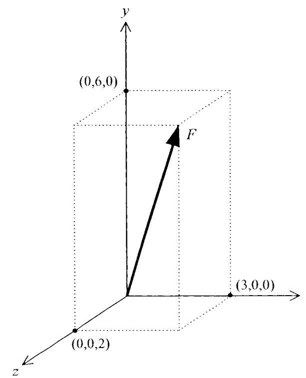


2.20 As 2.19, but now with $F = F_a = 5$ kN.

Resolving a force in space into its components (Section 2.2.1)

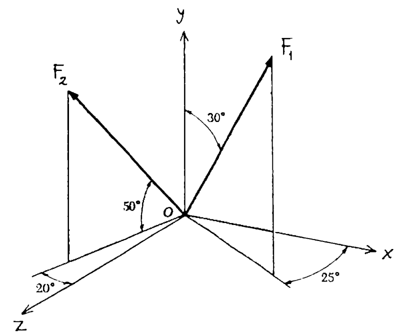
2.21 *Question*:

a. Determine the components in the x, y and z direction for force F shown when $F = 35$ kN.
b. Calculate the angles α_x, α_y and α_z that F makes with respectively the x, y and z axis.



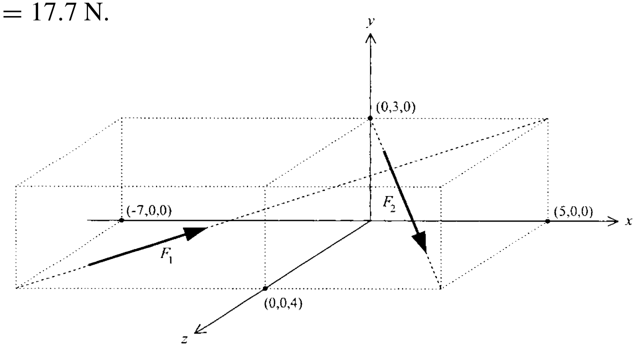
2.22 *Question*:
For the forces shown, determine the components in the x, y and z direction and the angles α_x, α_y and α_z that they make with respectively the x, y and z axis:

a. $F_1 = 1250$ N.
b. $F_2 = 1500$ N.



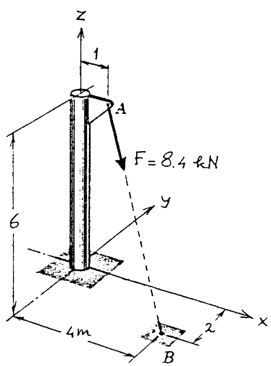
2.23 *Question*:
For the forces shown, determine the components in the x, y and z direction and the angles α_x, α_y and α_z that they make with respectively the x, y and z axis:

a. $F_1 = 6.5$ N;
b. $F_2 = 17.7$ N.



2.24 Cable AB exerts a tensile force $F = 8.4$ kN at A on the console.

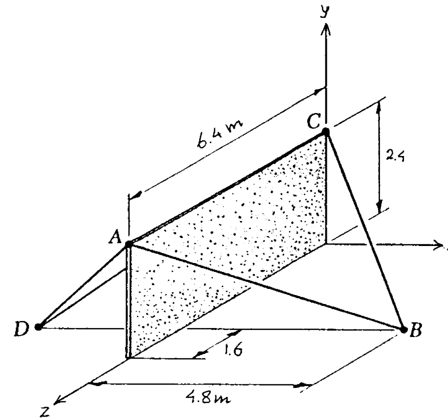
Question:
Determine the x, y and z component of the force at A.



2.25 A prefabricated concrete wall is temporarily kept in place by cables.

Question:

a. If there is a tensile force of 3.5 kN in cable AB, determine the components of the force that cable AB exerts on the wall in A.
b. If there is a tensile force of 4.5 kN in cable BC, determine the components of the force that cable BC exerts on the wall in C.

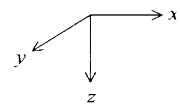


Resultant of the forces on a particle in space (Section 2.2.2)

2.26 The components of a force are

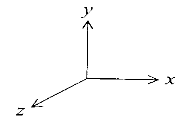
$$F_x = +7.5 \text{ kN}, \quad F_y = +17.5 \text{ kN} \quad \text{and} \quad F_z = -10 \text{ kN}.$$

Question:
Determine the magnitude and direction of this force.



2.27 The components are given in the table below for three forces F_1, F_2 and F_3.

	F_x (kN)	F_y (kN)	F_z (kN)
F_1	–60	20	100
F_2	30	50	–80
F_3	90	20	–60



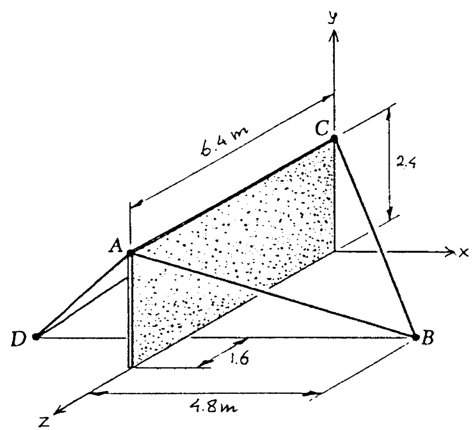
Question:
Determine the magnitude and direction of the resultant of these three forces.

2.28 Three forces F_a, F_b and F_c in the origin O of the xyz coordinate system are aimed respectively at the points A$(-1, 2, 4)$, B$(3, 0, -3)$ en C$(2, -2, 4)$.

Question:
Determine the magnitude and direction of the resultant of these three forces if $F_a = 200$ N, $F_b = 50$ N and $F_c = 150$ N.

2.29 A prefabricated concrete wall is kept in place temporarily by cables. There is a tensile force in the cables AB and BC of respectively 7.0 kN and 6.0 kN.

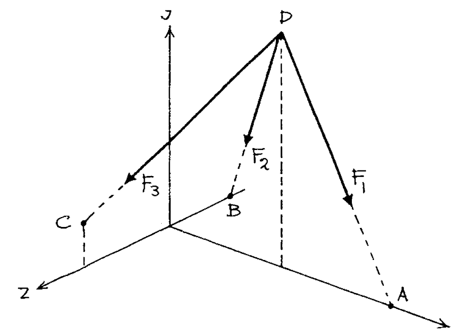
Question:
Determine the magnitude and direction of the force that the cables AB and BC exert jointly on anchor B.



2.30 Three forces are exerted on D$(5, 10, 0)$, namely $F_1 = 3$ kN, $F_2 = 4$ kN and $F_3 = 5$ kN. Coordinates in m.

F_1 is aimed at A$(10, 0, 0)$,
F_2 is aimed at B$(0, 0, -3)$ and
F_3 is aimed at C$(0, 2, 4)$.

Question:
Determine the magnitude and direction of the resultant of these three forces.



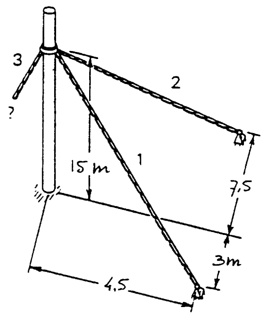
2.31 A force F is acting on the origin O of the xyz coordinate system. The force has an angle of 150° with respect to the z axis. The components in the x and y direction are respectively $F_x = 4$ kN and $F_y = 3$ kN.

Question:
a. Determine F.
b. Determine the component F_z.
c. Determine the direction cosines for F.

2.32 A narrow steel mast is supported at its top by three tight cables. There is a tensile force of 5.9 kN in cables 1 and 2. Cable 3 makes an angle of 30° with the mast. The forces that the three cables exert on the mast have a vertical resultant. Assume that all the forces are aimed at a single point.

Question:

a. Determine the magnitude of the tensile force in cable 3.
b. Determine where cable 3 is anchored at ground level.

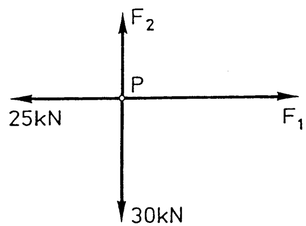


Equilibrium of a particle in a plane (Sections 2.3.1 and 2.3.2)

2.33 Particle P is subject to five forces, four of which are shown. The particle is in equilibrium. $F_1 = 40$ kN and $F_2 = 20$ kN.

Question:
The fifth force is acting:

a. top right;
b. top left;
c. bottom right;
d. bottom left.

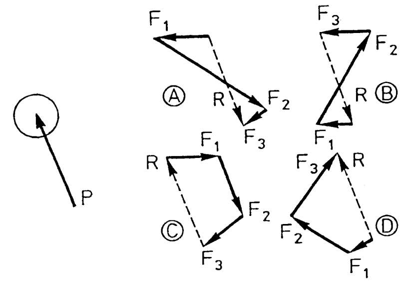


2.34 As 2.33, but now with $F_1 = F_2 = 35$ kN.

2.35 As 2.33, but now with $F_1 = 10$ kN and $F_2 = -15$ kN.

2.36 A particle is subject to three forces F_1, F_2 and F_3 with resultant R. The body is kept in equilibrium by an additional force P.

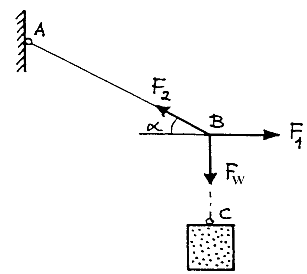
Question:
Which of the force polygons for F_1, F_2 and F_3 and R is correct?



2.37 Cable ABC is carrying a block in C with mass $m = 50$ kg. At point B of the cable the forces F_1, F_2 and F_w are exerted. F_w represents to the weight of the block. The system is in equilibrium.

Question:

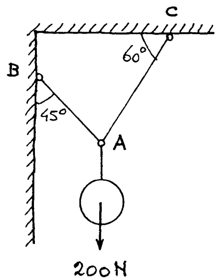
a. Determine F_1 if $\tan\alpha = 0.5$.
b. Determine F_1 as a function of α; represent this in a graph.



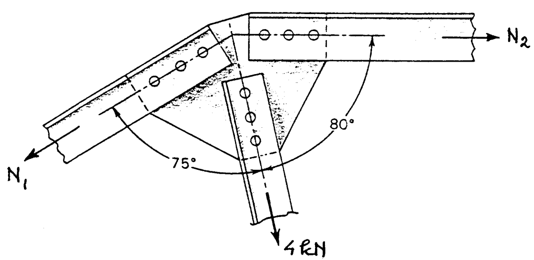
2.38 A lamp with mass 200 N is hung from two wires.

Question:

a. Determine and draw all the forces exerted on joint A.
b. Draw the force polygon for the equilibrium in joint A.



2.39 Three forces are exerted on the joint of a truss. The system is in equilibrium.



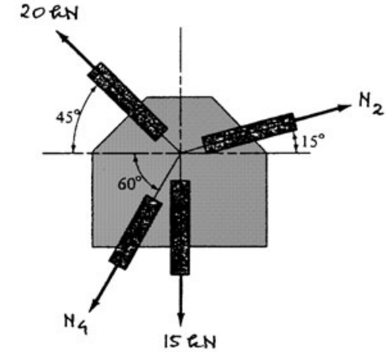
Question:

a. Draw the force polygon for the equilibrium in the joint.
b. Determine the forces N_1 and N_2. Are they tensile or compressive forces?

2.40 Four forces act on the joint of a truss. The system is in equilibrium.

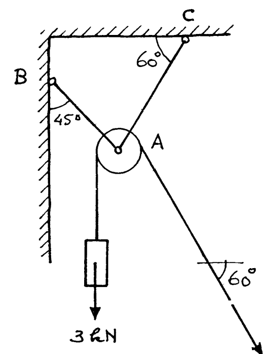
Question:

a. Draw the force polygon for the equilibrium of the joint.
b. Determine forces N_2 and N_4. Are they tensile or compressive forces?



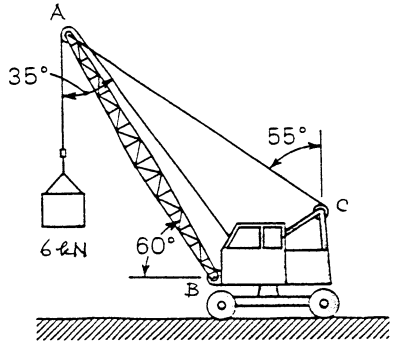
2.41 A pulley is hanging on two bars. A weight of 3 kN is hanging on a cable that is fed over the pulley. The tensile force in the cable is equal on both sides of the pulley (this can be shown with knowledge from Chapter 3).

Question:
From the force equilibrium of the pulley, determine the forces that the bars exert on the pulley in the situation shown.



2.42 Top A of the crane is schematised as a particle. The tensile force in the cable is equal on both sides of the pulley (this can be shown with knowledge from Chapter 3). The force that jib AB exerts on particle A has a line of action along AB.

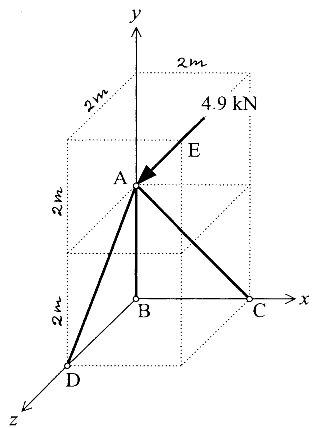
Question:
Draw the particle and determine and draw all the forces that are acting on it. Also draw the force polygon for the equilibrium of the particle.



Equilibrium of a particle in space (Section 2.3.2)

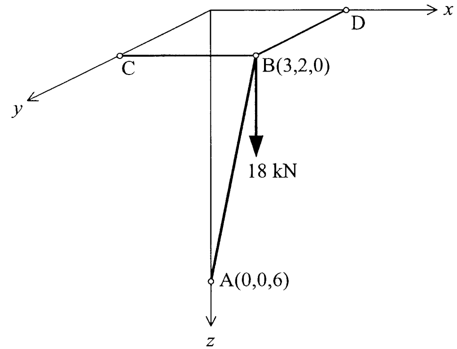
2.43 Three bars joined at A can transfer forces only in the direction of their axes. Joint A is loaded by the force shown of 4.9 kN, with its line of action through E.

Question:
Determine the forces that the bars exert on joint A. Are the forces in the members tensile or compressive forces?



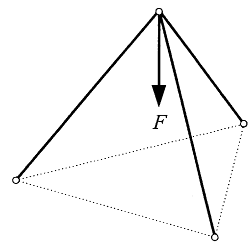
2.44 A derrick AB is mounted as a hinge in A(0, 0, 6) and is supported in B(3, 2, 0) by two horizontal wires BC and BD, parallel to the x and y axis respectively. The vertical loading in B is 18 kN. Coordinates in m.

Question:
Determine the forces exerted by AB, BC and BD on joint B if their lines of action are along the axes shown.



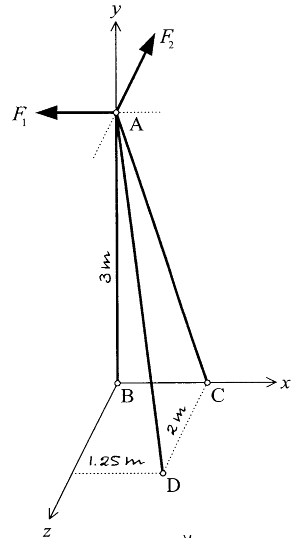
2.45 The hoisting device consists of three 4.5-metre poles. The three feet are in an equilateral triangle with sides of 3.9 metres. The device bears a vertical load of $F = 13$ kN. The poles can transfer forces only in the direction of their axes.

Question:
Determine the compressive forces in the poles.



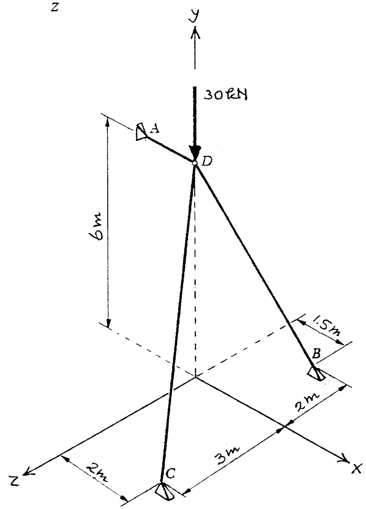
2.46 Three bars linked at A can transfer forces only in the direction of their axes. Joint A is loaded by the two forces shown $F_1 = 3.75$ kN and $F_2 = 5.25$ kN, parallel to the x and z axis respectively.

Question:
Determine the forces that the bars exert on joint A. Are the forces in the members tensile or compressive?



2.47 Three bars linked at D can transfer forces only in the direction of their axes. Joint D is loaded by a vertical force of 30 kN.

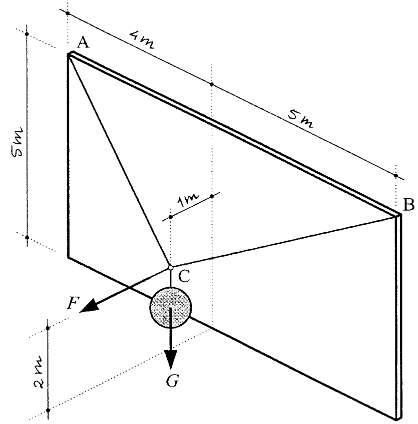
Question:
Determine the forces that the bars exert on joint D. Are they tensile or compressive?



2.48 A load of weight $G = 9$ kN is hanging at C on the cables AC and BC. The cables are joined to the corners of a vertical wall. A horizontal force F, perpendicular to the wall keeps the block in the position shown.

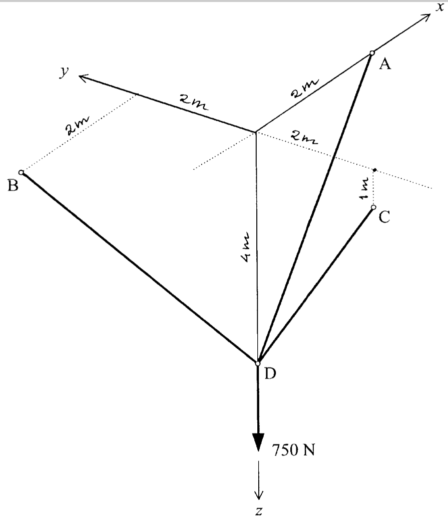
Question:

a. Determine the magnitude of force F on the basis of the equilibrium in joint C.
b. Determine the forces in the cables.



2.49 At D, a weight of 750 N is hanging from three wires. The points of attachment A and B are in the horizontal xy plane; point of attachment C is 1 metre below.

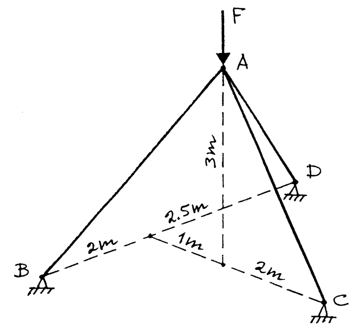
Question:
Determine the forces in the wires.



2.50 The hoisting device consists of three bars joined at A that can transfer forces only in the direction of their axes. Joint A is loaded by a vertical force F. The compressive force may be no larger than 12 kN in any of the bars.

Question:

a. Determine the maximum load the device can bear.
b. Determine the forces in the bars under this maximum load; are they tensile or compressive?



3 Statics of a Rigid Body

So far, we have discussed the behaviour of a particle with negligibly small dimensions, for which one can assume that all forces act on the particle at the same point. In this chapter, we will show that for the equilibrium of a rigid body with certain dimensions, the point of application (or actually the lines of action) of the various forces are of critical importance.

The equilibrium of a particle demands that the resultant of all forces be zero. This condition is also necessary for a body, but is not sufficient. Forces on a body can together form a *couple* that will try to turn it. In this chapter, we will define the *moment of a couple* as well as the *moment of a force*. Equilibrium demands that a body does not rotate. In addition to the *force equilibrium* of a body, if it is to be in equilibrium, it must also be in *moment equilibrium*.

In the first instance, in order to keep the discussion simple, we will look only at coplanar forces. Section 3.1 addresses compounding and resolving forces and moments, while Section 3.2 looks at the equilibrium of a body in a plane.

When considering equilibrium, we can consider forces as sliding vectors. In the spatial discussion in Section 3.3, we will talk about the fact that moments of a force and of a couple are vectors. The chapter ends with Section 3.4, in which we look at equilibrium equations for a body in space.

The discussions relate to rigid bodies. In reality, there are no rigid bodies, as all solids are deformable. Most construction material deforms so little, however, that for equilibrium of a body, it can often be considered as non-deformable.[1]

3.1 Coplanar forces and moments

3.1.1 Motion of a rigid body

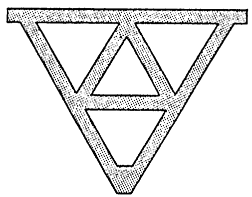


Figure 3.1 An element from the so-called "nabla beam" over the Haringvliet sluices, part of the Delta works in the Netherlands.

If several forces act on a body with particular dimensions, they can have various points of application. For the motion (the equilibrium) of a body, it is certainly important *where* the forces act. For example, with a billiard ball, it makes a difference whether one strikes the ball on the left or on the right. And if you want to lift the construction element in Figure 3.1, it makes a great difference whether you lift it from one of the upper corners or from the middle. Only in the latter case, on the basis of symmetry, can you expect the construction element not to rotate.

The movement of a rigid body differs from that of a particle in the sense that we also have to take the *rotation* of the body into consideration.

If we investigate the free motion of a rigid body, under the action of forces with zero resultant, there is a particular point that moves with uniform speed in a straight line (or is and stays at rest). This point about which the body

---

[1] There are exceptions. For example, a *stability investigation* – an investigation into the *reliability of the equilibrium* – investigates how the distribution of forces changes as a result of deformation of the structure. In such cases, one has to relate the equilibrium to the deformed geometry, however small the deformations might be, and the structure may no longer be considered rigid. This topic falls outside the scope of this book.

can perform further rotations is called the *mass centre*, MC.[1]

Without addressing the theory, we will cover four examples of how a rigid body, which originally is at rest, starts to move if it is subject to forces. In order to keep the discussion simple, we will confine our attention to cases in which all the forces are coplanar.

1. The body is subject to two equal and opposite forces with the same line of action, see Figure 3.2a. The state of movement does not change: if the body is at rest it remains at rest. The two forces are *in equilibrium*. The equilibrium is not influenced by the location of the points of application of the forces on their common line of action. The forces can be moved along their lines of action without any effect on the motion.

2. The body is subject to a force of which the line of action passes through the mass centre MC (see Figure 3.2b).
   The mass centre MC will move in a straight line as if it were a particle in which the entire mass is concentrated. No rotation occurs: the body performs a *translation*. The effect of the force does not change when the point of application is chosen elsewhere on the line of action of the force.

3. The body is subject to two equal and opposite forces with parallel lines of action (see Figure 3.2c).
   The mass centre MC remains at rest, but the body starts to *rotate* about an axis through MC perpendicular to the plane in which the forces are applied. For the progression of the movement, it now matters whether the forces *maintain their direction*, or *turn with* the body.

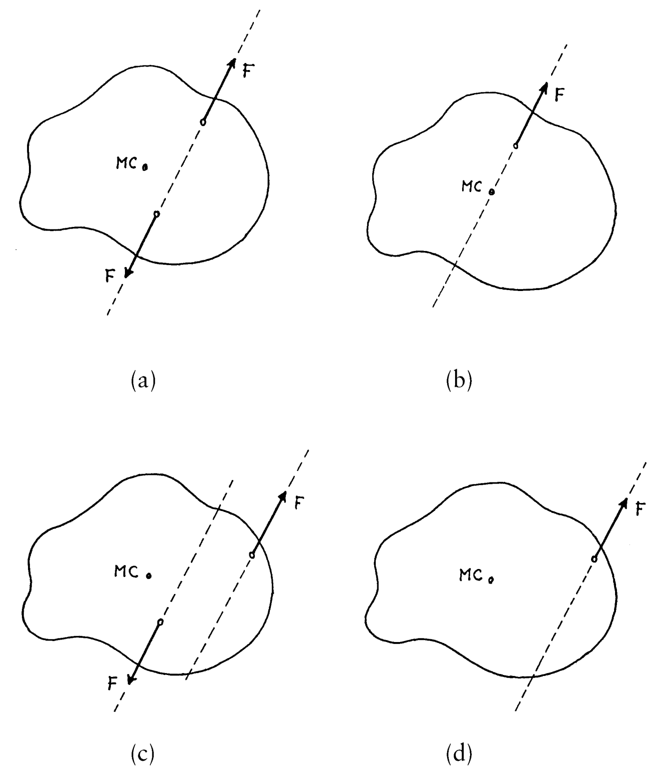


Figure 3.2 Equilibrium or motion of a body subject to forces: (a) equilibrium, (b) translation, (c) rotation about MC; (d) rotation and translation.

1 Since in a homogenous gravitational field the *centre of gravity* and *mass centre* of a body are the same, both names are often used interchangeably.

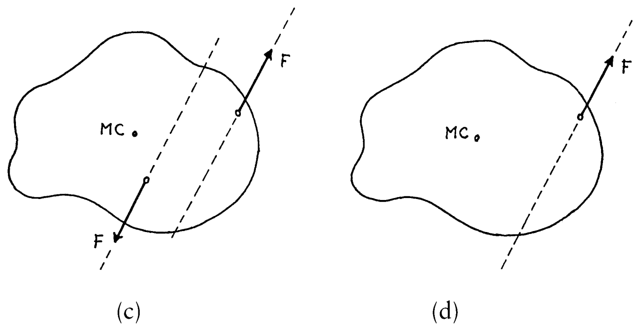


Figure 3.2 Equilibrium or motion of a body subject to forces: (c) rotation about MC; (d) rotation and translation.

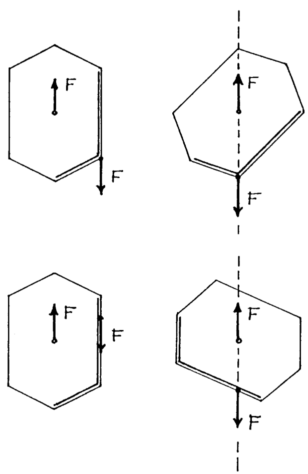


Figure 3.3 Forces that maintain their direction during rotation are fixed vectors: the final position depends on the location of the points of application.

If the forces turn with the body, it does not matter where they are exerted on their lines of action and they can be moved along their lines of action. If the forces maintain their direction, such as forces resultant from the gravitational field, they cannot be moved along their lines of action.

In Figure 3.3 this is illustrated by a plate subject to a pair of forces. One of the forces acts on the middle of the plate, the other on a point on the edge. Under the influence of these forces, the plate will move to a state of equilibrium, in which the lines of action coincide. The final position depends on where the forces are applied.

If one limits oneself to the so-called *instantaneous movement* of the body, or in other words the movement immediately after the application of the forces when the rotations are still very small, then the difference noted disappears, and the forces may be moved along their lines of action. The difference also disappears if one investigates the equilibrium of a body at rest, a situation without rotation.

4. A force acts on a body, and the line of action does not pass through the mass centre MC (see Figure 3.2d).
   The mass centre will start to move as if the force were applied directly to MC, and the body will also rotate about MC. The body experiences both a *translation* and a *rotation*.

Conclusion: *For the equilibrium (or instantaneous movement) of a rigid body, it does not matter at which point of its line of action a force is applied. The force on a rigid body can therefore be seen as a sliding vector. Although physically impossible, one can therefore also allow a force to "apply itself" to a point outside the body.*

Note: In investigating the *deformation* or *phenomena inside a body*, one cannot move a force along its line of action, and the force must be considered as a fixed vector.

In the bar in Figure 3.4, one can clearly see what happens if one changes the points of application of the two equal and opposite forces $F_1 = F$ and $F_2 = F$, with a common line of action. As far as the equilibrium is concerned, it is irrelevant where F_1 and F_2 are applied, while it certainly makes a difference to what happens "*internally*" and for the *deformation of the bar*: the upper bar is loaded by a tensile force and will lengthen, while the lower bar is loaded by compression, and will shorten.

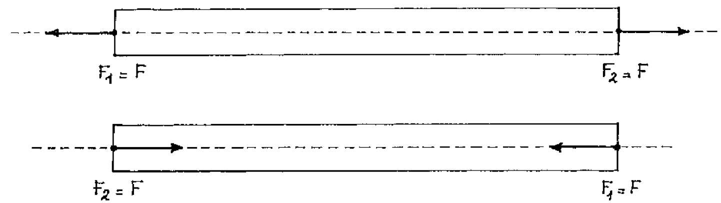


Figure 3.4 When considering equilibrium, one can shift forces along their lines of action. This is not permitted for considerations of what happens "internally".

3.1.2 Graphical composition of non-parallel forces

In the previous section it was stated that when considering the (instantaneous) movement and equilibrium of a rigid body, one can shift the forces along their lines of action. This means that it is possible to determine the resultant R of the two forces F_1 and F_2 in Figure 3.5 graphically by shifting them both to the intersection of their lines of action, and then applying the parallelogram rule. The resultant R is an *imaginary* force that with respect to the equilibrium of the body has the same effect as the two forces F_1 and F_2 together. We say that R is *statically equivalent* to F_1 and F_2.

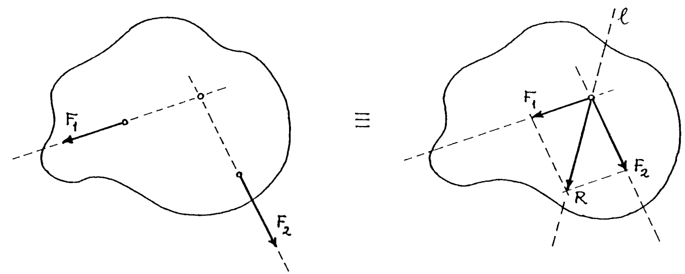


Figure 3.5 Compounding two forces using the parallelogram rule.

Besides the magnitude and direction of the resultant, we also find the location of its line of action ℓ. It is pointless talking about the point of application, only its line of action is fixed.

The magnitude and direction of the resultant can also be determined in a force polygon (see Figure 3.6). The line of action is determined by realising that it has to pass through the intersection of the lines of action of the forces to be compounded. Note that here the line of action of the resultant is entirely outside the body!

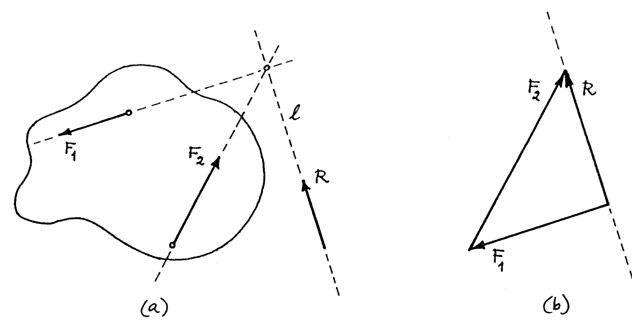


Figure 3.6 Compounding two forces using (a) line of action figure and (b) force polygon.

If several forces have to be compounded together, this can be done in phases by first determining the resultant of two forces, then compounding it with the third force, and so forth. This procedure is shown in Figure 3.7a.

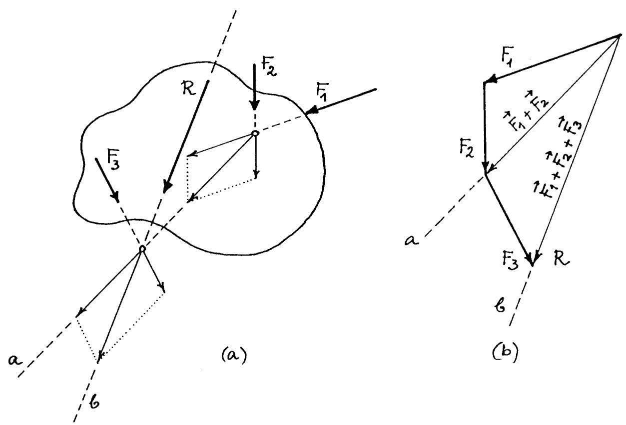


Figure 3.7 Compounding several forces using (a) a line of action figure and (b) force polygon.

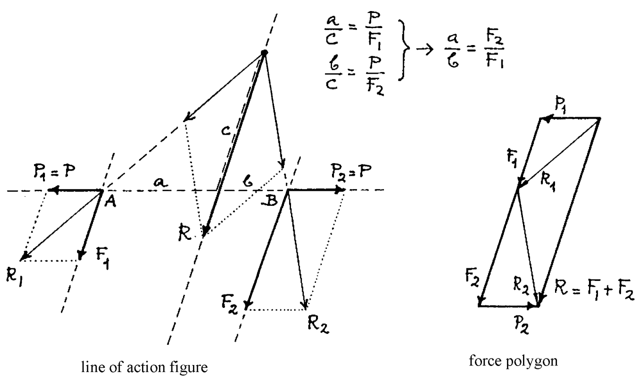


Figure 3.8 Compounding two parallel forces F_1 and F_2 graphically by adding the equilibrium system $P_1 = P$ and $P_2 = P$. Here we use for the forces the visual notation.

The magnitude and direction of the resultant can also be found quickly using a *force polygon*, as in Figure 3.7b. The force polygon does not provide information about the location of the line of action, however. To find the line of action, one would have to revert to Figure 3.7a. This figure is referred to as the *line of action figure*.

For more than two forces, using the line of action figure becomes laborious, and the analytical approach is clearly preferable (see Section 3.1.7). To determine the magnitude and direction of the resultant, the force polygon can still be useful.

3.1.3 Graphical composition of parallel forces

If the forces F_1 and F_2 are almost parallel, or parallel, one can determine the magnitude and direction of the resultant R graphically in a force polygon, although the graphical construction of its line of action (the line of action figure) becomes difficult as the intersection of the lines of action is far away or even at infinity.

In Figure 3.8, F_1 and F_2 are two parallel forces. The body on which the forces act is not shown. A graphical construction of the line of action is possible by having two equal yet opposite forces $P_1 = P$ and $P_2 = P$ apply to point A on the line of action of F_1, and to point B on the line of action of F_2, with AB as their common line of action. The magnitude of P can be chosen arbitrarily.

Since P_1 and P_2 together form an equilibrium system, the combined effect of the forces F_1, F_2, P_1 and P_2 is equal to that of only F_1 and F_2.

If R_1 is the resultant of F_1 and P_1, and R_2 is the resultant of F_2 and P_2, then the line of action of the resultant of all the forces, that is the resultant R of F_1 and F_2, passes through the intersection of the lines of action of R_1 and R_2.

From the graphical construction, one can see that the line of action of the resultant R of two parallel forces F_1 and F_2, acting in the same direction, is between their lines of action, nearer the larger force, and such that the distances a and b to the lines of action of F_1 and F_2 respectively are reversed proportionally to the magnitudes of these forces (see Figure 3.9a):

$$\frac{a}{b} = \frac{F_2}{F_1}.$$

If the two parallel forces F_1 and F_2 have opposite directions, then the resultant R has the same direction as the larger of the two forces, and the line of action of R is outside the lines of action of F_1 and F_2 on the side of the larger force. Now too, the distances a and b from the line of action of R to the lines of action of F_1 and F_2 are reversed proportionally to the magnitude of these forces (see Figure 3.9b).

In conclusion, for the resultant R of two parallel forces F_1 and F_2:
- R is in the direction of the larger force;
- the line of action of R is closer to the larger force;
- R is between F_1 and F_2 if these forces are in the same direction;
- R is outside F_1 and F_2 if these forces have opposite directions.

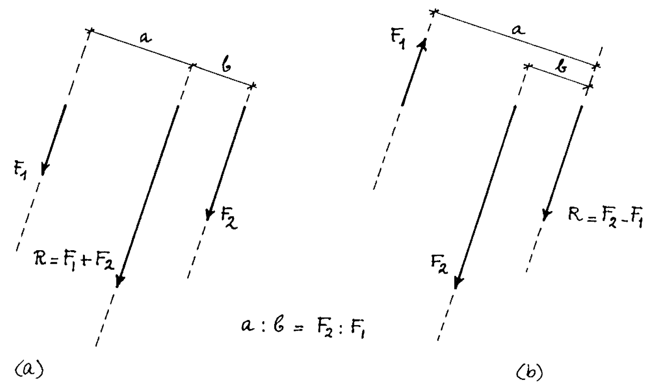


Figure 3.9 The resultant R of two parallel forces F_1 and F_2, in (a) the same and (b) opposite directions.

3.1.4 Moment of a couple

Figure 3.10 shows the special case of two equal and opposite parallel forces $F_1 = F$ and $F_2 = F$. If we want to graphically compound these forces in the way described above, using two equal and opposite additional forces $P_1 = P$ and $P_2 = P$, with the common line of action AB, we again find two equal and opposite parallel forces R_1 and R_2 (see Figure 3.11).

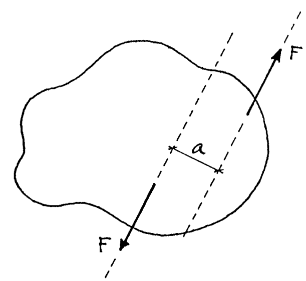


Figure 3.10 The pair of forces F forms a couple. a is the couple arm. The product Fa is the moment of the couple.

It is impossible to compound the pair of forces F into a single force. We call such a pair of forces a couple. The product of the magnitude of F of the forces and the distance a between the lines of action is called the *moment of the couple*. As symbol for this quantity we use the letter T:

$$T = Fa.$$

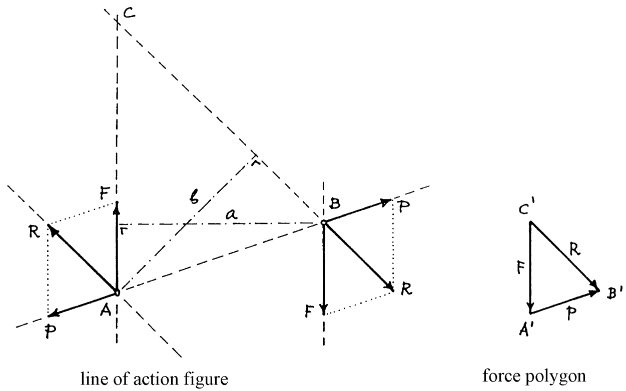


Figure 3.11 The result of a couple does not change if one replaces it by another couple with the same moment and the same direction of rotation: $Fa = Rb$.

a is referred to as the *couple arm* and is always measured perpendicularly to the lines of action.

The two forces $R_1 = R$ and $R_2 = R$ also form a couple. Here the moment of the couple is

$$T = Rb,$$

b is the couple arm.

Since $P_1 = P$ and $P_2 = P$ form an equilibrium system, the effect of the couple caused by the forces R with arm b is equal (*statically equivalent*) to the effect of the couple formed by the forces F with arm a. The moment of the couple is therefore the same for both:

$$T = Fa = Rb.$$

This can also be derived from line of action figure in Figure 3.11.

Consider triangle ABC; its area is

$$\text{area ABC} = \tfrac{1}{2}a \cdot \text{AC} = \tfrac{1}{2}b \cdot \text{BC},$$

so that

$$\frac{a}{b} = \frac{\text{BC}}{\text{AC}}.$$

Triangle ABC, from the line of action figure, is geometrically similar to force triangle A′B′C′, so that the corresponding sides are proportional:

$$\frac{\text{BC}}{\text{AC}} = \frac{\text{B}'\text{C}'}{\text{A}'\text{C}'} = \frac{R}{F}.$$

On combining these two equations we deduce that

$$\frac{a}{b} = \frac{R}{F},$$

which is equivalent to

$$Fa = Rb.$$

Conclusion: *The effect of a couple on the equilibrium of a body does not change if you replace it by another couple with the* same moment *and* the same direction of rotation.

The magnitude of the moment of a couple determines the state of rotation of the body. In addition to a *magnitude*, the moment also has a *direction of rotation*. The sign for the direction of rotation is linked to the coordinate system, see the sign convention in Section 1.3.2.

In the xy coordinate system shown, the moment of the couple in Figure 3.12a is

$$T_z = -Fa = -12 \text{ kNm}.$$

The letter T is given the index z, which indicates the normal of the plane in which the couple acts.

In Figure 3.12b the moment of the same couple is in another coordinate system:

$$T_y = +Fa = +12 \text{ kNm}.$$

In Figure 3.12c, the couple is represented by a curved arrow. In this *visual notation* the arrow indicates the direction of rotation of the moment and includes a value. The same conventions apply as for the visual notation of

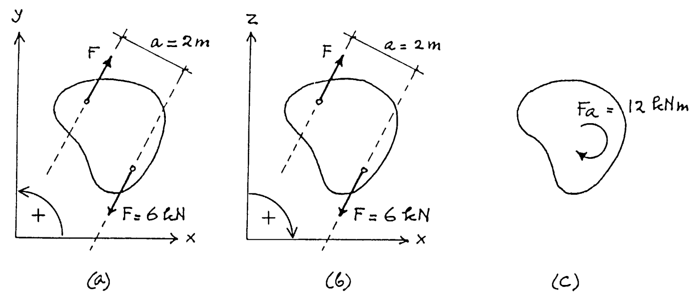


Figure 3.12 Three times the same couple: (a) $T_z = -12$ kNm, (b) $T_y = +12$ kNm, (c) the couple using visual notation.

a force (see Section 1.3.6).

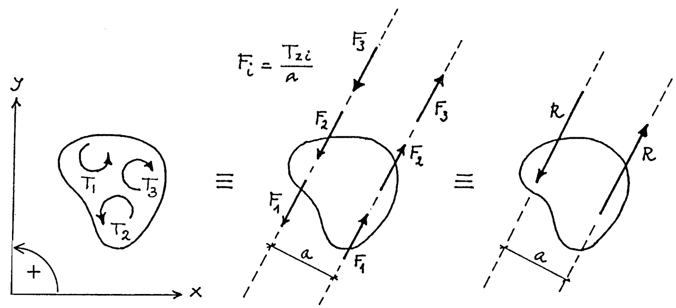


Figure 3.13 The moment of the resultant couple is found by adding the moments of the couples to be compounded (algebraically).

Couples can be compounded in many different ways. In Figure 3.13, the couples operating in the xy plane, T_1, T_2 and T_3, have been compounded by replacing them by equal couples of which the forces have common lines of action. If $T_1 = 12$ kNm, $T_2 = 6$ kNm and $T_3 = 10$ kNm, and the distance a between the lines of action is 4 m, then, in the coordinate system shown,

$$T_{z;1} = +T_1 = +12 \text{ kNm} = F_1 a \Rightarrow F_1 = T_{z;1}/a = +3 \text{ kN},$$

$$T_{z;2} = +T_2 = +6 \text{ kNm} = F_2 a \Rightarrow F_2 = T_{z;2}/a = +1.5 \text{ kN},$$

$$T_{z;3} = -T_3 = -10 \text{ kNm} = F_3 a \Rightarrow F_3 = T_{z;3}/a = -2.5 \text{ kN}.$$

Note: The force F_3 has the value 2.5 kN and acts opposite to the direction shown in Figure 3.13.

The moment of the resultant couple is

$$T_z = Ra = (F_1 + F_2 + F_3) \cdot a = \sum_{i=1}^{3} T_{z;i} = 8 \text{ kNm}.$$

If all the couples are exerted in the same plane, the moment of the resultant couple is found by compounding the couple moments simply by adding them together.

The example shows that the couples form an equilibrium system if the sum of their moments is zero (because $R = 0$).

3.1.5 The moment of a force about a point

The *moment of a force* about a point A is defined as the product of magnitude F of the force and the perpendicular distance a from point A to the line of action of the force. The sign of the moment is plus or minus, depending on whether the force F turns the body in the positive or negative direction of rotation about A.

For Figure 3.14, the moment of force F with respect to A is seen as positive as F causes a rotation about A in the positive direction of rotation in the xy plane:

$$T_z|\mathrm{A} = +Fa = +(10\ \mathrm{kN})(4\ \mathrm{m}) = +40\ \mathrm{kNm}.$$

The same force F causes the body to rotate about B in the negative direction of rotation. The moment of F about B is therefore negative:

$$T_z|\mathrm{B} = -Fb = -(10\ \mathrm{kN})(5\ \mathrm{m}) = -50\ \mathrm{kNm}.$$

The moment of the force F about a point C located on its line of action, is zero:

$$T_z|\mathrm{C} = 0.$$

For a force, in contrast to a couple, one has to specify the point about which the moment is being calculated. Here, this is done by including the point in question, after a vertical line, in the expression for the moment.

Figure 3.15 shows a single force F acting at point B. Now introduce two equal and opposite forces $F_1 = F$ and $F_2 = F$ acting at point A. Since F_1 and F_2 together form an equilibrium system, the single force F at B is statically equivalent to the three forces F at B and F_1 and F_2 at A. F at B and $F_2 = F$ at A together form a couple with moment Fa. The force $F = 7$ kN at B is therefore statically equivalent with a force $F = 7$ kN at B and a couple with moment $Fa = 21$ kNm.

Conclusion: *The moment of a force F about a point A is equal to the moment of the couple one has to add when moving the force parallel to its a line of action to A.*

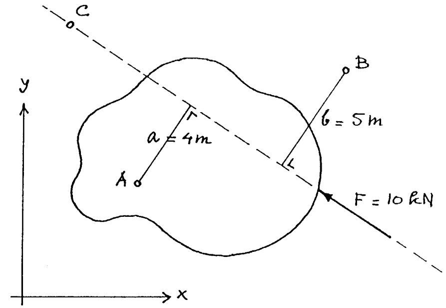


Figure 3.14 The moment of a force with respect to a point is defined as the product of the magnitude of the force and the perpendicular distance to its line of action.

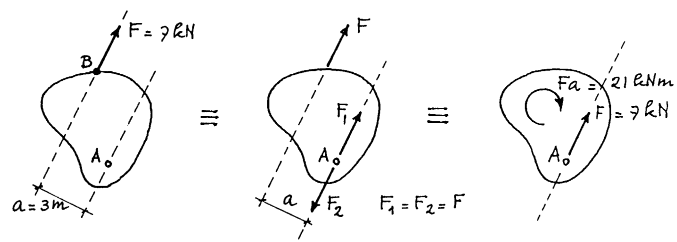


Figure 3.15 The moment of the force F about point A is equal to the moment of the couple that one has to add to the force if one shifts it parallel to its line of action through A.

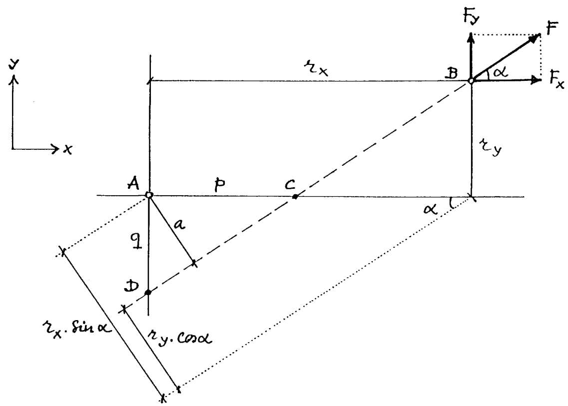


Figure 3.16 The moment of the force F about point A is equal to the sum of the moments about A of its components: $T_z|\mathrm{A} = Fa = F_y r_x - F_x r_y$.

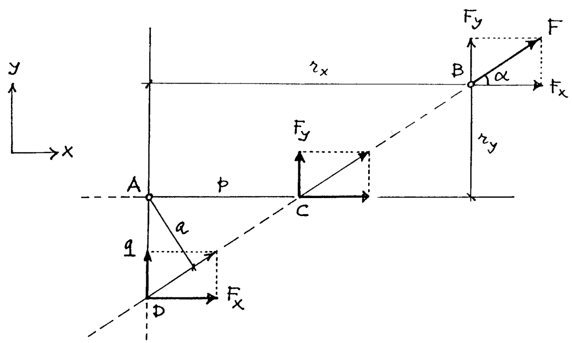


Figure 3.17 The moment of a force does not change if the force is shifted along its line of action: $T_z|\mathrm{A} = Fa = F_y p = F_x q$.

The moment of a force F in the xy plane about a point A in the same plane can be calculated in a variety of ways (see Figure 3.16). The components of F are

$$F_x = F\cos\alpha,$$

$$F_y = F\sin\alpha.$$

If the force is applied in a point B, then

$$r_x = x_\mathrm{B} - x_\mathrm{A},$$

$$r_y = y_\mathrm{B} - y_\mathrm{A}.$$

From the figure one can derive

$$a = r_x \sin\alpha - r_y \cos\alpha.$$

For the moment of F about A applies

$$T_z|\mathrm{A} = Fa = F(r_x \sin\alpha - r_y\cos\alpha) = F_y r_x - F_x r_y.$$

$F_y r_x$ is the moment of the component F_y about A, and $-F_x r_y$ is the moment of the component F_x about A. This shows that the moment of a force F about a point A is equal to the sum of the moments about A of its components.

Since the moment of a force does not change if the force is moved along its line of action, it is sometimes useful to shift the force to point C or D (see Figure 3.17). In this case, the moment of F about A is

$$T_z|\mathrm{A} = Fa = F_y p = F_x q.$$

Example

The moment about A of the force at B in Figure 3.18 can now be calculated as follows:

- Force multiplied by the distance to its line of action:

$$T_z|\text{A} = -(2\sqrt{5}\ \text{kN})(2\sqrt{5}\ \text{m}) = -20\ \text{kNm}.$$

- Force in B resolved into its components:

$$T_z|\text{A} = -(4\ \text{kNm})(3\ \text{m}) - (2\ \text{kNm})(4\ \text{m}) = -20\ \text{kNm}.$$

- Force shifted to C:

$$T_z|\text{A} = -(2\ \text{kN})(10\ \text{m}) = -20\ \text{kNm}.$$

- Force shifted to D:

$$T_z|\text{A} = -(4\ \text{kN})(5\ \text{m}) = -20\ \text{kNm}.$$

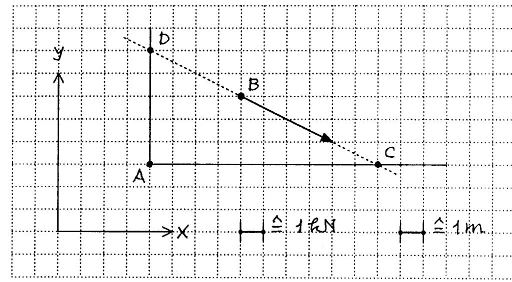


Figure 3.18 The moment about A of the force at B can be calculated in various ways.

3.1.6 Moment theorems

In Figure 3.19, R is the resultant of the forces F_1 and F_2:

$$R_x = F_{x;1} + F_{x;2},$$

$$R_y = F_{y;1} + F_{y;2}.$$

In order to be able to determine the moment of F_1 and F_2 about an arbitrary point A, both forces are shifted to the intersection of their lines of action. In the previous section, it was shown that the moment of a force about an arbitrary point is equal to the sum of the moments of its components about that point. Therefore, for the moment of F_1 and F_2 about A it is true that:

$$\begin{aligned}\sum T_z|\text{A} &= (T_z|\text{A due to } F_1) + (T_z|\text{A due to } F_2)\\ &= (F_{y;1}r_x - F_{x;1}r_y) + (F_{y;2}r_x - F_{x;2}r_y)\\ &= (F_{y;1} + F_{y;2})r_x - (F_{x;1} + F_{x;2})r_y\end{aligned}$$

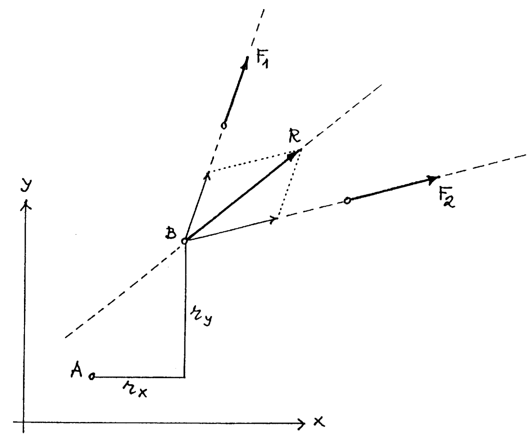


Figure 3.19 The sum of the moments of F_1 and F_2 about an arbitrary point A is equal to the moment of the resultant R about that point A. This is known as Varignon's First Theorem.

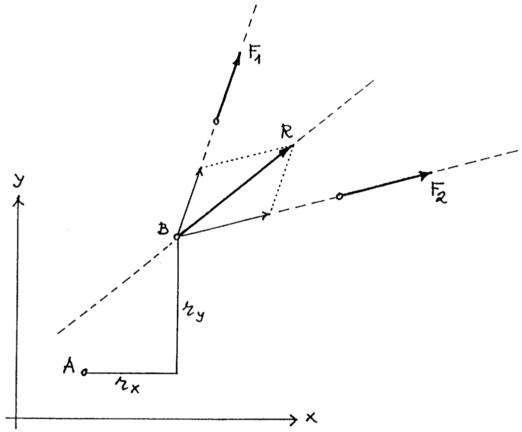


Figure 3.19 The sum of the moments of F_1 and F_2 about an arbitrary point A is equal to the moment of the resultant R about that point A. This is known as Varignon's First Theorem.

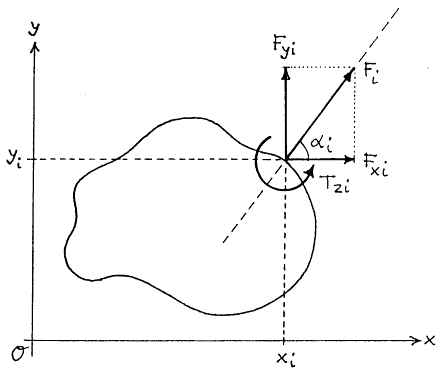


Figure 3.20 A body loaded by couples $T_{z;i}$ and forces F_i, with components $F_{x;i}$ and $F_{y;i}$, at points i ($i = 1, 2, 3, \ldots$).

$$= R_y r_x - R_x r_y$$

$$= T_z|\text{A due to } R.$$

Conclusion: *If two forces F_1 and F_2 have a resultant R, the sum of the moments of F_1 and F_2 about an arbitrary point A is equal to the moment of the resultant R about that point A. This is called* Varignon's First Moment Theorem.[1] *The theorem also applies if F_1 and F_2 have parallel lines of action.*

If the two forces F_1 and F_2 together form a couple, the sum of the moments of F_1 and F_2 is independent of the point with respect to which the moment is determined. This sum of moments is equal to the moment of the couple. This is known as the *Varignon's Second Moment Theorem.*

Varignon's momentary theorems can be applied repeatedly if several forces act in the same plane. This results in the following *General Moment Theorem*:
The sum of the moments of a number of forces distributed in a plane, about an arbitrary point A in that plane, is *either* equal to the moment of the resultant force about that point or equal to the moment of the resultant couple.

3.1.7 Compounding forces and moments analytically

Compounding coplanar forces and couples analytically is now relatively simple. Each of the forces F_i ($i = 1, 2, \ldots$) can be resolved into the components $F_{x;i}$ and $F_{y;i}$, and for each of these forces, we can now determine the moment about an arbitrary point A. In fact, this means that all the forces are shifted to point A with addition of a couple (see Section 3.1.5). If we place the origin O of the coordinate system at A, and x_i and y_i are the

---

[1] Pujol Varignon (1654–1722) was a French mathematician.

coordinates of the point of application of force F_i (or of another point on the line of action of F_i), then (see Figure 3.20)

$$R_x = \sum F_{x;i} = \sum F_i \cos\alpha_i,$$

$$R_y = \sum F_{y;i} = \sum F_i \sin\alpha_i,$$

$$\sum T_z|\text{O} = \sum\{(F_{y;i}x_i - F_{x;i}y_i) + T_{z;i}\}.$$

The sum of the moments also includes the moments of the (concentrated) couples $T_{z;i}$ that may be applied on the body.

For the (instantaneous) movement or the equilibrium of a rigid body, one may replace the force system by a single resultant force R at O together with a couple $\sum T_z|\text{O}$ (see Figure 3.21a).

The resultant force R at O can be compounded with the couple $\sum T_z|\text{O}$ into a single force R by shifting it parallel to itself to a line of action at a perpendicular distance a from O (see Figure 3.21b):

$$a = \frac{\sum T_z|\text{O}}{R}.$$

The line of action of R can also be found as follows. Imagine that (x, y) is an arbitrary point on the line of action of R (see Figure 3.21b). According to the moment theorem,

$$\sum T_z|\text{O} = R_y x - R_x y.$$

The values for $\sum T_z|\text{O}$, R_x and R_y are known, while those of x and y are unknown. This expression therefore also provides the equation for the line of action of R. The line of action of R intersects the x axis at

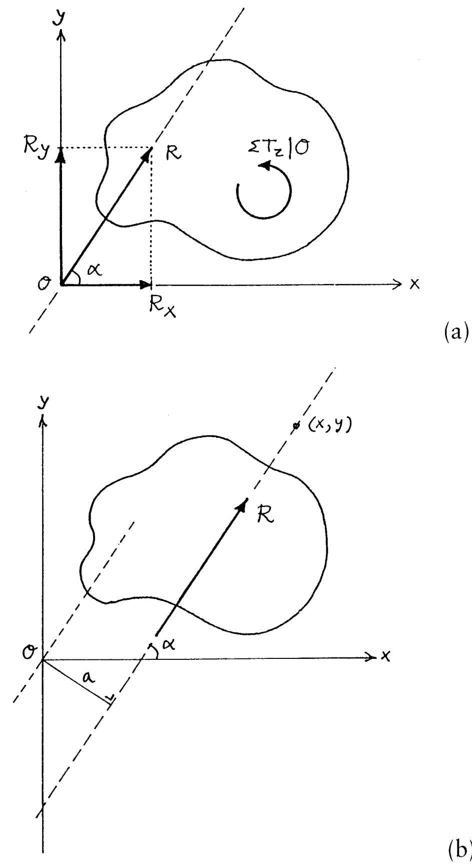


Figure 3.21 (a) The resultant force R at O and the associated couple $\sum T_z|\text{O}$ are statically equivalent to (b) a force R at a distance $a = (\sum T_z|\text{O})/R$ from O.

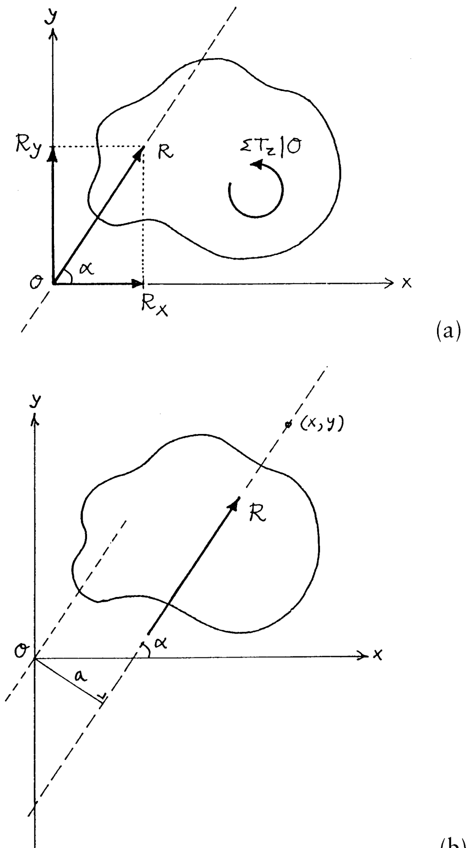


Figure 3.21 (a) The resultant force R at O and the associated couple $\sum T_z|\mathrm{O}$ are statically equivalent to (b) a force R at a distance $a = (\sum T_z|\mathrm{O})/R$ from O.

$$x = \frac{\sum T_z|\mathrm{O}}{R_y}; \; y = 0,$$

and the y axis at

$$x = 0; \; y = -\frac{\sum T_z|\mathrm{O}}{R_x}.$$

A special case is when $R = 0$ and $\sum T_z|\mathrm{O} \neq 0$. In this case, there is no resultant force, while there is a resultant couple. When also $\sum T_z|\mathrm{O} = 0$, then there is equally no resultant couple and the forces together form an equilibrium system.

To summarise, with respect to the resultant of a system of forces and couples, one can distinguish the following cases:

- $R \neq 0$ and $\sum T_z|\mathrm{O} \neq 0$
  There is a resultant force, and the line of action does not pass through O.
- $R \neq 0$ and $\sum T_z|\mathrm{O} = 0$
  There is a resultant force of which the line of action passes through O.
- $R = 0$ and $\sum T_z|\mathrm{O} \neq 0$
  There is no resultant force, but there is a resultant couple.
- $R = 0$ and $\sum T_z|\mathrm{O} = 0$
  The forces and couples together form an equilibrium system.

Example
Three forces and a couple are exerted on the triangular block in Figure 3.22a. The magnitude and the direction of the forces can be found in the diagram, as can the direction of couple T. The magnitude of the couple is 30 kNm.

Question:
Determine the magnitude, direction, and line of action of the resultant force on the block.

Solution:
For convenience sake, the units (kN and/or m) are not always shown in the interim calculations. For the components of the resultant force R applies

$$R_x = \sum_{i=1}^{3} F_{x;i} = -10 + 30 + 0 = +20\ \text{kN},$$

$$R_y = \sum_{i=1}^{3} F_{y;i} = 0 + 20 - 40 = -20\ \text{kN}$$

so that

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{20^2 + (-20)^2} = 20\sqrt{2}\ \text{kN}.$$

The magnitude and direction of R and of its components can of course also be determined graphically by using a force polygon (see Figure 3.22b).

The moment about O of the three forces and the couple is

$$\begin{aligned}
\sum T_z|\text{O} = &+10 \times 6 && \text{(for } F_1\text{)}\\
&+(20 \times 6 - 30 \times 3) && \text{(for } F_2\text{, resolved into its components)}\\
&-40 \times 4 && \text{(for } F_3\text{)}\\
&+30 && \text{(for } T\text{)}\\
= &-40\ \text{kNm}.
\end{aligned}$$

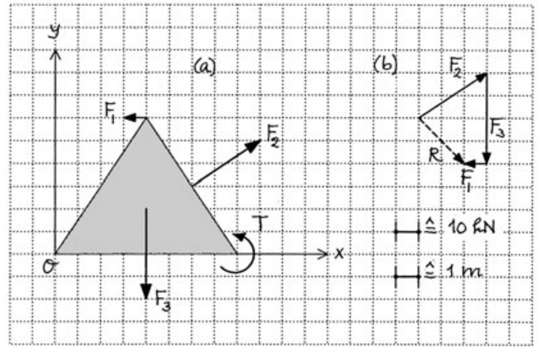


Figure 3.22 (a) A triangular block subject to three forces and a couple; (b) the resultant force R on the block, determined using a force polygon.

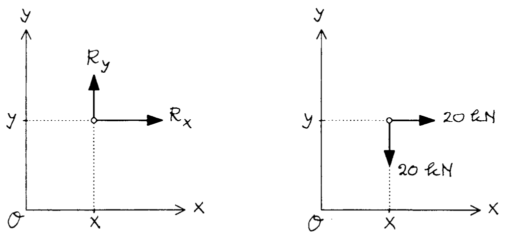


Figure 3.23 The resultant force R at a point (x, y) of its line of action: (a) in its components R_x and R_y and (b) in components as they act in reality.

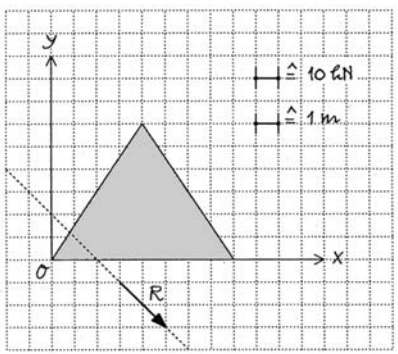


Figure 3.24 The resultant R and its line of action.

The resultant R must have the same moment about O as the three forces and the couple. Imagine (x, y) is a point on the line of action of R, with components R_x and R_y (see Figure 3.23a). Then

$$\sum T_z|\text{O} = -40 \text{ kNm} = R_y x - R_x y.$$

With $R_x = +20$ kN and $R_y = -20$ kN, this gives the following equation for the line of action of the resultant R:

$$-40 \text{ kNm} = (-20 \text{ kN})x - (+20 \text{ kN})y \Rightarrow x + y = 2 \text{ m}.$$

Of course it is also possible to depict R_x and R_y as in Figure 3.23b, according to the actual magnitude and direction. This figure immediately gives the expression shown above fore the line of action of R. Figure 3.24 shows the resultant R with its line of action.

Note: If one performs the calculation using a picture, all the unknown quantities that are related to the coordinate system in that picture have to be shown *positively*. In Figure 3.23a that would be x, y, R_x and R_y, in Figure 3.23b this only relates to x and y.

3.1.8 Resolving a force along given lines of action graphically

A force F, with given magnitude, direction, and line of action, can be resolved along three given lines of action a, b and c, which do not intersect in one point, into the forces F_a, F_b and F_c (see Figure 3.25a).

Here

$$\vec{F} = \vec{F}_\text{a} + \vec{F}_\text{b} + \vec{F}_\text{c},$$

so that

$$(\vec{F} - \vec{F}_\mathrm{a}) = (\vec{F}_\mathrm{b} + \vec{F}_\mathrm{c}).$$

$(\vec{F} - \vec{F}_\mathrm{a})$ and $(\vec{F}_\mathrm{b} + \vec{F}_\mathrm{c})$ are equal and therefore have the same line of action. The line of action of $(\vec{F} - \vec{F}_\mathrm{a})$ passes through the intersection A of the lines of action of $\vec{F}$ and $\vec{F}_\mathrm{a}$. The line of action of $(\vec{F}_\mathrm{b} + \vec{F}_\mathrm{c})$ passes through the intersection $\mathrm{S_{ab}}$ of the lines of action b and c. Therefore, $\mathrm{AS_{bc}}$ is the line of action of both $(\vec{F} - \vec{F}_\mathrm{a})$ and $(\vec{F}_\mathrm{b} + \vec{F}_\mathrm{c})$. $\vec{F}$ at A can now be resolved into $\vec{F}_\mathrm{a}$ with line of action a and $(\vec{F}_\mathrm{b} + \vec{F}_\mathrm{c})$ with line of action $\mathrm{AS_{bc}}$. Subsequently $(\vec{F}_\mathrm{b} + \vec{F}_\mathrm{c})$ at $\mathrm{S_{bc}}$ can be resolved into $\vec{F}_\mathrm{b}$ and $\vec{F}_\mathrm{c}$. This is shown graphically in Figure 3.25b in a single force polygon.

The order in which $\vec{F}$ is resolved is irrelevant. In Figure 3.26 $\vec{F}$ is first resolved at B into $\vec{F}_\mathrm{b}$ and $(\vec{F}_\mathrm{a} + \vec{F}_\mathrm{c})$ and subsequently $(\vec{F}_\mathrm{a} + \vec{F}_\mathrm{c})$ at $\mathrm{S_{ac}}$, the intersection of the lines of action a and c, is resolved into $\vec{F}_\mathrm{a}$ and $\vec{F}_\mathrm{c}$. The force polygon now has a different shape, as the forces were resolved in a different order, but the result is the same.

The name *Culmann*[1] is associated with this graphical method in the literature.

3.1.9 Resolving a force along given lines of action analytically

Resolving F into three forces F_a, F_b and F_c along given lines of action a, b, and c, can of course also be done analytically. Of the many possible methods, the method below is based on Varignon's first moment theorem:

[1] Karl Culmann (1821–1881), a German engineer, was involved in the design and construction of important railway bridges and was especially known for his graphical methods for calculating structures.

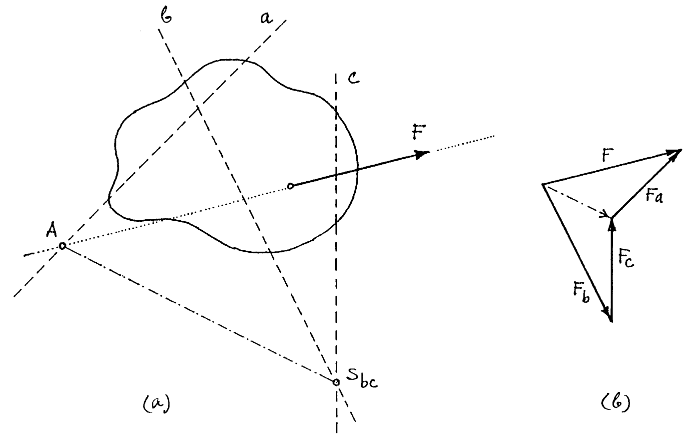


Figure 3.25 Resolving the force F graphically along three given lines of action; (a) line of action figure and (b) force polygon.

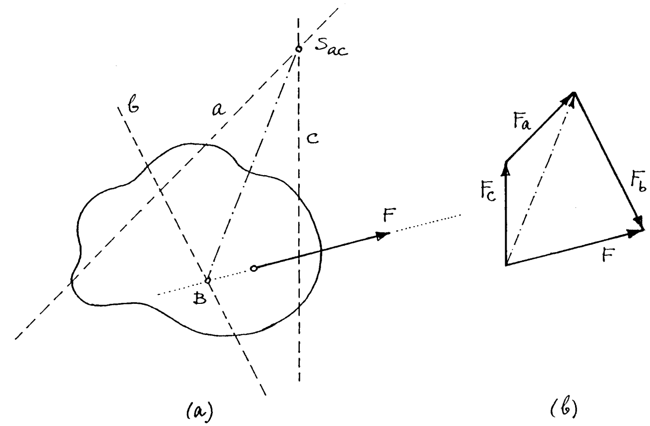


Figure 3.26 Resolving the force F graphically along three given lines of action; (a) line of action figure and (b) force polygon.

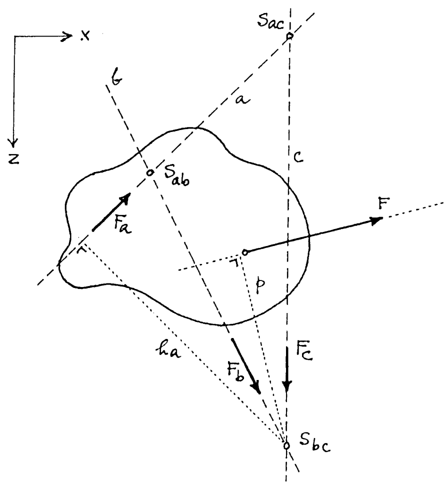


Figure 3.27 Analytically resolving force F along three given lines of action.

the moment of F about an arbitrary point is equal to the sum of the moments of F_a, F_b and F_c about that same point.

In Figure 3.27, the directions of the as yet unknown forces F_a, F_b, and F_c have been assumed. In addition, a coordinate system has been assumed in order to be able to indicate the sign of the moments (the direction of rotation).

If the moment theorem is applied with respect to S_{bc}, the intersection of the lines of action of F_b and F_c, then these forces do not contribute to the sum of the moments, and one can determine F_a directly:

$$\sum T_y|S_{bc} = -F \cdot p = -F_a \cdot h_a \Rightarrow F_a = \frac{p}{h_a} F.$$

Note: The signs are related to the xz coordinate system shown.

By applying the moment theorem in the same way with respect to S_{ac} and S_{ab} respectively, we also find F_a and F_c directly.

Since the direction of rotation of F about S_{ab} is opposite to that of F_c about S_{ab} the value of F_c will be negative. This means that the force F_c works opposite to the direction assumed in Figure 3.27.

The analytical approach can also be used for resolving a couple into three forces along given lines of action.

Example
The block in Figure 3.28a is subject to the three forces F_a, F_b and F_c, along given lines of action a, b and c. The resultant is the couple T with the direction shown in the figure.

Question:
Determine the three forces if $T = 80$ kNm.

Solution:
In Figure 3.28b an assumption was made with respect to the directions of the forces. In the coordinate system given

$$\sum T_z|\mathrm{A} = -\tfrac{4}{5}F_\mathrm{a} \times (4\ \mathrm{m}) = -T = -80\ \mathrm{kNm} \;\Rightarrow\; F_\mathrm{a} = +25\ \mathrm{kN},$$

$$\sum T_z|\mathrm{B} = +F_\mathrm{b} \times (4\ \mathrm{m}) = -T = -80\ \mathrm{kNm} \;\Rightarrow\; F_\mathrm{b} = -20\ \mathrm{kN}.$$

The minus sign in the latter answer shows that the force F_b acts in the opposite direction to that assumed in Figure 3.28b.

From $\sum T_z|\mathrm{C} = -80$ kNm we can derive F_c directly. Finding the location of C, the intersection of the lines of action, takes some calculation. The force F_c is therefore easier to find since the resultant force is zero:

$$\sum F_x = \tfrac{3}{5}F_\mathrm{a} + F_\mathrm{c} = \tfrac{3}{5} \times (25\ \mathrm{kN}) + F_\mathrm{c} = 0 \Rightarrow F_\mathrm{c} = -15\ \mathrm{kN}.$$

Apparently the direction of F_c was also falsely assumed. Figure 3.28c shows the forces as they are actually exerted on the block. It would indeed not be difficult to determine the correct directions of the forces prior to making the calculation.

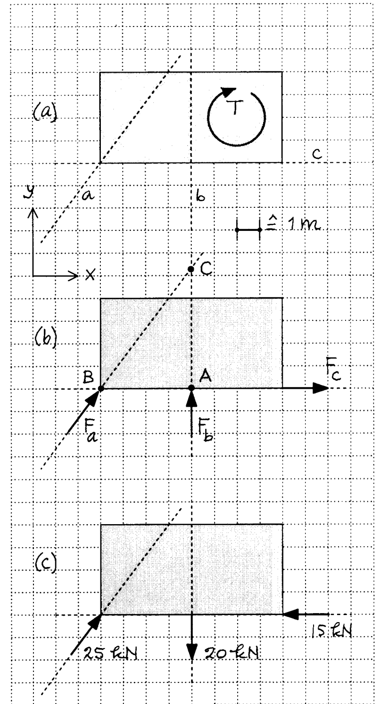


Figure 3.28 Resolving a couple into three forces along given lines of action: (a) the couple T and the lines of action a, b and c; (b) the assumed directions of the forces F_a, F_b and F_c; (c) the forces as they have to act on the block in reality if they are to be statically equivalent to the couple.

3.2 Equilibrium of a rigid body in a plane

For the (instantaneous) motion of a rigid body, the system of forces exerted on it can be replaced by a single force at an arbitrary point and a couple. When considering the motion of the body, it is preferable to choose the mass centre as that point, as the motion can then be split into a *translation* due to the force, and a *rotation* due to the couple (see Section 3.1.1).

From the above, it follows that a rigid body is in equilibrium if for all the forces exerted on it, the resultant force and the resultant couple are zero. The *equilibrium conditions* for a rigid body, only subject to forces in the xy plane, are:

$$\sum F_x = 0,$$
$$\sum F_y = 0,$$
$$\sum T_z = 0.$$

The summation symbol means that all contributions of the forces acting on the body have to be added.

The first two equations stand for the *force equilibrium* in respectively the x and y direction, and express that there is no resultant force. The third equation stands for the *moment equilibrium*, and expresses that the forces together do not form a resultant couple. Here, the moment with respect to an arbitrary point has to be determined for all the forces, and the moments have to be added together.

If (concentrated) couples are applied to the body, schematically represented by curved arrows, their moments of course also have to be included in the moment summation. The equations for the force equilibrium are not influenced by these couples.

For particles (with negligibly small dimensions), the force equilibrium is a necessary and sufficient condition for equilibrium. For rigid bodies (with finite measurements) the force equilibrium is a necessary but not sufficient condition for equilibrium; since a body can rotate, another condition is required, namely the moment equilibrium.

3.2.1 Equilibrium equations

In a plane, the equilibrium of a body is assured if it meets two conditions for the force equilibrium and one condition for the moment equilibrium:

$$\sum F_x = 0,$$
$$\sum F_y = 0,$$
$$\sum T_z = 0.$$

These *equilibrium equations* in a plane can be replaced by three arbitrary linear combinations, on the condition that these combinations are independent. Three of these combinations are mentioned separately below:

1. The condition of force equilibrium in two mutually perpendicular directions can be replaced by the condition that of all the forces, the sum of the components in two arbitrary directions is zero.
2. The equilibrium can also be described by three moment conditions with respect to three points A, B and C that are not in a straight line:

$$\sum T_z|\mathrm{A} = 0,$$
$$\sum T_z|\mathrm{B} = 0,$$
$$\sum T_z|\mathrm{C} = 0.$$

That these three equations are sufficient to ensure equilibrium can be shown as follows (see Figure 3.29). Each system of coplanar forces and couples can be replaced by either a *resultant force*, or a *resultant couple*. If $\sum T_z|\mathrm{A} = 0$, there is no resultant couple. There could still be a resultant force of which the line of action must pass through A. If $\sum T_z|\mathrm{B} = 0$, the line of action of the resultant force must also pass through B. If C is not located on AB (the line of action of the resultant force), and $\sum T_z|\mathrm{C} = 0$, the resultant force can only be zero.

3. The equilibrium can also be formulated by two moment conditions with respect to two points A and B and an equation for the force equilibrium in a direction that is not perpendicular to AB:

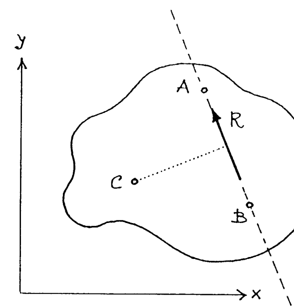


Figure 3.29 The relationships $\sum T_z|\mathrm{A} = 0$ and $\sum T_z|\mathrm{B} = 0$ imply that there is no resultant couple and that, if there is a resultant force R, its line of action is along AB.

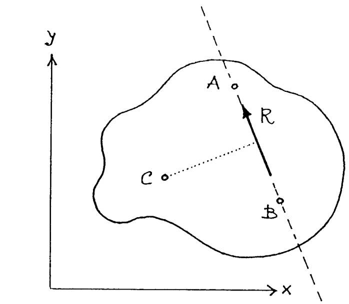


Figure 3.29 The relationships $\sum T_z|\text{A} = 0$ and $\sum T_z|\text{B} = 0$ imply that there is no resultant couple and that, if there is a resultant force R, its line of action is along AB.

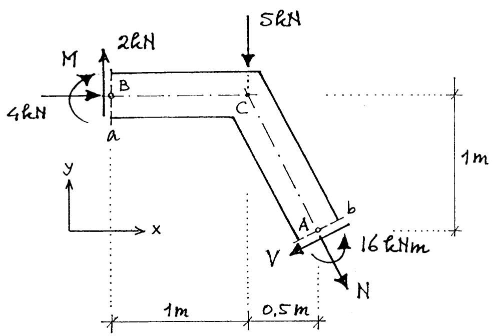


Figure 3.30 Corner joint in a frame; the three unknown section forces M, V and N can be deduced from the equilibrium.

$$\sum T_z|\text{A} = 0,$$

$$\sum T_z|\text{B} = 0,$$

$$\sum F_x = 0 \text{ (the } x \text{ direction may not be perpendicular to AB).}$$

The relationships $\sum T_z|\text{A} = 0$ and $\sum T_z|\text{B} = 0$ imply that there is no resultant couple and that, if there is a resultant force, its line of action coincides with AB (see Figure 3.29). The resultant force is zero if the condition for force equilibrium is met in the direction AB, or in another direction that is not perpendicular to AB.

The equilibrium conditions can therefore be formulated in various ways. For a manual calculation, one always has to look for equilibrium equations that are as simple as possible in order to limit the amount of calculation. When using a computer for the calculation, the systematics and the general applicability of the set up of the calculation (the program) are more important than the number of calculations involved and the laborious character of the calculations.

Example
Figure 3.30 shows the corner joint of a frame. The joint is loaded at C by a vertical force of 5 kN. So-called section forces act on the cross-sectional planes a and b. They act in the centre lines shown. The system is in equilibrium.

Question:
Determine the three unknown section forces M, V and N (with the correct sign for the directions shown).[1]

---

[1] M (bending moment), V (shear force) and N (normal force) are section forces. Their nomenclature and sign conventions will be revealed in Chapter 10.

Solution:
The two unknown section forces V and N are determined using the two equations for the force equilibrium. For the coordinate system shown,

$$\sum F_x = +(4\ \text{kN}) - \tfrac{2}{5}\sqrt{5} \times V + \tfrac{1}{5}\sqrt{5} \times N = 0,$$

$$\sum F_y = +(2\ \text{kN}) - (5\ \text{kN}) - \tfrac{1}{5}\sqrt{5} \times V - \tfrac{2}{5}\sqrt{5} \times N = 0.$$

These are two equations with two unknowns. The solution is

$$V = +\sqrt{5}\ \text{kN} \ \text{ and } \ N = -2\sqrt{5}\ \text{kN}.$$

It would also be possible to construct a closed force polygon and to derive the forces from there. This is shown in Figure 3.31. The force of $2\sqrt{5}$ kN on the line of action of N is active in an opposite direction to that shown in Figure 3.30. That is why there is a minus sign in the expression for N.

M is found using the equation for the moment equilibrium about an arbitrary point. If A is selected, the contribution of V and N to the moment is zero, and M can be found even if V and N are still unknown:

$$\sum T_z|\text{A} = -M - (4\ \text{kN})(1\ \text{m}) - (2\ \text{kN})(1.5\ \text{m}) + \\ +(5\ \text{kN})(0.5\ \text{m}) + (16\ \text{kNm}) = 0 \Rightarrow M = 11.5\ \text{kNm}.$$

If M had been calculated first, one would be able to derive V directly afterwards from, for example, the moment equilibrium about C:

$$\sum T_z|\text{C} = -(11.5\ \text{kNm}) - (2\ \text{kN})(1\ \text{m}) - V \times \left(\tfrac{1}{2}\sqrt{5}\ \text{m}\right) + (16\ \text{kNm}) = 0.$$

This again gives $V = +\sqrt{5}$ kN. As such, there are several ways to derive the unknown forces.

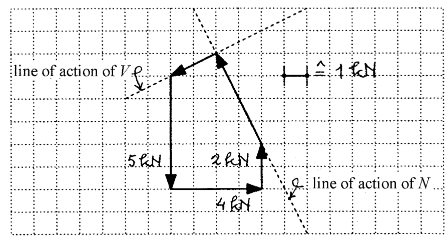


Figure 3.31 The closed force polygon represents the force equilibrium for the corner joint in the frame.

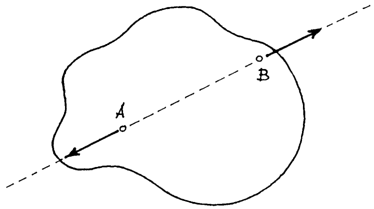


Figure 3.32 A body subject to two forces at two points.

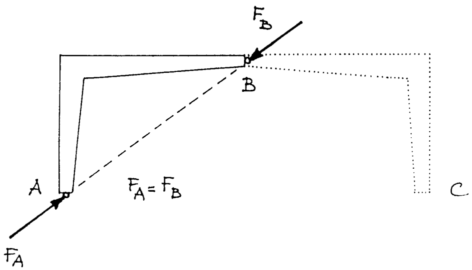


Figure 3.33 The left section of the three-hinged frame ABC, as an example of a body subject to two forces at two points.

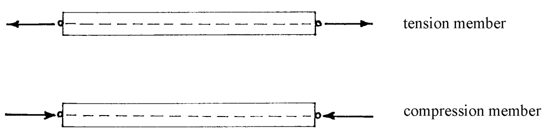


Figure 3.34 Two-force members are straight bars that can transfer forces only along their so-called bar axes.

The various guises of the equilibrium equations offer an important opportunity for performing *control calculations*. Checking results is necessary not only for manual calculations, but also for computer calculations.

3.2.2 Particular cases of equilibrium

In the analysis of the transfer of forces in structures, certain equilibrium systems are quite common. For a good insight into the behaviour of a structure, it is important to be able to quickly recognise three more or less particular cases of equilibrium. They are covered below.

1. A body subject to two forces at two points (see Figure 3.32).

A body subject to two forces can be in equilibrium only if both forces:
- have the same line of action,
- have the same magnitude, and
- have opposite directions.

If these three conditions are not all met, the two forces together form either a resultant force or a resultant couple, and the system will not be in equilibrium.

Figure 3.33 shows the left part AB of a so-called *three-hinged frame*. The foundation exerts a force F_A at A on AB, while the right part BC of the frame exerts a force F_B at B on AB. If we neglect the weight of the frame, the part AB of the frame is in equilibrium only if both forces F_A and F_B are equal and opposite, with AB as the common line of action.

Certain construction elements are intentionally designed to this type of force transfer. These are straight bars, only subject to a force at both ends (see Figure 3.34). Such bars, which can transfer forces only along their so-called bar axis, are called *two-force members*. Depending on whether they are loaded by tensile or compressive forces, they are also referred to as tension members or compression members.

When analysing structures, one must be able to recognise two-force members quickly. Structures made solely of two-force members are called *trusses*. Figure 3.35 is an example of a truss. In this truss, bar CD is loaded by compression. Calculating the forces in a truss is covered in detail in Chapter 9. It can be noted at this stage that, in a joint, the bars that come together exert forces on one another on the basis of the law of action and reaction. It is therefore possible for several forces to be exerted concurrently on the end of a bar. For example, the two compression forces on the ends of bar CD in Figure 3.35 are in fact the resultants of several forces.

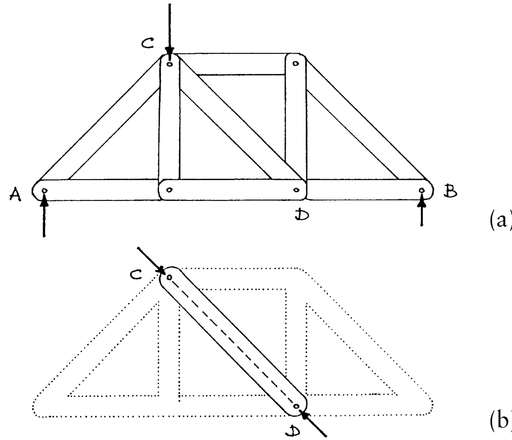


Figure 3.35 (a) The truss as a structure of two-force members; (b) the forces at the ends of compression member CD are the resultants of several forces.

Example

Two forces are exerted on the body in Figure 3.36a: F_A is exerted on A, F_B is exerted on B. Of F_A, only the horizontal component of 28 kN is given. The body is in equilibrium.

Question:
The magnitude and direction of F_B.

Solution:
If two forces are exerted on a body, the body can only be in equilibrium if the two forces have a common line of action, an equal magnitude and an opposite direction. In vector notation: $\vec{F}_A = -\vec{F}_B$. From the moment equilibrium about A, it follows that the common line of action of F_A and F_B is along AB (see Figure 3.36b). In that case, the horizontal component of F_B is $(4/5)F_B$. From the horizontal force equilibrium, it follows that:

$$\sum F_x = -(28 \text{ kN}) + \tfrac{4}{5}F_B = 0 \Rightarrow F_B = 35 \text{ kN}.$$

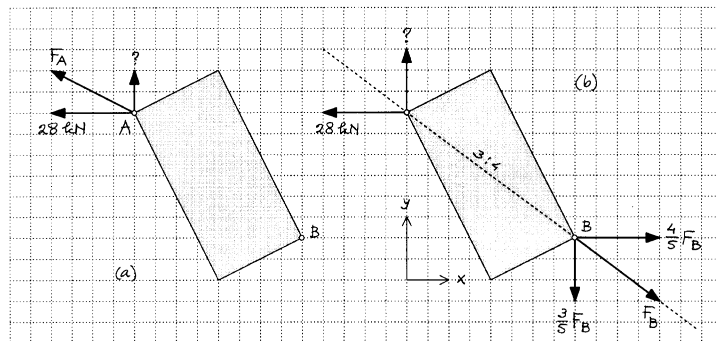


Figure 3.36 The body, subject to two forces F_A and F_B at A and B, is in equilibrium if these forces are equal and opposite and have the common line of action AB.

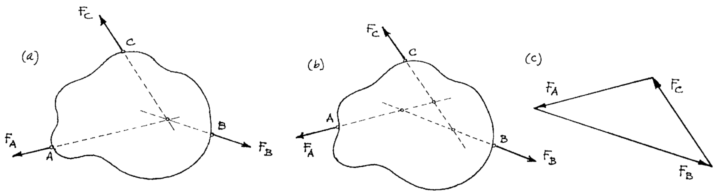


Figure 3.37 A body subject to three forces at three points: (a) moment equilibrium exists; (b) there is no moment equilibrium; (c) the closed force polygon shows that both bodies are in force equilibrium.

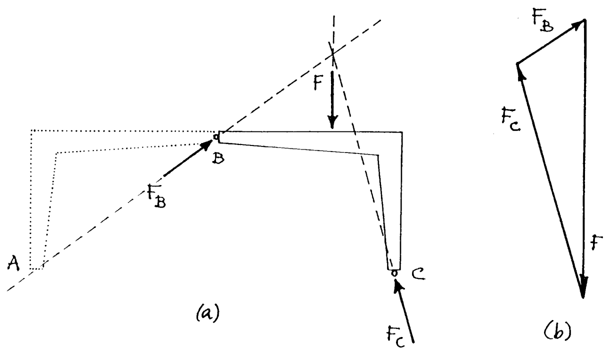


Figure 3.38 The right part of the three-hinged frame ABC as an example of a body subject to three forces at three points: (a) moment equilibrium exists because the lines of action pass through a single point and (b) there is force equilibrium because the forces form a closed force polygon.

2. A body subject to forces at three points (see Figure 3.37).

A body that is subject to three forces can be in equilibrium only if
- the three forces are coplanar,
- the forces form a closed force polygon (*force equilibrium*),[1] and
- their lines of action pass through a single point (*moment equilibrium*).

The same closed force polygon (c) is applicable for both bodies (a) and (b) in Figure 3.37: there is therefore force equilibrium in both cases. In case (a) there is moment equilibrium. This is easily checked by determining the moment of the three forces about the intersection of the three lines of action: none of the forces contribute to the sum of the moments. There is no moment equilibrium in case (b). The system of forces forms a resultant couple. The magnitude of the couple is determined by deriving the sum of the moments about the intersection of two lines of action.

Figure 3.38 shows the right-hand part BC of the *three-hinged frame*, mentioned earlier. This part of the frame is loaded by the vertical force F shown. In addition, the left frame part AB is exerting a force F_B at B on BC and the foundation is exerting a force F_C at C on BC. Moment equilibrium is only possible if the lines of action of the three forces F, F_B and F_C pass through a single point. The force equilibrium exists if the three forces form a closed force polygon.

Example
The block in Figure 3.39, loaded by two forces in C, is kept in equilibrium by the three forces A_h, A_v and B_v.

Question:
Determine these three forces and check the moment equilibrium and the

[1] It should be noted that three forces in space can only form a closed force polygon if they are acting in the same plane. The first condition is therefore actually superfluous as a result of the second.

force equilibrium graphically.

Solution:
The three unknown forces are determined using the three equilibrium equations:

$$\sum F_x = A_\mathrm{h} + (4\ \mathrm{kN}) = 0,$$

$$\sum F_y = A_\mathrm{v} + B_\mathrm{v} - (6\ \mathrm{kN}) = 0,$$

$$\sum T_z|\mathrm{A} = +B_\mathrm{v} \times (6\ \mathrm{m}) - (6\ \mathrm{kN})(8\ \mathrm{m}) - (4\ \mathrm{kN})(3\ \mathrm{m}) = 0.$$

The first equation gives $A_\mathrm{h} = -4$ kN, the third gives $B_\mathrm{v} = 10$ kN, and the second equation gives $A_\mathrm{v} = -4$ kN.

A_v can also be found directly from the moment equilibrium about B:

$$\sum T_z|\mathrm{B} = -A_\mathrm{v} \times (6\ \mathrm{m}) - (6\ \mathrm{kN})(2\ \mathrm{m}) - (4\ \mathrm{kN})(3\ \mathrm{m}) = 0 \Rightarrow A_\mathrm{v} = -4\ \mathrm{kN}.$$

The fact that A_h and A_v are negative means that they act in a direction opposite to the directions given in Figure 3.39.

In Figure 3.40a, the forces are depicted as they act on the block in reality. The block is subject to forces at three points:

- the resultant of the two forces at C, with line of action c,
- the force B_v at B, with line of action b, and
- the resultant of the forces A_h and A_v at A, with line of action a.

Graphical check of the moment equilibrium (see Figure 3.40a):
For a body subject to three forces, the lines of action of the three forces have to pass through a single point. This condition is met.

Graphical check of the force equilibrium (see Figure 3.40b):
There is force equilibrium if all the forces acting on the block form a closed force polygon. This is the case.

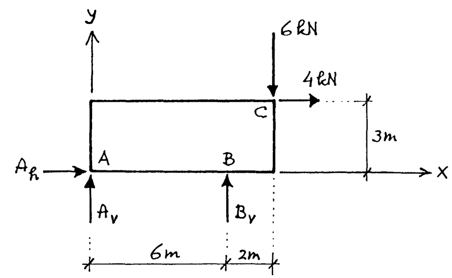


Figure 3.39 A block, loaded by two forces at C, is kept in equilibrium by the three forces A_h, A_v and B_v.

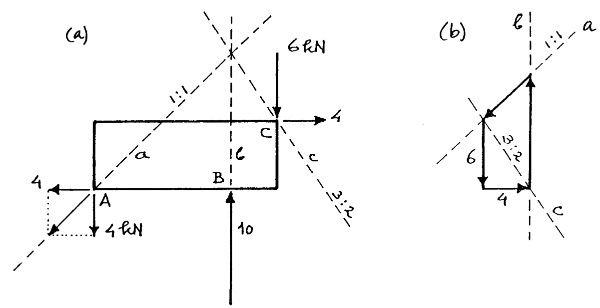


Figure 3.40 (a) Forces are exerted on the block at three points of which the lines of action pass through a single point, so that there is moment equilibrium; (b) since all the forces exerted on the block form a closed force polygon there is also force equilibrium.

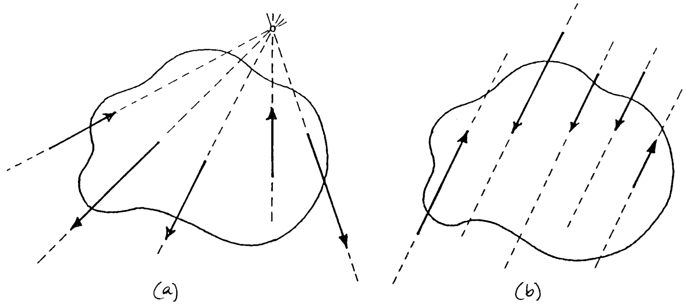


Figure 3.41 A body subject to several forces. (a) All the lines of action pass through a single point: there is moment equilibrium. (b) All the lines of action are parallel: there is force equilibrium in the direction perpendicular to these lines of action.

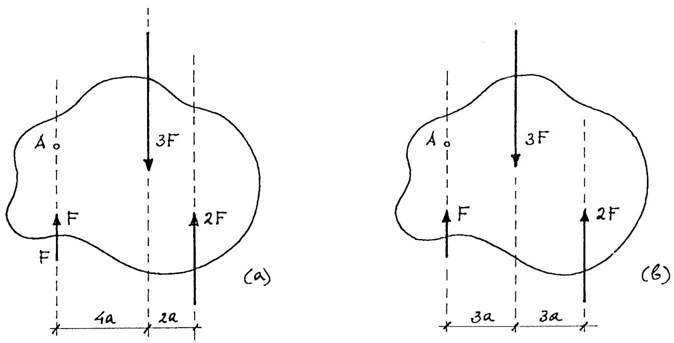


Figure 3.42 Both bodies are in force equilibrium; (a) there is moment equilibrium; (b) there is no moment equilibrium: the forces together form a couple (anti-clockwise) with magnitude $3Fa$.

3. A body subject to several forces of which the lines of action all pass through a single point (Figure 3.41a) or are all parallel (Figure 3.41b).

If for all the forces on a body the lines of action intersect at a single point, the moment equilibrium of the body is assured (see Figure 3.41a). The force equilibrium needs further investigation.

If for all the forces on a body the lines of action are parallel, the force equilibrium is assured in the direction perpendicular to the lines of action (see Figure 3.41b).

The force equilibrium in other directions, and the moment equilibrium needs further investigation.

Example
In Figure 3.42, both bodies are in force equilibrium. If one investigates the moment equilibrium by determining the sum of the moments of the forces, for example about A, it turns out that in case (a) the system is in moment equilibrium, while it is not in moment equilibrium in case (b). In case (a) the forces form an equilibrium system. In case (b), the forces form a couple acting anti-clockwise with magnitude $3Fa$. Nothing can be said about the sign associated with the direction of the couple until a coordinate system is chosen.

3.3 Forces and moments in space

So far, we looked at the equilibrium of a body only in the simple case in which all the forces and couples act in one plane. The moment was taken about a point in the same plane. In this section we look at the general three dimensional case. Here we have to define the concept of moment of forces and couples more generally.

3.3.1 Moment of a force about a point

Imagine a force $\vec{F}$ in space, with point of application B (see Figure 3.43). The moment $\vec{T}$ of this force about point A is now defined as the *vector product* (cross product) of the *position vector* $\vec{r}$, from A to B, and the *force vector* $\vec{F}$:

$$\vec{T} = \vec{r} \times \vec{F}.$$

The vector product of two vectors $\vec{r}$ and $\vec{F}$ is a vector with magnitude $rF \sin\theta$ and perpendicular to both $\vec{r}$ and $\vec{F}$. Here r and F are the magnitudes of $\vec{r}$ and $\vec{F}$ respectively, and θ is the smaller angle between the vectors $\vec{r}$ and $\vec{F}$ when both are drawn outwards from the same point.

There are two useful rules for finding the *direction* of the moment vector $\vec{T}$. The first is that it corresponds to the direction in which a *corkscrew* (with a right-hand screw) moves when the handle is turned from the first vector $\vec{r}$ to the second vector $\vec{F}$ through the angle θ (that is the direction of the rotation that the moment will cause about A) (see Figure 3.44). If necessary, the vectors will have to be shifted to the intersection of their lines of action. An alternative for finding the direction of $\vec{T}$ is the so-called *right-hand rule*: if one bends the fingers of the right hand to form a fist in the direction of the rotation that $\vec{F}$ would cause about A, then the thumb points in the direction of the moment vector.

In Figure 3.44, the vectors $\vec{r}$ and $\vec{F}$ are in the xy plane. The moment vector $\vec{T}$ is then parallel to the z axis. The figure also shows the perpendicular line AC from point A to the line of action of $\vec{F}$. The length of line segment AC is $r \sin\theta$, and the magnitude of the vector product is therefore equal to the product of the magnitude of the force and the distance from point A to the line of action of the force.[1] This corresponds to the definition of the

[1] Note that $\vec{T}$ is again independent of the location of the point of application B on the line of action of $\vec{F}$.

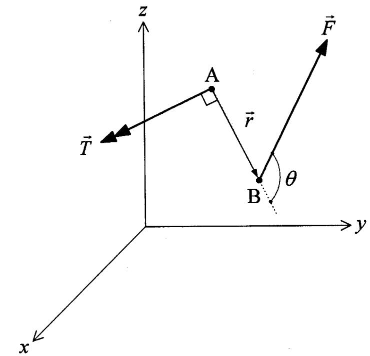


Figure 3.43 The moment of the force $\vec{F}$ about a point A is defined as the vector product $\vec{T} = \vec{r} \times \vec{F}$; the moment vector $\vec{T}$ is perpendicular to the plane through $\vec{r}$ and $\vec{F}$.

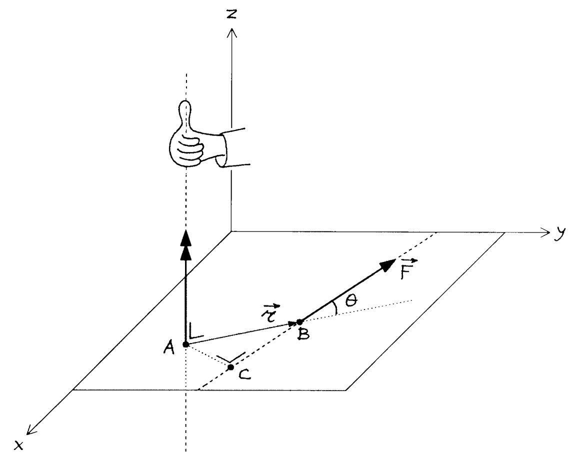


Figure 3.44 The direction of moment vector $\vec{T}$ is determined by the corkscrew rule or the right-hand rule.

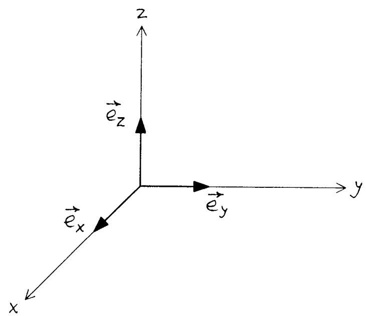


Figure 3.45 The unit vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$.

moment of a force about a point, as given in Section 3.1.5. In that section, the limitation to forces and points in the same plane was essential. Here the definition is more general.

In order to distinguish a *moment vector* from a *force vector* in an illustration, the moment vector is often given a double arrow point.

The vector product can also be effectively described by defining vectors according to components. For example:

$$\vec{r} = r_x\vec{e}_x + r_y\vec{e}_y + r_z\vec{e}_z,$$

$$\vec{F} = F_x\vec{e}_x + F_y\vec{e}_y + F_z\vec{e}_z.$$

For the vector products of the mutually perpendicular unit vectors as shown in Figure 3.45, the following relationships apply on the basis of the definition of a vector product:

$$\vec{e}_x \times \vec{e}_y = -\vec{e}_y \times \vec{e}_x = \vec{e}_z,$$

$$\vec{e}_y \times \vec{e}_z = -\vec{e}_z \times \vec{e}_y = \vec{e}_x,$$

$$\vec{e}_z \times \vec{e}_x = -\vec{e}_x \times \vec{e}_z = \vec{e}_y,$$

and

$$\vec{e}_x \times \vec{e}_x = \vec{e}_y \times \vec{e}_y = \vec{e}_z \times \vec{e}_z = 0.$$

The components of $\vec{T} = \vec{r} \times \vec{F}$ are therefore

$$T_x = r_y F_z - r_z F_y,$$

$$T_y = r_z F_x - r_x F_z,$$

$$T_z = r_x F_y - r_y F_x.$$

This definition for the components of the moment vector $\vec{T}$ is a generalisation of the definition of T_z as given in Section 3.1.5.

For the moment vector $\vec{T}$ and its components T_x, T_y and T_z it is again preferable to mention the point A about which the moment was determined, such as $\vec{T}|\mathrm{A}$, $T_x|\mathrm{A}$, and so forth.

An alternative notation for the moment vector $\vec{T}$ is

$$\vec{T} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}.$$

The components of $\vec{T}$ are found by developing the *determinant*.

Example

For the force $F = 65$ kN in Figure 3.46, the line of action ℓ passes through the points A(4, 0, 0) and B(0, 12, 3). The coordinates are expressed in metres.

Question:

Determine the moment of the force about point C(2, 6, 6).

Solution:

The units used are kN and m; they are not always shown in interim calculations.

First the components F_x, F_y and F_z are determined (see Section 2.2.1). Vector $\overline{\mathrm{AB}}$ (pointing from A to B) has the same direction as the force $\vec{F}$. If $\overline{\mathrm{AB}}$ is hereafter referred to as $\vec{d}$, then

$$\vec{d} = d_x\vec{e}_x + d_y\vec{e}_y + d_z\vec{e}_z = (-4\vec{e}_x + 12\vec{e}_y + 3\vec{e}_z)\ \mathrm{m},$$

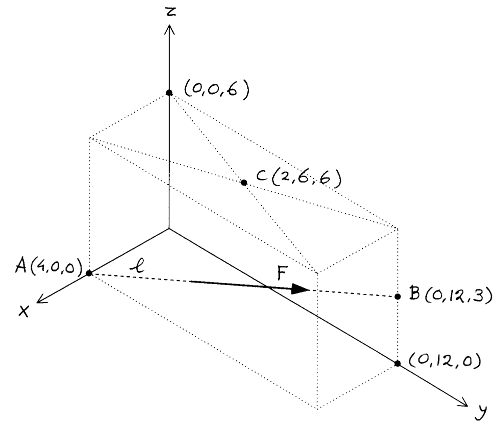


Figure 3.46 The line of action ℓ of force $F = 65$ kN passes through the points A(4, 0, 0) and B(0, 12, 3). The question relates to the moment of the force about point C(2, 6, 6). The coordinates are expressed in metres.

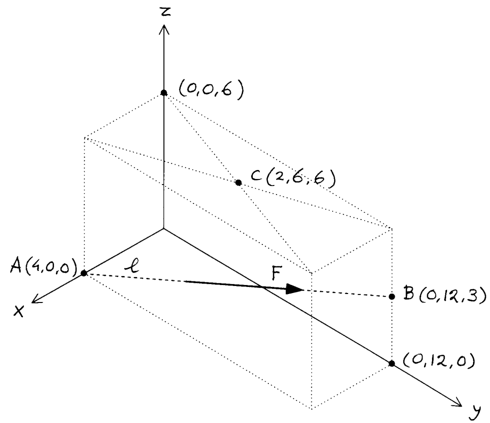


Figure 3.46 The line of action ℓ of force $F = 65$ kN passes through the points A(4, 0, 0) and B(0, 12, 3). The question relates to the moment of the force about point C(2, 6, 6). The coordinates are expressed in metres.

and

$$d = |\vec{d}| = \sqrt{(-4)^2 + 12^2 + 3^2} = 13 \text{ m}.$$

Since the direction cosines of $\vec{F}$ and $\vec{d}$ are equal

$$\cos\alpha_x = \frac{F_x}{F} = \frac{d_x}{d} \Rightarrow F_x = F\frac{d_x}{d} = 65 \times \tfrac{-4}{13} = -20 \text{ kN},$$

$$\cos\alpha_y = \frac{F_y}{F} = \frac{d_y}{d} \Rightarrow F_y = F\frac{d_y}{d} = 65 \times \tfrac{12}{13} = +60 \text{ kN},$$

$$\cos\alpha_z = \frac{F_z}{F} = \frac{d_z}{d} \Rightarrow F_z = F\frac{d_z}{d} = 65 \times \tfrac{3}{13} = +15 \text{ kN}.$$

$\vec{F}$ can now be defined according to its components:

$$\vec{F} = (-20\vec{e}_x + 60\vec{e}_y + 15\vec{e}_z) \text{ kN}.$$

Imagine $\vec{F}$ is exerted at point A, then

$$\vec{r} = \overline{\text{CA}} = r_x\vec{e}_x + r_y\vec{e}_y + r_z\vec{e}_z = (+2\vec{e}_x - 6\vec{e}_y - 6\vec{e}_z) \text{ m},$$

and for the moment of $\vec{F}$ with respect to C

$$\vec{T}|\text{C} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ +2 & -6 & -6 \\ -20 & +60 & +15 \end{vmatrix}.$$

This gives the following components:

$$T_x|\text{C} = r_yF_z - r_zF_y = (-6 \text{ m})(+15 \text{ kN}) - (-6 \text{ m})(+60 \text{ kN})$$
$$= +270 \text{ kNm},$$

$$T_y|\mathrm{C} = r_z F_x - r_x F_z = (-6\text{ m})(-20\text{ kN}) - (+2\text{ m})(+15\text{ kN})$$
$$= +90\text{ kNm},$$
$$T_z|\mathrm{C} = r_x F_y - r_y F_x = (+2\text{ m})(+60\text{ kN}) - (-6\text{ m})(-20\text{ kN})$$
$$= 0\text{ kNm}.$$

To show that the moment of force $\vec{F}$ with respect to C is independent of the point of application on its line of action, the following represents an example in which $\vec{F}$ is exerted at B. In that case

$$\vec{r} = \overline{\mathrm{CB}} = (-2\vec{e}_x + 6\vec{e}_y - 3\vec{e}_z)\text{ m}.$$

The result of

$$\vec{T}|\mathrm{C} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -2 & +6 & -3 \\ -20 & +60 & +15 \end{vmatrix}$$

does indeed give the same values:

$$T_x|\mathrm{C} = r_y F_z - r_z F_y = (+6\text{ m})(+15\text{ kN}) - (-3\text{ m})(+60\text{ kN})$$
$$= +270\text{ kNm},$$
$$T_y|\mathrm{C} = r_z F_x - r_x F_z = (-3\text{ m})(-20\text{ kN}) - (-2\text{ m})(+15\text{ kN})$$
$$= +90\text{ kNm},$$
$$T_z|\mathrm{C} = r_x F_y - r_y F_x = (-2\text{ m})(+60\text{ kN}) - (+6\text{ m})(-20\text{ kN})$$
$$= 0\text{ kNm}.$$

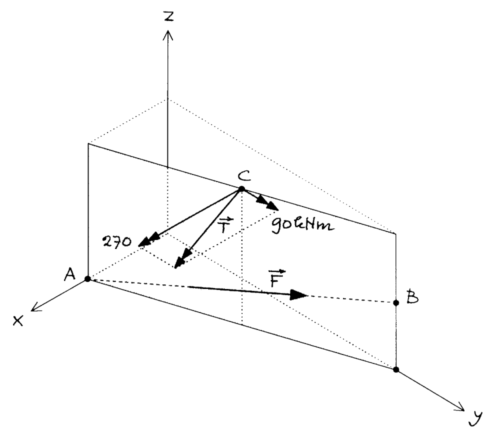


Figure 3.47 The components of vector $\vec{T}$ for the moment of $\vec{F}$ about point C. The moment vector $\vec{T}$ is perpendicular to plane ABC.

Figure 3.47 shows the components of the moment vector in C. The moment vector $\vec{T}$ lies in the horizontal plane through C. Further consideration shows

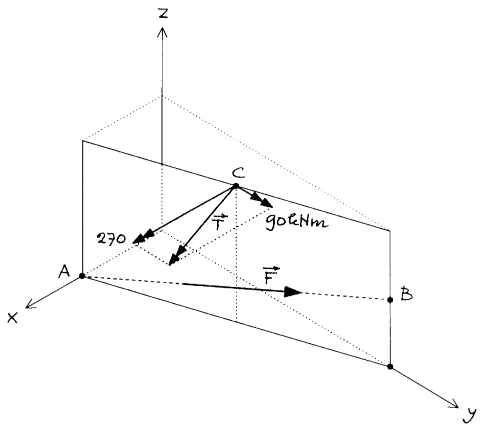


Figure 3.47 The components of vector $\vec{T}$ for the moment of $\vec{F}$ about point C. The moment vector $\vec{T}$ is perpendicular to plane ABC.

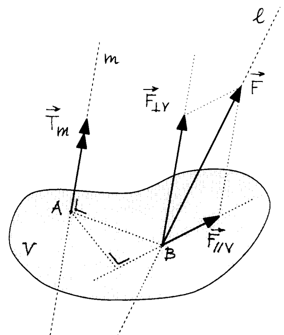


Figure 3.48 The moment of the force $\vec{F}$ about line m is defined as the moment $\vec{T}_m$ of the projection $\vec{F}_{//V}$ of $\vec{F}$ on a plane V perpendicular to m about point A, the intersection of V and m.

that $\vec{T}$ is indeed perpendicular to plane ABC. The magnitude of the resultant moment about C is

$$T|\text{C} = \sqrt{270^2 + 90^2} = 90\sqrt{10} = 284.6 \text{ kNm}.$$

3.3.2 Moment of a force about a line

Figure 3.48 shows a force $\vec{F}$ with line of action ℓ, and a line m. The lines ℓ and m will generally cross one another and not be perpendicular. Imagine V is an arbitrary plane perpendicular to m. The lines ℓ and m intersect the plane V at B and A respectively. In Figure 3.48, it has been assumed that $\vec{F}$ is applied at B. As shall become clear in a moment, $\vec{F}$ may also be applied elsewhere on ℓ.

$\vec{F}$ can be resolved into a component $\vec{F}_{\perp V}$ perpendicular to plane V and so parallel to m, and a component $\vec{F}_{//V}$ in plane V. If $\vec{F}$ is not applied at B, $\vec{F}_{//V}$ is the projection of $\vec{F}$ on V. The line of action of $\vec{F}_{//V}$ is the projection of the line of action ℓ of $\vec{F}$ on V. Wherever one places the plane V perpendicular to m, the line of action of $\vec{F}_{//V}$ always remains the same.

The moment $\vec{T}_m$ of the force $\vec{F}$ about line m has now been defined as the moment of the projection $\vec{F}_{//V}$ of $\vec{F}$ on a plane V perpendicular to m with respect to the intersection A of V and m.

For the components of $\vec{T}|\text{A}$, the moment of $\vec{F}$ about point A in a xyz coordinate system, we have earlier derived that

$$T_x|\text{A} = r_y F_z - r_z F_y,$$

$$T_y|\text{A} = r_z F_x - r_x F_z,$$

$$T_z|\text{A} = r_x F_y - r_y F_x.$$

Here one recognises the moment about three lines through A, parallel to the

x, y and z axis respectively.

Comment:
For a moment about the origin O of the coordinate system or a moment about one of the coordinate axes, the point O is generally omitted in the representation of the moment.

Example
The curved beam AB in Figure 3.49 is loaded at B by a force of which the components are defined with respect to magnitude and direction in the figure.

Question:
Find the moment about the x, y and z axis respectively of the force(s) at B.

Solution:

$$T_x = +(25\text{ kN})(3\text{ m}) - (50\text{ kN})(1\text{ m}) = +25\text{ kNm},$$

$$T_y = +(40\text{ kN})(1\text{ m}) - (25\text{ kN})(2\text{ m}) = -10\text{ kNm},$$

$$T_z = +(50\text{ kN})(2\text{ m}) - (40\text{ kN})(3\text{ m}) = -20\text{ kNm}.$$

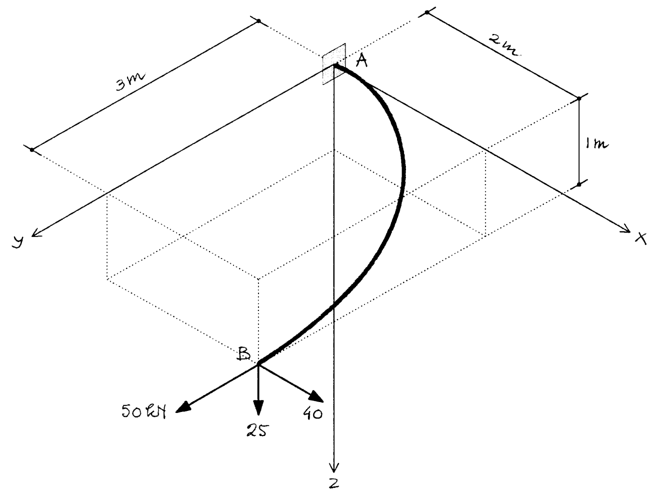


Figure 3.49 A curved beam AB is loaded at B by the three components of a force.

3.3.3 Moment of a couple

Two parallel forces that are equal and opposite form a couple (see Section 3.1.4). Figure 3.50 shows two forces $\vec{F}_1 = \vec{F}$ and $\vec{F}_2 = -\vec{F}$, forming a couple in space.[1]

For the moment of the couple about a point A we have

$$\vec{T}|\mathrm{A} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times F_2 = \vec{r}_1 \times \vec{F} + \vec{r}_2 \times (-\vec{F}) = (\vec{r}_1 - \vec{r}_2) \times \vec{F}$$
$$= \vec{r} \times \vec{F}.$$

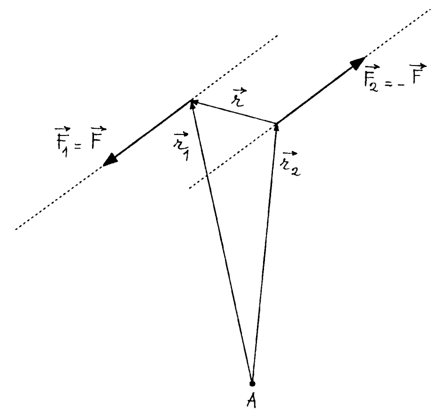


Figure 3.50 The moment of the couple about a point A is $\vec{T} = \vec{r} \times \vec{F}$. This moment is independent of the location of point A.

[1] There is no resultant force, for $\vec{F}_1 + \vec{F}_2 = 0$.

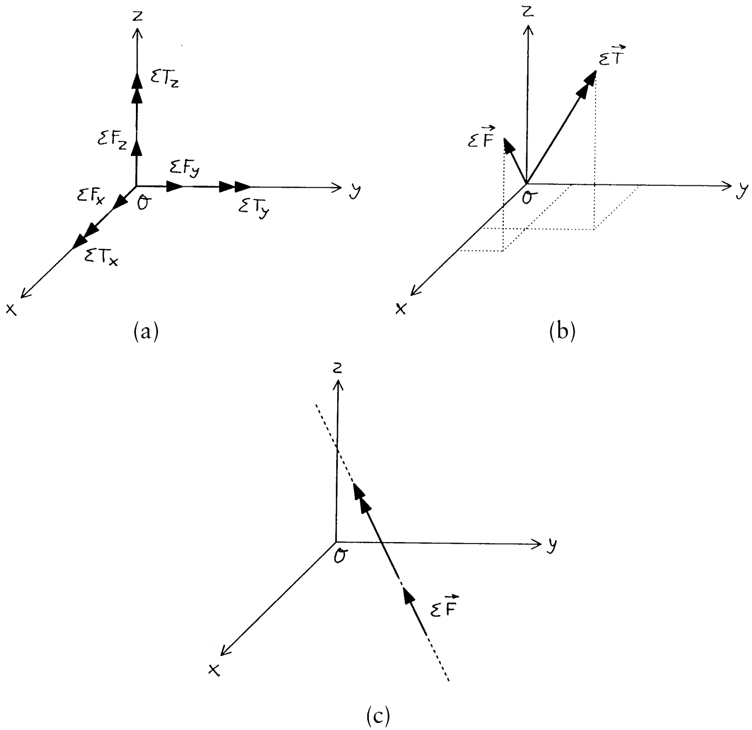


Figure 3.51 (a) The components of a resultant force $\sum \vec{F}$ in O and a resultant couple $\sum \vec{T}$. (b) The resultant force vector and the resultant moment vector need not necessarily have the same direction. (c) By shifting the resultant force $\sum \vec{F}$ parallel to itself one can provide that the resultant moment vector has the same direction as the force vector. The combination of a force and a moment of which the vectors have the same direction is called a *screw*.

The moment of the couple is equal to the moment of the one force about an arbitrary point on the line of action of the other force. The moment is independent of the location of point A about which it was originally determined. This means that the moment of a couple is a *free vector*. The moment vector of the couple is perpendicular to the plane in which the couple acts.

3.3.4 Compounding forces and couples

Compounding forces and couples in space is analytically relatively simple. Each of the forces F_i ($i = 1, 2, \ldots, n$) can be resolved into the components $F_{x;i}$; $F_{y;i}$; $F_{z;i}$ and for each of these forces, one can determine the moment with respect to an arbitrary point A. In fact, this means that all the forces with the addition of a couple, are shifted to that point A (see Section 3.1.5).

If we place the origin O of the coordinate system at A, and x_i, y_i, z_i are the coordinates of the point of application of force F_i (or of another point on the line of action of F_i), then:

$$\sum F_x = \sum_{i=1}^{n} F_{x;i}, \qquad \sum T_x = \sum_{i=1}^{n} \{(y_i F_{z;i} - z_i F_{y;i}) + T_{x;i}\},$$

$$\sum F_y = \sum_{i=1}^{n} F_{y;i}, \qquad \sum T_y = \sum_{i=1}^{n} \{(z_i F_{x;i} - x_i F_{z;i}) + T_{y;i}\},$$

$$\sum F_z = \sum_{i=1}^{n} F_{z;i}, \qquad \sum T_z = \sum_{i=1}^{n} \{(x_i F_{y;i} - y_i F_{x;i}) + T_{z;i}\}.$$

The moment sum also includes the moments of any (concentrated) couples T_i that act on the body.

$\sum F_x$, $\sum F_y$ and $\sum F_z$ are the components of the resultant force $\sum \vec{F}$ in O while $\sum T_x$, $\sum T_y$ and $\sum T_z$ are the components of a resultant couple $\sum \vec{T}$ (see Figure 3.51a). The resultant force vector $\sum \vec{F}$ at O and the resultant moment vector $\sum \vec{T}$ need not necessarily have the same direction (see Figure 3.51b).

By shifting the resultant force $\sum \vec{F}$ parallel to itself one can provide that the resultant force vector and moment vector have the same direction (see Figure 3.51c). The combination of a force and a moment of which the vectors have the same direction is called a *screw*.[1]

The following represents three examples that relate to the determination of the resultant of a number of forces and/or couples.

Example 1

A flat slab of $6 \times 5\ \text{m}^2$ in the horizontal xy plane is loaded by six vertical forces (see Figure 3.53a). The grid lines are 1 m apart.

Question:

Determine the resultant force R as to magnitude and direction and the location at which it acts on the slab.

Solution:

The units used are kN and m. The units are omitted in the interim calculations.

The x and y components of all the forces given are zero, as are their moments about the z axis, therefore

$$\sum F_x = 0,$$

$$\sum F_y = 0,$$

$$\sum T_z = 0.$$

[1] Reducing a system of forces and couples into a screw is an interesting academic problem, but is of little practical use and therefore not covered in further detail.

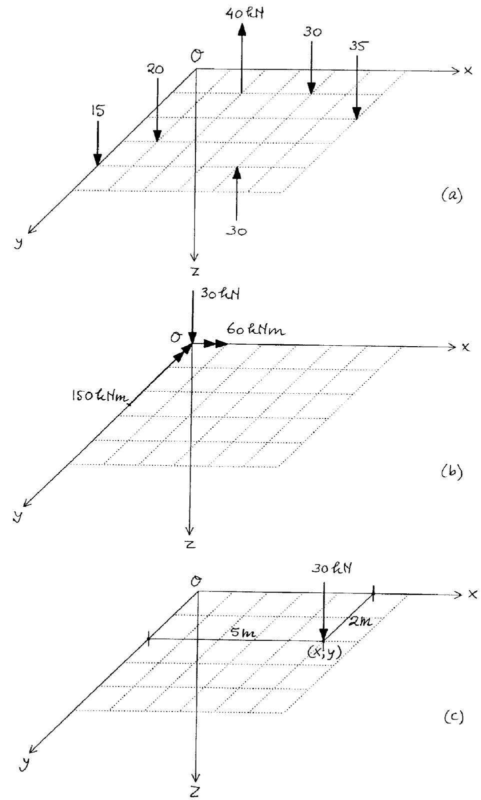


Figure 3.52 (a) A flat slab of $6 \times 5\ \text{m}^2$ in the horizontal xy plane is loaded by six vertical forces. The grid lines are 1 m apart. (b) The system of forces is statically equivalent with a vertical force $R = 30$ kN at O pointing downwards, together with two couples of 150 kNm and 60 kNm of which the moment vectors are along the x and y axis respectively, or (c) with only a force $R = 30$ kN at the point ($x = 5$ m, $y = 2$ m).

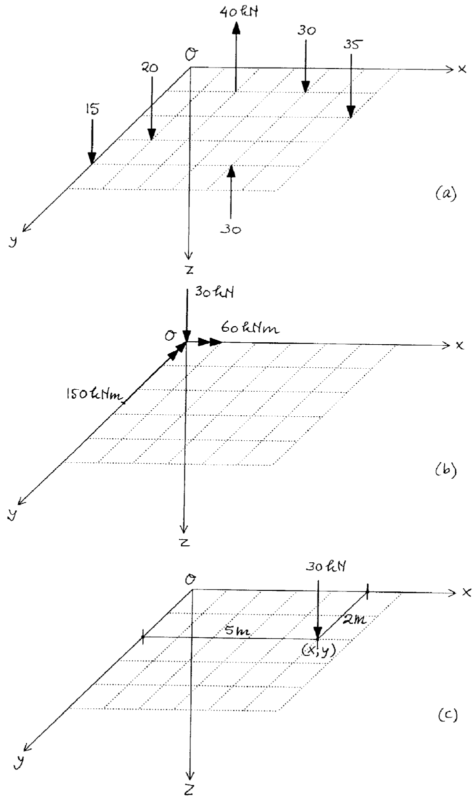


Figure 3.52 (a) A flat slab of 6×5 m^2 in the horizontal xy plane is loaded by six vertical forces. The grid lines are 1 m apart. (b) The system of forces is statically equivalent with a vertical force $R = 30$ kN at O pointing downwards, together with two couples of 150 kNm and 60 kNm of which the moment vectors are along the x and y axis respectively, or (c) with only a force $R = 30$ kN at the point ($x = 5$ m, $y = 2$ m).

In addition,

$$\sum F_z = +15 + 20 - 40 + 30 - 30 + 35 = +30 \text{ kN},$$

$$\sum T_x = -40 \times 1 + 30 \times 1 + 35 \times 2 + 20 \times 3 + 15 \times 4 - 30 \times 4$$

$$= +60 \text{ kNm},$$

$$\sum T_y = 15 \times 0 - 20 \times 1 + 40 \times 2 - 30 \times 4 + 30 \times 4 - 35 \times 6$$

$$= -150 \text{ kNm}.$$

So the system of forces can be replaced by a downward force $R = 30$ kN at O, together with two couples of 150 kNm and 60 kNm of which the moment vectors are along the x and y axis respectively (see Figure 3.53b).

Since the moment vectors are perpendicular to force R, they can be eliminated by shifting R to another point of application. Imagine (x, y) is the new point of application (see Figure 3.53c). We can find (x, y) from the condition that $R = 30$ kN has to generate the same moment about the x and y axis as all the forces together, so that

$$\sum T_x = Ry = 60 \text{ kNm} \quad \Rightarrow \quad y = \frac{60 \text{ kNm}}{R} = \frac{60 \text{ kNm}}{30 \text{ kN}} = 2 \text{ m},$$

$$\sum T_y = -Rx = -150 \text{ kNm} \quad \Rightarrow \quad x = \frac{150 \text{ kNm}}{R} = \frac{150 \text{ kNm}}{30 \text{ kN}} = 5 \text{ m}.$$

Example 2

In Figure 3.53a, a flat slab of 6×5 m^2 in the horizontal xy plane is loaded by five vertical forces. The distance between the grid lines is 1 m.

Question:

Determine the resultant of this set of forces.

Solution:

The units used are kN and m. The units are omitted in the interim calculations.

When determining the resultant of this system of parallel forces, only the force sum in the z direction and the moment sum about the x and y axis are relevant:

$$\sum F_y = +25 - 15 + 20 - 45 + 15 = 0 \text{ kN},$$

$$\sum T_x = 25 \times 0 + 15 \times 1 - 20 \times 3 - 15 \times 4 + 15 \times 4 + 45 \times 4$$

$$= +75 \text{ kN},$$

$$\sum T_z = +25 \times 6 - 15 \times 2 + 20 \times 5 - 45 \times 3 + 15 \times 1$$

$$= +100 \text{ kN}.$$

There is no resultant force, but there is a resultant couple T of which the moment vector is in the xy plane (see Figure 3.53b). Its magnitude is

$$T = \sqrt{75^2 + 100^2} = 125 \text{ kNm}.$$

The resultant couple acts in a plane perpendicular to the moment vector.

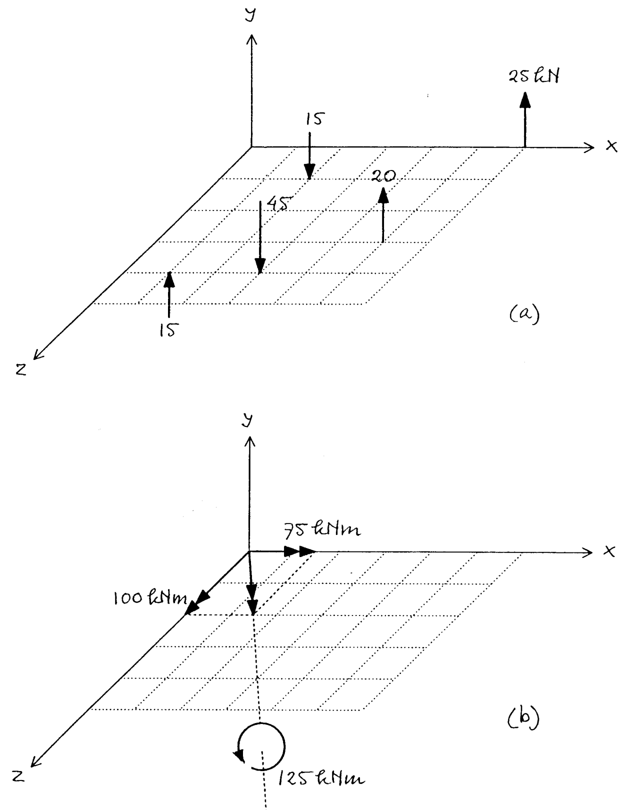


Figure 3.53 (a) A flat slab of 6×5 m^2 in the horizontal xz plane is loaded by five vertical forces. The grid lines are 1 m apart. (b) There is no resultant force, but there is a resultant couple of which the moment vector is in the xz plane. The resultant couple acts in a plane perpendicular to the moment vector.

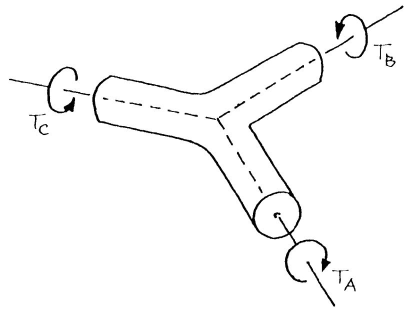


Figure 3.54 A junction of three coplanar tubes that are rigidly connected at equal angles of 120°. The tubes are loaded (*by torsion*) by the couples T_A, T_B and T_C.

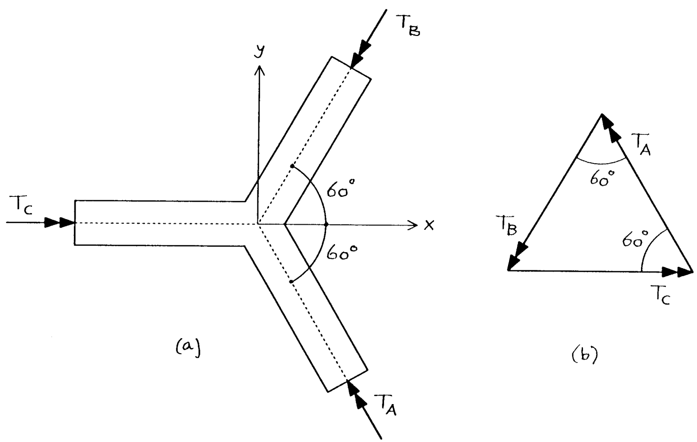


Figure 3.55 (a) The couples acting on the junction represented by their moment vectors. (b) If there is no resultant couple, the three moment vectors must form a closed polygon.

Example 3

Figure 3.54 shows a junction of three coplanar tubes that are rigidly connected at equal angles of 120°. The tubes are loaded (by *torsion*) by the couples T_A, T_B and T_C. The resultant couple on the junction is zero.

Question:
How large are the couples T_A and T_B if $T_C = 75$ Nm?

Solution:
In Figure 3.55a, the couples are represented by their moment vectors. The three vectors are in the xy plane, the plane in which the tubes are located. The resultant moment on the junction is zero if the three vectors form a closed polygon, analogous to the closed force polygon for force equilibrium. The equilateral triangle in Figure 3.55b gives

$$T_A = T_B = T_C = 75 \text{ Nm}.$$

This can of course also be determined analytically. If there is no resultant couple, then

$$\sum T_x = -\tfrac{1}{2}T_A - \tfrac{1}{2}T_B + T_C = -\tfrac{1}{2}T_A - \tfrac{1}{2}T_B + (75 \text{ Nm}) = 0,$$

$$\sum T_y = +\tfrac{1}{2}T_A\sqrt{3} - \tfrac{1}{2}T_B\sqrt{3} = 0.$$

The result of these two equations is again

$$T_A = T_B = 75 \text{ Nm}.$$

3.4 Equilibrium of a rigid body in space

Generalising the equilibrium equations for a rigid body is relatively simple. After all, equilibrium demands that both the resultant force and the resultant moment about an arbitrary point A are zero. This means that the following requirements have to be met by the forces and moments exerted on a rigid body at rest:

$$\sum F_x = 0,$$

$$\sum F_y = 0,$$

$$\sum F_z = 0,$$

$$\sum T_x|\mathrm{A} = 0,$$

$$\sum T_y|\mathrm{A} = 0,$$

$$\sum T_z|\mathrm{A} = 0.$$

The first three equations state that there is *force equilibrium* in the x, y and z directions respectively, and that the body is therefore not subject to translation acceleration. The latter three equations define that there is *moment equilibrium* at A about lines parallel to respectively the x, y and z axis, and that the body is not subject to rotational acceleration.

The following examples address the equilibrium of a body in space.

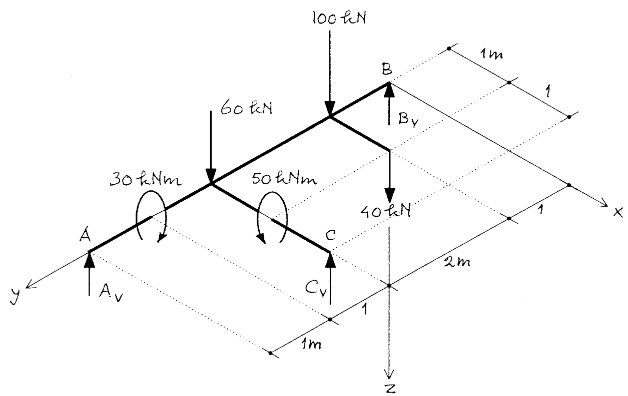


Figure 3.56 A structure consisting of a system of mutually perpendicular beams in the horizontal xy plane that is loaded perpendicularly to its plane by a number of forces and couples. The unknown forces A_v, B_v and C_v have to be derived from the equilibrium.

Example 1

The structure in Figure 3.56 consists of a number of mutually perpendicular beams in the horizontal xy plane that are loaded at the locations shown by three vertical forces of respectively 40, 60 and 100 kN and by two couples of 30 and 50 kNm. The structure is kept in equilibrium by the three vertical forces A_v, B_v and C_v.

Question:
Determine these three unknown forces.

Solution:
Since all the forces are parallel to the z axis,

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0.$$

The moment vectors of both couples are in the xy plane, so that in addition

$$\sum T_z = 0.$$

To determine the three unknown forces, we can use the following three equilibrium equations:

$$\sum F_z = 0, \quad \sum T_x = 0 \quad \text{and} \quad \sum T_y = 0.$$

By choosing the equilibrium equations carefully, and by applying them in a carefully chosen order, it is sometimes possible to cut back on the amount of calculation needed.

C_v is derived directly from $\sum T_y = 0$:

$$\sum T_y = -(30\text{ kNm}) - (40\text{ kN}) \times (1\text{ m}) + C_v \times 2 = 0$$

$$\Rightarrow C_v = +35\text{ kN}.$$

Next, we find A_v directly from $\sum T_x = 0$:

$$\sum T_x = -A_\mathrm{v} \times (5\ \mathrm{m}) + (60\ \mathrm{kN}) \times (3\ \mathrm{m}) + (50\ \mathrm{kNm})$$
$$-C_\mathrm{v} \times (3\ \mathrm{m}) + (100\ \mathrm{kN}) \times (1\ \mathrm{m}) + (40\ \mathrm{kN}) \times (1\ \mathrm{m}) = 0$$
$$\Rightarrow A_\mathrm{v} = +53\ \mathrm{kN},$$

after which B_v follows directly from $\sum F_z = 0$:

$$\sum F_z = +\{(100 + 40 + 60)\ \mathrm{kN}\} - A_\mathrm{v} - B_\mathrm{v} - C_\mathrm{v} = 0$$
$$\Rightarrow B_\mathrm{v} = +112\ \mathrm{kN}.$$

Figure 3.57 shows the forces A_v, B_v and C_v as they act on the structure in reality.

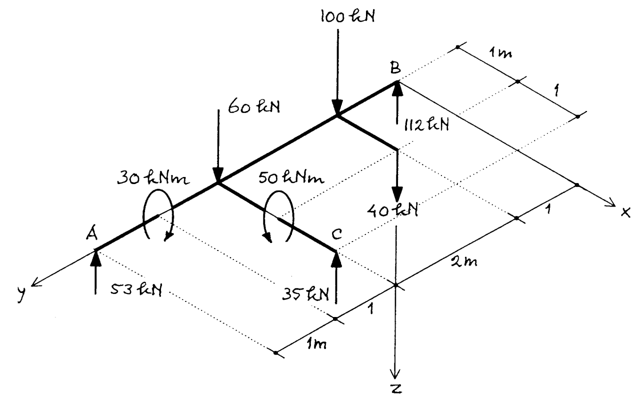


Figure 3.57 The forces A_v, B_v and C_v as they are actually acting on the structure.

To check, one could also have a look at the moment equilibrium at a point other than the origin, such as point A:

$$\sum T_x|\mathrm{A} = -\{(60 - 35)\ \mathrm{kN}\} \times (2\ \mathrm{m}) + (50\ \mathrm{kNm}) -$$
$$-\{(100 + 40)\ \mathrm{kN}\} \times (4\ \mathrm{m}) + (112\ \mathrm{kN}) \times (5\ \mathrm{m}) = 0.$$

The moment equilibrium is also met about a line through A parallel to the x axis.

Example 2
In Figure 3.58, a cube with edge length a and weight G is kept in equilibrium by the six forces F_1 to F_6. For the angle α between the lines of action of the forces applies $\tan\alpha = 3/4$.

Question:
Determine the six forces F_1 to F_6 if $a = 1$ m and $G = 24$ kN.

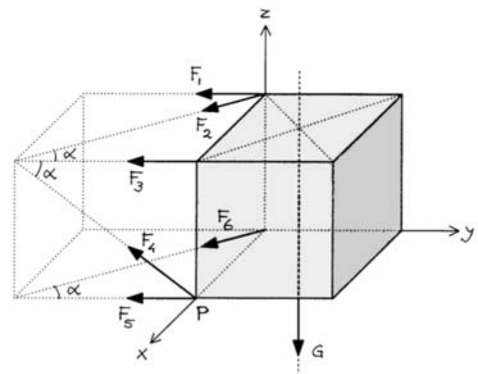


Figure 3.58 A cube with edge length a and weight G is kept in equilibrium by six forces F_1 to F_6. For the angle α between the lines of action of the forces applies $\tan\alpha = 3/4$.

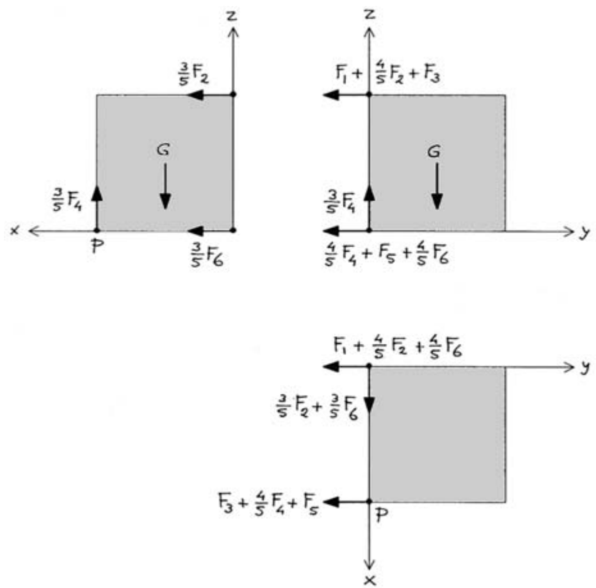


Figure 3.59 All the forces acting on the cube projected on the three coordinate planes.

Solution:
When writing down the equilibrium equations, it can sometimes be useful to project all the forces on the three coordinate planes (see Figure 3.59). In doing so, the forces F_2, F_4 and F_6 are resolved into components according to the coordinate directions. Using Figure 3.59 one finds

$$\sum F_x = \tfrac{3}{5}F_2 + \tfrac{3}{5}F_6 = 0, \tag{a}$$

$$\sum F_y = F_1 + \tfrac{4}{5}F_2 + F_3 + \tfrac{4}{5}F_4 + F_5 + \tfrac{4}{5}F_6 = 0, \tag{b}$$

$$\sum F_z = \tfrac{3}{5}F_4 - G = 0, \tag{c}$$

$$\sum T_x = \left(F_1 + \tfrac{4}{5}F_2 + F_3\right) \cdot a - G \cdot \tfrac{1}{2}a = 0, \tag{d}$$

$$\sum T_y = \tfrac{3}{5}F_2 \cdot a - \tfrac{3}{5}F_4 \cdot a + G \cdot \tfrac{1}{2}a = 0, \tag{e}$$

$$\sum T_z = -\left(F_3 - \tfrac{4}{5}F_4 + F_5\right) \cdot a = 0. \tag{f}$$

Equation (c) gives

$$F_4 = \tfrac{5}{3}G = 40 \text{ kN}.$$

Using this, one finds from equation (e)

$$F_2 = \tfrac{5}{6}G = 20 \text{ kN}$$

and then from equation (a)

$$F_6 = -\tfrac{5}{6}G = -20 \text{ kN}.$$

Determining the forces F_1, F_2 and F_3 from the three remaining equations (b), (d) and (f) demands some arithmetic. Sometimes one can reduce the amount of calculation by looking at the moment equilibrium about another point. Also here:

$$\sum T_z|\mathrm{P} = -\left(F_1 + \tfrac{4}{5}F_2 + \tfrac{4}{5}F_6\right)\cdot a = 0 \qquad \text{(g)}$$

so that

$$F_1 = 0.$$

Equation (d) now gives

$$F_3 = -\tfrac{1}{6}G = -4\ \text{kN}.$$

Finally, equation (f) gives

$$F_5 = -\tfrac{7}{6}G = -28\ \text{kN}.$$

Figure 3.60 depicts the forces (in kN) as they are acting on the cube in reality. The forces F_3, F_5 and F_6 act in directions opposite to those shown in Figure 3.58.

By using alternative equilibrium equation (g), equation (b) for the force equilibrium in y direction was not used, and can be used as a check. With the forces expressed in kN this gives

$$\begin{aligned}\sum F_y &= F_1 + \tfrac{4}{5}F_2 + F_3 + \tfrac{4}{5}F_4 + F_5 + \tfrac{4}{5}F_6\\ &= 0 + \tfrac{4}{5}\times 20 - 4 + \tfrac{4}{5}\times 40 - 28 - \tfrac{4}{5}\times 20 = 0.\end{aligned}$$

The conditions for force equilibrium in y direction are met.

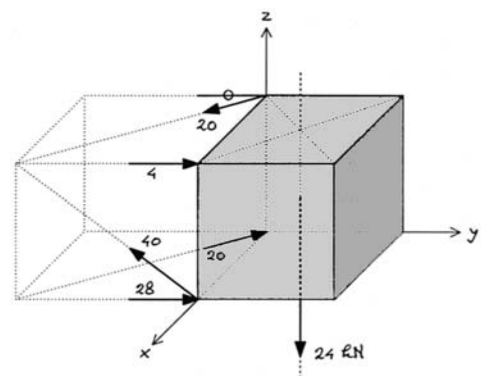


Figure 3.60 The forces (in kN) as they are acting on the cube.

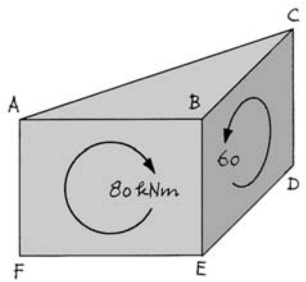


Figure 3.61 The cube, which has been halved diagonally, is subject to a couple of 80 kNm in plane ABEF and a couple of 60 kNm in plane BCDE. The body is kept in equilibrium by a couple on the diagonal plane ACDF.

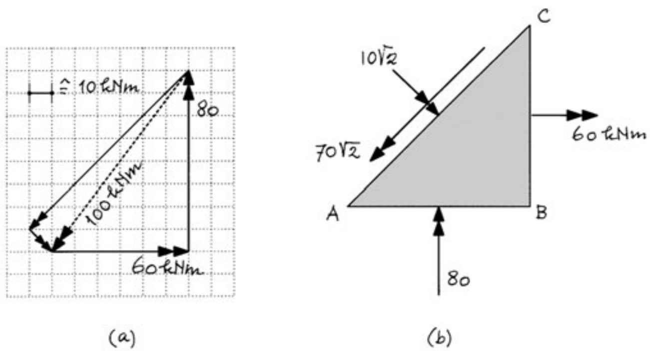


Figure 3.62 (a) If there is moment equilibrium, the moment vectors form a closed polygon. (b) Top view of the diagonally-halved cube with the moment vectors acting on it.

Example 3

The cube that has been halved diagonally in Figure 3.61 is subject to a couple of 80 kNm in plane ABEF and a couple of 60 kNm in plane BCDE. The directions are shown in the figure. The body is kept in equilibrium by a couple on the diagonal plane ACDF.

Question:
Determine the magnitude of that couple and resolve it into a component in plane ACDF and a component perpendicular to plane ACDF.

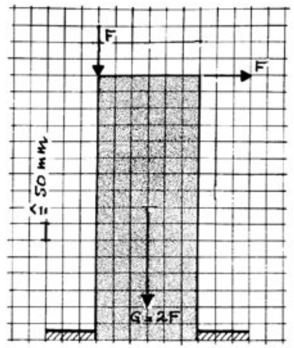
Solution:
There is moment equilibrium if the moment vectors form a closed polygon. The polygon in Figure 3.62a shows that a couple of 100 kNm is acting on plane ACDF. Of this couple, the moment vector has a component perpendicular to plane ACDF of $10\sqrt{2}$ kNm and a component along plane ACDF of $70\sqrt{2}$ kNm. Figure 3.62b shows the top view for the halved cube, with all the moment vectors that act on it.

When interpreting these results, one should remember that the moment vector is perpendicular to the plane on which the couple is exerted. The component of the couple that *is acting in the diagonal plane* has a moment vector perpendicular to that plane and is $10\sqrt{2}$ kNm. The component of the couple *that is acting perpendicular to the diagonal plane* has its moment vector in that plane and is $70\sqrt{2}$ kNm.

3.5 Problems

Compounding forces graphically (Sections 3.1.2 and 3.1.3)

3.1 The line of action of the resultant of the two forces F and the weight $G = 2F$ of the block intersect the right-hand side of the block at a distance a from the top.

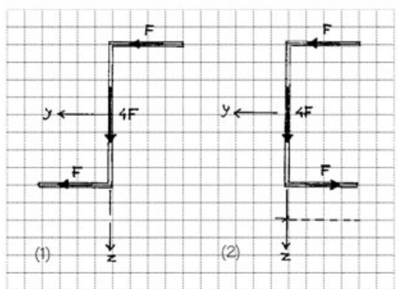


Question:
How large is a?

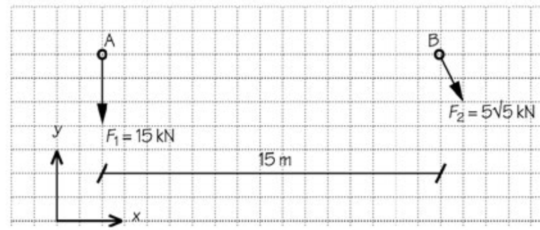
3.2: 1–2 The forces shown are exerted in the web and flange of a thin-walled profile. Length scale: 1 square $\equiv$ 25 mm.

Questions:

a. Using a force polygon determine the magnitude and direction of the resultant of these forces.
b. How large are the components of the resultant in the yz coordinate system shown?
c. Using a line of action figure, determine the location of the line of action of the resultant; where does this line of action intersect the y axis?



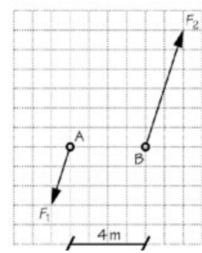
3.3 The forces F_1 and F_2 are exerted on a body at points A and B. The body is not shown. Force scale: 1 square $\equiv$ 5 kN. Length scale: 1 square $\equiv$ 1 m.



Question:
Using a force polygon, determine the magnitude and direction of the resultant of both forces graphically, and in a line of action figure determine the location of the line of action.
Hint: use additional forces at A and B of magnitude 15 kN.

3.4 The two parallel forces F_1 and F_2 are exerted on a body at A and B. The body is not shown. Force scale: 1 square $\equiv$ 10 kN. Length scale: 1 square $\equiv$ 1 m.

Question:
Determine graphically (using a force polygon), the magnitude and direction of the resultant of both forces, and (using a line of action figure) the location of the line of action.
Hint: use additional forces at A and B of magnitude 40 kN.

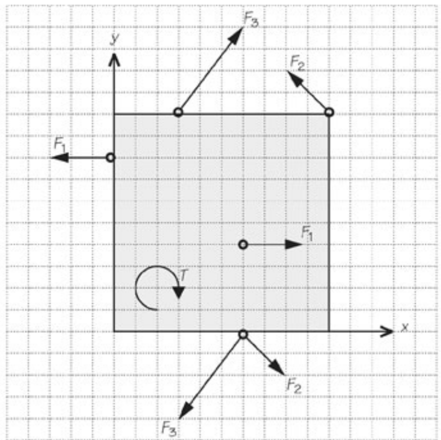


Moment of a couple (Section 3.1.4)

3.5 A block is subject to four couples in the xy plane. Force scale: 1 square $\equiv$ 1 kN. Length scale: 1 square $\equiv$ 1 m.

Question:
Find in the xy coordinate system the moment of:

a. the couple formed by the pair of forces F_1;
b. the couple formed by the pair of forces F_2;
c. the couple formed by the pair of forces F_3;
d. the couple T if T = 10 kNm;
e. the resultant couple.

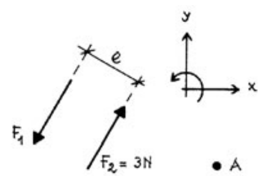


Moment of a force about to a point (Section 3.1.5)

3.6 F_1 and F_2 are statically equivalent to a couple.

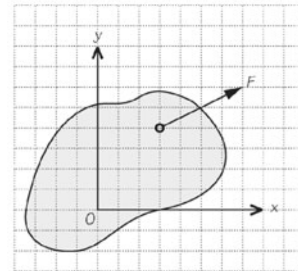
Questions:

a. How large is the distance e between both forces if the moment of F_1 about A is +2160 Nmm and the moment of F_2 about A is −1620 Nmm.
b. How large is the distance e between both forces if the moment of F_1 about a point B is +2160 Nmm and the moment of F_2 about the same point B is +1620 Nmm.



3.7 A force F is exerted on the body at A. Force scale: 1 square $\equiv$ 1 kN. Length scale: 1 square $\equiv$ 1 m.

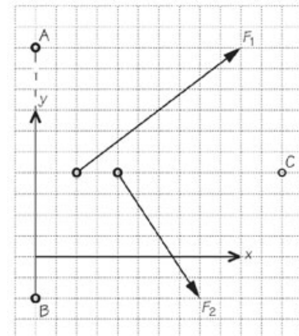
Question:
In four ways (!), calculate the moment of F with respect to the origin O of the xy coordinate system shown.



3.8 Find the forces F_1 and F_2 have magnitudes 250 and 180 kN respectively. Length scale: 1 square $\equiv$ 0.5 m.

Question:
The moment about A, B, and C respectively of:

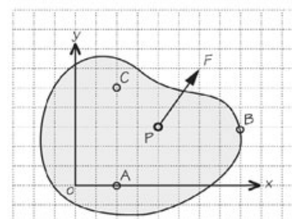
a. F_1;
b. F_2;
c. the resultant of F_1 and F_2.



3.9 For rigid bodies, a force may be shifted parallel to its line of action with the addition of a couple. Force scale: 1 square $\equiv$ 1 kN. Length scale: 1 square $\equiv$ 1 m.

Question:
How large is the moment of that couple if the force F at P is shifted respectively to:

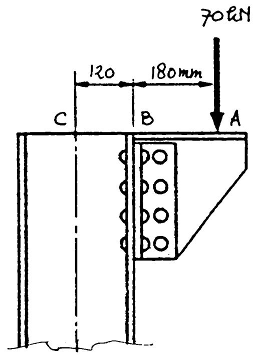
a. A.
b. B.
c. C.
d. O.



3.10 A console in a column is loaded at A by a vertical force of 70 kN.

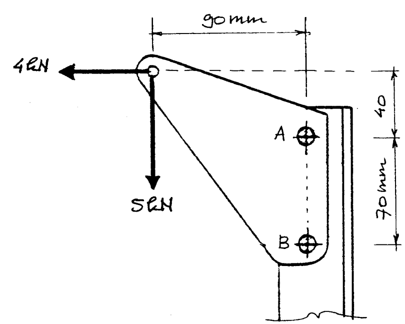
Questions:

a. Replace the force at A by a force at B and a couple.
b. Replace the force at A by a force at C and a couple.



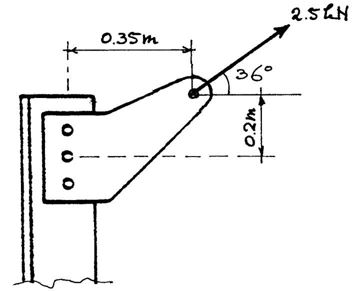
3.11 A console is subject to a horizontal force of 4 kN and a vertical force of 5 kN. In order to calculate the forces on the bolts A and B, the load is shifted to a point exactly halfway between A and B with the addition of a couple.

Question:
Determine the magnitude and direction of the couple.

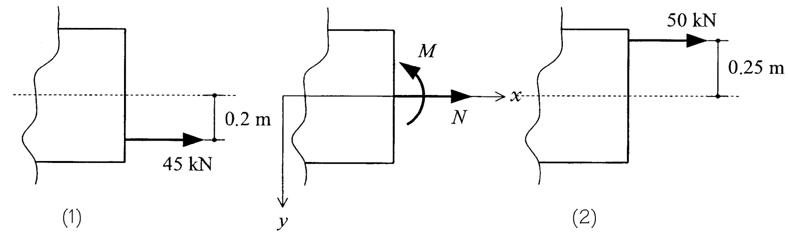


3.12 The console shown is fixed to a column by three bolts. In order to calculate the bolted connection, the load on the console is replaced by a horizontal and a vertical force at the point of the middle bolt, together with a couple.

Question:
The magnitude and direction of the forces and of the couple.



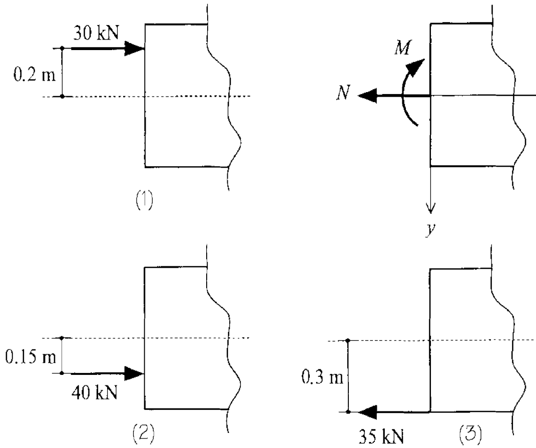
3.13: 1–2 In the left-hand and right-hand figures, a cross-section is subject to an eccentrically-applied tensile force. This force is statically equivalent to a normal force N and a bending moment M. The positive directions of N and M are shown in the middle figure.



Question:
Determine N and M, with the correct sign. Also depict N and M as they act in reality, and include their values.

Comment: N (normal force) and M (bending moment) are so-called section forces. Their nomenclature and sign conventions will be discussed further in Chapter 10.

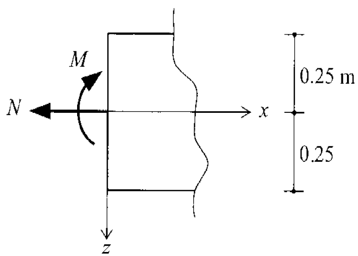
3.14: 1–3 As problem 3.13.



3.15 The section forces $N = 150$ kN and $M = 21$ kN are acting in a cross-section. They can be replaced by a single force acting at a distance e_z from the x axis, whereby it is assumed that e_z is positive if this force is acting on the positive side of the x axis ($z > 0$).

Questions:

a. Depict N and M as they are acting on the cross-section in reality and include their values.
b. Determine e_z with the correct sign.
c. Depict the force that is statically equivalent to N and M. Include its magnitude, direction and point of application.



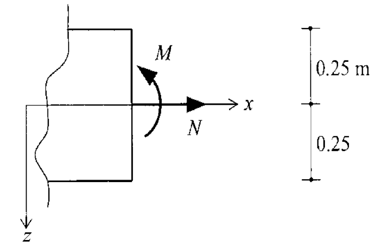
3.16: 1–2 As problem 3.15, but with

1. $N = -1$ kN and $M = +150$ Nm.
2. $N = +42$ kN and $M = -10.5$ kNm.

3.17 The section forces $N = -35$ kN and $M = +10.5$ kNm are acting in a cross-section. They can be replaced by a single force acting at a distance e_z from the x axis, whereby it is assumed that e_z is positive if this force is acting on the positive side of the x axis ($z > 0$).

Questions:

a. Depict N and M as they are acting in the section in reality and include their values.
b. Determine e_z with the correct sign.
c. Depict the force that is statically equivalent to N and M. Include its magnitude, direction and point of application.

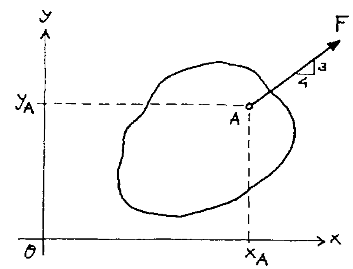


3.18: 1–2 As problem 3.17, but with

1. $N = -25$ kN and $M = -20$ kNm.
2. $N = +42$ kN and $M = -10.5$ kNm.

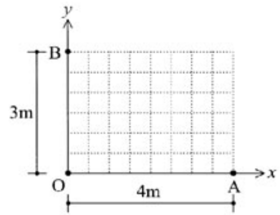
3.19 A is subject to a force $F = 100$ kN. The moment of this force about O is $T_z|O = 300$ kNm.

Question:
Where does the line of action intersect the x axis and the y axis respectively?



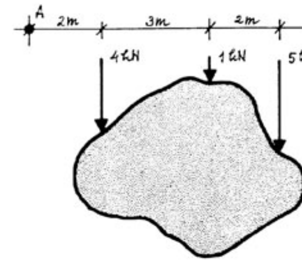
3.20 For a force F the line of action passes through the points A and B. The moment of F about O is $T_z|O = 6$ kNm.

Question:
Determine the components F_x and F_y.



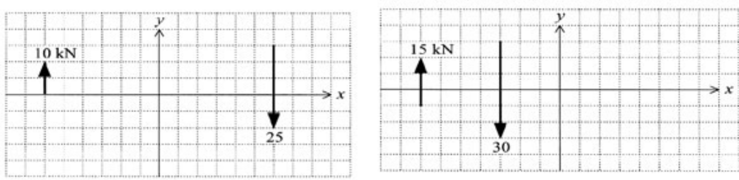
3.22 The resultant of the three parallel forces exerted on the body is R.

Question:
Determine the distance of the line of action of R to point A.



Compounding forces and couples analytically (Section 3.1.7)

3.21: 1–2 The two forces are equivalent to a single force R. Force scale: 1 square ≡ 5 kN. Length scale: 1 square ≡ 1 m.

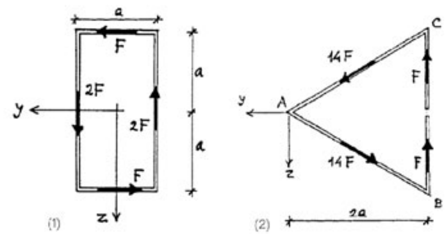


(1) (2)

Question:
Where does the line of action R intersect the x axis?

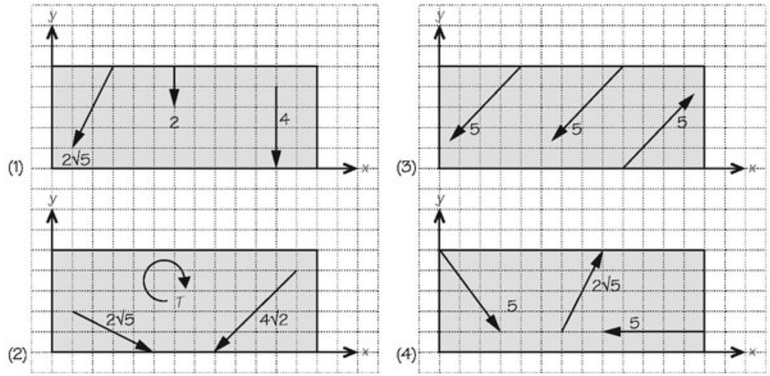
a. $x = -1$ m,
b. $x = +1$ m,
c. $x = +6$ m,
d. $x = +14$ m.

3.23: 1–2 The forces shown act on a thin-walled cross-section.



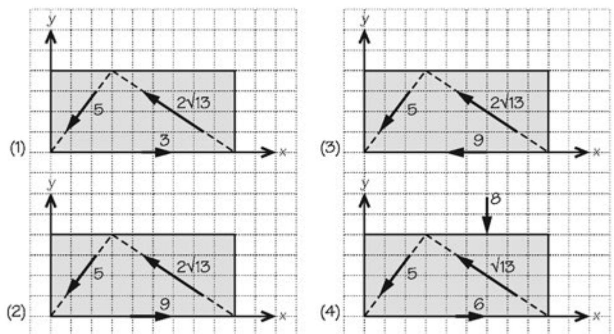
Question:
Determine the line of action, magnitude, and direction of the resultant of these forces.

3.24: 1–4 A number of forces act on a block. In case (2), there is also a couple $T = 36$ kNm. Force scale: 1 square ≡ 1 kN. Length scale: 1 square ≡ 1 m.



Question:
Determine the resultant.

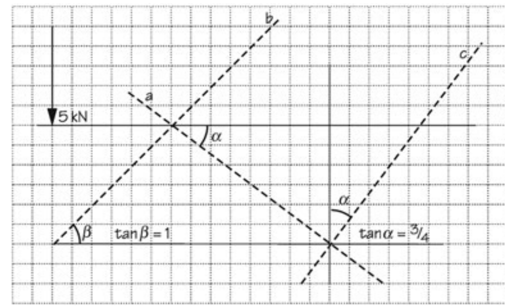
3.25: 1–4 A block is subject to three forces. The forces are not drawn to scale; the values are shown in kN. Length scale: 1 square ≡ 1 m.



Question:
Determine the resultant.

Resolving a force (couple) along three given lines of action (Sections 3.18 and 3.19)

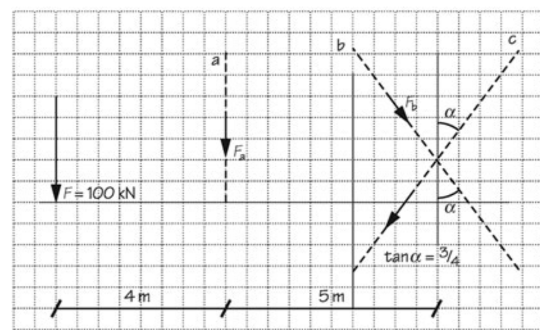
3.26 The force F is replaced by the three forces F_a, F_b and F_c with given lines of action a, b and c.



Question:
Determine the forces F_a, F_b and F_c:

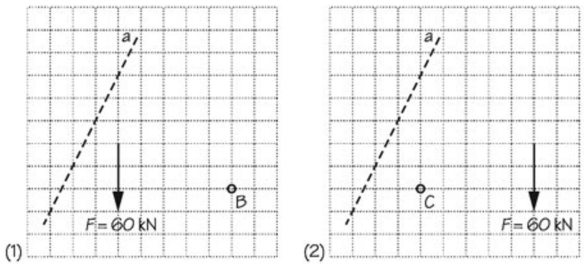
a. graphically;
b. analytically.

3.27 Force F is resolved into the components F_a, F_b and F_c with given lines of action a, b and c.



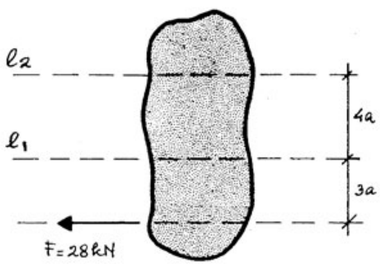
Question:
Find the magnitudes and directions of F_a, F_b and F_c.

3.28: 1–2 The force $F = 60$ kN is replaced by a force along line of action a and a force through point B, respectively point C.



Question:
Determine the magnitudes and directions of these forces.

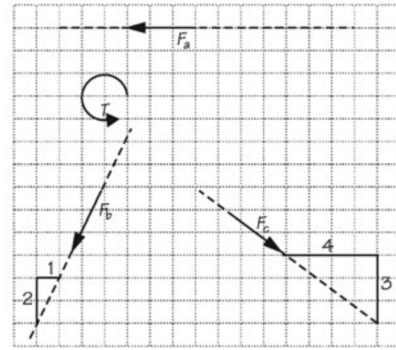
3.29 The force $F = 28$ kN is resolved into two parallel forces F_1 and F_2 with lines of action ℓ_1 and ℓ_2.



Question:
Determine the magnitudes and directions of the forces F_1 and F_2.

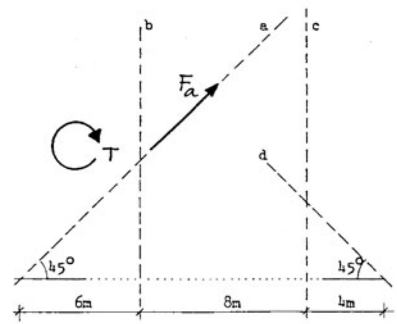
3.30 A couple $T = 110$ kNm is resolved into the forces F_a, F_b and F_c with given lines of action a, b and c. Length scale: 1 square $\equiv$ 1 m.

Question:
Determine F_a, F_b and F_c.



3.31 The couple $T = 60$ kNm is the resultant of four forces F_a, F_b, F_c and F_d with given lines of action a, b, c, and d. The magnitude and direction of the force F_a is given: $F_a = 30\sqrt{2}$ kN.

Question:
Determine F_b, F_c and F_d.

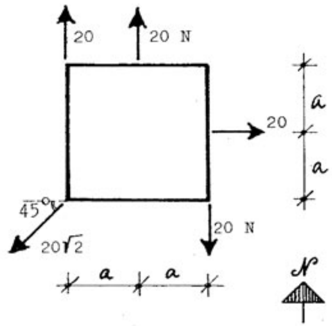


Equilibrium of a rigid body in a plane (Section 3.2)

3.32 A block is subject to the forces shown in the horizontal plane. The north direction is shown.

Question:
Which of the following statements about the forces exerted on the block is true?

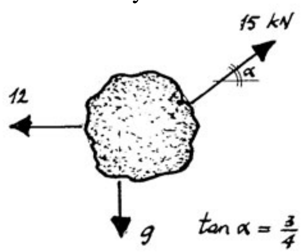
a. They comply with the three equilibrium conditions.
b. They form a couple together.
c. Their resultant points south-west.
d. Their resultant points east.



3.33 The three forces shown are exerted on the body.

Question:
Which statement about the body is true?

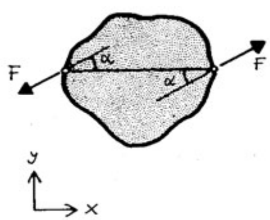
a. There is moment equilibrium.
b. There is force equilibrium.
c. There is no equilibrium.
d. There is equilibrium.



3.34 A body is subject to two parallel forces F.

Question:
Which statement is true?

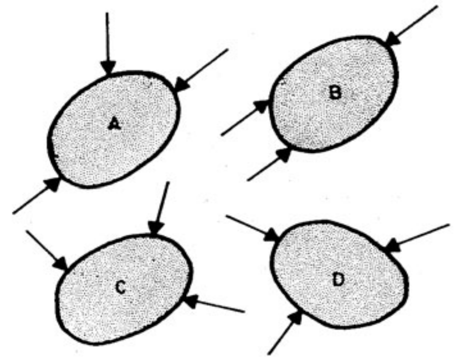
a. $\sum F_x \neq 0$.
b. $\sum F_y \neq 0$.
c. $\sum T_z \neq 0$.
d. The body is in equilibrium.



3.35 For two of the bodies shown, the equilibrium depends on the magnitude of the forces. For the other two, it is absolutely certain that they are not in equilibrium (the weights of the bodies are neglected).

Question:
Which of the two bodies are definitely not in equilibrium?

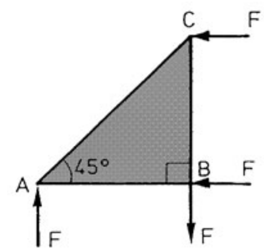
a. A and B.
b. A and C.
c. A and D.
d. B and C.



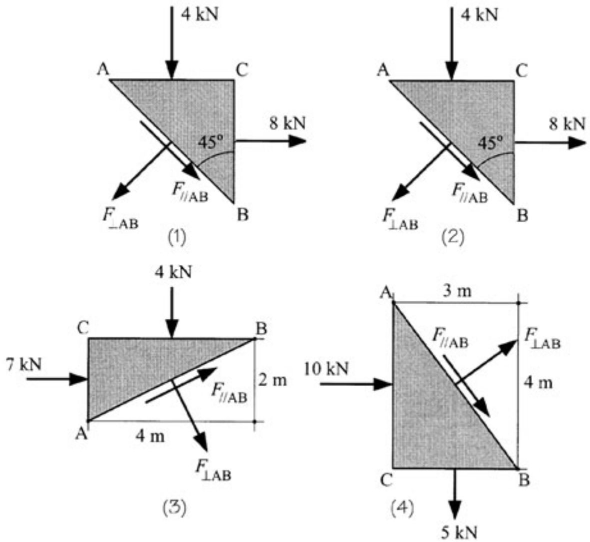
3.36 A triangular plate ABC is subject to four forces each with magnitude F (in the plane of the plate) that are *not* in equilibrium with one another. A fifth force is required to ensure equilibrium.

Question:
The line of action of the fifth force passes through:

a. A.
b. B.
c. C.
d. None of the points A, B and C.



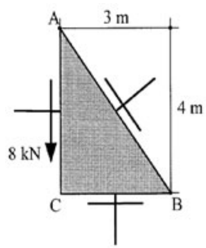
3.37: 1–4 The forces shown act on the edges of the triangular plate ABC. Their points of application are in the middle of the edges. The system is in equilibrium.



Question:
Determine $F_{//AB}$ and $F_{\perp AB}$, with the correct sign. Also depict how the forces are acting in reality and include their values.

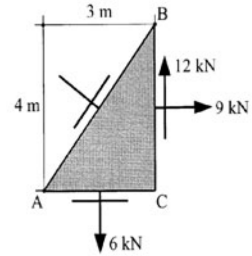
3.38 Of the six forces that act on the middle of the edges of the triangular plate ABC, one is known.

Question:
Which of the remaining five forces can be determined?

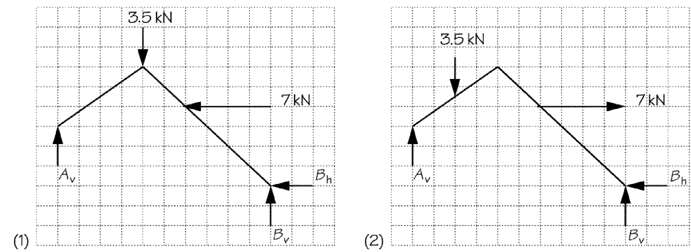


3.39 Of the six forces that act on the middle of the edges of the triangular plate ABC, three are given.

Question:
Determine the other three forces.



3.40: 1–2 A roof structure, loaded by the forces shown of 7 kN and 3.5 kN, is kept in equilibrium by the forces A_v, B_v and B_h. Length scale: 1 square $\equiv$ 0.5 m.



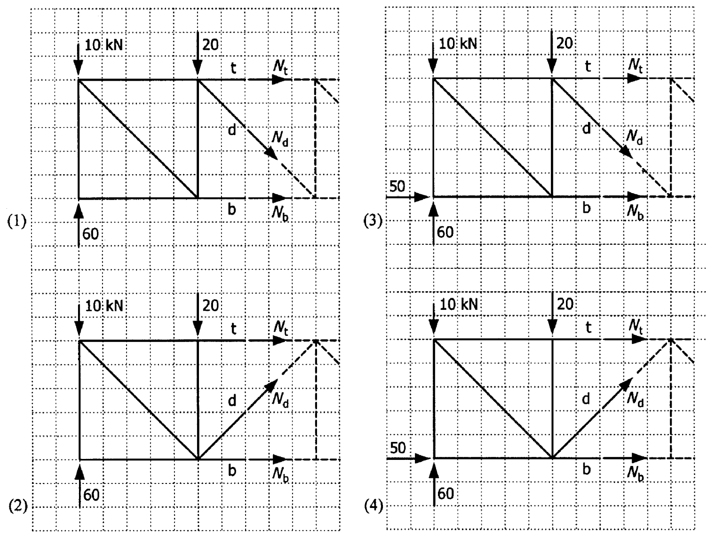
Question:
Determine the forces A_v, B_v and B_h.

3.41: 1–4 The part isolated (cut) from a so-called *truss* shown in the figure is in equilibrium. The truss is subject to the forces shown. The values are in kN. Length scale: 1 square ≡ 1 m.

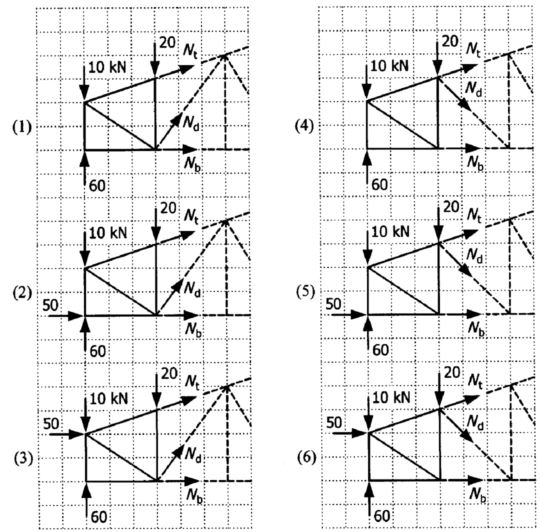
Question:
Determine the forces:

a. N_t in top chord member t;
b. N_d in diagonal member d;
c. N_b in bottom chord member b.

Comment: Trusses and calculating the truss forces N are covered in further detail in Chapter 9.



3.42: 1–6 The part isolated from a so-called *truss* shown in the figure is in equilibrium under the influence of the forces shown. The values are shown in kN. Length scale: 1 square ≡ 1 m.



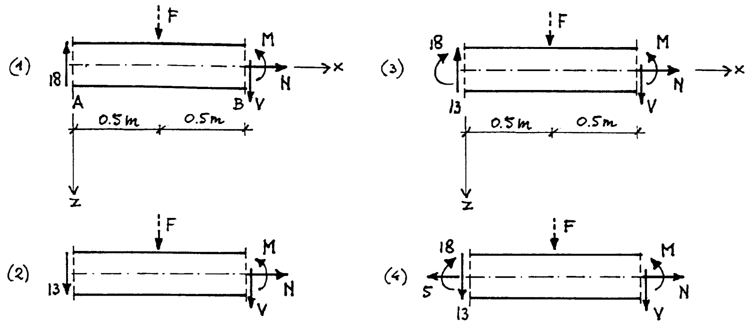
Question:
Determine the forces:

a. N_t in top chord member t;
b. N_d in diagonal member d;
c. N_b in bottom chord member b.

3.43: 1–4 A segment AB of length 1 m is isolated (cut away) from a *beam*. The section forces shown act in cross-section A. The forces are shown in kN and the couples (so-called *bending moments*) in kNm. The segment is in equilibrium.

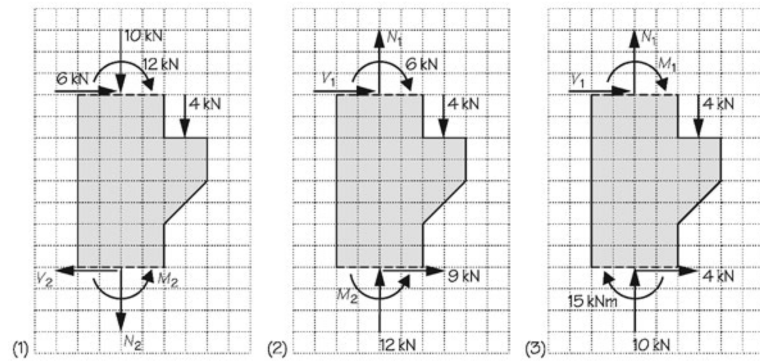
Question:
Determine the section forces N, V and M in cross-section B if:

a. the beam is not loaded between A and B;
b. the beam is loaded in the middle of AB by a vertical force F of 10 kN.



Comment: N (normal force), M (bending moment) and V (shear force) are so-called section forces. Their nomenclature and sign conventions are covered in further detail in Chapter 10.

3.44: 1–3 The body shown has been cut away from a column with console. The section forces N, V and M act at the central axis of the column. The console is subject to a force of 4 kN. The body is in equilibrium. Length scale: 1 square ≡ 0.1 m.

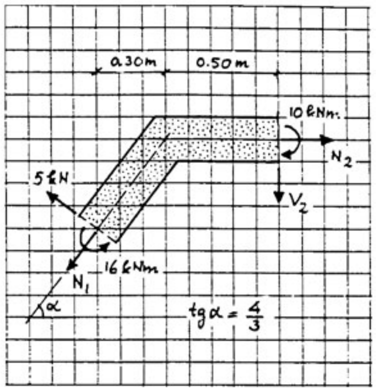


Question:
Determine the unknown section forces.

3.45 The section forces shown act on the cross-sections of the corner isolated from a portal frame. They act at the centre lines. The corner joint is in equilibrium. There is no loading between the cross-sections.

Questions:

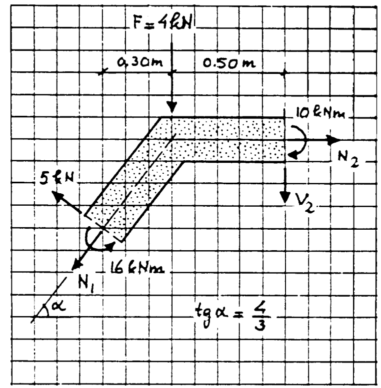
a. determine the (normal) force N_1;
b. determine the (normal) force N_2;
c. determine the (shear) force V_2.



3.46 The section forces shown act on the cross-sections of the corner isolated from a portal frame. They act at the centre lines. The corner is additionally loaded by a vertical force F of 4 kN.

Questions:

a. determine the (normal) force N_1;
b. determine the (normal) force N_2;
c. determine the (shear) force V_2.
d. Which of the three forces N_1, N_2 and V_2 is independent of the magnitude of the vertical load F on the corner? Provide reasoning for the answer.



4 Structures

To construct is to put together *structural elements* to create a structure, a *cohesive whole* that meets previously-determined demands. The structural elements are linked to one another by means of *joints*. The structure is linked to its normally fixed environment through *supports*. In this chapter, we will address a number of types of structural elements, joints, supports and structures. We will consider only two-dimensional structures.

In addition to the user requirements, which relate to the function of the structure, there are also mechanical demands (strength and stiffness), requirements relating to the structure itself (such as rate of construction, availability of the material), design requirements (representation), requirements relating to the physical components of the structure (such as climate control, warmth and sound insulation), and last but not least, economic requirements. Any contradictory requirements have to be weighed against one another wisely. To do so, a methodical approach is needed. Designing a structure is therefore anything but a random process.

As far as the mechanical section of a structure is concerned (strength and stiffness), an attempt must always be made to make the most efficient use of the specific properties of the *structural elements*.

In Section 4.1, we distinguish between a *particle element*, a *line element*, a *surface element*, and a *spatial element*.

Section 4.2 addresses the joints between structural elements, and more particularly the *hinged joint* and the *rigid joint*. We will also look at the total number of unknown *interaction forces*, so that at a later stage we can identify whether or not the forces in a structure can or cannot be calculated using solely the equilibrium equations.

As far as supports are concerned, we will look at the number of *degrees of freedom* (possible movement) in the support, and at the *support reactions* that a support can generate. Section 4.3 looks at *bar supports*, *roller supports*, *hinged supports*, and *fixed supports*.

Many spatial structures can be seen as a system of planar structures constructed from line elements. Investigating such planar frames is therefore certainly worth the effort. Based on matters such as the type of loading, the nature of the joints, and the external appearance, Section 4.4 defines a number of planar frames.

Structures are supported in such a way that all free movement is restricted. This type of structure is referred to as a *kinematically determinate* or *immovable* structure. If there are too few supports, or if they are not applied effectively, the structure, or a part of it, will have a degree of freedom that cannot be restrained. The structure is no longer immmovable. The structure is then said to be *kinematically indeterminate*, or is referred to as a *mechanism*.

If it is possible to define all the support reactions and interaction forces in a structure using solely equilibrium equations, it is called a *statically determinate structure*. If there are too many unknown forces to determine them based on the equilibrium, the structure is said to be *statically indeterminate*. To determine the forces in a statically indeterminate structure, the deformation of the structure must be taken into account, which is beyond the scope of this book.

The last part of the chapter, Section 4.5, looks at the *kinematic/static (in)determinacy* of planar structures.

4.1 Structural elements

As far as structural mechanics is concerned (strength and stiffness), one always tries to make the most efficient use of the specific properties of a limited number of building blocks, or *structural elements*. The way of modelling in structural mechanics allows one to distinguish the following four types of structural elements:

- *Particle element* (see Figure 4.1a)
  All dimensions of the element are negligibly small with respect to those of other elements.
- *Line element* (see Figure 4.1b)
  Two of the dimensions of the element (those of the cross-section) are considerably smaller than the third dimension (the length).
- *Surface element* (see Figure 4.1c)
  One dimension of the element (the thickness) is considerably smaller than the other two dimensions (the length and width).
- *Spatial element* (see Figure 4.1d)
  All the dimensions of the element are of the same order of magnitude as those of other elements and are therefore not negligible.

4.1.1 Particle element

A *particle element* (Figure 4.1a) is a zero-dimensional structural element: all dimensions are negligibly small with respect to those of other elements. The dimensions of the element play a subordinate role. This is addressed further in Section 4.1.5. Also, see Section 4.2, in which particle elements are used for modelling hinged and fixed joints.

4.1.2 Line element

A *line element* (Figure 4.1b) is a one-dimensional structural element: two of the dimensions of the element (those of the cross-section) are significantly smaller than the third dimension (the length).

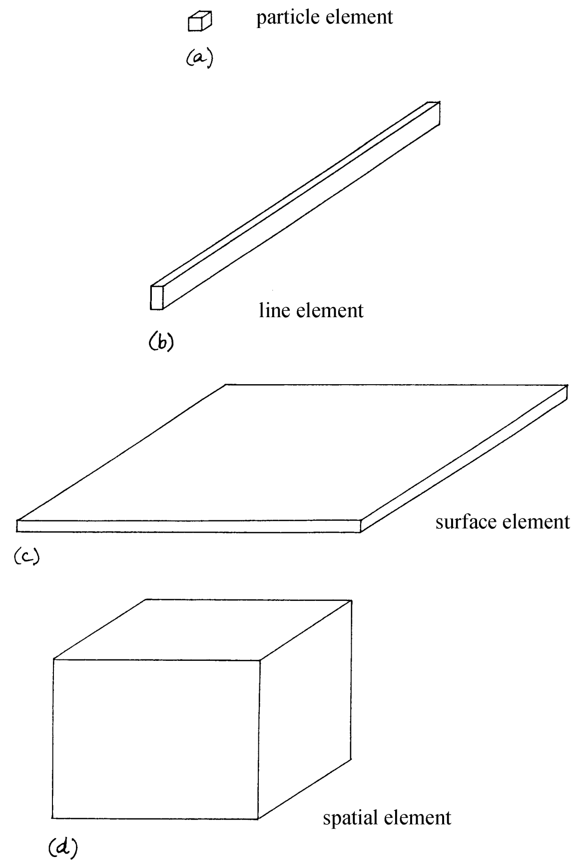


Figure 4.1 Structural elements: (a) particle element, (b) line element, (c) surface element (d) spatial element.

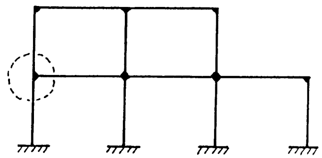


Figure 4.2 The model of a structure made of line elements; the joints between the line elements are particle elements (joints).

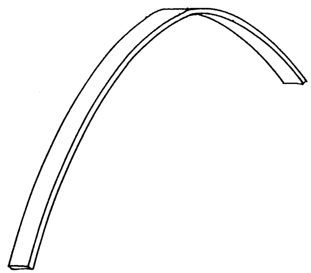


Figure 4.3 A (rigid) curved line element is called an arch.

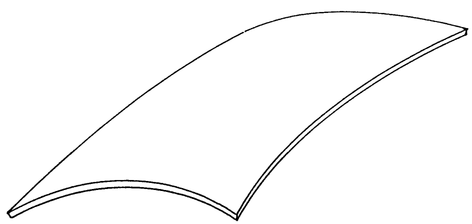


Figure 4.4 A (rigid) curved surface element is called a shell.

By using simplified assumptions in the smallest directions (those of the cross-section), all the properties of the line element can be assigned to a single line, the so-called *axis* of the line element. In mechanics, a bar which in reality is three-dimensional, can often be modelled by a one-dimensional member; the member is depicted by a single line: the *bar axis*.

Figure 4.2 represents the mechanical diagram of a structure constructed from line elements.

Line elements with a straight axis as known by a wide range of names, such as *bar*, *beam*, *joist*, *girder*, *column*, *post* and *member*. The nomenclature sometimes relates to the position of the line element in the structure: horizontal (beam, joist, girder) or vertical (column, post, stay). Hereafter, we will refer to a line element in general as a *member*.

An (inflexible) curved line element is known as an *arch*, see Figure 4.3. A line element without a particular shape is a *cable*: cables adapt to the loading.

4.1.3 Surface element

A *surface element* (Figure 4.1c) is a two-dimensional structural element: one dimension (the thickness) is small with respect to the other two dimensions (the length and width).

The behaviour of this element, which in reality is three-dimensional, can be described sufficiently accurately by means of a two-dimensional model by making simplified assumptions with respect to the thickness. In the two-dimensional model, all the properties of the element are assigned to a plane. This *reference plane* is sometimes also called the *central plane*. In a mechanical diagram, only the reference plane (without thickness) of the surface element is depicted.

With *plates*, the reference plane is a flat plane. With *shells*, the reference plane is curved (see Figure 4.4). If the reference plane does not have its own shape, but adapts to the loading, it is called a *membrane* or *film*. Plates are also given other names, such as *slab*, *floor*, *wall* and *disc*.

4.1.4 Spatial element

A *spatial element* (Figure 4.1d) is a three-dimensional structural element: all the dimensions are of the same order of magnitude as those of other elements. In a more general sense, a spatial element can be defined as an element for which the model of a particle, line, or surface element does not suffice.

4.1.5 Modelling structural elements

It was stated above that the difference between the four kinds of structural elements is the result of the *modelling method*, and that this strongly depends on the information sought by the research or calculation.

To illustrate, refer to the concrete bar structure and its model in Figure 4.2. The model has been created to investigate the mechanical behaviour of the structure as a whole. The lines in the diagram represent the beams and columns, which have been schematised as *line elements*. The beams and columns are rigidly joined to one another. In the model, these joints are represented as *particle elements* (capable of transferring both forces as well as concentrated couples).

In Figure 4.5, the circled joint between the beam and outer column has been elaborated. Further investigation shows that there is a complex interplay of forces in the joint; the concrete transfers the compressive forces, and the reinforcement bars transfer the tensile forces. This type of investigation is critical for detailed modelling of the joint. Can the concrete transfer the compressive forces; how much reinforcement is required for transferring the tensile forces, and where should this reinforcement be placed?

When we are studying the behaviour of a structure as a whole, we can

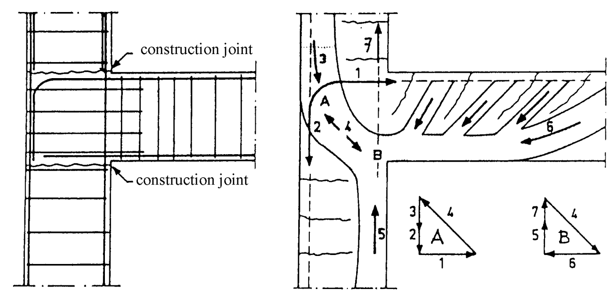


Figure 4.5 In a detailed study, a joint should be modelled as a spatial structural element.

model joints as particle elements. When we are studying detailed behaviour of a joint, it must be modelled as a spatial element.

In principle, all structural elements are three-dimensional, and therefore are spatial elements; modelling them as particle, line, or surface elements always means that some information and accuracy is lost. This is acceptable as long as the model of the structure gives results close enough to the actual structure. If there is too much discrepancy, the model will have to be modified to include more detail.

The justification of the models used below derives from satisfactory results obtained over many years.

4.2 Joints between structural elements

Two bodies can be joined together in a wide variety of ways. For joints between structural elements, in the same plane, there are two kinds:

- *Hinged joints* (hinges);
- *Fixed joints* (entirely rigid or infinitely rigid joints).

In a *hinged joint*, or *hinge*, the joined parts cannot translate with respect to one another, but can rotate freely with respect to one another. In a *rigid joint*, the joined parts cannot translate with respect to one another, nor can they rotate with respect to one another. The forces that the structural elements exert on one another in a joint are referred to as *interaction forces* or *joint forces*.

Hinges will always have a certain amount of resistance to rotation, even if only due to the occurrence of friction. If this resistance is limited, the joint can be idealised as a frictionless hinge. When the resistance to rotation in a joint is very large, the joint tends to be represented as infinitely stiff. The reality will always lie between these two extremes.

Spring joints are joints in which the magnitude of the acting interaction forces is related to the deformation in the joint. These will not be covered here.

4.2.1 Hinged joints

In Figure 4.6a, the bodies (1) en (2) are joined by a hinge at S. In the figure, the *hinge* is depicted as a small open circle. The bodies are able to rotate freely with respect to one another about the hinge S, but cannot translate with respect to one another. The bodies can exert only forces on one another at S; they cannot exert any couple.

Dissecting a body into its joints, and at the same time depicting the forces that are exerted on the body in the joints, is referred to as *isolating* the body; the diagram so formed is called the *free body diagram*.

In Figure 4.6b, both bodies have been isolated from one another and the forces that the bodies exert on one another in the joint are shown. Based on Newton's third law of action and reaction, these *interaction forces* are equal and opposite (see Section 1.4.1). In other words: $S^{(1)} = S^{(2)} = S$.

In the hinged joint shown, there are two unknowns: the magnitude of the *hinge force*[1] S and the direction of its line of action. We could also select the two components S_h and S_v as unknowns.

A joint comes about by some means of joining. In hinged joints, this could be a pin or axis, perpendicular to the plane shown, about which both bodies can rotate, and through which they can exert forces on one another.

In Figure 4.6c, the pin has also been isolated in S for both bodies (1) and (2). The pin is seen as a *particle*, even though it is shown as a body in

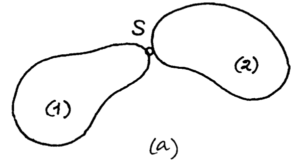


Figure 4.6a (a) Two bodies joined in a hinge at S.

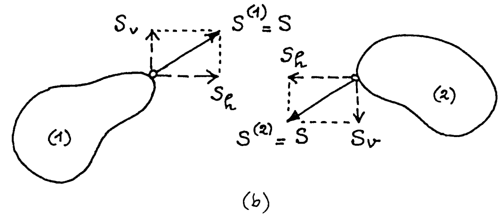


Figure 4.6b The forces acting on each body at the hinged joint S.

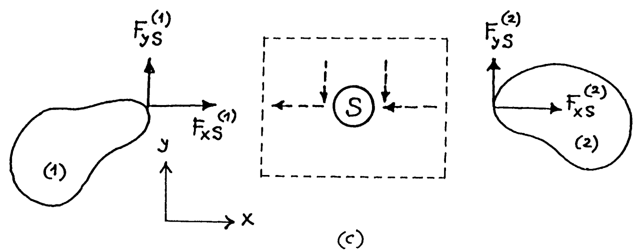


Figure 4.6c In hinged joint S, not only have both bodies been isolated, but so has joint S which should be seen as a particle element. The forces shown are the interaction forces between the two bodies and joint S.

1 Although we are talking about a hinge force S (singular) in reality it concerns a *pair of forces* (plural).

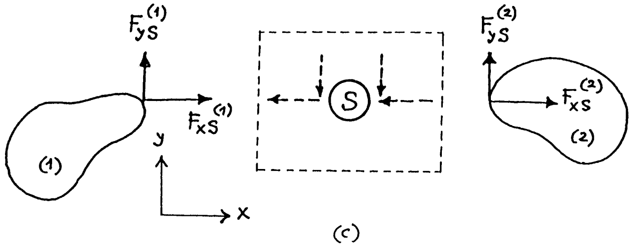


Figure 4.6c In hinged joint S, not only have both bodies been isolated, but so has joint S which should be seen as a particle element. The forces shown are the interaction forces between the two bodies and joint S.

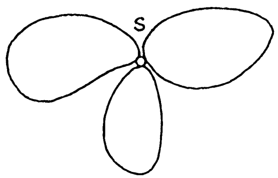


Figure 4.7 Three bodies hinged together at joint S.

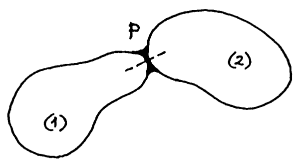


Figure 4.8a Two bodies joined rigidly at P.

the figure, in this case a *circle*. This circle is also known as the *connection* between the bodies (1) and (2).

Imagine $F^{(1)}_{x;\mathrm{S}}$; $F^{(1)}_{y;\mathrm{S}}$ and $F^{(2)}_{x;\mathrm{S}}$; $F^{(2)}_{y;\mathrm{S}}$ are the forces exerted through the connection (pin S) in the xy coordinate system given on body (1) and body (2) respectively.[1] This makes four unknown forces. Based on Newton's third law, equal and opposite forces are exerted on the connection. If the system of bodies is in equilibrium, then each of the parts must be in equilibrium, including the connection. The force equilibrium of the connection therefore gives

$$\sum F_x = -F^{(1)}_{x;\mathrm{S}} - F^{(2)}_{x;\mathrm{S}} = 0,$$

$$\sum F_y = -F^{(1)}_{y;\mathrm{S}} - F^{(2)}_{y;\mathrm{S}} = 0.$$

There are therefore two linear relationships between the four unknown forces $F^{(1)}_{x;\mathrm{S}}$, $F^{(1)}_{y;\mathrm{S}}$, $F^{(2)}_{x;\mathrm{S}}$ and $F^{(2)}_{y;\mathrm{S}}$, so that two of the four unknowns can be eliminated, leaving two independent interaction forces in the hinged joint:

$$F^{(1)}_{x;\mathrm{S}} = -F^{(2)}_{x;\mathrm{S}}\ (= S_\mathrm{h}),$$

$$F^{(1)}_{y;\mathrm{S}} = -F^{(2)}_{y;\mathrm{S}}\ (= S_\mathrm{v}).$$

The formal approach described here to determine the number of unknown (independent) interaction forces in a hinged joint seems rather complicated if you compare it to the simple approach in Figure 4.6b. The formal approach, however, offers clear benefits if more than two bodies are joined together at the hinge.

---

[1] The upper index indicates the body on which the force is exerted.

For example, in Figure 4.7, three bodies are joined at a hinge S. Six interaction forces act on the hinge. The force equilibrium of the hinge gives two linear relationships between these six unknowns, so that we are left with $6 - 2 = 4$ independent interaction forces in S.

4.2.2 Fixed joints

Fixed joints are also referred to as *rigid joints*.[1]

The two bodies (1) and (2) in Figure 4.8a are rigidly joined at P. The fixed joint is depicted in the figure as a thickening at P. The joint at P ensures that the bodies cannot translate nor rotate with respect to one another. The joint could be realised as a pin that, in the plane of the figure, is stuck into both bodies.

Both bodies have been isolated in Figure 4.8b. Three unknown interaction forces P_h, P_v and P_m are exerted in P. Although P_m stands for the two equal and opposite *couples*, it is referred to as a *force* when *generalising*.

If more than two bodies are rigidly connected at a joint, as in Figure 4.9a, the easiest way of finding the number of unknown (independent) interaction forces is the formal approach, in which the joint is also isolated. The joint is seen as a *particle* that in addition to forces can now also transfer *concentrated couples*.

In Figure 4.9b, the bodies (1), (2) and (3) and the *joint* have been isolated. Since it can transfer couples, the connection has been depicted as a *square*.

In the xy coordinate system shown, $F^{(e)}_{x;\mathrm{P}}$, $F^{(e)}_{y;\mathrm{P}}$ and $F^{(e)}_{z;\mathrm{P}}$ are the (generalised) forces that are exerted through the connection at P on body (e) $(e = 1, 2, 3)$. Based on Newton's third law, equal and opposite forces are exerted on the connection, making a total of nine unknown forces. If the

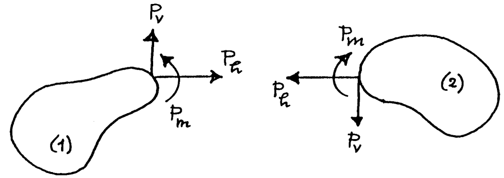


Figure 4.8b The three interaction forces between both bodies isolated at P.

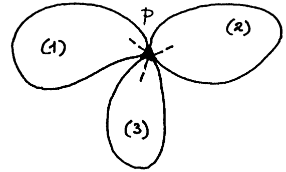


Figure 4.9a Three bodies rigidly conneced at P.

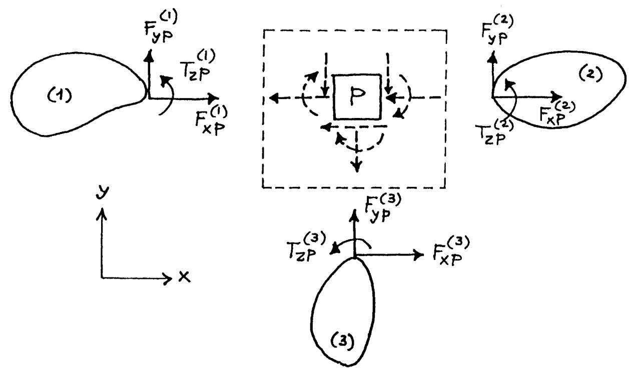


Figure 4.9b The interaction forces between the three bodies and the joint P shown as a particle element.

[1] This is actually an incomplete definition. It is preferable to refer to an *infinitely stiff joint*.

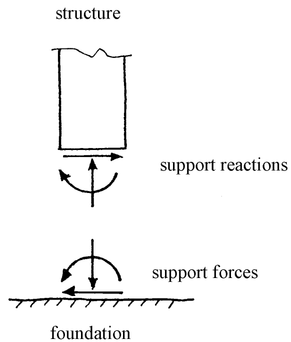


Figure 4.10 The interaction forces in a support are pairs of forces. The forces that act on the foundation are called support forces or support actions, and the equal and opposite forces acting on the structure are called support reactions.

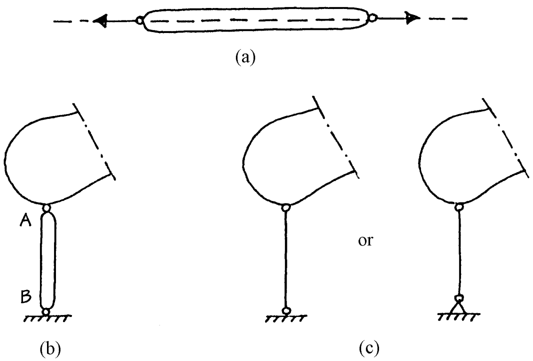


Figure 4.11 (a) A two-force member is a straight bar that is joined at both ends with a hinge to its surroundings and is loaded only by forces at its ends. A two-force member can transfer only forces of which the line of action passes through both hinges. (b) A bar support. (c) Model of a bar support.

system of bodies is in equilibrium, the connection is also in equilibrium. There are three equilibrium equations for the connection: two for the force equilibrium and one for the moment equilibrium. These equilibrium equations give three linear relationships between the nine unknowns, so that $9 - 3 = 6$ independent interaction forces remain at the fixed joint between the three bodies.

4.3 Supports

Most structures are not free-floating, but are joined to a *fixed* environment. The joints between the structure and its fixed environment are called *supports*.

The interaction forces that act in the supports on the structure are known as *support reactions*. They act in the direction in which displacement of the structure is prevented. The forces that the structure exerts on the supports (for example on the foundation) are called *support forces* or *support actions*. The support forces are equal and opposite to the support reactions (see Figure 4.10).

We will look at four types of supports:
- bar supports;
- roller supports;
- hinged supports;
- (fully) fixed supports.

4.3.1 Bar supports

A *two-force member* is a straight bar which is joined to its environment at both ends by a hinge, and is loaded only by forces at the ends. From the moment equilibrium it follows that such members can transfer forces only when the line of action passes through both hinges (see Figure 4.11a).

In a *bar support* the two-force member is used as a *link* between the structure and the immovable environment (see Figure 4.11b). Figure 4.11c is a model of the bar support: the bar support is depicted as a single line between the two hinges. The immovable environment is generally shown by means of a hatched area.

In Figure 4.12, the two-force member has been isolated at hinges A and B. The position of the two-force member (the line joining both hinges) fixes the line of action of the interaction forces F. Only the magnitude of F (with its sign for the correct direction) is unknown.

When the body moves, point A is forced to follow a circle with centre B by the two-force member (see Figure 4.13a). If the displacement remains very small with respect to the length of the two-force member (which is generally the case), then the arc is almost the same as the tangent at A to the circle (Figure 4.13b). Note that the diagram of the structure is much smaller than the actual structure, and also the displacement is strongly magnified in the diagram.

The bar support at A prevents displacement in the direction of the bar. Displacement in the direction perpendicular to the bar is free (Figure 4.13b), as is a rotation of the body about A.

Imagine that in the free displacement of the body, $u_{x;\mathrm{A}}$ and $u_{y;\mathrm{A}}$ are displacements of A in the x and y directions, and that $\varphi_{z;\mathrm{A}}$ is the rotation of the body about A. *Generalising*, the *rotation* is called *a motion*. For a bar support at A (with the bar in the y direction) the *generalised motions* are:

$$
\begin{aligned}
u_{x;\mathrm{A}} &= \text{unknown} \quad \text{(free motion)},\\
u_{y;\mathrm{A}} &= 0 \quad \text{(prescribed motion)},\\
\varphi_{z;\mathrm{A}} &= \text{unknown} \quad \text{(free motion)}.
\end{aligned}
$$

The bar support prevents free motion of point A by exerting forces on it.

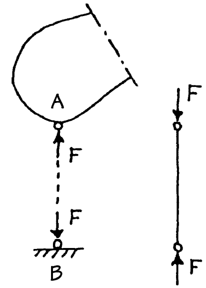


Figure 4.12 The two-force member, isolated from the body and support, with the interaction forces.

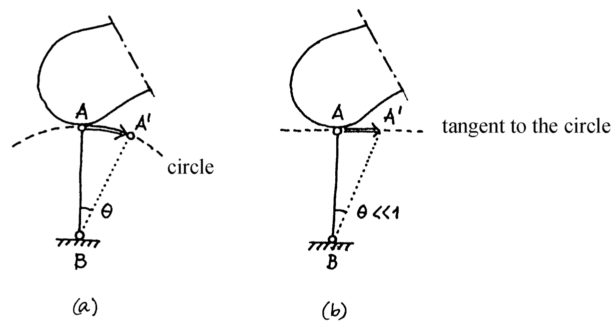


Figure 4.13 (a) If the body moves, the bar AB forces point A to follow a circular path with centre B. (b) If the motion remains small with respect to the length of the bar, the arc can be approximated by the tangent at A to the circle.

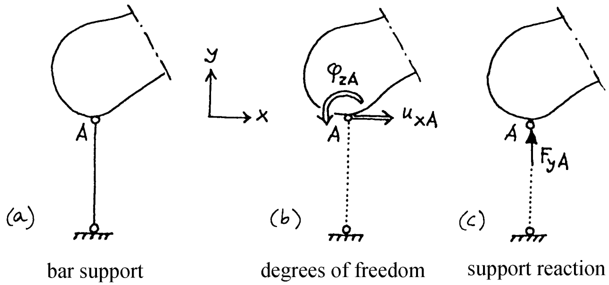


Figure 4.14 (a) The bar support has (b) two degrees of freedom (a rotation and a displacement perpendicular to the bar) and gives (c) one support reaction (a force in the direction of the bar).

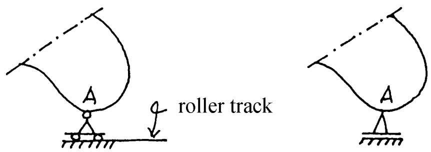


Figure 4.15 Model of a roller support.

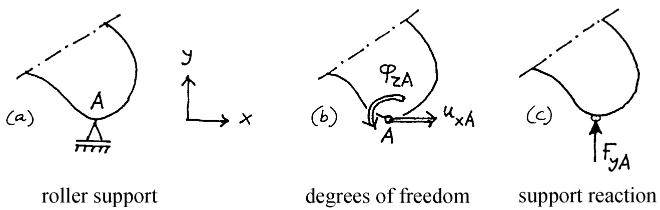


Figure 4.16 (a) The roller support has (b) two degrees of freedom (a rotation and a movement along the rolling surface) and generates (c) one support reaction (a force perpendicular to the roller track).

For the *generalised forces* at A in the given coordinate system:

$$F_{x;\mathrm{A}} = 0 \quad \text{(prescribed force),}$$
$$F_{y;\mathrm{A}} = \text{unknown} \quad \text{(free force),}$$
$$T_{z;\mathrm{A}} = 0 \quad \text{(prescribed force).}$$

The free (freely adjustable) motions are called the *degrees of freedom* at the support; the free (freely adjustable) force is the *support reaction*. A bar support therefore has two degrees of freedom and generates one support reaction.

If a motion is prescribed, the associated force is unknown, and *vice versa*. This is true not only for bar supports but also for all other supports discussed below. The total number of degrees of freedom and support reactions is therefore always three for a support (in a plane). In Figure 4.14 the degrees of freedom and support reactions are shown. Sometimes, motions are depicted by means of open arrows, while forces are depicted by closed arrows.

4.3.2 Roller supports

Figure 4.15 is a schematic representation of a *roller support*. For a roller support at A, the body can move parallel to the so-called *roller track*, and can also rotate freely about A. Only motion of A perpendicular to the roller track is prevented; this is the direction in which the interaction force is exerted.

For the roller support in Figure 4.16, with the roller track parallel to the x axis, the following applies for the motion at A:

$$u_{x;\mathrm{A}} = \text{unknown} \quad \text{(free motion),}$$
$$u_{y;\mathrm{A}} = 0 \quad \text{(prescribed motion),}$$
$$\varphi_{z;\mathrm{A}} = \text{unknown} \quad \text{(free motion).}$$

The following applies for the forces in A in the coordinate system given:

$$F_{x;\mathrm{A}} = 0 \quad \text{(prescribed force)},$$
$$F_{y;\mathrm{A}} = \text{unknown} \quad \text{(free force)},$$
$$T_{z;\mathrm{A}} = 0 \quad \text{(prescribed force)}.$$

The roller support therefore has two degrees of freedom and generates one support reaction. Note the parallel with a bar support!

Figure 4.18 shows a steel roller support used in older bridge structures. Due to the continuous sideways movement, the roller can end up askew after a while. To prevent this happening, the roller is provided with a *tooth* structure on its sides (comparable to a cogwheel). In order to prevent displacement in the z direction, a *groove* is sometimes cut into the roller that fits over an open *ridge* in the *rail* and *bearing pedestal*.

This example of a steel roller support provides a good picture of how it works. Roller supports can be made of materials other than steel, but then as *sliding supports*. Examples include supports made of rubber or plastics (neoprene), occasionally in combination with Teflon to reduce friction.

The roller support shown can transfer only compressive forces and no tensile forces. This is not a problem as long as the loading generates only compressive forces in the support. Such a load could be, for example, the ever-present weight of the structure. Generally speaking, the weight of a structure, such as a bridge, is sufficiently large to ensure that the roller support is continuously loaded by compressive forces. If a roller support also has to be able to transfer tensile forces, special structural provisions have to be made.

It is assumed here that a roller support can transfer both tensile and compressive forces.

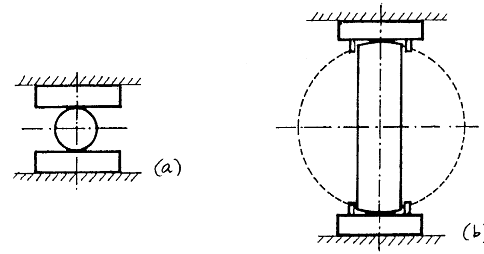


Figure 4.17 (a) An example of a simple roller support. (b) If the roller is large enough and the movements and rotations remain small, a large part of the roller can be omitted. The roller support changes into a bar support.

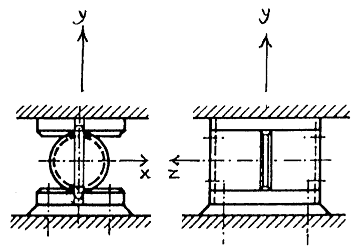
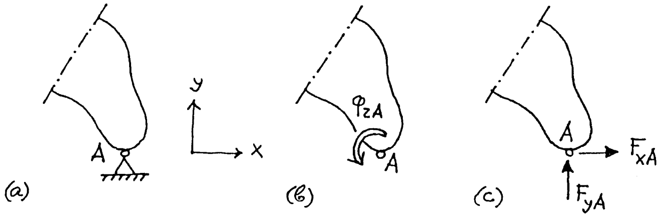


Figure 4.18 A steel roller support.



hinged support degree of freedom support reactions

Figure 4.19 (a) The hinged support has (b) one degree of freedom (a rotation) and generates (c) two support reactions (the two components of a force).

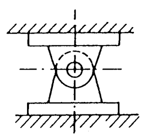


Figure 4.20 Simple example of a hinged support.

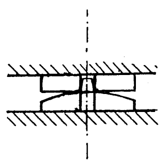


Figure 4.21 A steel hinged support, previously used in smaller bridges. The horizontal movement is prevented by the pin.

4.3.3 Hinged supports

A *hinged support* is a hinge between the structure and its immovable environment (see Section 4.2.1). A hinged support is modelled in Figure 4.19a (the open circle is often omitted). In a hinged support at A, the displacement of the body at A is prevented and the body can only rotate about A. The support cannot transfer a couple, but can transfer a force. The interaction force is unknown with respect to both magnitude and direction.

For the hinged support in Figure 4.19 with the coordinate system shown, the following applies for motion at A:

$$
\begin{aligned}
u_{x;\mathrm{A}} &= 0 && \text{(prescribed motion),}\\
u_{y;\mathrm{A}} &= 0 && \text{(prescribed motion),}\\
\varphi_{z;\mathrm{A}} &= \text{unknown} && \text{(free motion).}
\end{aligned}
$$

and for the forces at A:

$$
\begin{aligned}
F_{x;\mathrm{A}} &= \text{unknown} && \text{(free force),}\\
F_{y;\mathrm{A}} &= \text{unknown} && \text{(free force),}\\
T_{z;\mathrm{A}} &= 0 && \text{(prescribed force).}
\end{aligned}
$$

A hinged support therefore has one degree of freedom (a rotation) and generates two support reactions (the two components of a force).

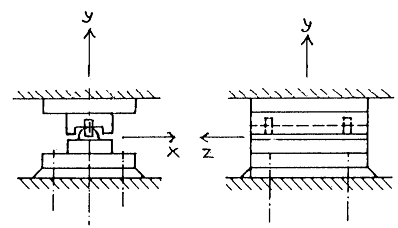


Figure 4.22 A steel hinged support as still found in many bridges today.

Figure 4.20 is a good example of a hinged support. Figure 4.21 shows how a steel hinged support can be used in small bridges. Horizontal motion is prevented by a *pin*. The steel hinged support in Figure 4.22 can transfer large forces and is an example of what is used in larger bridges. Like roller supports, hinged supports can be made from materials other than steel, or from a combination of materials Although the supports in Figures 4.21 and 4.22 can transfer only compressive forces, it is assumed below that hinged supports can also transfer tensile forces.

4.3.4 Fixed supports

A fixed support is an infinitely stiff or rigid joint between a body and its environment, see also Section 4.2.2. Figure 4.23a is a model of a fixed support (the dotted line is generally omitted). At A, the fixed support prevents both the displacement and rotation of the body. In fixed supports, all motion is prescribed: fixed supports therefore have no degrees of freedom. A fixed support has three support reactions, see Figure 4.23: two forces and a so-called *fixed-end moment*.

The balcony (cantilever beam) in Figure 4.24a is an example of a fixed supported structure. Another example is the support in Figure 4.24b of a concrete column on a concrete foundation, constructed as a single, monolithic whole.

In many cases, a fixed support will not fully prevent rotation. Such a support is incomplete and is referred to as a *spring support* if the magnitude of the rotation is related to the magnitude of the fixed-end moment. We will always refer to a *fully fixed support* below.

4.3.5 Free support

Frequently, a beam, such as a floor beam, is placed directly on the masonry or concrete. Here, a roller support or hinged support described above is not

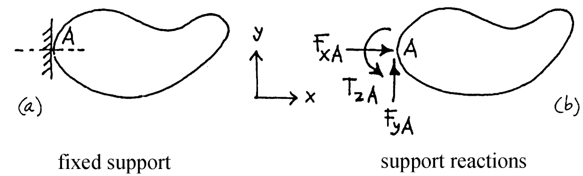


Figure 4.23 (a) The fixed support has no degrees of freedom and generates (b) three support reactions (the two components of a force and a fixed-end moment).

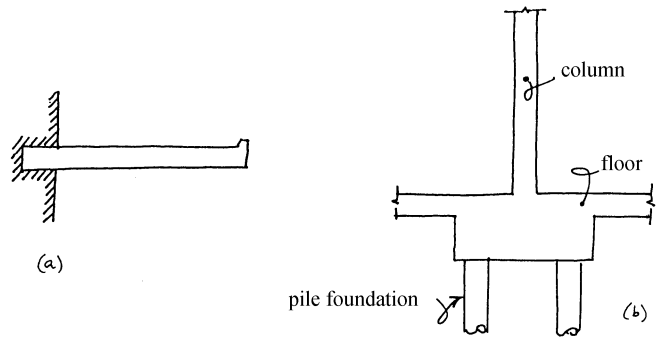


Figure 4.24 Examples of fixed supports: (a) a balcony and (b) a concrete column that forms a single monolithic whole with the concrete foundation.

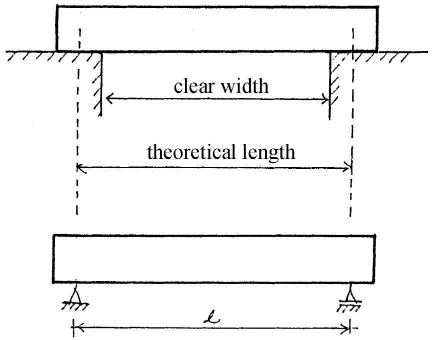


Figure 4.25 The simply supported beam modelled as a beam with a hinged and roller support.

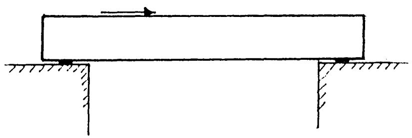


Figure 4.26 A beam supported on rubber blocks and loaded by a horizontal force.

used. Sometimes, the function of the roller is fulfilled by a slide layer of steel felt, Teflon, or other suitable material.

In practice, this sort of beam is often referred to as *freely supported* or simply supported, and is generally modelled as a beam on a hinge and a roller (see Figure 4.25).

In the event of vertical loading, it is arbitrary on which side the roller or hinge is placed. The model of a freely supported beam must however be performed with the necessary reserve if it relates to support reactions as a result of a horizontal load. For example, in the beam in Figure 4.26, that is supported on rubber blocks at both ends and which is loaded by a horizontal brake force, the model of a *free support* leads to incorrect (horizontal) support reactions.

4.4 Planar structures

A spatial structure can often be viewed as a system of planar structures composed of line elements. It is therefore certainly worth investigating the properties of such planar structures in more detail. Based amongst other things on the nature of the joints and the external appearance, various types of planar structures can be distinguished.

4.4.1 Modelling structures

In mechanics, a structure is a three-dimensional cohesive whole of structural elements that has to be able to resist external influences (the loads).

In many cases, structures appear to have been designed and built in such a way that the loads are transferred to the foundation via certain planes. In such cases, the three-dimensional structure can be modelled as a system of so-called *planar structures* (or two-dimensional structures). This is illustrated using two examples.

The first example is the bridge in Figure 4.27a. The loading by the traffic is transferred from the plane of the road deck to the vertical walls. These walls are in practice the spanning structure and transfer the load via the supports to the abutments, which subsequently transfer it to the foundation.

Surface elements (plates) can be used for the road deck and the walls; and together they form a so-called *trough bridge*. If the transverse measurements of the bridge are small compared to the span, the bridge can be modelled as a line element, or in other words, a bar with a U-section.

In order to limit the use of material and thereby reduce the self-weight that has to be carried, the surface elements can be replaced by planar structures made of line elements, as has been done in Figure 4.27b for the vertical walls.

The second example is the apartment building in Figure 4.28a. The structure consists of only surface elements. The vertical floor loading is transferred to the vertical walls and from there is transferred to the foundation. The horizontal wind loading is also distributed across the floors via the walls to the foundation.

Figure 4.28b represents the same building, but now all the horizontal and vertical surface elements in the main load-bearing structure have been replaced by planar structures made up of beams and columns. Although the structure now consists of only line elements, the transfer of forces is mostly unchanged and occurs through the same planes as in Figure 4.28a.

These examples illustrate that spatial structures can be composed of planar structures that consist of line elements. It is therefore certainly worth the effort of further investigating these types of planar structures.

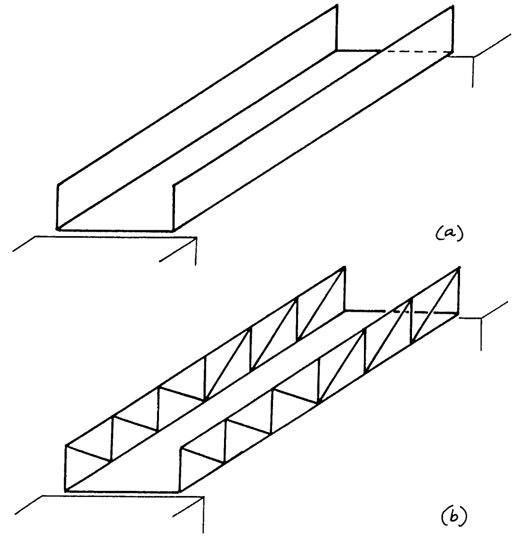


Figure 4.27 (a) A trough bridge, composed of surface elements; (b) the same bridge with the walls replaced by trusses.

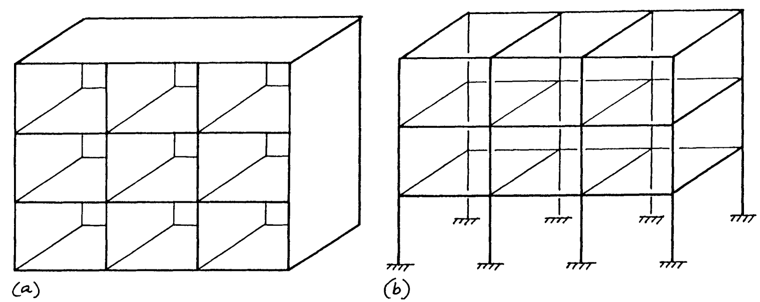
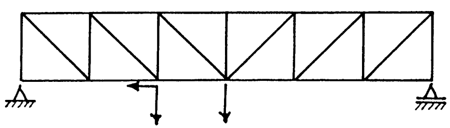
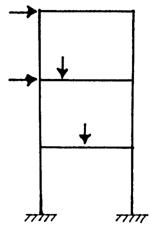


Figure 4.28 (a) An apartment building constructed of only surface elements; (b) the same building with the main load-bearing structure constructed of beams and columns.



(a) planar truss



(b) frame

Figure 4.29 (a) A truss with by definition solely hinged joints and (b) a frame with by definition exclusively rigid joints.

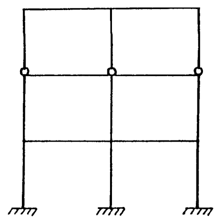


Figure 4.30 If there are also hinged joints in a frame, these have to be clearly indicated by means of open circles.

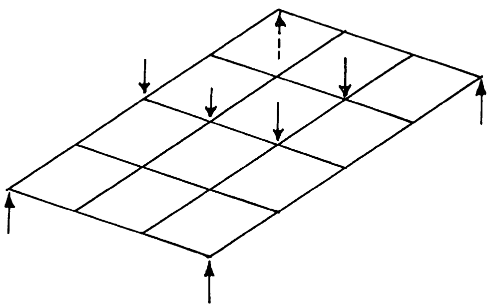


Figure 4.31 A beam grillage.

4.4.2 Planar trusses and frames

Planar *trusses* and *frames* are planar structures that are loaded *in their plane* (see Figure 4.29).

The difference between a truss and a frame is determined by the nature of the joints in the connections.

- in a truss, the bars are joined together by *hinges at all the connections*;[1]
- in frames, *all the joints are fixed and entirely stiff.*

The truss in Figure 4.29a appeared in the bridge in Figure 4.27b. The open circles, which represent the hinged joints, are generally omitted as in a truss all the joints are by definition hinged. The structure in Figure 4.29b is a frame. You will recognise part of the building in Figure 4.28b here, with the vertical floor loading and the horizontal wind loading. Sometimes the stiffnesses of the joints are accentuated by thickenings in the connections, but generally they are omitted. If there are also hinged joints in a frame, they have to be clearly depicted by means of open circles. This is the case in Figure 4.30, which could represent a building made of concrete, on which a steel floor was placed at a later stage.

4.4.3 Beam grillages

Beam grillages are planar structures that are loaded *normal to their plane*, see Figure 4.31. A beam grillage consists of two cooperative beam layers: beams and girders. The beams and girders are generally placed in two mutually perpendicular directions.

Beams grillages are often used as floor structures in bridges and buildings. Lock doors are also sometimes built as a system of beams and girders. A

[1] In Chapter 9, which addresses calculations related to trusses, another demand is covered, namely that the load has to be exerted only at connections.

façade made of posts and girders (columns and beams), with perpendicular wind loading, can sometimes also be seen as a beam grillage.

Calculating the forces and deformations in a beam grillage is in fact a three-dimensional problem. For information about the spatial character, refer to Section 3.3.4, examples 1 and 2.

4.4.4 Frames

Frames are planar, bent beams structures that are *loaded in the plane of the structure*. Such structures are often used to cover a space (warehouse, sports arena, and so forth).

Figure 4.32 shows a number of simple examples of *frames*. In Figure 4.33, both fixed supports have been replaced by hinged supports, so that the structure is now referred to as a two-hinged frame. If the structure with hinged supports itself consists of two parts joined by a hinge, this is referred to as a three-hinged frame (see Figure 4.34). If the beam structure is not bent but arched, then the structure in Figure 4.35a is called a two-hinged arch, and the structure in Figure 4.35b a three-hinged arch.

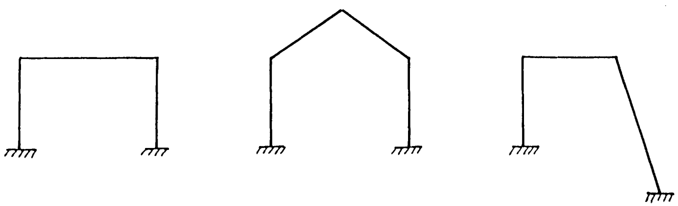


Figure 4.32 Examples of fixed frames.

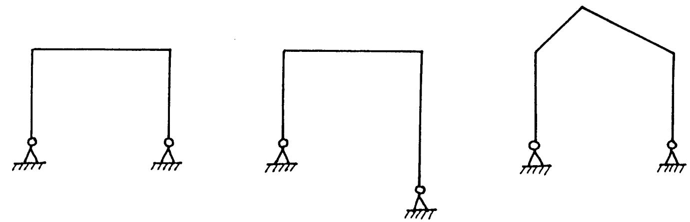


Figure 4.33 Examples of two-hinged frames.

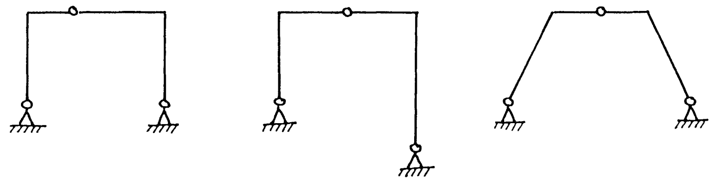


Figure 4.34 Examples of three-hinged frames.

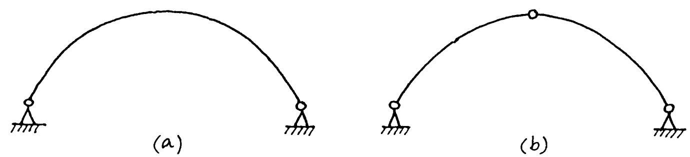


Figure 4.35 (a) A two-hinged arch and (b) a three-hinged arch.

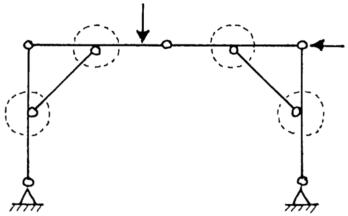


Figure 4.36 A shored frame.

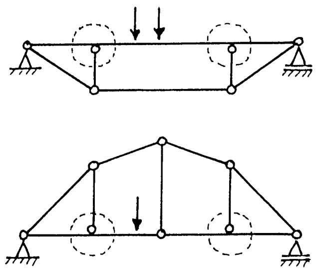


Figure 4.37 Examples of trussed beams.

4.4.5 Special structures

It will be clear that a wide range of planar structures can be constructed using line elements. Two types of structure not mentioned in the earlier categories are shown here. The structure in Figure 4.36 is called a *shored frame*. The structures in Figure 4.37 go by the name of trussed beams.

Although the structures in Figures 4.36 and 4.37 include hinges at all the connections, none of these structures are trusses. A characteristic of a truss is that *all the ends* of the members that merge in a connection are hinged together. This is not the case in the circled connections. Here the hinge is attached to the *outside* of a so-called *continuous beam*, and is *not fitted internally* in the beam.

4.5 Kinematic/static (in)determinate structures

Structures are supported in such a way that all free movements are prevented. This type of structure is known as *kinematically determinate* or *immovable*. If there are too few supports, or if they are not applied effectively, the structure, or part of the structure, will have a certain freedom of movement that is not resisted. The structure is then no longer immovable. This type of structure is called *kinematically indeterminate*, or is called a *mechanism*. If it is possible to calculate all the support reactions and interaction forces for a kinematically determinate structure using only equilibrium equations, it is called a statically determinate structure. If there are too many unknown forces to be able to determine them from the equilibrium, the structure is said to be *statically indeterminate*. In order to determine the forces in a statically indeterminate structure, the deformation of the structure must be taken into account.

4.5.1 Kinematically (in)determinate supported rigid structures

A *dimensionally stable structure* or *self-contained structure* is a structure that, isolated from its supports, retains its shape. If we neglect the deformations that occur, a self-contained structure can be seen as a rigid body. In a plane, a rigid body has *three degrees of freedom*: two components of a translation and one rotation, see Figure 4.38a.

In Figure 4.38b, the block is supported by a bar (two-force member) and is free to move in the direction perpendicular to the bar (on the condition that the movements remain small, see Section 4.3.1) and can rotate about A. The bar support at A reduces the three degrees of freedom of the body to two.

The freedom of movement can be limited further with a second bar support, for example at B (see Figure 4.38c). The movement that the body can now perform, with (minor) movement at A and B perpendicular to the bars, can be interpreted as a rotation about the so-called (*instantaneous*) *centre of rotation* RC, which is located on the intersection of the two bars.[1] With two bar supports the number of degrees of freedom of the block has been reduced to one.

The last degree of freedom, the rotation about RC, can be removed with a third bar support, for example at C (see Figure 4.38d). The three bars now prevent all possible movement. However the body is pulled or pushed, it remains where it is. This is referred to as the body having an *immovable* or *kinematically determinate* support.

Three bar supports (at least) are required for an immovable or kinematically

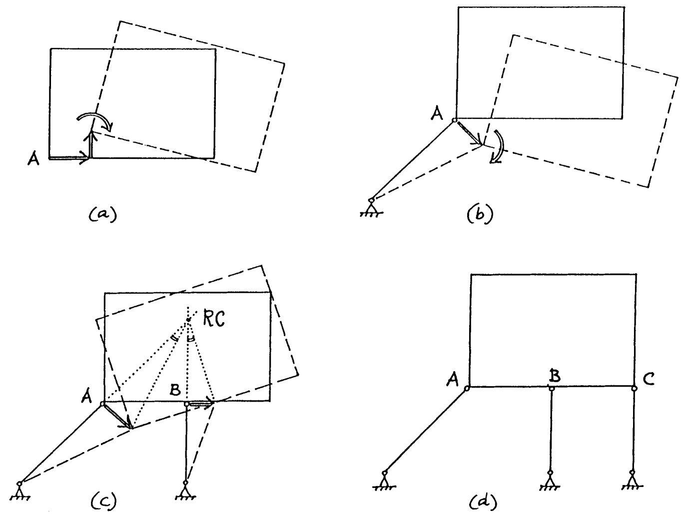


Figure 4.38 In a plane, a rigid body has three degrees of freedom; (b) due to the bar support at A this number is reduced to two; (c) two bar supports act as a hinged support at the centre of rotation RC, the point of intersection of the two bars; there is only one degree of freedom left: the rotation about RC; (d) with three bar supports, the body is immovable or kinematically determinate.

---

1 The fact that the centre of rotation RC is a fixed point is true only if the rotation is still small. When considering Figure 4.42, one should not be confused by the fact that the displacements have been drawn to a large scale with respect to the dimensions of the structure.

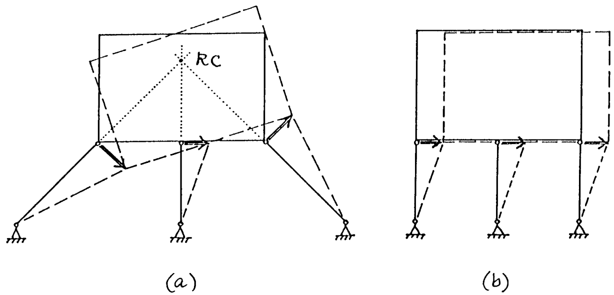


Figure 4.39 Examples of movable or kinematically indeterminate supports: the support permits (a) a rotation about RC and (b) a movement perpendicular to the supporting bars.

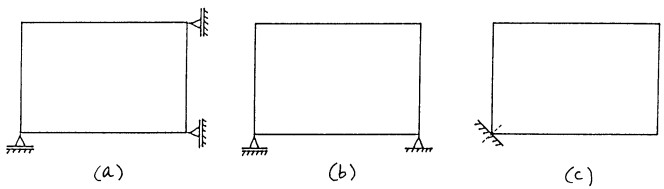


Figure 4.40 Examples of immovable or kinematically determinate supports with (a) three rollers, (b) a roller and a hinge, and (c) a fixed support.

determinate support of a rigid body. The bars may not all intersect in one point, or all be parallel, as is shown in Figure 4.39. In the support in Figure 4.39a, all bars intersect at the rotation centre RC, allowing the body to rotate. This support is *movable* or *kinematically indeterminate*. The support in Figure 4.39b, in which all bars are parallel to one another (intersect at a point at infinite distance), is also kinematically indeterminate, as the body is free to move in the direction perpendicular to the bar supports.

The similarities between bar supports and roller supports were repeatedly pointed out in Section 4.2. Figure 4.38c also shows that two bar supports act as a hinged support at the rotation centre RC, the intersection of the two bars.

An immovable support of a rigid body is therefore also possible with three roller supports, as in Figure 4.40a, or with a roller and hinged support, as in Figure 4.40b. It should be clear that a fixed support of a rigid body, as in Figure 4.40c, also is an immovable support.

4.5.2 Statically (in)determinate supported rigid structures

Instead of investigating the freedom of movement of a body, it is possible to determine also how many support reactions would be needed to keep the body in equilibrium under all imaginable loading conditions.

The support *reactions* adapt to the loading (the *action*) until equilibrium is reached. The unknown support reactions must therefore meet the conditions of the three equilibrium equations (in a plane) that apply to a rigid body. With three support reactions, there is an equal number of unknowns as equilibrium equations and, with the exception of a number of special cases which will be addressed later on, the support reactions can be deduced directly from the equilibrium. The support is then referred to as *statically determinate*.

An example is the rectangular block in Figure 4.41a, supported by three bars (two-force members). The resultant of the loading on the block is the force R, with components R_x and R_y (not shown).

In Figure 4.41b the block has been isolated from its supports and the unknown support reactions F_1, F_2, and F_3 are shown. For equilibrium, the following have to apply:

$$\sum F_x = -\tfrac{1}{2}\sqrt{2} \cdot F_1 + R_x = 0,$$

$$\sum F_y = -\tfrac{1}{2}\sqrt{2} \cdot F_1 - F_2 - F_3 + R_y = 0,$$

$$\sum T_z|\mathrm{D} = -F_3 \cdot b + R \cdot a = 0.$$

To keep the equation for the moment equilibrium transparent, it has been related to the intersection D of the lines of action of F_1 and F_2; here a is the perpendicular distance from D to the line of action of R.

In matrix notation, the three equilibrium equations are

$$\begin{bmatrix} \tfrac{1}{2}\sqrt{2} & 0 & 0 \\ \tfrac{1}{2}\sqrt{2} & 1 & 1 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} R_x \\ R_y \\ R \cdot a \end{bmatrix}.$$

We could wonder whether this system of three linear equations with three unknowns has a unique solution under all imaginable loading (i.e. for all possible values of R and a).

One way to find out is simply to try to solve the set of equations; we find that

$$F_1 = R_x\sqrt{2},$$

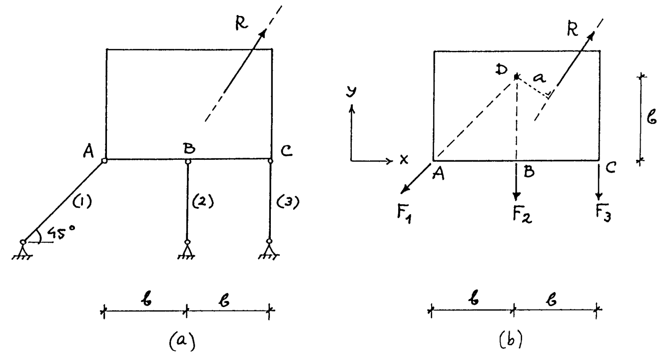


Figure 4.41 (a) A block supported by three bars; (b) the isolated block.

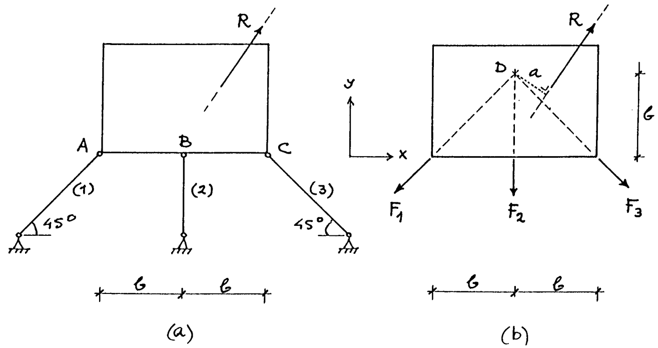


Figure 4.42 (a) A block supported by three bars; (b) the support is kinematically indeterminate as no moment equilibrium about D is possible.

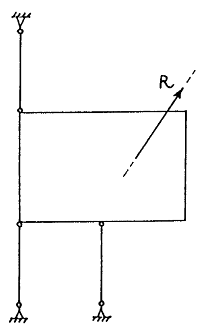


Figure 4.43 The support is kinematically indeterminate as no horizontal force equilibrium is possible.

$$F_2 = -R_x + R_y - \frac{a}{b}R,$$

$$F_3 = +\frac{a}{b}R.$$

The values of F_1 to F_3 exist for all values of a and R. The support is therefore *kinematically determinate* (equilibrium is possible with any arbitrary loading) and *statically determinate* (the support reactions can be determined from the equilibrium).

A more general answer is found in linear algebra: there is a unique solution if the determinant of the coefficient matrix is not equal to zero. This is indeed the case in this example:

$$\text{Det} = \tfrac{1}{2}\sqrt{2} \cdot b \neq 0.$$

The set of equations cannot be solved if the determinant of the coefficient matrix is zero. The figures in the coefficient matrix are determined by the manner in which the body is supported. The fact that the determinant is zero means that, from a physical perspective, the support is kinematically indeterminate.

In order to illustrate this, the bar support (3) in Figure 4.42 has been placed at an angle. With this type of support, the three equilibrium conditions can be represented by

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & 1 & +\frac{1}{2}\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} R_x \\ R_y \\ R \cdot a \end{bmatrix}.$$

The determinant of the coefficient matrix is now zero.

In the last equilibrium equation, the moment equilibrium about D, the condition $R \cdot a \neq 0$ cannot be met. Neither can the support reactions for $R \cdot a = 0$ be determined. The method of support in Figure 4.42 allows a rotation about D and is therefore *kinematically indeterminate*. This is in line with what we determined in the previous section for a support on three bars that pass through a single point.

The support in Figure 4.43, on three parallel bars, is also kinematically indeterminate. No force equilibrium is possible in the direction normal to the bars and the block is able to move in that direction. With less than three support reactions, there are more equilibrium equations than unknowns. Here the support is also kinematically indeterminate: the conditions for all three equilibrium equations cannot be met for arbitrary loading. In Figure 4.44 in case (a) moment equilibrium is not possible and a rotation occurs about A. In case (b) horizontal force equilibrium is not possible, and the block will move horizontally.

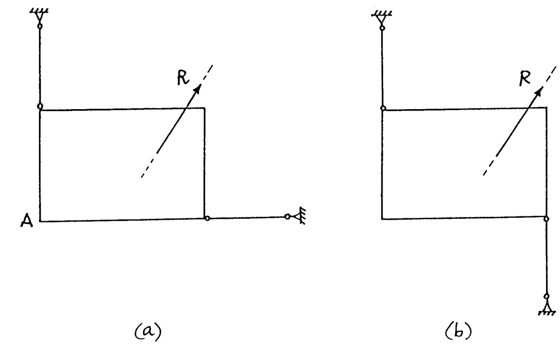


Figure 4.44 With less than three support reactions, the support is kinematically indeterminate: (a) moment equilibrium is not possible about A; (b) horizontal force equilibrium is not possible.

With more than three bar supports, such as in Figure 4.45, which do not all pass through a single point and are not all parallel, the support is immovable or kinematically determinate. The number of unknown support reactions is now larger than the available number of equilibrium equations and a unique solution is impossible. In fact, there is an infinity of solutions that satisfy the equilibrium equations. An immovable support of a rigid body or self-contained structure with more than three support reactions is therefore referred to as being *statically indeterminate* (or *hyperstatic*). In a statically indeterminate support, the support reactions cannot be deduced directly from the equilibrium, and the deformation of the structure will also have to be included in the consideration.

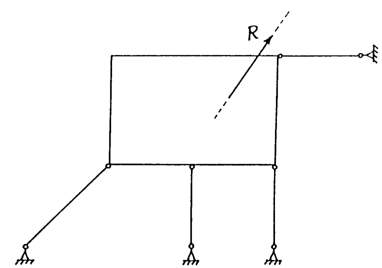


Figure 4.45 If the support of a rigid body or self-contained structure has more than three support reactions, then the support is statically indeterminate.

To summarise, for a rigid body or self-contained structure with r support reactions:

$r < 3$ the support is kinematically indeterminate (movable);

$r \geq 3$ the support is kinematically determinate (immovable), unless all the support reactions pass through a single point or are parallel to one another.

$r \geq 3$ is therefore a necessary, but not sufficient condition for kinematically determinate support of a rigid body or self-contained structure.

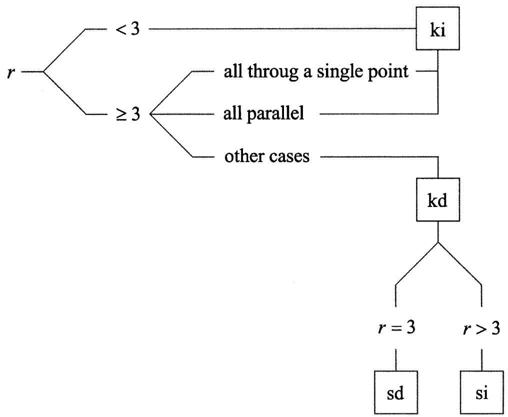


Figure 4.46 The relationship between the number of bar supports r (support reactions) and the **k**inematic/**s**tatic (**in**)**d**eterminacy of the support of a rigid body or self-contained structure.

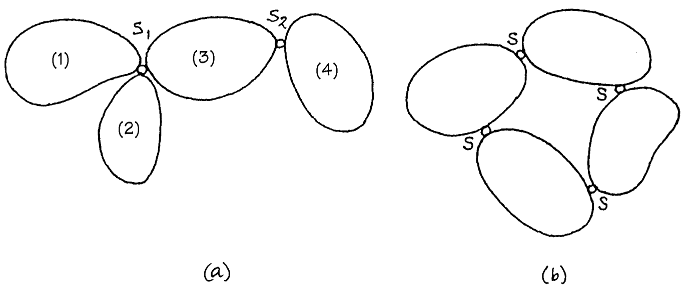


Figure 4.47 Compound structures: (a) with 6 degrees of freedom; (b) with 4 degrees of freedom.

If the support is kinematically determinate, the following distinctions are also possible:

$r = 3$ the support is statically determinate: all the support reactions follow directly from the equilibrium;

$r > 3$ the support is statically indeterminate: the three equilibrium equations are not enough to determine all the support reactions.

The statements above are summarised in Figure 4.46.

4.5.3 Kinematically/statically (in)determinate supported compound structures

So far, we have looked only at *dimensionally stable structures* or *self-contained structures*. In this section, we will look at *dimensionally unstable structures* or *compound structures*. Isolated from its supports, compound structures are unable to retain their shape, as the composite parts can move with respect to one another.

Figure 4.47 shows two examples of compound structures, without their supports. The structures consist of a number of rigid (or self-contained) parts (sub-structures), which are capable of rotating with respect to one another at the hinged joints S. For immovable or kinematically determinate supports, more than three bar supports (support reactions) are now required. The immovable support of the compound structure in Figure 4.47a needs at least six bar supports. Body (1) can be fixed with three bars (Figure 4.48a). Here, S_1 has become a hinged support for the bodies (2) and (3). For each of these bodies, one bar is sufficient to fix them (Figure 4.48b). Now only body (4) can still rotate around S_2, which can be prevented with a sixth bar (Figure 4.48c). To achieve an immovable support, more than six bar supports could also be used of course; six is the minimum required.

The number of bar supports (support reactions) required for an immovable support can also be deduced, as in Section 4.5.2, by analysing the equilibrium and comparing the number of unknowns (support reactions and interaction forces) with the number of equilibrium equations available.

Imagine

$r =$ number of support reactions,

$v =$ number of interaction forces,

$e =$ number of equilibrium equations,

and

$$n = r + v - e.$$

n is equal to the difference between the number of unknowns $(r + v)$ and the number of available equilibrium equations e.

If all the support reactions have been applied *effectively*, the following two cases can be distinguished:

- $n < 0$ – *the structure is kinematically indeterminate*
  The total number of unknown forces $r + v$ is smaller than the number of available equilibrium equations e. This means that the equilibrium equations cannot be solved for arbitrary loading. The structure may move under certain loads. The number of support reactions r is too small to remove all the degrees of freedom. The support is therefore kinematically indeterminate (movable). A kinematically indeterminate structure is also referred to as a *mechanism*.
  The negative value of n is equal to the number of *degrees of freedom* (movement possibilities) of the structure (or the mechanism).
- $n \geq 0$ – *the structure is kinematically determinate*
  For an immovable support (or kinematically determinate structure) it would seem that $n \geq 0$.

For kinematically determinate structures $(n \geq 0)$, one can distinguish between two cases:

- $n = 0$ – *the structure is statically determinate*
  The number of unknown forces $r + v$ is equal to the number of available

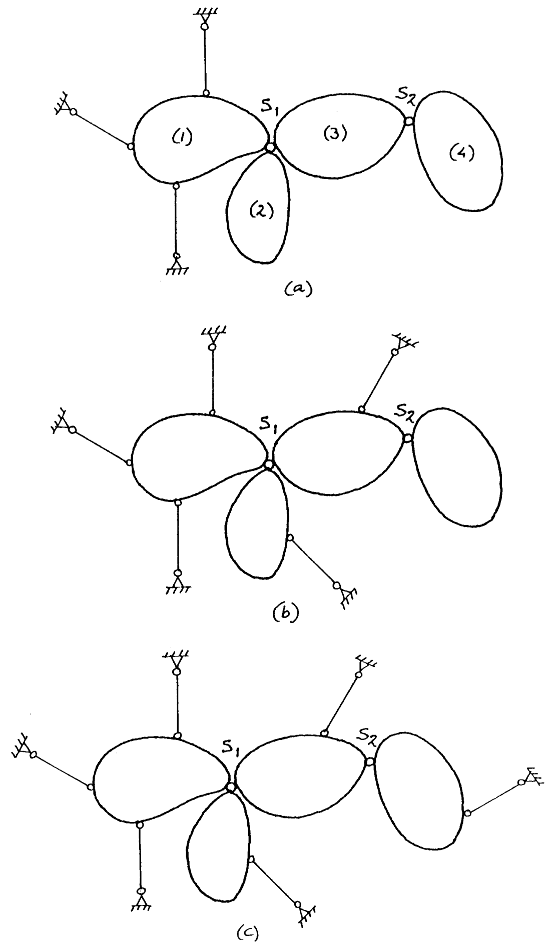


Figure 4.48 Each effectively applied bar support reduces the number of degrees of freedom by one.

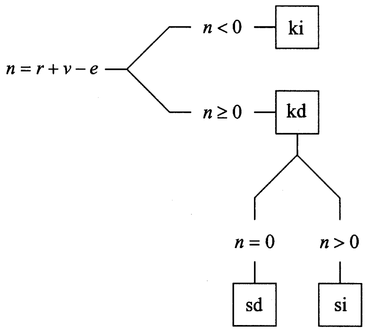


Figure 4.49 Summary of the **k**inematic/**s**tatic (**in**)**d**eterminacy of a structure.

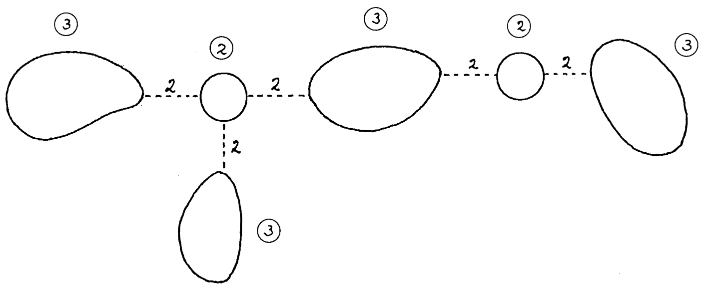


Figure 4.50 When isolating a compound structure, it falls apart into a number of (self-contained) sub-structures and a number of hinged joints.

equilibrium equations. All the support reactions and interaction forces can be determined on the basis of the equilibrium. The structure is statically determinate.

- $n > 0$ – *the structure is statically indeterminate*
  The number of unknowns is greater than the number of available equilibrium equations. An infinity of solutions satisfy the equilibrium equations. The structure is statically indeterminate. The value of n is called the *degree of static indeterminacy*.

Figure 4.49 provides a summary of these statements.

For the compound structure in Figure 4.47a, we will now determine with how many (*effectively placed*) support reactions the structure can be supported in an immovable way. To do so, it will be assumed that the self-contained sub-structures do not directly exert forces on one another, but that they do so via the *joints*. When isolated, the compound structure therefore falls apart into a number of sub-structures and a number of joints (see Figure 4.50).

There are two interaction forces at every joint between a sub-structure and a (hinged) joint. In the figure, the connections are shown by dotted lines, and the number of interaction forces is shown. There are a total of five joints, which brings the total number of unknown interaction forces to

$$v = 5 \times 2 = 10.$$

Each self-contained sub-structure gives three equilibrium equations (force equilibrium and moment equilibrium) and each hinged joint gives two equilibrium equations (only force equilibrium). These numbers are included in the circles in Figure 4.50.

With four sub-structures and two joints, the total number of available equilibrium equations becomes

$$e = 4 \times 3 + 2 \times 2 = 16.$$

Without support reactions ($r = 0$) this gives

$$n = r + v - e = 0 + 10 - 16 = -6.$$

The compound structure in Figure 4.47a therefore has six *degrees of freedom*.

The *minimum* number of required support reactions r for a kinematically determinate structure follows from the condition $n = 0$:

$$n = r + v - e = 0 \Rightarrow r = e - v = 16 - 10 = 6.$$

For an immovable support, six *effectively applied* bar supports (support reactions) are therefore sufficient. This is in line with what was found earlier (see Figure 4.48c): for, each bar support removes one degree of freedom.

We have frequently used the phrase *effectively applied* bar supports. In Figure 4.48c, all the bars have been applied effectively. If in Figure 4.48 the bar supports were not placed effectively, for example by using all the bars to support body (1), the structure would remain kinematically indeterminate even though the condition $n \geq 0$ is met.

The condition $n \geq 0$ for a kinematically determinate structure is not a sufficient condition, as it is always possible to apply the supports (ineffectively) so that the structure remains kinematically indeterminate. One must be aware of this.

An example of the above is the structure in Figure 4.51, in which three hinges are on a straight line. Imagine that the hinged joint S and both bodies are isolated. The hinged joint gives two equilibrium equations and each body gives three. The total number of available equilibrium equations is

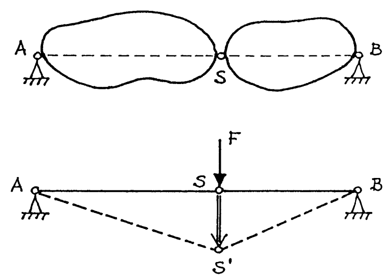


Figure 4.51 This structure, with three hinges in line at A, S and B, allows (minor) movement at S, and is therefore kinematically indeterminate.

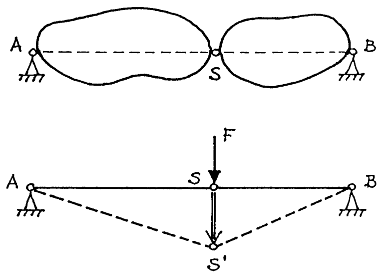


Figure 4.51 This structure, with three hinges in line at A, S and B, allows (minor) movement at S, and is therefore kinematically indeterminate.

then

$$e = 2 + 2 \times 3 = 8.$$

The number of interaction forces between joint S and both bodies is

$$v = 4.$$

A hinged support can provide two support reactions. For two hinged supports (A and B), it therefore applies that

$$r = 4.$$

This gives

$$n = r + v - e = 4 + 4 - 8 = 0.$$

Further investigation shows however that in case of a load normal to the line through the three hinges, the conditions for moment equilibrium cannot be met. For example, a vertical force at S can never create an equilibrium with the horizontal(!) support reactions at A and B. For such a load, the structure in S will allow (minor) movement. The structure is therefore kinematically indeterminate, even though $n = 0$.

4.5.4 Static (in)determinancy of a frame

A frame is a structure constructed of members that are connected to one another at *rigid* or *hinged* joints. In order to be able to determine the static (in)determinacy for a kinematically determinate frame, we use the procedure based on a consideration of equilibrium from the previous section: all members and joints in the structure are isolated. Joints are also assumed at the supports.

This is illustrated in Figure 4.52a; all bars and joints have been isolated in Figure 4.52b.

Each member gives three equilibrium equations (force equilibrium and moment equilibrium).

Two different types have to be distinguished as far as the joints are concerned:

- Joints on which only forces can be exerted (fully hinged[1] joints); they are shown as *circles*, and give two equilibrium equations (force equilibrium).
- Joints on which both couples and forces can be exerted (rigid[2] and incompletely hinged[3] joints); they are shown as *squares*, and give three equilibrium equations (force equilibrium and moment equilibrium).

The number of equilibrium equations that the members and joints introduce is included as a circled value in Figure 4.52b.

The four bars therefore give $4 \times 3 = 12$ equilibrium equations, the two fully hinged joints give $2 \times 2 = 4$ equilibrium equations while the other two joints give $2 \times 3 = 6$ equilibrium equations. The total number of available equilibrium equations is therefore

$$e = 12 + 4 + 6 = 22.$$

The connections between the members and the joints are shown in Figure 4.52b by means of dashed lines; also the number of interaction forces is shown. Here, we have to distinguish between the following:

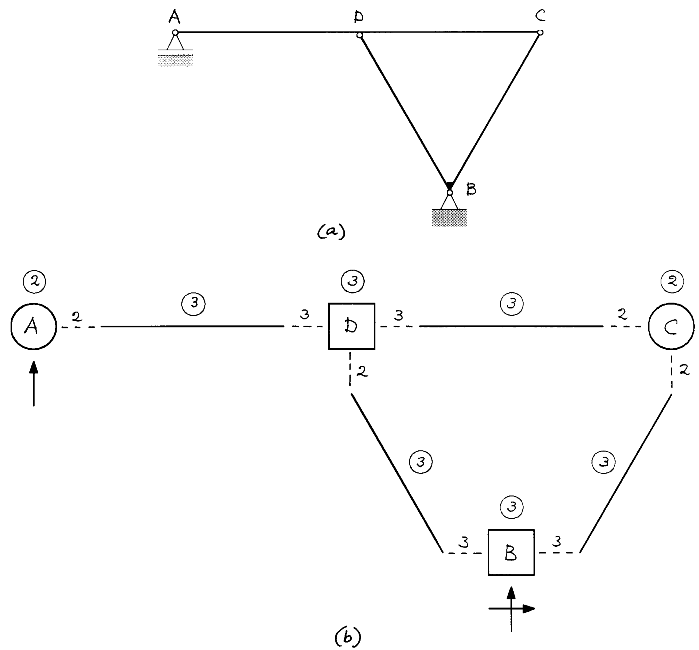


Figure 4.52 (a) A structure of which in (b) all the members and joints have been isolated. Joints have also been assumed at the supports. Joints only subject to forces are shown as circles; joints that can also be subject to couples are shown as squares.

[1] All the bars that meet at the joint are connected to the joint by a hinge.
[2] All the bars that meet at the joint are connected to the joint rigidly.
[3] Of all the bars that meet at the joint, some are connected to the joint by a hinge, and some are connected rigidly.

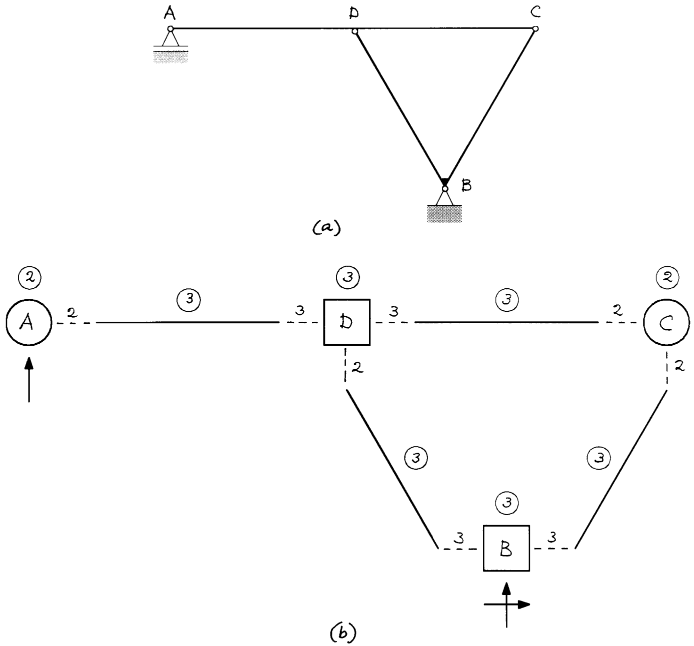


Figure 4.52 (a) A structure of which in (b) all the members and joints have been isolated. Joints have also been assumed at the supports. Joints only subject to forces are shown as circles; joints that can also be subject to couples are shown as squares.

- Hinged connections between the end of the member and the connection – two interaction forces are acting here, and
- Rigid connections between the end of the bar and the joint – three interaction forces are acting here.

With four hinged connections between member and joint and four rigid connections, the total number of interaction forces is

$$v = 4 \times 2 + 4 \times 3 = 20.$$

In Figure 4.52b the support reactions that can act on joints A and B are also shown. The roller support provides one support reaction, and the hinged support provides two. The total number of support reactions is therefore

$$r = 1 + 2 = 3.$$

For the difference n of the number of unknown forces (support reactions and interaction forces) and the number of available equilibrium equations, we arrive at

$$n = r + v - e = 3 + 20 - 22 = 1.$$

This means that the structure is statically indeterminate to the first degree: there is one unknown too many to be able to derive all the support reactions and interaction forces directly from the equilibrium.

By isolating the structure into all its smallest parts (members and joints) the procedure used can be laborious and prone to calculation errors. The static indeterminacy can often be found more quickly and with fewer calculations by releasing the structure into a number of larger parts. This is illustrated with help of the frame in Figure 4.53a.

With the section in Figure 4.53b, the structure falls apart into two self-contained parts. There are three equilibrium equations available per part.

Together both parts give $e = 2 \times 3 = 6$ equilibrium equations. The section was introduced across three members. Three unknown interaction forces are acting in each section. The total number of interaction forces is therefore: $v = 3 \times 3 = 9$. The hinged support gives two support reactions and the roller support gives one. Together that makes $r = 2 + 1 = 3$ unknown support reactions. The numbers of interaction forces and support reactions are shown in the figure.

The degree of static indeterminacy (the number of unknowns too many) is therefore

$$n = r + v - e = 3 + 9 - 6 = 6.$$

More generally speaking we can say that

$$n = r + v - 3s$$

in which

r = number of support reactions,

v = number of interaction forces in the section(s) applied,

s = number of rigid sections (sub-structures).

In this way, we find in Figure 4.53c that

$$n = (3 + 4 \times 3) - 3 \times 3 = 6.$$

And for the three sections in Figure 4.53d

$$n = (3 + 7 \times 3) - 3 \times 6 = 6.$$

A condition for an accurate result is that the section(s) has/have to be applied in such a way that the sub-structures are *singly-cohesive*. This means

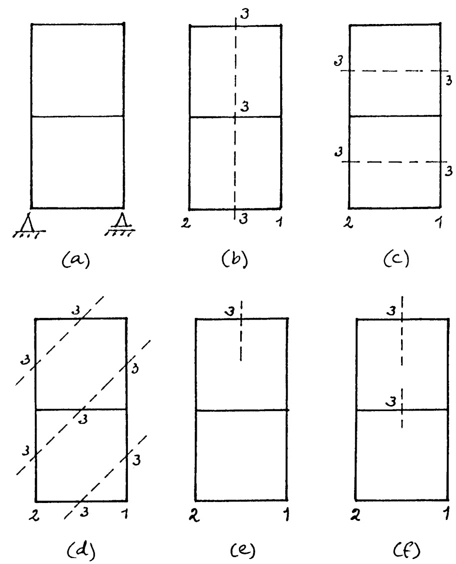


Figure 4.53 (a) An internally statically indeterminate frame; (b) to (d) with the sections shown, all the sub-structures are singly-cohesive; (e) with the section shown, the frame remains multiply-cohesive; (f) with these two sections, the frame becomes singly-cohesive.

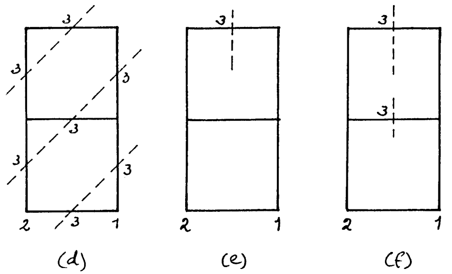


Figure 4.53 Internally statically indeterminated structure; (d) with the sections shown, all the sub-structures are singly-cohesive; (e) with this section, the frame remains multiply-cohesive; (f) with these two sections, the frame becomes singly-cohesive.

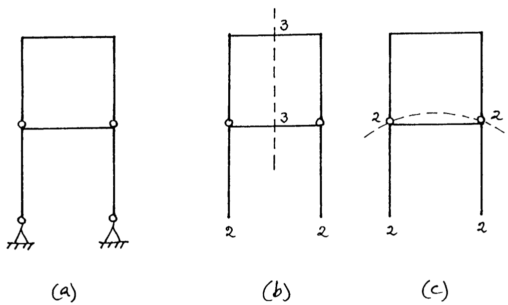


Figure 4.54 (a) A frame structure; (b) with the section shown the sub-structures are not self-contained; (c) with a section across the hinges, the sub-structures are self-contained.

that the cohesion in the sub-structure has to be such that for an arbitrary section across any member, the sub-structure has to fall apart into two new self-contained (or rigid) parts.

For example, it is not possible to determine the static indeterminacy for the section in Figure 4.53e. The structure is not singly-cohesive, as the extra 'cut', in Figure 4.53f does not make the structure fall apart into two new self-contained (or rigid) parts. In contrast, the static indeterminacy can be found for the two 'cuts' in Figure 4.53f. This structure is singly-cohesive, as each extra 'cut' over any member makes the structure fall apart into two new parts.

The degree of static indeterminacy is

$$n = (3 + 2 \times 3) - 3 \times 1 = 6.$$

Note that the support reactions of the six-fold *statically indeterminate structure* in Figure 4.53 can be found directly from the equilibrium equations. The support of the structure therefore is *statically determinate*. A statically indeterminate structure for which one can find the support reactions directly from the equilibrium is also said to be *internally statically indeterminate*.

If, as in Figure 4.54a, there are hinged joints in a structure, you have to be aware whether the parts into which you split the structure are self-contained and retain their shape. In that respect, the section in Figure 4.54b is not effective. You should choose the section across the hinges here, see Figure 4.54c. The degree of static indeterminacy is

$$n = (4 + 4) - 3 \times 2 = 2.$$

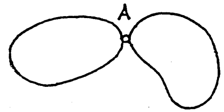
There is no simple recipe to determine the degree of static indeterminacy quickly. The approach depends on the insight into how forces are transferred within structures; this insight develops with experience.

4.6 Problems

Joints between structural elements (Section 4.2)

4.1 Two bodies are joined in hinge A.

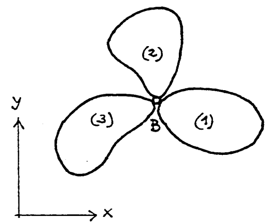
Question:
How many (independent) interaction forces are there at A?



4.2 Three bodies (1), (2), and (3) are connected at B by a hinge. The bodies exert forces on one another via joint B. The joint is modelled as a particle element.

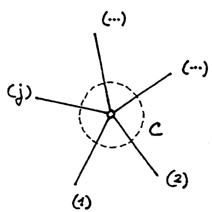
Question:

a. Isolate the bodies at B, draw all the interaction forces acting between the bodies and joint B, and name them in the xy coordinate system shown.
b. How many equilibrium equations are available for the joint?
c. How many independent interaction forces are there at B?



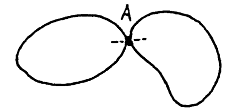
4.3 In joint C, j bars are connected by a hinge.

Question:
Derive the relationship between the number of joined bars j and the number of independent interaction forces i at C.



4.4 In A, two bodies are connected rigidly.

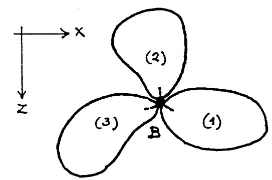
Question:
How many (independent) interaction forces are there at A?



4.5 The three bodies (1), (2), and (3) are rigidly connected at B. The bodies exert forces on one another via joint B. The joint is modelled as a particle element.

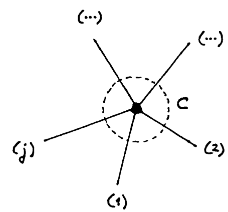
Question:

a. Isolate the bodies at B, draw all the interaction forces between the bodies and joint B, and name them in the xz coordinate system shown.
b. How many equilibrium equations are there for the joint?
c. How many independent interaction forces are there at B?

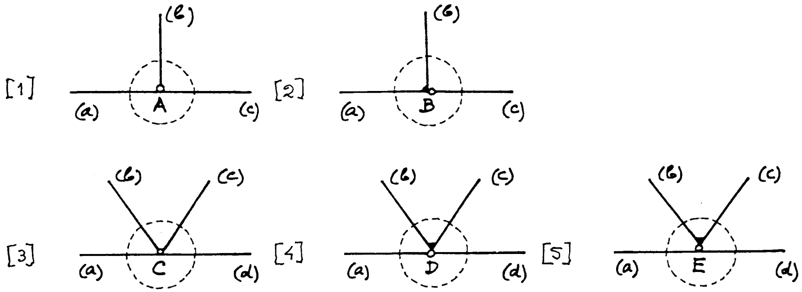


4.6 At joint C, j bars are rigidly connected.

Question:
For joint C, derive the relationship between the number of independent interaction forces i and the number of rigidly joined bars j.



4.7: 1–5 A number of bars are connected in a variety of ways at a joint.



Question:

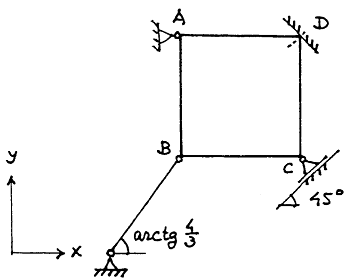
a. Determine the number of connection forces between each of the bar ends and the joint.
b. Determine the number of equilibrium equations available for the joint.
c. Determine the number of independent connection forces at the joint.

Supports (Section 4.3)

4.8 A square block ABCD is supported at its four corners as shown. If the block is loaded, it will deform.

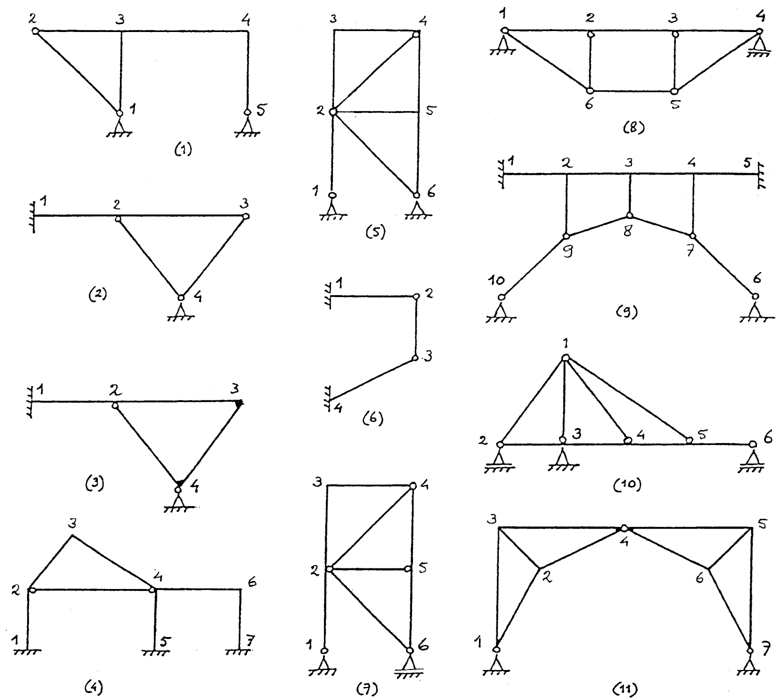
Question:

a. How many displacements, and in which directions, do the supports at the corners permit?
b. How many displacements, and in which directions, are prescribed at the corners by the supports?



c. How many and which forces can develop freely at the supports?
d. How many and which forces are prescribed in the corners by the supports?

4.9: 1–11 A number of structures are shown.

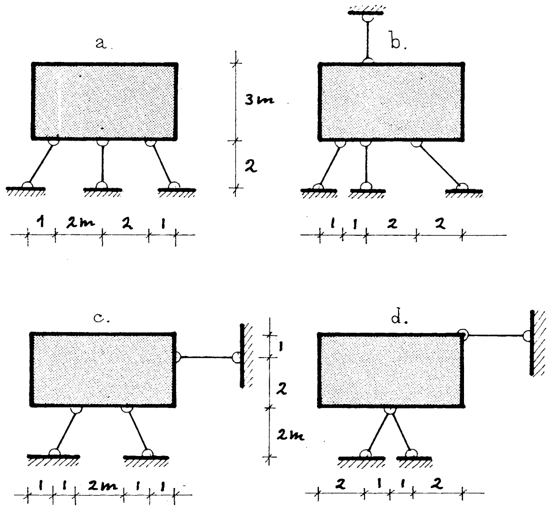


Question:

a. What types of support are being used?
b. How many and which support reactions will these supports supply?

Kinematic/static (in)determinacy of structures (Section 4.5)

4.10 A block is supported in four different ways.



Question:
Which support method is not effective?

4.11: 1–16 A rectangular block is supported in a variety of ways.

Question:

a. Determine whether the support is kinematically determinate (kd) or kinematically indeterminate (ki).

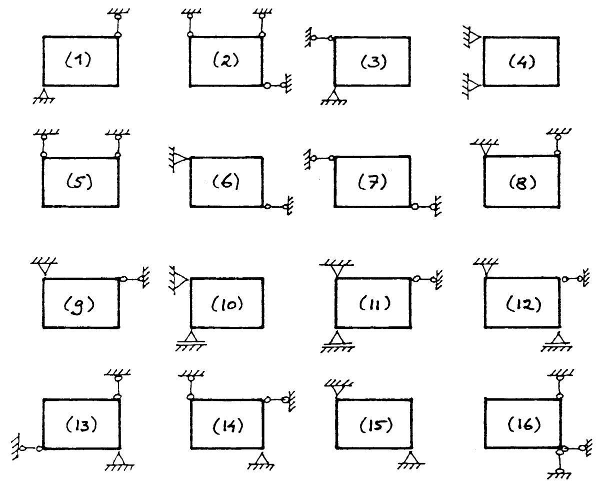
Do so in two different ways:

– first investigate the freedom of movement for the method of support given, and
– secondly count the number of support reactions present, and if there are enough, determine whether the support reactions are situated properly (and can form an equilibrium system with an arbitrary loading).

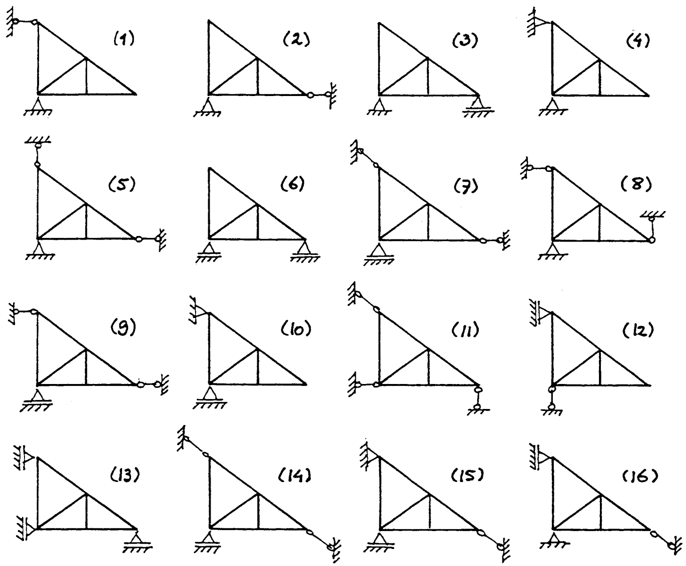
b. If the support is kinematically indeterminate, give the number of degrees of freedom v.

c. If the support is kinematically determinate, indicate whether the support is statically determinate (sd) or statically indeterminate (si).

d. If the support is statically indeterminate, give the degree of static indeterminacy n.



4.12: 1–16 A rigid truss is supported in a variety of ways.



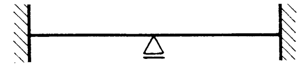
Question:

a. For each of the cases, determine whether the support is kinematically determinate (kd) or kinematically indeterminate (ki).
b. If the support is kinematically indeterminate, give the number of degrees of freedom v.
c. If the support is kinematically determinate, indicate whether the support is statically determinate (sd) or statically indeterminate (si).
d. If the support is statically indeterminate, give the degree of static indeterminacy n.

4.13 Which of the following statements is true for the beam shown?

The beam is:

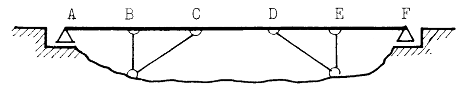
a. Kinematically determinate.
b. Statically determinate.
c. Statically indeterminate to the fourth degree.
d. Statically indeterminate to the seventh degree.



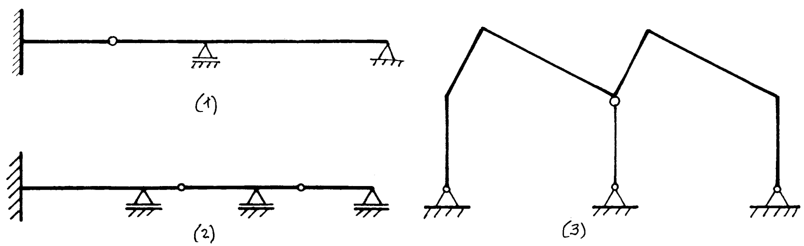
4.14 A bridge beam is resting on roller supports at A and F, and on bar supports at B, C, D, and E.

Question:
What is the degree of static indeterminacy of this structure?

a. 2.
b. 3.
c. 4.
d. 5.



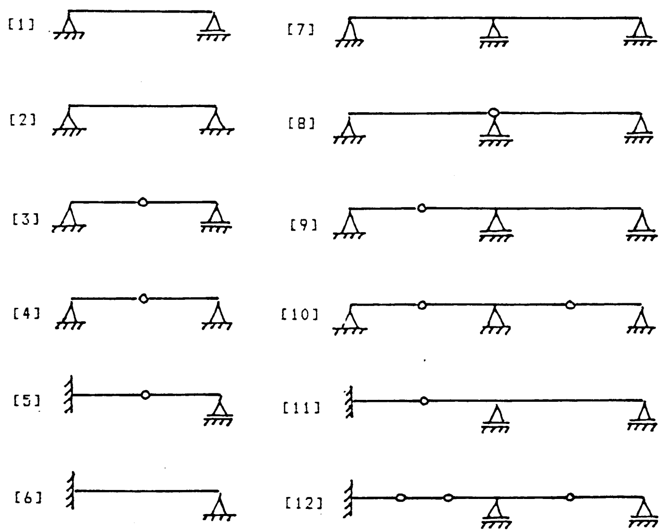
4.15: 1–3



Which statement applies to the structure shown? The structure is:

a. Statically determinate.
b. Statically indeterminate to the first degree.
c. Statically indeterminate to the second degree.
d. Statically indeterminate to the third degree.
e. Statically indeterminate to the fifth degree.
f. Statically indeterminate to the sixth degree.

4.16: 1–12 A number of beams and hinged beams are supported as shown.

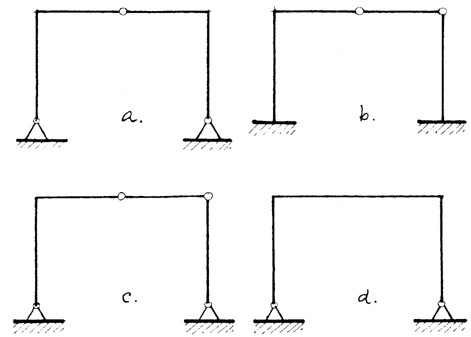


Questions:

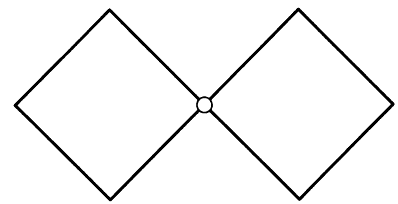
a. Is the structure kinematically determinate or indeterminate? If the structure is kinematically indeterminate, show the movement (displacements) that can occur freely. If the structure is kinematically determinate, go to question b.

b. Is the structure statically determinate or indeterminate? If the structure is statically indeterminate, give the degree of static indeterminacy n.

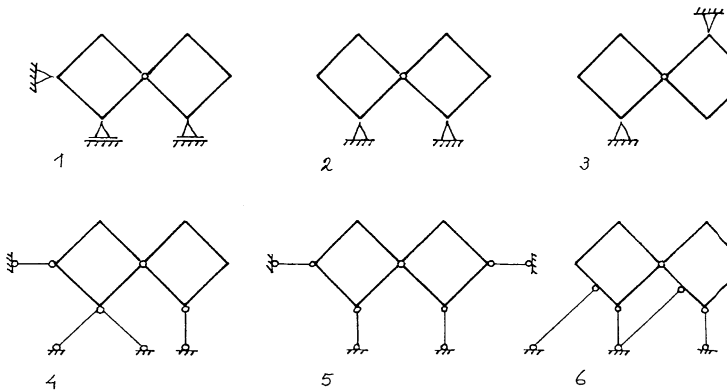
4.17 *Question*:
Which structure is kinematically determinate and statically indeterminate?



4.18 *Question*:
Show that for a kinematically determinate support of the two blocks connected by a hinge, four support reactions are required.



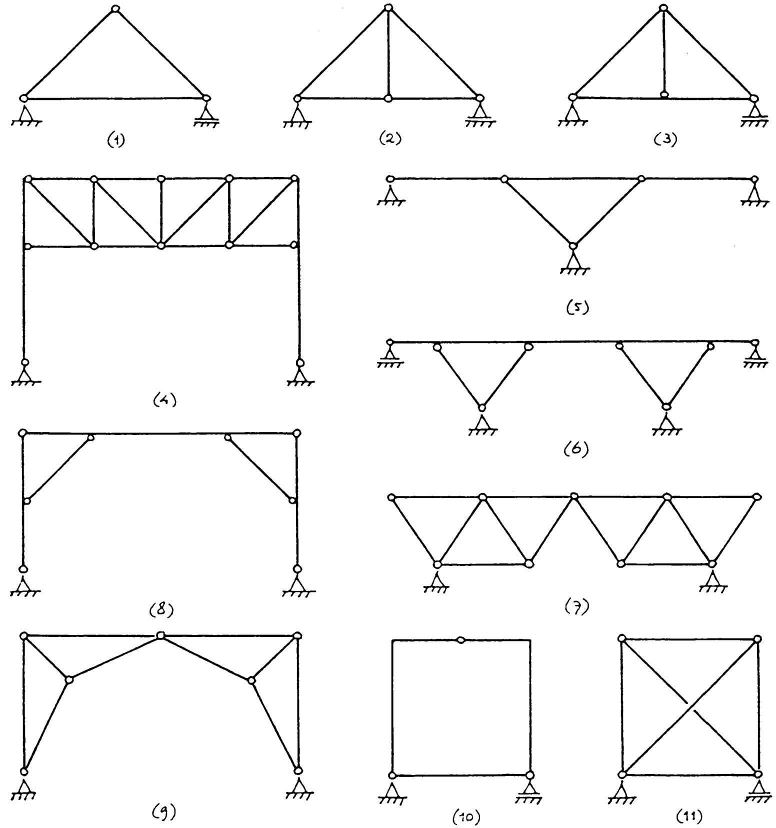
4.19: 1–6 Two square blocks are connected by a hinge and supported in a variety of ways.



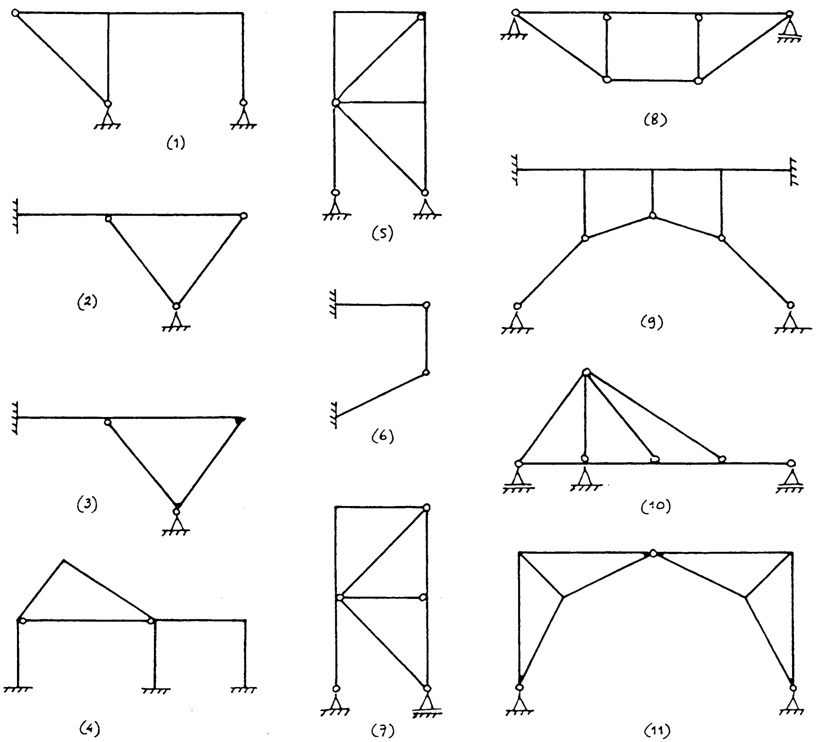
Question:
Determine whether the number of two-force members for the given method of support is sufficient.

4.20: 1–11 A number of kinematically determinate structures are shown.
Question:

a. Is the structure a truss or not?
b. Is the structure statically determinate or indeterminate?
c. If the structure is statically indeterminate, give the degree of static indeterminacy.



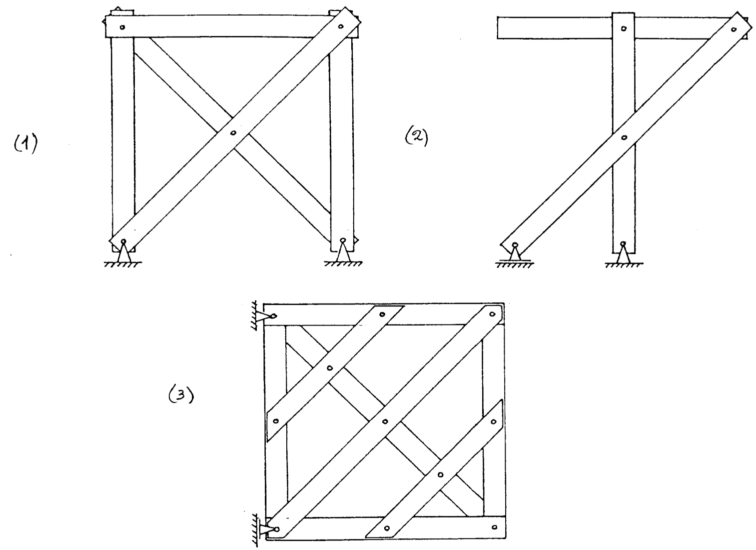
4.21: 1–11 A number of kinematically determinate bar structures are shown.



Question:

a. Is the structure statically determinate or indeterminate?
b. If the structure is statically indeterminate, indicate the degree of static indeterminacy.

4.22: 1–3 The structures shown are constructed from a number of planks. All the joints are hinges.



Question:

a. Is the structure a truss?
b. Is the structure statically determinate or indeterminate?
c. If the structure is statically indeterminate, give the degree of static indeterminacy.

5 Calculating Support Reactions and Interaction Forces

In this chapter, we will see how to calculate support reactions and interaction forces in statically determinate bar-type structures from the equilibrium equations using a number of examples.

For compound structures, if you write down all the available equilibrium equations and then try to solve the system, you soon end up with a large number of calculations. To prevent this, you have to select the equilibrium equations in a sensible order, preferably in such a way that an unknown force can be calculated directly with each new equilibrium equation.

The strategy for determining all the forces as efficiently as possible depends to a large degree on the type of structure. For this reason, in addition to self-contained structures, we will also look at compound and related structures, such as hinged beams, three-hinged frames (with or without tie-rods), shored structures and trussed beams.

The loading remains limited to a few point loads. In one case, the structure is loaded by a concentrated couple.

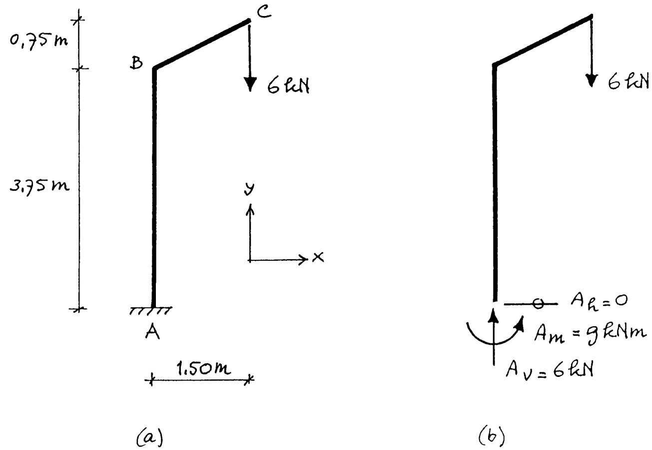


Figure 5.1 (a) A light mast fixed at A with (b) the support reactions.

5.1 Self-contained structures

In this section, we will use five examples to show how, for statically determinate self-contained structures, it is possible to determine the support reactions and interaction forces directly from the equilibrium.

Example 1
The light mast ABC in Figure 5.1a is fixed at A and is loaded at C by a vertical force of 6 kN.

Question:
Draw the support reactions at A as they are expected to act and determine them.

Solution:
No horizontal loading is being exerted on the mast. The horizontal support reaction at A is therefore zero. The vertical support reaction at A must generate an equilibrium with the vertical force of 6 kN, and will therefore be pointed upwards. In order to determine the fixed-end moment, the isolated structure is considered to be pinned at A. The load causes a clockwise rotation about A. The fixed-end moment has to prevent this rotation and will therefore act counter-clockwise. The support reactions are shown in Figure 5.1b. The equilibrium equations are

$$\sum F_x = A_\mathrm{h} = 0,$$

$$\sum F_y = -(6\text{ kN}) + A_\mathrm{v} = 0,$$

$$\sum T_z|\mathrm{A} = -(6\text{ kN}) \times (1.5\text{ m}) + A_\mathrm{m} = 0.$$

The solution is

$$A_\mathrm{h} = 0,\ \ A_\mathrm{v} = 6\text{ kN}\ \ \text{and}\ \ A_\mathrm{m} = 9\text{ kNm}.$$

The fact that the solutions found are positive confirms the correctness of the directions assumed for these support reactions.

Note that the support reaction A_v and the force of 6 kN at C together form a couple that is in equilibrium with the fixed-end moment A_m.

Example 2
In Figure 5.2a, a block with a weight of 60 kN is supported on three bars.

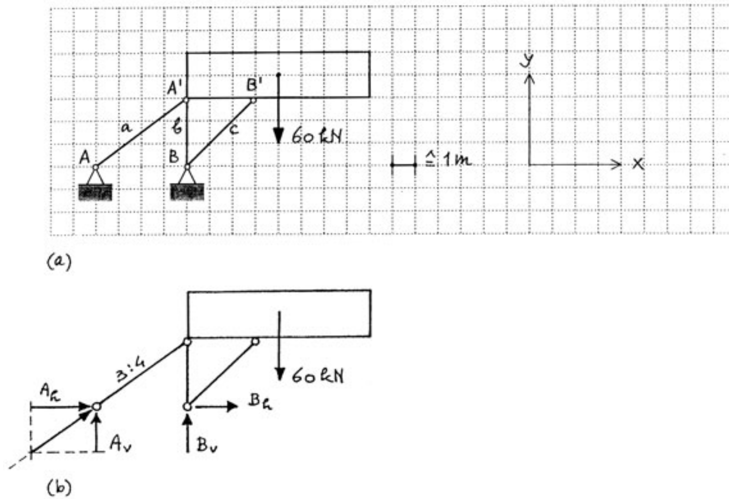


Figure 5.2 (a) A block supported on three bars; (b) the assumed directions of the support reactions at A and B.

Questions:

a. Determine the support reactions at A and B.
b. Determine the forces $N^{(a)}$, $N^{(b)}$ and $N^{(c)}$ in the bars[1] with the correct sign for tension and compression, based on the convention that a force N as tensile force is positive and as compressive force is negative.

Solution (units in kN and m):
a. Figure 5.2b shows the support reactions. The directions of A_h and A_v are such that the line of action of their resultant coincides with two-force member (a). For the others, the directions of the support reactions have been assumed arbitrarily.

On the basis of the slope of bar (a), it follows that $A_h/A_v = 4/3$, or $A_h = (4/3)A_v$. A_v can be determined using the moment equilibrium about B:

$$\sum T_z|\mathrm{B} = -A_v \times 4 - 60 \times 4 = 0 \Rightarrow A_v = -60 \text{ kN}$$

so that

$$A_h = \tfrac{4}{3}A_v = -80 \text{ kN}.$$

[1] The upper index indicates the relevant bar. The brackets can be omitted as they do not create any confusion.

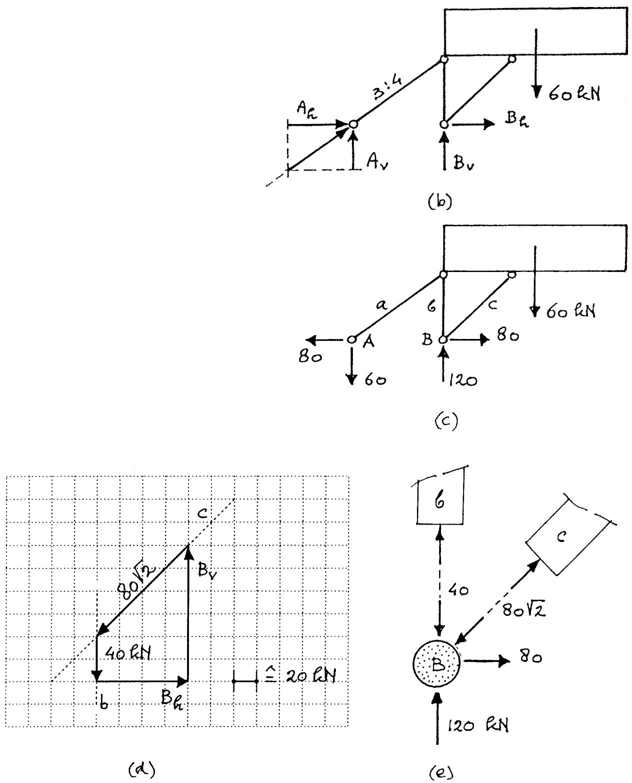


Figure 5.2 (b) The assumed directions of the support reactions in A and B; (c) the support reactions as they are actually acting; (d) the closed force polygon for the force equilibrium of joint B; (e) isolated joint B with all the forces acting on it.

B_h is found from the horizontal force equilibrium:

$$\sum F_x = A_h + B_h = -80 + B_h = 0 \Rightarrow B_h = 80 \text{ kN}.$$

B_v follows from the vertical force equilibrium:

$$\sum F_y = A_v + B_v - 60 = -60 + B_v - 60 = 0 \Rightarrow B_v = 120 \text{ kN}.$$

B_v can also be determined from the moment equilibrium about A.

In Figure 5.2c, the support reactions are shown as they act in reality. Only the direction of the support reactions at A was falsely assumed.

b. Figure 5.2c shows directly that a tensile force is acting in bar (a):

$$N^{(a)} = \sqrt{A_h^2 + A_v^2} = \sqrt{80^2 + 60^2} = 100 \text{ kN}.$$

The forces in the bars (b) and (c) can be determined from the force equilibrium of joint B. The force polygon in Figure 5.2d shows that bar (b) exerts a force of 40 kN on joint B. This force "pushes" against the joint. Figure 5.2e shows the interaction forces between bar and joint. In bar (b), there is a compressive force $N^{(b)} = -40$ kN. Bar (c) is exerting a force of $80\sqrt{2}$ kN on joint B, also a compressive force, so that $N^{(c)} = -80\sqrt{2}$ kN.

Alternative solution (units in kN and m):
The questions a and b are now answered in reverse order.

b. In Figure 5.3a, the block has been isolated at A′ and B′. $N^{(a)}$, $N^{(b)}$ and $N^{(c)}$ are the (tensile) forces that the bars are exerting on the block. In Figure 5.3b, they have been resolved into their components.

$N^{(c)}$ follows from the moment equilibrium about A′:

$$\sum T_z|\mathrm{A}' = -60 \times 4 - \tfrac{1}{2}\sqrt{2}\,N^{(c)} \times 3 = 0 \Rightarrow N^{(c)} = -80\sqrt{2}\ \mathrm{kN}.$$

$N^{(a)}$ now follows from the horizontal force equilibrium:

$$\begin{aligned}\sum F_x &= -\tfrac{4}{5}N^{(a)} - \tfrac{1}{2}\sqrt{2}N^{(c)}\\ &= -\tfrac{4}{5}N^{(a)} - (-80) = 0 \Rightarrow N^{(a)} = 100\ \mathrm{kN}.\end{aligned}$$

Finally, $N^{(b)}$ can be derived from the vertical force equilibrium:

$$\begin{aligned}\sum F_y &= -60 - \left(N^{(b)} + \tfrac{3}{5}N^{(a)}\right) - \tfrac{1}{2}\sqrt{2}N^{(c)}\\ &= -60 - (N^{(b)} + 60) - (-80) = 0 \Rightarrow N^{(b)} = -40\ \mathrm{kN}.\end{aligned}$$

$N^{(a)}$ is a tensile force; $N^{(b)}$ and $N^{(c)}$ are compressive forces (see Figure 5.3c).

a. The support reactions at A and B now follow from the force equilibrium of the joints A and B. Figure 5.3d shows the forces that the bars are exerting on the joints. It is not difficult to see that the support reactions are acting in the directions shown in the figure. This gives

$$A_h = 80\ \mathrm{kN},\ A_v = 60\ \mathrm{kN},\ B_h = 80\ \mathrm{kN}\ \text{ and }\ B_v = 40 + 80 = 120\ \mathrm{kN}.$$

Example 3
Figure 5.4 represents a schematisation of a *retaining wall on piles* shown as a two-dimensional problem. Assume the piles are exclusively transferring forces in their longitudinal direction. In that case they can be considered two-force members. The resultant of the total loading carried by the piles

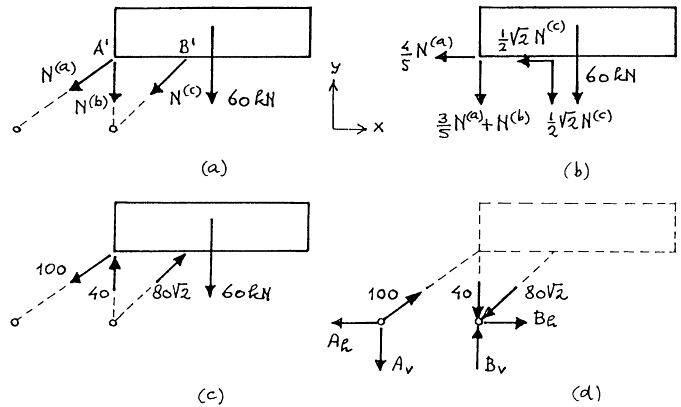


Figure 5.3 (a) The block isolated at A′ and B′, assuming that all the bars are tension members; (b) the forces acting on the block resolved into horizontal and vertical components; (c) the forces actually exerted by the bars on the block; (d) the support reactions at A and B are found from the force equilibrium of the joints A and B.

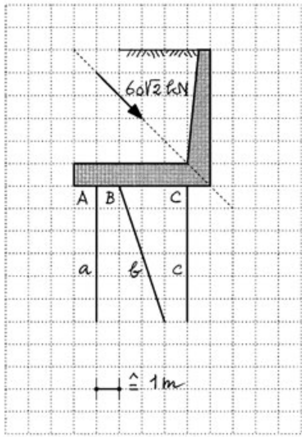


Figure 5.4 A retaining wall on piles.

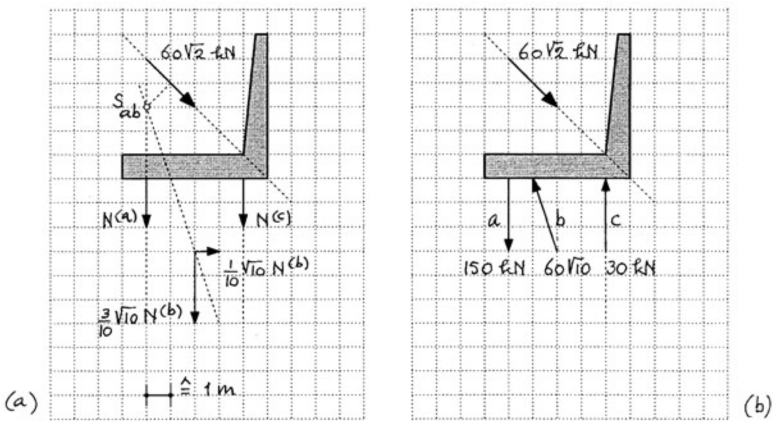


Figure 5.5 (a) The isolated retaining wall, in which it has been assumed that all the piles exert tensile forces on the bottom plate; (b) the pile forces as they really act on the bottom plate.

is a force of $60\sqrt{2}$ kN, of which the direction and line of action are shown in the figure.

Question:
Determine the pile forces, with the correct sign for tension and compression:

a. analytically;
b. graphically.

Solution:
a. Analytical method (units in kN and m):
In Figure 5.5a, the retaining wall has been isolated and the (tensile) forces $N^{(a)}$, $N^{(b)}$ and $N^{(c)}$ that the piles exert on the bottom plate are shown. To keep the picture simple, $N^{(b)}$ has been shifted somewhat along its line of action.

$N^{(b)}$ follows from the horizontal force equilibrium:

$$\sum F_x = \tfrac{1}{10}\sqrt{10}\, N^{(b)} + 60 = 0 \Rightarrow N^{(b)} = -60\sqrt{10}\,\text{kN}.$$

$N^{(c)}$ can be found from the moment equilibrium about A. One could also take the moment equilibrium about intersection $\mathrm{S_{ab}}$ of the piles (a) and (b), which works faster in this case:

$$\sum T_z|\mathrm{S_{ab}} = -60\sqrt{2} \times \sqrt{2} - N^{(c)} \times 4 = 0 \Rightarrow N^{(c)} = -30\,\text{kN}.$$

Finally, $N^{(a)}$ can be derived from the vertical force equilibrium:

$$\sum F_y = -N^{(a)} - \tfrac{3}{10}\sqrt{10}\, N^{(b)} - N^{(c)} - 60 = 0 \Rightarrow N^{(a)} = +150\,\text{kN}.$$

Figure 5.5b shows the forces as they act on the structure. In pile (a) there is a tensile force while there is a compressive force in piles (b) and (c).

b. Graphical method (see Section 3.1.8):
The pile forces can also be found graphically. Imagine $\vec{F}^{(a)}$, $\vec{F}^{(b)}$ and $\vec{F}^{(c)}$ are the forces that the piles exert on the structure. These forces have to be in equilibrium with the load $\vec{F}$, so that:

$$\vec{F}^{(a)} + \vec{F}^{(b)} + \vec{F}^{(c)} = \vec{F}$$

or

$$\vec{F}^{(a)} + \vec{F}^{(b)} = \vec{F} - \vec{F}^{(c)}.$$

In addition to the force equilibrium there also has to be moment equilibrium. Therefore $(\vec{F}^{(a)} + \vec{F}^{(b)})$ and $(\vec{F} - \vec{F}^{(c)})$ have a common line of action. The line of action of $(\vec{F}^{(a)} + \vec{F}^{(b)})$ passes through S_{ab} and that of $(\vec{F} - \vec{F}^{(c)})$ passes through P, see the line of action figure in Figure 5.6a. The common line of action is therefore PS_{ab}.

Since $(\vec{F}^{(a)} + \vec{F}^{(b)})$ and $\vec{F}^{(c)}$ are in equilibrium with $\vec{F}$ in P, $(\vec{F}^{(a)} + \vec{F}^{(b)})$ and $\vec{F}^{(c)}$ can be obtained from a force polygon (see Figure 5.6b). $(\vec{F}^{(a)} + \vec{F}^{(b)})$ can then be resolved in S_{ab} into $\vec{F}^{(a)}$ and $\vec{F}^{(b)}$. The force polygon in Figure 5.6b now shows:

$$\vec{F}^{(a)} = 150\text{ kN}\downarrow;\ \vec{F}^{(b)} = 60\sqrt{10}\text{ kN}\uparrow \quad \text{and} \quad \vec{F}^{(c)} = 30\text{ kN}\uparrow.$$

These are the forces that the piles exert on the retaining wall. Translated into the pile forces with the correct sign for tension and compression one now finds:

$$N^{(a)} = +150\text{ kN};\ N^{(b)} = -60\sqrt{10}\text{ kN} \quad \text{and} \quad N^{(c)} = -30\text{ kN}.$$

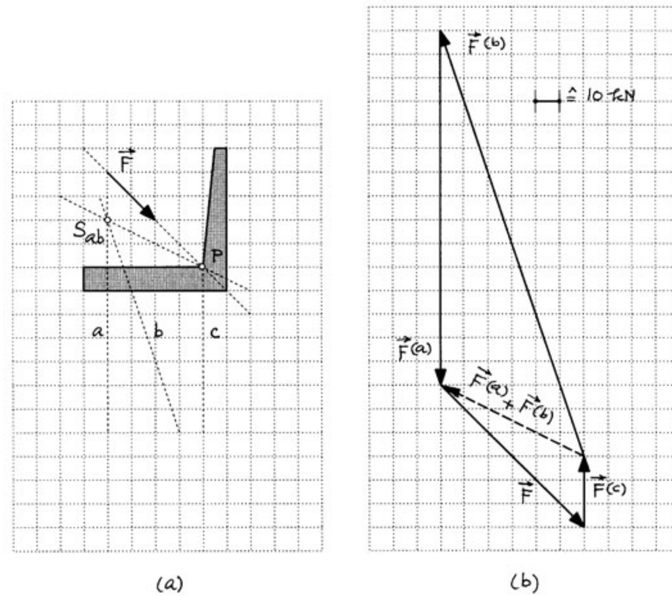


Figure 5.6 Graphical method for finding the pile forces: (a) line of action figure and (b) force polygon.

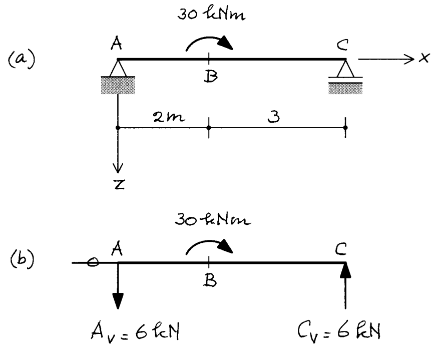


Figure 5.7 (a) A beam loaded by a couple at joint B; (b) the isolated beam with its support reactions.

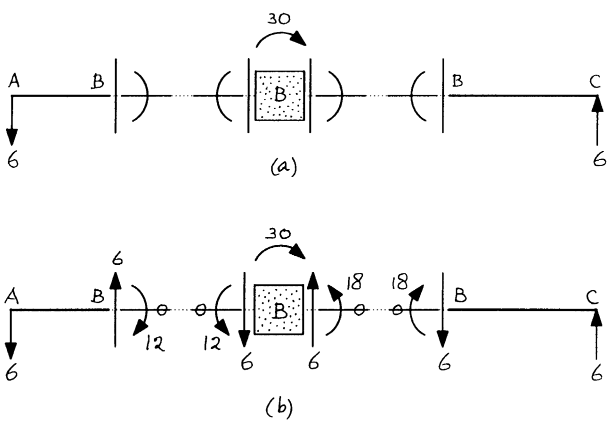


Figure 5.8 (a) The interaction forces between joint B and members AB and BC can be found from the equilibrium of these members; (b) the interaction forces as they are actually acting; joint B must meet the conditions of force and moment equilibrium.

Example 4

The simply supported beam ABC in Figure 5.7a consists of the members AB and BC that are connected rigidly in joint B. The beam is loaded at joint B by a couple of 30 kNm.

Question:
Isolate joint B and draw all the forces[1] acting on it.

Solution:
From the horizontal force equilibrium, it follows that the horizontal support reaction A is zero. Only vertical support reactions are therefore acting at A and C. In order to find the direction of the vertical support reaction at C, one considers the isolated beam to be pinned by a hinge at A. Due to the couple of 30 kNm, beam ABC will try to rotate clockwise about A. The vertical support reaction in C must prevent this rotation and therefore acts upwards (see Figure 5.7b).

The vertical equilibrium requires that the vertical support reactions at A and C must be of equal magnitude and opposite direction. The vertical support reaction A_v at A therefore acts downwards (see Figure 5.7b).

C_v and A_v are found with the following equilibrium equations:

$$\sum T_y|\mathrm{A} = -(30\ \mathrm{kNm}) + C_\mathrm{v} \times (5\ \mathrm{m}) = 0 \ \Rightarrow C_\mathrm{v} = 6\ \mathrm{kN},$$
$$\sum F_z = -A_\mathrm{v} + C_\mathrm{v} = 0 \qquad \Rightarrow A_\mathrm{v} = 6\ \mathrm{kN}.$$

In Figure 5.8a, the members AB and BC have been isolated at joint B. In this figure, the calculated support reactions are shown, as are (without indicating their direction) the currently unknown interaction forces[2] be-

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[1] The forces are intended here in a *generalised* sense (see Section 4.2.2).

[2] Remember that three interaction forces act at a rigid connection (see Section 4.2.2).

tween the members and the joint. The forces that are exerted at B on the member ends are found from the force and moment equilibrium of respectively member AB and BC. The law of action and reaction requires that the member ends exert equal and opposite forces on joint B. Figure 5.8b shows the interaction forces according to their direction and magnitude (forces in kN and moments in kNm).

Check: At joint B the force and moment equilibrium is satisfied.

Example 5

Of the bar-type structure in Figure 5.9, parts AC, BC and DC are rigidly connected at joint C.

Questions:

a. Determine and draw the support reactions.
b. Graphically check the force and moment equilibrium.
c. Isolate AC, BC, and DC at joint C and draw all the support reactions and interaction forces.

Solution:

a. The support reactions are found from the three equilibrium equations for the structure as a whole. The result is shown in Figure 5.10a.

b. The lines of action of the three (resulting) forces at A, B and D intersect at one point. This means that there is moment equilibrium. In Figure 5.10b these forces form a closed force polygon; there is therefore also force equilibrium.

c. In Figure 5.11a, all the parts connected at joint C have been isolated. The forces acting at C on AC can be determined using the known support reactions at A. Equal and opposite forces are acting on joint C. The forces between joint C and the parts BC and CD can be calculated in the same way. The result is shown in Figure 5.11b.

Check: At joint C, the force and moment equilibrium is satisfied.

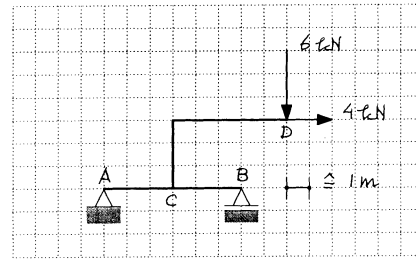


Figure 5.9 A structure of which the parts AC, BC and DC are connected rigidly at joint C.

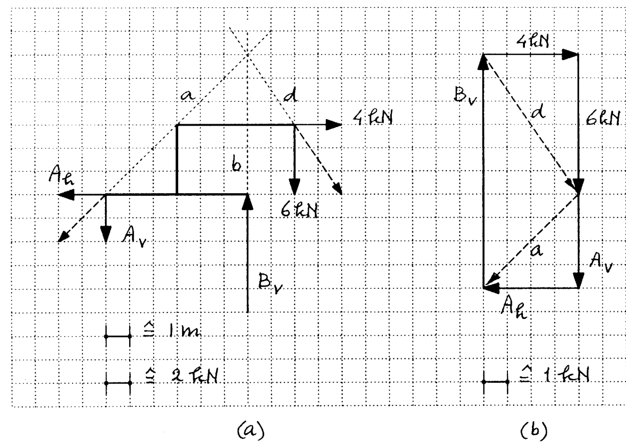


Figure 5.10 (a) Graphical check of the moment equilibrium: the lines of action of the three resulting forces at A, B and D pass through a single point; (b) graphical check of the force equilibrium: all the forces form a closed force polygon.

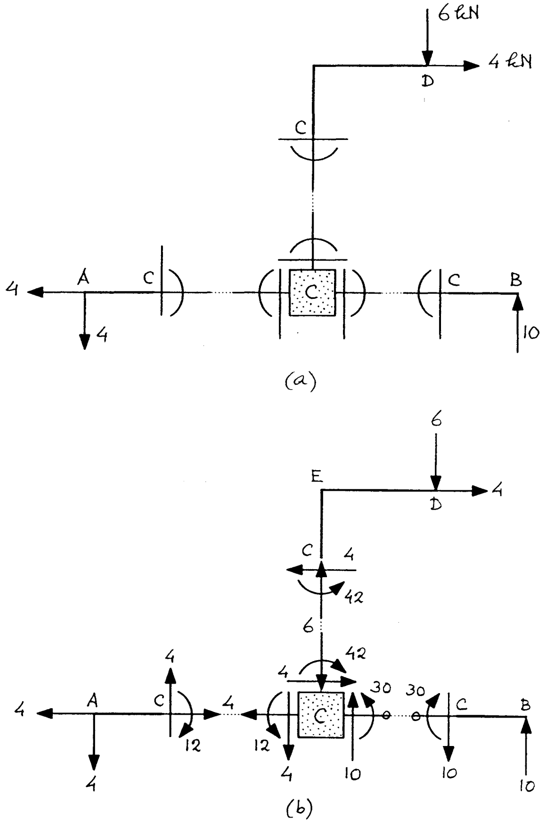


Figure 5.11 (a) The interaction forces between joint C and parts AC, BC and DC can be found from the equilibrium of these parts; (b) the interaction forces as they are really acting; joint C must meet the conditions of the force and moment equilibrium.

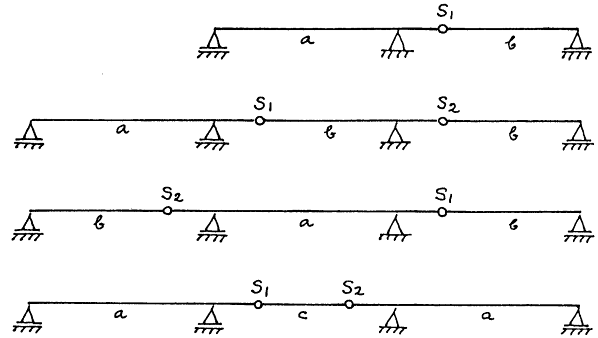


Figure 5.12 Hinged beams.

5.2 Hinged beams

A *hinged beam* is a structure in which several beams are linked through consecutive hinges. Figure 5.12 shows examples of hinged beams. Hinged beams are found in roof girders and bridges.

In Figure 5.12, the beams with an overhang are depicted with (a). These beams are referred to as being supported at *fixed* points. The beams (b) and (c) are sometimes referred to as being supported at *floating* points, as they rest on the non-fixed supporting points S_1 and/or S_2. Beam (c) is called a suspended beam; it can be placed at a later stage during construction.

Statically determinate hinged beams are also known as *Gerber beams* after the German *Gerber*,[1] who first used this type of structure in the second half of the 19th century.

---

1 Heinrich Gerber (1832–1912), German engineer.

By choosing an adequate place for the hinges, it is possible to influence the force distribution in the structure positively. However, you have to make sure that the structure does not become kinematically indeterminate, as for example in Figure 5.13.

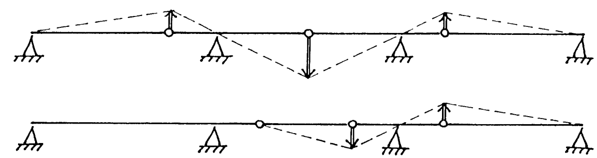


Figure 5.13 With too many hinges, or inadequate placement, the structure becomes kinematically indeterminate and changes into a mechanism.

A possibility for hinge S_1 in a bridge structure is shown in Figure 5.14. The right-hand part (the suspended beam) is supported at a hinge on the left-hand part. In this example, the hinge works only if the right-hand part exerts a downward force onto the left-hand part. This requirement is usually fulfilled as a result of the relatively large dead weight of the suspended beam.

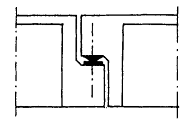


Figure 5.14 Example of a hinge in a bridge structure.

From now on we assume that all hinges in a hinged beam can transfer both tensile and compressive forces.

Example 1
The hinged beam in Figure 5.15a consists of parts AS and CS, which are connected at a hinge in S.

Questions:
a. Determine the support reactions.
b. Determine the forces exerted on hinge S.

Solution (units in kN and m):
a. There are three equilibrium equations available for the structure. With the directions assumed for the support reactions in Figure 5.15b, the following applies for the given xy coordinate system:

$$\sum F_x = A_\mathrm{h} = 0, \tag{a}$$

$$\sum F_y = -40 - 60 + A_\mathrm{v} + B_\mathrm{v} + C_\mathrm{v} = 0, \tag{b}$$

$$\sum T_z|A = -40 \times 4 - 60 \times 12 + B_\mathrm{v} \times 8 + C_\mathrm{v} \times 16 = 0. \tag{c}$$

The moment equilibrium for the entire structure can also be applied for a point other than A.

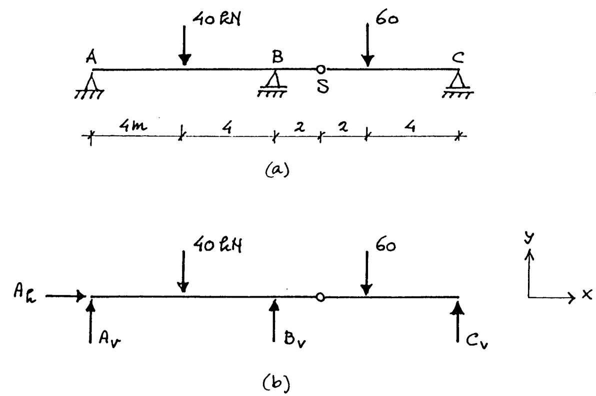


Figure 5.15 (a) A hinged beam on three supports; (b) the assumed directions for the support reactions.

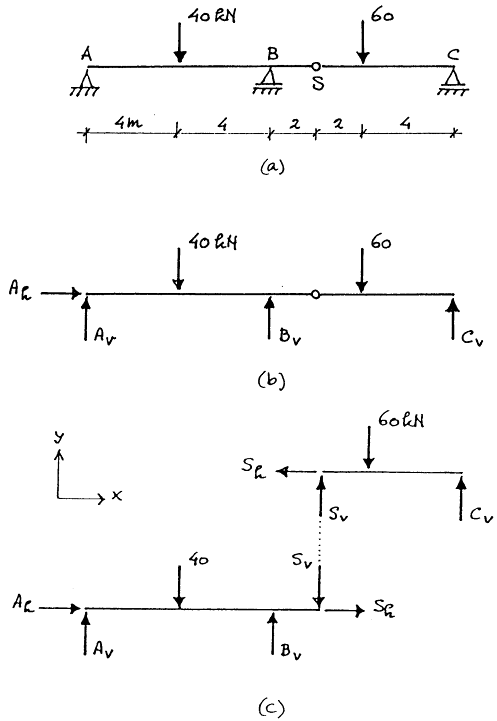


Figure 5.15 (a) A hinged beam on three supports; (b) the assumed directions for the support reactions. (c) The hinge forces in S.

The three equations (a) to (c) are insufficient for finding all the support reactions. A fourth equation is required. This equation relates to the property that no couple can be transferred at hinge S. If parts AS and CS are isolated at S, we are left with the interaction forces S_h and S_v (see Figure 5.15c). The missing equation is now found from the moment equilibrium about S of one of the individual parts.

For the left-hand part AS one finds[1]

$$\sum T_z^{(AS)}|S = 40 \times 6 - A_v \times 10 - B_v \times 2 = 0 \tag{d}$$

and for the right-hand part CS

$$\sum T_z^{(CS)}|S = -60 \times 2 + C_v \times 6 = 0. \tag{e}$$

Both equations (d) and (e) are of equal value, but it should be clear that equation (e) is preferable as it is simpler.

The support reactions are therefore most easily found as follows:

$$\begin{array}{lll}
\text{(e)} \ \sum T_z^{(CS)}|S = 0 & \Rightarrow C_v = 20 \text{ kN}, \\
\text{(c)} \ \sum T_z^{(AC)}|A = 0 & \Rightarrow B\text{v} = 70 \text{ kN}, \\
\text{(b)} \ \sum F_y^{(AC)} = 0 & \Rightarrow A_v = 10 \text{ kN}, \\
\text{(a)} \ \sum F_x^{(AC)} = 0 & \Rightarrow A_h = 0.
\end{array}$$

It seems that the correct direction was assumed for all the support reactions. The support reactions are shown in Figure 5.16.

1 In $\sum T_z^{(AS)}|S = 0$, the upper index indicates the part AS to which the equilibrium equation relates. This notation is particularly useful if the equilibrium has to be written down for the various parts of the same structure.

b. The hinge forces follow from the equilibrium of the separate parts. Taking the right-hand part CS in Figure 5.16 we find

$$\sum F_x^{(\mathrm{CS})} = -S_\mathrm{h} = 0 \qquad \Rightarrow S_\mathrm{h} = 0,$$

$$\sum F_z^{(\mathrm{CS})} = S_\mathrm{v} - 60 + 20 = 0 \Rightarrow S_\mathrm{v} = 40\ \mathrm{kN}.$$

The same values are found from the force equilibrium for the left-hand part AS.

S_h and S_v are the forces that are acting at S on AS and CS. The forces acting on the hinged joint S are the same magnitude, but of opposite direction (see Figure 5.17).

Alternative solution:
The *floating* supported part CS can be seen as a beam, supported on a roller and a hinge (see Figure 5.18). The support reactions at S and C follow from the equilibrium of CS:

$$\sum F_x^{(\mathrm{CS})} = 0 \quad \Rightarrow S_\mathrm{h} = 0,$$

$$\sum T_z^{(\mathrm{CS})}|\mathrm{C} = 0 \Rightarrow S_\mathrm{v} = 40\ \mathrm{kN},$$

$$\sum T_z^{(\mathrm{CS})}|\mathrm{S} = 0 \Rightarrow C_\mathrm{v} = 20\ \mathrm{kN}.$$

With S_h and S_v we now know the load on the overhang of ABS and we can determine the support reactions at A and B:

$$\sum T_z^{(\mathrm{AS})}|\mathrm{A} = 0 \Rightarrow B_\mathrm{v} = 70\ \mathrm{kN},$$

$$\sum T_z^{(\mathrm{AS})}|\mathrm{B} = 0 \Rightarrow A_\mathrm{v} = 10\ \mathrm{kN},$$

$$\sum F_x^{(\mathrm{AS})} = 0 \quad \Rightarrow A_\mathrm{h} = 0.$$

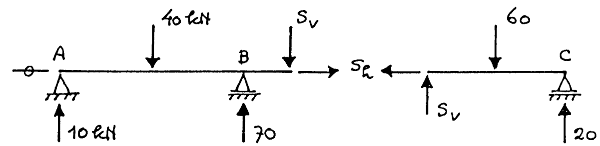


Figure 5.16 The support reactions as they really act; the hinge forces follow from the equilibrium of AS or CS.

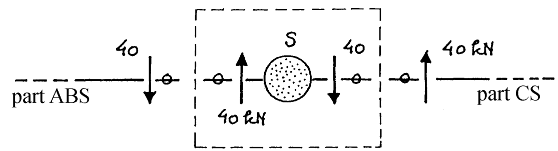


Figure 5.17 The forces acting on the isolated hinged joint S.

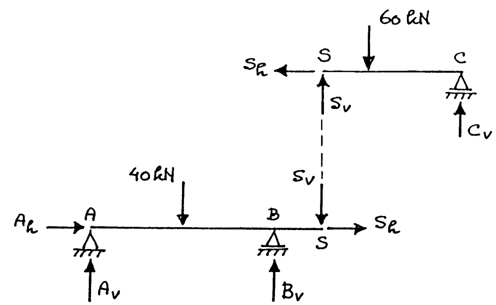


Figure 5.18 The support reactions and interaction forces can also be found by first working out the equilibrium of SC and then the equilibrium of AS.

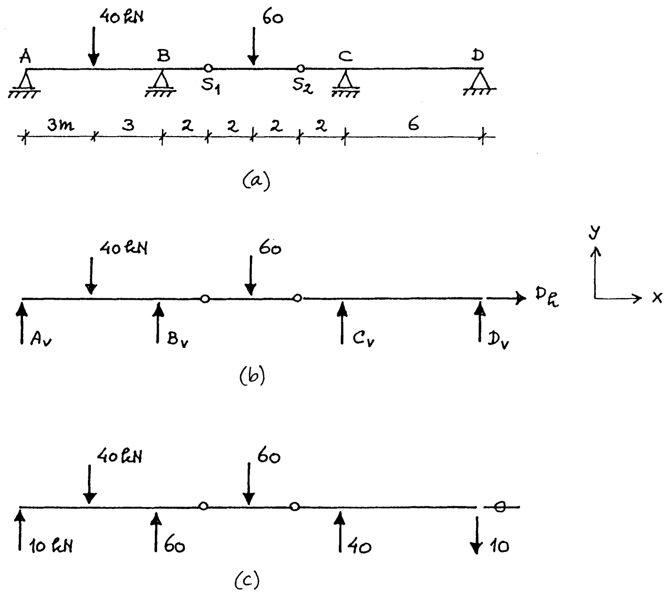


Figure 5.19 (a) A hinged beam with four supports; (b) the assumed directions of the support reactions; (c) the support reactions as they really act.

Example 2
The hinged beam in Figure 5.19a is given.

Question:
Determine the support reactions.

Solution (units in kN and m):
In Figure 5.19b, the following applies for the assumed directions of the support reactions in the given coordinate system, and for the system as a whole:

$$\sum F_x = D_\mathrm{h} = 0, \tag{a}$$

$$\sum F_y = -40 - 60 + A_\mathrm{v} + B_\mathrm{v} + C_\mathrm{v} + D_\mathrm{v} = 0, \tag{b}$$

$$\begin{aligned} \sum T_z|\mathrm{A} = &-40 \times 3 - 60 \times 10 \\ &+B_\mathrm{v} \times 6 + C_\mathrm{v} \times 14 + D_\mathrm{v} \times 20 = 0. \end{aligned} \tag{c}$$

We have three equations with five unknowns. The two missing equations are found from the condition that the hinges S_1 and S_2 cannot transfer couples. Therefore the following applies for the isolated part S_2D:

$$\sum T_z^{(\mathrm{S_2D})}|\mathrm{S_2} = C_\mathrm{v} \times 2 + D_\mathrm{v} \times 8 = 0. \tag{d}$$

and for the isolated part S_1D:

$$\sum T_z^{(\mathrm{S_1D})}|\mathrm{S_1} = -60 \times 2 + C_\mathrm{v} \times 6 + D_\mathrm{v} \times 12 = 0. \tag{e}$$

Here the moment equilibrium has been associated with the parts to the right of the hinges. One could just as well look at the moment equilibrium of the parts to the left of both hinges, although doing so would involve more calculations.

To summarise, a good strategy for solving this is as follows:

$$\text{(a)}\quad \sum F_x^{(\mathrm{AD})} = 0 \qquad \Rightarrow D_\mathrm{h} = 0\,\mathrm{kN},$$

$$\left.\begin{array}{ll}\text{(e)} & \sum T_z^{(\mathrm{S_1D})}|\mathrm{S_1} = 0 \\[2ex] \text{(d)} & \sum T_z^{(\mathrm{S_2D})}|\mathrm{S_2} = 0\end{array}\right\} \Rightarrow C_v = 40\,\mathrm{kN} \;\text{ and }\; D_\mathrm{v} = -10\,\mathrm{kN},$$

$$\text{(c)}\quad \sum T_z^{(\mathrm{AD})}|\mathrm{A} = 0 \qquad \Rightarrow B_\mathrm{v} = 60\,\mathrm{kN},$$

$$\text{(b)}\quad \sum F_y^{(\mathrm{AD})} = 0 \qquad \Rightarrow A_\mathrm{v} = 10\,\mathrm{kN}.$$

Figure 5.19c shows the support reactions as they act in reality. Apparently, only the direction of D_v was initially assumed falsely.

Alternative solution:
The most efficient approach however is to first look at the moment equilibrium of the suspended beam $\mathrm{S_1S_2}$ (see Figure 5.20):

$$\sum T_z^{(\mathrm{S_1S_2})}|\mathrm{S_1} = 0 \qquad \Rightarrow \; S_{2;\mathrm{v}} = 30\,\mathrm{kN},$$

$$\sum T_z^{(\mathrm{S_1S_2})}|\mathrm{S_2} = 0 \qquad \Rightarrow \; S_{1;\mathrm{v}} = 30\,\mathrm{kN},$$

With $S_{1;\mathrm{v}}$ and $S_{2;\mathrm{v}}$, we know the vertical forces that the suspended beam exerts on the overhangs of beams $\mathrm{AS_1}$ and $\mathrm{S_2D}$. For these beams, the vertical support reactions can be determined from the moment equilibrium:

$$\sum T_z^{(\mathrm{AS_1})}|\mathrm{B} = 0 \qquad \Rightarrow \; A_\mathrm{v} = 10\,\mathrm{kN},$$

$$\sum T_z^{(\mathrm{AS_1})}|\mathrm{A} = 0 \qquad \Rightarrow \; B_\mathrm{v} = 60\,\mathrm{kN},$$

$$\sum T_z^{(\mathrm{S_2D})}|\mathrm{D} = 0 \qquad \Rightarrow \; C_\mathrm{v} = 40\,\mathrm{kN},$$

$$\sum T_z^{(\mathrm{S_2D})}|\mathrm{C} = 0 \qquad \Rightarrow \; D_\mathrm{v} = -10\,\mathrm{kN},$$

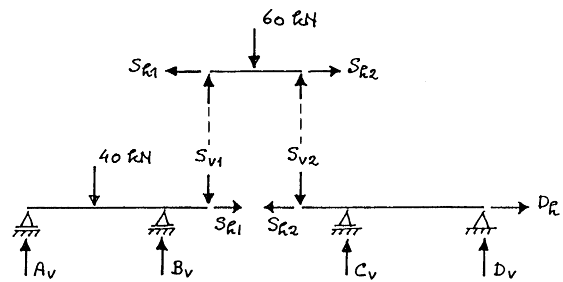


Figure 5.20 The support reactions can also be found by first working out the moment equilibrium of suspended beam $\mathrm{S_1S_2}$ and then the equilibrium of AS and DS.

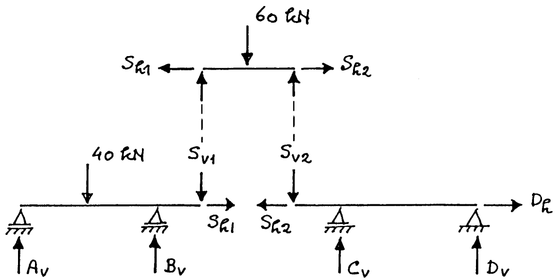


Figure 5.20 The support reactions can also be found by first working out the moment equilibrium of suspended beam S_1S_2 and then the equilibrium of AS and DS.

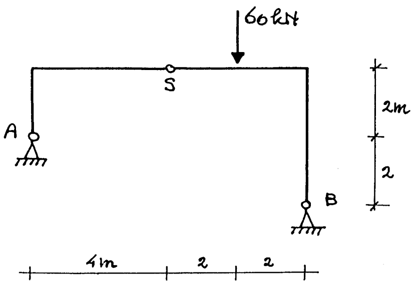


Figure 5.21 (a) A three-hinged frame with the hinge bearings at different levels.

Finally, the horizontal force equilibrium for each of the structural members gives

$$S_{1;\mathrm{h}} = S_{2;\mathrm{h}} = D_\mathrm{h} = 0 \text{ kN}.$$

5.3 Three-hinged frames

Figure 5.21a is an example of a *three-hinged frame*. The frame consists of two self-contained parts AS and BS that are connected at S by means of a hinge, and are supported at A and B by a hinge. The whole is statically determinate. Three-hinged frames are often used as covering structures. They were previously mentioned in Sections 3.2.2 and 4.4.4.

A three-hinged frame has four unknown support reactions. In order to be able to calculate these, we need four equilibrium equations. Three of these are found from the equilibrium of the structure as a whole. The fourth equation follows from the condition that the hinged joint at S cannot transfer a couple.

Example 1
In the three-hinged frame in Figure 5.21a, the hinge bearings at A and B are at different levels. The frame is loaded by a vertical force of 60 kN that acts on the right-hand part BS.

Questions:
a. Determine the support reactions.
b. Determine the forces that parts AS and BS in S exert on one another.
c. Perform a graphical check of the equilibrium.

Solution (units in kN and m):

a. For the given coordinate system and the directions assumed for the support reactions in Figure 5.21b the following applies for the structure as a whole:

$$\sum F_x^{(\mathrm{ASB})} = A_\mathrm{h} - B_\mathrm{h} = 0, \tag{a}$$

$$\sum F_y^{(\mathrm{ASB})} = -60 + A_\mathrm{v} + B_\mathrm{v} = 0, \tag{b}$$

$$\sum T_z^{(\mathrm{ASB})}|A = -60 \times 6 - B_\mathrm{h} \times 2 + B_\mathrm{v} \times 8 = 0. \tag{c}$$

The missing fourth equation is found from the moment equilibrium about S of one of the separate parts AS or BS. For the left-hand part AS one finds

$$\sum T_z^{(\mathrm{AS})}|\mathrm{S} = A_\mathrm{h} \times 2 - A_\mathrm{v} \times 4 = 0. \tag{d}$$

For the right-hand part BS, one finds

$$\sum T_z^{(\mathrm{BS})}|\mathrm{S} = -60 \times 2 - B_\mathrm{h} \times 4 + B_\mathrm{v} \times 4 = 0. \tag{e}$$

The equations (d) and (e) are equivalent. Either of them is sufficient for calculating the support reactions in combination with the equations (a) to (c). The other equation can then be used to check the values found.

Equation (e) is preferable in finding the solution as, in combination with equation (c), it leads directly to the support reactions at B:

$$B_\mathrm{h} = 20\ \mathrm{kN};\ \ B_\mathrm{v} = 50\ \mathrm{kN}.$$

From (a) and (b) we find

$$A_\mathrm{h} = 20\ \mathrm{kN};\ \ A_\mathrm{v} = 10\ \mathrm{kN}.$$

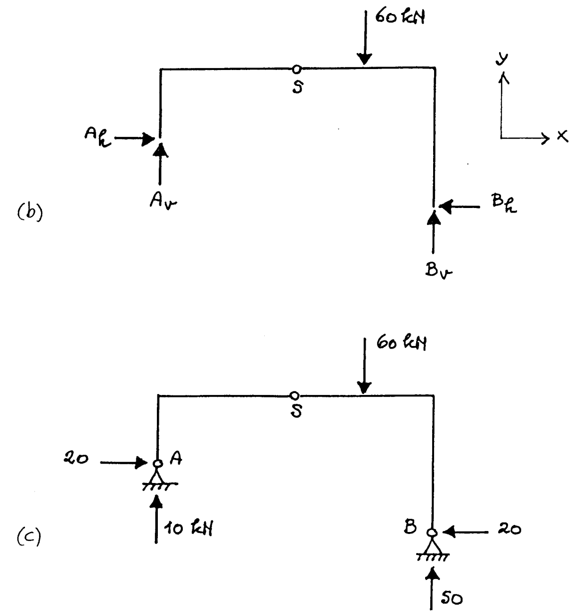


Figure 5.21 (b) The assumed directions for the support reactions; (c) the support reactions as they really act.

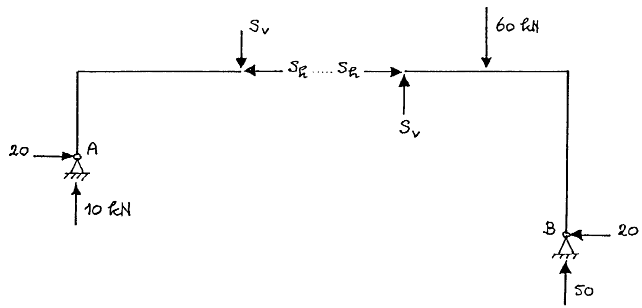


Figure 5.22 The interaction forces at S follow from the force equilibrium of AS or BS.

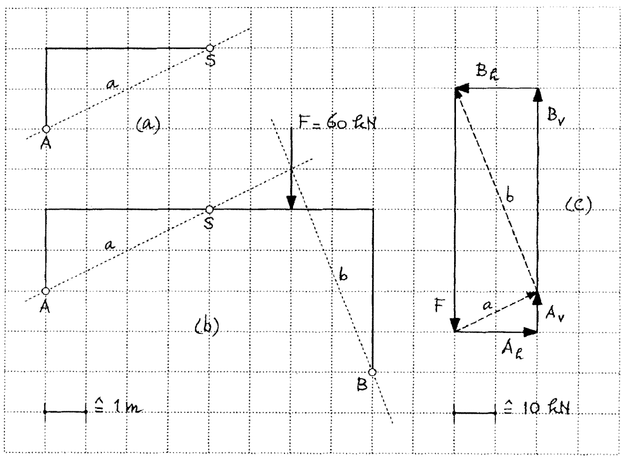


Figure 5.23 (a) The left frame half AS is in equilibrium if the two forces at A and S are equal and opposite and have a common line of action; (b) the three-hinged frame is in moment equilibrium if the lines of action of force F and the support reactions at A and B intersect in a single point; (c) the three-hinged frame is in force equilibrium if force F and the support reactions at A and B form a closed force polygon.

The support reactions are shown in Figure 5.21c. Since there is only vertical and no horizontal loading, the horizontal support reactions are equal and opposite.

Check: The solution is true in equation (d).

b. The forces that parts AS and BS in S exert on one another (the interaction forces at S) follow from the force equilibrium of one of the separate parts AS or BS (see Figure 5.22). The force equilibrium for the left-hand part AS gives

$$S_h = 20 \text{ kN} \quad \text{and} \quad S_v = 10 \text{ kN}.$$

The same values follow from the force equilibrium for the right-hand part BS. This therefore offers an opportunity for checking.

c. Since the load only acts on one half of the frame, one can also easily check the solution graphically (see Section 3.2.2).

Only two forces are acting on the left-hand part AS: the support reaction at A and the hinge force at S. The left-hand part AS can be in equilibrium only if the two forces that act on AS at A and S are equal and opposite. Both forces must also have the same line of action (see Figure 5.23a). The line of action of the support reaction at A will therefore pass through S and is thus determined.

Three forces are acting on the entire frame (the two support reactions at A and B and the load) that together have to form an equilibrium system. This is possible only if the lines of action of the three forces intersect in a single point (if not, there is no moment equilibrium). The line of action of the support reaction at B must therefore pass through the intersection of the line of action of the point load and the known line of action of the support reaction at A (see Figure 5.23b).

With the known lines of action for both support reactions, the magnitude and direction can be found by means of the force polygon in Figure 5.23c.

The figure shows that the support reactions at A and B correspond in magnitude and direction with those calculated previously.

Example 2
The left-hand column of the three-hinged frame from the previous example is extended in such a way that the hinge bearings at A and B are at equal level (see Figure 5.24a). The load remains unchanged.

Questions:
a. Determine the support reactions at A and B.
b. Determine the forces that AS and BS at S exert on one another.
c. Determine the forces acting on joint D.

Solution (units in kN and m):
a. For a three-hinged frame with the hinge bearings at equal level, the vertical support reactions can be determined directly from the moment equilibrium of the structure as a whole.

With the directions assumed for the support reactions in Figure 5.24b the following applies for the given coordinate system for the frame as a whole:

$$\sum T_z^{(\mathrm{ASB})}|\mathrm{A} = -60 \times 6 + B_\mathrm{v} \times 8 = 0 \Rightarrow B_\mathrm{v} = 45\ \mathrm{kN}, \qquad \text{(a)}$$

$$\sum T_z^{(\mathrm{ASB})}|\mathrm{B} = 60 \times 2 - A_\mathrm{v} \times 8 = 0 \quad \Rightarrow A_\mathrm{v} = 15\ \mathrm{kN}. \qquad \text{(b)}$$

One of these equations for the moment equilibrium can be replaced by the equation for the vertical force equilibrium.

The horizontal force equilibrium of the structure as a whole gives

$$\sum F_x^{(\mathrm{ASB})} = A_\mathrm{h} - B_\mathrm{h} = 0. \qquad \text{(c)}$$

Since there is no horizontal loading, the horizontal support reactions are equal and opposite. The magnitude of the horizontal support reactions follow from the moment equilibrium about S of one of the parts AS or BS.

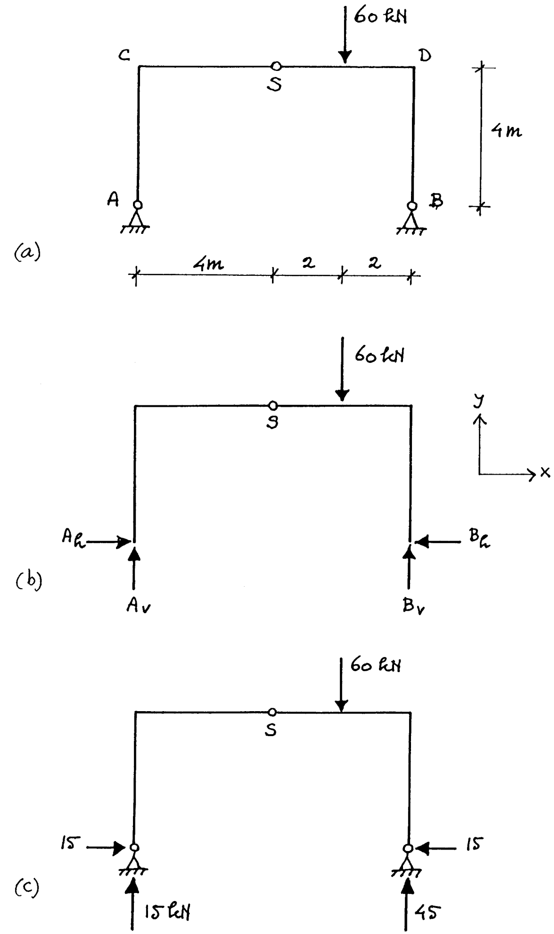


Figure 5.24 (a) A three-hinged frame with the hinged supports at the same level; (b) the assumed directions for the support reactions; (c) the support reactions as they really act.

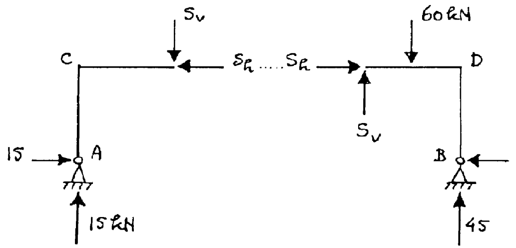


Figure 5.25 The hinge forces at S follow from the force equilibrium of AS or BS.

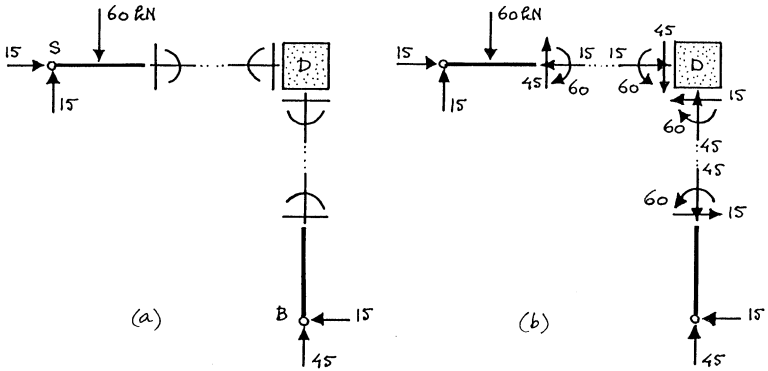


Figure 5.26 (a) The interaction forces between joint D and members SD and BD are found from the equilibrium of the separate members; (b) the interaction forces as they really act.

If one selects the left-hand part AS, this gives

$$\sum T_z^{(\mathrm{AS})}|\mathrm{S} = A_\mathrm{h} \times 4 - A_\mathrm{v} \times 4 = 0 \tag{d}$$

or, if one assumes the right-hand part BS

$$\sum T_z^{(\mathrm{BS})}|\mathrm{S} = -60 \times 2 - B_\mathrm{h} \times 4 + B_\mathrm{v} \times 4 = 0. \tag{e}$$

Both equations are equivalent. The solution is

$$A_\mathrm{h} = B_\mathrm{h} = 15 \text{ kN}.$$

All the support reactions are shown in Figure 5.24c.

b. The interaction forces in hinge S follow from the force equilibrium of AS or BS (see Figure 5.25). The equilibrium of the left-hand part AS gives

$$S_\mathrm{h} = S_\mathrm{v} = 15 \text{ kN}.$$

Check: For these forces, the left-hand part BS is also in equilibrium.

c. To find the forces acting on joint D, the joint is isolated (see Figure 5.26a). There are three interaction forces acting between joint D and member SD. The magnitude of these forces is found from the equilibrium of member SD. In the same way, one can use the equilibrium of BD to find the magnitude of the three interaction forces between joint D and member BD. The result is shown in Figure 5.26b.

Check: Joint D has to meet the conditions of force and moment equilibrium.

5.4 Three-hinged frames with tie-rod

The previous section shows that a vertical load on a three-hinged frame generates not only vertical, but also horizontal support reactions (see Figures 5.27a and 5.27b). Horizontal forces on foundations in soft soil often cause problems. To reduce the horizontal forces on the foundation, one can decide to link the bearings A and B of the three-hinged frame by means of a so-called tie-rod. In this way a self-contained structure is created that can be supported by a roller and a hinge (see Figure 5.27c). This is referred to as a *three-hinged frame with tie-rod*. Tie-rod AB ensures that the roller support B stays in place and carries the horizontal support reactions. Vertical loading now generates exclusively vertical support reactions.

Whether rod AB is subject to tension or compression depends on the loading. The name *tie-rod* indicates that such a structure is used only if tension can be expected in the rod.

Example

In Figure 5.28, a vertical and a horizontal load is acting on a three-hinged frame with tie-rod.

Questions:

a. Determine the support reactions.
b. Determine the force in rod AB.
c. Determine the interaction forces at S.
d. Determine the forces acting on joint A.

Solution (units in kN and m):

a. In Figure 5.29a, the structure has been isolated from its supports. The support reactions follow from the equilibrium of the structure as a whole. For the directions assumed for A_h, A_v and B_v we find

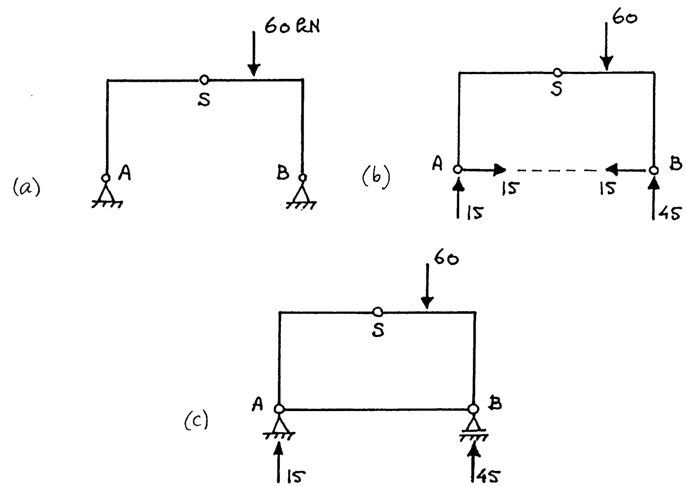


Figure 5.27 (a) A vertical load on a three-hinged frame gives (b) not only vertical but also horizontal support reactions; (c) by linking the bearings A and B of the three-hinged frame by a tie rod, the horizontal support reactions can be eliminated.

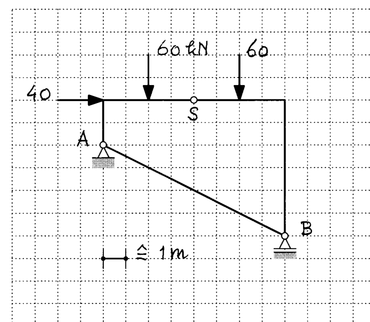


Figure 5.28 A three-hinged frame with tie rod, with a horizontal and vertical load.

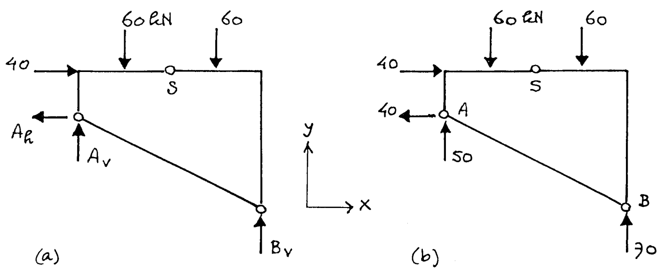


Figure 5.29 (a) The assumed directions of the support reactions; (b) the support reactions as they really act.

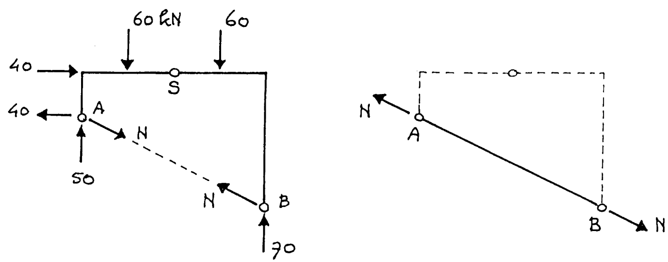


Figure 5.30 Three-hinged frame ASB and rod AB isolated from one another, assuming that a tensile force N acts in rod AB.

$$\sum F_x^{(\mathrm{ASB})} = 40 - A_\mathrm{h} = 0 \qquad \Rightarrow A_\mathrm{h} = 40\ \mathrm{kN},$$

$$\sum F_y^{(\mathrm{ASB})} = -60 - 60 + A_\mathrm{v} + B_\mathrm{v}$$

$$= -60 - 60 + A_\mathrm{v} + 70 = 0 \ \Rightarrow A_\mathrm{v} = 50\ \mathrm{kN},$$

$$\sum T_z^{(\mathrm{ASB})}|\mathrm{A} = -40 \times 2 - 60 \times 2$$

$$-60 \times 6 + B_\mathrm{v} \times 8 = 0 \ \Rightarrow B_\mathrm{v} = 70\ \mathrm{kN}.$$

The support reactions are shown in Figure 5.29b.

b. To calculate the force in rod AB, it is isolated from ASB in Figure 5.30. We can immediately recognise a two-force member in rod AB: the rod is loaded only by forces at its ends A and B and can therefore be in equilibrium only if these forces are equal and opposite with AB as common line of action. It is assumed that a tensile force N acts in rod AB.

The magnitude of N follows from the moment equilibrium about S of one of the parts AS or BS. In Figure 5.31a both parts have been isolated at S. In order to simplify the calculation, N has been resolved into a horizontal component N_h and a vertical component N_v:

$$N_\mathrm{h} = \tfrac{2}{5}\sqrt{5}\,N,$$

$$N_\mathrm{v} = \tfrac{1}{5}\sqrt{5}\,N.$$

Taking the right-hand part BS we find

$$\sum T_z^{(\mathrm{BS})}|\mathrm{S} = -60 \times 2 + 70 \times 4 - N_\mathrm{h} \times 6 + N_\mathrm{v} \times 4$$

$$= 160 - \tfrac{8}{5}\sqrt{5}\,N = 0$$

from which it follows that

$$N = 20\sqrt{5}\,\text{kN}$$

and

$$N_\text{h} = 40\ \text{kN},$$

$$N_\text{v} = 20\ \text{kN}.$$

Since N is positive, the force in rod AB is indeed a tensile force.

The equation for the moment equilibrium about S can be simplified by shifting N along its line of action to a convenient position, for example to the point vertically under S. In that case it follows that

$$\sum T_z^{(\text{SB})}|\text{S} = -60 \times 2 + 70 \times 4 - N_\text{h} \times 4 = 0 \Rightarrow N_\text{h} = 40\ \text{kN}.$$

Check: For the value determined for N, the left-hand part AS must also satisfy the conditions for moment equilibrium:

$$\sum T_z^{(\text{AS})}|\text{S} = 0.$$

c. The hinge forces at S follow from the force equilibrium of the left-hand or right-hand part of the frame. With the directions of S_h and S_v assumed in Figure 5.31a we find for the right-hand part BS

$$\sum F_x^{(\text{BS})} = -N_\text{h} - S_\text{h} = -40 - S_\text{h} = 0 \ \Rightarrow S_\text{h} = -40\ \text{kN},$$

$$\sum F_z^{(\text{BS})} = -60 + 70 + N_\text{v} + S_\text{v}$$

$$= -60 + 70 + 20 + S_\text{v} = 0 \quad \Rightarrow S_\text{v} = -30\ \text{kN}.$$

Clearly the wrong direction was assumed in Figure 5.31a for both hinge forces. Figure 5.31b shows all the forces as they act in reality.

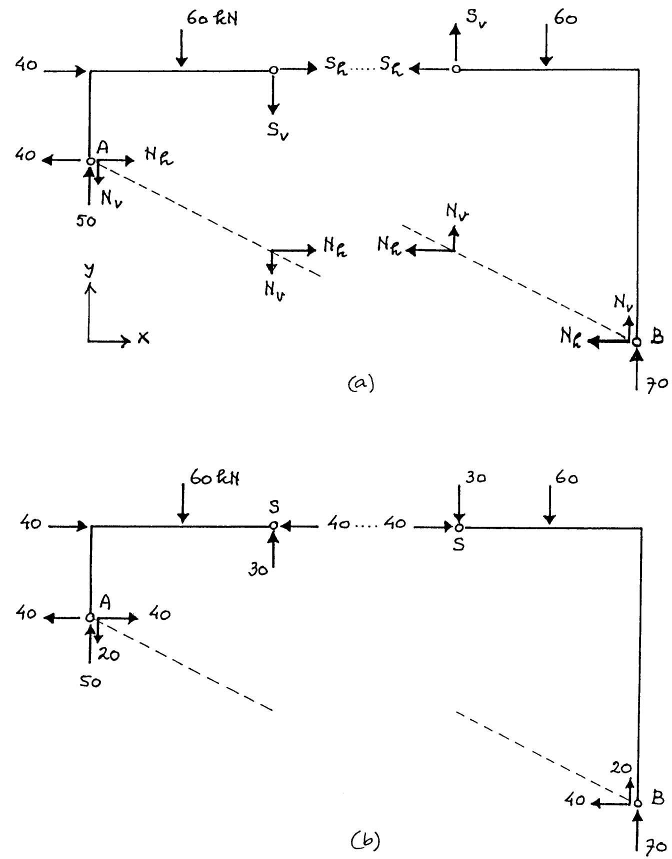


Figure 5.31 (a) The magnitude of N follows from the moment equilibrium of one of the frame halves about S, after which the interaction forces at S follow from the force equilibrium of the frame halves; (b) all the forces as they really act on the frame halves.

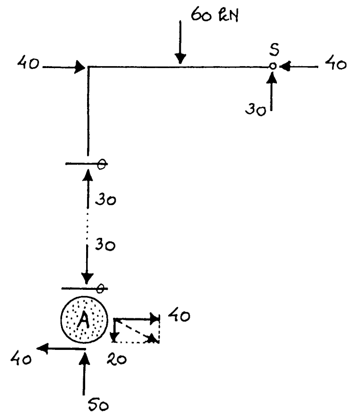


Figure 5.32 The forces acting on joint A and frame half AS.

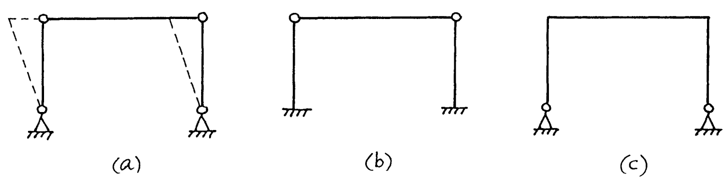


Figure 5.33 (a) This portal-like structure with only hinged joints is kinematically indeterminate and can tilt. To prevent tilting one can (b) fix the columns or (c) replace the hinged joints between the columns and the beams by rigid joints.

Check: With the hinge forces calculated, the left-hand frame part AS must also be in force equilibrium.

d. The following forces are acting on joint A:

- the support reactions $A_h = 40$ kN and $A_v = 50$ kN;
- the force N exerted by the tie-rod AB, with components $N_h = 40$ kN and $N_v = 20$ kN;
- the forces exerted by the left-hand frame part AS.

The last-mentioned forces can be found from the force equilibrium of joint A. All the forces on the joint are shown in Figure 5.32.

Check: The part AS isolated from joint A has to be in force equilibrium.

Note that here the horizontal load of 40 kN is transferred via a long detour to the support at A.

5.5 Shored structures

The portal-like structure in Figure 5.33a, with only hinged joints, is kinematically indeterminate. The structure can *tilt*. To prevent this, one can fix one or more of the columns (Figure 5.33b). Or one can replace one or more of the hinges between column and beam by rigid connections (Figure 5.33c). It is also possible to prevent the construction from tilting by applying so-called *shoring bars*, indicated in Figure 5.34 with the letter s.

If the shoring bar s in Figure 5.34b can transfer only compressive forces, a single shoring bar is not enough. The shoring bar applied does prevent tilting to the left, as in Figure 5.33a (the shoring bar has to shorten and therefore comes under pressure), but not tilting to the right (the shoring bar would be subject to tensile pressure, and may fall or come loose). In that case, two shoring bars would be required, as shown in Figure 5.34c.

The solution with shoring bars, also known simply as *shoring*, stems from the time when stiff corner joints were hard to achieve. You will often find them in (older) timber structures.

An example of this is the wooden roof structure in Figure 5.35a. This type of structure, still often used at the turn of the century, is called a *mansart roof truss*.[1] Figure 5.35b gives the structural model.

Strut B ensures that the horizontal forces are transferred to the beam layer that operates as a *tie-rod*. Strut B, in combination with the *hammer beam* C, can be seen as a shore that ensures a certain restraint of *rafter* A, in the same way as the shoring bar in Figure 5.34a, but in this case placed on the inside. *Brace* G fixes the corner between rafter A and *collar beam* D. They operate like the shoring bar in Figure 5.34b.

Figure 5.35a clearly shows that brace G is connected to rafter A and collar beam D by means of *toothed joints*. Since toothed joints work only under pressure, the upper struts can transfer only compressive forces. For the shoring bars, one still often refers to bars that are loaded by compressive forces.

Shoring bars are used not only to make a structure kinematically determinate, but also to influence the force flow positively, as the *shores* F in Figure 5.35. These shores provide additional support to rafter A, which can therefore be made lighter.

Shoring is found not only in old structures. Shores are still used to influence force flow positively, so that less material is required to meet the demands of strength and rigidity.

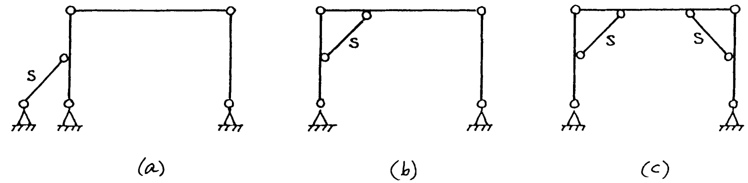


Figure 5.34 (a) A fixed end and (b) a rigid corner connection, both created by using a shoring bar. (c) If the shoring bar can transfer only compressive forces, two shoring bars are required to prevent the tilting to the left and to the right.

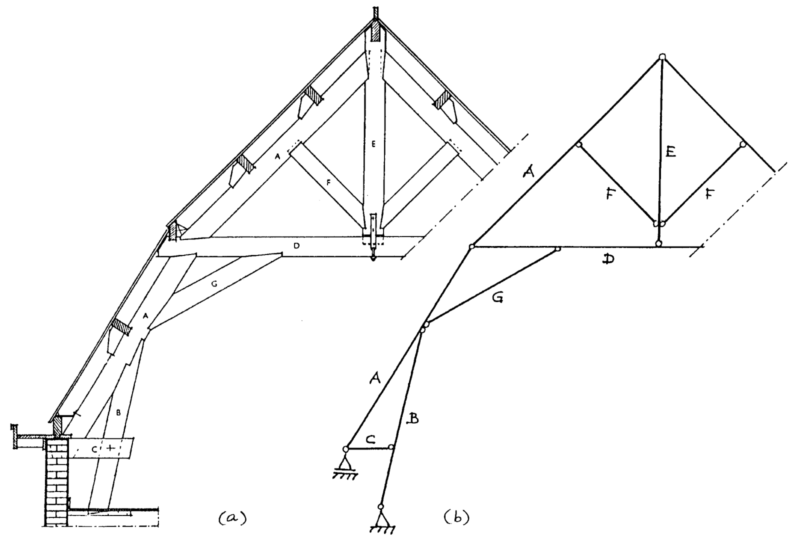


Figure 5.35 (a) A Mansart truss with (b) the structural model.

[1] Named after Jules Hardouin Mansart (1646–1708), French architect. He built the Dôme des Invalides in Paris and major sections of the palace in Versailles.

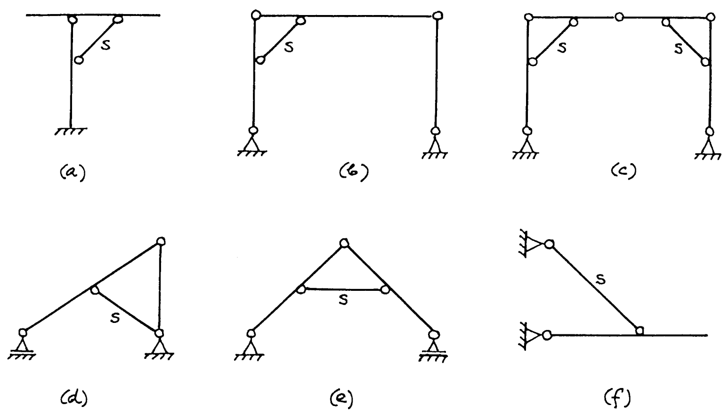


Figure 5.36 Examples of shored structures. In examples (e) to (f) one refers to a tie rod rather than a shoring bar.

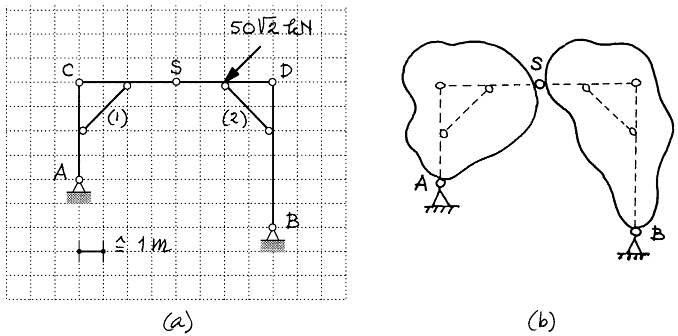


Figure 5.37 (a) A shored three-hinged frame. (b) A three-hinged frame in principle consists of two self-contained parts that are connected by a hinge at S and supported by hinges at A and B.

The following examples will be limited to statically determinate structures. It is assumed that shoring bars can transfer both tensile and compressive forces.

Figure 5.36 shows a number of statically determinate shored structures. In cases (e) and (f) one refers to a *tie-rod*[1] rather than to a *shoring bar*, even though the tie-rod is actually fulfilling the role of a shore.

Example 1
The shored structure in Figure 5.37a is loaded by the force $F = 50\sqrt{2}$ kN.

Questions:

a. Determine the support reactions.
b. Determine the forces in the shoring bars (with the correct sign for tension and compression).
c. Determine all the forces acting on bar SD.

Solution:
a. You will recognise a three-hinged frame in the structure. There are two self-contained parts that are connected in a hinge at S and are supported by hinges at A and B (see Figure 5.37b). The structure in Figure 5.37a is therefore also referred to as a *shored three-hinged frame*. The support reactions can be derived in the standard way for a three-hinged frame (see Section 5.3). The calculation, which will be left to the reader, leads to the support reactions shown in Figure 5.38.

b. The shoring bars are loaded only by forces at the end of the bars and therefore act as two-force members. Suppose that a tensile force $N^{(1)}$ acts in the left shoring bar (1) and a tensile force $N^{(2)}$ in the right shoring bar (2). In Figure 5.39, AC and BD have been isolated. The unknown interaction forces at C and D are not shown here.

---

[1] Since the vertical weight causes tension in these bars.

It is now possible to deduce $N^{(1)}$ from the moment equilibrium of AC about C:

$$\sum T_z^{(AC)}|C = +(40\sqrt{2}\,\text{kN})(2\sqrt{2}\,\text{m}) + N^{(1)} \times (\sqrt{2}\,\text{m}) = 0$$

so that

$$N^{(1)} = -80\sqrt{2}\,\text{kN}.$$

There is a compressive force in shoring bar (1).

In the same way, one can find $N^{(2)}$ from the moment equilibrium of DB about D:

$$\sum T_z^{(BD)}|D = +(10\sqrt{2}\,\text{kN})(3\sqrt{2}\,\text{m}) - N^{(2)} \times (\sqrt{2}\,\text{m}) = 0$$

so that

$$N^{(2)} = 30\sqrt{2}\,\text{kN}.$$

Shoring bar 2 is a tension bar.

To demonstrate clearly how the shoring bars act on frame ASB, the frame and the shoring bars have been isolated from one another in Figure 5.40.

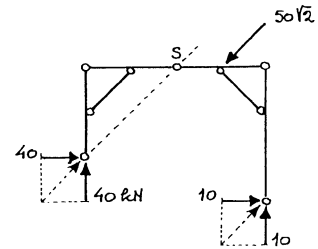


Figure 5.38 The support reactions of the frame as they really act.

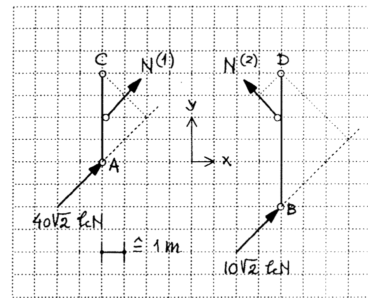


Figure 5.39 The isolated columns AC and BD. The unknown interaction forces at C and D are not shown here.

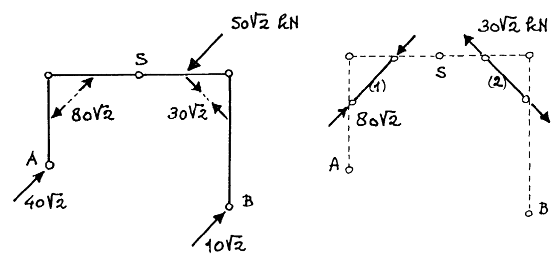


Figure 5.40 To see which forces the shoring bars and frame are exerting on one another, they have been isolated.

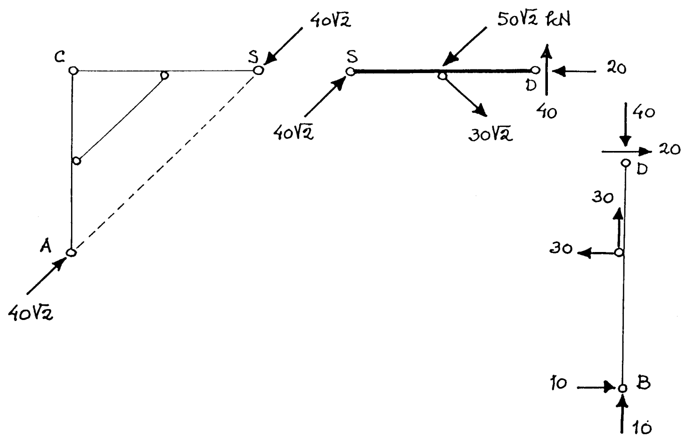


Figure 5.41 The forces acting on SD are found from the equilibrium of the isolated parts.

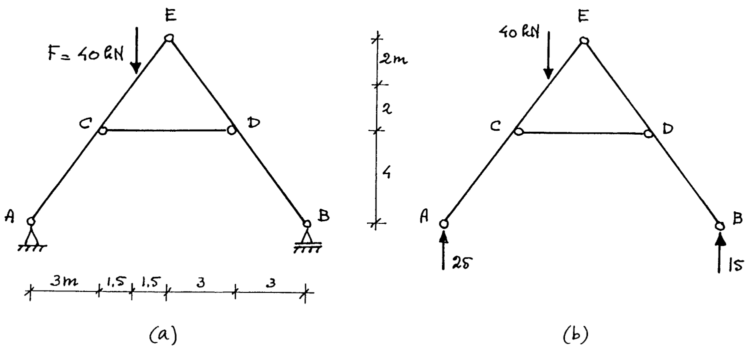


Figure 5.42 (a) A structure loaded by a vertical force of 40 kN on the left-hand rafter, with (b) its support reactions.

c. The force acting at S on SD is equal to the support reaction at A. The force that shoring bar (2) exerts on SD is also known. Still unknown are the components of the force exerted on SD at D. These are found via the force equilibrium of column BD (see Figure 5.41).

Check: SD must be in equilibrium.

Example 2

The structure in Figure 5.42a is loaded on rafter ACE by a vertical force $F = 40$ kN.

Questions:

a. Determine the support reactions.
b. Determine the force in bar CD (with the correct sign for tension and compression).
c. Determine the hinge force at E.

Solution:

a. The support reactions follow directly from the equilibrium of the structure as a whole. There are only vertical support reactions. They are shown in Figure 5.42b.

b. Suppose the tensile force in CD is $N^{(CD)}$. In Figure 5.43, CD has been isolated from AEB. The magnitude of $N^{(CD)}$ follows from the moment equilibrium about E of one of the rafters AE or BE. The unloaded rafter BE is simpler with respect to the amount of arithmetic:

$$\sum T_z^{(BE)}|E = -N^{(CD)} \times (4\text{ m}) + (15\text{ kN}(6\text{ m}) = 0 \Rightarrow N^{(CD)} = 22.5\text{ kN}$$

CD is a tension member.

c. The hinge force at E is subsequently found from the force equilibrium of one of the rafters AE or BE. Again, the unloaded right-hand rafter BE is preferable. In Figure 5.44a, BE has been isolated, and the result of the calculation is shown.

The forces acting on BDE at D and E can also be determined graphically. The lines of action b and d are known (see Figure 5.44a). Line of action e of the hinge force at E must pass through the intersection of b and d (moment equilibrium of a body subjected to three forces). In a force polygon, one can now determine the forces at D and E that ensure equilibrium with the support reaction at B (see Figure 5.44b).

Check: The left-hand rafter ACE must also be in equilibrium. You can see immediately that there is force equilibrium in Figure 5.45. To check the moment equilibrium, write down the moment equation for all the forces about an arbitrary point.

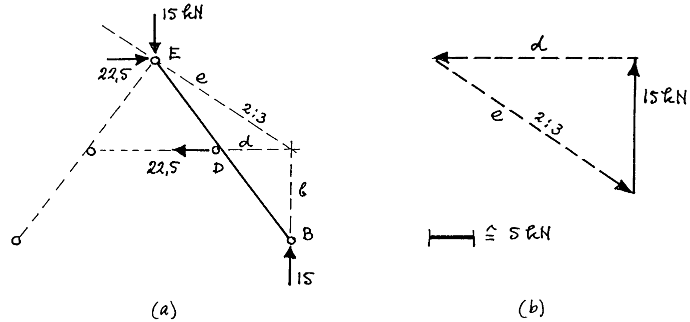


Figure 5.44 Graphical determination of the forces acting at D and E on the right-hand rafter BDE: (a) line of action figure and (b) force polygon.

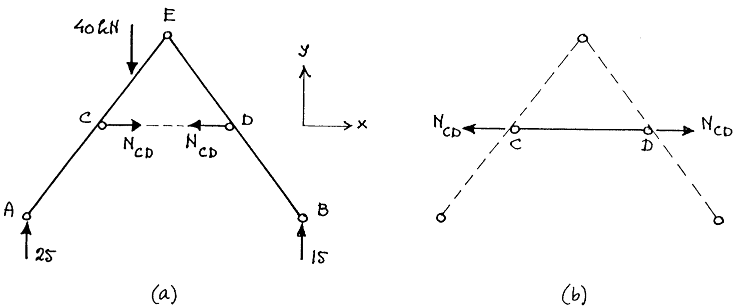


Figure 5.43 To see how rafter AEB and bar CD exert forces on one another, they have been isolated.

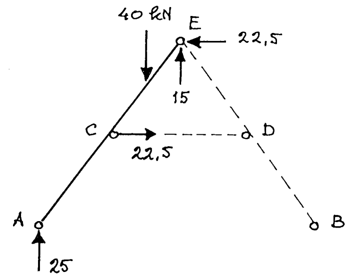


Figure 5.45 The forces acting on the left-hand rafter ACE.

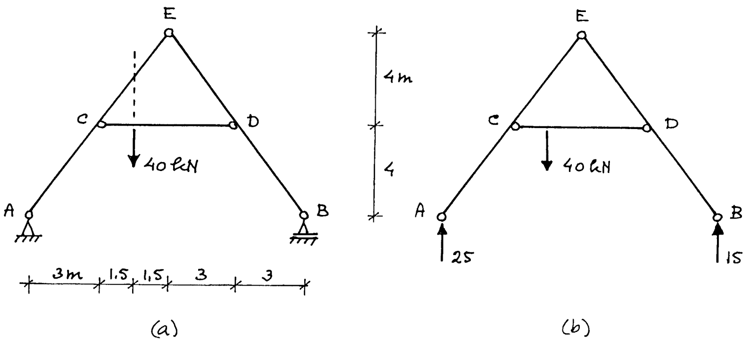


Figure 5.46 (a) A structure loaded by a vertical force of 40 kN on the tie rod, with (b) its support reactions. For self-contained structures the support reactions do not change if one shifts a loading force along its line of action; on the other hand the interaction forces do change.

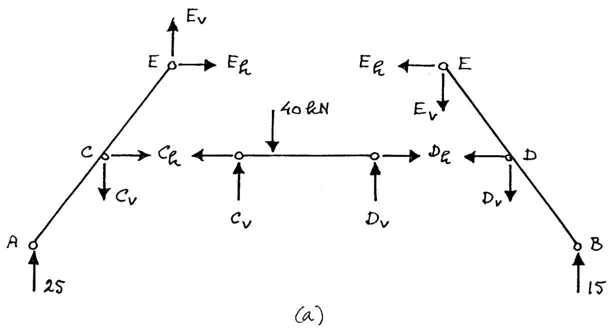


Figure 5.47 (a) The interaction forces between the isolated parts AE, BE and CD. The interaction forces C_v and D_v are found from the moment equilibrium of CD.

Example 3

Figure 5.46a uses the same structure as in Example 2, except that this time, the vertical force $F = 40$ kN has been shifted along its line of action to a point of application on member CD.

Questions:

a. Determine the support reactions.
b. Determine the forces acting on the isolated parts ACE, BDE, and DE.
c. Perform a graphical check of the moment equilibrium for each of the parts.

Solution:

a. The support reactions are the same as those in example 2. They are shown in Figure 5.46b. Note that for a self-contained structure, the support reactions do not change if one shifts a force along its line of action. The forces *within* the structure do change, however, as is shown below.

b. In Figure 5.47a, the various structural parts have been isolated, and all the interaction forces are shown.

First look at the equilibrium of CD. From the moment equilibrium about C follows

$$D_\mathrm{v} = 10\ \text{kN}.$$

From the moment equilibrium about D follows

$$C_\mathrm{v} = 30\ \text{kN}.$$

The horizontal force equilibrium gives

$$C_\mathrm{h} = D_\mathrm{h}.$$

Next look at the right-hand rafter AE (see Figure 5.47b). The moment equilibrium about E gives

$$C_{\mathrm{h}} = 15 \text{ kN}$$

so that

$$D_{\mathrm{h}} = 15 \text{ kN}.$$

The force equilibrium gives

$$E_{\mathrm{h}} = -15 \text{ kN},$$
$$E_{\mathrm{v}} = 5 \text{ kN}.$$

The direction of E_{h} was obviously assumed falsely.

In Figure 5.48, all the interaction forces are shown as they act in reality.

Check: BE must also meet the conditions of the force and moment equilibrium. Figure 5.48 shows that the force equilibrium conditions are satisfied. Only the moment equilibrium has to be checked.

c. If three forces act on a body, there is moment equilibrium only if the lines of action of the forces intersect at one point. In Figure 5.48, this check for moment equilibrium has been performed for each of the structural parts.

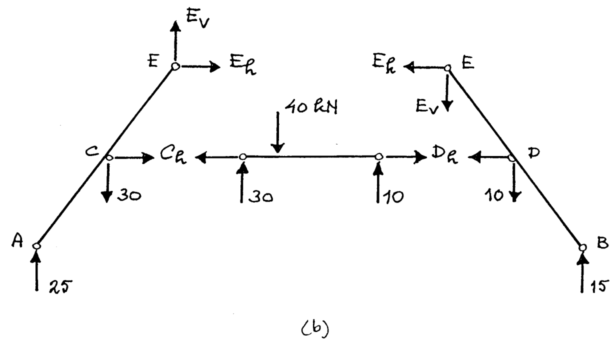


Figure 5.47 (b) The equilibrium of AE and BE is then used to find the other interaction forces.

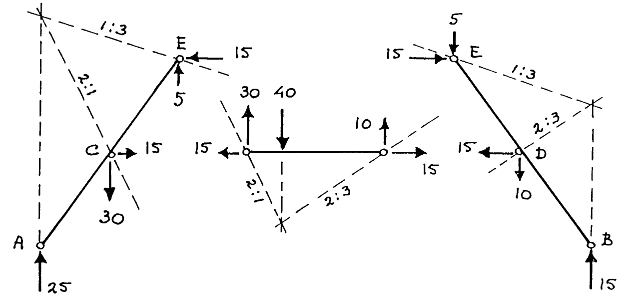


Figure 5.48 Graphical check of the moment equilibrium of AE, CD and BE: in all the cases, the lines of action of the three (resulting) forces pass through a single point.

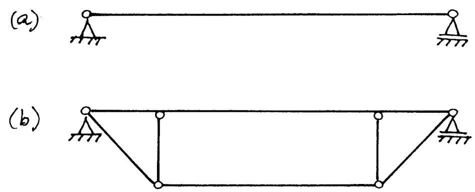


Figure 5.49 (a) A beam and (b) a trussed beam.

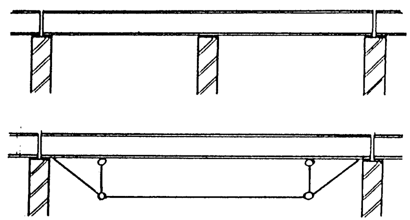


Figure 5.50 After a dividing wall has been demolished, the bearing capacity of a beam can be restored by introducing intermediate supports.

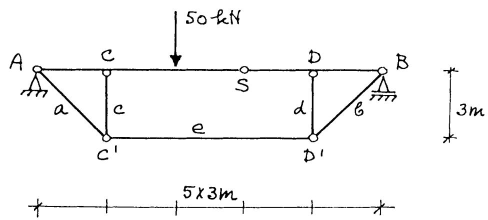


Figure 5.51 A statically determinate trussed beam.

5.6 Trussed beams

The bearing capacity of the beam in Figure 5.49a can be increased by introducing intermediate supports. These structures are referred to as *trussed beams* when these intermediate supports are realised by a bar system applied directly to the beam (see Figure 5.49b).

Trussed beams are used in simple bearing structures and for auxiliary structures in the construction industry (formwork bearers). You may also see them in restoration activities when, for example, after a dividing wall has been demolished, the bearing capacity of the floor beams is no longer adequate, as a result of the enlarged span (see Figure 5.50).

In the examples given, the trussed beams are (internally) statically indeterminate. In the following will address only statically determinate structures.

Example
The trussed beam ASB in Figure 5.51 consists of the two beam segments AS and SB joined by a hinge at S. The structure is loaded by a vertical force of 50 kN.

Questions:

a. Determine the support reactions.
b. Determine the forces in the bars (a) to (e) (with the correct sign for tension and compression).
c. Draw the forces acting on beam segments AS and SB.
d. Draw the forces acting on joint D.

Solution:
a. The support reactions follow directly from the equilibrium of the structure as a whole. They are shown in Figure 5.52.

b. The bars (a) to (e) are loaded only at their ends. They are therefore two-force members. Note: ACS and SDB are not two-force members!

In Figure 5.52, the isolated structure has been dissected across bar (e) and the hinged joint at S. Suppose there is a tensile force in bar (e) of $N^{(e)}$. The magnitude of $N^{(e)}$ follows from the moment equilibrium about S of the left-hand or right-hand part. The simpler equation is obtained with the unloaded right-hand part:

$$\sum T_z^{(SB)}|S = (20\text{ kN})(6\text{ m}) - N^{(e)} \times (3\text{ m}) = 0 \Rightarrow N^{(e)} = 40\text{ kN}.$$

Bar (e) is therefore a tension member.

The moment equilibrium of the left-hand part about S can be used to check the solution.

The forces in the bars (a) and (c) follow from the force equilibrium of joint C′. In Figure 5.53 these forces have been determined using a force polygon. In bar (a) there is a tensile force, while there is a compressive force in bar (c):

$$N^{(a)} = 40\sqrt{2}\text{ kN} \quad \text{and} \quad N^{(c)} = -40\text{ kN}.$$

In the same way, the force equilibrium of joint D′ gives

$$N^{(b)} = 40\sqrt{2}\text{ kN} \quad \text{and} \quad N^{(d)} = -40\text{ kN}.$$

c. Figure 5.54 shows all the forces acting on the beam segments AS and SB. The components of the hinge force S follow from the force equilibrium of the part to the left or to the right of S.

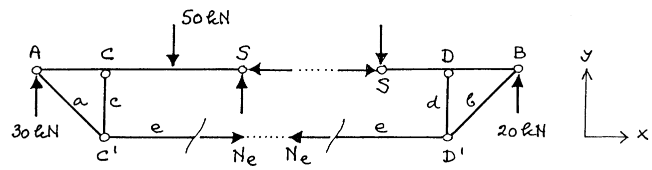


Figure 5.52 The trussed beam, isolated from its supports, has been "cut" across hinged joint S and bar (e). It has been assumed that bar (e) is a tension member.

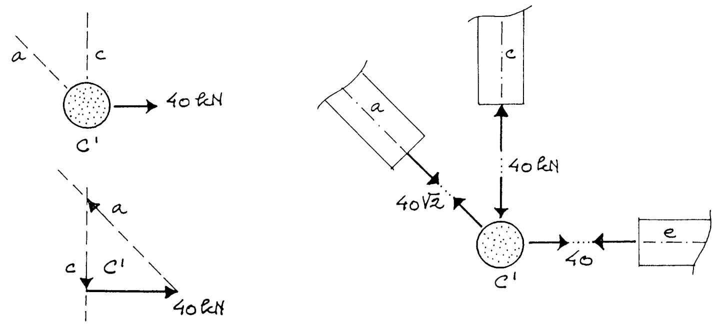


Figure 5.53 The forces in the bars (a) and (c) follow from the force equilibrium of joint C′: in bar (a) there is a tensile force and in bar (c) there is a compressive force.

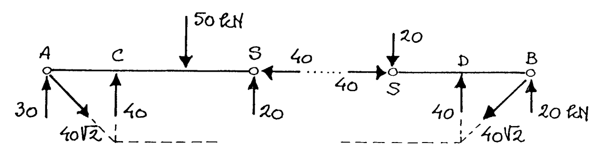


Figure 5.54 The isolated beam segments AS and SB, with all the forces as they really act.

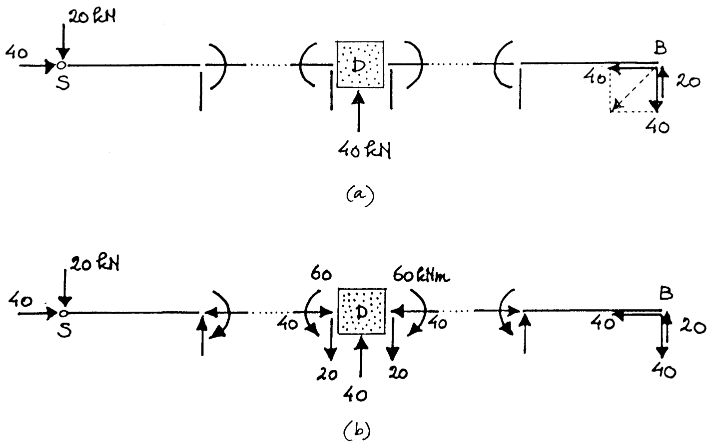


Figure 5.55 (a) Joint D isolated from the beam segments SD and DB. Three interaction forces are acting in the rigid connections. These can be found from the equilibrium of the separate beam segments. (b) The interaction forces as they really act. Joint D satisfies the conditions for force and moment equilibrium.

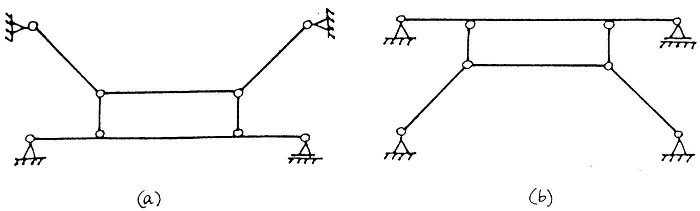


Figure 5.56 A beam (a) suspended from and (b) leaning upon a strengthening bar system.

d. In Figure 5.55a, joint D has been isolated from SD and DB. Three interaction forces are acting in the rigid connections. The forces exerted by joint D on SD and DB can be found from the equilibrium of these parts. Equal and opposite forces act on joint D (see Figure 5.55b).

Check: Joint D is in force equilibrium and in moment equilibrium.

5.7 Strengthened beams

The strengthened beams in Figure 5.56 are in many ways comparable to trussed beams. An important difference is that in here the strengthening bar system is supported outside the beam. In Figure 5.56a the beam is suspended from the strengthening bar system, in Figure 5.56b the beam is leaning upon it.

These structures are used in bridges. They are used also as auxiliary structures during building activities.

The structures in Figure 5.56 are statically indeterminate to the first degree. In the following we will address only statically determinate examples.

Example
The structure in Figure 5.57 is loaded by a vertical force of 40 kN.

Questions:
a. Determine the support reactions at A and B.
b. Determine the forces in bars (1) to (3) and (a) to (d).
c. Draw the forces acting on the hinged joint S.

Solution (units in kN and m):
a. This compound structure has five support reactions:

- two at hinged support A,
- one at roller support B,
- one at hinge A′, and
- one at hinge B′.

The three equilibrium equations for the structure as a whole are not sufficient for finding the five unknown support reactions. The solutions have to be found by means of the strengthening bar system.

Bars (1) to (3) and (a) to (d) are all two-force members. If one of the bar forces is known, all the others follow from the force equilibrium of the joints S′, C′ and D′. This is shown graphically in Figure 5.58 on the assumption that there is a tensile force N in bar (2):

$$N^{(2)} = N.$$

The force equilibrium of joint S′ then gives

$$N^{(b)} = N^{(c)} = \tfrac{1}{2}\sqrt{17}\,N.$$

The force equilibrium of joint C′ gives

$$N^{(1)} = \tfrac{3}{2}N \quad \text{and} \quad N^{(a)} = 2\sqrt{2}\,N,$$

while the equilibrium of joint D′ gives

$$N^{(3)} = \tfrac{3}{2}N \quad \text{and} \quad N^{(d)} = 2\sqrt{2}\,N.$$

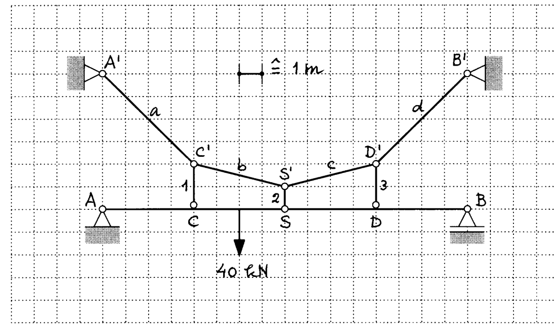


Figure 5.57 A statically determinate strengthened beam.

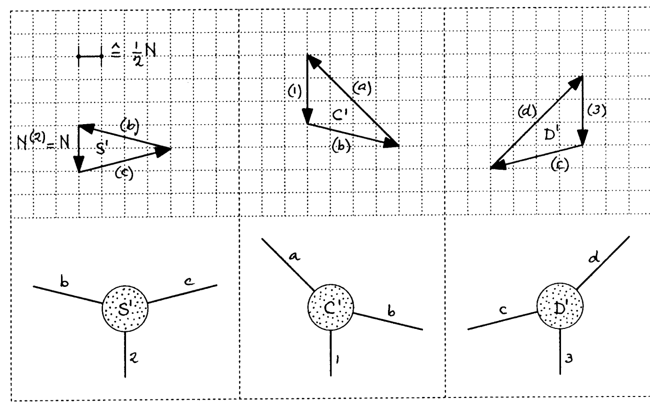


Figure 5.58 Assuming that a tensile force N acts in bar (2), the force equilibrium of the hinged joints S′, C′ and D′ can be used to express the forces in the bars (a) to (d) and (1) and (3) in terms of N.

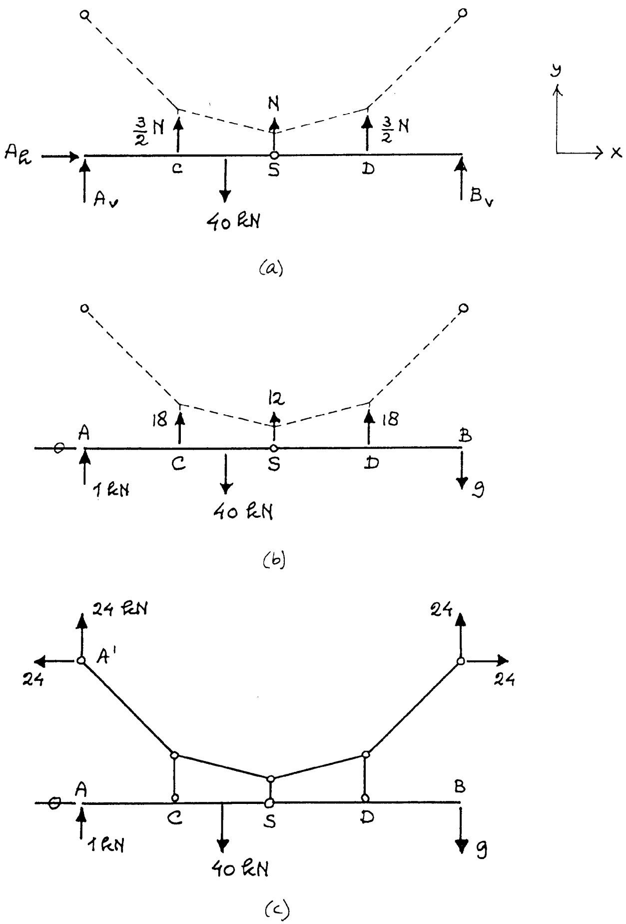


Figure 5.59 (a) The isolated beam ASB. (b) All the forces acting on the beam ASB as they really act. (c) The entire structure with all the support reactions.

In Figure 5.59a, hinged beam ASB has been isolated and all the forces acting on it are shown. The horizontal force equilibrium of the hinged beam as a whole then gives

$$A_h = 0.$$

The vertical support reactions A_v and B_v, and the unknown force N are calculated in the same way as for a hinged beam (see Section 5.2).

For the beam as a whole applies

$$\begin{aligned}\sum T_z|A &= \tfrac{3}{2}N \times 4 + N \times 8 + \tfrac{3}{2}N \times 12 + B_v \times 16 - 40 \times 6 \\ &= 32N + 16B_v - 240 = 0.\end{aligned} \tag{a}$$

For the right-hand section SB

$$\sum T_z|S = 4 \times \tfrac{3}{2}N + 8 \times B_v = 0. \tag{b}$$

These two equations with N and B_v as unknowns give

$$N = 12 \text{ kN} \quad \text{and} \quad B_v = -9 \text{ kN}.$$

The vertical support reaction at B therefore acts opposite to the direction assumed in Figure 5.59a.

The vertical support reaction at A follows from the vertical force equilibrium of beam ASB as a whole:

$$\sum F_y = A_v + \tfrac{3}{2}N + N + \tfrac{3}{2}N + B_v - 40 = 0$$

so that

$$A_v = -4N - B_v + 40 = -4 \times 12 - (-9) + 40 = 1 \text{ kN}.$$

Figure 5.59b shows all the forces on beam ASB as they really act.

b. The forces in the bars (1) to (3) and (a) to (d) were previously expressed in terms of N (see Figure 5.58). With $N = 12$ kN the result is

$$N^{(1)} = N^{(3)} = \tfrac{3}{2}N = 18 \text{ kN},$$

$$N^{(2)} = N = 12 \text{ kN},$$

$$N^{(a)} = N^{(d)} = 2\sqrt{2}\,N = 24\sqrt{2} \text{ kN},$$

$$N^{(b)} = N^{(c)} = \tfrac{1}{2}\sqrt{17}\,N = 6\sqrt{17} \text{ kN}.$$

All bar forces are tensile forces.

Figure 5.59c gives the entire structure with all the support reactions.

Check: The structure as a whole satisfies the conditions of the force and moment equilibrium.

c. In Figure 5.60a, the beam segments AS and BS and the hinged joint S have been isolated. The values of all the known forces are shown. The forces acting on joint S are found via the equilibrium of the segments AS and SB. They are shown in Figure 5.60b.

Check: Joint S is in equilibrium.

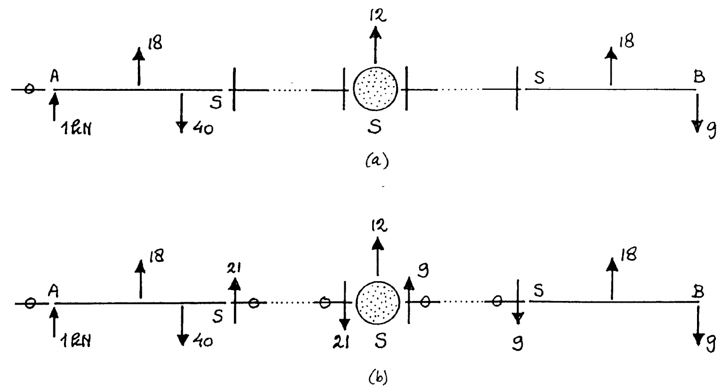


Figure 5.60 (a) The hinged joint S isolated from the beam segments AS and SB. The interaction forces can be found from the equilibrium of the segments AS and SB. (b) All the forces as they really act. Joint S is in equilibrium.

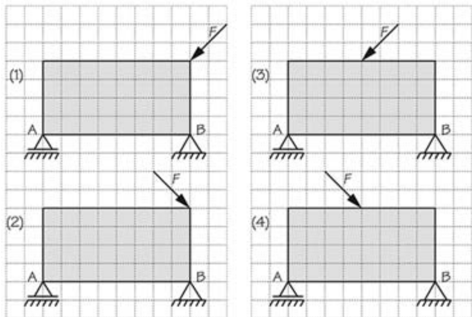
5.8 Problems

Self-contained structures (Section 5.1)

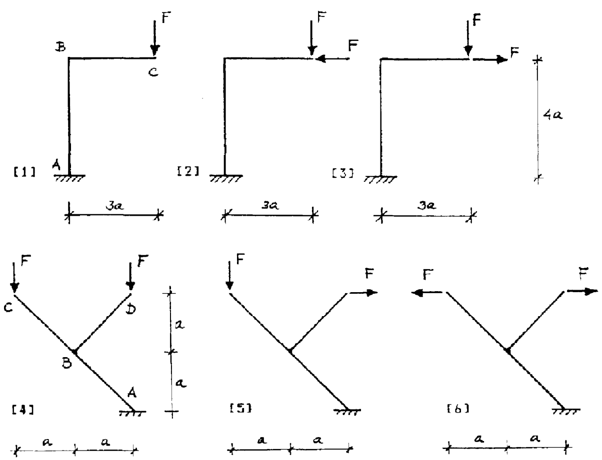
5.1: 1–4 A block is supported on a roller at A and a hinge at B. The block is loaded by a force $F = 20\sqrt{2}$ kN. Length scale: 1 square ≡ 1 m.

Question:
Determine the support reactions at A and B:

a. analytically;
b. graphically.



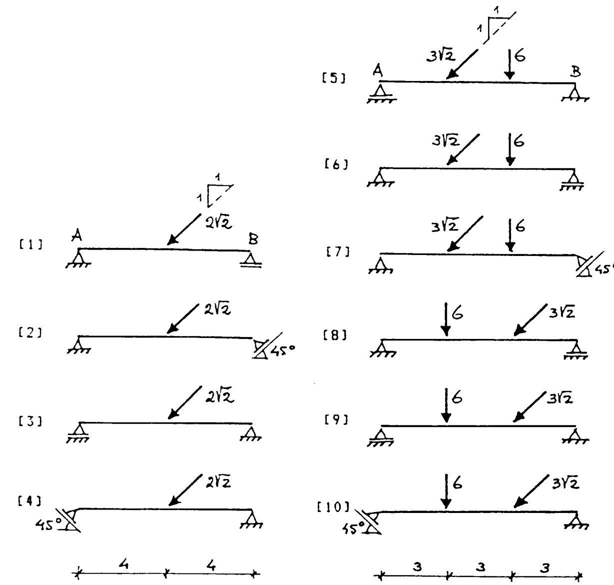
5.2: 1–6 Given a number of fixed structures.



Questions:

a. In which directions would you expect the support reactions at A to act?
b. Determine the support reactions at A, working with the directions assumed in (a).
c. For which support reactions did you assume the wrong direction?
d. Draw all the support reactions as they act in reality.

5.3: 1–10 A number of beams are supported on a hinge and a roller. The dimensions are given in m, the forces are in kN.

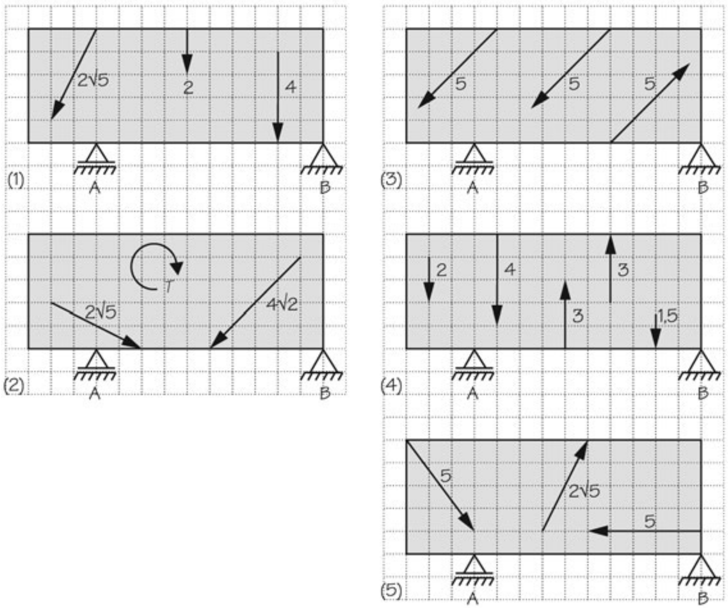


Questions:

a. Determine the support reactions analytically.
b. Check the answers graphically (if possible).

5.4: 1–5 A block is supported on a roller at A and a hinge at B. A number of forces act on the block. In case 2, a couple $T = 36$ kNm also acts on the block. Force scale: 1 square ≡ 1 kN; length scale: 1 square ≡ 1 m.

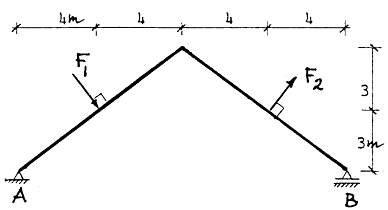
Question: Determine the support reactions at A and B.



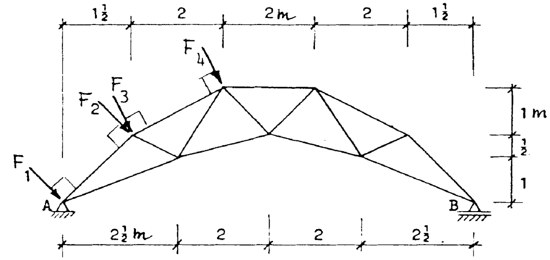
5.5 A roof structure is loaded by wind forces:

$$F_1 = 5.6 \text{ kN},$$
$$F_2 = 2.8 \text{ kN}.$$

Question:
Determine the support reactions at A and B.

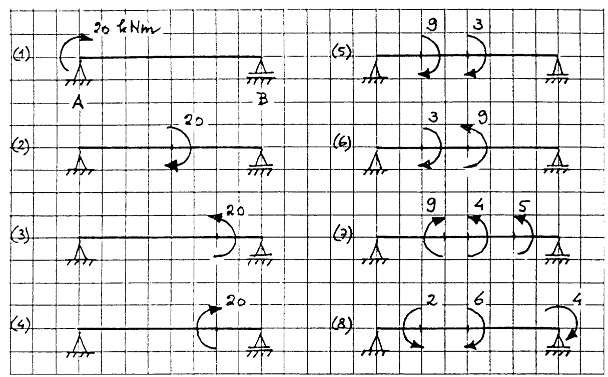


5.6 A truss arch is loaded by wind forces: $F_1 = F_2 = 750\sqrt{2}$ kN, $F_3 = F_4 = 500\sqrt{5}$ kN.



Question:
Determine the support reactions at A and B.

5.7: 1–8 The simply supported beam AB is loaded in various ways by couples. The magnitude of the couples is shown in kNm. Length scale: 1 square ≡ 1 m.

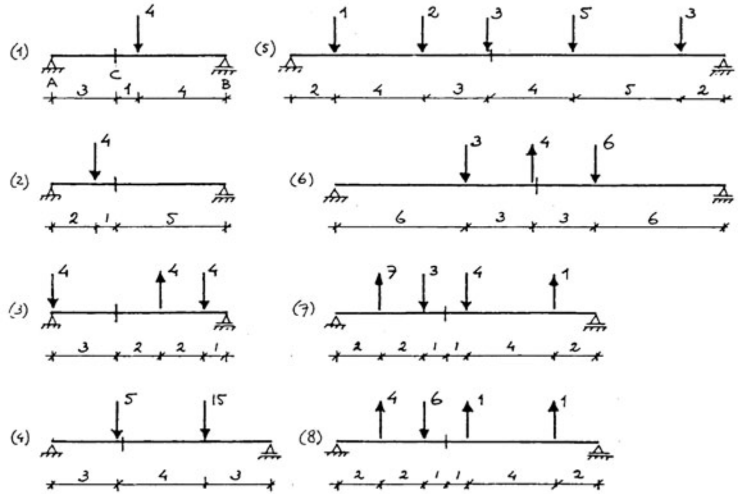


Question:
Find the support reactions at A and B.

5.8: 1–8 A number of beams simply supported at A and B are composed of the segments AC and BC that are rigidly connected at C. The location of joint C is shown in the figure by means of a vertical dash. The forces are given in kN, the lengths in m.

Questions:

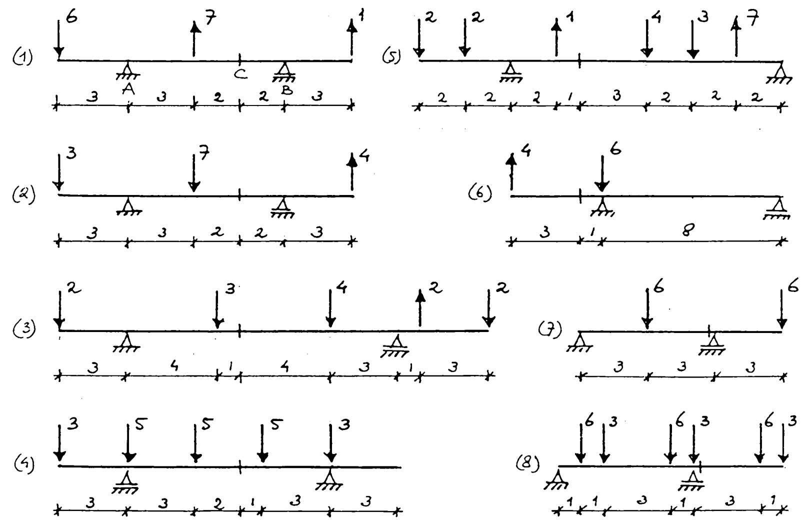
a. Determine the support reactions.
b. Determine the interaction forces at C; draw these forces as they act at C on segments AC and BC.
c. Draw the forces as they really act on joint C.



5.9: 1–6 A number of cantilever beams, simply supported at A and B, are composed of two segments that are rigidly connected at C. The location of joint C is shown in the figure by means of a vertical dash. The forces are given in kN, the lengths are in m.

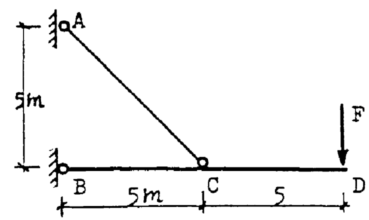
Questions:

a. Determine the support reactions.
b. Determine the interaction forces at C; draw these forces as they act at C on the segments AC and BC.
c. Draw the forces as they really act on joint C.

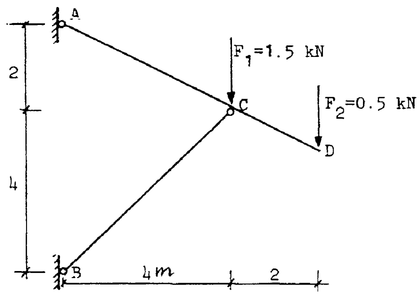


5.10 *Question*:
Determine the support reactions at A and B.

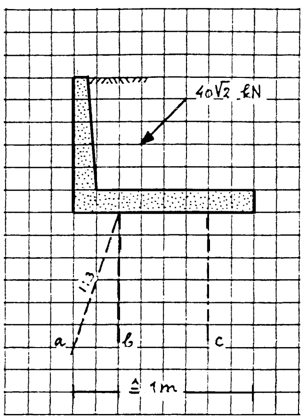
a. graphically;
b. analytically.

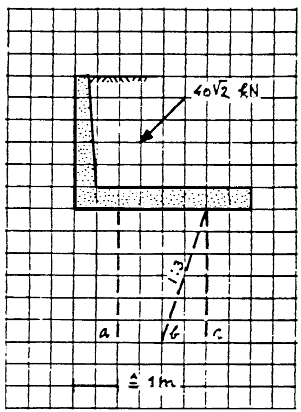


5.11 *Questions*:
For porch ACD determine the support reactions at A and B due to:
a. only F_1;
b. only F_2;
c. both F_1 and F_2.



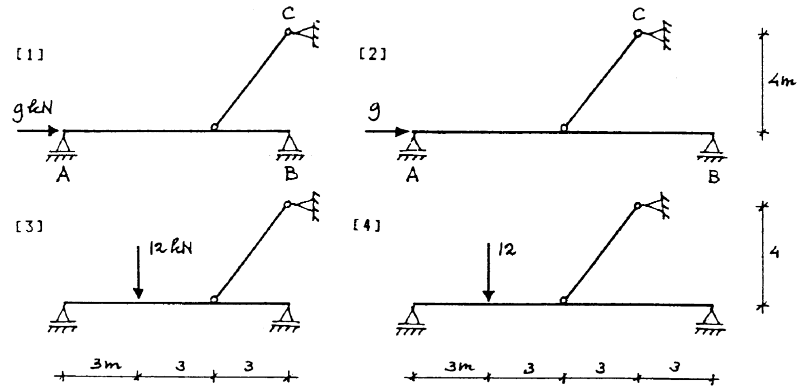
5.12: 1–2 You are given a retaining wall on piles. Assume the piles are only transferring forces in their longitudinal direction. The resultant of all the loads that the piles have to bear is a force of $40\sqrt{2}$ kN. The direction and line of action are given in the figure. Length scale: 2 squares ≡ 1 m.

[1]


[2]


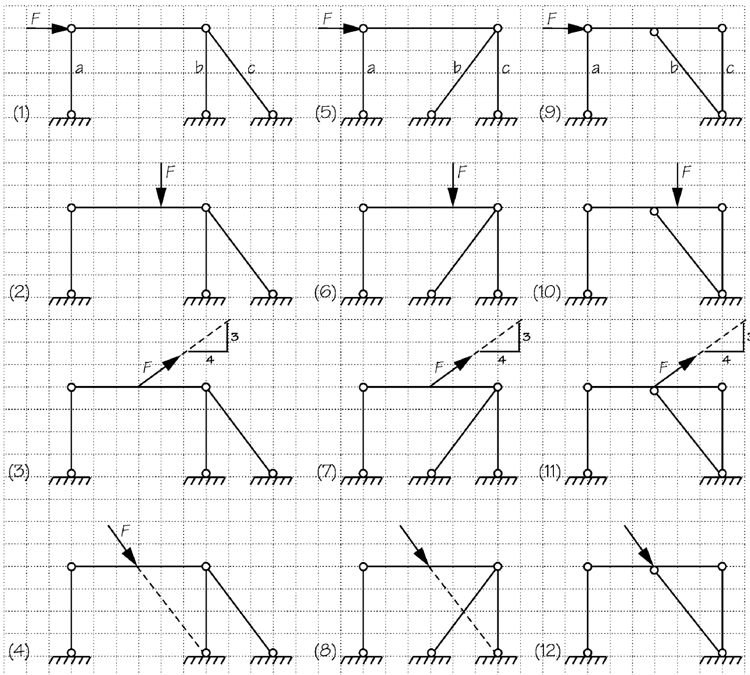
Question:
Find the pile forces with the correct signs for tension and compression (a tensile force is positive and a compression force is negative).

5.13: 1–4



Questions:
a. Make a realistic assumption about the directions of the support reactions at A, B and C.
b. Determine these support reactions.
c. Draw the support reactions as they really act and include relevant values.
d. If possible, check the calculated support reactions graphically.

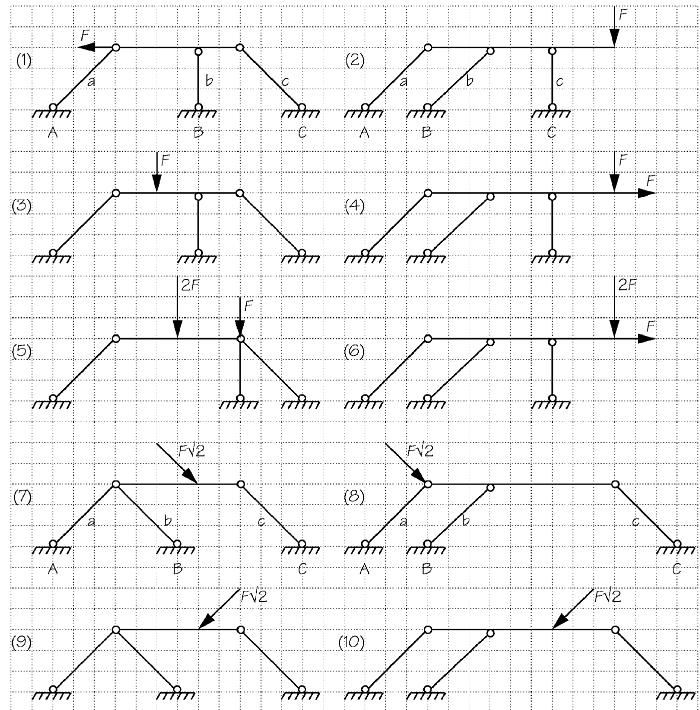
5.14: 1–12 A beam, loaded by a force $F = 30$ kN, is supported by the three bars a, b and c. Length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions at A, B and C.
b. Determine the forces in the beams, with the correct sign.
c. Isolate the beam, draw all the forces as they really act on it, and check the equilibrium.

5.15: 1–10 A beam is supported by the bars a, b and c. The load of the beam is expressed in the force $F = 30$ kN. Length scale: 1 square ≡ 1 m.

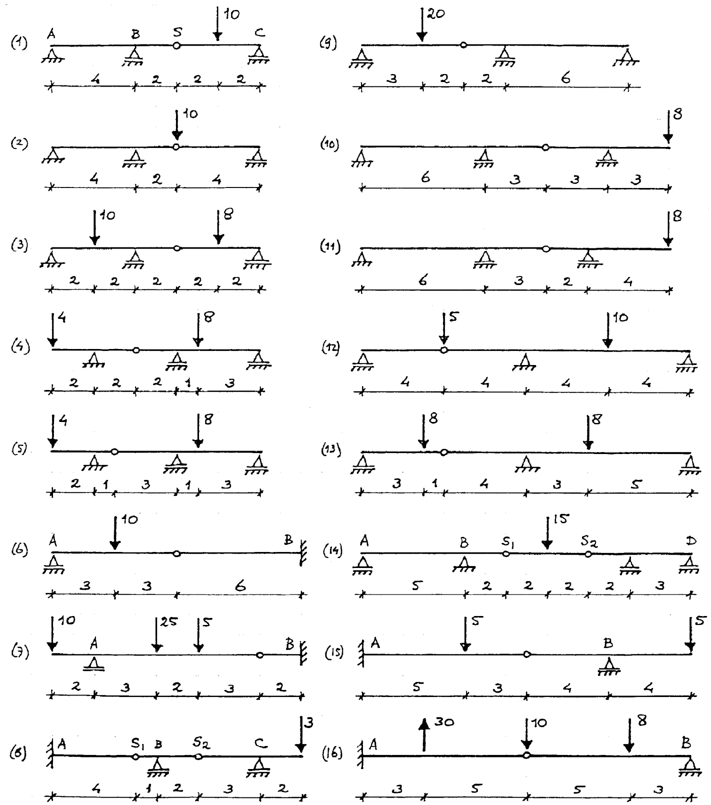


Questions:

a. Determine the support reactions at A, B and C.
b. Determine the forces in the bars, with the correct sign.
c. Isolate the beam, draw all the forces as they really act on it, and check the equilibrium.

Hinged beams (Section 5.2)

5.16: 1–16 The dimensions of the hinged beams are given in m, the forces are in kN.



Questions:

a. Determine the support reactions.
b. Isolate all the beam segments and draw the forces as they really act on these segments.
c. Check the force and moment equilibrium of the structure as a whole.

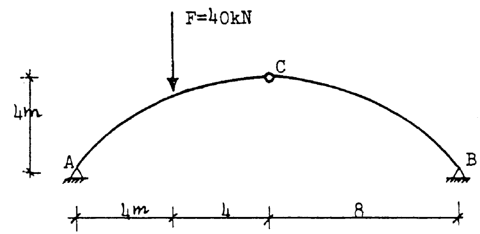
Three-hinged frames (Section 5.3)

5.17 Three-hinged arch ACB is loaded by a force $F = 40$ kN.

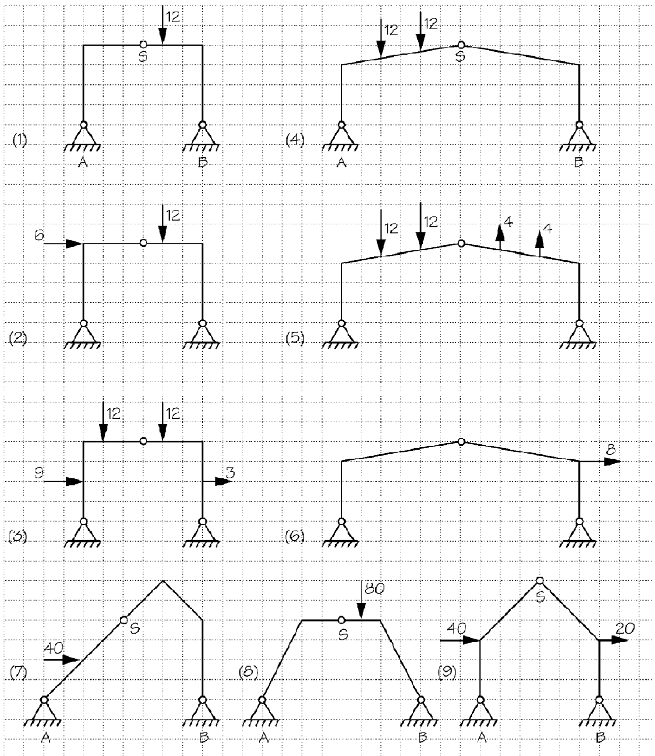
Questions:

Determine the support reactions at A and B:

a. graphically;
b. analytically.



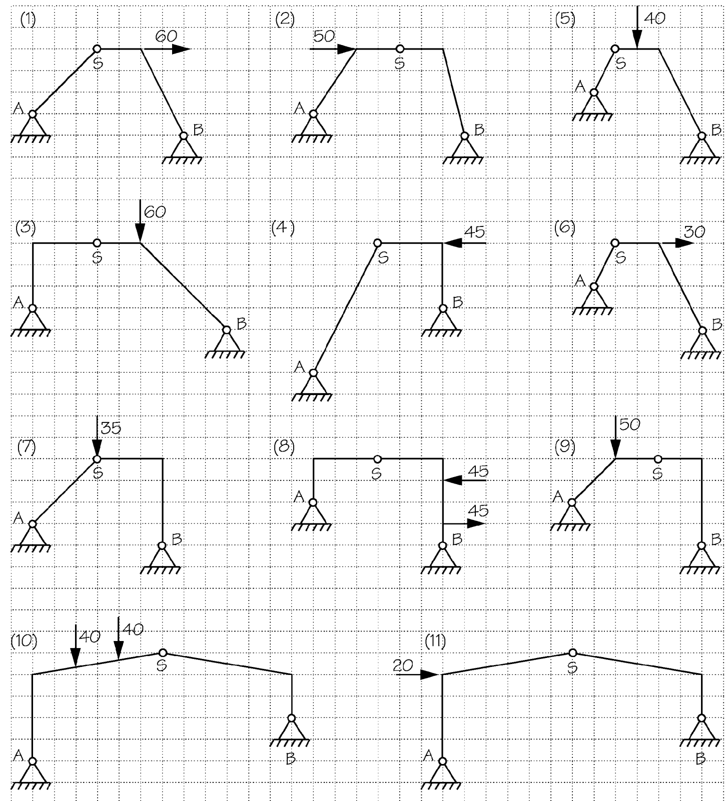
5.18: 1–9 The figure shows a number of three-hinged frames with loads expressed in kN. Length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions.
b. Isolate both frame halves and draw all the forces as they really act on them.

5.19: 1–11 The figure shows a number of three-hinged frames with the loading expressed in kN. Length scale: 1 square ≡ 1 m.

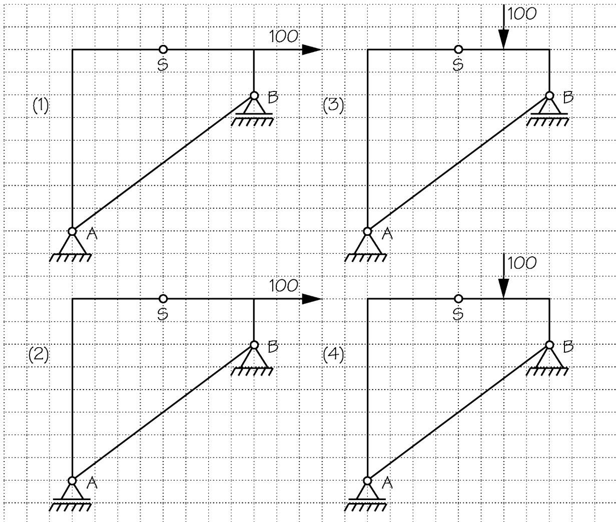


Questions:

a. Determine the support reactions.
b. Graphically check the support reactions (if possible).
c. Isolate both frame halves and draw all the forces as they really act on them.

Three-hinged frames with tie rods (Section 5.4)

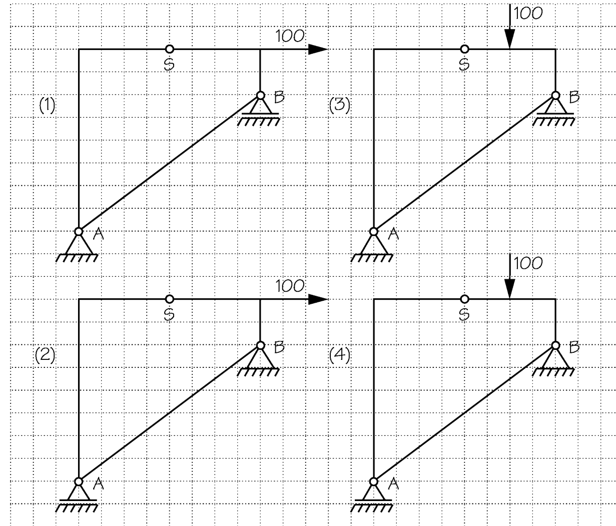
5.20 The structural parts AB, AS and BS are connected by hinges at A, B and S. The load is in kN; length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions.
b. Determine the force in rod AB, with the correct sign.
c. Determine the forces acting on the isolated parts AB, AS and BS.
d. Determine the forces acting on the isolated joints A, B, and S.

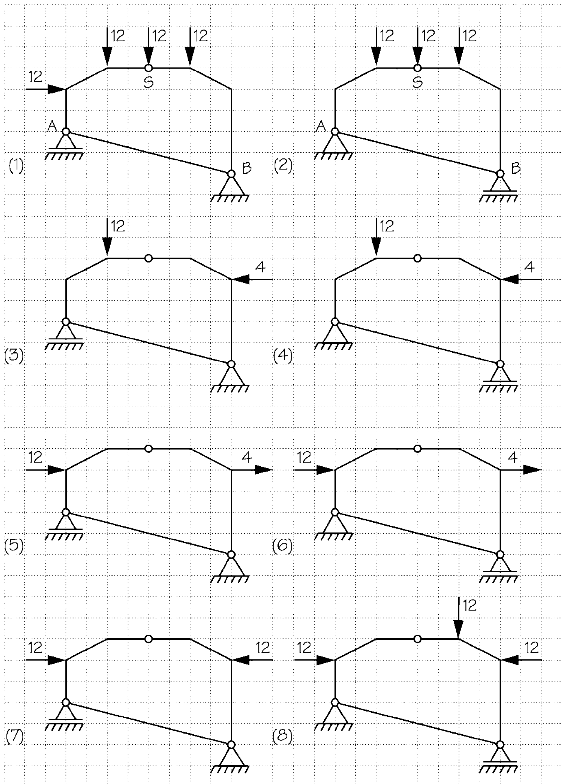
5.21: 1–4 The structural parts AB, AS and BS are connected by hinges at A, B and S. The load is given in kN; length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions.
b. Determine the forces in rod AB, with the correct sign.
c. Determine the forces acting on the isolated parts AB, AS and BS.
d. Determine the forces acting on the isolated joints A, B and S.

5.22: 1–12 The structural parts AB, AS and BS are connected by hinges at A, B and S. The load is given in kN; length scale: 1 square ≡ 1 m.

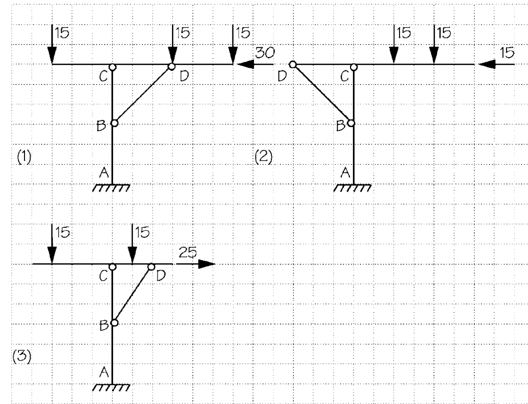


Questions:

a. Determine the support reactions.
b. Determine the forces in rod AB, with the correct sign.
c. Determine the forces acting on the isolated parts AB, AS and BS.
d. Determine the forces acting on the isolated joints A, B and S.

Shored structures (Section 5.5)

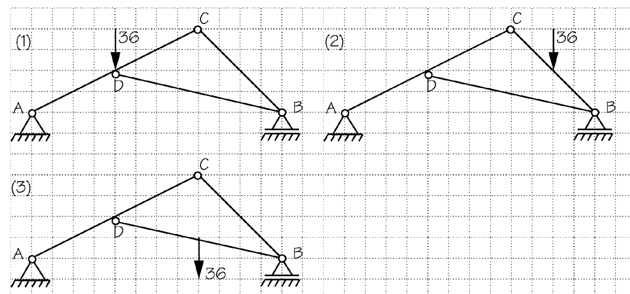
5.23: 1–3 The forces are given in kN; length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions.
b. Determine the force in the shoring bar, with the correct sign.
c. Determine the forces acting on the isolated joints B and C.

5.24: 1–3 The load is given in kN; length scale: 1 square ≡ 1 m.

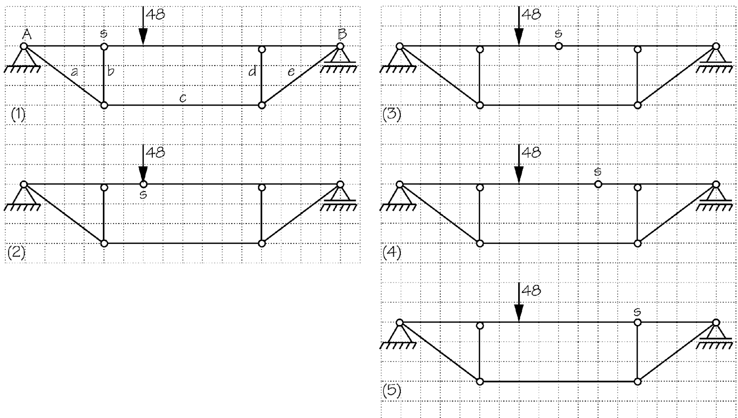


Questions:

a. Determine the support reactions at A and B.
b. Determine the force in bar BD, with the correct sign.
c. Isolate all bars and draw all the forces really acting on them.

Trussed beams (Section 5.6)

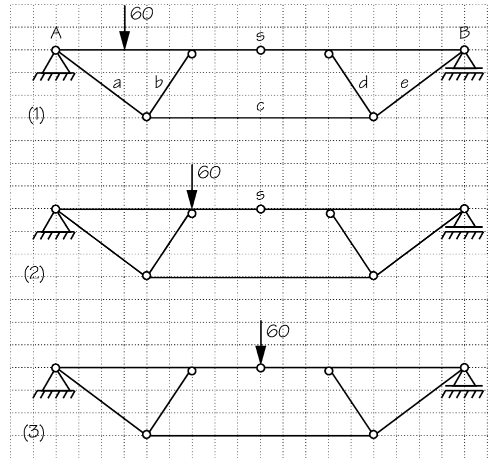
5.25: 1–5 The forces are given in kN; length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions at A and B.
b. Determine the forces in bars a to e, with the correct sign.
c. Isolate the beam sections AS and BS and draw all the forces acting on them.
d. Isolate joint B, draw all the forces really acting on it, and check the force equilibrium using a force polygon.

5.26: 1–3 The forces are given in kN; length scale: 1 square ≡ 1 m.



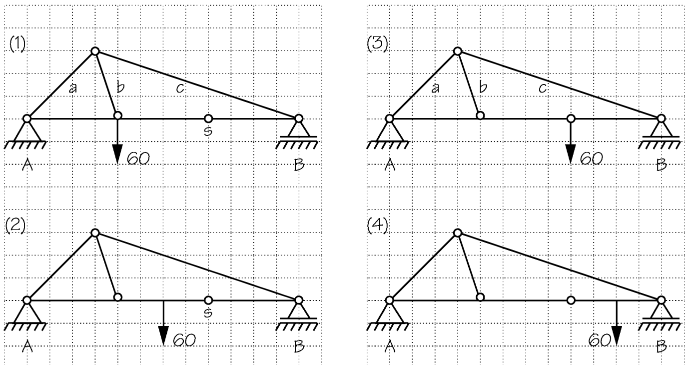
Questions:

a. Determine the support reactions at A and B.
b. Determine the forces in bars a to e, with the correct signs.
c. Isolate beam segments AS and BS and draw all the forces really acting on them.
d. Isolate joint B, draw all the forces really acting on it, and check the force equilibrium using a force polygon.

5.27: 1–4 The forces are given in kN; length scale: 1 square ≡ 1 m.

Questions:

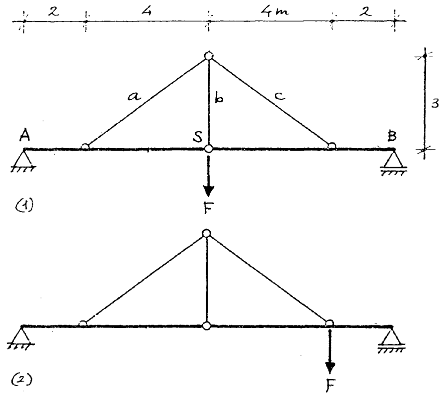
a. Determine the support reactions at A and B.
b. Determine the forces in bars a to c, with the correct signs.
c. Isolate beam segments AS and BS and draw all the forces really acting on it.
d. Isolate joint A, draw all the forces really acting on it, and check the force equilibrium using a force polygon.



5.28: 1–2 Trussed beam ASB is loaded by a vertical force $F = 48$ kN.

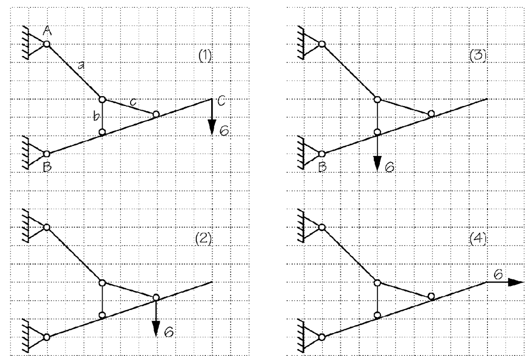
Questions:

a. Determine the support reactions at A and B.
b. Determine the forces in bars a to e, with the correct signs.
c. Isolate beam segments AS and BS, and draw all the forces really acting on them.



Strengthened beams (Section 5.7)

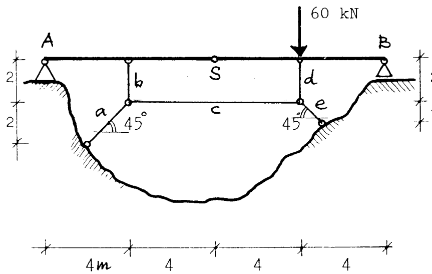
5.29: 1–4 The forces are given in kN; length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions at A and B.
b. Determine the forces in bars a to c, with the correct signs.
c. Isolate beam BC, draw all the forces really acting on it, and check the force and moment equilibrium.

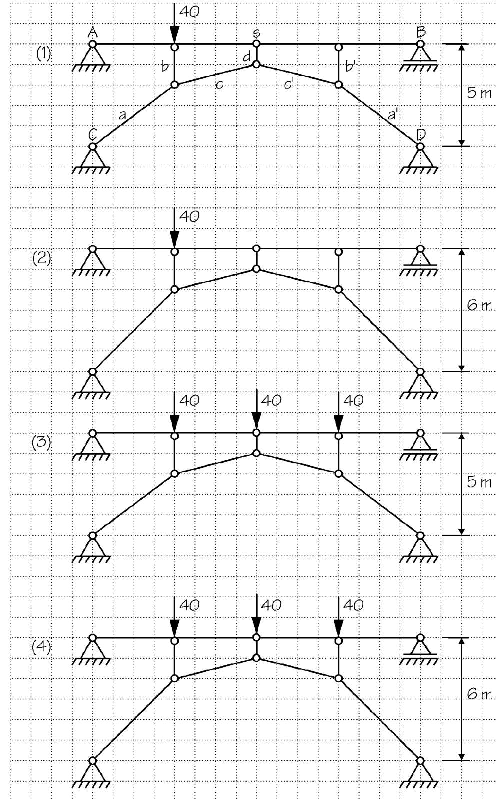
5.30 The queen post truss is loaded at stay d by a vertical force of 60 kN.



Questions:

a. Determine the forces in bars a to e, with the correct signs.
b. Isolate beam segments AS and BS and draw all the forces really acting on them.
c. Determine the support reactions at A and B.

5.31: 1–4 The forces are given in kN; length scale: 1 square ≡ 1 m.

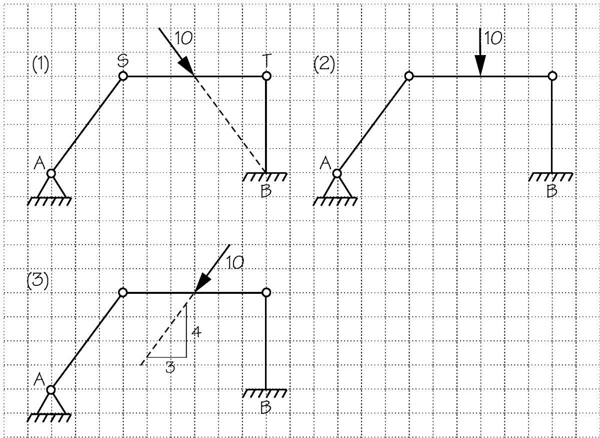


Questions:

a. Determine the forces in bars a to d and a′ to c′, with the correct signs.
b. Isolate beam segments AS and BS and draw all the forces really acting on them.
c. Determine the support reactions at A, B, C and D.

Various compound structures

5.32: 1–3 The forces are given in kN. Length scale: 1 square ≡ 1 m.



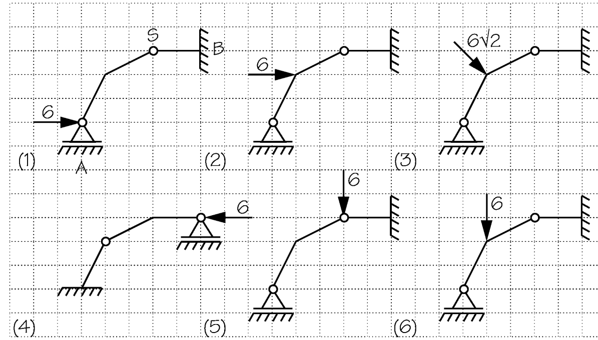
Questions:

a. Determine the support reactions.
b. Isolate all the structural members at the supports and hinged joints, and draw all the forces really acting on them.

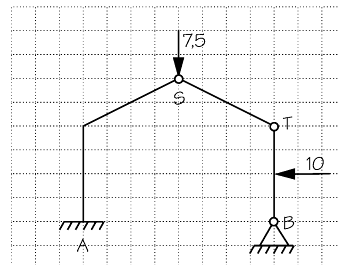
5.33: 1–6 The forces are given in kN. Length scale: 1 square ≡ 1 m.

Questions:

a. Determine the support reactions.
b. Isolate all the structural members at the supports and hinged joints, and draw all the forces really acting on them.



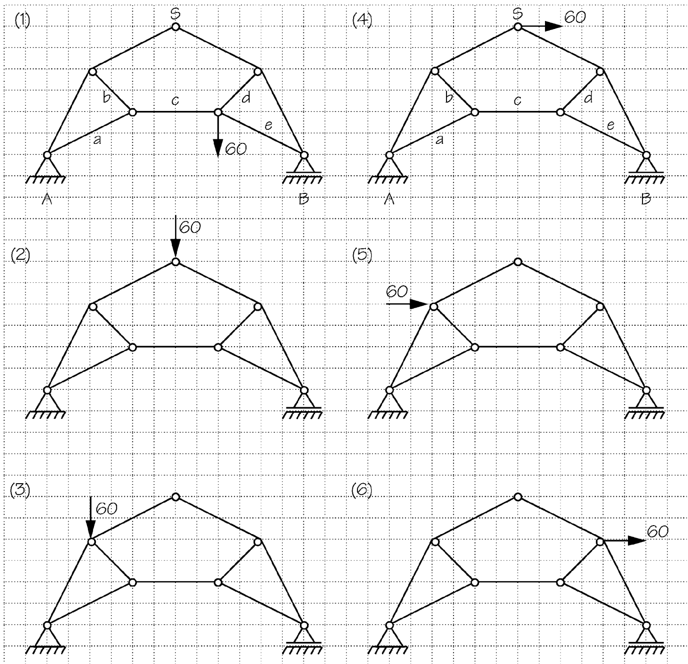
5.34 Given a frame with the loads in kN. Length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions at A and B.
b. Isolate all the structural members at the supports and hinged joints, and draw all the forces really acting on them.

5.35: 1–6 The forces are given in kN; length scale: 1 square ≡ 1 m.



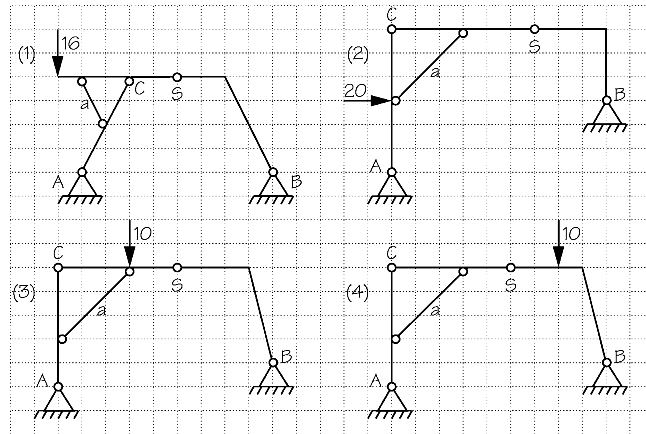
Questions:

a. Determine the support reactions at A and B.
b. Determine the forces in bars a to e, with the correct signs.
c. Determine the interaction forces at S, as they act on AS and BS.

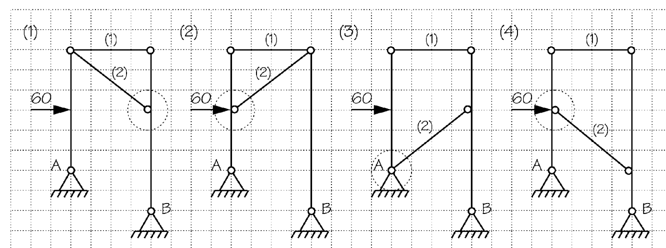
5.36: 1–4 The forces are given in kN; length scale: 1 square ≡ 1 m.

Questions:

a. Determine the support reactions.
b. Graphically check the support reactions.
c. Determine the force in shoring bar a, with the correct sign.
d. Determine the hinge forces at S, as they act on CS and BS.



5.37: 1–4 The forces are given in kN; length scale: 1 square ≡ 1 m.



Questions:

a. Determine the support reactions.
b. Determine the force in bars 1 and 2, with the correct signs.
c. Isolate the circled joint and draw all the forces acting on it.

6 Loads

All influences acting on a structure can be considered as loads. In mechanics, we generally restrict ourselves to loads that occur as a result of forces and prescribed deformations or displacements. In doing so, we make a distinction between *static* and *dynamic loads*. This is covered in Section 6.1.

In vibration-sensitive structures, dynamic loads can generate far greater forces and deformations than one would find from a static calculation. Dynamic calculations are more complex than static calculations and are beyond the scope of this book.

For traditional structures, *regulations* or *codes* are prescribed with respect to loads and loading combinations. These are based on experience, measurements and common sense. Section 6.2 briefly describes the loads mentioned in the regulations. For special structures, the regulations are often not sufficient, and loading analyses may demand extensive study.

Whereas up until now a load has consisted of one or more concentrated forces, this chapter addresses *distributed loads*; we distinguish between *volume loads, surface loads*, and *line loads*.

A system of forces on a structure (which is considered a rigid body) can, for equilibrium purposes, be replaced by its resultant. The same applies for a distributed load. Section 6.3 addresses how to calculate the resultant of a

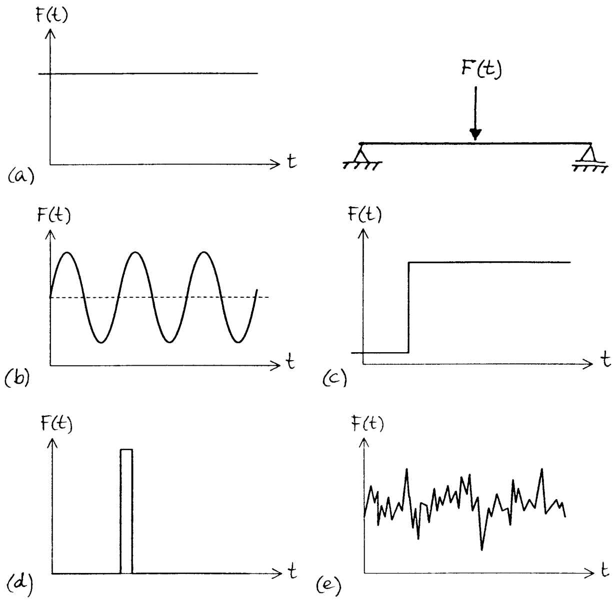


Figure 6.1 (a) A load that does not change with time is called a static load. Dynamic loads are loads that change in time, such as (b) periodic loads, (c) suddenly applied loads, (d) loads of short duration and collision phenomena (impact loads), and (e) stochastic loads.

distributed load. Line loads on a member will be treated more extensively.

How the load is determined depends to a great extent on the manner in which a structure or structural element is modelled. For example, the dead weight of a bar depicted as a line element is not treated as a volume load but rather as a line load. In the same way, the dead weight of a slab (plane element) is considered as surface load. This issue is illustrated in Section 6.4 using a simple building.

Section 6.5 addresses the concept of stress. The transfer of forces in and between materials is the result of extremely small interactions between adjacent particles. Spreading all the forces evenly over a section leads to the concept of *stress*.

6.1 Loads in mechanics

6.1.1 Influences on structures

All influences that can act on a structure can be considered as loads. In general, we distinguish between the following:

- *Loads due to forces*
  This could for example be the weight of traffic on a bridge. In addition to the traffic, the bridge must also be able to carry its dead weight.
- *Loads due to prescribed deformations or displacements*
  The settlement of a support is an example of a prescribed displacement. Other examples are the influences of temperature changes, shrinkage and creep.
- *Loads due to other influences*
  If the structure is located in an aggressive environment in which the material is affected, this effect on the material can be seen as a load. Fire is also seen as a load.

Structures have to be designed and constructed in such a way that they offer sufficient resistance to all these influences so that the function of the structure is not endangered in any way.

In mechanics, we generally restrict ourselves to loading by forces, and prescribed deformations and displacements. A further distinction here is that between *static* and *dynamic* loads.

6.1.2 Static and dynamic loads

If a load due to forces or prescribed displacements does not change (or changes very little) in time, as in Figure 6.1a, it is called a *static load*. In contrast, *dynamic loads* do change with time, as in Figures 6.1b to 1e.

The wave action on a structure at sea and the forces exerted by a machine on its foundations are examples of dynamic loading by *forces*. Another example of a dynamic load is an earthquake. In an earthquake, one refers to a *prescribed displacement*: the earth starts to move and the structure is forced to follow the movement of the earth via its foundations.

In general, one can distinguish between four different types of dynamic loading:

- *Periodic loads* (Figure 6.1b)
  This type of load is caused, for example, by rotating machines, ringing bells, eddies in a stream, or people jumping on a floor.
- *Suddenly-applied loads* (Figure 6.1c)
  This could include a load resulting from a snapping wire (see Figure 6.2a). Another example is the cableway in Figure 6.2b, which was used at the Haringvliet[1] dam to unload concrete blocks.
- *Loads of short duration and collision phenomena (impact loads)* (Figure 6.1d)

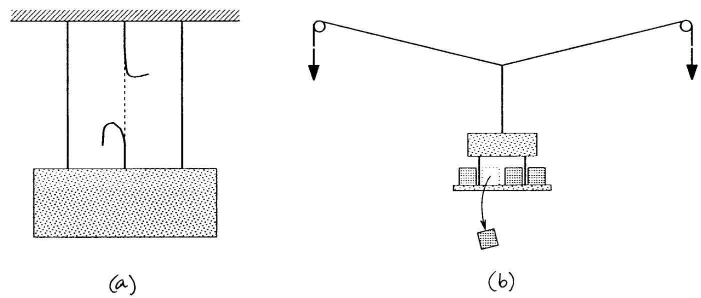


Figure 6.2 (a) When a block is suspended by three wires, and one of the wires suddenly breaks, there is a sudden change in the loading on the remaining wires. (b) Model of the cableway used to close the Haringvliet. Discarding a concrete block causes a sudden change in the loading on the cable gondola.

[1] A see arm. The enclosure of the Haringvliet is one of the Delta Works in the Netherlands.

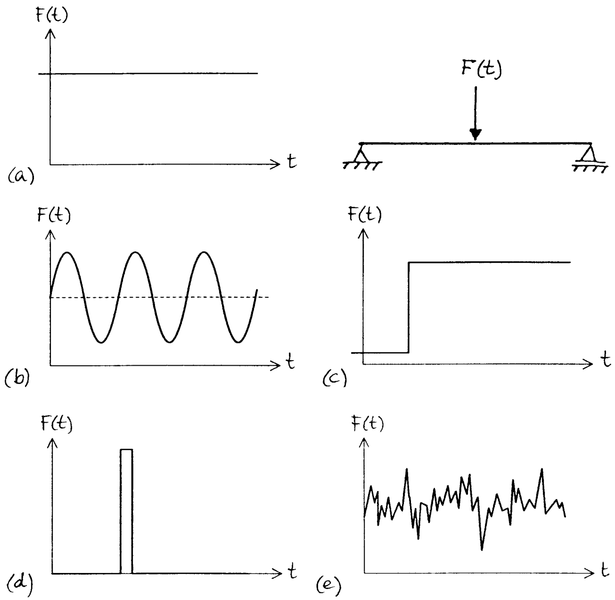


Figure 6.1 (a) A load that does not change with time is called a static load. Dynamic loads are loads that change in time, such as (b) periodic loads, (c) suddenly applied loads, (d) loads of short duration and collision phenomena (impact loads), and (e) stochastic loads.

Examples include explosions, wave impact, gusts of wind, or a falling pile hammer on a pile.

- *Stochastic loads* (Figure 6.1e)
  This includes loads of a variable and unpredictable character, such as those resulting from wind, waves, traffic or earthquakes.

In vibration-sensitive structures, dynamic loads can generate much larger forces and deformations than one would find from a static calculation. This will be illustrated using the simply supported beam in Figure 6.3a, which in the middle of the span has to carry a block with mass m and weight $G = mg$. To simplify matters, the mass of the beam will be disregarded.

The block is suspended from the cable, and touches the beam without resting on it. If the block is carefully placed on the beam by letting out the cable very slowly, both the vertical support reactions will slowly increase to $\frac{1}{2}G$, after which they do not change in time (see Figure 6.3b). The load is static.

It would also be possible to have the weight of the block act on the beam suddenly, not by slowly letting out the cable, but by cutting it. The beam with the block will then start to vibrate around the *static equilibrium position* (see Figure 6.3c). The vertical support reactions are now twice as large (albeit of short duration) as in the case with the static loading.[1] As a result of the ever-present *damping*, the amplitude of the vibration will decrease in time, and the block will finally come to rest in the static equilibrium position, as indicated in Figure 6.3d.

Due to a suddenly applied load, the forces in the structure are twice as large as would be determined by means of a static calculation. If the block is dropped from a certain height, the acting forces are even larger.

In the case of a *periodic load* (soldiers walking in step across a bridge, people jumping up and down on a floor, bells ringing in towers, foundations

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[1] The evidence cannot be given at this stage and is beyond the scope of this book.

for machines, turbines, engines, and so forth) the structure must be designed in such a way that the *natural frequencies*[1] of the structure clearly differ from the *frequency of the loading*. If this is not the case, there is the danger of *resonance*, in which the forces and deformations in the structure can become extremely large.

Dynamic calculations are more complex than static calculations. In regulations and codes, the dynamic influences have often been taken into account by increasing the load so that a static calculation is enough. For example, the static load of people on floors is approximately 3 kN/m^2. Due to the movement of the people, the load changes by frequencies of 1 to 2.5 Hz.[2] The resulting forces are approximately twice the static value. Regulations therefore prescribe that a static equivalent of about 6 kN/m^2 has to be taken into account.[3]

6.1.3 Volume loads, surface loads, line loads, and point loads

So far, we have always imagined that loads are concentrated forces that have their points of application on the structure. In reality, a force never acts on a single point, but acts across a particular area. The following distinctions are made, depending on the dimension of the area of application:

- *Volume loads* (forces per volume; N/m^3)
  For example: a material's dead weight.
- *Surface loads* p (forces per surface area; N/m^2)

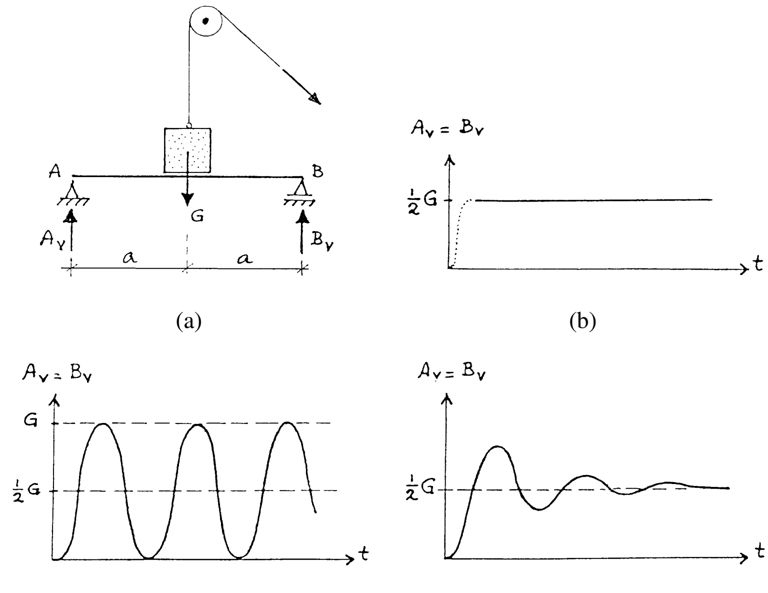

Figure 6.3 (a) A simply supported beam has to carry a block with weight G in the middle of the span. Initially the block is hanging from a cable and touches the beam without resting on it. (b) The vertical support reactions under static conditions after gently letting out the cable. (c) If the cable is not let out slowly, but is cut, the beam with the block starts to vibrate and the vertical support reactions (albeit of short duration) are twice as large as they are under static conditions. (d) As a result of the ever-present damping, the amplitude of the vibration decreases in time, and the block finally ends up at rest in the static equilibrium position.

[1] A natural frequency is a frequency with which (part of) a structure can vibrate freely.

[2] The unit of frequency (Hz = hertz = s^{-1}) is named after Heinrich Rudolf Hertz (1857–1894), German physicist.

[3] In the regulations, the *value for the load* is found from the *characteristic load* by multiplying this by a *load factor* (see Section 6.2.5).

Example: wind and snow loading, gas, liquid, and earth pressures.
- *Line loads* q (forces per length; N/m)
  Example: the weight of a dividing wall on a floor.
- *Point loads*, or concentrated forces F (N)

Volume loads, surface loads, and line loads are referred to as *distributed loads*. In equilibrium analysis, distributed loads may be replaced by their resultant. Section 6.3 addresses the calculation of the magnitude and line of action of this resultant.

6.2 Loads in regulations

For traditional structures, *regulations* or *codes* are prescribed with respect to loads and loading combinations. These have been created on the basis of experience, measurements and common sense. For special structures, the regulations are generally insufficient and the load analysis may demand extensive study.

In the regulations, two important main groups are generally distinguished:
- *dead loads*
- *live loads*

The live load due to the (vertical) traffic load on bridges is known as a *moving load*.

6.2.1 Dead loads

Dead loads are loads that are always present for the entire lifecycle of the structure. The dead load can often be determined quite easily and accurately.

Examples of dead loads include:
- *Dead weight*
  This is the weight of the (bearing) structural element under consideration.

- *Permanent loads*
  This is the weight of non-bearing elements that rest permanently on the structural element under consideration, either directly or indirectly. Examples include the weight of the insulation plates and waterproof roofing material for a roof or the weight of the topping of a bridge deck.
- *Loads due to prestressing*

The effect of the dead load can sometimes be most unfavourable *during construction*, when the structure has not yet been completed and the dead load is not yet present everywhere. A similar situation can occur when the structure is being *converted* or *demolished*.

6.2.2 Live loads (buildings)

Live loads are loads that do not act permanently on the structural element in question. At times, they are present, while at other times they are absent. It is often not as easy to determine the magnitude of live loads as it is to determine dead loads. The values prescribed in the regulations are the result of many years of experience and research.

Live loads include snow on a roof, people on a dance floor, goods in a warehouse or traffic on a bridge. Traffic loads are referred to as *moving loads* (see Section 6.2.3).

In calculations, one has to assume the most unfavourable situation.

In order to simplify calculations, regulations often prescribe the live load on floors, balconies, stairs, roofs, porches, and so forth, in three different guises:

- A uniformly distributed surface load p;
- A uniformly distributed line load q;
- A concentrated load F (a force acting on a small area).

For the live load prescribed for floors, the weight of a standard inventory

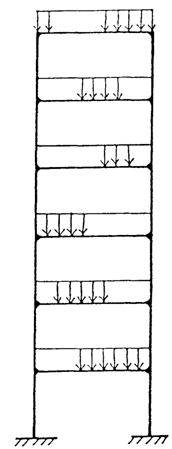

Figure 6.4 Since it is unlikely that the floors in a building are all maximally loaded at the same time, the live load may in certain cases be reduced.

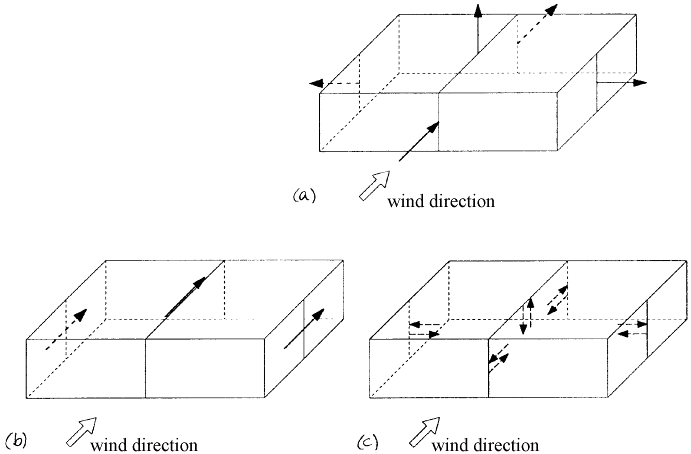

Figure 6.5 Wind loads are distributed loads. The direction of the load is shown by means of an open arrow: (a) wind pressure and wind suction, (b) wind friction, and (c) overpressure and underpressure.

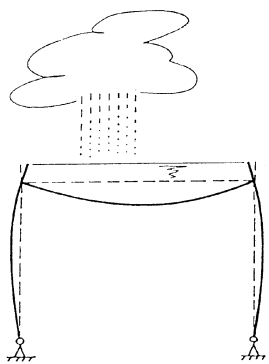

Figure 6.6 For flat roofs, one has to take the risk of water accumulation into account: the deflection of the roof due to the rain water allows the storage of an increasing amount of water. If the roof is not sufficiently rigid, it may eventually collapse.

is included in addition to the weight of people. Furthermore, the dynamic effects of walking, jumping, dancing, stamping, and so forth, are taken into account as well. The line loads and concentrated loads are introduced as they may occur during removals.

The live load has to be calculated separately for machines, archives, and so forth.

Since it is unlikely that in a building all the floors are maximally loaded at the same time, as in Figure 6.4, the live load can be reduced in certain cases.

For roofs where local *snow accumulation* is possible, the associated load concentration has to be taken into account. If the wind loading is predominant, the snow as well as people or tools on the roof can be ignored.

Wind loading is also a live load, but is generally defined separately. A distinction is made between:

- Wind pressure and wind suction (Figure 6.5a);
- Wind friction (Figure 6.5b);
- Overpressure and underpressure (Figure 6.5c).

For *rainwater*, the load of gutters and rainwater pipes filled with water as a result of blockages have to be taken into account.

For flat roofs, the possibility that the water cannot drain away has to be considered. This incurs the risk of *water accumulation*: the deflection of the roof due to the water allows for the storage of an increasing amount of water (see Figure 6.6). If the roof is not sufficiently rigid, this can result in its collapse at times of continuing rainfall.

6.2.3 Live loads (bridges)

Vertical live loads on bridges due to traffic are referred to as *moving loads*. In regulations, this load is a uniformly distributed surface load together with a limited number of concentrated loads (see Figure 6.7).

The uniformly distributed load is a representation of the actual load that can occur over large lengths. This load becomes more important for longer spans.

The system of point loads, with the underlying part of the uniformly distributed load, represents the load caused by a few very heavy trucks or locomotives. This load is important for bridge elements of limited length.

It may occur that certain structural elements are loaded more unfavourably if the load is omitted over a certain length. For this reason, the fact that the traffic load on bridges may be missing along that length has to be considered.

In the hinged beam in Figure 6.8, for example, the vertical support reaction at A as a compression force has its maximum when field AB is loaded and field BC is unloaded. In this case, the support reaction is qa. In the event of full loading, the support reaction in A is half the magnitude. The maximum tensile force that support A has to transfer is $\frac{1}{2}qa$, and occurs when only field BC is loaded.

For *railway bridges*, the train is always a continuous load, even though it can consist partially of empty carriages. A lower load is prescribed for the empty carriages.

The influence of impacts and vibrations are taken into account by multiplying the moving loads by an *impact factor* S ($S > 1$).

In longer *traffic bridges*, it is increasingly less likely that the maximum moving load occurs, unless there is a traffic jam. In that case, the impact factor will lead to a too heavy load, and a reduction is justified. This reduction is achieved by multiplying the traffic loading by a *load factor* B ($B < 1$).

For *railway bridges*, there is no load factor.

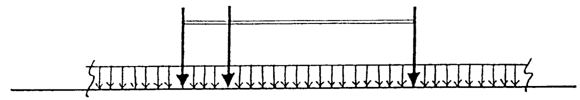

Figure 6.7 In regulations, the mobile load is seen as a uniformly distributed surface load together with a limited number of concentrated loads.

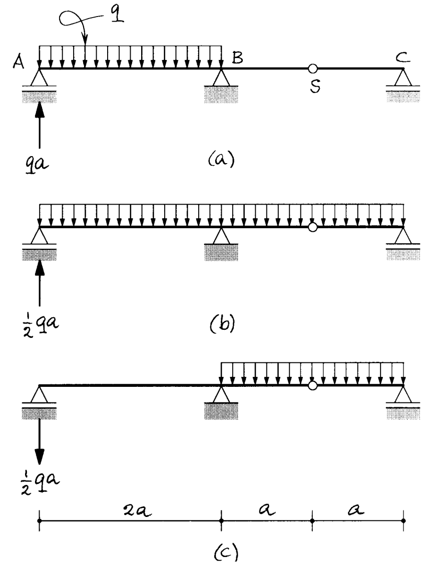

Figure 6.8 For bridges, one has to take into account that certain structural elements are loaded less favourably if the load is omitted over a certain length. (a) The vertical support reaction at A as a compressive force is a maximum when field AB is loaded and field BC is unloaded. (b) Due to a full load, the support reaction at A is half as large. (c) The largest tensile force that support A has to transfer occurs when only field BC is loaded.

In addition to the vertical traffic loads, horizontal loads such as brake forces and wind loads have to be taken into account.

6.2.4 Limit states

For each structure, it has to be shown that it is *reliable* (safe), and will not collapse prematurely, and that the structure meets the requirements related to *serviceability*.

In order to be able to check a structure on these various aspects, the concept of *limit state* was introduced. A limit state is a state in which the structure just meets (or just does not meet) certain demands regarding the structure. A distinction is made between two groups of limit states, which are directly related to the concepts reliability and serviceability:

- *ultimate limit states*
- *serviceability limit states*

Ultimate limit states
If a load is gradually increased, a moment arises at which the structure will collapse, for example because its strength limit is reached (exceeded), or because its equilibrium is no longer reliable (instability). Limit states used to test a structure for its reliability (or more generally speaking, structural safety) are referred to as *ultimate limit states*, or also as *failure states*.

Serviceability limit states
If a structure is insufficiently rigid, this can negatively influence its serviceability. Examples include doors that start to jam if the deformations become too large, and windows that may shatter. Another example is a floor that sags too much. This sort of floor elicits feelings of insecurity and is unusable, even if there is no risk of failure. Annoying cracking can also lead to a situation in which a structure is no longer serviceable (leakage through the cracks or corrosion of the reinforcement in reinforced concrete). Limit states used to check a structure for its serviceability are referred to as *serviceability limit states*.

When checking ultimate limit states, the structure is subjected to an *overload*. When checking *serviceability limit state*, the load at *serviceability level* is used.

The following section provides a brief summary of how, with design codes, the loads (and strength) that have to be used in the calculations are determined.

6.2.5 Characteristic values and design values

With loads and (material) strengths, it is not possible to indicate their precise values beforehand. In practice, they are subject to dispersion. One can only indicate with what *probability* certain values will occur. Loads and strengths are therefore *stochastic quantities*.[1]

Stochastic quantities can be defined by means of a *probability density function*, of which the *normal distribution* is the best-known. Most stochastic quantities that play a part in the assessment of the behaviour of a structure follow the normal distribution. Figure 6.9 shows the curve for the normal distribution of a quantity x. This curve, also known as the *Gauss curve*,[2] is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

The normal distribution is characterised by two parameters: the *mean value* μ and the *standard deviation* σ. The curve is in the shape of a bell, with a vertical symmetry axis and two points of inflection, and approaches zero for $x \to -\infty$ and $x \to +\infty$. The mean value μ coincides with the symmetry

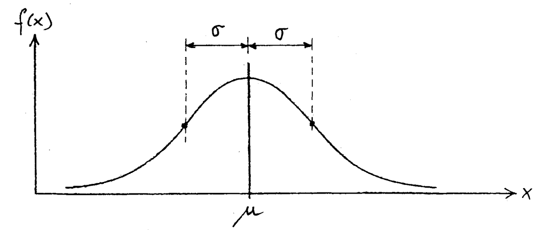

Figure 6.9 The normal distribution or Gauss curve is characterised by the mean value μ and standard deviation σ.

[1] From the Greek $\sigma\tau o\chi\alpha\xi o\mu\alpha\iota$ (to guess, to suspect).

[2] Carl Friedrich Gauss (1777–1855), German mathematician and astronomer.

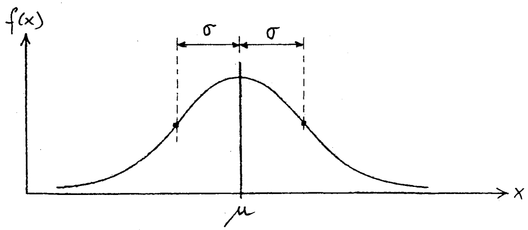

Figure 6.9 The normal distribution or Gauss curve is characterised by the mean value μ and standard deviation σ.

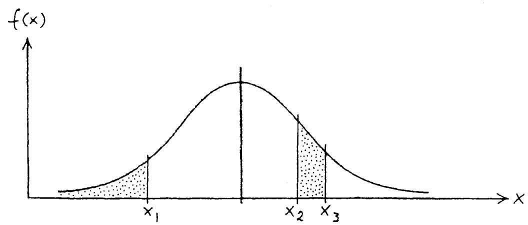

Figure 6.10 The probability that the value of x is smaller than x_1 is equal to the area under the curve for $x < x_1$; the probability of a value of x between x_2 and x_3 is equal to the area under the curve between x_2 and x_3.

axis. The standard deviation σ is the distance from the symmetry axis to the points of inflection.

The probability $P(x < x_1)$ that the value of x is smaller than x_1 is equal to the surface area under the Gauss curve for $x < x_1$ (see Figure 6.10):

$$P(x < x_1) = \int_{-\infty}^{x_1} f(x)\,\mathrm{d}x.$$

There are tables available for this integral.

The probability of a value of x between x_2 and x_3 is equal to the area under the Gauss curve between x_2 and x_3.

The total area under the curve is equal to 1: there is a probability of 100% that the value of x lies between $-\infty$ and $+\infty$.

The probability $P(x < x_1)$ can be shown in various ways. The area under the curve gives a value (smaller or equal to 1), such as

$$P(x < x_1) = 0.0025 = 2.5 \times 10^{-3}.$$

This value can also be written as a ratio:

$$P(x < x_1) = 1 : 400.$$

The probability is also often shown as the percentage of the total area under the probability density curve (which is equal to 1). In this example, the probability is

$$0.0025 \times 100\% = 0.25\%.$$

The ultimate limit state is a check for strength. This means that, on the one hand, the strength R of the structure has to be determined, and that on the other, we have to determine the load S. In the building regulations, the procedure used is based on the so-called *characteristic values* for strength and load.

The *characteristic strength* R_k is defined as the strength that is *exceeded* with a probability of 95%; in other words, the strength is therefore less than the characteristic strength in 5% of all cases (see Figure 6.11a).

The *characteristic load* S_k is defined as the load that with a probability of 95% is *not exceeded* throughout the lifetime of the structure; only 5% of all occurring loads are larger than the characteristic load (see Figure 6.11b).

The symbols R for *strength* and S for *load* are used internationally.[1] They have a broad meaning. *Strength R* (generally) relates to the largest forces and stresses that can be transferred by a structure, such as the *admissible tensile force* in a tie-rod, or the *compressive strength* of the material. *Load S* (generally) relates to the force or stress exerted on the structure (or part of a structure), or in other words, the *acting tensile force* in the tie-rod or the *acting compressive stress* in the material.

The strength R must not be smaller than the load S. With respect to the characteristic values, this means:

$$R_k \geq S_k.$$

In this situation, however, the probability of failure is considered too great. In order to reduce this probability, the calculation is not carried out with the characteristic values, but with a lesser strength R_d and a larger load S_d, known as the design values.[2]

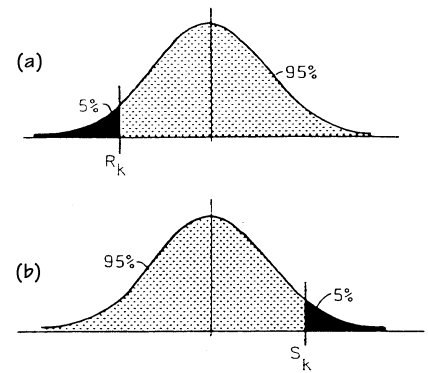

Figure 6.11 (a) The characteristic strength R_k is the strength that is *exceeded* with a 95% probability; in only 5% of all cases, the strength is therefore less than the characteristic strength. (b) The characteristic load S_k is the load that is *not exceeded* throughout the lifecycle of the structure with 95% probability; only 5% of all occurring loads are larger than the characteristic load.

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[1] From French: *R* of *Résistance* (resistance, stamina) and *S* from *Sollicitation* (load).

[2] The index <u>*d*</u> is derived from *design*.

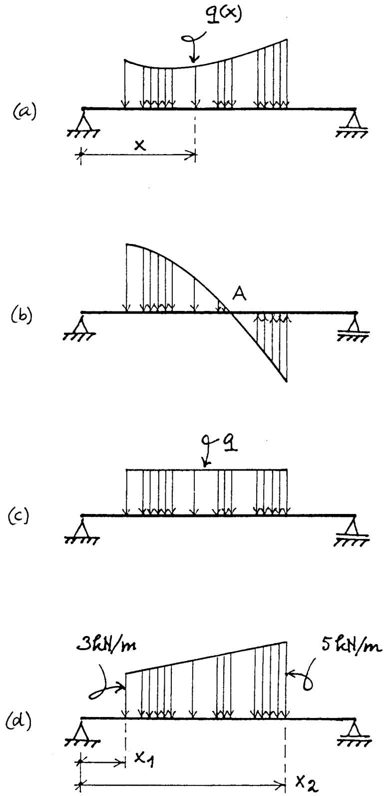


Figure 6.12 Distributed loads normal to the member axis: (a) a distributed load $q(x)$ as a function of x; (b) a distributed load that changes direction at A; (c) a uniformly distributed load; (d) a linearly distributed load.

The *design value for strength* is derived from the characteristic strength by dividing it by a *material factor* γ_R:

$$R_d = \frac{R_k}{\gamma_R}.$$

The material factor accounts for insecurities in construction. As such, steel has a lower material factor than, for example, cast in situ concrete.

The *design value of the load* is derived from the characteristic load by multiplying it by a *load factor* γ_S:

$$S_d = \gamma_S S_k.$$

Amongst other things, the magnitude of the load factor depends on the type of load (dead or live, and whether its effect is favourable or unfavourable), the safety class (office building or shed), and the limit state in question. To check an *ultimate limit state*, the structure is subjected to an overload, and the design value of the load is larger than the characteristic value. To check a *serviceability limit state*, the *load at serviceability level* is used: in this case, the design value of the load is equal to the characteristic value.

Structures are considered sufficiently strong if the design value of the strength is not smaller than the design value of the load:

$$R_d \geq S_d$$

or

$$\frac{R_k}{\gamma_R} \geq \gamma_S S_k.$$

Each limit state has its own load factor. For information concerning load factors and material factors, please refer to the regulations, building codes and relevant books.

In this book, all the examples use only design values.

6.3 Working with distributed loads

When working with a distributed load, it can sometimes be useful to replace it (temporarily) by its resultant. This section addresses the calculation of the resultant of a distributed load. Most attention is devoted to line loads on a member.

6.3.1 Resultant of a line load on a member

Figure 6.12a provides a schematic representation of a *line load* q on (a part of) a member. The direction of the distributed load is shown by means of arrows. The load in Figure 6.12a is acting normal to the member axis and is a function of x. Other examples of distributed loads acting normal to the member axis are shown in Figures 6.12b to 6.12d.

In the special case that the distributed load is constant, we refer to a *uniformly distributed load* (see Figure 6.12c).

The distributed load in Figure 6.12d is known as a *linearly distributed load*; it varies linearly from $q(x_1) = 3$ kN/m to $q(x_2) = 5$ kN/m.

A distributed load can also act in the direction of the member axis. Figure 6.13, for example, shows the uniformly distributed load q on a column as a result of its dead weight.

A distributed load q, acting at an angle to a member, can be resolved into directions parallel to and normal to the member axis (see Figure 6.14). In the xz coordinate system shown, $q_x = q \cos\alpha$ and $q_z = q \sin\alpha$ are called the components of q.

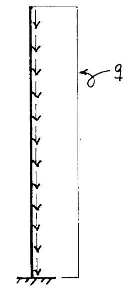


Figure 6.13 A uniformly distributed load in the direction of the member axis.

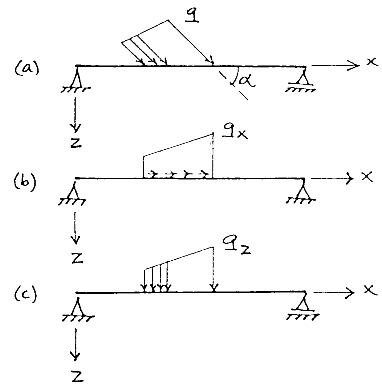


Figure 6.14 (a) A distributed load, at an angle to a member, can be resolved into components (b) parallel, and (c) normal to the member axis.

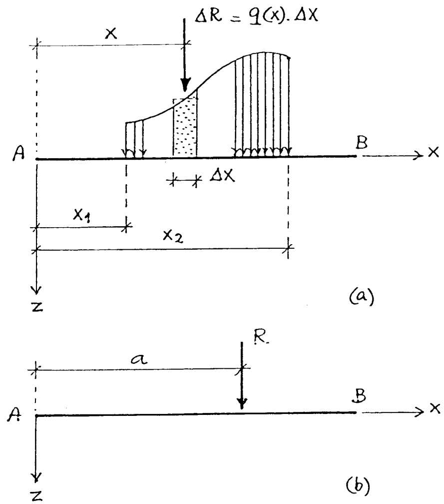


Figure 6.15 (a) An arbitrarily distributed load $q(x)$ normal to the member axis. A small force $\Delta R = q(x)\Delta x$ acts on a small element at length Δx. The magnitude of the resultant R of the distributed load is equal to the sum of all parallel forces ΔR. (b) The magnitude of the resultant R is equal to the area of the load diagram; its line of action passes through the centroid of the load diagram.

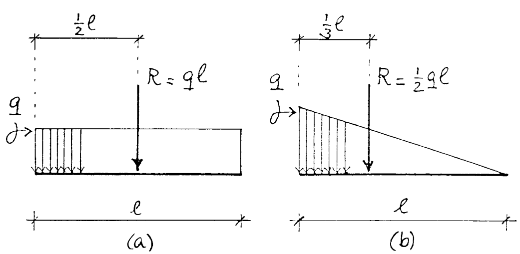


Figure 6.16 Magnitude and line of action of resultant R with (a) a rectangular, and (b) a triangular load diagram.

Note: the distributed load has the dimension of force per length. So far, the length was always measured *along the member axis*. With inclined members, the distributed load is also sometimes expressed per length projected on the (horizontal) ground surface, for example in the case of a snow load. See Example 3 in this section.

When considering the equilibrium of a system of forces on a structure, considered as a rigid body, we can replace the system of forces by its resultant.

In Figure 6.15, an arbitrarily distributed force $q(x)$ is acting normal to the axis of member AB between $x = x_1$ and $x = x_2$. A small force ΔR is acting on a small element at length Δx:

$$\Delta R = q(x)\Delta x.$$

The magnitude of the resultant R of the distributed force is equal to the sum of all small parallel forces ΔR:

$$R = \sum \Delta R = \sum q(x)\Delta x = \int_{x_1}^{x_2} q(x)\,\mathrm{d}x.$$

Conclusion: *The magnitude of R is equal to the area enclosed by the load diagram.*

The line of action of R is found using Varignon's Moment Theorem: the resultant R and the distributed load $q(x)$ have to produce the same moment about an arbitrary point. The moment about point A, for example, gives

$$aR = \sum (x\Delta R) = \int_{x_1}^{x_2} xq(x)\,\mathrm{d}x$$

so that

$$a = \frac{\int_{x_1}^{x_2} x q(x)\, \mathrm{d}x}{R} = \frac{\int_{x_1}^{x_2} x q(x)\, \mathrm{d}x}{\int_{x_1}^{x_2} q(x)\, \mathrm{d}x}.$$

By definition, a is the x coordinate of the *centroid* of the load diagram.[1]

Conclusion: *The line of action of R passes through the centroid of the load diagram.*

Figures 6.16a and 6.16b give the magnitude and location of the resultant for a rectangular and triangular load diagram respectively.

Example 1

Determine the vertical support reactions at A and B of the simply supported beam AB in Figure 6.17a, with a distributed load that increases linearly from 4 kN/m at A to 12 kN/m at B.

Solution (units kN and m):
For the distributed load, with $\ell = 6$ m, applies

$$q(x) = 4 + 8\frac{x}{\ell} \text{ kN/m}.$$

The resultant R of the distributed load is

$$R = \int_0^{\ell} q(x)\, \mathrm{d}x = \int_0^6 \left(4 + 8\frac{x}{6}\right) \mathrm{d}x = \left(4x + 4\frac{x^2}{6}\right)\Bigg|_{x=0}^{x=6} = 48 \text{ kN}.$$

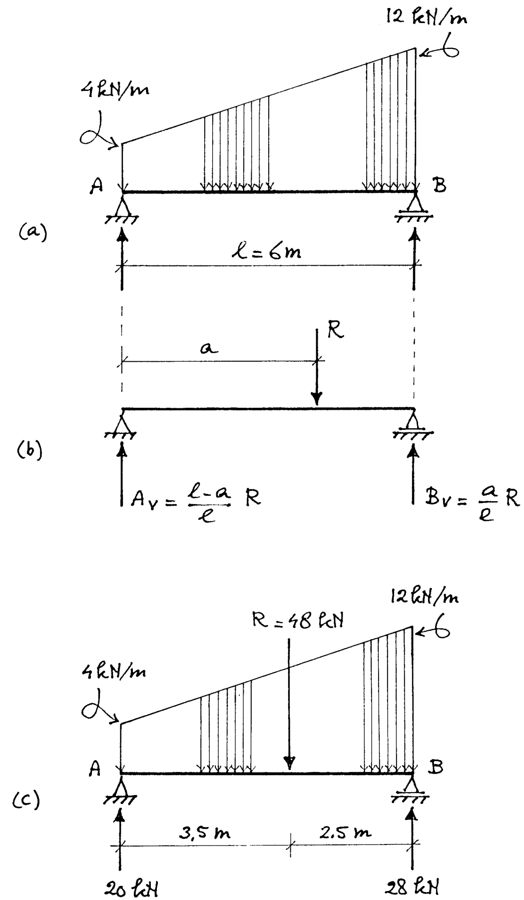


Figure 6.17 (a) A simply supported beam with a linearly distributed load. (b) The support reactions due to resultant R. (c) The magnitude and line of action of resultant R of the distributed load and the associated support reactions.

[1] Volume 2, *Stresses, Deformations, Displacements*, addresses the calculation of centroids. Here it is assumed that the reader is aware of the location of centroids in simple figures.

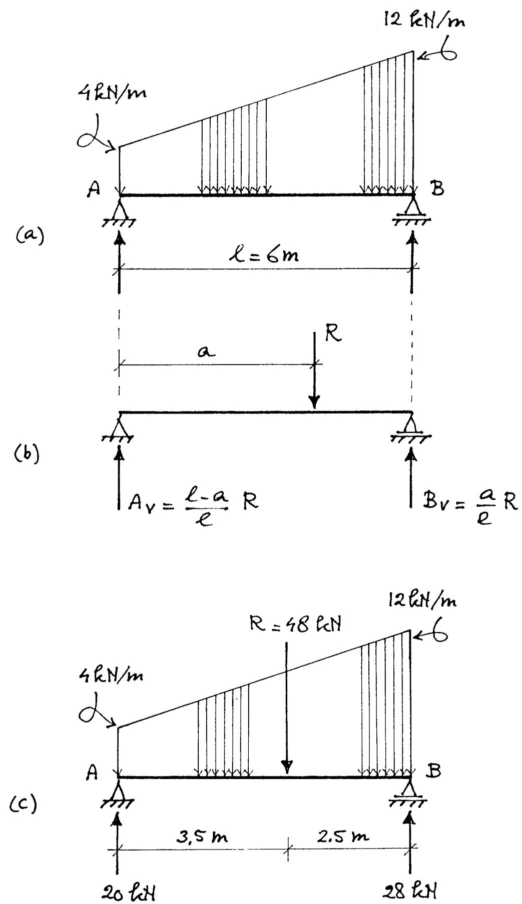


Figure 6.17 (a) A simply supported beam with a linearly distributed load. (b) The support reactions due to resultant R. (c) The magnitude and line of action of resultant R of the distributed load and the associated support reactions.

One can also determine R directly from the area of the trapezoidal load diagram:

$$R = \tfrac{1}{2} \times 6 \times (4 + 12) = 48 \text{ kN}.$$

In Figure 6.17b, the distributed load has been replaced by the resultant R. The line of action of R is determined by:

$$aR = \int_{x_1}^{x_2} xq(x)\,\mathrm{d}x = \int_0^6 \left(4x + 8\frac{x^2}{6}\right) \mathrm{d}x$$

$$= \left(2x^2 + 8\frac{x^3}{18}\right)\Bigg|_{x=0}^{x=6} = 168 \text{ kNm}$$

so that

$$a = \frac{168}{R} = \tfrac{168}{48} = 3.5 \text{ m}.$$

The magnitude and location of the resultant R of the distributed load are shown in Figure 6.17c.

The vertical support reactions at A and B are now found from the moment equilibrium about B and A respectively:

$$\sum T|\text{B} = 0 \Rightarrow A_\text{v} = \tfrac{2.5}{6} \times 48 = 20 \text{ kN},$$

$$\sum T|\text{A} = 0 \Rightarrow B_\text{v} = \tfrac{3.5}{6} \times 48 = 28 \text{ kN}.$$

A distributed load q may also be split up into loads q_1 and q_2, as in Figure 6.18, where the individual influences may be added. It always holds that

$$R = \int_{x_1}^{x_2} q(x)\,\mathrm{d}x = \int_{x_1}^{x_2} q_1(x)\,\mathrm{d}x + \int_{x_1}^{x_2} q_2(x)\,\mathrm{d}x = R_1 + R_2$$

and

$$aR = \int_{x_1}^{x_2} xq(x)\,\mathrm{d}x = \int_{x_1}^{x_2} xq_1(x)\,\mathrm{d}x + \int_{x_1}^{x_2} xq_2(x)\,\mathrm{d}x = a_1R_1 + a_2R_2.$$

The fact that the influences of the individual loads can be added is referred to as the *principle of superposition*. This principle is based on the fact that the relationships between the various quantities are linear.

If a load diagram can be split into a number of simpler diagrams, such as a number of rectangles and triangles, the abovementioned approach often leads to a result more quickly. This is illustrated in the next example.

Example 2
For the simply supported beam in Figure 6.19, the trapezoidal load diagram has been split into triangles and/or rectangles in four different ways.

Question:
Show that the same support reactions at A and B are found in all four cases.

Solution (units in kN and m):
In Figure 6.19a, the trapezoidal load has been split up into two triangular loads. The determination of the support reactions is shown below:

$$R_1 = \tfrac{1}{2} \times 6 \times 4 = 12\ \mathrm{kN},$$

$$R_2 = \tfrac{1}{2} \times 6 \times 12 = 36\ \mathrm{kN},$$

$$\sum T|\mathrm{B} = 0 \Rightarrow A_\mathrm{v} = \tfrac{4}{6}R_1 + \tfrac{2}{6}R_2 = 8 + 12 = 20\ \mathrm{kN},$$

$$\sum T|\mathrm{A} = 0 \Rightarrow B_\mathrm{v} = \tfrac{2}{6}R_1 + \tfrac{4}{6}R_2 = 4 + 24 = 28\ \mathrm{kN}.$$

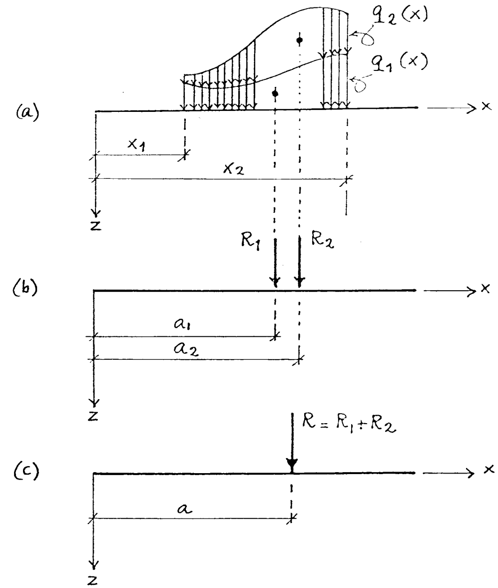


Figure 6.18 Principle of superposition: One can (a) split a distributed load q into loads q_1 and q_2, with (b) resultants R_1 and R_2, and (c) add their individual influences.

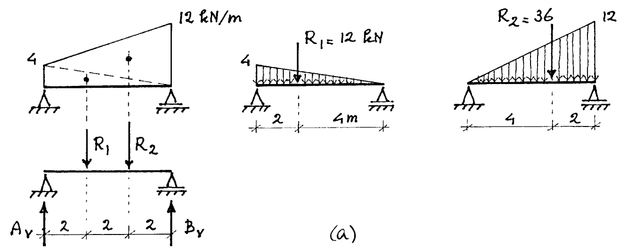


Figure 6.19 A simply supported beam with the trapezoidal load diagram split up into two triangles.

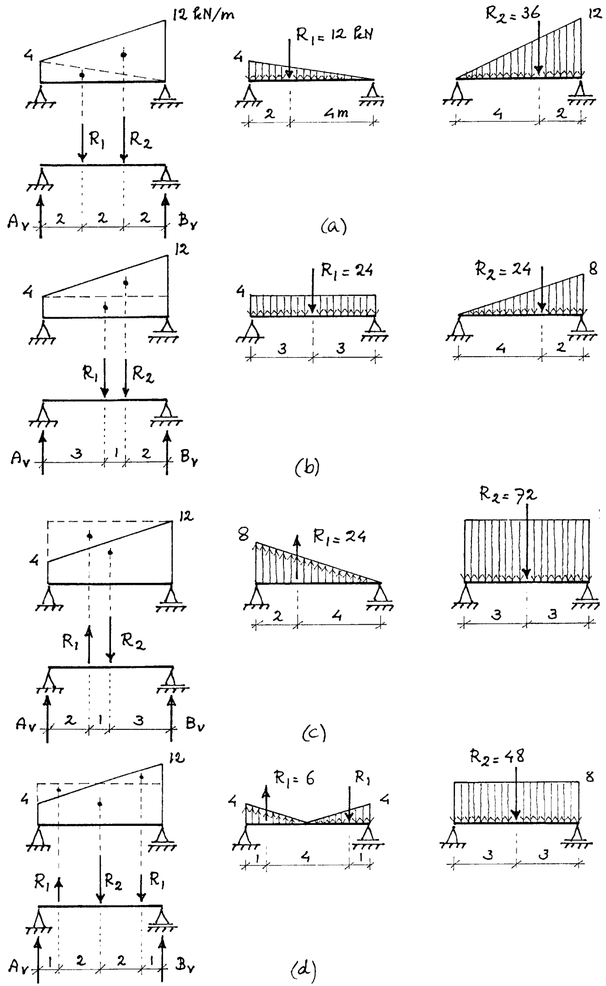


Figure 6.19 A simply supported beam with the trapezoidal load split up in four different ways into triangles and/or rectangles to determine the support reactions.

In Figure 6.19b, the trapezoidal load is divided into a rectangular and a triangular load diagram:

$$R_1 = 6 \times 4 = 24 \text{ kN},$$

$$R_2 = \tfrac{1}{2} \times 6 \times 8 = 24 \text{ kN},$$

$$\sum T|\mathrm{B} = 0 \Rightarrow A_\mathrm{v} = \tfrac{3}{6}R_1 + \tfrac{2}{6}R_2 = 12 + 8 = 20 \text{ kN},$$

$$\sum T|\mathrm{A} = 0 \Rightarrow B_\mathrm{v} = \tfrac{3}{6}R_1 + \tfrac{4}{6}R_2 = 12 + 16 = 28 \text{ kN}.$$

The trapezoidal load can be split into a uniformly distributed load and a triangular load in many other ways, as for example in Figure 6.19c:

$$R_1 = \tfrac{1}{2} \times 6 \times 8 = 24 \text{ kN},$$

$$R_2 = 6 \times 12 = 72 \text{ kN},$$

$$\sum T|\mathrm{B} = 0 \Rightarrow A_\mathrm{v} = -\tfrac{4}{6}R_1 + \tfrac{3}{6}R_2 = -16 + 36 = 20 \text{ kN},$$

$$\sum T|\mathrm{A} = 0 \Rightarrow B_\mathrm{v} = -\tfrac{2}{6}R_1 + \tfrac{3}{6}R_2 = -8 + 36 = 28 \text{ kN}.$$

If the trapezoidal load is split as shown in Figure 6.19d, it follows that

$$R_1 = \tfrac{1}{3} \times 3 \times 4 = 6 \text{ kN},$$

$$R_2 = 6 \times 8 = 48 \text{ kN},$$

$$\sum T|\mathrm{B} = 0 \Rightarrow A_\mathrm{v} = -\tfrac{5}{6}R_1 + \tfrac{3}{6}R_2 + \tfrac{1}{6}R_1 = -5 + 24 + 1 = 20 \text{ kN},$$

$$\sum T|\mathrm{A} = 0 \Rightarrow B_\mathrm{v} = -\tfrac{1}{6}R_1 + \tfrac{3}{6}R_2 + \tfrac{5}{6}R_1 = -1 + 24 + 5 = 28 \text{ kN}.$$

Irrespective of how the load diagram is split, the support reactions are always the same.

Example 3

Part of a roof modelled as the line element in Figure 6.20a is supported by a hinge at A, and a roller with vertical roller track at B. The member is loaded by three uniformly distributed (line) loads, of which the load diagrams are shown in Figures 6.20b to 6.20d:

- the *dead weight* $q_{\rm dw}$ (vertical force per length measured along the member axis);
- a *snow load* $q_{\rm sn}$ (vertical force per horizontally measured length);
- a *wind load* $q_{\rm w}$ normal to the member axis (force per length measured along the member axis).

Unlike dead weight and wind load, the snow load is given as a load per length projected on the horizontal ground plane. The load diagram for snow in Figure 6.20c is drawn differently therefore.

Question:
Determine the support reactions at A and B for all three loads.

Solution:
When calculating the support reactions, we can replace the distributed loads by their resultants. The dead weight and the wind load act over a length of $15a$, while the snow load acts over a length of $12a$, so that

$$R_{\rm dw} = 15aq_{\rm dw},$$

$$R_{\rm sn} = 12aq_{\rm sn},$$

$$R_{\rm w} = 15aq_{\rm w}.$$

The resultants and their lines of action are shown in Figures 6.20e to 6.20g. The same figures also show the associated support reactions.

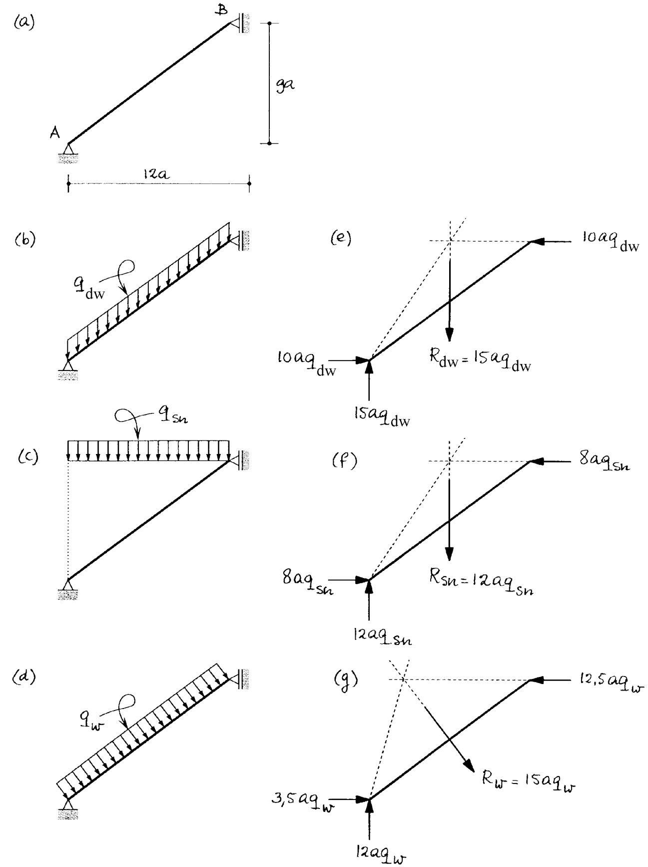


Figure 6.20 (a) Part of a roof, modelled as a line element, is loaded by (b) its dead weight, (c) snow, and (d) wind. The resultant and support reactions due to (e) the dead weight, (f) snow load, and (g) wind load.

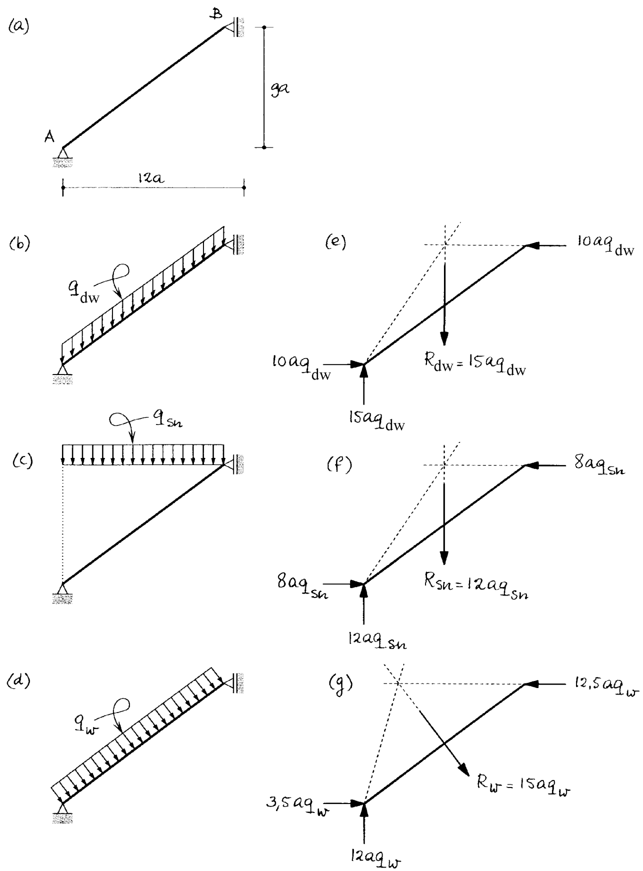


Figure 6.20 (a) Part of a roof, modelled as a line element, is loaded by (b) its dead weight, (c) snow, and (d) wind. The resultant and support reactions due to (e) the dead weight, (f) snow load, and (g) wind load.

Table 6.1

	R	$A_v(\uparrow)$	$A_h(\rightarrow)$	$B_h(\leftarrow)$
dead weight	$R_{dw} = 15aq_{dw}$	$15aq_{dw}$	$10aq_{dw}$	$10aq_{dw}$
snow	$R_{sn} = 12aq_{sn}$	$12aq_{sn}$	$8aq_{sn}$	$8aq_{sn}$
wind	$R_w = 15aq_w$	$12aq_w$	$3.5aq_w$	$12.5aq_w$

All the values are shown Table 6.1.

The reader is asked to verify the correctness of the support reactions.

Example 4

The hinged beam in Figure 6.21a carries a uniformly distributed load of 6 kN/m over part CD.

Question:
Determine the support reactions.

Solution:
In an equilibrium system, a distributed load may be replaced by its resultant. Therefore, when looking at the equilibrium of the hinged beam as a whole, we can use the resultant of the entire distributed load (see Figure 6.21b). For the directions assumed for the support reactions, this gives

$$\sum F_x^{(\text{ASD})} = -A_h = 0 \Rightarrow A_h = 0,$$

$$\sum F_z^{(\text{ASD})} = -A_v - B_v - C_v + (36 \text{ kN}) = 0, \qquad \text{(a)}$$

$$\sum T_y^{((\text{ASD})}|\text{A} = +B_v \times (4 \text{ m}) + C_v \times (8 \text{ m}) - (36 \text{ kN})(5 \text{ m}) = 0. \qquad \text{(b)}$$

The two equations (a) and (b) are not sufficient to determine all vertical support reactions. The additional equation required is found from the moment equilibrium of parts SA or SD about hinge S. In this case it is not possible to work with the resultant in Figure 6.21b; this resultant has to be replaced by the resultants of the distributed loads on the individual parts (see Figure 6.21c).

With equations (a) and (b), an efficient way of obtaining results is to consider the moment equilibrium of SD:

$$\sum T_y^{(\mathrm{SD})}|\mathrm{S} = +C_\mathrm{v} \times (2\ \mathrm{m}) - (12\ \mathrm{kN})(1\ \mathrm{m}) = 0. \tag{c}$$

From (c) we find

$$C_\mathrm{v} = +6\ \mathrm{kN}$$

which then gives the following from (b) and (a)

$$B_\mathrm{v} = +33\ \mathrm{kN},$$

$$A_\mathrm{v} = -3\ \mathrm{kN}.$$

Figure 6.21d shows the support reactions in the directions in which they are really acting. Only the direction of the vertical support reaction at A was assumed falsely.

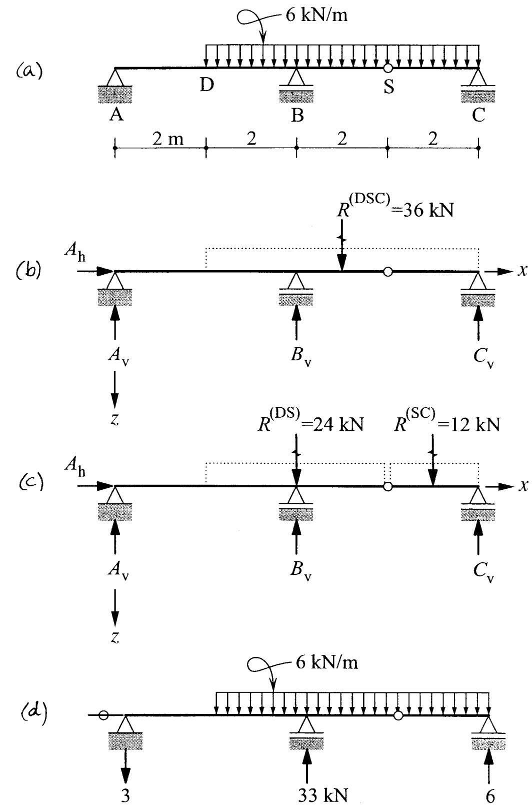


Figure 6.21 (a) Hinged beam with uniformly distributed load. (b) For the equilibrium of the structure as a whole, the total distributed load can be replaced by its resultant. (c) For the equilibrium of the separate parts, each part has its own resultant, and one may no longer use the resultant of the total distributed load. (d) Support reactions.

6.3.2 Resultant of a surface load on a plate

In Figure 6.22a, a plate in the xy plane is loaded normal to its plane by an arbitrarily distributed load $p(x, y)$.

The resultant of the distributed load on a small area ΔA is a small force ΔR:

$$\Delta R = p(x, y)\Delta A.$$

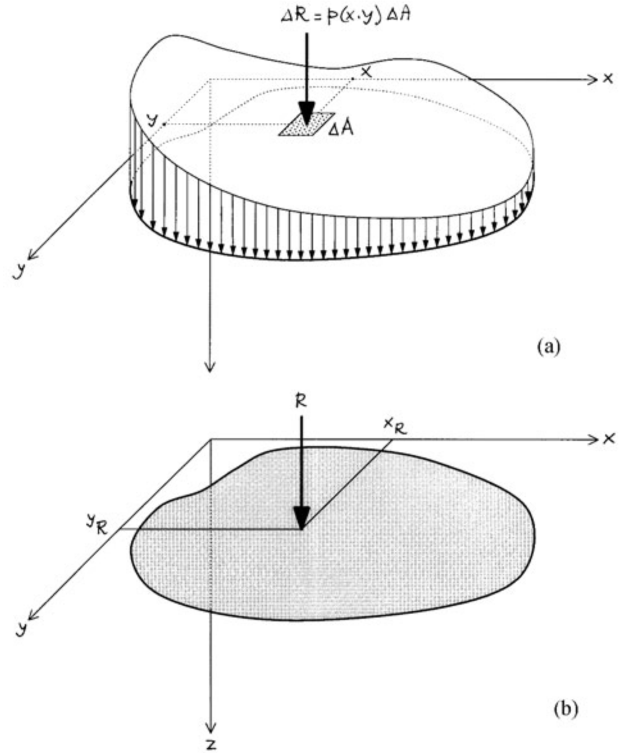


Figure 6.22 (a) A plate in the xy plane is loaded normal to its plane by an arbitrarily distributed load $p(x, y)$. A small force $\Delta R = p(x, y)\Delta A$ is acting on a small area ΔA. (b) The resultant R of the distributed load and the location (x_R, y_R) where the line of action of R intersects the xy plane. The magnitude of R is equal to the volume of the load diagram. The line of action of R passes through the centroid of the load diagram.

The magnitude of the resultant R of the distributed load is equal to the sum of all parallel forces ΔR:

$$R = \sum \Delta R = \sum p(x, y)\Delta A = \int_A p(x, y)\,\mathrm{d}A.$$

Conclusion: *The magnitude of R is equal to the volume of the load diagram.*

The location (x_R, y_R) of the line of action of R is found using Varignon's theorem[1] (see Figure 6.22b):

$$\sum T_y = -x_R R = -\sum (x\Delta R) = -\int_A xp(x, y)\,\mathrm{d}A,$$

$$\sum T_x = +y_R R = +\sum (y\Delta R) = +\int_A yp(x, y)\,\mathrm{d}A$$

so that

$$x_R = \frac{\int_A xp(x, y)\,\mathrm{d}A}{R} = \frac{\int_A xp(x, y)\,\mathrm{d}A}{\int_A p(x, y)\,\mathrm{d}A},$$

$$y_R = \frac{\int_A yp(x, y)\,\mathrm{d}A}{R} = \frac{\int_A yp(x, y)\,\mathrm{d}A}{\int_A p(x, y)\,\mathrm{d}A}.$$

By definition, x_R and y_R are the x and y coordinates of the centroid of the load diagram.

[1] See also Examples 1 and 2 in Section 3.3.4.

Conclusion: *The line of action of R passes through the centroid of the load diagram.*

Example

In Figure 6.23a, a rectangular plate in the xy plane, with an area $A = ab$, is loaded by a distributed load normal to its plane:

$$p(x, y) = \hat{p}\frac{x(b-y)}{ab}.$$

Question:

Determine the magnitude of the resultant R and the coordinates $(x_R; y_R)$ where the line of action of R intersects the xy plane.

Solution:

The magnitude of the resultant R is equal to the volume of the load diagram:

$$\begin{aligned} R &= \int_A p(x, y)\,\mathrm{d}A = \int_0^a \int_0^b \hat{p}\frac{x(b-y)}{ab}\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{\hat{p}}{ab}\int_0^a x\,\mathrm{d}x \int_0^b (b-y)\,\mathrm{d}y \\ &= \frac{\hat{p}}{ab}\frac{x^2}{2}\bigg|_0^a \left(by - \frac{y^2}{2}\right)\bigg|_0^b \\ &= \frac{\hat{p}ab}{4}. \end{aligned}$$

The line of action of R passes through the centroid of the load diagram (see Figure 6.23b). For the coordinates (x_R, y_R) of the centroid, the formulas derived earlier can be used.

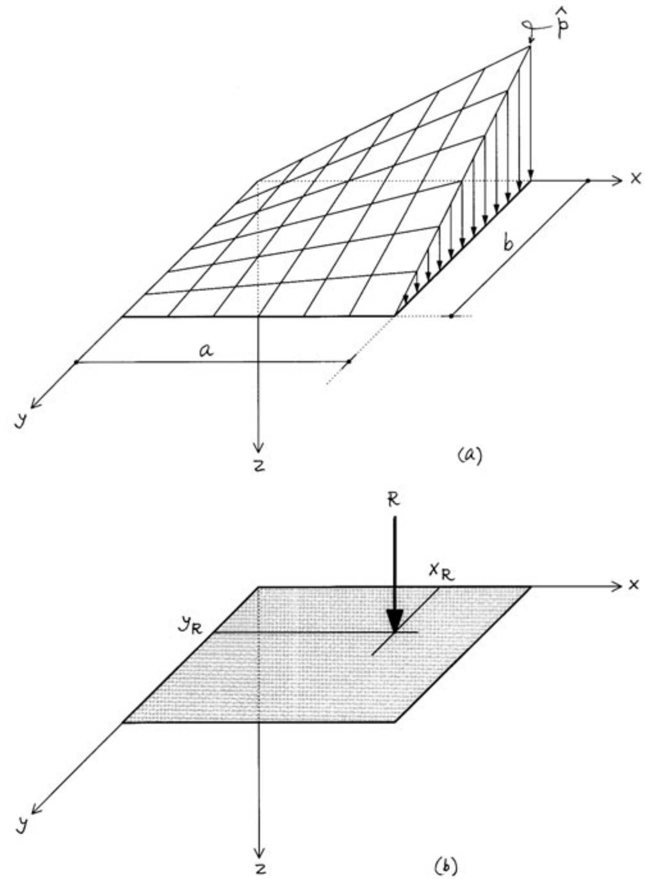


Figure 6.23 (a) A rectangular plate in the xy plane is loaded normal to its plane by a distributed load. (b) The resultant R of the distributed load and the location (x_R, y_R) where the line of action of R intersects the xy plane.

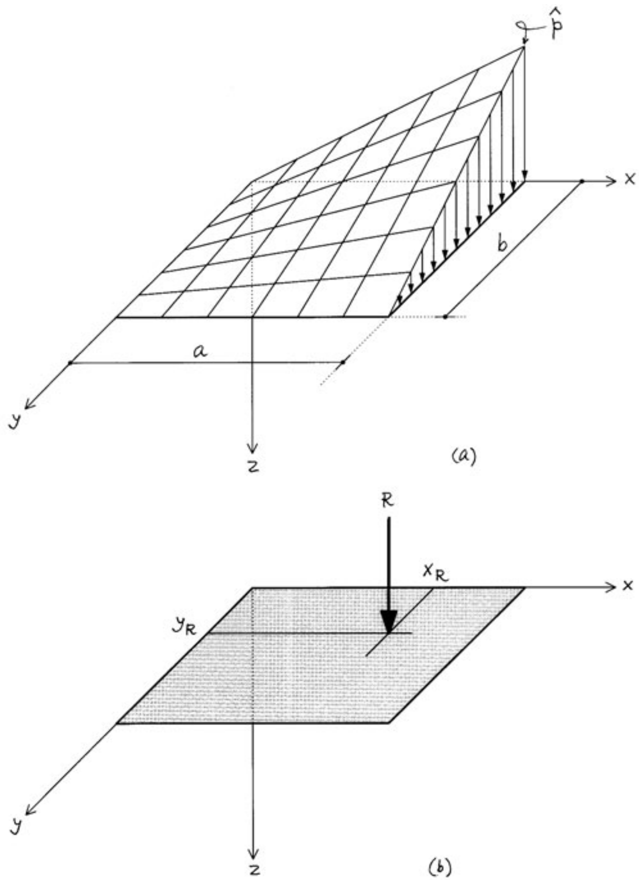


Figure 6.23 (a) A rectangular plate in the xy plane is loaded normal to its plane by a distributed load. (b) The resultant R of the distributed load and the location (x_R, y_R) where the line of action of R intersects the xy plane.

It is also possible to start at once with Varignon's theorem:

$$\sum T_y = -x_R R = -\int_A x p(x, y)\,\mathrm{d}A,$$

$$\sum T_x = +y_R R = +\int_A y p(x, y)\,\mathrm{d}A.$$

This gives

$$x_R = \frac{\displaystyle\int_A x p(x, y)\,\mathrm{d}A}{R} = \frac{\dfrac{\hat{p}}{ab}\displaystyle\int_0^a x^2\,\mathrm{d}x \int_0^b (b - y)\,\mathrm{d}y}{\dfrac{\hat{p}ab}{4}}$$

$$= \frac{\dfrac{\hat{p}}{ab} \times \dfrac{a^3}{3} \times \dfrac{b^2}{2}}{\dfrac{\hat{p}ab}{4}} = \tfrac{2}{3}a,$$

$$y_R = \frac{\displaystyle\int_A y p(x, y)\,\mathrm{d}A}{R} = \frac{\dfrac{\hat{p}}{ab}\displaystyle\int_0^a x\,\mathrm{d}x \int_0^b (by - y^2)\,\mathrm{d}y}{\dfrac{\hat{p}ab}{4}}$$

$$= \frac{\dfrac{\hat{p}}{ab} \times \dfrac{a^2}{2} \times \dfrac{b^3}{6}}{\dfrac{\hat{p}ab}{4}} = \tfrac{1}{3}b.$$

6.4 Modelling load flow

How the load is taken into account depends greatly on the way in which a structure or structural element is modelled. For example, the dead weight of a member modelled as a line element is not considered as volume load, but rather as a line load. In the same way, the dead weight of a plate (surface element) will be taken into account as a surface load.

This will be demonstrated in an example using the simple concrete building in Figure 6.24a. For this example we will investigate how the vertical loads on the building are transferred to the foundation.

The building consists of two frames that, 4 metres apart, carry the roof slabs. Figure 6.24b shows one of the frames. Each frame consists of an 8-metre beam that is simply supported at both ends by a column. The 5-metre columns are rigidly joined to a square footing, located at a certain depth below ground level. The dead weight of the roof slabs is 2 kN/m^2. The weight of the waterproof roof covering and insulation is set at 0.3 kN/m^2. In addition, a live load of 0.5 kN/m^2 is taken into account. The total load on the roof slabs is therefore

$$p = (2\ \text{kN/m}^2) + 0.3\ \text{kN/m}^2) + (0.5\ \text{kN/m}^2) = 2.8\ \text{kN/m}^2.$$

Figure 6.25a shows a load of 2.8 kN acting on a square metre. If one takes an arbitrary strip of the roof of 1-metre width, the total load on the strip would be

$$(4\ \text{m})(1\ \text{m})(2.8\ \text{kN/m}^2) = 11.2\ \text{kN}.$$

Each beam carries half of this, or in other words, 5.6 kN over a 1-metre length, (see Figure 6.25b). Over the full length, the beam is therefore loaded by a uniformly distributed load of 5.6 kN/m. We also have to include the dead weight of the beam. If we assume a dead weight of 6 kN/m, the total

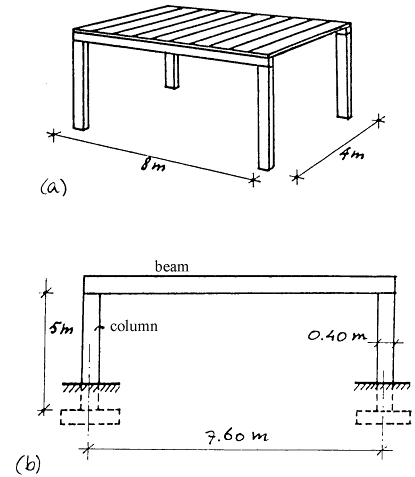


Figure 6.24 (a) A simple concrete building consisting of two frames, covered by roof slabs. (b) Each frame consists of an 8-metre beam that at each end is simply supported on a column. The 5-metre columns are rigidly joined to a square footing, which is located at a certain depth below ground level.

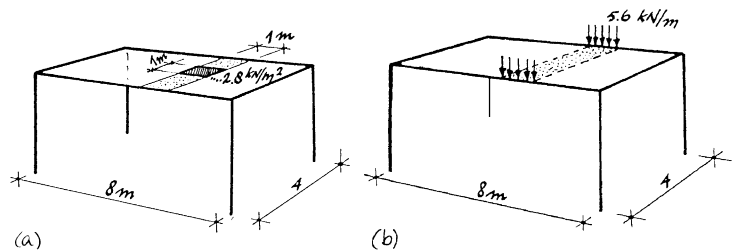


Figure 6.25 (a) The total roof load is 2.8 kN/m^2; (b) this generates a uniformly distributed load on the beam equal to 5.6 kN/m.

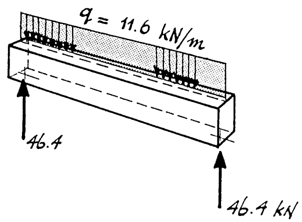


Figure 6.26 All the forces acting on the isolated beam. The distributed load is composed of the roof load and a dead weight of 6 kN/m. The support reactions of 46.4 kN have to be provided by the columns.

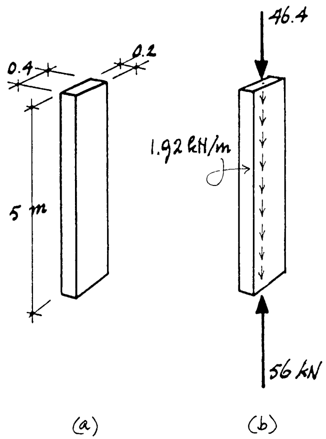


Figure 6.27 (a) The column dimensions and (b) all the forces acting on the isolated column. The dead weight of the column is a uniformly distributed load parallel to the column axis.

load on the beam is (see Figure 6.26)

$$q = (5.6\ \text{kN/m}) + (6\ \text{kN/m}) = 11.6\ \text{kN/m}.$$

The beam is simply supported. The support reactions, which have to be provided by the columns, amount to

$$\tfrac{1}{2} \times (11.6\ \text{kN/m})(8\ \text{m}) = 46.4\ \text{kN}.$$

Equal and opposite forces are acting on the columns.

Figure 6.27a shows the cross-sectional dimensions of the columns. With mass density $\rho = 2400$ kg/m, the specific weight of concrete is

$$\rho g = (2400\ \text{kg/m}^3) \times (10\ \text{N/kg}) = 24000\ \text{N/m}^3) = 24\ \text{kN/m}^3.$$

For the cross-sectional dimensions of the column in Figure 6.27a, the dead weight per length is

$$(0.2\ \text{m})(0.4\ \text{m})(24\ \text{kN/m}^3) = 1.92\ \text{kN/m}.$$

This is a uniformly distributed load acting in the direction of the column axis (see Figure 6.27b).

The total dead weight of the column is

$$(5\ \text{m})(1.92\ \text{kN/m}) = 9.6\ \text{kN}.$$

At its base, the column has to be kept in equilibrium by a force of

$$(46.4\ \text{kN}) + (5\ \text{m})(1.92\ \text{kN/m}) = 56\ \text{kN}.$$

An equally large, opposite force is acting on the footing of the column. If the footing is square, and has the dimensions given in Figure 6.28a, the dead weight of the footing is

$$(0.8 \text{ m})(0.8 \text{ m})(0.3 \text{ m})(24 \text{ kN/m}^3) = 4.6 \text{ kN}.$$

The earth pressure on the bottom of the footing has to be in equilibrium with the force of 56 kN from the column, and the footing's dead weight of 4.6 kN (see Figure 6.28b)

$$(56 \text{ kN}) + (4.6 \text{ kN}) = 60.6 \text{ kN}.$$

If the earth pressure is uniformly distributed, it equals

$$p = \frac{60.6 \times 10^3 \text{ N}}{(800 \text{ mm})(800 \text{ mm})} = 0.095 \text{ N/mm}^2.$$

In general, the earth pressure is not uniformly distributed. The value given for p is then referred to as the *average earth pressure*.

How the load exerted by the footing is transferred further into the ground, is a problem addressed by the special field of *soil mechanics*.

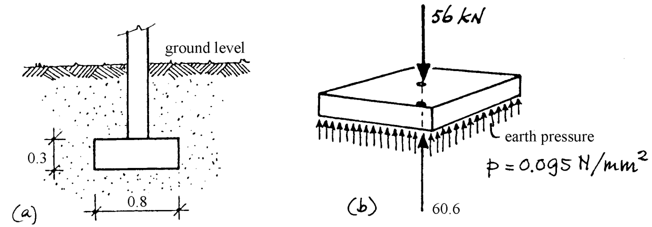


Figure 6.28 (a) The dimensions of the square footing. (b) On the bottom of the footing, the earth pressure has to provide an equilibrium with the force of 56 kN from the column and the footing's dead weight of 4.6 kN. The resultant of the earth pressure is therefore 60.6 kN.

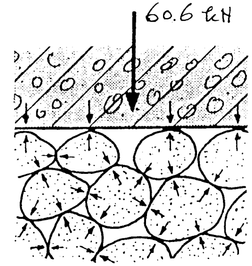


Figure 6.29 In reality, the earth pressure on the footing consists of a very large number of small forces provided by the grains of soil. Spreading all these forces evenly into a distributed surface load implies an idealisation of reality.

6.5 Stress concept; normal stress and shear stress

In reality, the earth pressure on the footing in Figure 6.29 consists of a very large number of very small forces provided by the grains of soil. Spreading all these forces evenly into a distributed surface load implies an idealisation of reality: the soil as a discontinuous material is replaced by a continuous material.

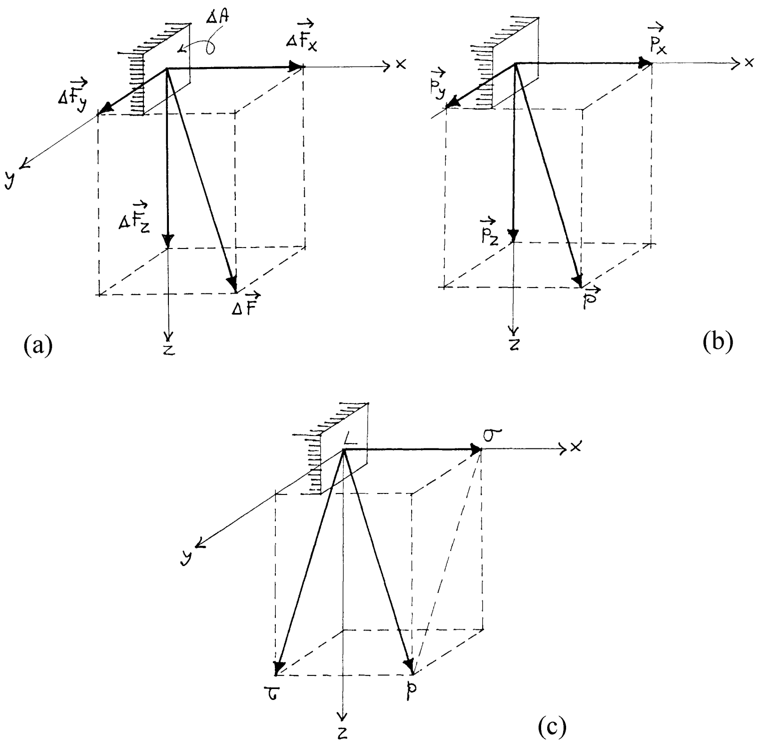


Figure 6.30 (a) Force $\Delta\vec{F}$ is the resultant of all the forces acting on a small yet finite area ΔA. (b) Stress vector $\vec{p}$ with its components. (c) Stress p (in visual notation) resolved into the normal stress σ perpendicular to the section plane and the shear stress τ in the section plane.

In fact, as a result of their atomic structure, all materials are discontinuous. The force flow in and between materials is the result of a very large number of small interactions between adjacent elementary particles.

Mathematically, the transfer of forces in and between materials is described using the concept *stress*. This concept is explained using Figure 6.30, in which a part of a body has been isolated from its environment.

Imagine that force $\Delta\vec{F}$ in Figure 6.30a is the resultant of all the small forces acting on a small, but finite area ΔA. As ΔA is chosen to be smaller, $\Delta\vec{F}$ is also smaller. The limit of the relationship between $\Delta\vec{F}$ and ΔA as ΔA approaches zero is defined as the *stress vector* $\vec{p}$:

$$\vec{p} = \lim_{\Delta A \to 0} \frac{\Delta\vec{F}}{\Delta A}.$$

When introducing the stress concept, one uses an idealised model of reality (a continuous material). The justification of this model is given *post hoc* by the agreement between model and reality. This agreement only exists if the stresses vary gradually. In areas with a major change in stresses (in the surroundings of *stress peaks*), one has to take into account the differences between model and reality.

The stress vector $\vec{p}$ (in space) has three components: p_x, p_y and p_z (see Figure 6.30b).

The stress vector can also be resolved into a component σ normal to the section plane and a component τ in the section plane. See Figure 6.30c, in which the visual notation is used. σ is known as *normal stress* and τ is referred to as *shear stress*.[1]

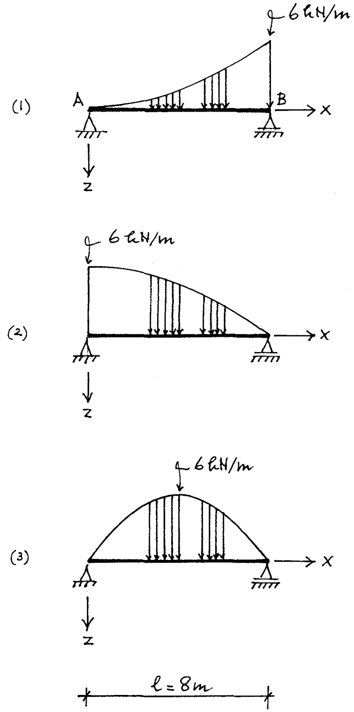
In mechanics, it is common practice to define the normal stress σ in solids as positive if it is a tensile stress. Sometimes, if dealing mainly with compressive stresses, it can be useful to define compressive stresses as positive. We often use a *prime* to indicate a change in sign. In such a case, $\sigma' = 300\ \text{N/m}^2$ means the same as $\sigma = -300\ \text{N/m}^2$. However, be aware that this notation is not always used, for instance in the cases of gas, liquid and earth pressures.

[1] The normal stress and shear stress are shown (for the present) as the components of a stress vector. Using the normal stress and shear stress to describe the *interaction* in the section plane, this presentation is not complete. In that case the normal vector on the section plane has to be considered. The complete definition is addressed in Chapter 10, where we look at the section forces in a member. Here it becomes clear that also for the shear stress it is possible to have an unequivocal sign convention.

6.6 Problems

Resultant of a line load on a member (Section 6.3.1)

6.1: 1–3 The same simply supported beam AB is carrying three parabolically distributed loads.



(1) $q(x) = 6\dfrac{x^2}{\ell^2}$ kN/m

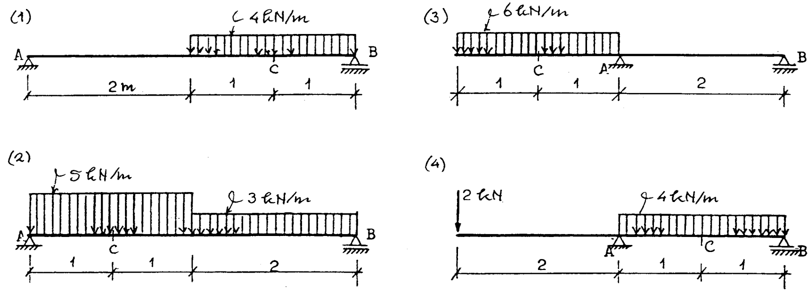
(2) $q(x) = 6\left(1 - \dfrac{x^2}{\ell^2}\right)$ kN/m

(3) $q(x) = 24\left(\dfrac{x}{\ell} - \dfrac{x^2}{\ell^2}\right)$ kN/m

Questions:

a. Determine the line of action and magnitude of the resultant of the distributed load.
b. Determine the support reactions at A and B.

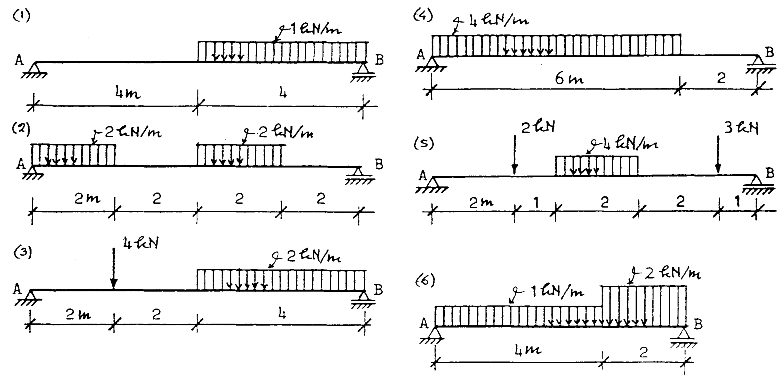
6.2: 1–4



Questions:

a. Determine the support reaction at A.
b. Determine the support reaction at B.
c. Determine the interaction forces at C.

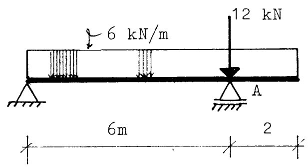
6.3: 1–6



Questions:

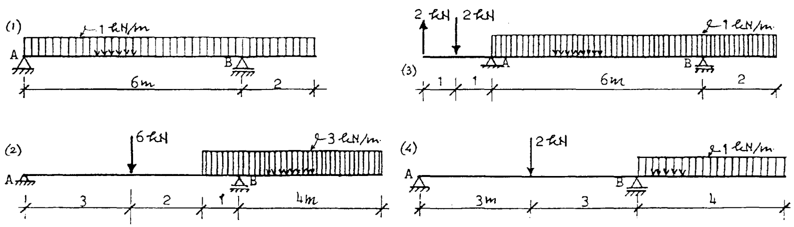
a. Determine the support reaction at A.
b. Determine the support reaction at B.

6.4



Question: Determine the support reaction at A.

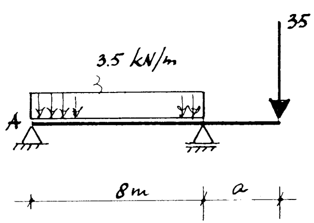
6.5: 1–4



Questions:

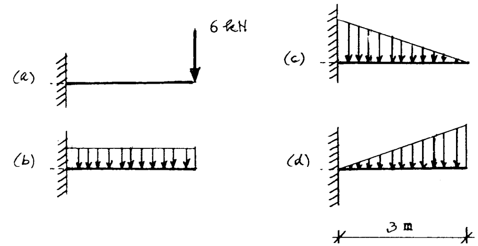
a. Determine the support reaction at A.
b. Determine the support reaction at B.

6.6



Question:
For which length *a* of the cantilever is the support reaction at A zero for the given load?

6.7 The same fixed beam is loaded in four different ways.

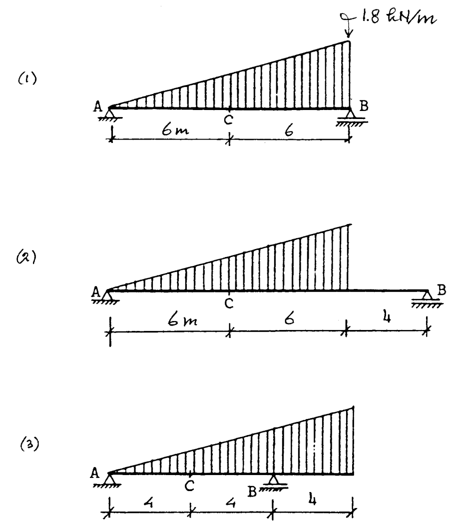


Question:
Determine the (peak value of the) distributed loads so that the fixed-end moment in all four cases is the same.

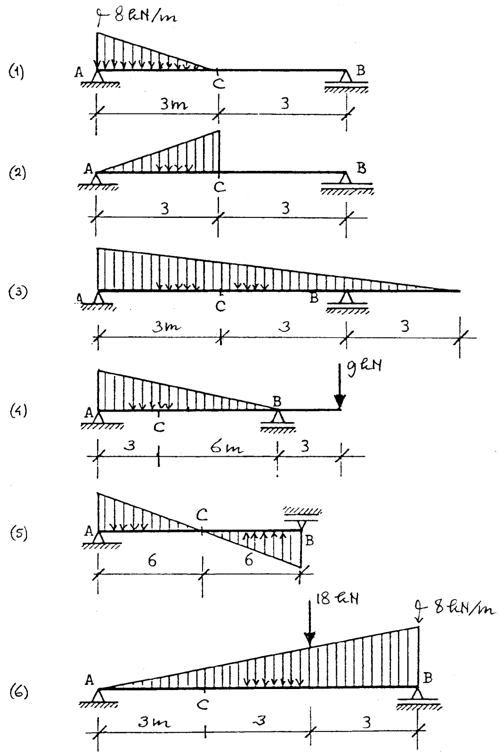
6.8: 1–3 Three beams are given with a linearly distributed load. The peak value of the distributed load is 1.8 kN/m for all cases.

Questions:

a. Determine the support reaction at A.
b. Determine the support reaction at B.
c. Determine the interaction forces at C.



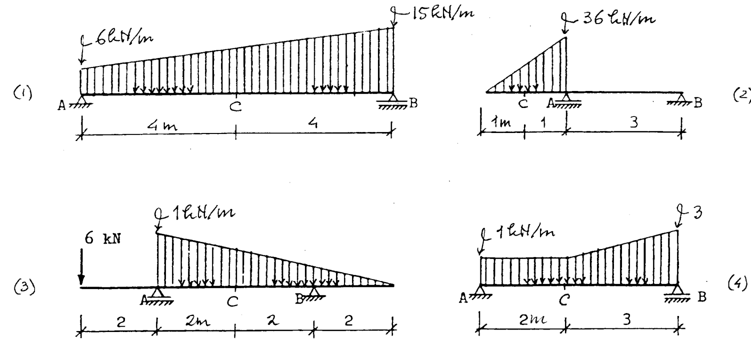
6.9: 1–6 A number of beams are given with a linearly distributed load and also a point load in two cases. The figures are not all shown to the same scale. The top value of the linearly distributed load is 8 kN/m in all cases. The magnitude of the point loads is given in the figure.



Questions:

a. Determine the support reaction at A.
b. Determine the support reaction at B.
c. Determine the interaction forces at C.

6.10: 1–4

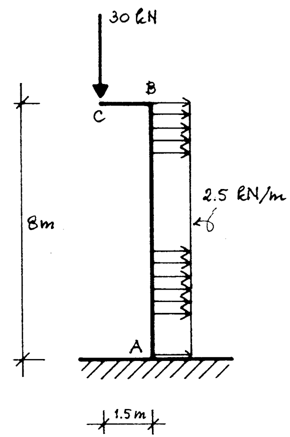


Questions:

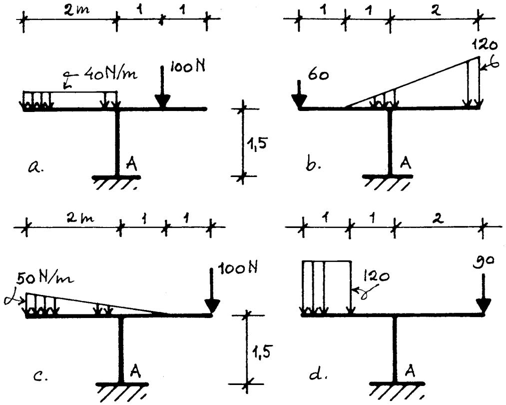
a. Determine the support reaction at A.
b. Determine the support reaction at B.
c. Determine the interaction forces at C.

6.11 The fixed member AB is loaded by an eccentric compressive force, and a uniformly distributed horizontal load.

Question:
Determine the support reactions at A.

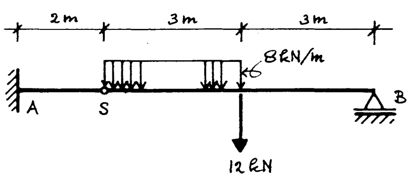


6.12 The same shelter is loaded in four different ways.



Question:
In which case is the fixed-end moment at most?

6.13 Hinged beam ASB is fixed at A.

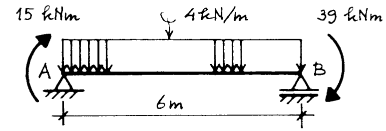


Question:
Determine the fixed-end moment at A.

6.14 The simply supported beam AB is carrying a uniformly distributed load over the entire length and is also loaded by couples at the supports.

Questions:

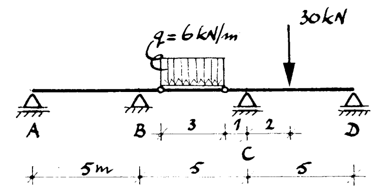
a. Determine the support reaction at A.
b. Determine the support reaction at B.



6.15 You are given a hinged beam.

Questions:

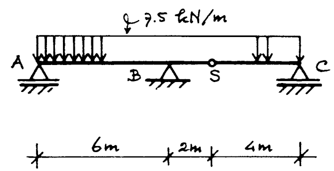
a. Determine the support reaction at A.
b. Determine the support reaction at D.
c. Determine the other support reactions.



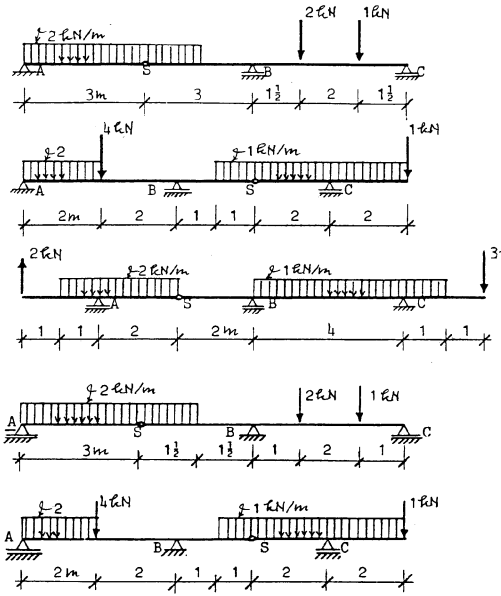
6.16 You are given a hinged beam.

Questions:

a. Determine the support reaction a A.
b. Determine the other support reactions.



6.17: 1–5 You are given five different hinged beams.

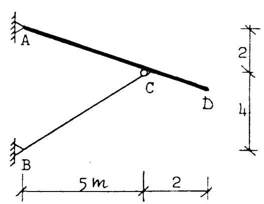


Question:
Determine the support reaction at A, B and C.

6.18 You are given a canopy roof ACD modelled as a line element.

Questions: Determine the horizontal and vertical support reactions at A and the force in member BC (with the correct sign) due to the following uniformly distributed loads on ACD:

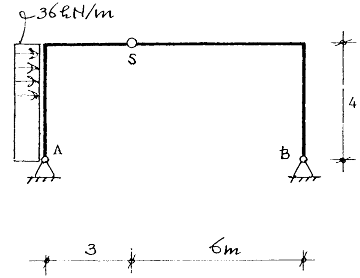
a. Dead weight of 2 kN/m.
b. Wind load of 3 kN/m.
c. Snow load of 4 kN/m.



6.19 A uniformly distributed horizontal load is acting on the left-hand post of the three-hinged frame ASB.

Questions:

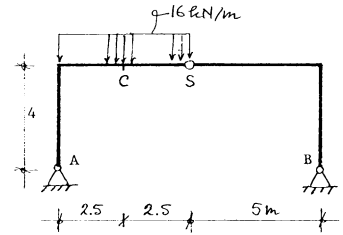
a. Determine the support reactions at A.
b. Determine the support reactions at B.



6.20 A uniformly distributed load is acting on the left-hand side of the three-hinged frame ASB.

Questions:

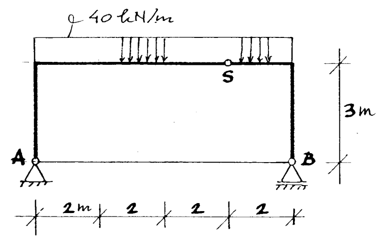
a. Determine the support reactions at A.
b. Determine the support reactions at B.
c. Determine the interaction forces at C.



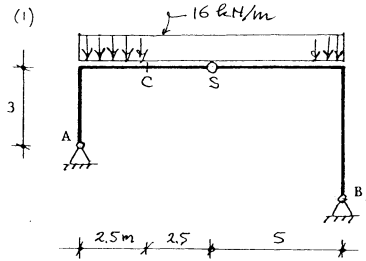
6.21 A three-hinged frame with tie-rod is carrying a uniformly distributed load of 40 kN/m.

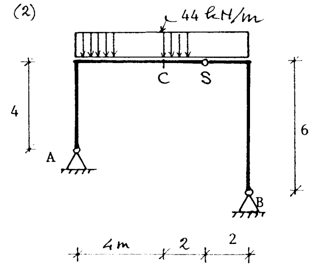
Questions:

a. Determine the support reaction at A.
b. Determine the force in tie-rod AB.

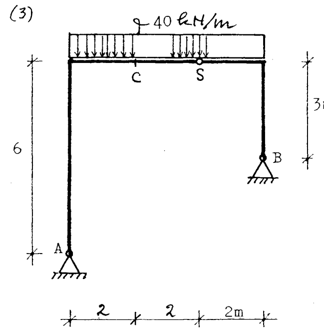


6.22: 1–3 You are given three different three-hinged frames with unequal post lengths.

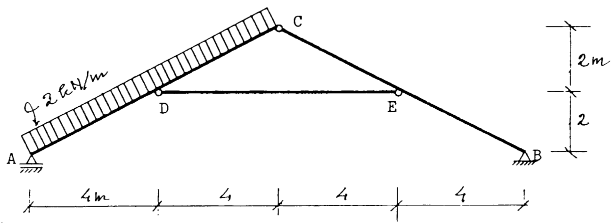




Questions:
a. Determine the support reactions at A.
b. Determine the support reactions at B.
c. Determine the interaction forces at C.



6.23

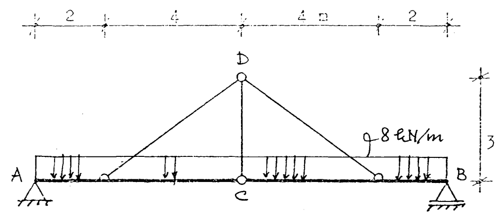


Questions:
a. Determine the support reactions at A and B.
b. Determine the force in bar DE, with the correct sign.

6.24 For the structure in problem 6.23, the roller and hinged support are exchanged.

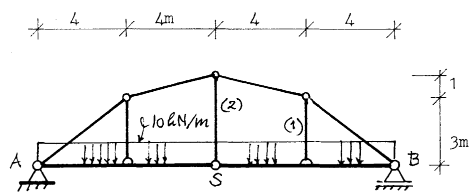
Questions:
a. Determine the support reactions at A and B.
b. Determine the force in bar DE, with the correct sign.

6.25 Trussed beam ACB is carrying over its entire length a uniformly distributed load of 8 kN/m.



Question:
Determine the force in bar CD. Is it a tensile force or a compressive force?

6.26 Trussed beam ASB is carrying over its entire length a uniformly distributed load of 10 kN/m.

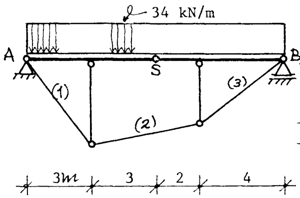


Questions:
a. Determine the force in bar 1.
b. Determine the force in bar 2.
c. Draw the closed force polygon for the equilibrium of joint B.

6.27 Trussed beam ASB is carrying over its entire length a uniformly distributed load of 34 kN/m.

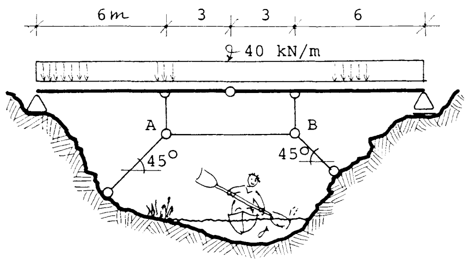
Questions:

a. Determine the force in bar 1.
b. Determine the force in bar 2.
c. Determine the force in bar 3.
d. Draw the closed force polygon for the equilibrium of joint A.
e. Draw the closed force polygon for the equilibrium of joint B.



6.28 You are given a queen post truss with a uniformly distributed load of 40 kN/m.

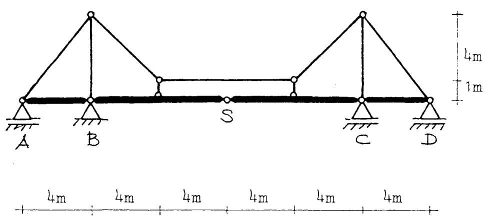
Question:
Determine the force in member AB. Is this a tensile force or a compressive force?



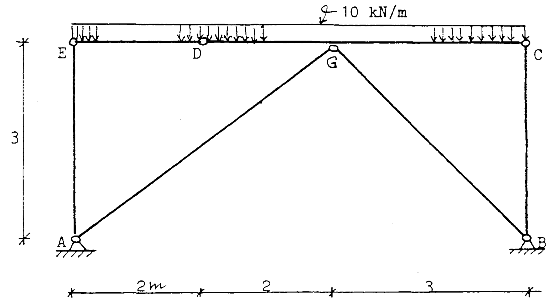
6.29 The dead weight of beam ABSCD is 125 kN/m.

Questions:

a. Determine the support reaction at A due to this dead weight.
b. Determine the other support reactions.



6.30 In the compound structure shown, ED and DC are connected by a hinge at D.

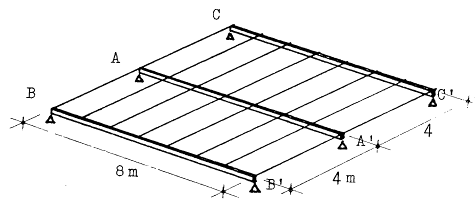


Questions:

a. Determine the vertical support reactions at A and B.
b. Determine the forces in the members AE and BC, with the correct signs.
c. Determine the forces in the members AG and BG, with the correct signs.
d. Determine the horizontal support reactions at A and B.
e. Draw the closed force polygon for the equilibrium of joint A.
f. Draw the closed force polygon for the equilibrium of joint B.

Modelling load flow (Section 6.4)

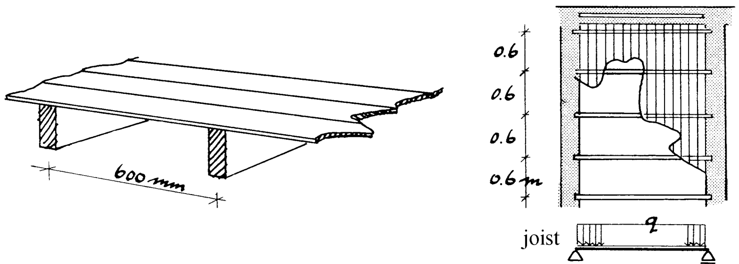
6.31 Steel beams AA′, BB′ and CC′ are carrying roof slabs. The dead weight of the roof slabs, together with the live load, equals 4 kN/m^2. The dead weight of the beams is estimated as 1 kN/m.



Questions:

a. Determine the (uniformly) distributed line load which has to be taken into account for beam AA′.
b. Determine the (uniformly) distributed line load which has to be taken into account for beam BB′.

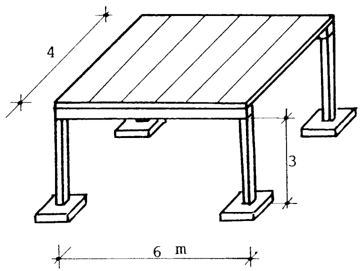
6.32 You are given a wooden joisting whereby the joists have a lateral distance of 0.6 m. The joists have a mass of 10 kg/m, and the floor has a mass of 10 kg/m^2. The live load is 1.5 kN/m^2. The load on an arbitrary joist (no edge joist) is modelled by as a uniformly distributed line load q. Let $g = 10$ N/kg.



Question:
How large is q?

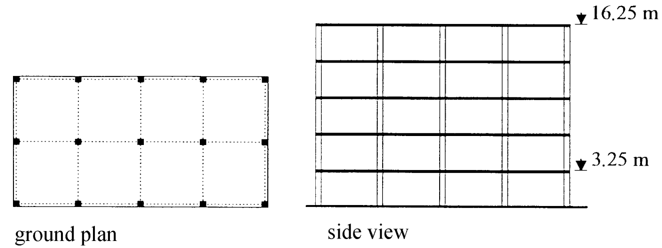
6.33 For the building on spread foundation the following is given:

- *Roof*
  Live load 500 N/m^2
  Dead load 300 N/m^2
  Dead weight 1500 N/m^2
- *Beams*
  Dead weight 3000 N/m
- *Columns*
  Dead weight 1500 N/m



Question:
Determine the load on one of the footings.

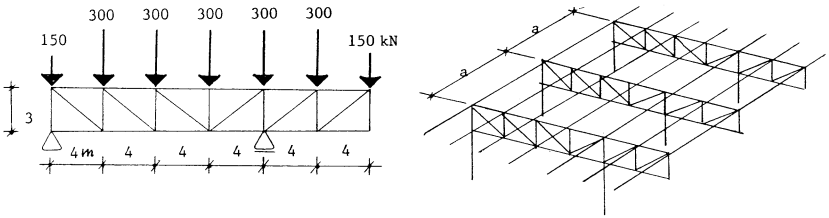
6.34 You are given a concrete skeleton with the columns on a grid of 5.5 × 5.5 m^2 and a height between floors of 3.25 m. The floors and the roof are 0.25 m thick. All the columns have cross-sectional dimensions of 0.5 × 0.5 m^2. The specific weight of concrete is 24 kN/m^3. The dead load is 1.5 kN/m^2.



Question:
Determine the load on the lower columns of the skeleton (or make a good estimation) due to the dead weight and dead load. Distinguish between:

a. a centre column,
b. an outer column, and
c. a corner column.

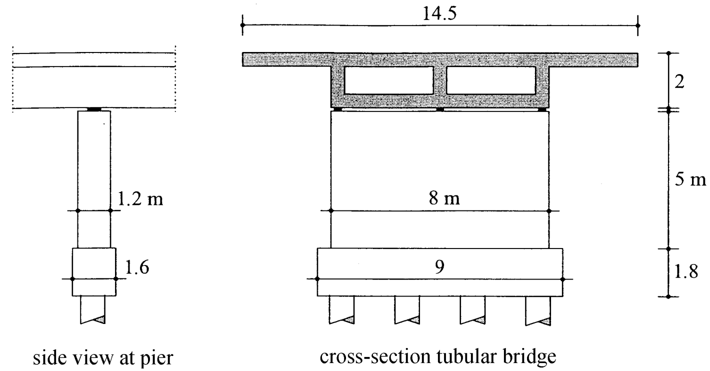
6.35 A roof truss is loaded by the forces shown. The truss spacing is a. The roof load (including the dead weight of the roof) is a uniformly distributed surface load p.



Question:
Which combination of truss spacing a and load p occur according to the given forces on the roof truss?

	Truss spacing a (m)	Load p (kN/m^2)
a.	4	22.5
b.	4	25
c.	3.75	20
d.	3	22.5
e.	3	25

6.36 For a concrete box girder bridge with a large number of spans, all the spans have the same length of 42 m. The piers have a pile foundation. The cross-sectional dimensions of the bridge and piers are shown in the figure. The box girder bridge has the same wall thickness of 0.4 m everywhere. The specific weight of concrete is 24 kN/m^3.



Questions:
Due to the dead weight, determine:

a. The load on the bridge modelled as a line element.
b. The load on a centre pier.
c. The load on a pile under the centre pier (assuming all piles are loaded equally).

7 Gas Pressure and Hydrostatic Pressure

Sometimes, an important part of the loading on a structure consists of gas pressure (such as with an air-supported hall or pneumatic structure) or hydrostatic pressure (such as with lock-gates, barrages or reservoirs). We look at this type of loading more closely in this chapter.

Because of the loose structure of material particles in stationary gases and fluids, there are no shear stresses. *As a result, the stresses in stationary gases and fluids always act normal to any bounding plane*.

In Section 7.1, we will show that, if there are no shear stresses, the stress at a particular point is independent of the orientation of the plane on which the stress is acting. This property is known as *Pascal's Law*. Such a stress situation is known as an *isotropic or spherical state of stress*.

Sections 7.2 and 7.3 provide examples of structures on which the loading is caused by gas and hydrostatic pressures respectively. The difference between the two is that the pressure in a gas is constant within the closed space in question.[1] In a fluid, the pressure increases linearly with depth due to its dead weight. The latter is referred to as a *hydrostatic pressure distribution*.

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[1] Other conditions apply when looking at the air pressure within the earth's atmosphere, for example; it depends on the distance to the surface of the earth, and is influenced by currents (wind).

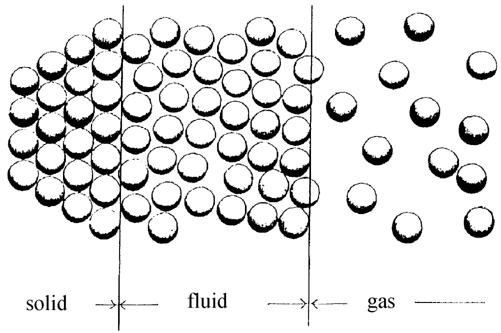

Figure 7.1 Because of the loose structure of the material particles in stationary gases and fluids, there are no shear stresses.

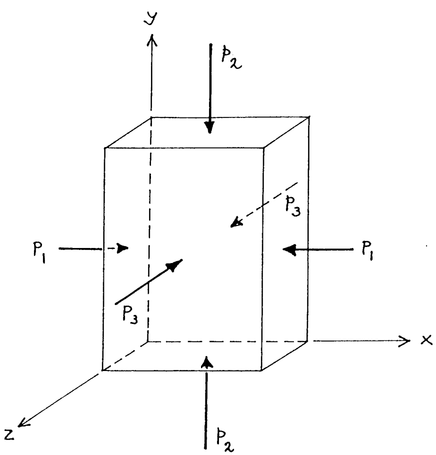

Figure 7.2 A small, rectangular volume element isolated from a gas or fluid, with compressive forces p_1; p_2; p_3 on its boundaries. Since there are no shear stresses, the stresses are normal to the sides in question. Here, the arrows should not be interpreted as forces.

7.1 Pascal's law – All-round pressure

Gases and fluids are distinct from solids in that they lack a solid shape. They can flow and adapt their shape to the environment. As such, gases do not have their own volume: all gas quantities distribute themselves throughout the available space. One of the reasons for this is their loose particle structure (see Figure 7.1).

Because of the weak bonding, gas and fluid particles can easily move with respect to one another. As a result, we could (rather boldly) state that no shear stresses can be transmitted in gases and fluids. This is, however, not the case with *flowing* gases and fluids; because of the differences in speed between adjacent layers, shear stresses can occur, although they are far weaker than in solids.

Below, it is assumed that no shear stresses occur in gases and fluids at rest. *This means that the stresses in stationary gases and fluids always act normal to any bounding plane*.

In Figure 7.2, a rectangular volume element has been isolated from a gas or fluid. Compressive stresses p_1; p_2; p_3 act on the boundary of the element. The volume element is so small that, for all the stresses on the boundary, it can be assumed that they are uniformly distributed. In that case, one does not have to draw the entire stress distribution, but a single arrow[1] is sufficient.

The condition that no shear stresses can act in the material implies that the stresses on the boundary of the volume element have to be of the same magnitude:

$$p_1 = p_2 = p_3.$$

---

[1] Note: the arrows here cannot be interpreted as forces.

To demonstrate this, a small triangular part has been isolated from the material parallel to the xy plane in Figure 7.3. The oblique side has an area ΔA. The area of the vertical side is therefore $\Delta A \cos\alpha$, while that of the horizontal side is $\Delta A \sin\alpha$. The triangular part is so small that, for all the stresses on the boundary, it can be assumed that they are uniformly distributed. Assume that a compressive stress p is acting on the oblique side. This stress acts normal to the side as there is no shear stress.

In Figure 7.4, the forces (force = stress × area) on the edges of the triangular part are shown. The lines of action of the forces pass through a single point. This means that there is moment equilibrium in the xy plane. Here it is assumed that the element is so small that its dead weight can be neglected.

The equations for the force equilibrium in respectively the x and y direction are

$$\sum F_x = p_1 \Delta A \cos\alpha - p\Delta A \cos\alpha = 0,$$

$$\sum F_y = p_2 \Delta A \sin\alpha - p\Delta A \sin\alpha = 0,$$

so that

$$p_1 = p_2 = p_3.$$

The result is independent of angle α.

In the same way, using the equilibrium of a triangular section parallel to the xz plane, we derive

$$p_1 = p_3 = p.$$

This means that the stress at a particular point is independent of the orientation of the plane that the stress acts on. This characteristic is known as

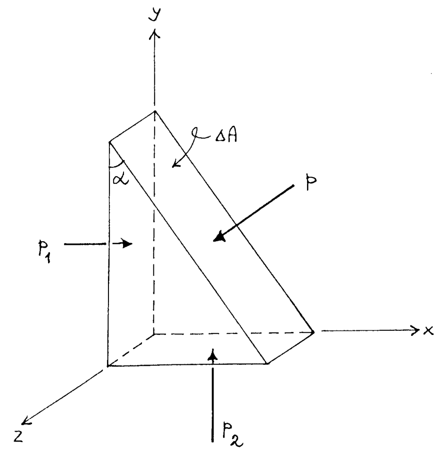

Figure 7.3 Stresses normal to the sides of a triangular volume element.

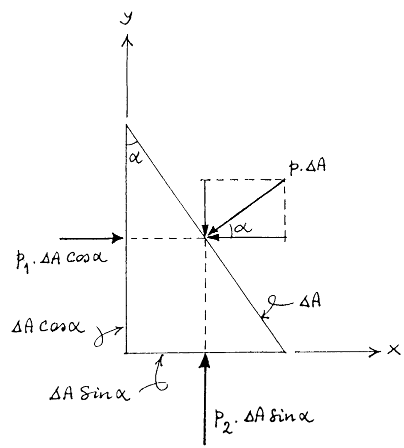

Figure 7.4 Forces on the sides of the triangular volume element.

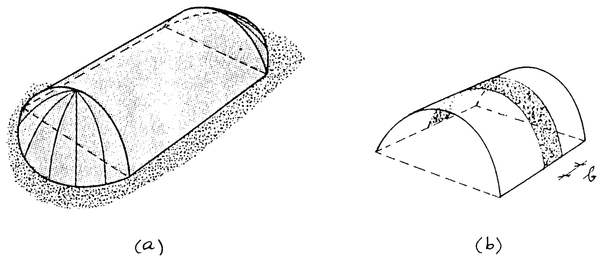

Figure 7.5 (a) A pneumatic structure consists of a membrane that maintains its shape through internal overpressure. (b) A strip with width b from the circular cylindrical midsection in more detail.

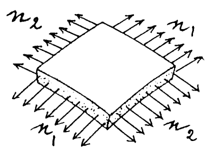

Figure 7.6 A membrane can transfer forces only in the direction of its curved plane.

Pascal's Law.[1] This state of stress is known as *isotropic* or *spherical*. With gases and fluids, in which only compressive stresses occur, we also speak of *all-round pressure*.

7.2 Working with gas pressures

The type of structure in Figure 7.5a which is sometimes used as a tennis hall, is an *air-supported hall* or *pneu*. Pneu is an abbreviation of *pneumatic structure*. This type of structure consists of a thin, flaccid skin (*membrane*), which can transfer tensile forces only in its curved plane (see Figure 7.6). The structure maintains its shape through internal *overpressure*.[2] The same holds, for example, for an inflated balloon, or the inner tube of a bike. We will look at three examples for this type of structure. In the first two examples the load is a gas pressure (the overpressure in the pneu). The third example concerns a body subjected to an all-round pressure.

Example 1

The pneu in Figure 7.5a consists of a circular cylindrical midsection that is closed by means of spherical ends. The diameter of the circular cylinder is r, the aperture angle is α, and the internal overpressure is p.

Question:

Determine the distributed support reactions (forces per length) for the circular cylindrical midsection of the pneu.

Solution:

In Figure 7.5b, a strip of width b has been isolated. This strip is modelled in Figure 7.7a as a curved line element with a distributed load pb. In addition

---

[1] Blaise Pascal (1623–1662), French mathematician, physicist and writer. With Fermat, he was one of the founders of the theory of probability. As a writer he is known for his Pensées, a collection of loose notes published posthumously.

[2] This is the difference between the pressure inside and outside the structure.

to the support reactions at A and B, no forces other than those shown in the figure act in the plane of the drawing.

Since a membrane can transfer forces only in its (curved) plane, the support reactions at A and B act along the tangents of the circular cross-section (see Figure 7.7b). Because of mirror symmetry, the support reactions at A and B are of equal magnitude. Assume these are tensile forces N. With an aperture angle α, the horizontal and vertical components of N are:

$$N_{\mathrm{h}} = N \cos\alpha,$$

$$N_{\mathrm{v}} = N \sin\alpha.$$

The vertical component N_{v} can be derived from the vertical force equilibrium. In doing so, a tricky point is that the distributed load pb changes direction. The calculation can, however, be considerably simplified by isolating the structure from its surroundings, not "*by itself*", but "*with content*" (see Figure 7.7c). The overpressure on plane AB (with width b) is equal to pb. If the dead weight of the gas and the membrane can be ignored, the equation for the vertical force equilibrium is

$$2N_{\mathrm{v}} - pb \cdot 2r \sin\alpha = 0$$

so that

$$N_{\mathrm{v}} = pbr \sin\alpha.$$

The horizontal component of N is then

$$N_{\mathrm{h}} = pbr \cos\alpha$$

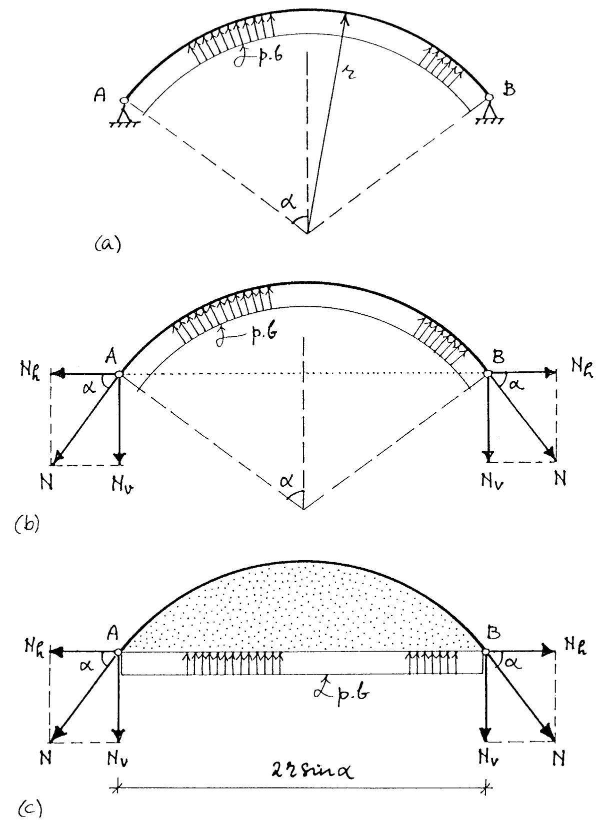

Figure 7.7 (a) A strip with width b from the cylindrical midsection of the pneu, modelled as a line element. (b) The support reactions at A and B. (c) The vertical component of the support reactions at A and B are found from the vertical equilibrium of the "strip with content".

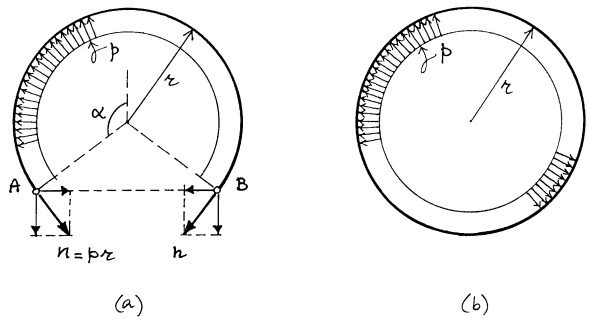

Figure 7.8 (a) The membrane force $n = pr$ in the circumferential direction is independent of the aperture angle α. (b) The formula $n = pr$ is also referred to as the boiler formula as calculating the force in the walls of a steam boiler was an important field of application.

Figure 7.9 A spherical pneu designed by Frei Otto to cover a settlement in Antarctica.

and the resulting support reaction is

$$N = pbr.$$

The calculation relates to a strip with width b. The requested support reactions per length are:[1]

$$n = \frac{N}{b} = pr.$$

Note that the magnitude of the force $n = pr$ is independent of the aperture angle α (see Figure 7.8a). Obviously, in the circular cylinder pneu, the (distributed) circumferential tensile forces have the same magnitude everywhere. The formula is also applicable for a closed ring (see Figure 7.8b, where $\alpha = 180°$) and is known as the *boiler formula*, as calculating forces in the walls of a steam boiler was an important field of application.

Example 2

The second example relates to a pneu designed by architect Frei Otto[2] to cover a settlement in Antarctica (see Figure 7.9). This design is discussed in his book "*Zugbeanspruchte Konstruktionen*".

The pneu is shaped like a segment of a sphere and rests on a concrete ring beam. The diameter of the sphere is $r = 2200$ m. The diameter of the ring beam is $r_{\text{beam}} = 1000$ m. The segment of the sphere is 240 m high (see Figure 7.10). The weight of the roof is 82 N/m^2. The pneu maintains its shape through an internal overpressure of 350 N/m^2.

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1 It is the convention to use a lower case letter for distributed forces.

2 Frei Otto (1925), German architect. Renowned designer of pneumatic structures and cable networks. One of his most famous designs was the roof of the Olympic Stadium in Munich (1972). Also see Chapter 14, Section 14.3, Example 4.

Questions:

a. Determine the weight of the ring beam so that the foundation is not subjected to tension.
b. Determine the compressive force in the ring beam.

Solution:

a. To calculate the forces in the pneu, it is assumed that the dead weight of the roof acts in the direction of the centre of the sphere instead of the centre of the earth. This assumption introduces only a minor discrepancy. In this case, the resulting overpressure in the pneu is

$$p = 350 - 82 = 268 \text{ N/m}^2.$$

This overpressure generates tensile forces n (forces per length) in the membrane. The vertical component n_v can be deduced from the vertical force equilibrium of the segment of the sphere "*with content*" (see Figure 7.11). Here, n_v acts on the circumference of the ring beam and the overpressure p acts on the area within the ring beam. The equilibrium equation is

$$n_\text{v} \cdot 2\pi r_\text{beam} - p \cdot \pi r_\text{beam}^2 = 0$$

so that

$$n_\text{v} = \frac{p \cdot \pi r_\text{beam}^2}{2\pi r_\text{beam}} = \tfrac{1}{2} p r_\text{beam}.$$

With $n_\text{v} = n \sin\alpha$ and $r_\text{beam} = r \sin\alpha$, the (distributed) tensile forces are

$$n = \tfrac{1}{2} pr.$$

Here too, the (distributed) tensile forces are independent of the aperture angle α. They are, however, half as large as the circumferential tensile forces in the circular cylindrical pneu from the previous example.

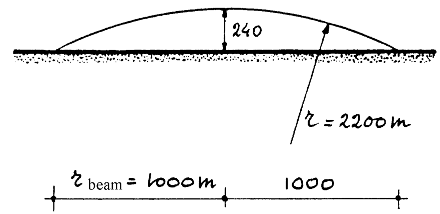


Figure 7.10 Dimensions of the spherical pneu.

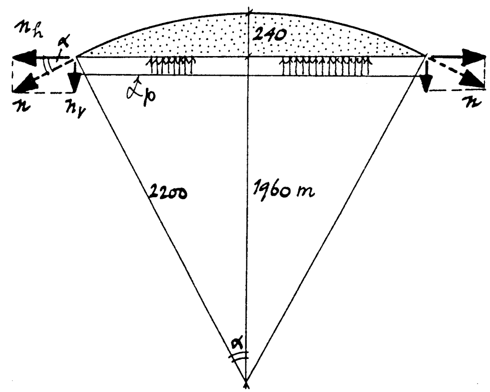


Figure 7.11 The vertical component of the membrane force is found from the vertical equilibrium of the sphere segment with content. The membrane force acts on the circumference of the ring beam. The overpressure p acts on the area within the ring beam.

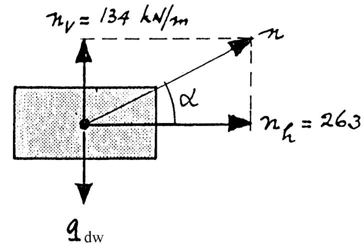


Figure 7.12 The forces acting on the ring beam.

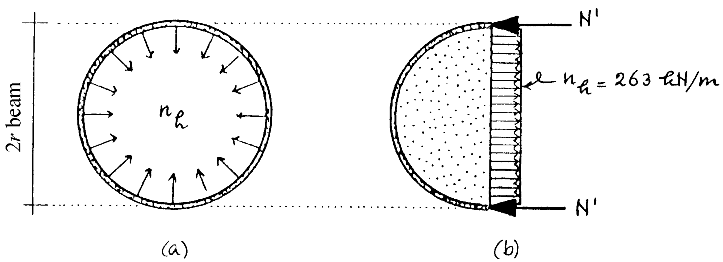


Figure 7.13 (a) Due to the horizontal component of the membrane force, the ring beam is pulled inwards on all sides. Here we can recognise the analogy of a closed ring with underpressure. (b) Compressive forces are generated in the ring. They can be determined using the boiler formula, or directly from the equilibrium of half a ring beam.

In this example

$$n = \tfrac{1}{2}pr = \tfrac{1}{2} \times (268\ \text{N/m}^2)(2200\ \text{m}) = 295\ \text{kN/m}.$$

The aperture angle α is (see Figure 7.11)

$$\alpha = \arccos\left(\frac{1960\ \text{m}}{2200\ \text{m}}\right) = 27°.$$

At the ring beam, the horizontal and vertical components of n are

$$n_\text{h} = n\cos\alpha = (295\ \text{kN/m}) \times \cos 27° = 263\ \text{kN/m},$$

$$n_\text{v} = n\sin\alpha = (295\ \text{kN/m}) \times \sin 27° = 134\ \text{kN/m}.$$

Figure 7.12 shows all the forces acting on the ring beam. These are the distributed force n, which the pneu exerts on the ring beam, and the dead weight q_dw of the ring beam (also a force per length).

The vertical component n_v tries to lift the ring beam. In order to prevent this, the dead weight q_dw has to be larger than $n_\text{v} = 134$ kN/m. If the ring beam is made of concrete, with a specific weight of 24 kN/m^3, then the cross-section A of the beam has to obey

$$q_\text{dw} = A \times (24\ \text{kN/m}^3) \geq n_\text{v} = 134\ \text{kN/m} \Rightarrow A \geq \frac{134\ \text{kN/m}}{24\ \text{kN/m}^3} = 5.6\ \text{m}^2.$$

The cross-section of the ring beam has to be at least 5.6 m^2.

b. Due to the horizontal forces n_h, the ring beam is pulled inwards from all sides (see Figure 7.13a). Here, you will recognise the loading case of the closed ring from Figure 7.8b, but now with an *underpressure* instead of an *overpressure*.

A compressive force N'[1] is formed in the ring. This can be calculated using the *boiler formula* from the previous example, or directly from the equilibrium of the half ring beam in Figure 7.13b:

$$N' = \frac{2r_{\text{beam}}n_{\text{h}}}{2} = r_{\text{beam}}n_{\text{h}} = (1000\text{ m})(263\text{ kN/m}) = 263\text{ MN}.$$

Comment: This force is relatively large for a concrete cross-section of 5.6 m^2. The compressive force in the ring may therefore call for a larger cross-section.

Example 3

A uniformly distributed load q is acting on the plane body in Figure 7.14. The load acts in the plane of the body along the entire outline and normal to the body.

Questions:

a. Show that the resultant of the distributed load on the body is zero, regardless of the shape of the body.
b. Determine the resultant of the load above section AB.

Solution:

a. In Figure 7.15a, a minor force ΔF is acting perpendicular to the given boundary element with small length Δs:

$$\Delta F = q\Delta s.$$

The horizontal and vertical components of ΔF are respectively

$$\Delta F_x = q\Delta s\cos\varphi,$$

[1] The convention is that N as tensile force is positive. The prime for a switch in sign indicates that compressive forces are now positive (see Section 6.5).

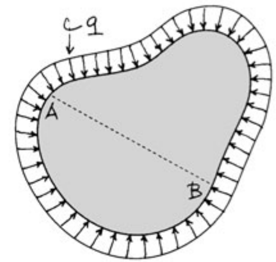


Figure 7.14 A plane body loaded over its entire outline by a uniformly distributed load normal to the body.

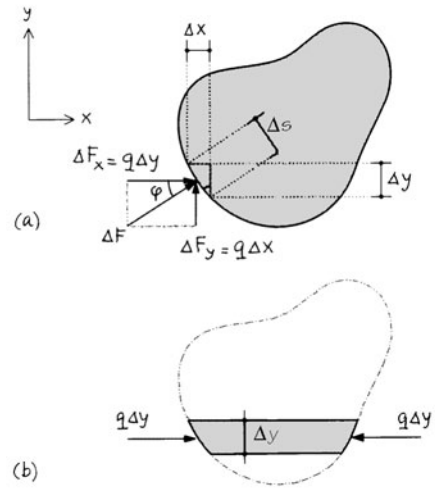


Figure 7.15 (a) As a result of the distributed load q, a small force $\Delta F = q\Delta s$ is acting on a small boundary element with length Δs. (b) The horizontal components of the load on the boundary elements of a horizontal strip are equal and opposite. Together they form an equilibrium system with zero resultant.

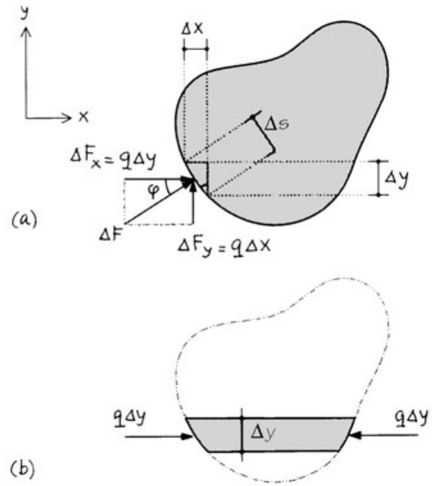


Figure 7.15 (a) As a result of the distributed load q, a small force $\Delta F = q\Delta s$ is acting on a small boundary element with length Δs. (b) The horizontal components of the load on the boundary elements of a horizontal strip are equal and opposite. Together they form an equilibrium system with zero resultant.

$$\Delta F_y = q\,\Delta s \sin\varphi.$$

Since $\Delta s\cos\varphi = \Delta y$ and $\Delta s\sin\varphi = \Delta x$, we can also write

$$\Delta F_x = q\,\Delta y,$$
$$\Delta F_y = q\,\Delta x.$$

The components ΔF_x and ΔF_y of force ΔF on boundary element Δs are equal to the product of the distributed load q and the projection of Δs on the y axis and the x axis respectively.

Figure 7.15b shows a horizontal strip from the body with a small width Δy. The horizontal components of the load on the boundary elements are equal and opposite. They form an equilibrium system with resultant zero. Since this applies to all the horizontal strips of which the body is composed, the resulting horizontal load on the body is zero.

By dividing the body into vertical strips, and looking at the vertical component of the load on the boundary elements, we can similarly deduce that the resulting vertical load on the body is zero.

Conclusion: *If a uniformly distributed load acts on a plane body in the plane of the body along its entire outline, and everywhere normal to the body, the load forms an equilibrium system with resultant zero.*

One can show that this is also true in three-dimensional cases: *If a uniformly distributed load acts on a body in space on its entire surface, and everywhere normal to the body, the load forms an equilibrium system with resultant zero.*

b. In Figure 7.16a, the part of the body above section AB has been isolated. Assume the resultant of the distributed load on the outside between A and B is R.

If a uniformly distributed load q is also applied to section AB, as in Figure 7.16b, the total load on the isolated part of the body forms an equilibrium system: the resultant R of the load on the outside of the body is equal and opposite to the resultant R_{section} of the load on the section. Therefore $R = R_{\text{section}} = qa$, in which a is the length of the section. The line of action of R coincides with the perpendicular bisector of AB.

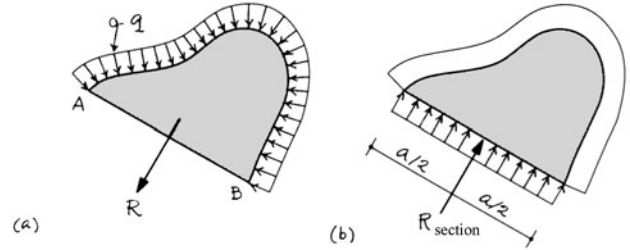


Figure 7.16 (a) The resultant R of the uniformly distributed load on the outside AB is equal and opposite to (b) the resultant R_{section} of an equally large uniformly distributed load on section AB.

7.3 Working with hydrostatic pressures

In a fluid at rest, the (all-round or isotropic) pressure increases linearly with depth. This can be derived from the vertical force equilibrium of the fluid column in Figure 7.17. Using density ρ of the fluid and the gravitational field intensity $g = 10$ N/kg the specific weight γ is

$$\gamma = \rho g.$$

The weight ΔG of the fluid column, with height z and cross-section ΔA is

$$\Delta G = \gamma z \Delta A.$$

At the base of the fluid column, there is a compressive force $p\Delta A$, the resultant of the compressive stresses p on the area ΔA. The vertical force equilibrium of the column (there are no shear stresses) now gives

$$p\Delta A = \Delta G$$

or

$$p = \frac{\Delta G}{\Delta A} = \frac{\gamma z \Delta A}{\Delta A} = \gamma z.$$

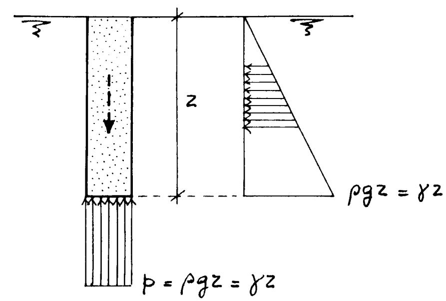


Figure 7.17 In a fluid at rest, the (all-round) pressure increases linearly with depth as a result of its dead weight. This is derived from the vertical force equilibrium of the fluid column.

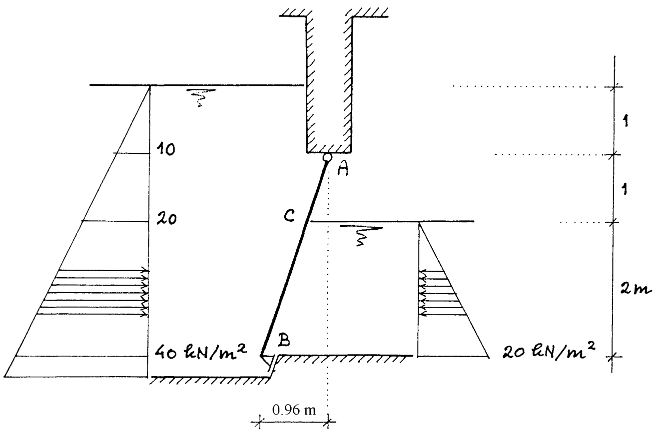


Figure 7.18 Longitudinal section of a channel with the 4-metre wide flap AB. The distribution of the water pressure is shown on both sides of the flap.

The isotropic compressive stress p increases linearly with depth z. This is referred to as a *hydrostatic pressure distribution*.

Below you will find a number of examples covering loads due to a hydrostatic pressure. We assume that in all cases the *fluid is at rest* and that the pressure distribution is *hydrostatic*. At any point the *hydrostatic pressure is equally large in all directions* (isotropic state of stress) and *always acts normal to the plane in question* (as there are no shear stresses).

Example 1
Figure 7.18 shows the longitudinal section of a channel with a 4-metre wide flap AB. The flap is supported at A by a hinge and is resting at B on a sill. The support in B can be seen as a roller support. The water level on both sides of the flap is shown in the figure. The density of water is 1000 kg/m^3. The gravitational field intensity is 10 N/kg.

Question:
Determine the support reactions at A and B due to the total water pressure. The dead weight of the flap should be ignored.

Solution:
The linear distribution of the water pressure on both sides of the flap is shown in Figure 7.18.

To the left of the flap, the water pressure at A is

$$(1000\ \mathrm{kg/m^3})(10\ \mathrm{N/kg})(1\ \mathrm{m}) = 10\ \mathrm{kN/m^2}$$

and at B it is

$$(1000\ \mathrm{kg/m^3})(10\ \mathrm{N/kg})(4\ \mathrm{m}) = 40\ \mathrm{kN/m^2}.$$

To the right of the flap, the water pressure at B is

$$(1000\ \mathrm{kg/m^3})(10\ \mathrm{N/kg})(2\ \mathrm{m}) = 20\ \mathrm{kN/m^2}.$$

In Figure 7.19a, the 4-metre wide flap is modelled as a line element, with line loads due to the water pressures normal to it.

To the left of the flap, the distributed load varies linearly from

$$(4\ \text{m})(10\ \text{kN/m}^2) = 40\ \text{kN/m at A},$$

to

$$(4\ \text{m})(40\ \text{kN/m}^2) = 160\ \text{kN/m at B}.$$

To the right of the flap, the load increases linearly from 0 at C to

$$(4\ \text{m})(20\ \text{kN/m}^2) = 80\ \text{kN/m at B}.$$

Figure 7.19b represents the load diagram for the resulting water pressure.

The length of flap AB is (see Figure 7.18)

$$\sqrt{(3\ \text{m})^2 + (0.96\ \text{m})^2} = 3.15\ \text{m}.$$

The distances BC and CA are respectively 2.10 m and 1.05 m.

To work quickly, the load diagram in Figure 7.20 has been placed horizontally and is split up into a number of areas for which the resultants can be easily calculated:

$$R_1 = (2.10\ \text{m}(80\ \text{kN/m}) = 168\ \text{kN},$$

$$R_2 = \tfrac{1}{2} \times (1.05\ \text{m})(80\ \text{kN/m}) = 42\ \text{kN},$$

$$R_3 = \tfrac{1}{2} \times (1.05\ \text{m})(40\ \text{kN/m}) = 21\ \text{kN}.$$

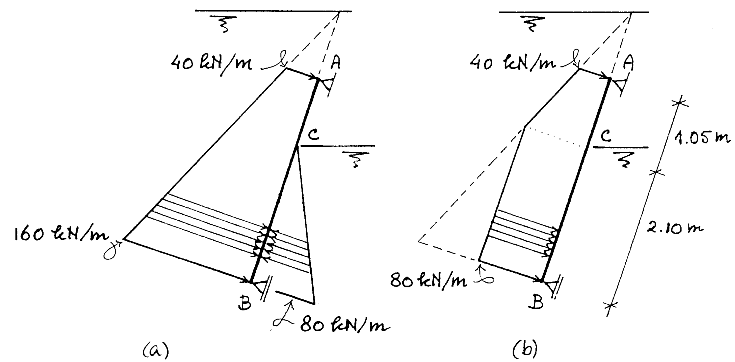


Figure 7.19 The 4-metre wide flap modelled as a line element with (a) the water pressure normal to the flap and (b) the resulting load diagram.

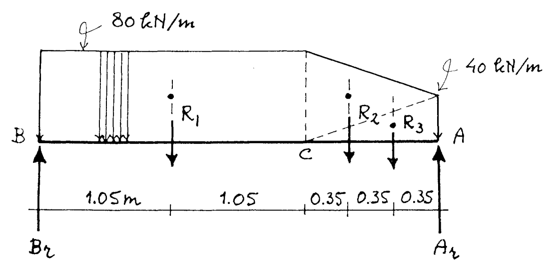


Figure 7.20 The load diagram split up into three areas for which the resultants are easy to find with respect to their magnitudes and lines of action.

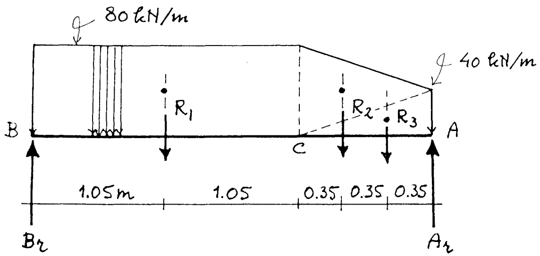


Figure 7.20 The load diagram split up into three areas for which the resultants are easy to find with respect to their magnitudes and lines of action.

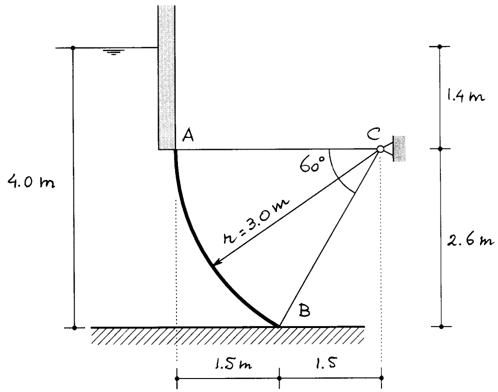


Figure 7.21 A moveable dam consisting of a circular cylindrical slide AB hinged at C and joined to an rigid vertical partition wall at A. There is no water to the right of the dam.

The support reaction A_r at A is found from the moment equilibrium of the flap about B:

$$A_r = \frac{(1.05 \text{ m}) \times R_1 + (2.45 \text{ m}) \times R_2 + (2.80 \text{ m}) \times R_3}{3.15 \text{ m}} = 107.3 \text{ kN}.$$

The support reaction B_r in B is found from the force equilibrium:

$$B_r = R_1 + R_2 + R_3 - A_r = 123.7 \text{ kN}.$$

Example 2

The moveable dam in Figure 7.21 consists of a circular cylindrical slide AB hinged at C and joining a rigid vertical partition wall at A. There is no water to the right of the dam. The specific weight of water is $\gamma_w = 10 \text{ kN/m}^3$. All other information required can be found in the figure.

Question:

Find the magnitude and direction of the resultant water pressure on a 1-metre strip from the circular cylindrical slide.

Solution:

In Figure 7.22, the 1-metre wide strip from the slide is modelled as a line element. The figure also shows the water pressure, increasing from 14 kN/m at top A of the slide to 40 kN/m near base B.

The water pressure is acting normal to the slide everywhere. In other words, all the forces on the slide pass through C, the centre of arc AB. Therefore, the resultant R of the total water pressure on the slide also passes through C.

To determine the resultant water pressure, please refer to Figure 7.23, which shows all symbols used. The water pressure as a function of φ is

$$q(\varphi) = \frac{1 + r\sin\varphi}{d} \cdot \hat{q}.$$

The resultant of the water pressure on a small part of the slide with length $r\,\mathrm{d}\varphi$ is a small force $\mathrm{d}F$:

$$\mathrm{d}F = q(\varphi) \cdot r\,\mathrm{d}\varphi$$

with components

$$\mathrm{d}F_x = \mathrm{d}F\cos\varphi = q(\varphi) r\cos\varphi\,\mathrm{d}\varphi,$$

$$\mathrm{d}F_y = \mathrm{d}F\sin\varphi = q(\varphi) r\sin\varphi\,\mathrm{d}\varphi.$$

The components R_x and R_y of the resulting water pressure are found by summing up all the contributions $\mathrm{d}F_x$, respectively $\mathrm{d}F_y$, over the length of slide AB. This summation is done by integrating between the limits $\varphi = 0$ and $\varphi = 60° = \pi/3$ rad:

$$R_x = \int_0^{\pi/3} q(\varphi) r\cos\varphi\,\mathrm{d}\varphi,$$

$$R_y = \int_0^{\pi/3} q(\varphi) r\sin\varphi\,\mathrm{d}\varphi.$$

Using the previously deduced expression for $q(\varphi)$ and the formulas in Table 7.1, the integrals are elaborated:

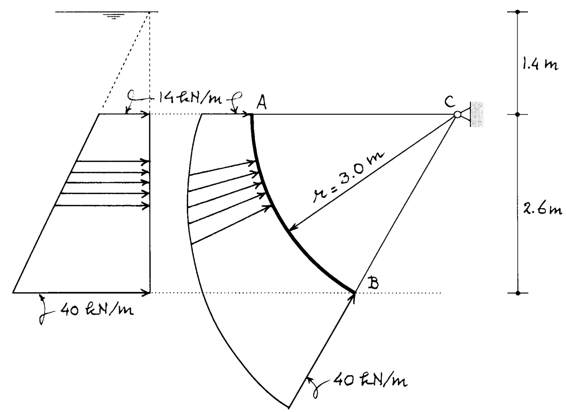


Figure 7.22 The distribution of the water pressure on a 1-metre wide strip from the slide.

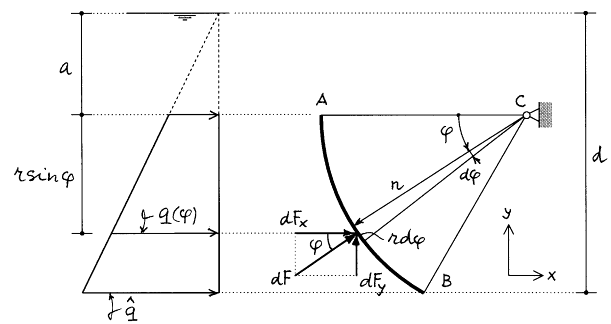


Figure 7.23 To calculate the resulting water pressure, all the data has been shown as symbols.

Table 7.1

$\int \sin\varphi \cos\varphi \, d\varphi = \frac{1}{2}\sin^2\varphi$
$\int \sin^2\varphi \, d\varphi = -\frac{1}{4}\sin 2\varphi + \frac{1}{2}\varphi$

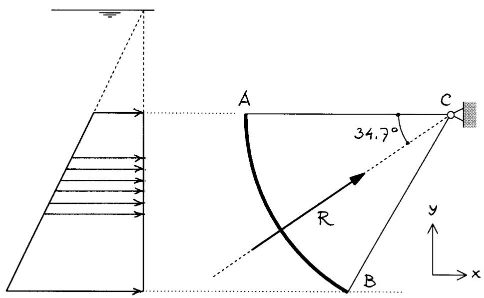


Figure 7.24 The resultant of the water pressure on the circular cylindrical slide passes through C, the centre of arc AB.

$$R_x = \frac{\hat{q}r}{d}\int_0^{\pi/3}(a + r\sin\varphi)\cos\varphi \, d\varphi = \frac{\hat{q}r}{d}\left[a\sin\varphi + \tfrac{1}{2}r\sin^2\varphi\right]_{\varphi=0}^{\varphi=\pi/3}$$

$$= \frac{\hat{q}r}{d}(a \times 0.86 + r \times 0.375),$$

$$R_y = \frac{\hat{q}r}{d}\int_0^{\pi/3}(a + r\sin\varphi)\sin\varphi \, d\varphi$$

$$= \frac{\hat{q}r}{d}\left[-a\cos\varphi + r\left(-\tfrac{1}{4}\sin 2\varphi + \tfrac{1}{2}\varphi\right)\right]_{\varphi=0}^{\varphi=\pi/3}$$

$$= \frac{\hat{q}r}{d}(a \times 0.5 + r \times 0.307).$$

By substituting $\hat{q} = 40$ kN/m, $r = 3.0$ m, $d = 4.0$ m and $a = 1.4$ m, we find

$$R_x = 70.1 \text{ kN},$$

$$R_y = 48.6 \text{ kN}.$$

The vertical component of the water pressure generates an *upward* force on the slide.

The resulting water pressure R on the 1-metre strip from the slide is shown in Figure 7.24:

$$R = \sqrt{(70.1 \text{ kN})^2 + (48.6 \text{ kN})^2} = 85.3 \text{ kN}.$$

The line of action, as shown earlier, passes through C and is at an angle of α to the horizontal:

$$\alpha = \arctan\left(\frac{48.6 \text{ kN}}{70.1 \text{ kN}}\right) = 34.7°.$$

Alternative solution:
Since the shape of the slide is actually rather simple, the question can also be answered without integrals. To do so, Figure 7.25 shows the isolated slide including water mass ADB. Assume that $R_{\text{h;w}}$ is the resultant of the horizontal water pressure on AD and $R_{\text{v;w}}$ is the resultant of the vertical water pressure on BD:

$$R_{\text{h;w}} = \tfrac{1}{2} \times (2.6\ \text{m})\{(14\ \text{kN/m}) + (40\ \text{kN/m})\} = 70.2\ \text{kN},$$

$$R_{\text{v;w}} = (1.5\ \text{m})(40\ \text{kN/m}) = 60\ \text{kN}.$$

Assume that G_{w} is the weight of the volume of water enclosed by ADB. We are looking at a 1-metre wide strip from the slide:

$$G_{\text{w}} = \gamma_{\text{w}} A^{(\text{ADB})}(1\ \text{m}).$$

Here $A^{(\text{ADB})}$ is the area of ADB. This is equal to the area of trapezium ADBC, reduced by the area of circle sector ABC. The area of trapezium ADBC is

$$A^{(\text{ADBC})} = \tfrac{1}{2} \times (2.6\ \text{m})\{(3.0\ \text{m}) + (1.5\ \text{m})\} = 5.85\ \text{m}^2.$$

The area of circle sector ABC, with an aperture angle of 60°, is equal to one sixth of the area of the entire circle:

$$A^{(\text{ABC})} = \frac{60^\circ}{360^\circ} \times \pi (3.0\ \text{m})^2 = 4.71\ \text{m}^2.$$

With

$$A^{(\text{ADB})} = A^{(\text{ADBC})} - A^{(\text{ABC})} = (5.85\ \text{m}^2) - (4.71\ \text{m}^2) = 1.14\ \text{m}^2$$

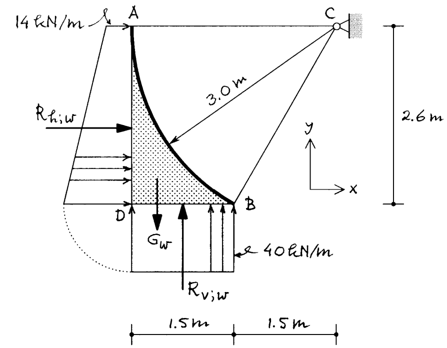


Figure 7.25 The resultant of the water pressure on the slide can also be found by looking at the forces acting on the isolated slide together with water mass ADB.

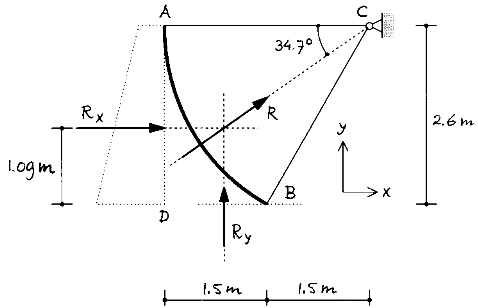


Figure 7.26 The lines of action of the horizontal and vertical component of the resulting water pressure on the slide. The horizontal component R_x is independent of the shape of the slide and can be directly found from the trapezoidal load diagram for the water pressure on the vertical AD.

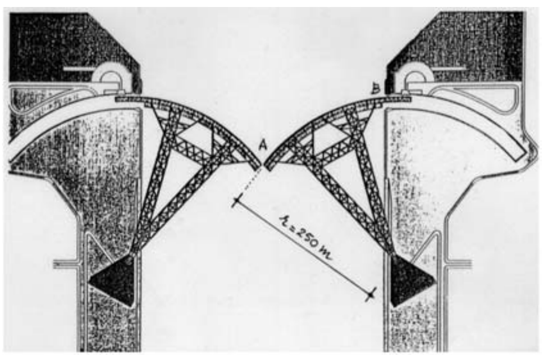


Figure 7.27 A storm barrier consisting of two sector doors with a radius $r = 250$ m.

one finds

$$G_w = \gamma_w A^{(ADB)}(1 \text{ m}) = (10 \text{ kN/m}^3)(1.14 \text{ m}^2)(1 \text{ m}) = 11.4 \text{ kN}.$$

The resulting water pressure on the slide is

$$R_x = R_{h;w} = 70.2 \text{ kN},$$

$$R_y = R_{v;w} - G_w = (60 \text{ kN}) - (11.4 \text{ kN}) = 48.6 \text{ kN}.$$

The results agree with those of the first calculation, with the exception of a minor difference in the magnitude of R_x. This is because in the alternative solution, the height of the slide ($= r \sin 60°$) was rounded off to 2.6 m.

From the alternative approach, one can conclude the following: *The resultant of the horizontal water pressure on the slide is independent of the shape of the slide and is exclusively determined by the height of the slide and the depth at which it is located under the water surface.*

Figure 7.26 shows the lines of action of R_x and R_y. The line of action of R_x can be found directly from the trapezoidal load diagram on AD.[1]

Example 3

At Hoek van Holland, near Rotterdam in the Netherlands, the Maeslantkering became operational in 1997. This storm barrier in the Nieuwe Waterweg consists of two sector doors with a radius $r = 250$ m (see Figure 7.27). The arc length of AB is 209.5 m. The door is 22.5 m in height. Figure 7.28 is a sketch of the longitudinal section of the door, with the water levels on both sides. The specific weight of water is $\gamma_w = 10.25$ kN/m^3. To simplify the question, the part of the door within the parking dock is ignored.

[1] The calculation is left to the reader. See Section 6.3.1, Example 1.

In addition, it is assumed that the water levels in front and behind the dam are present over the entire length of arc AB and that the pressure distribution on both sides is hydrostatic.

Question:
Determine the resulting horizontal water pressure on part AB of the right-hand sector door.

Solution:
With a specific weight of $\gamma_w = 10.25\ \mathrm{kN/m^3}$, the water pressure increases for each metre of depth by $10.25\ \mathrm{kN/m^2}$. At the base of the door, the water pressure on the sea-side is

$$22 \times (10.25\ \mathrm{kN/m^2}) = 225.5\ \mathrm{kN/m^2},$$

while on the river-side it is

$$14 \times (10.25\ \mathrm{kN/m^2}) = 143.5\ \mathrm{kN/m^2}.$$

Figure 7.29 shows the distribution of the horizontal water pressure on a 1-metre wide vertical strip of the door. The horizontal water pressure on the door is independent of the shape of the door.[1] The resultant of the horizontal water pressure on the 1-metre wide vertical strip is

$$\tfrac{1}{2} \times (22\ \mathrm{m})(225.5\ \mathrm{kN/m}) - \tfrac{1}{2} \times (14\ \mathrm{m})(143.5\ \mathrm{kN/m}) = 1476\ \mathrm{kN}.$$

We have shown, therefore, that per metre in the circumferential direction, the door is subject to a force of 1476 kN. In other words, the horizontal water pressure on the door consists of a uniformly distributed load

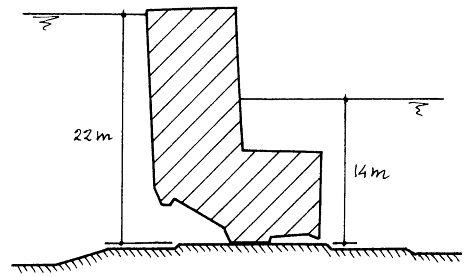


Figure 7.28 A sketch of the cross-section of the door, with the water levels on both sides.

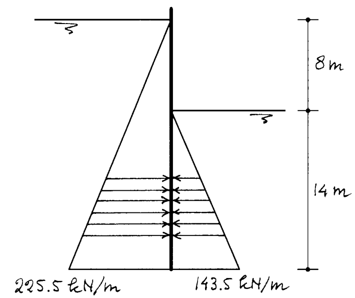


Figure 7.29 The distribution of the horizontal water pressure on a 1-metre wide vertical strip from the door. The horizontal water pressure is independent of the shape of the door.

1 See the previous example.

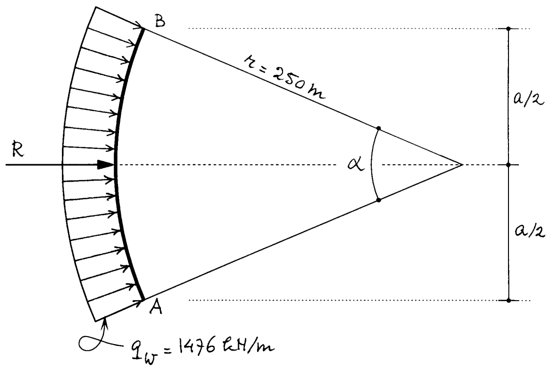


Figure 7.30 A horizontal force of 1476 kN is acting on the door per metre in the circumferential direction. In other words, the horizontal water pressure on the door consists of a uniformly distributed load $q_\mathrm{w} = 1476$ kN/m in radial direction.

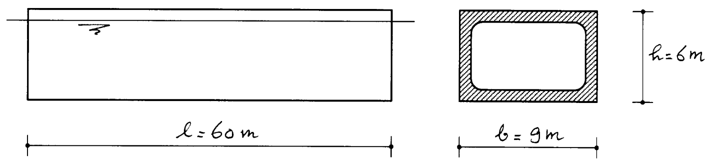


Figure 7.31 A floating element of a two-track metro tunnel, ready to be transported to the sinking site.

$q_w = 1476$ kN/m in radial direction (see Figure 7.30).

With an arc length of 209.5 m for AB and a radius of $r = 250$ m, the aperture angle α is

$$\alpha = \frac{\text{arc length AB}}{2\pi r} \cdot 360^\circ = \frac{209.5 \text{ m}}{2\pi \times (250 \text{ m})} \times 360^\circ = 48^\circ.$$

The resultant R of the horizontal water pressure on arc AB is equal to the resultant of the horizontal water pressure on chord AB (see Section 7.2, Example 2). This gives:

$$R = q_\mathrm{w} a = q_\mathrm{w} 2r \sin(\alpha/2)$$
$$= (1476 \text{ kN/m}) \times 2 \times (250 \text{ m}) \times \sin 24^\circ = 300 \text{ MN}.$$

Example 4
Figure 7.31 represents an element of a floating two-track metro tunnel, ready to be transported to its sinking site. The tunnel element is considered a rigid body.

Dimensions: length $\ell = 60$ m, width $b = 9$ m and height $h = 6$ m.
Dead weight of the tunnel element: $q_\mathrm{dw} = 524$ kN/m.
Weight of each of the temporary bulkheads: $F_\mathrm{head} = 235$ kN.
Specific weight of water: $\gamma_\mathrm{w} = 10$ kN/m^3.

Questions:

a. Determine the water pressure at the base of the tunnel element.
b. Determine the resultant of the horizontal water pressure on a bulkhead.

Solution:
a. The total dead weight R_{dw} of the tunnel element is

$$R_{\mathrm{dw}} = q_{\mathrm{dw}}\ell + 2F_{\mathrm{head}}$$
$$= (524\ \mathrm{kN/m})(60\ \mathrm{m}) + 2 \times (235\ \mathrm{kN}) = 31910\ \mathrm{kN}.$$

Figure 7.32 shows the distribution of the water pressures on the tunnel element. With a specific weight γ_{w}, the water pressure p_{w} at a depth d is

$$p_{\mathrm{w}} = \gamma_{\mathrm{w}} d.$$

The vertical water pressure on the base of the tunnel element gives an upward force $R_{\mathrm{v;w}}$:

$$R_{\mathrm{v;w}} = p_{\mathrm{w}} b\ell = \gamma_{\mathrm{w}} d b\ell.$$

The upward force is equal to the weight of the displaced water.

The tunnel element will sink in the water until the upward force is in equilibrium with the total dead weight R_{dw}:

$$R_{\mathrm{dw}} = R_{\mathrm{v;w}} = \gamma_{\mathrm{w}} d b\ell$$

so that

$$d = \frac{R_{\mathrm{dw}}}{\gamma_{\mathrm{w}} b\ell} = \frac{31910\ \mathrm{kN}}{(10\ \mathrm{kN/m^3})(9\ \mathrm{m})(60\ \mathrm{m})} = 5.91\ \mathrm{m}.$$

The water pressure at the base of the tunnel element is therefore

$$p_{\mathrm{w}} = \gamma_{\mathrm{w}} d = (10\ \mathrm{kN/m^3})(5.91\ \mathrm{m}) = 59.1\ \mathrm{kN/m^2}.$$

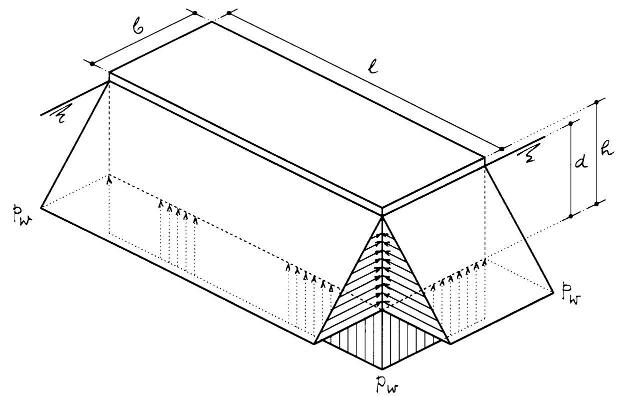


Figure 7.32 The distribution of the water pressures on the tunnel element.

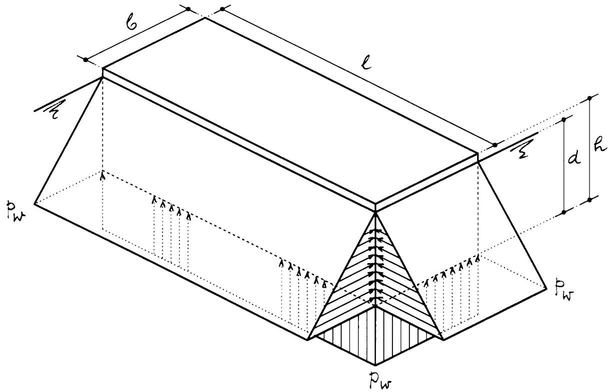


Figure 7.32 The distribution of the water pressures on the tunnel element.

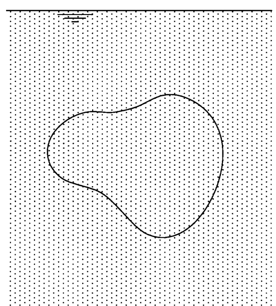


Figure 7.33 If the fluid is in equilibrium, the vertical component of the hydrostatic pressures on the outside of a contained space has to provide an upward force that equals the weight of the fluid within the contained space.

b. The resultant $R_{h;w}$ of the horizontal water pressure on a bulkhead is equal to the volume of the load diagram (see Figure 7.32):

$$R_{h;w} = \tfrac{1}{2} p_w b d = \tfrac{1}{2} \times (59.1\ \text{kN/m}^2)(9\ \text{m})(5.91\ \text{m}) = 1572\ \text{kN}.$$

In the calculation, it was noted that the vertical water pressure on the tunnel element exerts an upward force that is equal in magnitude to the weight of the displaced water. This is not a coincidence, but applies in general, regardless of the shape of the body, and is known as *Archimedes' Law*.[1] The general proof can be found below.

Take a contained space of arbitrary shape within a fluid (see Figure 7.33). If there is an equilibrium, the vertical component of the hydrostatic pressures has to provide an upward force on the outside of the contained space that equals the weight of the fluid within the contained space. The upward force does not change if the contained space is taken up by a body.

Conclusion: *A body in a fluid is exposed to an upward force that is equal to the weight of the displaced volume of fluid.*

Example 5

In the water-retaining wall in Figure 7.34 there is a circular partition of radius r. The centroid C of the partition is at a depth z_C.

Question:
Determine the resultant R of the water pressure on the partition.

Solution:
The water pressure on the partition varies linearly. At a depth z the water pressure is $\rho g z$, whereby ρ is the density of water, and g is the gravitational

---

[1] Archimedes (287–212 BC), Greek scientist from Syracuse. He addressed issues relating to integral calculus and was one of the founders of statics (equilibrium of solids) and hydrostatics (equilibrium of fluids).

field intensity (see Figure 7.35a).

For the partition, take a very narrow horizontal strip dz at depth z (see Figure 7.35b). The width of the strip is $b(z)$. The water pressure on this narrow strip is constant and equal to $\rho g z$. This contribution dR of the strip to the resulting water pressure R on the strip is

$$dR = \rho g z \cdot b(z) \cdot dz, \tag{1}$$

whereby

$$b(z) = 2r\cos\varphi, \tag{2}$$

$$z = z_C - r\cos\varphi, \tag{3}$$

$$dz = \frac{dz}{d\varphi}\,d\varphi = r\sin\varphi\,d\varphi. \tag{4}$$

Substitute (2) to (4) into (1) and we find

$$dR = 2\rho g r^2 (z_C - r\cos\varphi)(\sin\varphi)^2\,d\varphi.$$

To find the resulting water pressure R, one has to sum up the contributions of all the strips. This is done by integrating between the limits $\varphi = 0$ and $\varphi = \pi$:

$$R = 2\rho g r^2 \int_0^\pi (z_C - r\cos\varphi)(\sin\varphi)^2\,d\varphi.$$

Using the formulas in Table 7.2 we find

$$\int_0^\pi z_C(\sin\varphi)^2\,d\varphi = z_C\left[-\tfrac{1}{4}\sin 2\varphi + \tfrac{1}{2}\varphi\right]_{\varphi=0}^{\varphi=\pi} = \frac{\pi}{2}z_C$$

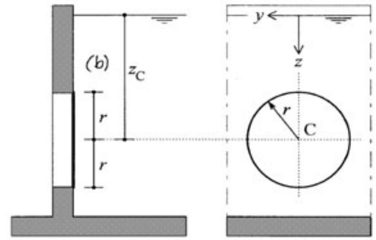


Figure 7.34 A circular partition in a water-retaining wall.

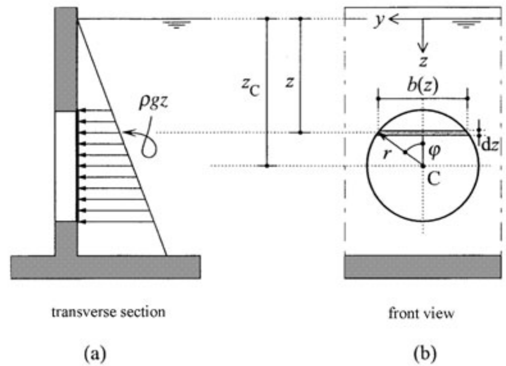


Figure 7.35 (a) The water pressure on the partition increases linearly with the depth. (b) The water pressure is constant on a small horizontal strip and is equal to $\rho g z$.

Table 7.2

$$\int (\sin\varphi)^2\,d\varphi = -\tfrac{1}{4}\sin 2\varphi + \tfrac{1}{2}\varphi$$

$$\int \cos\varphi(\sin\varphi)^2\,d\varphi = -\tfrac{1}{3}(\sin\varphi)^3$$

Table 7.2

$$\int (\sin\varphi)^2\,\mathrm{d}\varphi = -\tfrac{1}{4}\sin 2\varphi + \tfrac{1}{2}\varphi$$

$$\int \cos\varphi(\sin\varphi)^2\,\mathrm{d}\varphi = -\tfrac{1}{3}(\sin\varphi)^3$$

and

$$\int_0^\pi r\cos\varphi(\sin\varphi)^2\,\mathrm{d}\varphi = r\left[-\tfrac{1}{3}(\sin\varphi)^3\right]_{\varphi=0}^{\varphi=\pi} = 0$$

so that

$$R = \rho g z_\mathrm{C}\cdot\pi r^2.$$

Conclusion: *The resulting water pressure on the partition is equal to the water pressure at the centroid, multiplied by the area of the partition.*

Although derived for a circular partition, this characteristic is generally applicable. The proof can be provided easily if one knows that the z coordinate of the centroid C of a plane figure with area A is defined as:[1]

$$z_\mathrm{C} = \frac{\displaystyle\int_A z\,\mathrm{d}A}{A}. \tag{5}$$

The resultant of the water pressure on a small area dA at depth z is

$$\mathrm{d}R = \rho g z\cdot\mathrm{d}A.$$

The resulting water pressure is found by summing up all the contributions dR for the entire area A. This is performed by integrating with respect to the area A:

$$R = \int_A \rho g z\,\mathrm{d}A = \rho g\int_A z\,\mathrm{d}A. \tag{6}$$

---

[1] Volume 2, *Stresses, Deformations, Displacements*, addresses the definition and calculation of centroids in detail. Here, it is assumed that readers know the location of the centroid for simple plane figures.

Definition (5) gives

$$\int_A z \, \mathrm{d}A = z_\mathrm{C} A. \tag{7}$$

Substitute (7) in (6) and we find

$$R = \rho g z_\mathrm{C} \cdot A.$$

Conclusion: *The resulting water pressure R on a plane figure is equal to the water pressure $\rho g z_\mathrm{C}$ at the point of centroid* C *of the figure, multiplied by the area A of the figure.*

Note: This does not give the line of action of the resultant R which passes through the *centroid of the load diagram* (see Section 6.3.2).

7.4 Summary

The various characteristics of gas pressures and hydrostatic pressures in this chapter are summarised below.

1. Since there are no shear stresses, the compressive forces in a gas and fluid always act normal to any bounding plane (see Section 7.1).

2. In a gas and fluid, the pressure at a particular point is independent of the orientation of the plane on which the pressure acts. It is also said that, in that point, the stress is of equal magnitude in all directions (isotropic or spherical state of stress) (see Section 7.1).

3. Gas pressure is constant in a contained volume.

4. If a uniformly distributed force acts on the entire area of a body, and

normal to that body, this load forms an equilibrium system, and the resultant is zero (see Section 7.2, Example 3).

5. In a fluid, pressure increases linearly with depth (hydrostatic pressure distribution) (see Section 7.3).

6. The resultant of the hydrostatic pressure on a flat plate is equal to the pressure at the centroid of the plate, multiplied by the area of the plate (see Section 7.3, Example 5).

7. The horizontal component of the resulting hydrostatic pressure on a body is equal to the resultant of the hydrostatic pressure on the horizontal projection of the body on a vertical plane (see Section 7.3, Example 2).

8. The vertical component of the resulting hydrostatic pressure on a body is an upward force that is equal to the weight of the volume of water displaced by the body (Archimedes' Law) (see Section 7.3, Example 4).

7.5 Problems

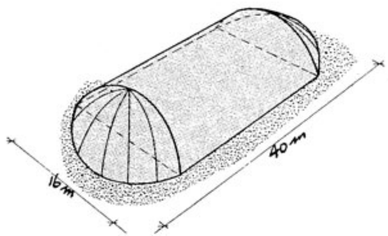
Remark: If necessary, assume that the gravitational field intensity is $g = 10$ N/kg.

Working with gas pressures (Section 7.2)

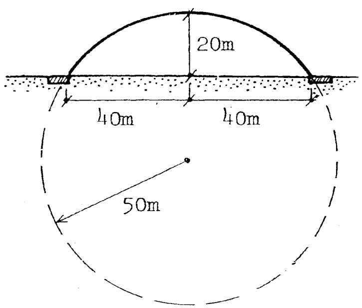
7.1 A cylindrical pneu with a length of 40 m and a width of 16 m has a semi-circular cross-section. The internal overpressure is 375 N/m^2.

Questions:

a. Determine the support reactions for the cylindrical part of the pneu.
b. Determine the membrane force in the circumferential direction at the cylindrical pneu.
c. Determine the force in the longitudinal direction of the cylindrical pneu.



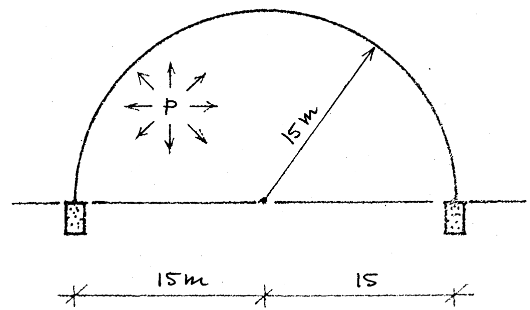
7.2 A spherical pneumatic hall is supported on a concrete ring beam. The internal overpressure is 350 N/m^2.



Questions:

a. Determine the membrane force in the pneu.
b. Determine the vertical forces that the pneu exerts on the ring beam.
c. Determine the horizontal forces that the pneu exerts on the ring beam.
d. Determine the (normal) force in the ring beam. Is it a tensile force or a compressive force?

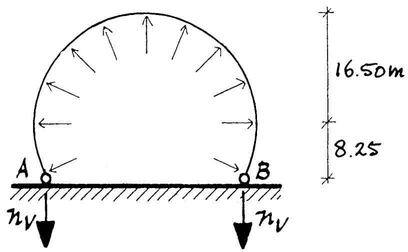
7.3 A pneumatic structure has the shape of a hemisphere with a radius of 15 m and an internal overpressure of 400 N/m^2. The forces in the pneu are transferred to a concrete ring beam. The weight of the ring beam ensures that the pneu is not lifted. The specific weight of concrete is 24 kN/m^3.



Questions:

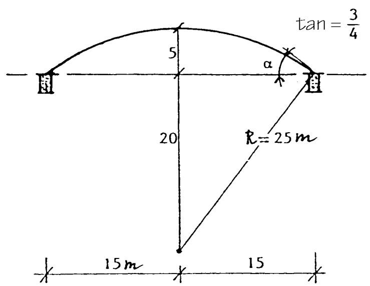
a. Determine the membrane force in the pneu.
b. Determine the normal pressure in the ring beam.
c. Determine the required diameter of the ring beam to prevent the pneu from lifting.

7.4 In a spherical pneu with radius 16.5 m there is an overpressure of 350 N/m^2. The horizontal support reactions are transferred by a ring belt applied around the pneu.



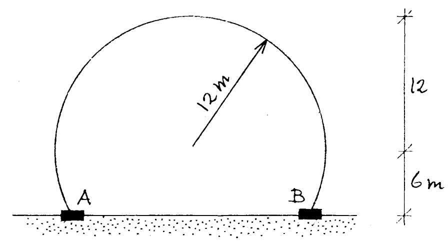
Questions:
a. Determine the membrane force in the pneu.
b. Determine the vertical support reactions n_V.
c. Determine the force in the ring belt.

7.5 A pneumatic structure, with a spherical shape, is connected to a circular ring beam that rests freely on the ground. The overpressure in the pneu is 400 N/m^2. The dead weight of the pneu can be neglected.



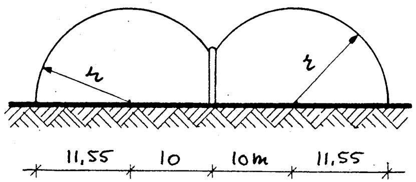
Questions:
a. Determine the membrane force in the pneu.
b. How large must the weight per metre of the ring beam be to prevent lifting?
c. Determine the normal force in the ring beam. Is this a tensile force or a compressive force?

7.6 A (long) cylindrical pneu with an internal pressure of 400 N/m^2 has a circular cross-section as indicated in the figure with a radius of 12 m. The weight of the concrete beams at A and B has to prevent the pneu from lifting. Tie-rods have been applied between the beams A and B every 2.5 m. The specific weight of concrete is 24 kN/m^3.



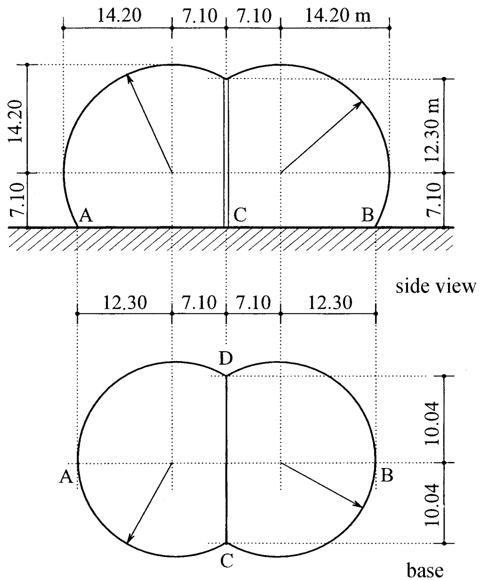
Questions:
a. Determine the membrane force (in the circumferential direction) in the pneu.
b. Determine the required cross-section of the beams to prevent lifting.
c. Determine the force in a tie-rod between A and B.

7.7 Two spherical pneus with radius $r = 11.55$ m have been placed adjacent to one another and joined. A cable has been placed over the pneus at the connection. The overpressure in the pneu is 400 N/m^2.



Questions:
a. Determine the membrane forces in the pneu.
b. Determine the forces that the pneus exert on the cable.
c. Determine the tensile force in the cable.

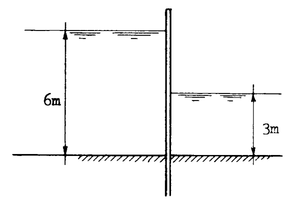
7.8 Two spherical pneus with a radius of 14.20 m have been placed adjacent to one another and joined. A cable has been laid over the pneus at the line of joining. The overpressure in the pneu is 410 N/m^2. The pneus are attached to concrete foundation beams.



Questions:
a. Determine the membrane forces in the pneu.
b. Determine the vertical support reactions for the pneu.
c. Determine the (normal) force in the ring beams. Are they tensile forces or compressive forces?
d. Determine the forces that the pneus exert on the cable.
e. Determine the tensile force in the cable.
f. Determine the (normal) force in beam CD. Is this a tensile force or a compressive force?

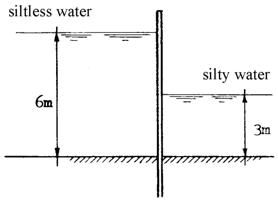
Working with hydrostatic pressures (Section 7.3)

7.9 A steel sheet-pile wall is fixed in a concrete floor. There is 6 m of water against one side of the wall, and 3 m on the other. Mass density of water: 1000 kg/m^3.



Questions:
a. Draw the distribution of the water pressure on the wall.
b. Determine the horizontal support reaction per metre wall.
c. Determine the fixed-end moment per metre wall.

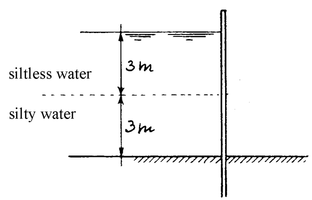
7.10 A steel sheet-pile wall is fixed in a concrete floor. There is 6 m of water against one side of the wall with a mass density of 1000 kg/m^3. On the other side, there is 3 m of water with, due to a high silt content, a mass density of 1400 kg/m^3.



Questions:
a. Draw the distribution of the water pressure on the wall.
b. Determine the horizontal support reaction per metre wall.
c. Determine the fixed-end moment per metre wall.

7.11 Like the previous question, but now with the silty water to the left, and the siltless water to the right of the wall. The mass density of the silty water is 1200 kg/m^3, and that without silt is 1000 kg/m^3.

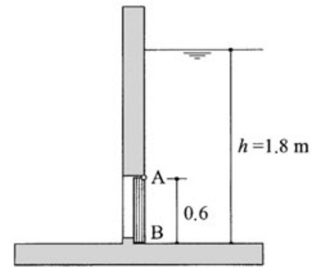
7.12 A steel sheet-piling is fixed in a concrete floor, and is retaining 6 m of water. The mass density of the upper 3 metres is 1000 kg/m^3. The lower three metres have a mass density of 1400 kg/m^3 as a result of the silt present.



Questions:
a. Draw the distribution of the water pressure on the sheet-piling.
b. Determine the horizontal support reaction per metre of sheet-piling.
c. Determine the fixed-end moment per metre sheet-piling.

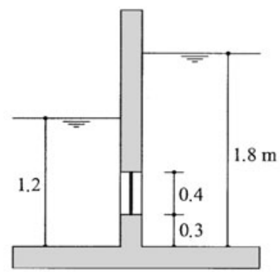
7.13 A water-retaining wall contains a square flap that is hinged at A, and supported by a sill at B.

Questions:
a. Draw the distribution of the water pressure against the wall.
b. Determine the resultant of the water pressure on the flap.
c. Determine the line of action of this resultant.
d. Determine the support reactions at A and B.



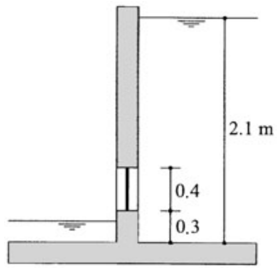
7.14 What is the water depth *h* if the total water pressure on the square flap from the previous question is 3.6 kN?

7.15 A connection between two reservoirs is sealed by means of a circular valve.



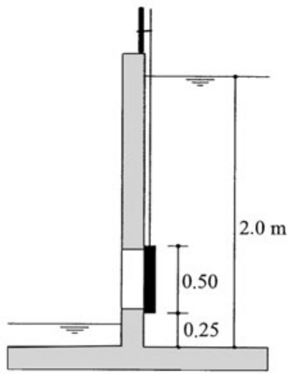
Questions:
a. Draw the distribution of the water pressure on both sides of the wall.
b. Draw the resulting water pressure on the wall.
c. Determine the resulting water pressure on the sealing valve.

7.16 A connection between two reservoirs is closed by means of a circular valve.



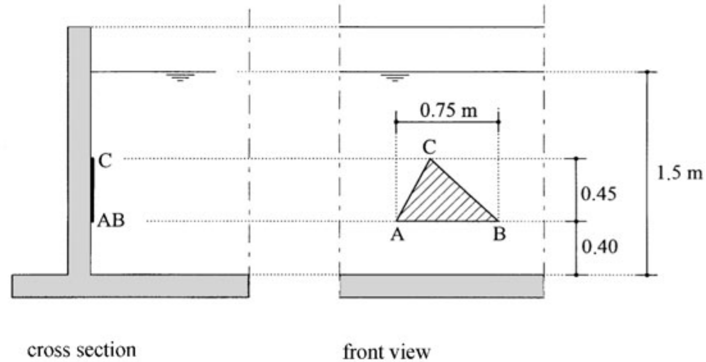
Question:
Determine the resulting water pressure on the valve.

7.17 An opening in a water-retaining wall is closed by means of a slide. The slide is 0.5 m high and 0.4 m wide.



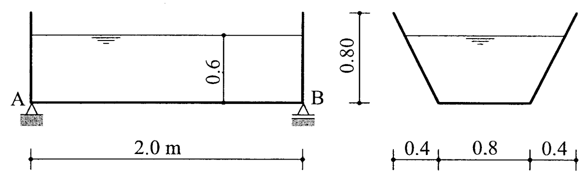
Question:
Determine the resulting water pressures on the slide.

7.18 You are given a water-retaining wall with the triangular area ABC as shown.



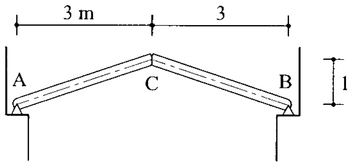
Questions:
a. Determine the resulting water pressure on triangle ABC
b. Determine the resulting water pressure on triangle ABC if base AB is above top C.

7.19 You are given the longitudinal section and the cross-section of a water trough.

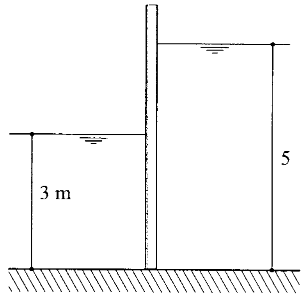


Questions:
a. Determine the support reactions at A and B.
b. Determine the resultant of the water pressure on an end-partition.

7.20 You are given a wooden mitre gate in a small lock. The depth of the water outside the lock is 5 m and 3 m inside.



top view

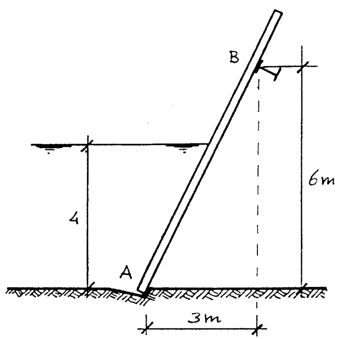


cross-section

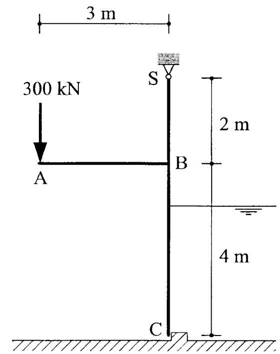
Questions:
a. Determine the resulting water pressures on door AC.
b. Determine the forces that the doors at A and B exert on the lock walls.
c. Determine the force that the doors at C exert on one another.

7.21 A barrage is made up of partitions that at base A are resting against a groove and at top B against an I-section. The I-section beam is supported in the walls of the barrage.

Questions:
a. Draw the distribution of the water pressure on the walls.
b. Determine the support reactions at A and B for a partition with a width of 1.5 m.

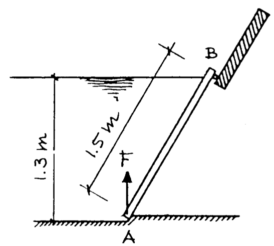


7.22 A lock door is supported by a hinge at S and is pressed against sill C by a force of 300 kN at A. There is only water to the right of the door. The lock door has a width of 4 m. The weight of the door can be ignored.



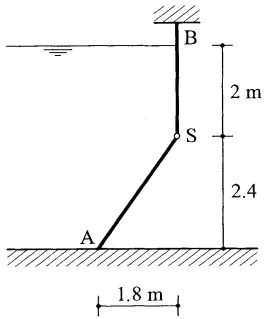
Question:
At which water level will the door open?

7.23 A barrage contains a flap AB with a width of 1 m. The flap is resting in a groove at A and is supported by a hinge at B.



Question:
Determine the vertical force F required to open flap AB.

7.24 In a barrage, flap AS is resting at A on an entirely flat base and is connected in a hinge at S with SB. The flap is 2.5 m wide. There is only water to the left of the barrage.



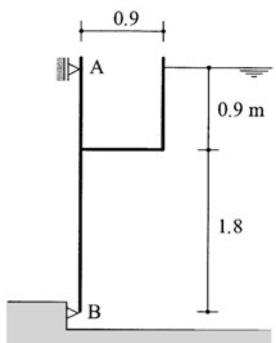
Questions:

a. Determine the (total) support reaction at A.
b. Determine the horizontal and vertical component of the hinge force at S. Also clearly indicate the directions.

7.25 A barrage with the shape shown is located in a 1-m wide channel.

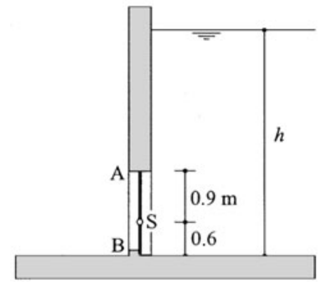
Questions:

a. Draw the distribution of the water pressure on the barrage.
b. Determine the support reactions at A and B.

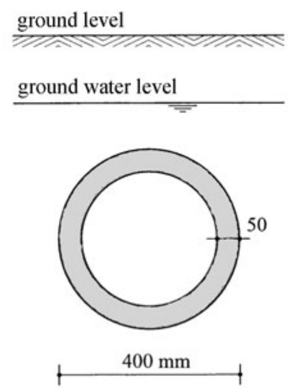


7.26 A barrage contains a flap AB that can rotate about a hinge at S.

Question:
Determine the water level h at which the flap will open.

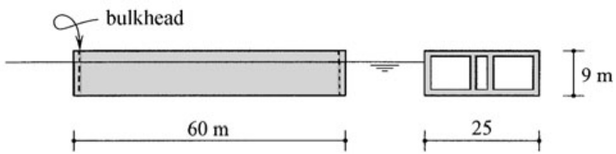


7.27 A concrete sewer pipe with Ø400 mm and a wall thickness of 50 mm is located in an area with sandy soil and a high water level. The pipe is located below ground water level. The specific weight of concrete is 24 kN/m^3.



Question:
Determine whether there is a danger of lifting if the weight of the soil above the pipe is not taken into account as a load.

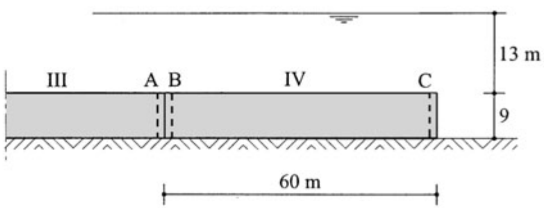
7.28 A concrete tunnel element is afloat, waiting to be transported to the sinking site. The element is 60 m long and has a cross-section of 25×9 m^2. The outer walls of the tunnel are 1.20 m thick, the inner walls are 0.75 m thick. The two temporary bulkheads each have a weight of 1320 kN. The specific weight of concrete is 25 kN/m^3, while that of water is 10 kN/m^3.



Questions:

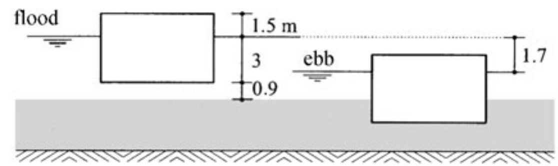
a. How much is the tunnel above water level?
b. How many litres of water have to be used to fill the ballast tanks to sink the tunnel element?

7.29 Once tunnel element IV has been sunk and placed at the correct level, the space between the bulkheads A and B is pumped empty. The tunnel has a rectangular cross-section of 25×9 m^2. The specific weight of the water is 10 kN/m^3.



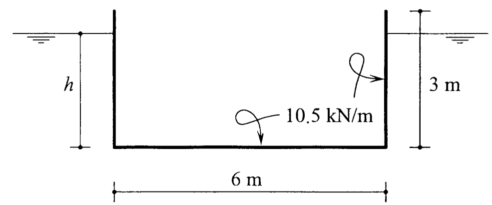
Question:
Determine the force that tunnel element IV exerts on element III?

7.30 At high tide, a barge with rectangular cross-section is 1.5 m above the water and 3.0 m below the water. At low tide, the water level is 1.7 m less, and the base of the barge ends up in a muddy layer of sediment. The muddy layer has a mass density of 1400 kg/m^3 and behaves like a liquid. The water above the muddy layer has a mass density of 1050 kg/m^3.



Question:
How much does the barge stick out of the water at low tide?

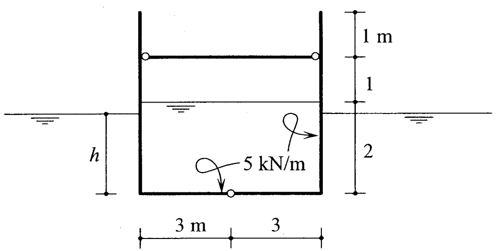
7.31 A 1-metre strip has been isolated from the length of a long barge and is modelled as a line element. The dead weight of the isolated line element (walls and base) is 10.5 kN/m.



Questions:

a. Determine depth h of the barge.
b. Draw the distribution of the water pressure on the walls and the base.
c. Isolate the base of the barge and draw all the forces acting on it.

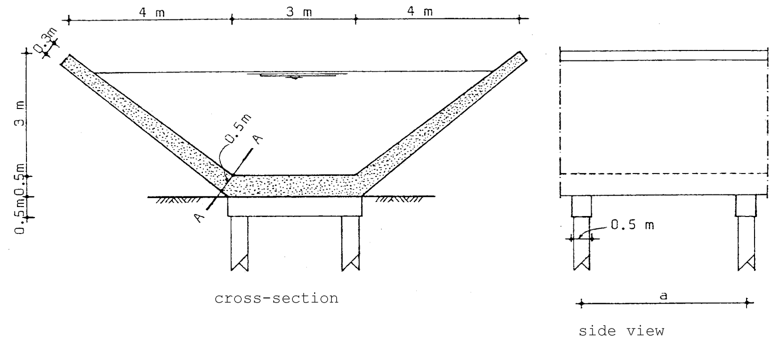
7.32 In the middle of the base of a long barge there is a continuous hinge. There are bars between the walls of the barge three metres above its base. The centre to centre distance of these bars is 2.5 m. The barge is filled with petroleum up to 2 m. The dead weight of the petroleum is 7.5 kN/m^3. A 1-metre strip has been isolated in the length of the barge and modelled as a line element. The dead weight of the isolated line element (walls and base) is 5 kN/m.



Questions:

a. Determine the depth h of the barge.
b. Draw the distribution of the hydrostatic pressures on walls and base.
c. Isolate a wall and draw all the forces acting on it.
d. Determine the force in a bar. Is it a tensile force or a compressive force?

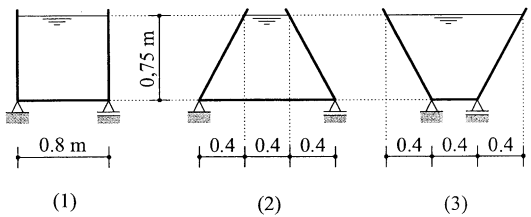
7.33 A sketch is shown with a number of estimated thicknesses of a long concrete channel for the transport of water. The water can rise to the upper edge of the channel The specific weight of concrete is 24 kN/m^3.



Questions:

a. Determine the centre to centre distance a of the cross-beams to an accuracy of 0.1 m, such that the piles are not loaded by more than 600 kN.
b. Determine the section forces (interaction forces) per metre in cross-section A–A of the channel.

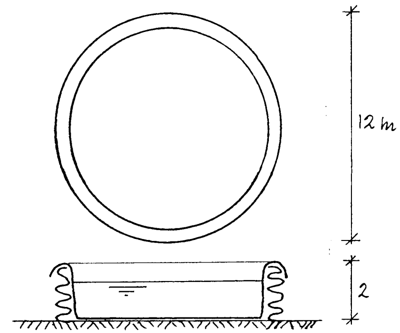
7.34: 1–3 A 1-metre strip has been isolated from a channel filled with water and is modelled as a line element. There are three different shapes of channel.



Questions:

a. Determine the support reactions.
b. Draw the distribution of the water pressure on the walls and the base.
c. Isolate the base and draw all the forces acting on it.

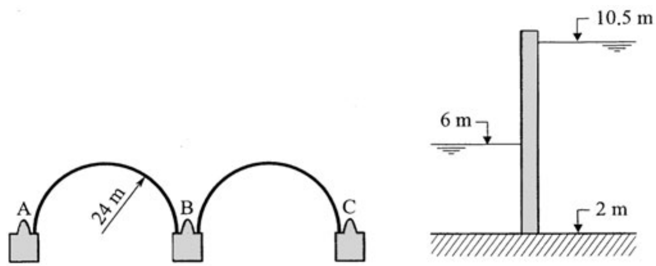
7.35 At a nursery, an open tank is being built to store water. The round tank has a diameter of 12 m and a height of 2 m. The wall of the tank consists of corrugated steel plates that are connected by means of bolts. The water retention is achieved by means of a membrane.



Questions:

a. Determine the (normal) force in the circumferential direction if the tank is three-quarters full. Is this a tensile force or a compressive force?
b. Determine the (normal) force in the circumferential direction if the tank is filled to the top.

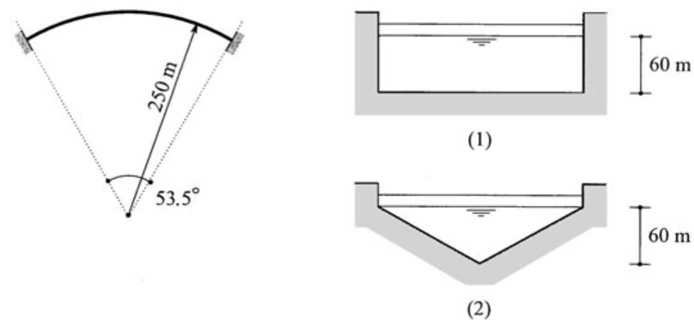
7.36 You are given two curved weirs. The weirs are a semi-circle and have a radius of 24 m. The bulging sides of the weirs are pointing downstream. The water levels are shown in the figure.



Questions:

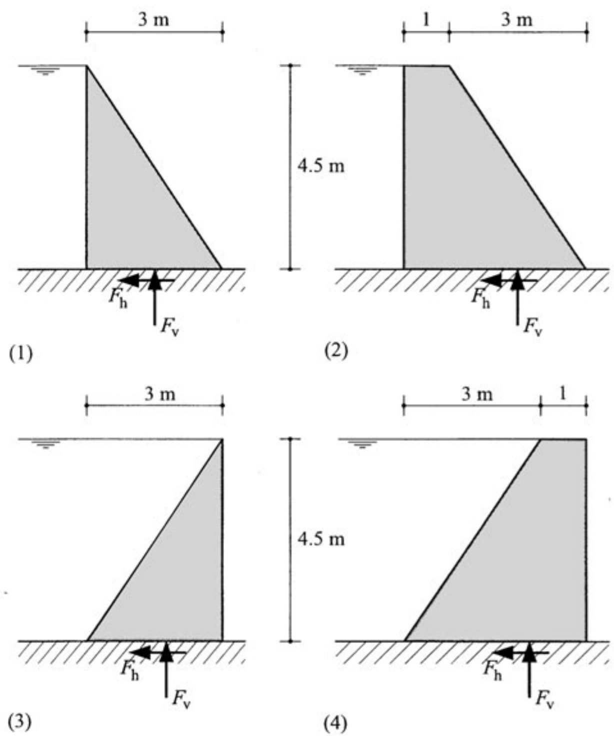
a. Determine the (normal) force in the circumferential direction of the weir. Is this a compressive force or a tensile force?
b. Determine the total force that the weirs AB and BC exert on pier B.

7.37: 1–2 Sixty metres of water is standing against a circular storage dam with a radius of 250 m and an aperture angle of 53.5°. In case (1), the transverse section of the closed valley is rectangular, while in case (2) it is an isosceles triangle.



Question:
Determine the horizontal resultant of the water pressure on the dam.

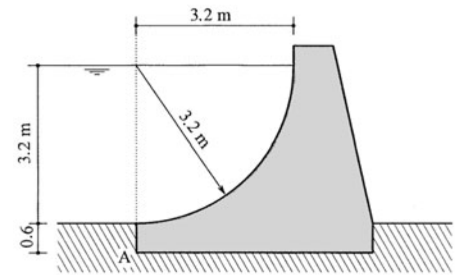
7.38: 1–4 A concrete wall is retaining 4.5 m of water. The support reactions F_h and F_v exerted by the foundation provide equilibrium. The specific weight of concrete is 24 kN/m^3 and that of water is 10 kN/m^3.



Questions (for 1 m retaining wall):

a. Determine support reaction F_h.
b. Determine support reaction F_v.
c. Determine the line of action of F_v.

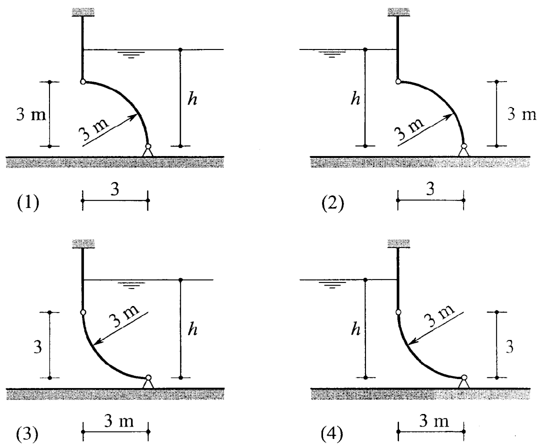
7.39 The cross-section of a retaining wall is circular on the water-retaining side.



Questions (for 1 m retaining wall):

a. Determine the vertical component of the water pressure.
b. Determine the horizontal component of the water pressure.
c. At which distance from A does the line of action of the resulting water pressure intersect the base of the retaining wall?

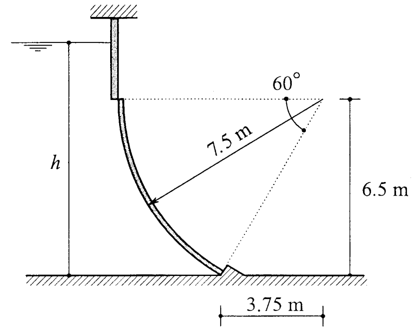
7.40: 1–4 A 5-metre wide dam is retaining a water level h. The dam is composed of a flat vertical wall and a circular cylindrical wall. The cross-section of the cylindrical wall is a quarter-circle with a radius of 3 m.



Question:
Determine the resultant of the water pressure on the 5-metre long circular cylindrical wall, with its line of action, when:

a. $h = 4.80$ m.
b. $h = 3.00$ m.
c. $h = 1.50$ m.

7.41 A circular cylindrical slide with a length of 20 m, a radius of 7.5 m and an aperture angle of 60°, is retaining a water level h.



Question:
Determine the resultant of the water pressure on the slide, with its line of action, if:

a. $h = 8.0$ m.
b. $h = 6.5$ m.
c. $h = 4.0$ m.

8 Earth Pressures

All civil engineering structures eventually come into contact with the soil for their foundations. For some structures, this contact is significant, such as for retaining walls, diaphragm walls, lock chambers, culverts, open tunnel sections, and tunnels. Here, an important part of the load consists of earth pressures.

When calculating the stresses in soil, we assume that the grains in the soil form a skeleton. The *grain skeleton* can transfer forces via the contact points between the grains. Section 8.1 addresses the concept *grain pressure*.

In dry soil, the forces are transferred by means of the grain skeleton. In wet soil, the water also plays a role. The water pressure combined with the grain pressure is the *earth pressure*. Section 8.2 concerns the calculation of vertical earth pressures.

In contrast to (stationary) gases and fluids, shear stresses do occur in soil. The shear stresses are transferred by the grain skeleton. The horizontal earth pressure is therefore in general not equal to the vertical earth pressure (there is no isotropic stress state). Section 8.3 addresses the determination of horizontal earth pressures.

The active and passive grain pressure are the two extreme values between which the horizontal grain pressure can vary. These limits occur when a slide plane develops and a soil mass slides (*active grain pressure*) or is

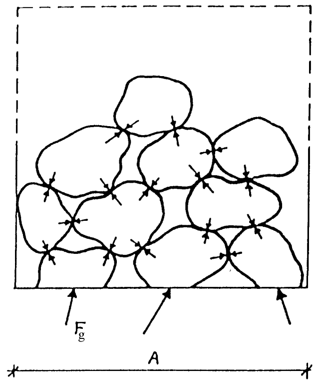


Figure 8.1 A small part of the grain skeleton in dry soil. The forces F_g are required to keep the grains on the section plane in equilibrium.

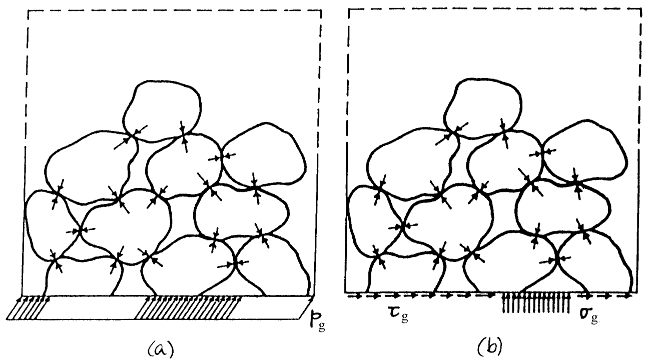


Figure 8.2 (a) The grain stress p_g is defined as the average grain force over a small area that is large compared to the grain diameter. (b) The grain stress p_g has a normal stress component σ_g that acts normal to the section plane and a shear stress component τ_g that acts along the section plane.

upset (*passive grain pressure*). This is covered in Sections 8.3.1 and 8.3.2 respectively.

An intermediate value is the *neutral grain pressure*, the horizontal grain pressure on an immovable rigid wall. The neutral grain pressure is covered in more detail in Section 8.3.3.

The explanations in this chapter remain highly elementary. Soil is a complex matter; its properties are so different from those of regular solids that soil is not part of applied mechanics, but has its own discipline: *soil mechanics*. Please therefore refer to text on soil mechanics for more detailed information.

8.1 Stresses in soil

Soil can be described as a collection of non-cohesive or mildly-cohesive, generally small particles of mineral or organic origin, in which the voids between the particles is entirely or partially filled with water or air. The solid particles are called *grains*. This definition can be taken literally for *sand*, but not too literally for *clay* or *peat* for example.

When calculating stresses in the soil, we assume that the grains in the soil form a skeleton. The *grain skeleton* can transfer forces via the contact areas between the grains. In dry soil, the forces are transferred via the grain skeleton; in wet soil, the water also plays a role.

To start with, we will look at dry soil. In Figure 8.1, part of the grain skeleton has been isolated from dry soil. The section plane, with area A cuts a number of grains. The forces F_g are required to keep the grains on the section, and thus the isolated grain skeleton in equilibrium. The *average compressive grain force* over a *small area* A, which is large compared to the grain diameter, is defined as the *grain pressure* p_g (see Figure 8.2a):

$$\vec{p}_{g} = \frac{\sum \vec{F}_{g}}{A}. \tag{1}$$

The grain pressure p_g has a normal stress component σ_g and a shear stress component τ_g. The *normal stress* σ_g acts normal to the section plane, while the *shear stress* τ_g acts in the section plane (see Figure 8.2b). For the sake of clarity, neither of the stresses have been shown in the figure along the entire length of the section.

Note that the convention from *soil mechanics* has been adopted, in which *compressive (normal) stresses* in the soil are *positive*.

The grain pressure p_g and the stress components σ_g and τ_g are a measure for the forces that are transferred via the grain skeleton.

In soil that is fully saturated with water, *water pressure* acts on the grains, in addition to the forces from the grain skeleton. If the contact surfaces between the grains are modelled as points, the water pressure acts on the entire surface of the grain. If the grains are so small that the difference between the water pressures at the top and at the base of the grain is negligible, it is said that the grain is subjected to an all-round water pressure σ_w. This load, as shown in Figure 8.3 for a single grain, forms an *equilibrium system*, regardless of the shape of the grain; see Section 7.2, Example 3 (Figure 7.14 and beyond).

In Figure 8.4, part of the grain skeleton has been isolated from the water-saturated soil. Here too, the section cuts a number of grains. In the figure, only the influence of the water pressure is shown. The forces in the grain skeleton that lead to grain pressure p_g have been omitted.

The cut grains on the horizontal section plane remain in equilibrium as long as the water pressure σ_w also acts *in the section plane on the grains*, as shown in Figure 8.4. Of course, the water pressure σ_w also acts in the (horizontal) section plane in the pores *between the grains*. If the water is stationary and all the pores are linked to one another, σ_w is the hydrostatic

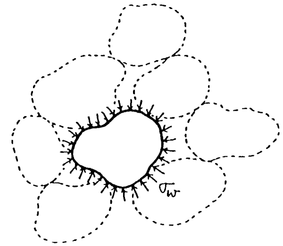


Figure 8.3 In soil saturated with water, an all-round water pressure σ_w acts on the grain. This load forms an equilibrium system, regardless of the shape of the grain.

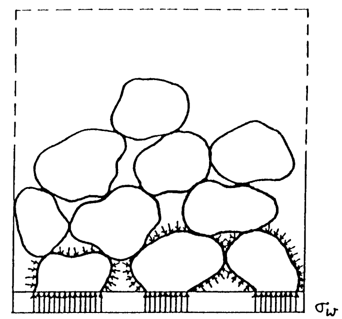


Figure 8.4 The water pressure on the grain skeleton in soil saturated with water. The cut grains on the section plane remain in equilibrium if the water pressure σ_w also acts on the grains in the section plane. Since the water is also present in the pores between the grains, there is an equal hydrostatic pressure σ_w across the entire (horizontal) section plane.

water pressure. This leads to the following conclusion:

Due to the water, the same hydrostatic water pressure σ_w acts over the entire area of the (horizontal) section plane.

The water pressure σ_w combined with grain pressure p_g, which represents a measure for the forces in the grain skeleton, is called the *earth pressure* p_e. The earth pressure has a normal stress component σ_e and a shear stress component τ_e:

$$\sigma_e = \sigma_g + \sigma_w, \tag{2}$$

$$\tau_e = \tau_g. \tag{3}$$

Note that the shear stresses in soil are exclusively transferred by the grain skeleton!

8.2 Vertical earth pressures

In an extensive area with a horizontal ground level (GL), the vertical earth pressures and grain pressures can be deduced from the vertical equilibrium of a soil prism (see Figure 8.5a).

No shear stresses act on the vertical sides of the soil prism. This can be deduced from symmetry considerations. In an unbounded region, each vertical section is a section of mirror symmetry. Mirror symmetry implies that if the shear stress is acting upward on the left-hand side, the shear stress must at the same time be acting upward on the right-hand side, as in Figure 8.6a. According to the principle of action and reaction, the shear stresses must however be acting in opposite directions, as in Figure 8.6b. Both conditions can be met only if these shear stresses are zero.

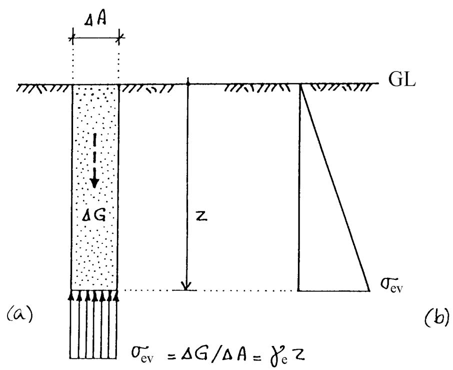


Figure 8.5 (a) The vertical earth pressure can be derived from the vertical force equilibrium of a vertical soil prism. (b) The vertical earth pressure $\sigma_{e;v}$ increases linearly with depth z.

The weight ΔG of the soil prism in Figure 8.5a, with a height z and cross-section ΔA is

$$\Delta G = \gamma_e z \Delta A. \tag{4}$$

Here, γ_e is the specific weight of the soil, including any water that may be present. For the vertical earth pressure at a depth z one finds

$$\sigma_{e;v} = \frac{\Delta G}{\Delta A} = \gamma_e z. \tag{5}$$

The vertical earth pressure increases linearly with depth z (see Figure 8.5b).

If the soil is *entirely dry*, the weight is carried entirely by the grain skeleton, and the vertical grain pressure is equal to the vertical earth pressure:

$$\sigma_{g;v} = \sigma_{e;v}. \tag{6}$$

If the soil is *entirely saturated with water* up to the ground level, the vertical grain pressure is no longer equal to the vertical earth pressure:

$$\sigma_{g;v} \neq \sigma_{e;v}. \tag{7}$$

In that case, the vertical grain pressure $\sigma_{g;v}$ is deduced from the vertical earth pressure $\sigma_{e;v}$ by reducing it by the water pressure σ_w:

$$\sigma_{g;v} = \sigma_{e;v} - \sigma_w.$$

If the water is stationary and all the pores are linked to one another, the water pressure increases hydrostatically from zero at ground level to $\gamma_w z$ at a depth z:

$$\sigma_w = \gamma_w z$$

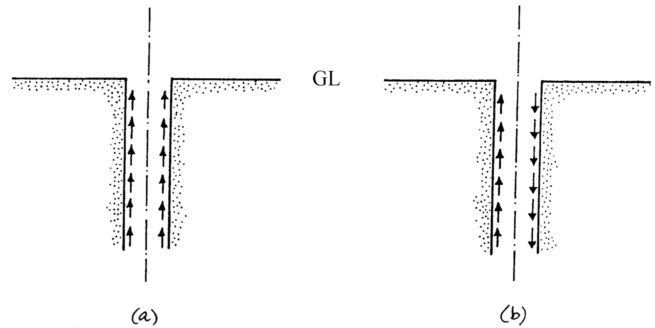


Figure 8.6 There are no shear stresses acting in the vertical planes of the soil prism. This can be deduced from the condition of mirror symmetry in combination with the principle of action and reaction. (a) Mirror symmetry means that if the shear stress is acting upwards on the left-hand side of a vertical section, it must also act in an upward direction on the right-hand side. (b) On the basis of the principle of action and reaction, the shear stresses on the left-hand and right-hand sides must have opposite directions. Both conditions can be met concurrently only if these shear stresses are zero.

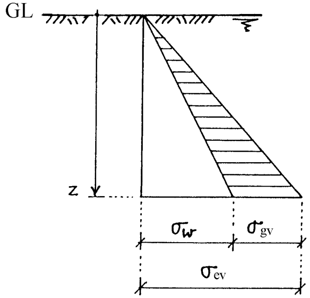


Figure 8.7 In fully saturated soil, the vertical grain pressure $\sigma_{g;v}$ is found from the vertical earth pressure $\sigma_{e;v}$ by subtracting it by the water pressure σ_w. The vertical grain pressure is shown by means of a hatching.

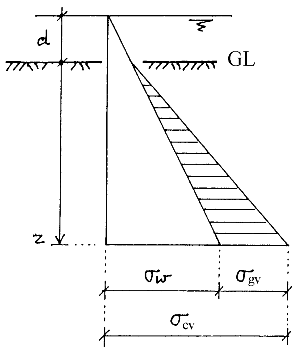


Figure 8.8 If in fully saturated soil the water level is increased to above ground level, this influences the vertical earth pressure $\sigma_{e;v}$ but not the vertical grain pressure $\sigma_{g;v}$.

so that

$$\sigma_{g;v} = \sigma_{e;v} - \sigma_w = \gamma_e z - \gamma_w z = (\gamma_e - \gamma_w)z. \tag{8}$$

In Figure 8.7, the contribution of the vertical grain pressure $\sigma_{g;v}$ in the vertical earth pressure is shown by means of a hatching.

If, with fully saturated soil, the water level is raised to above the ground level, this influences the vertical earth pressures, but not the vertical grain pressures (see Figure 8.8):

$$\begin{aligned} \sigma_{e;v} &= \gamma_w d + \gamma_e z, \\ \sigma_{g;v} &= \sigma_{e;v} - \sigma_w = \gamma_w d + \gamma_e z - \gamma_w(d+z) = (\gamma_e - \gamma_w)z. \end{aligned} \tag{9}$$

An extensive, uniformly distributed load p on the ground level increases both the vertical earth pressure and the vertical grain pressure by an amount p. The water pressure does not change (see Figure 8.9):

$$\begin{aligned} \sigma_{e;v} &= p + \gamma_e z, \\ \sigma_{g;v} &= \sigma_{e;v} - \sigma_w = p + (\gamma_e - \gamma_w)z. \end{aligned} \tag{10}$$

Example
Figure 8.10a shows a package with three soil layers. In the figure, the specific weight of the soil is given for each layer. The water level is 3 metres below ground level. A uniformly distributed load of 15 kN/m^2 is acting on the ground level.

Question:
Determine the distribution of the vertical earth pressure and grain pressure.

Solution (units in kN and m):
First the vertical earth pressure $\sigma_{e;v} = p + \gamma_e z$ is determined. Second, we determine the water pressure $\sigma_w = \gamma_w z$. The vertical grain pressure is found by subtracting the water pressure from the earth vertical pressure.

Table 8.1

z (m)	$\sigma_{e;v}$ (kN/m^2)	σ_w (kN/m^2)	$\sigma_{g;v} = \sigma_{e;v} - \sigma_w$ (kN/m^2)
0	15	0	15
3	$15 + 16 \times 3 = 63$	0	63
8	$63 + 20 \times 5 = 163$	$10 \times 5 = 50$	113
10	$163 + 18 \times 2 = 199$	$10 \times 7 = 70$	129

The calculation is shown in Table 8.1.

Figure 8.10b shows the distribution of the vertical earth pressure, split according to grain pressure (hatched) and water pressure (not hatched). If one is only interested in the distribution of the vertical grain pressure, one can also use the expression (10) given above:

$$\sigma_{g;v} = p + (\gamma_e - \gamma_w)z.$$

The calculation is summarised in Table 8.2.

Table 8.2

z (m)	$\gamma_e - \gamma_w$ (kN/m^3)	$\sigma_{g;v}$ (kN/m^2)
0	–	$p = 15$
3	$16 - 0 = 16$	$15 + 16 \times 3 = 63$
8	$20 - 10 = 10$	$63 + 10 \times 5 = 113$
10	$18 - 10 = 8$	$113 + 8 \times 2 = 129$

Figure 8.10c shows the distribution of the vertical grain pressure separately.

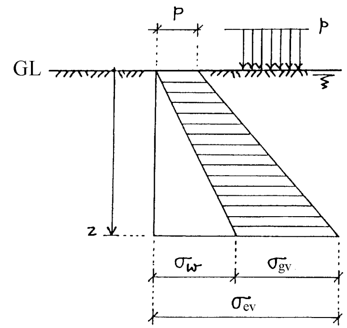


Figure 8.9 An extensive, uniformly distributed terrain load p on the ground level increases both the vertical earth pressure and the vertical grain pressure by an amount p. The water pressure does not change.

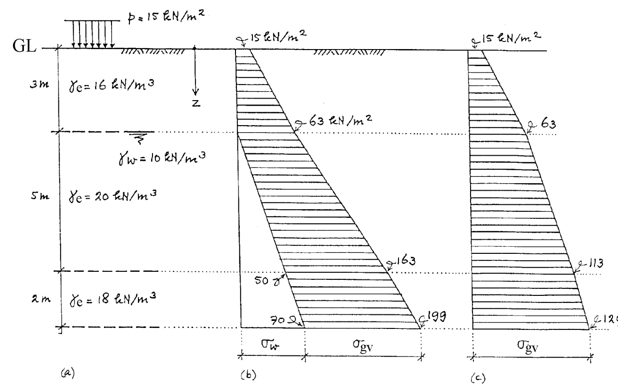


Figure 8.10 (a) The specific weight of the soil is given for three soil layers. The water level is 3 m below ground level. A uniformly distributed load of 15 kN/m^2 is acting on the ground level. (b) The distribution of the vertical earth pressure, split into grain pressure (hatched) and water pressure (not hatched). (c) The distribution of the vertical grain pressure shown separately.

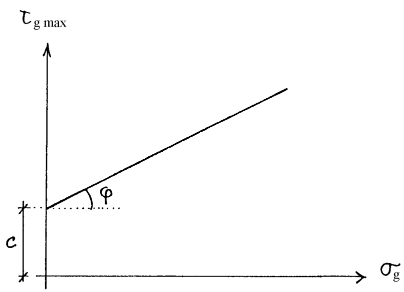


Figure 8.11 The maximum shear stress $\tau_{g;max}$ that the grain skeleton can transfer in a section is dependent on the normal stress σ_g in that section according to $\tau_{g;max} = c + \sigma_g \tan\varphi$. Here c is the cohesion and φ is the angle of internal friction.

Comment: In this section, the following were not taken into account:

- Deviation in the hydrostatic distribution of the water pressure due to the presence of poorly-permeable layers;
- The influence of groundwater currents;
- The influence of time on a range of phenomena (time effects).

For these issues, please refer to the specialist field of soil mechanics.

8.3 Horizontal earth pressures

In contrast to (stationary) gases and fluids, shear stresses can occur in soil. The shear stresses are transferred by the grain skeleton. The horizontal earth pressure is therefore in general not equal to the vertical earth pressure (there is no isotropic state of stress). To simplify the problem, we will for the moment consider only *dry soil*. The grain pressures then remain equal to the earth pressures.

It is often not easy to calculate the horizontal grain pressure. It is possible, however, to indicate the limits to which it is bound. These limits are determined by the maximum shear stress that the grain skeleton can transfer. The *maximum shear stress* $\tau_{g;max}$ in a section depends on the normal stress σ_g in that section:

$$\tau_{g;max} = c + \sigma_g \tan\varphi.$$

Here c is the *cohesion*[1] and φ is the *angle of internal friction*. This relationship is shown in Figure 8.11.

---

[1] Cohesion is the resistance to sliding resulting from a certain bond between the soil particles because of sticking and tangling, the influence of capillary water, and/or the hooking of particles with an irregular shape (hook resistance).

To keep matters simple, we will consider only grainy matter, such as sand, for which cohesion c is practically zero, so that

$$\tau_{g;\max} = \sigma_g \tan\varphi.$$

In order to find the extreme values of the horizontal grain pressure, a triangular slice OPQ is isolated from the soil (see Figure 8.12).

No shear stresses are acting on the vertical boundaries (section OP and the front and back of the slice). This was shown by means of symmetry in Section 8.2.

Suppose grain stresses σ_g and τ_g are acting on the oblique section PQ, and horizontal grain pressure $\sigma_{g;h}$ is acting in the vertical section OP. Vertical grain pressure $\sigma_{g;v}$ is acting on the horizontal section OQ. From the moment equilibrium in the plane of the drawing, about the middle of PQ, it follows that no shear stresses can be acting in the horizontal section OQ. Check it!

If shear stress τ_g on the oblique section PQ has reached its maximum $\tau_{g;\max}$, the soil mass will *slide*. Here one can distinguish two situations, depending on the direction in which the soil slides, and therefore the direction of the shear stress $\tau_{g;\max}$. Figure 8.13 shows both cases; the situation after sliding is shown by a dashed line.

The soil element in Figure 8.13a is *sliding* (moving downwards); the shear stress in the *slide plane* PQ is acting upwards. In Figure 8.13b, the soil element is *upset* (moving upwards) and the shear stress in the *slide plane* PQ is directed downwards. For both cases with given $\sigma_{g;v}$ we can look for the angle α for which the horizontal grain pressure $\sigma_{g;h}$ is extreme.

It is conventional to express the horizontal grain pressure in the vertical grain pressure using a coefficient K:

$$\sigma_{g;h} = K\sigma_{g;v}.$$

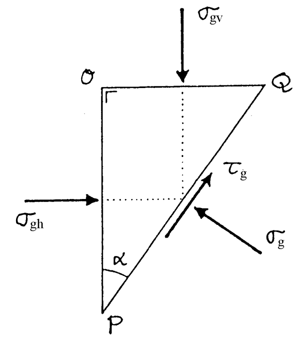


Figure 8.12 The grain stresses on a triangular soil element.

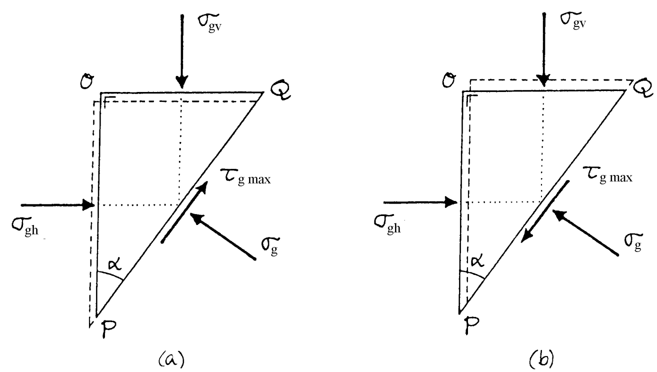


Figure 8.13 The maximum shear stress $\tau_{g;\max}$ is reached when the soil slides. One can distinguish two cases: (a) the soil element slides (moves downwards) and the shear stress in the slide plane PQ works upwards, and (b) the soil element is upset (moves upwards) and the shear stress in the slide plane PQ is directed downwards. The condition after sliding or upsetting is shown with a dashed line.

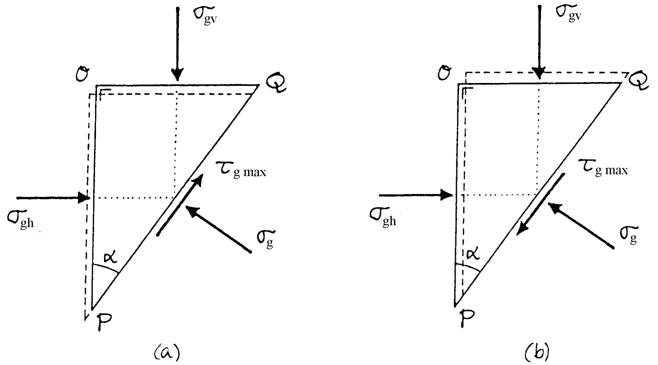


Figure 8.13 The maximum shear stress $\tau_{g;max}$ is reached when the soil slides. One can distinguish two cases: (a) the soil element slides (moves downwards) and the shear stress in the slide plane PQ works upwards, and (b) the soil element is upset (moves upwards) and the shear stress in the slide plane PQ is directed downwards. The condition after sliding or upsetting is shown with a dashed line.

Figure 8.14 The coefficient K_a and K_p for the active and passive soil pressure respectively, shown on a number line.

The *slide* in Figure 8.13a leads to the so-called *active earth pressure*:

$$\sigma_{g;h} = K_a \sigma_{g;v}.$$

The active earth pressure is the largest horizontal pressure for which the soil yields sideways (with smaller pressures the soil will certainly yield).

The *upset* in Figure 8.13b leads to the so-called *passive earth pressure*:

$$\sigma_{g;h} = K_p \sigma_{g;v}.$$

The passive earth pressure is the smallest horizontal pressures for which the soil is upset (with larger pressure, the soil will certainly be upset).

The distinction between *active* and *passive* is derived from the way in which the soil mass acts on its surroundings: active when (part of) a structure yields under the influence of earth pressure, and passive when the soil offers resistance to the displacement of (part of) a structure.

The *active earth pressure* on a wall has the same direction as the one in which the wall yields; the *passive earth pressure* acts opposite to the direction in which the wall moves.

Passive earth pressure can be expected to be greater than active earth pressure. It will always require greater effort (pressure) to upset the soil (passive earth pressure) than to resist sliding (active earth pressure). Figure 8.14 contains the coefficients for active and passive earth pressures on a number line.

With the coefficients K_a and K_p for the active and passive earth pressures respectively, we have determined the *extreme limits for the horizontal earth pressure* (in dry soil). These limits occur when a *slide plane* can develop. If this is not the case, the horizontal earth pressure lies somewhere between both limits. One of the intermediate values is the *neutral earth pressure*. This is the horizontal earth pressure on an entirely rigid wall, which does not move. This may include heavy retaining walls, lock walls, or tunnel walls. One assumes that (in dry soil) the neutral earth pressure is also in proportion to the vertical earth pressure, and refers to the associated coefficient as K_0.

The coefficients mentioned here for active, passive and neutral earth pressure relate to the normal stresses in *dry soil*, or in other words, the normal stresses in the grain skeleton (grain pressures). The earth pressure in *soil saturated with water* is found by superposing the water pressure on the normal stress in the grain skeleton.

The following sections address the magnitude of the coefficients for active, passive, and neutral earth pressure.

8.3.1 Active earth pressure

The coefficient K_a for active earth pressure is derived using the triangular slice of soil in Figure 8.15a.

If the area of side PQ is equal to ΔA, then the area of side OP is equal to $\Delta A \cos\alpha$ and that of side OQ is equal to $\Delta A \sin\alpha$. The forces on the three sides of the soil element are therefore

OP: $\sigma_{g;h}(\Delta A \cos\alpha)$,

OQ: $\sigma_{g;v}(\Delta A \sin\alpha)$,

PQ: $\sigma_g \Delta A$ and $\tau_{g;max}\Delta A$.

The equations for the force equilibrium in the plane of the drawing are easy to derive from the closed force polygon in Figure 8.15b.

Force equilibrium normal to PQ:

$$\sigma_g \Delta A - \sigma_{g;h}(\Delta A \cos\alpha)\cos\alpha - \sigma_{g;v}(\Delta A \sin\alpha)\sin\alpha = 0.$$

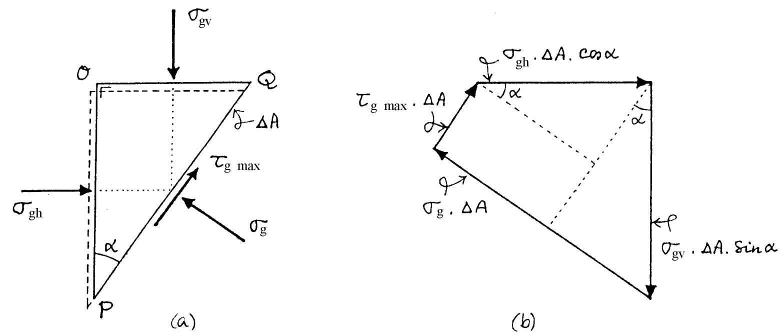


Figure 8.15 (a) The grain stresses on a triangular soil element that is sliding (moving downwards). (b) The equations for the force equilibrium in the plane of the drawing can be derived easily from the fact that the force polygon is closed.

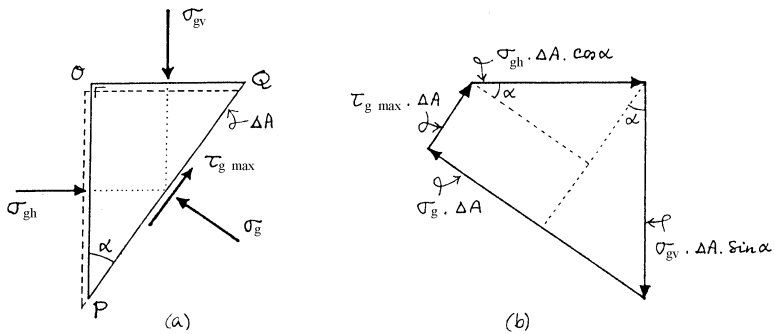


Figure 8.15 (a) The grain stresses on a triangular soil element that is sliding (moving downwards). (b) The equations for the force equilibrium in the plane of the drawing can be derived easily from the fact that the force polygon is closed.

Force equilibrium parallel to PQ:

$$\tau_{g;\max}\Delta A + \sigma_{g;h}(\Delta A\cos\alpha)\sin\alpha - \sigma_{g;v}(\Delta A\sin\alpha)\cos\alpha = 0.$$

From the two equations above, one can find the stresses in the slide plane PQ:

$$\sigma_g = +\sigma_{g;h}\cos^2\alpha + \sigma_{g;v}\sin^2\alpha,$$
$$\tau_{g;\max} = -\sigma_{g;h}\sin\alpha\cos\alpha + \sigma_{g;v}\sin\alpha\cos\alpha.$$

Since the shear stress has its maximum, also

$$\tau_{g;\max} = \sigma_g\tan\varphi.$$

In this expression, substitute those found before for σ_g and $\tau_{g;\max}$:

$$-\sigma_{g;h}\sin\alpha\cos\alpha + \sigma_{g;v}\sin\alpha\cos\alpha = (+\sigma_{g;h}\cos^2\alpha + \sigma_{g;v}\sin^2\alpha)\tan\varphi.$$

This gives the coefficient for the active earth pressure[1]:

$$K_a = \frac{\sigma_{g;h}}{\sigma_{g;v}} = 1 - \frac{2\sin\varphi}{\sin(2\alpha+\varphi)+\sin\varphi}.$$

We are looking for the largest horizontal earth pressure for which the soil *slides*. In other words, for which value of α does K_a have its maximum?

K_a has its maximum if the function

$$f = \sin(2\alpha+\varphi)$$

1 See Appendix 8.1 at the end of this chapter.

is a maximum, therefore, $2\alpha + \varphi = \frac{\pi}{2}$. For the angle α that the slide plane PQ makes with the vertical, one finds

$$\alpha = \frac{\pi}{4} - \frac{\varphi}{2}.$$

For this value of α

$$f = \sin(2\alpha + \varphi) = \sin(\pi/2) = 1$$

which is a maximum. This also means that K_a and therefore the horizontal earth pressure $\sigma_{g;h}$ reaches a maximum for this value of α.

By substituting the value of α found in the derived expression for K_a, one can immediately determine the value of the coefficient for the active earth pressure:

$$K_a = \frac{1 - \sin\varphi}{1 + \sin\varphi}.$$

For sand, for example, with $\varphi = 30°$, one finds

$$K_a = \frac{1 - \sin 30°}{1 + \sin 30°} = \tfrac{1}{3}.$$

In this case, the angle that the associated slide plane PQ makes with the vertical is

$$\alpha = 45° - \frac{30°}{2} = 30°.$$

Example
A sheet piling is located in an impermeable layer of clay 4.5 metres below ground level (see Figure 8.16). The specific weight of the dry soil behind

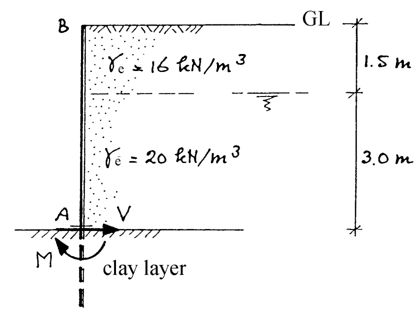


Figure 8.16 A sheet-pile wall is located in a relatively impermeable layer of clay 4.5 m below ground level. The groundwater level behind the sheet piling is 1.5 m below ground level. The specific weight γ_e of the soil (dry and wet) behind the wall is given in the figure.

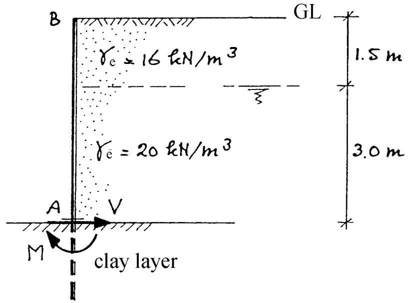


Figure 8.16 A sheet-pile wall is located in a relatively impermeable layer of clay 4.5 m below ground level. The groundwater level behind the sheet piling is 1.5 m below ground level. The specific weight γ_e of the soil (dry and wet) behind the wall is given in the figure.

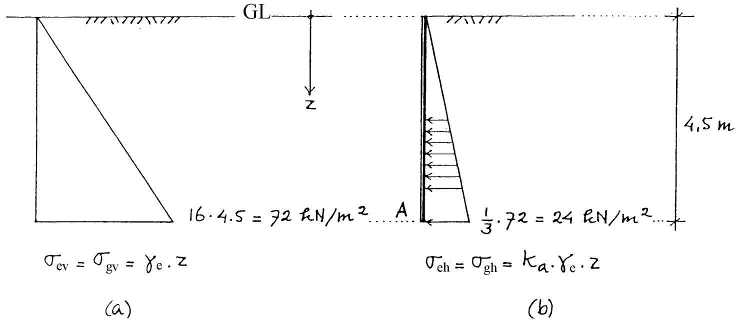


Figure 8.17 The distribution of (a) the vertical earth pressure and (b) the horizontal earth pressure on the sheet piling when the earth behind the sheet piling is entirely dry.

the sheet-pile wall is 16 kN/m^3. The same soil, fully saturated with water, has a specific weight of 20 kN/m^3. The angle of internal friction is $\varphi = 30°$. There is no cohesion.

Question:
Determine the "shear force" V and the "bending moment" M acting in cross-section A on a 1-metre wide vertical strip AB from the sheet piling. The following two cases must be distinguished:

a. The soil behind the sheet piling (not as shown in Figure 8.16) is entirely dry.
b. The groundwater level behind the sheet piling is 1.5 metres below the ground level.

Solution:
a. Due to the deformation of the sheet piling caused by the earth pressure, it will move slightly, and the soil may slide. The active earth pressure is then acting on the dam wall. With $\varphi = 30°$, the coefficient for active earth pressure is

$$K_a = \frac{1 - \sin 30°}{1 + \sin 30°} = \tfrac{1}{3}.$$

If the soil behind the sheet piling is entirely dry, the earth pressures are equal to the grain pressures. The horizontal earth pressure is 1/3 of the vertical earth pressure, so that

$$\sigma_{e;v} = \sigma_{g;v} = \gamma_e z,$$

$$\sigma_{e;h} = \sigma_{g;h} = K_a \gamma_e z = \tfrac{1}{3}\gamma_e z.$$

Figure 8.17a shows the distribution of the vertical earth pressure, and Figure 8.17b shows the distribution of the horizontal earth pressure. The horizontal earth pressure on the sheet piling increases linearly from 0 at the ground level to $\frac{1}{3} \times (16\ \text{kN/m}^3)(4.5\ \text{m}) = 24\ \text{kN/m}^2$ at A.

Now isolate a vertical strip with a width of 1 metre from the sheet pile, and model it as a line element. The horizontal load on the line element is equal to the horizontal earth pressure multiplied by the width of 1 metre, and therefore increases linearly from 0 at ground level to

$$(24\ \text{kN/m}^2)(1\ \text{m}) = 24\ \text{kN/m}$$

at A (see Figure 8.18).

The resultant R of the load is

$$R = \tfrac{1}{2} \times (4.5\ \text{m})(24\ \text{kN/m}) = 54\ \text{kN}.$$

In cross-section A, there must be acting a shear force V and a (bending) moment M. From the equilibrium of part AB cut from the wall, with the directions as shown in Figure 8.18 we find

$$V = R = 54\ \text{kN},$$

$$M = (54\ \text{kN})(1.5\ \text{m}) = 81\ \text{kNm}.$$

These are the requested forces at A that act on the 1-metre wide vertical strip AB isolated from the sheet piling.

b. If there is water behind the wall, the earth pressure is composed of a water pressure and a grain pressure. Figure 8.19a shows the distribution of the vertical earth pressure $\sigma_{e;v}$, including the contribution of the water pressure σ_w. If the water pressure is subtracted from the vertical earth pressure, it gives the vertical grain pressure $\sigma_{g;v}$ (see Figure 8.19b). The horizontal (active) grain pressure $\sigma_{g;h}$ is equal to the vertical grain pressure $\sigma_{g;v}$ multiplied by $K_a = 1/3$ (see Figure 8.19c).

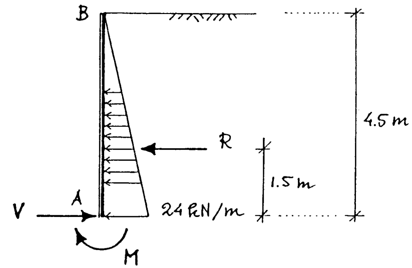


Figure 8.18 The horizontal load on a vertical strip of sheet piling that is 1-metre wide modelled as a line element for when the soil behind the sheet piling is entirely dry. Shear force V and the bending moment M in cross-section A are also shown.

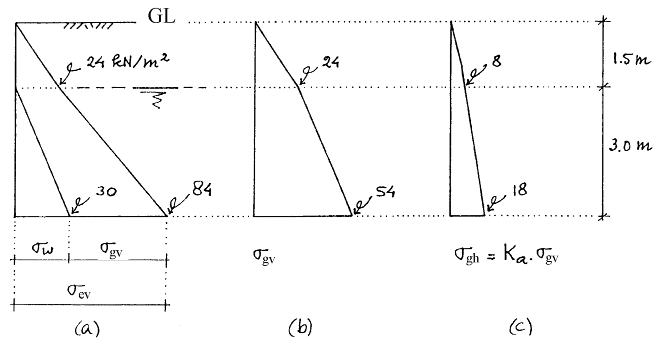


Figure 8.19 If there is water behind the sheet piling, the soil pressure is composed of a water pressure and a grain pressure. (a) The distribution of the vertical earth pressure $\sigma_{e;v}$, including the contribution of the water pressure σ_w. (b) If the water pressure is subtracted from the vertical earth pressure, one finds the vertical grain pressure $\sigma_{g;v}$. (c) The horizontal (active) grain pressure $\sigma_{g;h}$ is equal to the vertical grain pressure $\sigma_{g;v}$ multiplied by $K_a = 1/3$.

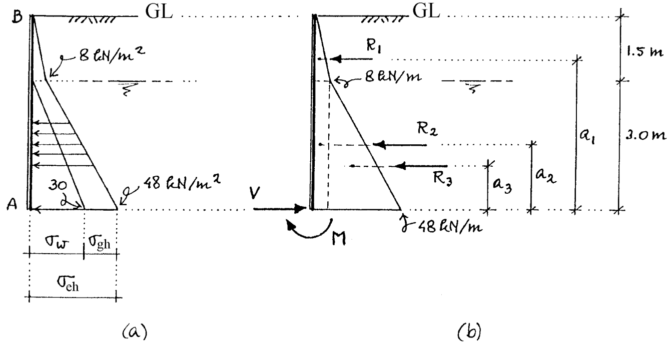


Figure 8.20 (a) The horizontal earth pressure $\sigma_{e;h}$ on the sheet piling, composed of the horizontal water pressure σ_w and the horizontal grain pressure $\sigma_{g;h}$. (b) The horizontal load on a 1-metre wide strip of sheet piling modelled as a line element, together with the shear force V and the bending moment M at A.

The sheet piling is subjected not only to the horizontal grain pressure, but also to water pressure. Figure 8.20a shows the distribution of the horizontal earth pressure $\sigma_{e;h}$, composed of the horizontal water pressure σ_w and the horizontal grain pressure $\sigma_{g;h}$.

Figure 8.20b shows the horizontal load acting on a 1-metre strip isolated from the sheet piling and modelled as a line element. The load diagram can be split into a rectangle and two triangles, of which the resultants R and their distances a to A are easy to calculate:

$$R_1 = 0.5 \times (1.5 \text{ m})(8 \text{ kN/m}) = 6 \text{ kN}, \quad a_1 = 3.5 \text{ m},$$

$$R_2 = (3 \text{ m})(8 \text{ kN/m}) = 24 \text{ kN}, \quad a_2 = 1.5 \text{ m},$$

$$R_3 = 0.5 \times (3 \text{ m})(40 \text{ kN/m}) = 60 \text{ kN}, \quad a_1 = 1.0 \text{ m}.$$

From the equilibrium of part AB of the sheet piling strip, we find with the directions of V and M as shown in Figure 8.20b:

$$V = R_1 + R_2 + R_3 = (6 + 24 + 60) \text{ kN} = 90 \text{ kN},$$

$$\begin{aligned} M &= R_1 a_1 + R_2 a_2 + R_3 a_3 \\ &= (6 \text{ kN})(3.5 \text{ m}) + (24 \text{ kN})(1.5 \text{ m}) + (60 \text{ kN})(1 \text{ m}) = 117 \text{ kNm}. \end{aligned}$$

These are the requested forces acting in A on the 1-metre wide vertical strip AB from the sheet piling.

In practice, one often uses a method developed by *Coulomb*[1] based on *flat slide planes*. Here, one assumes that the pressure on the wall is caused by a

[1] Charles Auguste de Coulomb (1736–1806), French scientist, known for his experiments in friction and electrostatic forces.

triangular piece of soil that slides along a flat slide plane (see Figure 8.21). One can deduce that the most dangerous slide plane (in the active case of a yielding wall) makes an angle

$$\alpha = \frac{\pi}{4} - \frac{\varphi}{2}$$

with the vertical. This turns out to be precisely the angle of the planes, as derived earlier, in which the shear stress is a maximum. The magnitude of the earth pressure according to Coulomb's method also agrees with the result above.

The agreement can best be understood by means of the sliding wedge of soil in Figure 8.21, assuming that there are no shear stresses in the horizontal and vertical planes (there is therefore also no wall friction). For simplicity, we will again assume that the soil is entirely dry.

One can recognise the triangular soil element we dealt with earlier on the edge of the slide plane, with stresses σ_g and $\tau_{g;max}$ on the oblique side. The weight of the vertical soil column determines the vertical grain pressure $\sigma_{g;v}$. The horizontal strip of soil transfers the horizontal grain pressure $\sigma_{g;h}$ to the wall.

The method with slide planes has the advantage that it is relatively easy to see that a terrain load on ground level increases the horizontal earth pressure on the wall only if it acts on the sliding soil wedge.

Comment: For the influence of wall friction, oblique walls, or oblique ground levels, as well as the necessary nuances in the examples shown here, please refer to a text book on the field of soil mechanics.

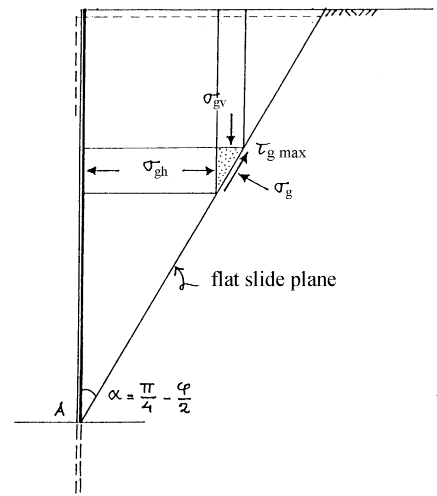


Figure 8.21 In practice, one often uses a method developed by Coulomb based on flat slide planes. Here one assumes that the pressure on the wall is caused by a triangular piece of soil that slides along a flat slide plane. One can recognise the triangular soil element from Figure 8.15a on the edge of the slide plane.

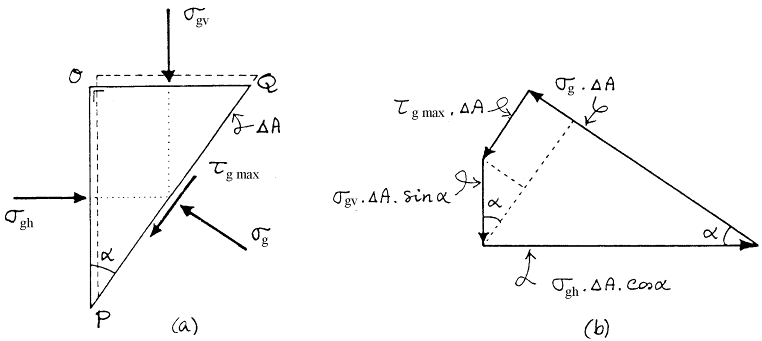


Figure 8.22 (a) The grain stresses on a triangular soil element that is upset (moving upwards). (b) The equations for the force equilibrium in the plane of the drawing can easily be derived from the fact that the force polygon is closed.

8.3.2 Passive earth pressure

The coefficient K_p for *passive earth pressure* can be derived using the triangular wedge of soil in Figure 8.22a, in the same way as the coefficient for active earth pressure in Section 8.3.1.

The equations for the force equilibrium in the plane of the drawing can be found from the closed force polygon in Figure 8.22b.

Force equilibrium perpendicular and parallel to PQ:

$$\sigma_g \Delta A - \sigma_{g;h}(\Delta A \cos\alpha)\cos\alpha - \sigma_{g;v}(\Delta A \sin\alpha)\sin\alpha = 0,$$

$$\tau_{g;max}\Delta A - \sigma_{g;h}(\Delta A \cos\alpha)\sin\alpha + \sigma_{g;v}(\Delta A \sin\alpha)\cos\alpha = 0$$

so that

$$\sigma_g = +\sigma_{g;h}\cos^2\alpha + \sigma_{g;v}\sin^2\alpha,$$

$$\tau_{g;max} = +\sigma_{g;h}\sin\alpha\cos\alpha - \sigma_{g;h}\sin\alpha\cos\alpha.$$

The shear stress is a maximum, therefore

$$\tau_{g;max} = \sigma_g \tan\varphi.$$

In this expression substitute the expressions found for σ_g and $\tau_{g;max}$:

$$+\sigma_{g;h}\sin\alpha\cos\alpha - \sigma_{g;v}\sin\alpha\cos\alpha = (+\sigma_{g;h}\cos^2\alpha + \sigma_{g;v}\sin^2\alpha)\tan\varphi.$$

This gives the coefficient for passive earth pressure[1]:

$$K_p = \frac{\sigma_{g;h}}{\sigma_{g;v}} = 1 + \frac{2\sin\varphi}{\sin(2\alpha - \varphi) - \sin\varphi}.$$

We are trying to find the smallest horizontal earth pressure for which the soil is *upset*. In other words, for which value of α is K_p a minimum?

K_p is a minimum if the function

$$f = \sin(2\alpha - \varphi)$$

is a maximum, therefore $2\alpha - \varphi = \frac{\pi}{2}$. For the angle α that the slide plane PQ makes with the vertical one finds

$$\alpha = \frac{\pi}{4} + \frac{\varphi}{2}.$$

For this value of α

$$f = \sin(2\alpha - \varphi) = \sin(\pi/2) = 1$$

and f indeed is a maximum. This means that K_p and therefore the horizontal grain pressure $\sigma_{g;h}$ has a minimum for this value of α.

The value of the coefficient K_p for passive earth pressure is found directly by substituting the value found for α in the expression derived for K_p:

$$K_p = \frac{1 + \sin\varphi}{1 - \sin\varphi}.$$

1 See Appendix 8.2 at the end of this chapter.

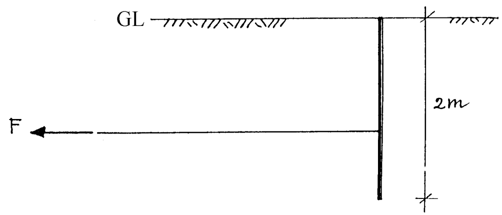


Figure 8.23 A square anchor plate in dry soil.

With sand, for instance, with $\varphi = 30°$, one finds

$$K_{\mathrm{p}} = \frac{1 + \sin 30°}{1 - \sin 30°} = 3.$$

In this case the coefficient K_{p} for passive earth pressure is 9 times larger than the coefficient K_{a} for active earth pressure! The angle α that the associated slide plane PQ makes with the vertical is

$$\alpha = 45° + \frac{30°}{2} = 60°.$$

Note that:

- The angle α that the slide plane makes with the vertical in the *active* case (when the wall yields) is always *smaller* than 45° and in the *passive* case (if the wall is upset) is always *larger* than 45°.
- There is a relationship between the coefficients for active and passive earth pressure, namely

  $$K_{\mathrm{a}} K_{\mathrm{p}} = 1.$$

Example

A square anchor plate is located in entirely dry soil (see Figure 8.23). The specific weight of the soil is 18 kN/m^3. The angle of internal friction is 24°. There is no cohesion.

Questions:

a. Determine the maximum anchor force that the plate can provide.
b. Determine the influence of a vertical terrain load of 20 kN/m^2 on the magnitude of the maximum anchor force.

Solution:

a. Due to the anchor force, the plate will tend to move to the left. An area of passive earth pressure develops in front of the plate, and an area of active earth pressure develops behind the plate. The coefficients K_{p} and

K_a, respectively for the active and passive earth pressure, are

$$K_p = \frac{1 + \sin 24°}{1 - \sin 24°} = 2.37,$$

$$K_a = \frac{1 - \sin 24°}{1 + \sin 24°} = 0.42.$$

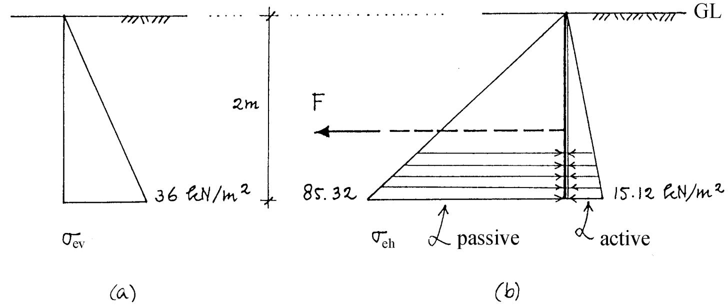


Figure 8.24 (a) The distribution of the vertical earth pressure and (b) the distribution of the passive and active (horizontal) earth pressure on the anchor plate.

Figure 8.24a shows the distribution of the vertical earth pressure. Derived from the vertical earth pressure, Figure 8.24b shows the distribution of the (horizontal) passive and active earth pressure on the plate. Taking into account the width of 2 metres of the plate, the resultants R_p and R_a of respectively the passive and active earth pressure are

$$R_p = (2\text{ m}) \times 0.5 \times (85.32\text{ kN/m}^2)(2\text{ m}) = 170.64\text{ kN},$$

$$R_a = (2\text{ m}) \times 0.5 \times (15.12\text{ kN/m}^2)(2\text{ m}) = 30.24\text{ kN}.$$

The maximum anchor force F that the plate can provide is

$$F = R_p - R_a = (170.64\text{ kN}) - (30.24\text{ kN}) = 140.4\text{ kN}.$$

b. Any *terrain load p* increases the vertical grain pressure in the soil and therefore also increases the active earth pressure (resultant R_a) and the passive earth pressure (resultant R_p).

Maximum anchor force means the anchor force that the plate can offer is *guaranteed*. In fact, we are therefore looking for the smallest value of

$$F = R_p - R_a.$$

The least favourable condition occurs when R_p is a minimum (no terrain load in the passive area) and R_a is a maximum (terrain load in the active area).

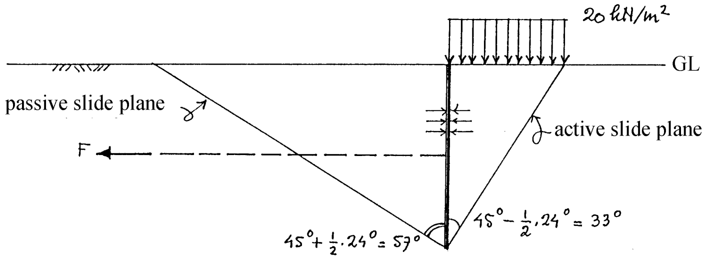


Figure 8.25 Using slide planes according to Coulomb's method, we quickly realise that a terrain load in the active area decreases the anchor force: the passive earth pressure remains unchanged while the active earth pressure increases across the entire plate.

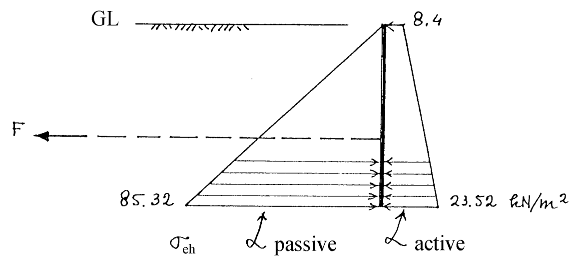


Figure 8.26 The distribution of the passive and active earth pressure on the anchor plate.

This unfavourable condition is shown in Figure 8.25, as well as the slide planes according to Coulomb's method. The associated distribution of the passive and active earth pressure is shown in Figure 8.26. The passive earth pressure remains unchanged, but the active earth pressure increases across the entire plate by

$$K_a p = 0.42 \times (20\ \text{kN/m}^2) = 8.4\ \text{kN/m}^2.$$

The resultant R_a of the active earth pressure is now

$$R_a = (30.24\ \text{kN}) + (2\ \text{m})(2\ \text{m})(8.4\ \text{kN/m}^2) = 63.84\ \text{kN}$$

so that the maximum anchor force is

$$F = R_p - R_a = (170.64\ \text{kN}) - (63.84\ \text{kN}) = 106.8\ \text{kN}.$$

The terrain load of 20 kN/m^2 in the area where the active earth pressures develop decreases the force that the plate can take by 33.6 kN, from 140.4 kN to 106.8 kN.

Note: When calculating the anchor force, we assume that the soil is upset over a width of 2 m (the width of the anchor plate). In reality, the width of the soil that is upset will be greater. For this and other discrepancies between the model and reality, please refer to a textbook on the field of soil mechanics.

8.3.3 Neutral earth pressure

With the coefficients K_a and K_p for the active and passive earth pressure respectively, we have defined the *extreme limits of the horizontal earth pressure*. These limits occur when a *slide plane* can develop. If this is not possible, the horizontal earth pressure is between these values.

One of these intermediate values is the *neutral earth pressure*. This is the horizontal earth pressure on an immovable wall, which may include heavy retaining walls, lock walls, or tunnel walls.

One assumes that the neutral earth pressure is also proportional to the vertical earth pressure $\sigma_{g;v}$ and we call the associated coefficient K_0:

$$\sigma_{g;h} = K_0 \sigma_{g;v}.$$

The value of K_0 will be between $K = K_a$ and $K = 1$ (when there are no shear stresses). One often *poses* that:

$$K_0 = 1 - \sin\varphi.$$

Figure 8.27 shows the coefficients K_a, K_p and K_0 on a number line. The numerical values included relate to soil with an angle of internal friction $\varphi = 30°$, such as sand, for instance.

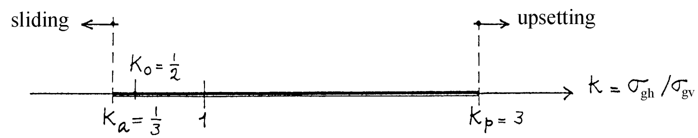


Figure 8.27 The coefficients K_a, K_p and K_0 for the active, passive, and neutral earth pressure respectively, depicted on a number line. The numerical values relate to soil with an angle of internal friction $\varphi = 30°$, like sand, for instance.

Appendix 8.1

Determining the coefficient K_a for active earth pressure:

$$\begin{aligned} K_a &= \frac{\sin\alpha\cos\alpha - \sin^2\alpha\tan\varphi}{\sin\alpha\cos\alpha + \cos^2\alpha\tan\varphi} = 1 - \frac{\cos\alpha^2\tan\varphi + \sin^2\alpha\tan\varphi}{\sin\alpha\cos\alpha + \cos^2\alpha\tan\varphi} \\ &= 1 - \frac{\sin\varphi}{\cos\alpha(\sin\alpha\cos\varphi + \cos\alpha\sin\varphi)} = 1 - \frac{\sin\varphi}{\cos\alpha\sin(\alpha+\varphi)} \\ &= 1 - \frac{2\sin\varphi}{\sin(2\alpha+\varphi) + \sin\varphi}. \end{aligned}$$

To determine the coefficient, we used the two trigoniometric equations shown below:

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta,$$

$$2 \sin\alpha \cos\beta = \sin(\alpha + \beta) + \sin(\alpha - \beta).$$

Appendix 8.2

Determining coefficient K_p for passive earth pressure:

$$\begin{aligned} K_\mathrm{p} &= \frac{\sin\alpha\cos\alpha + \sin^2\alpha\tan\varphi}{\sin\alpha\cos\alpha - \cos^2\alpha\tan\varphi} = 1 + \frac{\cos\alpha^2\tan\varphi + \sin^2\alpha\tan\varphi}{\sin\alpha\cos\alpha - \cos^2\alpha\tan\varphi} \\ &= 1 + \frac{\sin\varphi}{\cos\alpha(\sin\alpha\cos\varphi - \cos\alpha\sin\varphi)} = 1 + \frac{\sin\varphi}{\cos\alpha\sin(\alpha - \varphi)} \\ &= 1 + \frac{2\sin\varphi}{\sin(2\alpha - \varphi) - \sin\varphi}. \end{aligned}$$

To determine the coefficient, we used the two trigoniometric equations shown below:

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta,$$

$$2 \cos\alpha \sin\beta = \sin(\alpha + \beta) - \sin(\alpha - \beta).$$

8.4 Problems

Unless indicated otherwise, the following apply for all questions:

- Specific weight of water: $\gamma_{\text{water}} = 10$ kN/m^3.
- Coefficient for active earth pressure: $K_a = 1/3$.
- Coefficient for passive earth pressure: $K_p = 3$.
- The soil has no cohesion.
- All levels are given in metres with respect to sea level (SL).

Vertical earth pressures (Section 8.2)

8.1 In an area with sandy soil, the groundwater is 1 m below ground level. The specific weight of dry sand is 15 kN/m^3. The pore volume of sand is 40%.

Questions:

a. Determine the specific weight of wet sand.
   Draw the distribution to 3 m under the ground level of:
b. the vertical earth pressure.
c. the vertical grain pressure.
d. the water pressure.
e. How do the vertical earth pressure, grain pressure and water pressure change if the groundwater rises by 0.5 m?
f. How do the vertical earth pressure, grain pressure and water pressure change if the groundwater drops by 0.8 m?

8.2 As problem 8.1, but now with a terrain load of 5 kN/m^2.

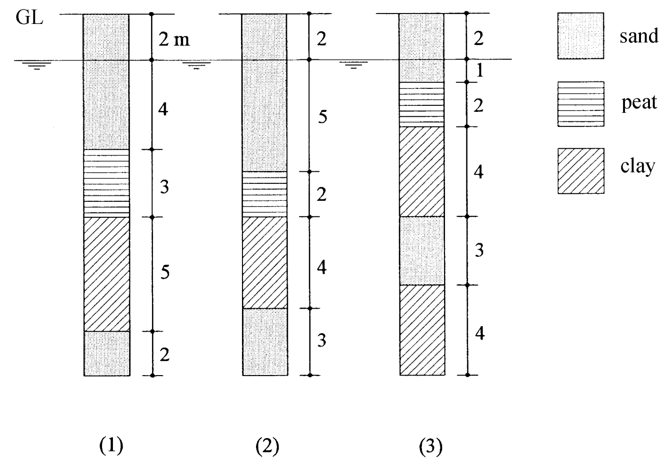
8.3 In an area with sandy soil, the groundwater is 1.2 m below ground level. Specific weight of the wet sand is 18 kN/m^3. The pore volume of the sand is 35%.

Questions:

a. Determine the specific weight of dry sand.
   Draw the distribution to 3 m under the ground level of:
b. the vertical earth pressure.
c. the vertical grain pressure.
d. the water pressure.
e. How do the vertical earth pressure, grain pressure and water pressure change if the groundwater rises by 0.6 m?
f. How do the vertical earth pressure, grain pressure and water pressure change if the groundwater drops by 0.8 m?

8.4 As problem 8.3, but now with a terrain load of 4 kN/m^2.

8.5: 1–3 An area consists of various soil layers. The groundwater is 2 m below ground level (GL).

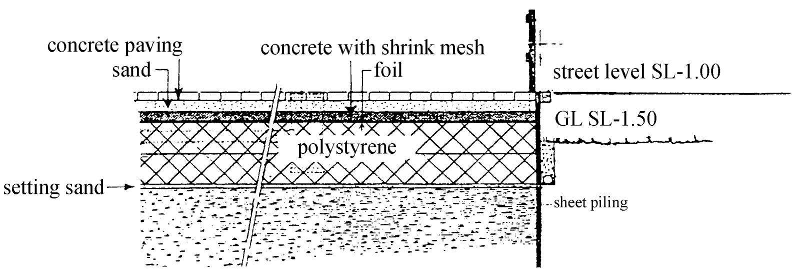


Material	Specific weight (kN/m^3)
dry sand	16
wet sand	20
wet peat	12
wet clay	18

Question:
Draw the distribution to 15 m under ground level of:
a. the vertical earth pressure.
b. the vertical grain pressure.
c. the water pressure.
e. How do the vertical earth pressure, grain pressure and water pressure change if the groundwater rises by 1.0 m?
f. How do the vertical earth pressure, grain pressure and water pressure change if the groundwater drops by 1.0 m?

8.6: 1–3 As problem 8.5, but now with a terrain load of 6 kN/m^2.

8.7 Soil mechanical research showed that with traditional preparation of a site for building works, one has to take into account a settlement of 2.0 m after 17.5 years. For this reason, a settlement-free raise was selected, using polystyrene.

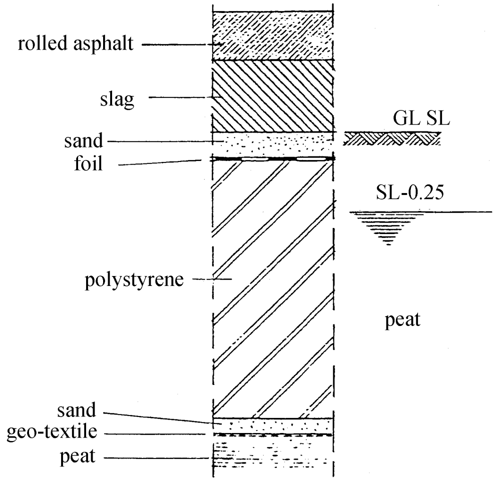


Existing situation	Level SL (m)	Material	Specific weight (kN/m^3)
ground level	–1.50	peat/dry	12
groundwater level	–2.10		

New situation	Specific weight (kN/m^3)
Street level SL–1.00 m	–
0.05 m concrete paving	23
0.15 m dry sand	14
foil	–
0.10 m concrete with shrink mesh	24
??? m polystyrene	0.2
0.05 m wet setting sand	17.5

Question:
How thick does the layer of polystyrene have to be if the earth pressure under the raise is the same as in the original situation?

8.8 Polystyrene is to be used for the construction of an access road in a peat area. The road structure is as follows (see the table below):
The weight of foil and geo-textile are negligible. Assume that polystyrene does not absorb water.
The ground level is at SL. The base of the slag layer is at the same height as the ground level. The average groundwater level is SL–0.25. The terrain load on the road is 6 kN/m^2.



Construction	m
rolled asphalt	0.17
slag	0.25
sand	0.10
foil	–
polystyrene	???
sand	0.10
geo-textile	–
peat	–

Material	Specific weight (kN/m^3)
rolled asphalt	24
slag	17
dry sand	15
wet sand	19
dry peat	13
wet peat	15
polystyrene	0.2

Questions:

a. How thick, with the average water level, does the layer of polystyrene have to be so that, for the road with the terrain load, the earth pressure under the geo-textile is equal to the earth pressure in the original situation?
b. To which height can the groundwater level rise so that the road (without terrain load) does not rise up?

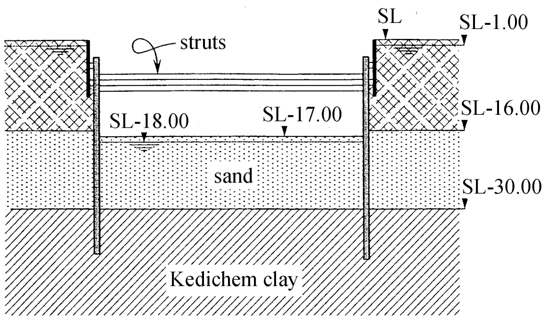
8.9 Two tubes of a tunnel for slow traffic under a river have been drilled using a special drilling machine. The external diameter of the concrete tunnel tube is 8.3 m, the walls are 0.3 m thick. In the river, the tunnel has a ground cover of 8.3 m. The riverbed consists of a layer of clay 1.5 m thick on sand. From the ground level, the south bank has a 2.0 m thick layer of clay on peat/sand. The groundwater level is 1.5 m under ground level. As vertical load, only the soil directly above the tube is taken into account.

Material	Specific weight (kN/m^3)
dry clay	14
wet clay	17
wet peat/sand	15
wet sand	18
concrete	24

Questions:

a. Determine the upward and downward forces on an empty tunnel tube under the riverbed, in kN/m (force per m length of the tunnel).
Check whether the tunnel will float.
b. How deep must the tunnel be built under the ground level of the south bank so that the difference between the upward and downward forces is 87.5 kN/m?

8.10 Concrete diaphragm walls have been used to build an underground railroad. The walls were poured into the clay. The clay layer is located under a layer of sand from SL–16.00 to SL–30.00. The building pit is dug to SL–17.00 and drained to SL–18.00. The groundwater is at SL–1.00.



Material	Specific weight (kN/m^3)
dry sand	14
wet sand	18
wet clay	17

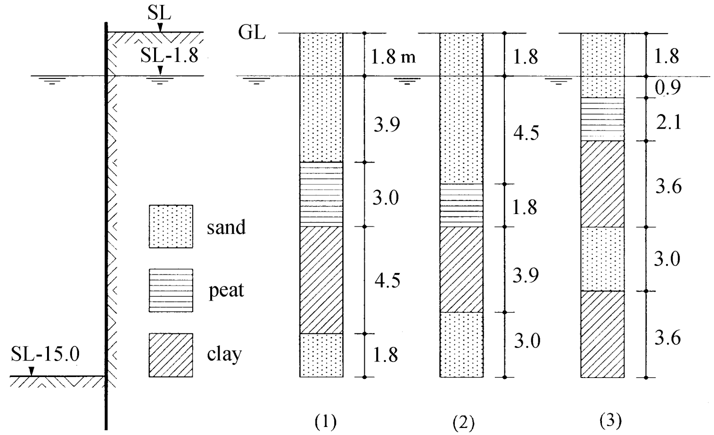
Questions:

a. How thick must the clay layer be so that after digging and draining in the building pit there is a grain pressure of 5 kN/m^2 at the base of the clay layer?
Hint: clay is highly impermeable.

b. At which thickness of the clay will the building pit burst?

Horizontal earth pressures (Section 8.3)

8.11: 1–3 You are given a sheet-pile wall alongside a quay. There are three different soil profiles.



Material	Specific weight (kN/m^3)
dry sand	16
wet sand	20
wet peat	12
wet clay	18

Questions:

a. Up to SL–15.00, draw the distribution of the horizontal water pressure to the left and the earth pressure to the right of the sheet pile. Split the earth pressure up into grain pressure and water pressure.

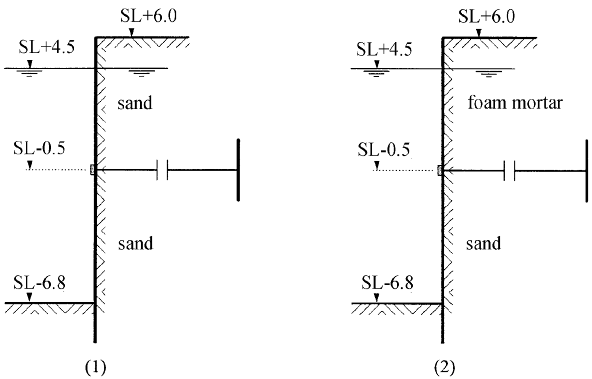
b. Draw the distribution of the resulting pressure on the sheet-pile wall.

c. Determine the resultant of the horizontal load on a 1-metre wide vertical strip of the wall in kN/m.

d. Also answer questions a to c with a terrain load of 12 kN/m^2.

8.12: 1–3 As problem 8.11, but now the water level to the left of the sheet piling is 1.5 m below the groundwater level to the right.

8.13: 1–2 Sand is located under the anchor bars of an anchored sheet piling in a port. The soil is filled up to ground level. To fill the site, there is a choice between sand and a (far more expensive) foam mortar. The terrain load is 20 kN/m^2.

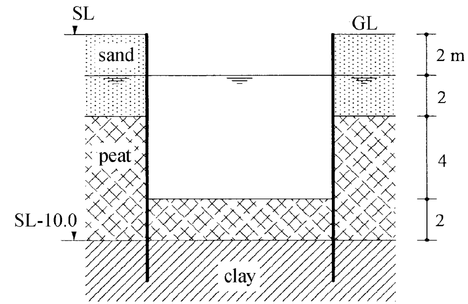


Material	Specific weight (kN/m^3)
dry sand	15
wet sand	18
dry foam mortar	2
wet foam mortar	6

Questions:

a. To SL–6.8, draw the distribution of the horizontal pressures to the left and to the right of the sheet piling.
b. Draw the distribution of the resulting horizontal pressure on the sheet piling.
c. Determine the resultant of the horizontal load on a 1-metre wide vertical strip of sheet piling in kN/m.
d. How does the horizontal load change if there is no terrain load?

8.14 A building pit is surrounded by a steel sheet piling and is still full of water. The soil profile is shown in the figure.

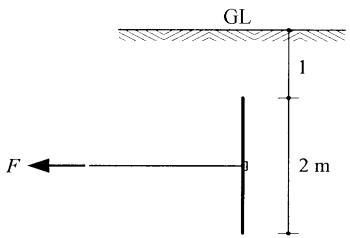


Material	Specific weight (kN/m^3)
dry sand	15
wet sand	18
wet peat	12

Questions:

a. Up to the clay layer, draw the distribution of the horizontal pressure on the outside of the sheet piling, split into grain pressure and water pressure.
b. Up to the clay layer, draw the distribution of the horizontal pressure on the inside of the sheet piling, split into grain pressure and water pressure.
c. Draw the distribution of the resulting horizontal load on the sheet piling.
d. Determine the resultant of the horizontal load on a 1-metre wide vertical strip of sheet piling in kN/m.
e. How does the resultant of the horizontal load on the sheet piling (in kN/m) change if the water level in the building pit is lowered to SL–8.0?

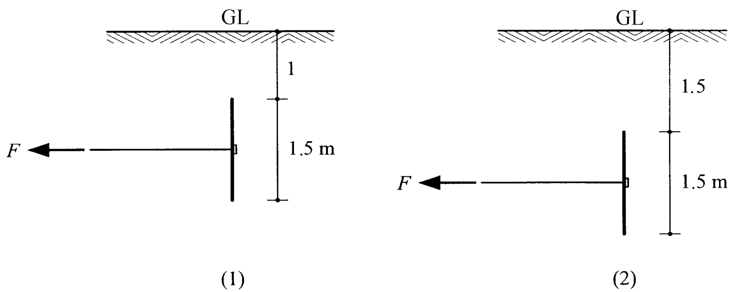
8.15 An anchor plate is located in dry soil, 1 m below ground level. The plate is 2 m high and 1.5 m wide. It can be assumed here that the anchor plate runs up to ground level. The specific weight of the soil is 15 kN/m^3.



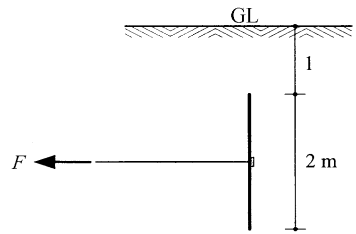
Questions:

a. Determine the maximum anchor force that the plate can provide.
b. Determine the influence of an terrain load of 18 kN/m on the magnitude of the anchor force.

8.16: 1–2 As 8.15, but now with an anchor plate that is 1.5 m high and 2 m wide, with the plate 1 m, respectively 1.5 m under ground level.



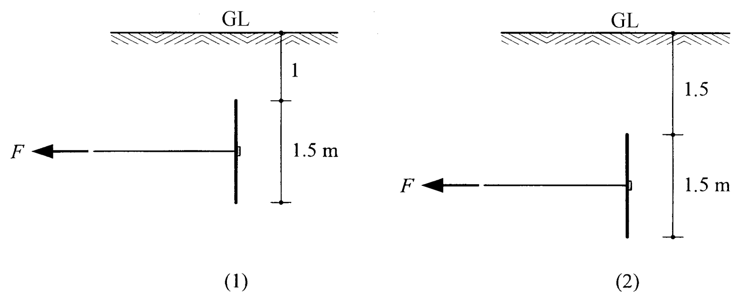
8.17 An anchor plate is located in dry soil, 1 m below ground level, and is 2 m high and 1.5 m wide. It can be assumed here that the anchor plate runs up to ground level. The specific weight of the soil is 15 kN/m^3. The angle of internal friction is 20°.



Questions:

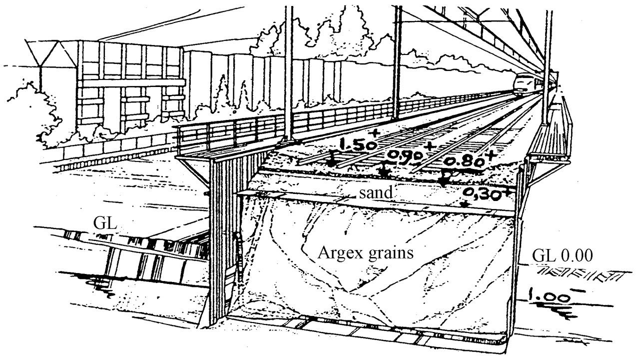
a. Determine the coefficient for active earth pressure.
b. Determine the coefficient for passive earth pressure.
c. Determine the maximum anchor force that the plate can provide.
d. Determine the influence of an terrain load of 18 kN/m on the magnitude of the anchor force.

8.18: 1–2 As 8.17, but now with an anchor plate that is 1.5 m high and 2 m wide, with the anchor plate 1 m, respectively 1.5 m under ground level.



Various questions

8.19 During the construction of a railway, part of the existing railway was moved to make room for the building pit of the new railway. In order to prevent settlement, it was decided to introduce a raise with light argex grains. The grains are sealed from the groundwater by means of a waterproof membrane and are kept dry by means of drainage.



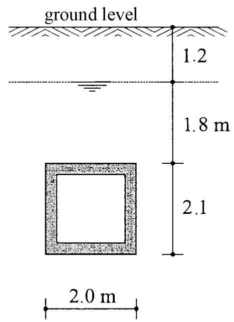
Construction in the railway:	Level SL (m)	Material	Specific weight (kN/m^3)
topside ballast bed	+1.50	ballast	18
topside gravel layer	+0.90	gravel	17.5
topside sand layer	+0.80	sand/dry	15
topside argex grains	+0.30	argex/dry	6.5

Next to the railway:	Level SL (m)	Material	Specific weight (kN/m^3)
ground level	0.00	soil/dry	14
groundwater level	–1.00	soil/wet	17

Questions:

a. Up to 3 m below ground level, draw the distribution of the vertical earth pressure outside the sheet-pile walls. In the diagram, indicate which part of the earth pressure is caused by the grain pressure and which is caused by the water pressure.
b. How deep must the waterproof membrane be with respect to SL, so that the vertical earth pressure directly under the membrane is equal to the earth pressure outside the sheet-pile walls?
c. Draw the distribution of the vertical earth pressure within the sheet-pile walls, to 3 m below ground level. In the diagram, indicate which part of the earth pressures is caused by the grain pressure and which is caused by the water pressure.
d. Up to 3 m below ground level, draw the distribution of the horizontal earth pressure on the outside of the sheet piling. In the diagram, indicate which part of the earth pressures is caused by the grain pressure and which is caused by the water pressure.

8.20 A concrete culvert with rectangular cross-section 2 m wide and 2.1 m high and the same wall thickness everywhere of 0.25 m, is located with its top side 3.0 m under ground level. The groundwater is located 1.2 m under ground level.



The specific weights are

$$\gamma_{\text{earth;dry}} = 17.5\ \text{kN/m}^3;$$

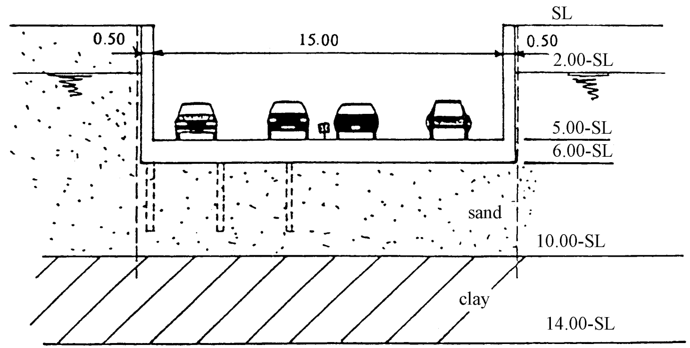
$$\gamma_{\text{earth;wet}} = 20\ \text{kN/m}^3;$$

$$\gamma_{\text{concrete}} = 24\ \text{kN/m}^3.$$

Question:
Determine the distribution of the earth pressure, split up into grain pressure and water pressure, on

a. the top side of the culvert.
b. the sides of the culvert.
c. the base of the culvert.

8.21 At a crossroad with unequal levels, one of the roads passes under the other in a trough structure. Sheet-pile walls are driven into the clay layer. Subsequently, the pit is dug down until SL–6.00 and the water in the pit is drained to SL–7.00. Once the trough structure is completed, the sheet-pile walls are removed.



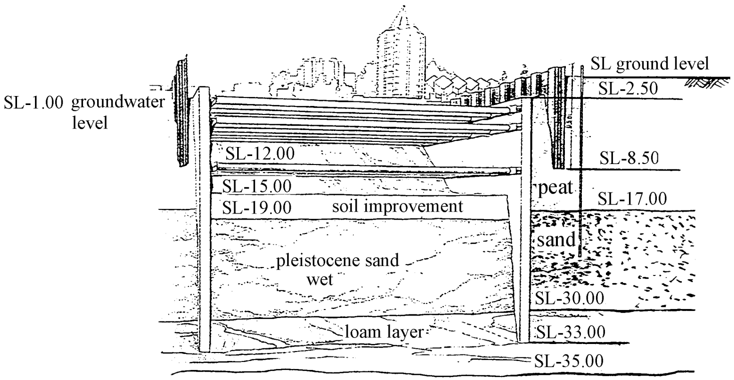
Material	Specific weight (kN/m^3)
dry sand	15
wet sand	20
wet clay	18
concrete	24

Questions:

a. To SL–14.00, draw the distribution of the vertical earth pressure. In the diagram, indicate which part of the earth pressure is caused by the grain pressure and which is caused by the water pressure.
b. Is the building pit safe from bursting once it has been dug out but prior to the placement of the concrete trough structure?
c. Show that, if there are no tension piles, the concrete trough will rise once the sheet-pile walls have been removed.
d. Draw the horizontal earth pressure on the wall of the trough structure. In the diagram, indicate which part of the earth pressure is caused by the grain pressure and which is caused by the water pressure.

8.22 Diaphragm walls have been used to build a tunnel. The diaphragm walls run to SL–33.00. The building pit has been dug out to SL–19.00, after which a soil improvement of 4 m was used to SL–1500. The lower struts are at SL–12.00. Ground level is at SL. The groundwater is at SL–1.00. The soil profile is as follows:

- 17 m peat: from SL to SL–17.00
- 13 m (Pleistocene) sand: from SL–17.00 to SL–30.00
- 5 m loam: from SL–30.00 to SL–35.00
- sand: from SL–35.00



Material	Specific weight (kN/m^3)
dry peat	13
wet peat	15
pleistoc. sand wet	19
loam	18
soil improvement	16

Questions:

a. To SL–35.00, draw the distribution of the vertical earth pressure adjacent to the building pit, split up into grain pressure and water pressure.

b. After applying the soil improvement in the building pit, how large is the grain pressure directly under the loam layer? The water in the building pit is at SL–19 .00

c. From SL–12.00 to SL–20.00, draw the distribution of the horizontal earth pressure on the outside of the diaphragm wall, split into grain pressure and water pressure.

9 Trusses

A *truss* is *by definition* a structure assembled with *straight bars* (*members*), which are connected by *hinged joints*, and loaded by forces which have their point of application at these joints.

In comparison to heavily-built structures, trusses need little material, and therefore have a relatively small dead weight. If we consider the use of little material, and the reduced costs for foundation because of the small dead weight, they can be cost saving. On the other hand, constructing trusses is often labour-intensive due to the complexity of the joints, and labour costs can be higher. Nevertheless, the total costs may be lower, and trusses can be an interesting type of structure from an economic perspective.

Trusses are often used in roof structures, bridges, cranes, and so forth. Scaffoldings are also often trusses.

Section 9.1 addresses the difference between a space truss and a plane truss. The rest of this chapter only looks at *plane trusses*. For this type of truss, all the members are located in the same plane, and the load acts in the plane of the truss. Section 9.1 also looks at the modelling of a structure as a truss, the nomenclature for the members in a truss, and the conventions used to label the joints and members.

Section 9.2 explains the relationship between the number of members and joints in a *simple* or *self-contained truss* and a *compound truss* respectively.

Next, the kinematic/static (in)determinacy of a truss is investigated and the relationship between the number of support reactions, members and joints is considered.

Calculating the member forces in a truss is addressed in Section 9.3. There are several methods for this, two of which are discussed:
- the method of sections;
- the method of joints.

In the *method of sections*, we make a suitable section in the truss, and calculate the member forces from the equilibrium of one of the bisected (isolated) parts. In the *method of joints*, we calculate the member forces from the equilibrium of the joints.

The methods mentioned are *manual calculation methods* and are applicable only to statically determinate trusses; they demand the necessary insight if they are to be used effectively. Sometimes it is useful to use both methods in combination.

Nowadays, we generally use computer programs to calculate trusses. Many of these programs use the so-called *displacement method*, which can be used for both statically determinate and statically indeterminate trusses.[1]

Even though increasing numbers of calculations are performed using computer programs, the manual calculation methods remain valuable, even if only because they can be used as a relatively simple check. This is true particularly for the method of sections, which offers a superb way of checking computer-based results. It allows us to check for errors that may be, for example, the result of incorrect data entry by the user.

[1] The displacement method not only uses the equilibrium relationships, but also the behaviour of the material (the constitutive relationships) and the compatibility of the structure (the kinematic relationships). The constitutive and kinematic relationships are covered in Volume 2: *Stresses, Deformations, Displacements*.

9.1 Plane trusses

This section addresses the difference between a space truss and a plane truss. From here on, we will look only at plane trusses, with all members in the same plane, and the load acting in the plane of the truss. We will also look at the way in which a structure is modelled as a truss, the nomenclature for the members in a truss, the various types of trusses, and the conventions for labelling the joints and members.

9.1.1 Plane and space trusses

A *truss* is defined as a structure constructed with *straight bars* (*members*), which are connected by *hinges* at so-called *joints*, and loaded by forces which have their point of application at these joints.

There are *plane* trusses and *space trusses*. In plane trusses, all the members are in the same plane, and forces only act in the plane of the structure. In space trusses, the members are not all in the same plane (see Figure 9.1).

Many space trusses in fact consist of plane trusses, such as the structure in Figure 9.1b. The load shown is transferred to the supports via the plane trusses ABCD and ABEG.

From here on, we will look only at plane trusses. The open circles, which indicate the hinged joints, will be omitted since in a truss all joints are *hinged by definition*.

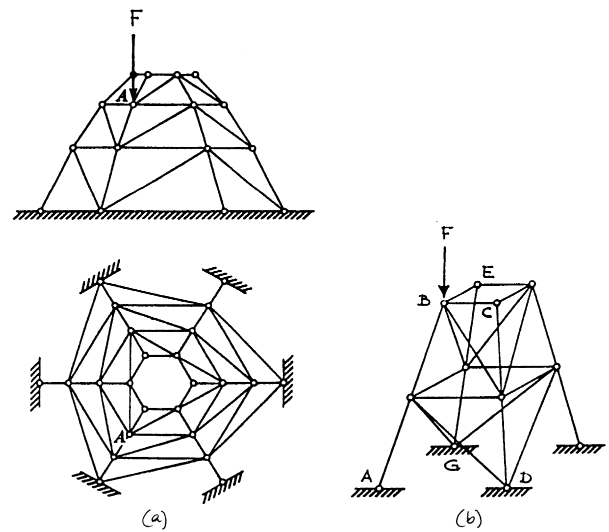


Figure 9.1 Space trusses. (a) Side view and top view of a truncated truss dome. (b) A space truss constructed from plane trusses.

9.1.2 Modelling a structure as a truss

Calculating a plane truss, hereafter referred to as truss, is based on the following assumptions:

- all members are straight;
- all members are connected at hinged joints;
- the load consists of forces that act in the plane of the structure and apply at the joints.

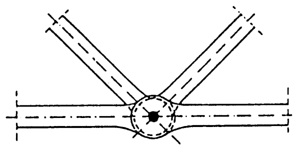


Figure 9.2 A hinged joint.

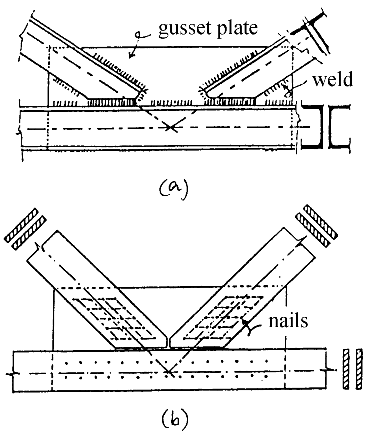


Figure 9.3 In trusses, the member axes intersect at one point. The members are usually rigidly connected to one another by a gusset plate. The joints (a) in a steel truss and (b) in a wooden truss are examples of this.

This implies that all the members in the truss behave as two-force members and can only transfer tensile and compressive forces between the joints (see also Sections 3.2.2 and Figure 3.35).

In the past, one tried to realise the connections in the joints as real hinges (see Figure 9.2). These days, all the members are rigidly connected, either directly or via a so-called *gusset plate*. Figure 9.3 shows two examples of a joint with a gusset plate: one made of steel (a) and the other made of wood (b). It is clear that these joints are not hinged. One can show, however, that whether or not the joints are hinged, this in fact has little impact on the force flow. A condition is, however, that the member axes intersect at the joints – clearly the case in Figure 9.3 – and that the load is applied at the joints.[1] This must be taken into account seriously when designing a truss.

Figures 9.4a to 9.4d show four structures with rigid joints, for which the load consists of forces that act at the joints. These structures behave as trusses, and can be calculated as such only if the structure remains kinematically determinate when all the rigid joints are replaced by hinged joints.

Figures 9.4e to 9.4h show the same structures as in (a) to (d), but now with hinged joints. After applying hinges, structures (a) and (b) are kinematically determinate and can therefore be considered trusses. With structures (c) and (d), a mechanism is formed after introducing hinged joints. They are now kinematically indeterminate and cannot be calculated as trusses. The force flow in these structures occurs mainly by bending.

The simple truss bridge in Figure 9.5 shows how to ensure that the load on the bridge ends up at the joints of the truss. The bridge consists of two *main beams* constructed as plane trusses. *Cross beams* have been introduced

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[1] The proof for this cannot be given at this stage, but is based on the characteristic that the members in a truss are relatively weak with respect to bending, and relatively stiff with respect to extension (changing length).

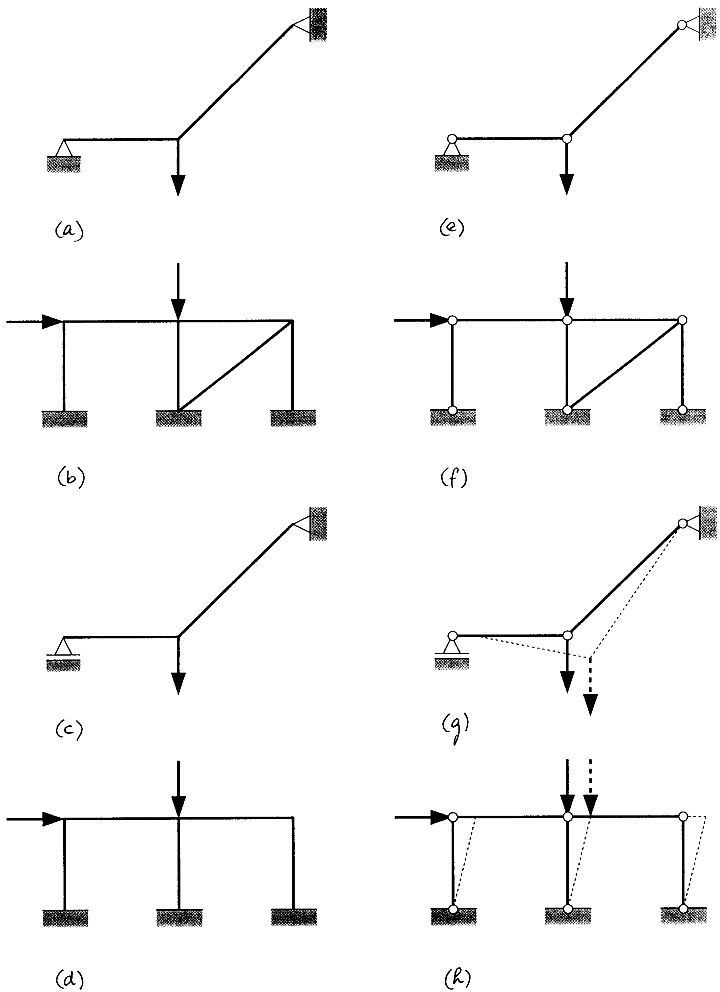

Figure 9.4

Figure 9.4 (a) to (d) Four structures with rigid joints and loaded by forces at the joints. (e) to (h) The same structures, but now all the rigid joints are replaced by hinged joints. With hinged joints (a) ad (b) are kinematically determinate and can be considered to be trusses. For (c) and (d), the use of hinged joints generates a mechanism; they cannot be considered trusses. The force flow in (c) and (d) occurs mainly by bending.

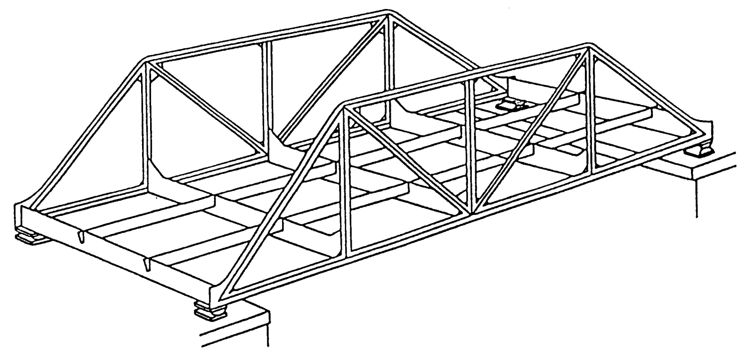

Figure 9.5 The structure of a simple truss bridge.

between the main beams, which are supported at the joints of the truss. Between the cross beams, *stringers* carry the *deck* (not shown). In this way, the traffic loading is directed via deck, stringers and cross beams as joint loads onto the trusses.

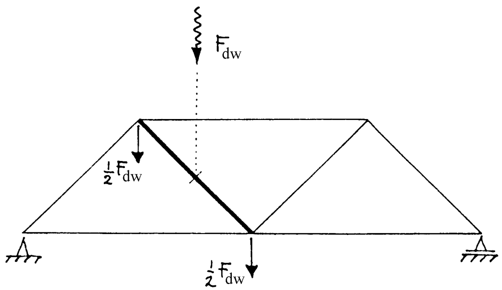

Figure 9.6 We assume that the dead weight of a truss applies in the joints. The total dead weight F_{dw} of a truss member is equally distributed over both adjacent joints.

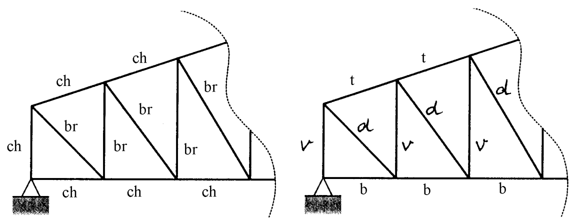

Figure 9.7 The members along the chord or circumference of the truss are chord members (ch), the others are known as bracing members (br). Chord members can be divided into top chord members (t) and bottom chord members (b), while for bracing members we distinguish between verticals (v) and diagonals (d). A vertical chord member is also referred to as a vertical.

One also often assumes that the *dead weight* of a truss applies at the joints. The total dead weight F_{dw} of a truss member is split up into two equal forces in both adjacent joints (see Figure 9.6). This is a rough model of reality, but since the dead weight is generally small with respect to the other loads that the truss has to bear, the deviations that occur are relatively small.

9.1.3 Nomenclature members and truss types

Figure 9.7 shows part of a truss. The letters show the names of the members in the truss. The members along the chord or perimeter of the truss are called *chord members* (ch), the others are referred to as *bracing members* (br). Chord members can be divided into *top chord members* (t) and bottom chord members (b). For bracing members, we distinguish between *verticals* (v) and *diagonals* (d), depending on whether the members are positioned vertically or obliquely. Vertical chord members are also referred to as verticals. In certain cases, one distinguishes between *rising diagonals* (rd) and *falling diagonals* (fd), depending on their position, seen from the perspective of the nearest support, towards the centre (see Figure 9.8).

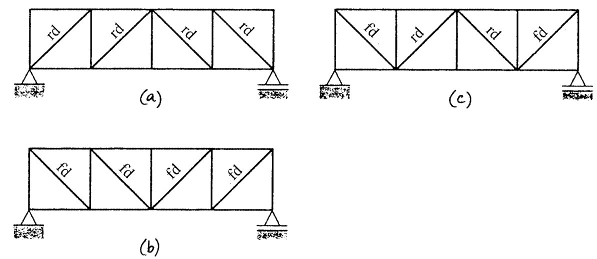

Figure 9.8 Trusses with (a) rising diagonals (rd), (b) falling diagonals (fd) and (c) alternating falling and rising diagonals.

In the following you will find a number of types of trusses. Several trusses have been named after their designer or after the region where they were developed. We will not discuss this nomenclature further, which differs per language area. We will also not address the benefits and disadvantage of the various trusses. We will briefly discuss only the motive for choosing rising or falling diagonals.

Figure 9.9 shows a number of trusses that are commonly used in roofs.

In a *Belgian truss* (a) the bracing consists of members at right angles to the top chord, and diagonals. In an *English truss* (*Howe truss*) (b) the bracing consists of verticals and diagonals. Trusses (c) and (d) have gently sloping top chords and alternating rising and falling diagonals. Truss (c) is suitable for a transom window. In a *Polonceau truss* (*Fink truss*) (e) one can recognise a three-hinged truss with a tie rod. Truss (f) is used in *saw tooth roofs*; glass is placed in the sheer sloping roof planes.

The truss in Figure 9.10 can be used in canopies and is therefore also referred to as a *canopy truss*.

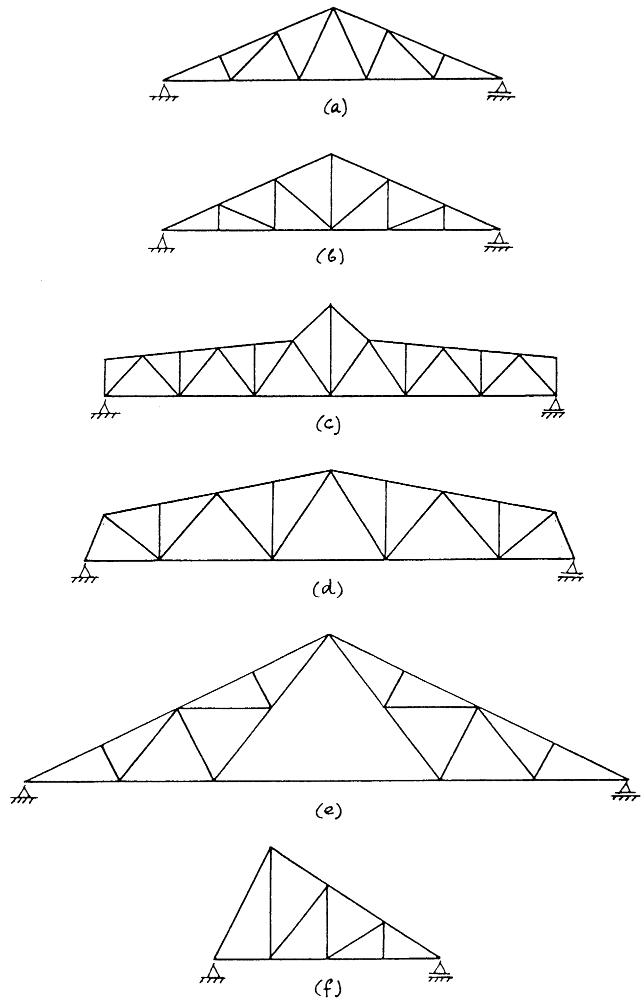

Figure 9.9 Trusses applied in roofs.

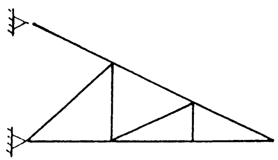

Figure 9.10 A canopy truss.

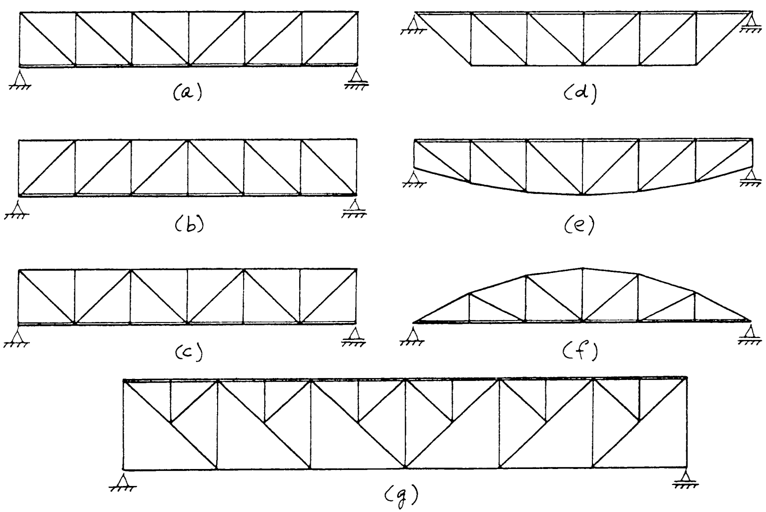

Figure 9.11 Trusses applied in bridges. The bridge deck is shown by means of a double line.

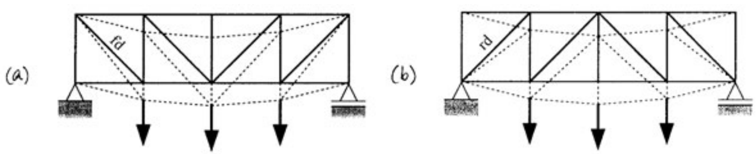

Figure 9.12 From the expected deformation due to a full load, we can deduce that (a) falling diagonals will extend and be subject to tension, and that (b) rising diagonals will shorten and be subject to compression.

You will find the trusses in Figure 9.11 in bridges. The deck is shown by means of a double line. Bridges (a), (b), (c) and (f) have a *lower deck*. The other bridges have an *upper deck*.

Since these trusses have the same function as a beam, they are often called *truss beams*. Trusses (a), (b), (c) and (g) are known as *parallel truss beams*, as a result of their parallel top and bottom chords. If the end verticals are omitted from a parallel beam, as in truss (d), the truss is referred to as a *trapezoidal truss beam*.

Truss beam (e) has a *curved bottom chord*. Truss beam (f) has a *curved top chord*. In a curved chord, the joints of the chord are located on a curve. The chord members are straight. The curve is often a parabola. This is known as a *parabolic truss beam* if the points of support are also part of the parabola, as in truss (f). If this is not the case, as in truss (e), it is called a *half-parabolic truss beam*.

Truss (g) is found in large spans. By creating an auxiliary truss within the main truss, additional points of support are created for the bridge deck, allowing the structure to be lighter.

Trusses (a) and (d) to (g) have falling diagonals, truss (b) has rising diagonals, and in truss (c) the diagonals alternate between falling and rising.

Due to the dead load, falling diagonals are loaded by tensile forces, and rising diagonals are loaded by compressive forces. This is shown in Figure 9.12 in a general sketch of the expected deformation of the truss beam subject to full loading. The falling diagonals in case (a) extend and are loaded by tensile forces. The rising diagonals in case (b) shorten and are loaded by compressive forces.

In steel trusses, falling diagonals (tension diagonals) are used most frequently, as (usually slender) steel members subject to compression run the risk of buckling. Preferably apply them as tension members.

In contrast, rising diagonals (compression diagonals) are most often used in wooden trusses, as in general a wooden joint is more suitable to transfer compressive forces rather than tensile forces.

An example close to home is the simple *garden gate* in Figure 9.13, with (a) a steel version (falling diagonal) and (b) a wooden version (rising diagonal). The wooden *lock-gate* in Figure 9.14 is another example. The wooden *diagonal strut* is a rising diagonal and acts as a compression member under influence of the dead weight of the gate. The wooden planking is facing the same way as the diagonal strut. The steel falling diagonal is a *tension bar*.

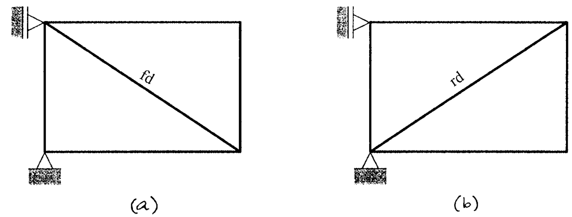

Figure 9.13 Model of (a) a steel garden gate with falling diagonal (tension diagonal) and (b) a wooden garden gate with rising diagonal (compression diagonal).

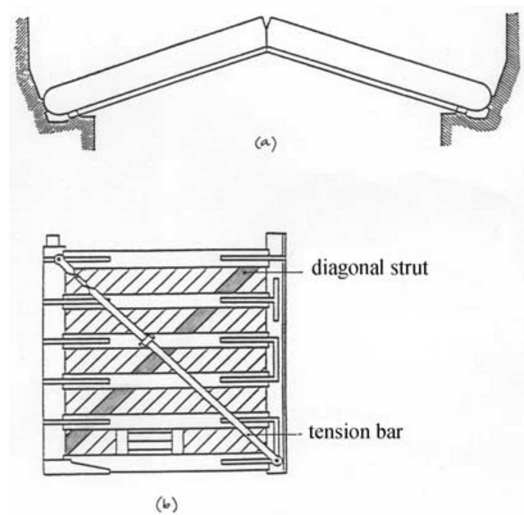

Figure 9.14 (a) The mitre gates of a simple navigation lock in closed position. (b) Interior view of the left-hand lock door, as found in older wooden mitre gates, with a diagonal strutt (compression diagonal) and a steel tension bar (tension diagonal).

9.1.4 Labelling joints and members

The joints in a truss are numbered or lettered (see Figure 9.15). The numbers or letters used to indicate the joints can be used as an index. For example, x_4; y_4 gives the x and y coordinates of joint 4, and $F_{x;\mathrm{C}}$ is the x component of force F on joint C. It is customary to use the joint label as *sub-index*.

Members are always numbered. The member numbers are often placed between brackets. For quantities that relate to a particular member, the member number is used as an *upper index*. The length ℓ of member (2) is recorded as $\ell^{(2)}$, and $N^{(1)}$ is the normal force in member (1).

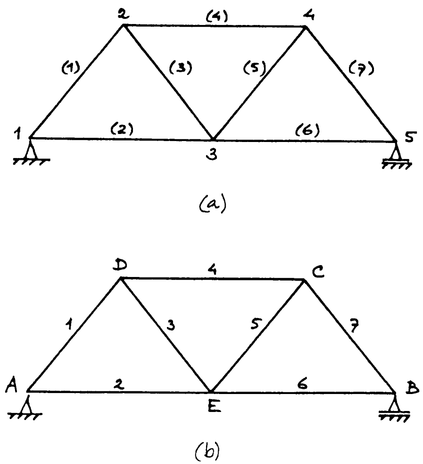

Figure 9.15 Joint and member numbering in (a) computer calculations and (b) manual calculations.

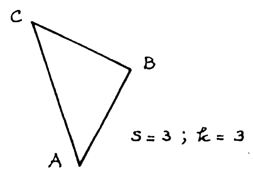

Figure 9.16 A triangle is the basic form of a simple or self-contained truss, defined as a truss that can retain its shape.

In computer calculations, it is customary to use the labelling in Figure 9.15a; computer programs can deal better with numbers than with letters. In manual calculations, the labelling in Figure 9.15b is used most. Occasionally, the brackets about the member numbers are omitted. Their context must then show whether ℓ^2 means "*the square of* ℓ", or "*the length of member* 2". If there is a chance of confusion, the member number has to remain between brackets.

9.2 Kinematically/statically (in)determinate trusses

In this section, we discuss the relationship between the number of members and joints in a simple or self-contained truss and a compound truss respectively. Subsequently a systematic procedure will be introduced to calculate the degree of kinematic/static (in)determinacy of a truss and the relationship between the number of support reactions, members, and joints.

9.2.1 Simple and compound trusses

A *simple* or *self-contained*[1] *truss* is defined as a truss that retains its shape. The basic element of a simple truss is the triangle with $s = 3$ members and $k = 3$ joints, like triangle ABC in Figure 9.16.

Unlike a triangle, a (hinged) quadrangle cannot retain its shape.[2] Figure 9.17 shows the displacements with respect to AG for quadrangle ABEG. One can imagine that BE is connected with AG via the two-force members AB and EG. The displacement of BE with respect to AG consists of a rotation about $\mathrm{RC}^{(\mathrm{BE})}$, the *centre of rotation* of BE, that coincides with

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1 The concept *self-contained* was covered earlier in Section 4.5.1.

2 The open circles for hinged joints are consistently omitted (see Section 9.1.1).

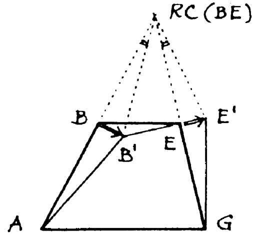


Figure 9.17 A hinged quadrangle cannot retain its shape.

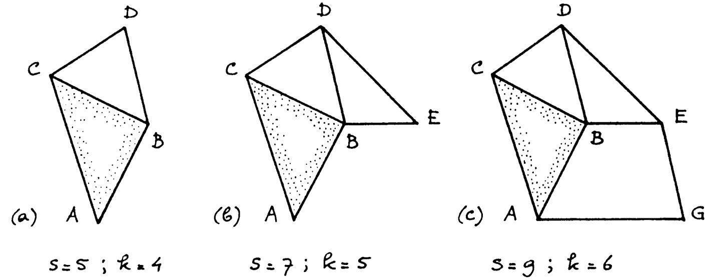


Figure 9.18 Based on a simple triangle, we can repeatedly create a new joint by adding two members.

the intersection of two-force members AB and EG. See also Section 4.5.1 and Figure 4.38c. When looking at the deformed quadrangle once more, it is important to note that the displacements are depicted large in the figure as compared to the length of the members.

The simplest way of constructing self-contained trusses is to start with a triangle, and, as in Figure 9.18, repeatedly create a new joint with two members. To retain its shape the truss does not have to consist only of triangles. For example, the quadrangle ABEG from Figure 9.17 is found again in the self-contained truss in Figure 9.18c.

Figure 9.19 shows a number of trusses that were constructed using this method. Trusses (a) and (b) consist entirely of triangles and are clearly self-contained. This is harder to determine for trusses (c) and (d) as they do not consist entirely of triangles.[1] They retain their shape however as they can be constructed from the dark triangle in the middle by repeatedly creating a new joint by adding two members to two existing joints.

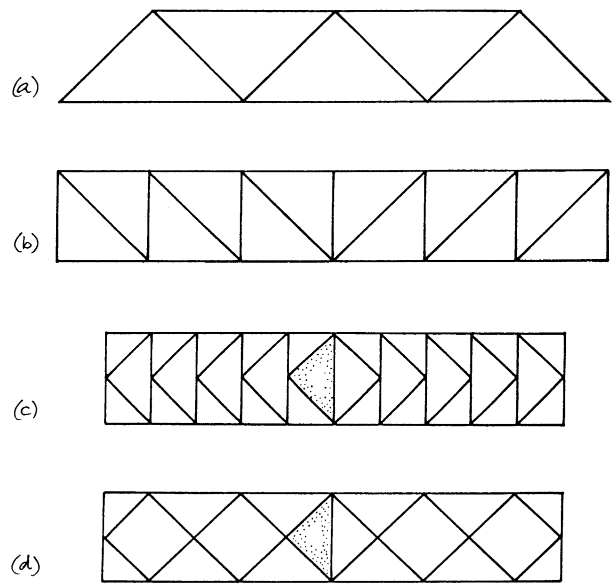


Figure 9.19 Simple trusses constructed in the way shown in Figure 9.18. In (c) and (d) we can start with the dark triangle in the middle. For all these trusses it holds that $s = 2k - 3$.

1 The eight triangles in truss (c) are not “real” triangles but quadrangles.

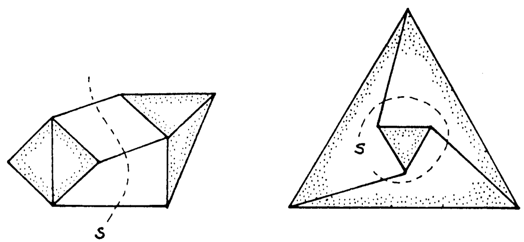


Figure 9.20 Simple trusses with a more complicated structure. The two dark self-contained parts are connected by three members. The formula $s = 2k - 3$ is also applicable to these trusses.

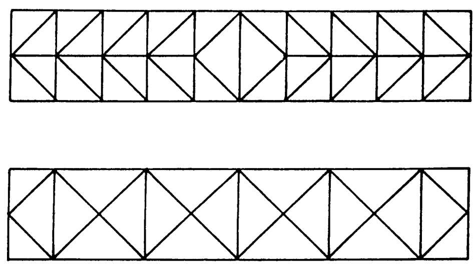


Figure 9.21 Simple trusses that contain more members than needed for being self-contained. For these trusses, it holds that $s > 2k - 3$.

The following relationship holds between the number of members s and the number of joints k for a truss created in the way described above:

$$s = 2k - 3.$$

This can be derived as follows. Three members are needed for the first three joints in the truss, which forms the first triangle. For the remaining $(k - 3)$ joints $2(k - 3)$ members are needed. The total number of members s is therefore:

$$s = 3 + 2(k - 3) = 2k - 3.$$

Figure 9.20 shows two examples of simple trusses that cannot be constructed as shown in Figure 9.18. They clearly have a more complicated structure. If we look more closely, we notice that the structures consist of two dark coloured simple trusses of the type described earlier, which are connected to one another by three members. The structures retain their shape only when the three members do not intersect at one point, and neither are parallel. In the figure, a section s has been introduced across the three members. The same formula $s = 2k - 3$ also applies to these more complicated trusses.

The formula $s = 2k - 3$ is a minimum condition for a truss that will retain its shape. By adding additional members to a simple truss, without creating new joints, the structure remains self-contained. In this way, the trusses in Figure 9.21 were created by adding additional members to trusses (c) and (d) in Figure 9.19. The trusses are still self-contained, but now the number of members is

$$s > 2k - 3$$

One would imagine that a truss is always self-contained if the number of members s is at least equal to $2k - 3$. This is a misconception, however, as is shown for the truss in Figure 9.22 with $s = 16$ and $k = 9$. This truss consists of two self-contained parts, which could both lose a member without losing their shape. Both parts are connected by means of a hinge, and can move with respect to one another. The structure is therefore *not self-contained*, although the number of members $s = 16$ is greater than $2k - 3 = 15$.

Hereafter, a *truss* that cannot retain its shape is referred to as a *compound truss*.[1]

The formula $s = 2k - 3$ is clearly not a good criterion for a truss that will retain its shape. One can say that for each self-contained truss, the following relationship must hold:

$$s \geq 2k - 3.$$

The reverse is not true, however. Not every truss with $s \geq 2k - 3$ is self-contained. This is demonstrated by the counterexample in Figure 9.22. The formula does not indicate the *functionalism* for which the various members were introduced. The formula $s \geq 2k - 3$ is a *necessary* although *insufficient* condition for a truss that will retain its shape.

To summarise:

$s < 2k - 3$ The truss is a compound truss (the truss cannot retain its form).

$s \geq 2k - 3$ Necessary condition for a self-contained truss, but not a sufficient condition. As a result of the application of inefficient members, the truss may still not be capable of retaining its shape. One can be sure only when the truss has been investigated from joint to joint.

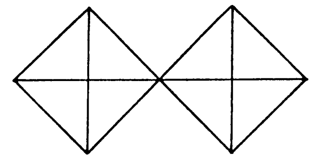


Figure 9.22 A truss that cannot retain its shape is called a compound truss.

[1] The literature often defines compound trusses as those of the type in Figure 9.20, but sometimes also those in Figure 9.22. Here, as in Section 4.5.3, a compound truss is defined as one that, when isolated from its supports, is not capable of retaining its shape.

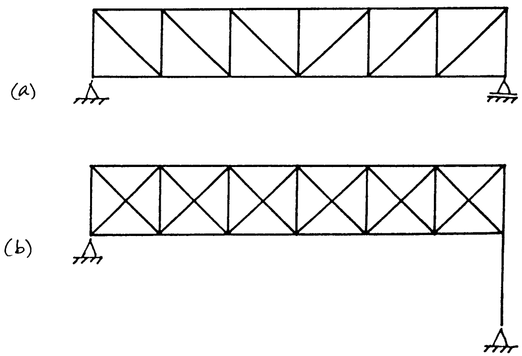


Figure 9.23 (At least) three support reactions are needed for an immovable support of a simple truss. Here they are provided by (a) a hinged and roller support and (b) a hinged and bar support.

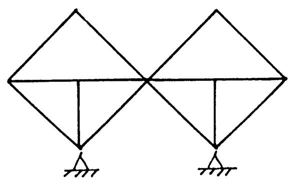


Figure 9.24 More than three support reactions are needed for an immovable support of a compound truss, as the internal degrees of freedom also have to be eliminated. In this case, four support reactions are required, provided by two hinged supports.

If the truss retains its shape, the following cases can be distinguished:

$s = 2k - 3$ The truss needs all the members to retain its shape.

$s > 2k - 3$ The truss can miss $s - (2k - 3)$ members without losing its capability to retain its shape. These members cannot be selected arbitrarily; they are determined by the way the truss is assembled.

9.2.2 Determining kinematic/static (in)determinacy

If a truss is supported so that is has no possibility of moving, the truss is defined as *immovable* or as *kinematically determinate*. This type of truss can resist all types of load. If a truss is to be kinematically determinate, it needs at least as many support reactions as degrees of freedom; one degree of freedom is removed for each support reaction (interaction force between truss and the immovable environment).

A simple truss may be considered as a rigid body. Since (in a plane) it has three degrees of freedom (one rotation and the two components of a translation), at least three independent support reactions are needed for immovability. For example by means of a hinged support together with a roller support or bar support, as shown in Figure 9.23.

Compound trusses can be seen as systems of rigid bodies that have a certain degree of freedom with respect to one another. The possible movements with respect to one another are known as the *internal degrees of freedom*. The immovability of a compound truss always needs more than three support reactions, as the internal degrees of freedom also have to be eliminated. In this way, the truss in Figure 9.24 is not shape-retaining in itself, as the two constituent parts can rotate with respect to one another. The two hinged supports ensure the kinematic determinacy of the truss.

In the examples, bar supports, roller supports, and hinged supports have been used. It should be clear that fixed supports are not used in trusses.

Instead of looking at the degrees of freedom for simple or compound trusses, we can also determine how many support reactions are needed to keep the truss in equilibrium under every imaginable load. This procedure was explained in Section 4.5.3 for an arbitrary structure. The answers are somewhat easier to determine for trusses as here the members can transfer only tensile and compressive forces between the joints. These forces are called normal forces. Normal forces are represented by means of a capital letter N. According to the *sign convention* N is positive for a tensile force, and negative for a compressive force.

The truss in Figure 9.25a is supported at A on a hinge and at B on a roller, and is loaded by the forces F_1; F_2; F_3. In Figure 9.25b, all the joints in the truss have been isolated. It has been assumed that all the member forces N are positive (all the members transfer tensile forces, and therefore pull at the joints).

An arbitrary truss has k joints, s members and r support reactions. The unknown force quantities in the truss are then the r support reactions and the s member forces. In total, there are therefore $(r + s)$ unknowns in the truss.

The equilibrium can be investigated for each joint. The conditions for moment equilibrium are automatically met as all the forces intersect at the joint. All that remains is the force equilibrium. Two equations can be created per joint. These contain both known forces (the loads) and unknown forces (member forces and support reactions). With k joints, there are therefore $2k$ equilibrium equations.

Let n be the difference between the number of unknown forces and the number of available equilibrium equations:

$$n = r + s - 2k.$$

If $n < 0$, there are more equations than unknowns. It is always possible to choose the (arbitrary) load in such a way that a number of the (redundant)

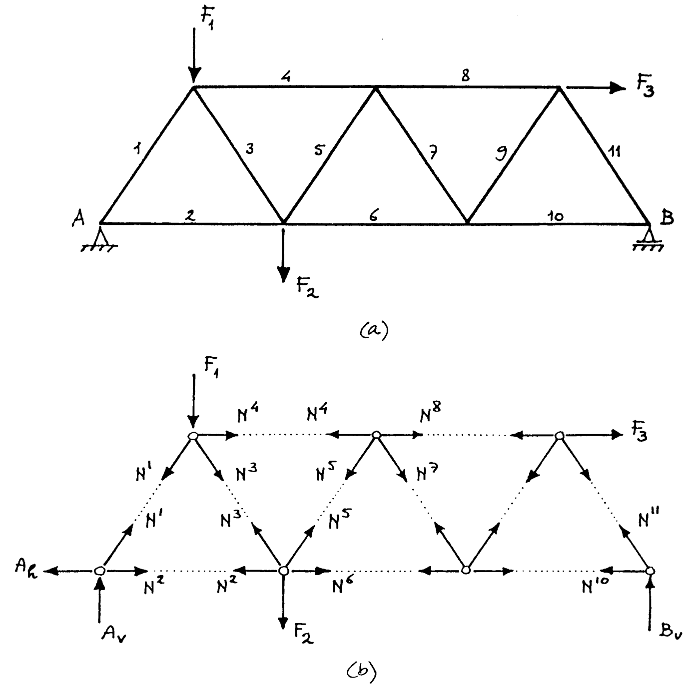


Figure 9.25 (a) A truss for which in (b) all the joints have been isolated. It has been assumed that the normal force N in each member is a tensile force. If so, all members pull at the joints.

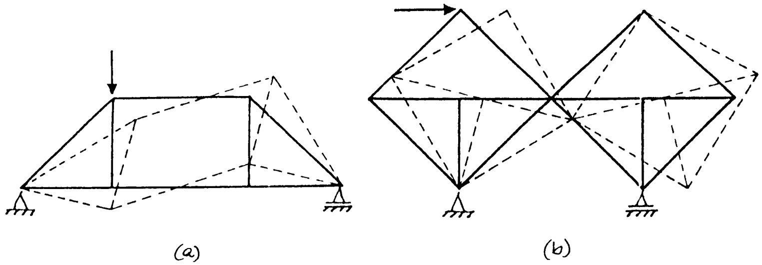


Figure 9.26 Kinematically indeterminate trusses or mechanisms. (a) $n = -1$; a diagonal member is missing in the middle. (b) $n = -1$; the hinged and roller support are insufficient to eliminate all possible movement of the compound truss.

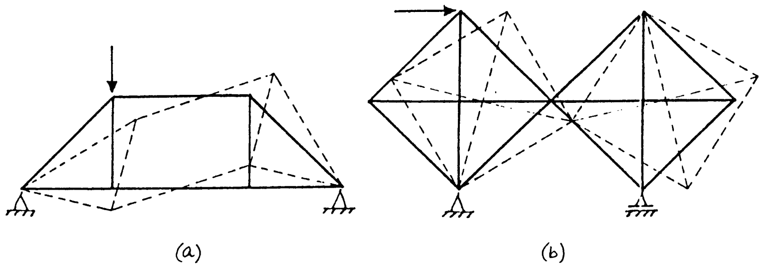


Figure 9.27 Kinematically indeterminate trusses or mechanisms with $n \geq 0$: (a) $n = 0$ and (b) $n = 1$.

equations become inconsistent. This means that, with that load, the equilibrium conditions cannot be met at all the joints. The truss is a mechanism and is kinematically indeterminate (not immovable). Examples are shown in Figure 9.26.

In Figure 9.26a (with $r = 3$, $s = 8$ and $k = 6$, and therefore $n = -1$) the kinematic indeterminacy results from the missing diagonal member in the centre field. In Figure 9.26b (with $r = 3$, $s = 14$ and $k = 9$, and therefore $n = -1$), the method of support is inadequate to remove all the degrees of freedom of the compound truss.

From the above, we can conclude that $n \geq 0$ is a *necessary condition* for kinematic determinacy. Since the value of n is the result of a calculation in which the *functionalism* of the members and supports present is not taken into account, this necessary condition is an *insufficient condition*. Even when $n \geq 0$, there is always the possibility that the structure is kinematically indeterminate. Examples of this are shown in Figure 9.27.

The structure in Figure 9.27a (with $r = 4$, $s = 8$, $k = 6$, and therefore $n = 0$) is the same as the structure in Figure 9.26a, except that the roller support is replaced by a hinged support. Since, for motion as a mechanism, the roller in Figure 9.26a remains in place, this change makes no difference whatsoever – the truss remains kinematically indeterminate.

In Figure 9.27b (with $r = 3$, $s = 16$, $k = 9$, and therefore $n = 1$) the support of the compound truss is equally inadequate as in Figure 9.26b. The only difference is that the two constituent parts now contain more members than required for retaining their shape.

The kinematic determinacy of a truss cannot be assessed based on a calculation alone; one always has to take the construction of the truss into account.

With $n = r + s - 2k$ the following is true for a truss:

$n < 0$ *The truss is kinematically indeterminate*. This is also known as a mechanism.

$n \geq 0$ Necessary but insufficient condition for a *kinematically determinate truss*. As a result of non-effective members and/or supports, the truss can still be kinematically indeterminate and a mechanism.

In kinematically determinate trusses, $n \geq 0$, and as in Section 4.5.3 we can distinguish the following cases:

$n = 0$ *The truss is statically determinate.*
The number of unknowns is equal to the number of available equilibrium equations. All unknowns (member forces and support reactions) can be derived directly from the equilibrium.

$n > 0$ *The truss is statically indeterminate.*
There are more unknowns than equilibrium equations. One or more of the member forces and/or support reactions cannot be determined directly from the equilibrium. In principle, there is an infinite number of solutions that satisfy the equilibrium conditions (the solution is undetermined). The correct solution can be found by taking into account the deformation behaviour of the structure. The surplus of unknowns, n, is known as the *degree of static indeterminacy*.

Figure 9.28 provides examples of statically determinate trusses.

The shape-retaining truss ABCD in Figure 9.28a is immovable supported by a hinge at A and a bar at B. The hinged support provides two support reactions, and the bar support provides one, so that $r = 3$. With $s = 25$ and $k = 14$, $n = 0$. The truss is therefore statically determinate. If the bar support is considered as one of the truss members, B′ has to be seen as a hinged support. In that case, $r = 4$, $s = 26$ and $k = 15$, and again $n = 0$.

In the simple truss in Figure 9.28b, the diagonal members cross one another. The truss is immovable, supported on a roller and by a hinge. Here $r = 3$, $s = 13$ and $k = 8$, so that $n = 0$. The truss is therefore statically determinate.

The compound truss in Figure 9.28c is also immovable supported. With $r = 4$, $s = 14$ and $k = 9$, $n = 0$. The truss is statically determinate.

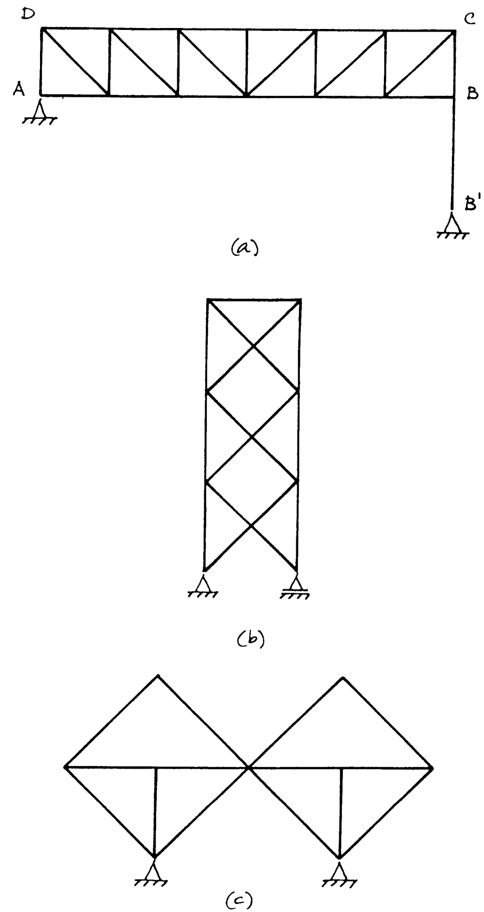


Figure 9.28 Statically determinate trusses.

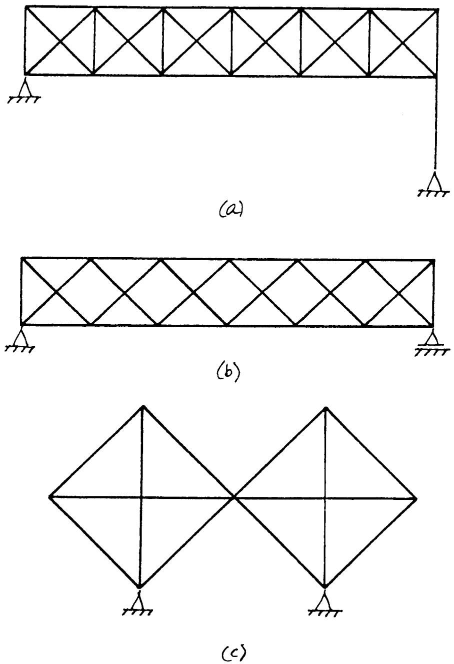


Figure 9.29abc Statically indeterminate trusses. Trusses (a) to (c) are supported with static determinacy. These trusses are also said to be externally statically determinate and internally statically indeterminate.

Figure 9.29 provides examples of statically indeterminate trusses. All the trusses are kinematically determinate. The degree of static indeterminacy can be determined with

$$n = r + s - 2k.$$

The truss in Figure 9.29a, in which the diagonal members cross one another, has a hinged support and a bar support. With $r = 3$, $s = 31$, $k = 14$, one finds $n = 6$. The truss is six-fold statically indeterminate. If we compare the truss with the statically determinate structure in Figure 9.28a, we see that the truss has 6 redundant diagonal members.

For the truss in Figure 9.29b, with crossing diagonals, $r = 3$, $s = 26$, $k = 14$ and so $n = 1$. The truss is therefore statically indeterminate to the first degree.

For the compound truss in Figure 9.29c, $r = 4$, $s = 16$ and $k = 9$, so that $n = 2$. A member could be omitted in each of two self-contained parts (see also Figure 9.28c).

The truss in Figure 9.29d is statically indeterminate to the first degree, with $r = 4$, $s = 19$, $k = 11$ and so $n = 1$. The structure can be made statically determinate by, for example, removing one of the roller supports. You could also remove an arbitrary top or bottom chord member.

For the truss in Figure 9.29e, $r = 6$, $s = 13$ and $k = 8$, so that $n = 3$. The truss is statically indeterminate to the third degree. The simple truss has three redundant support reactions and/or members.

In statically determinate trusses, all the force members and support reactions can be determined directly from the equilibrium. This is not possible for statically indeterminate trusses. Sometimes, for statically indeterminate trusses, it is possible to find all the support reactions from the equilibrium equations, but not all the member forces. Examples of this type of truss are shown in (a) to (c) in Figure 9.29. The support of these trusses is statically determinate. Their static indeterminacy is caused by redundant members in their self-contained parts. These types of trusses are also known as *externally statically determinate* and *internally statically indeterminate*.

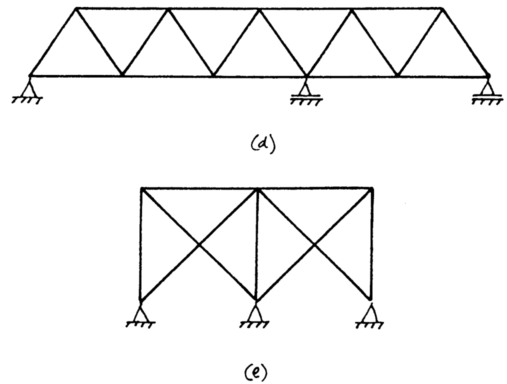


Figure 9.29de Statically indeterminate trusses.

9.3 Determining member forces

There are various methods for calculating member forces in statically determinate trusses. We will look at two:

- the method of sections;
- the method of joints.

In the *method of sections*, one introduces a suitable section across the truss and calculates the member forces from the equilibrium of one of the isolated parts. In the *method of joints*, we consistently determine the member forces from the equilibrium of the joints.

9.3.1 Method of sections

In the *method of sections*, the member forces in a (statically determinate) truss are determined by introducing a *section* and investigating at the equilibrium of one of the isolated parts. Since there are only three equilibrium equations available, you have to select a section such that there are no more than three unknowns. In general, the support reactions have to be determined previously. The method is demonstrated using a number of examples.

Example 1

The first example relates to the truss beam in Figure 9.30, with parallel top chord and bottom chord. The load consists of the two vertical forces shown in the figure of respectively 120 kN and 40 kN.

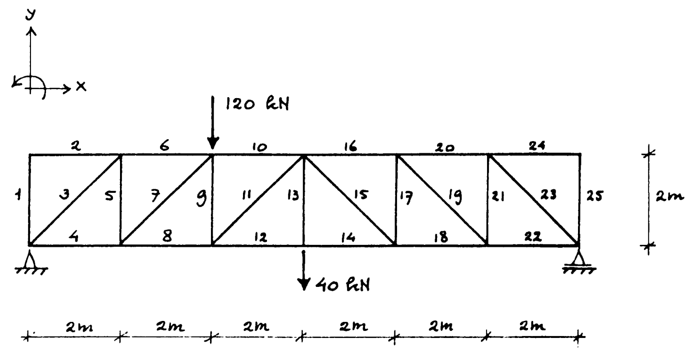


Figure 9.30 Truss, with parallel top and bottom chord, for which the forces in members 6 to 9 and in member 13 have to be calculated using the method of sections.

Question:

Determine the forces in the members 6 to 9 and in member 13, with the correct signs for tension and compression. In the calculation, use the

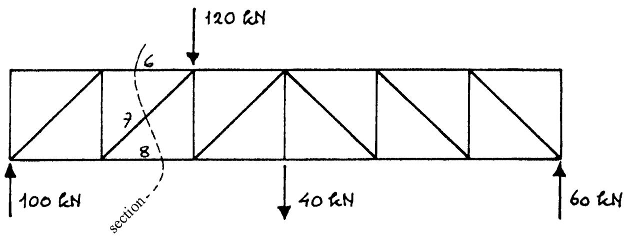


Figure 9.31 The isolated truss with support reactions. To calculate the forces in members 6, 7 and 8, a section is introduced across these members in the truss.

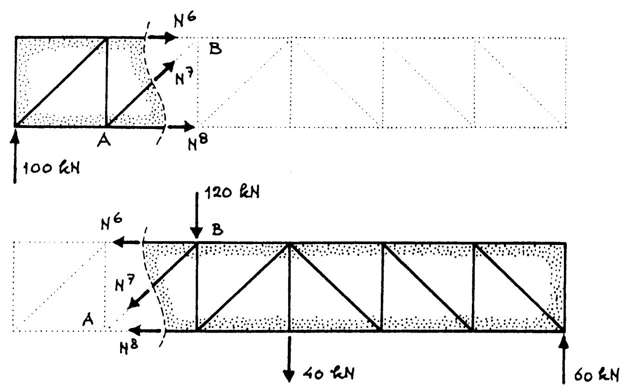


Figure 9.32 The isolated parts to the left and right of the section across members 6, 7 and 8. The interaction forces, the normal forces N, are shown as tensile forces because tensile forces are by definition positive.

coordinate system shown.

Solution:
In Figure 9.31, the truss has been isolated and the support reactions are shown. For calculating the forces in members 6, 7 and 8, we introduce a section across these members.

In Figure 9.32, the parts to the left and to the right of the section have been isolated. The as yet unknown member forces N^6, N^7 and N^8 are introduced as tensile forces. Here we use the sign convention that the normal force in a member is positive when it is a tensile force. If the member has to transfer a compressive force, this will become clear later through a negative value for the normal force N.

The normal force N^6 in member 6 is most easily determined by looking at the moment equilibrium of the left-hand part about intersection A of members 7 and 8:

$$\sum T_z|\mathrm{A} = -(100\text{ kN})(2\text{ m}) - N^6 \times (2\text{ m}) = 0 \Rightarrow N^6 = -100\text{ kN}.$$

The minus sign shows that member 6 is a *compression member*. The 100 kN force is therefore acting opposite to the direction shown in Figure 9.32.

Instead of the left-hand part, we can also look at the right-hand part. From the moment equilibrium about A of the right-hand part, it follows that

$$\sum T_z|\mathrm{A} = -(120\text{ kN})(2\text{ m}) - (40\text{ kN})(4\text{ m}) + (60\text{ kN})(10\text{ m}) + \\ + N^6 \times (2\text{ m}) = 0.$$

Of course, $N^6 = -100$ kN also here, except that it took a little more work to find the answer as more forces are acting on the right-hand part than on the left-hand part.

When calculating the member forces, it does not make a difference whether you look at the equilibrium on the left-hand side or the right-hand side of the section. It is sensible to choose the part that offers the simplest calculation.

The force in member 7 is most easily determined from the vertical force equilibrium for the part to the left of the section:

$$\sum F_y = (100\text{ kN}) + \tfrac{1}{2}N^7\sqrt{2} = 0 \Rightarrow N^7 = -100\sqrt{2}\text{ kN}.$$

Diagonal member 7 is also a *compression member*. Calculating this member force is easy as the parallel top and bottom chords members do not have a vertical component.

Member force N^8 is most easily determined from the moment equilibrium of the left-hand part about the intersection of members 6 and 7:

$$\sum T_z|\text{B} = -(100\text{ kN})(4\text{ m}) + N^8 \times (2\text{ m}) = 0 \Rightarrow N^8 = +200\text{ kN}.$$

Bottom chord member 8 is a *tension member*.

To check the above, we calculate whether there is horizontal force equilibrium in the left-hand part:

$$\begin{aligned}\sum F_x &= N^6 + \tfrac{1}{2}\sqrt{2} \times N^7 + N^8\\ &= (-100\text{ kN}) + \tfrac{1}{2}\sqrt{2} \times (-100\sqrt{2}\text{ kN}) + (200\text{ kN}) = 0.\end{aligned}$$

With the values found for N^6, N^7 and N^8, the conditions for horizontal force equilibrium are indeed satisfied.

Please note the parallel with calculating the support reaction for a structure on three bar supports, as in Examples 2 and 3 in Section 5.1.

The forces in the other members of the truss can be determined in the same way. For example, we find the force in member 9 by introducing a section across members 8, 9 and 10, as shown in Figure 9.33. From the vertical

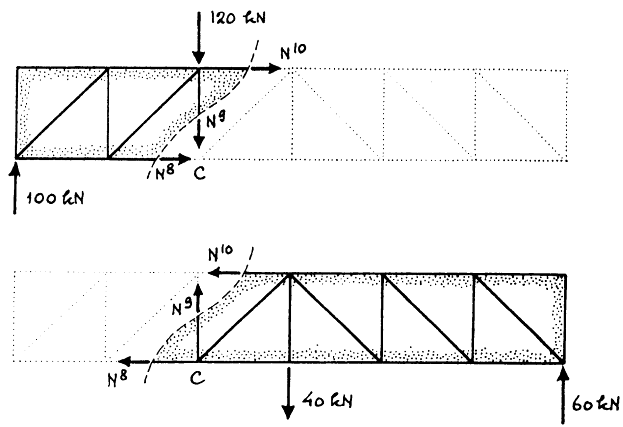


Figure 9.33 The section for calculating the forces in members 8, 9 and 10.

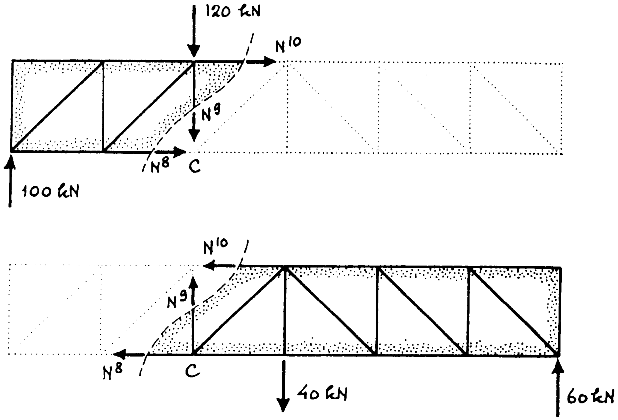


Figure 9.33 The section for calculating the forces in members 8, 9 and 10.

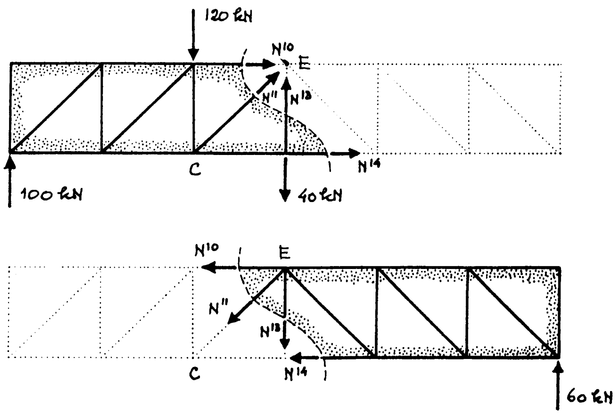


Figure 9.34 The section across member 13 cuts four members, one too many to be able to determine all the member forces from the equilibrium. With this section, we can only find N^{14} from the moment equilibrium about E.

force equilibrium for the left-hand part it follows that

$$\sum F_y = (100\text{ kN}) - (120\text{ kN}) - N^9 = 0 \Rightarrow N^9 = -20\text{ kN}.$$

Member 9 is a compression member.

When determining the force in member 13, the problem arises that a section across member 13 cuts more than three members. The section in Figure 9.34. for example, cuts through members 10, 11, 13 and 14, which is one too many to be able to determine all the forces from the equilibrium. If one of the member forces N^{10} of N^{11} is known, then it is possible to determine the other three. We therefore look at a second section to first determine one of the forces N^{10} or N^{11}.

From the moment equilibrium about C of one of the isolated parts in Figure 9.33 we find

$$N^{10} = -200\text{ kN}.$$

Using this information, we find from the moment equilibrium about C of one of the isolated parts in Figure 9.34 (here we select the right-hand part) the force in member 13 is

$$\sum T_z|\text{C} = -N^{10} \times (2\text{ m}) + (60\text{ kN})(8\text{ m}) - N^{13} \times (2\text{ m}) = 0$$
$$\Rightarrow N^{13} = +40\text{ kN}.$$

By chance, we can also find the force in member 13 using an easier method, namely by using the section in Figure 9.35 across the members 12, 13 and 14. N^{13} is found from the vertical force equilibrium of one of the parts. Here, we actually determine the force N^{13} from the force equilibrium of joint D, where three members come together, of which two in a direct line.

Note that it is not possible to determine the section forces N^{12} and N^{14} from

Table 9.1 Member forces Example 1.

Mem. no. *i*	N^i (kN)	Mem. no. *i*	N^i (kN)
1	0	14	+180
2	0	15	$-60\sqrt{2}$
3	$-100\sqrt{2}$	16	−120
4	+100	17	+60
5	+100	18	+120
6	−100	19	$-60\sqrt{2}$
7	$-100\sqrt{2}$	20	−60
8	+200	21	+60
9	−20	22	+60
10	−200	23	$-60\sqrt{2}$
11	$+20\sqrt{2}$	24	0
12	+180	25	0
13	+40		

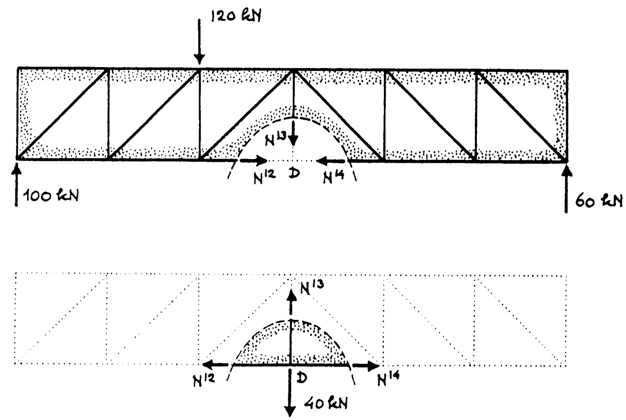


Figure 9.35 With this section across three members, we actually isolate joint D. N^3 follows directly from the vertical force equilibrium of the joint. We do not need to know N^{12} and N^{14} to do so.

the equilibrium of one of the parts isolated in Figure 9.35. It is possible to determine only that $N^{12} = N^{14}$ from the horizontal force equilibrium, but we cannot determine their magnitude.

Table 9.1 provides a summary of all the member forces in the truss.

Example 2

The second example relates to the truss in Figure 9.36, with non-parallel top and bottom chords. The load consists of a single vertical force of 120 kN.

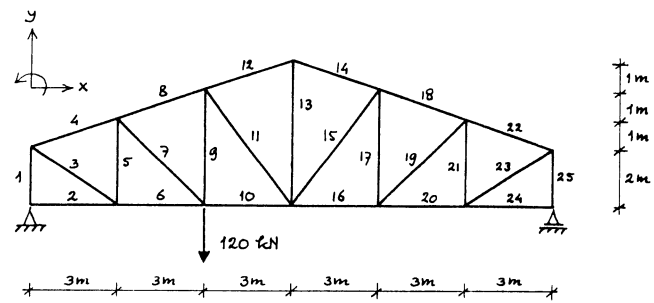


Figure 9.36 Truss with non-parallel top and bottom chord.

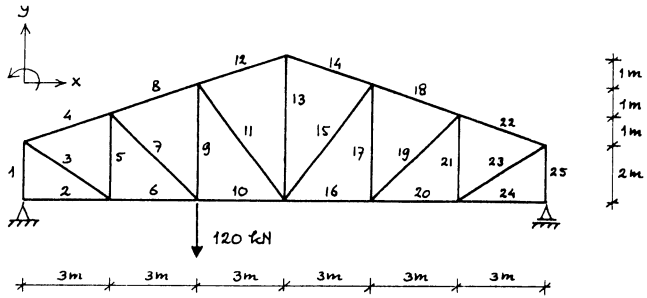


Figure 9.36 Truss with non-parallel top and bottom chord.

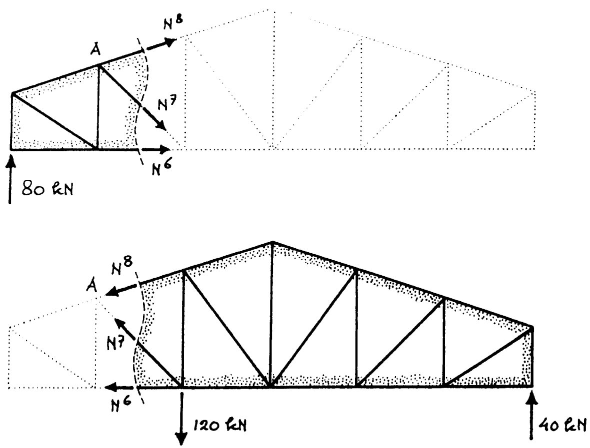


Figure 9.37 Section for calculating the forces in members 6, 7 and 8.

Question:
Determine the forces in members 6 to 9 and in member 13, with the correct sign for tension and compression. Use the coordinate system given.

Solution:
We first determine the support reactions. For the left-hand and right-hand support reactions, we find 80 and 40 kN respectively, both vertically and directed upwards.

For determining the three unknown member forces N^6, N^7 and N^8, a section has been introduced across members 6, 7, and 8 in Figure 9.37, and the parts on both sides of the section have been isolated. The member forces N^6, N^7 and N^8 follow from the equilibrium of one of the parts to the left or right of the section. The force N^6 is most easily determined. This follows directly from the moment equilibrium about intersection A of members 7 and 8. For the left-hand part, we find

$$\sum T_z|\mathrm{A} = -(80\ \mathrm{kN})(3\ \mathrm{m}) + N^6 \times (3\ \mathrm{m}) = 0 \Rightarrow N^6 = +80\ \mathrm{kN}.$$

Member 6 is a tension member.

If we use the right-hand section, the equation for the moment equilibrium about A demands a little more effort:

$$\sum T_z|\mathrm{A} = (40\ \mathrm{kN})(15\ \mathrm{m}) - (120\ \mathrm{kN})(3\ \mathrm{m}) - N^6 \times (3\ \mathrm{m}) = 0.$$

Of course, this way round we also find a tensile force of 80 kN in member 6.

The force in member 7 is found from the moment equilibrium about intersection B of the members 6 and 8 (see Figure 9.38a), where only the left-hand part is shown:

$$\sum T_z|B = (80 \text{ kN})(6 \text{ m}) - N^7 \times (6\sqrt{2} \text{ m}) = 0 \Rightarrow N^7 = +40\sqrt{2} \text{ kN}.$$

Member 7 is a tension member.

If the distance of point B to the line of action of N^7 is difficult to find (not the case here), force N^7 can be shifted along its line of action to a more suitable position.[1] In Figure 9.38b, N^7 has been shifted to point C, where it has been resolved into components. The equation for the moment equilibrium about B now only contains the vertical component of N^7:

$$\sum T_z|B = (80 \text{ kN})(6 \text{ m}) - \left(\tfrac{1}{2}N^7\sqrt{2}\right)(12 \text{ m}) = 0.$$

As found earlier, this gives $N^7 = +40\sqrt{2}$ kN. If N^6 is known, N^7 can also be determined from the moment equilibrium about a point other than B on the line of action of N^8, such as about point D.

For the left-hand part we find (see Figure 9.38b)

$$\sum T_z|D = -(80 \text{ kN})(6 \text{ m}) + N^6 \times (4 \text{ m}) + \left(\tfrac{1}{2}N^7\sqrt{2}\right)(4 \text{ m}) = 0.$$

With $N^6 = +80$ kN, we find $N^7 = +40\sqrt{2}$ kN, as expected. This sort of approach can offer benefits if the intersection B of the members 6 and 8 is far away or is difficult to find.

The force in member 8 is found from the moment equilibrium about intersection C of the members 6 and 7. Here, it is useful that force N^8 can be shifted along its line of action to point D (see Figure 9.39). For the left-hand part we find

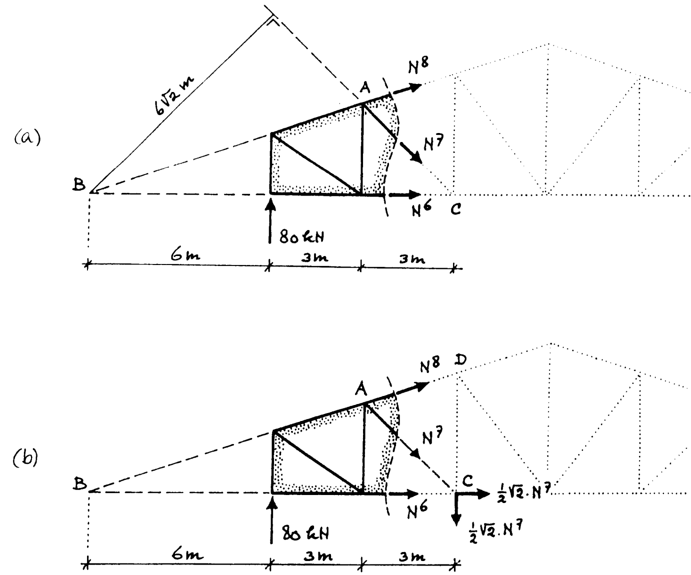


Figure 9.38 N^7 is found from the moment equilibrium about B. (a) Here we have to determine the distance from B to the line of action of N^7. (b) We can also shift N^7 to C and there resolve it into a horizontal and vertical component.

[1] See also Section 3.1.5 with Figure 3.17.

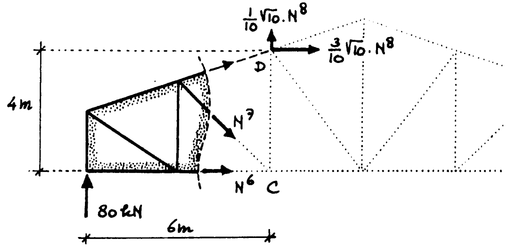


Figure 9.39 N^8 is found from the moment equilibrium about C.

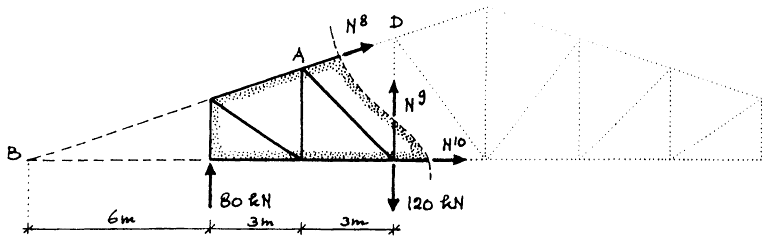


Figure 9.40 Section for calculating the force in member 9. N^9 follows from the moment equilibrium about B. As an interim step, we can also first determine N^{10} from the moment equilibrium about D, and then determine N^9 from the moment equilibrium about A.

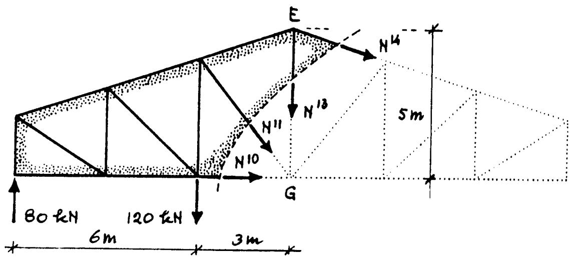


Figure 9.41 This section across member 13 intersects four members. We can only determine N^{14} from the moment equilibrium about G.

$$\sum T_z|\mathrm{C} = -(80\text{ kN})(6\text{ m}) - \left(\tfrac{3}{10}N^8\sqrt{10}\right)(4\text{ m}) = 0$$
$$\Rightarrow N^8 = -40\sqrt{10}\text{ kN}.$$

N^8 is a compressive force.

To verify the three values we have determined for N^6, N^7 and N^8, we can check whether the conditions for force equilibrium are satisfied for the left-hand part. This leads to the following two equations:

$$\sum F_x = \tfrac{3}{10}N^8\sqrt{10} + \tfrac{1}{2}N^7\sqrt{2} + N^6 = 0,$$
$$\sum F_y = \tfrac{1}{10}N^8\sqrt{10} - \tfrac{1}{2}N^7\sqrt{2} + (80\text{ kN}) = 0.$$

The values we found indeed meet these equilibrium conditions.

To determine the force in member 9, a section has been introduced in Figure 9.40 across the members 8, 9 and 10. The force N^9 follows directly from the moment equilibrium about the intersection B of the members 8 and 10. Written out in full, the left-hand part gives

$$\sum T_z|\mathrm{B} = (80\text{ kN})(6\text{ m}) - (120\text{ kN})(12\text{ m}) + N^9 \times (12\text{ m}) = 0$$
$$\Rightarrow N^9 = +80\text{ kN}.$$

Member 9 is a tension member.

If determining the location of point B is complicated (not the case here) you could also first determine N^{10} from the moment equilibrium about D and then derive N^9 from the moment equilibrium about A, for example:

$$\sum T_z|\mathrm{D} = -(80\text{ kN})(6\text{ m}) + N^{10} \times (4\text{ m}) = 0 \Rightarrow N^{10} = +120\text{ kN},$$

$$\sum T_z|\text{A} = -(80\text{ kN})(3\text{ m}) - (120\text{ kN})(3\text{ m})$$
$$+ N^{10} \times (3\text{ m}) + N^9 \times (3\text{ m}) = 0 \Rightarrow N^9 = +80\text{ kN}.$$

When determining the force in member 13, we again encounter the problem that a section across member 13 cuts four members (see Figure 9.41). In this case, the problem cannot be easily solved from the equilibrium of joint E isolated in Figure 9.42. In order to find member force N^{13} from the force equilibrium of joint E, we first have to know one of the member forces N^{12} or N^{14}.

Here there is a special case, in which we can determine the force in member 14 by means of the section in Figure 9.41, even though it passes over four members. Since, in this section, three of the four unknown member forces intersect at point G, the fourth force, in this case N^{14}, can be derived directly from the moment equilibrium about G. This gives (for the part shown to the left of the section, with force N^{14} moved to point E)

$$\sum T_z|\text{G} = -(80\text{ kN})(9\text{ m}) + (120\text{ kN})(3\text{ m}) - \tfrac{3}{10}N^{14}\sqrt{10} \times (5\text{ m}) = 0$$

so that $N^{14} = -24\sqrt{10}$ kN. The horizontal force equilibrium of joint E (Figure 9.42) now gives

$$N^{12} = N^{14} = -24\sqrt{10}\text{ kN}.$$

The vertical force equilibrium gives

$$N^{13} = -\tfrac{1}{10}\sqrt{10} \times (N^{12} + N^{14}) = +48\text{ kN}.$$

Member 13 is therefore a tension member.

Table 9.2 provides a summary of all the member forces in the truss.

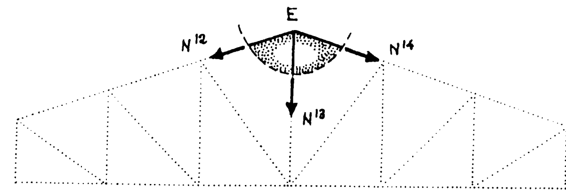


Figure 9.42 Once N^{14} is known we can find N^{13} from the equilibrium of joint E.

Table 9.2 Member forces Example 2.

Mem. no. *i*	**N^i (kN)**	**Mem. no. *i***	**N^i (kN)**
1	−80	14	075.89
2	0	15	+20
3	+96.15	16	+60
4	−84.33	17	−20
5	−53.33	18	−63.25
6	+80	19	+28.28
7	+56.57	20	+40
8	−126.49	21	−26.67
9	+80	22	−42.16
10	+120	23	+48.07
11	−80	24	0
12	−75.89	25	40
13	+48		

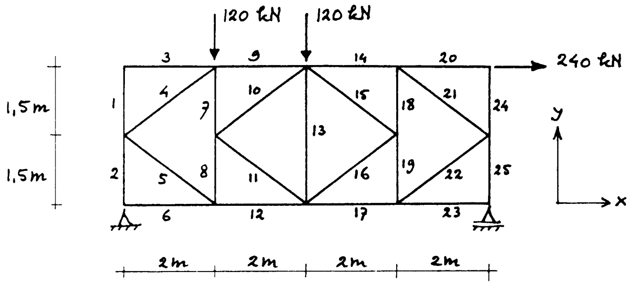


Figure 9.43 A K-truss.

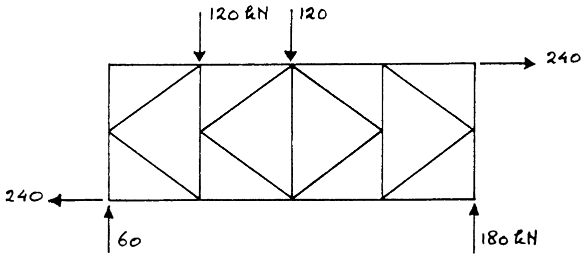


Figure 9.44 The isolated K-truss with support reactions.

Example 3

The third example relates to the somewhat more complicated truss in Figure 9.43, a so-called *K-truss*. This type of truss is sometimes used as *wind bracing* in bridges. Here, the K-truss has four fields and is loaded by two vertical forces of 120 kN and a horizontal force of 240 kN.

Question:
Determine the forces in members 7 to 13, with the correct sign for tension and compression. In the calculations, use the coordinate system given.

Solution:
In Figure 9.44, the truss has been isolated and the support reactions have been shown. Using the method of sections, we now encounter the difficulty that, for most of the members, no section can be found that intersects only three members. Sometimes it is possible to determine a member force if the section passes through more than three members, but in most cases, additional information is required that has to be obtained by selecting a section in a clever way, or by considering a combination of sections. Since the top chord and bottom chord members are easiest to determine, we will start with them.

To determine the normal force N^9 in top chord member 9, we introduce a section across members 5, 6, 7 and 9. Figure 9.45 shows only the part to the left of the section. Four unknown member forces are acting in the section. Since the lines of action of the forces N^5, N^6 and N^7 intersect one another at point A, only one force is unknown in the equation for the moment equilibrium about A, which can be determined directly. This gives

$$\sum T_z|\mathrm{A} = -(60\ \mathrm{kN})(2\ \mathrm{m}) - N^9 \times (3\ \mathrm{m}) = 0 \rightarrow N^9 = -40\ \mathrm{kN}.$$

Member 9 is a compression member.

The same equation is found from the moment equilibrium about A of the part to the left of the section over members 7, 8, 9 and 12 (see Figure 9.46). This section offers the advantage that the forces in the members of both the top chord and the bottom chord can be found. In this way, force N^{12} in member 12 follows directly from the moment equilibrium about B:

$$\sum T_z|B = -(240\text{ kN})(3\text{ m}) - (60\text{ kN})(2\text{ m}) + N^{12} \times (3\text{ m}) = 0$$

$$\Rightarrow N^{12} = +280\text{ kN}.$$

A tensile force is acting in member 12.

The section in Figure 9.46 has the additional benefit that the values found for N^9 and N^{12} can be checked using the horizontal force equilibrium of the isolated part, without having to know the forces in the diagonal members or verticals:

$$\sum F_x = -(240\text{ kN}) + N^9 + N^{12}$$

$$= -(240\text{ kN}) + (-40\text{ kN}) + (280\text{ kN}) = 0.$$

The isolated section in Figure 9.46 therefore meets the conditions for horizontal force equilibrium.

With the section in Figure 9.46, we can quickly determine the forces in the top chord member 9 and bottom chord member 12, but not the forces in the verticals 7 and 8. The forces in these verticals are found from the equilibrium of joints A and B, but we do not have enough information to do so yet.

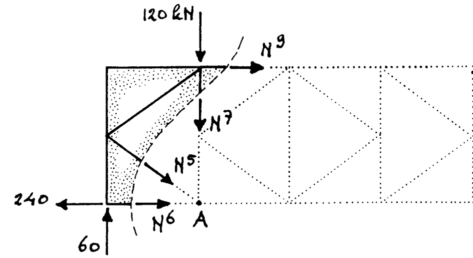


Figure 9.45 Section for determining the force in member 9. The force is found from the moment equilibrium about A.

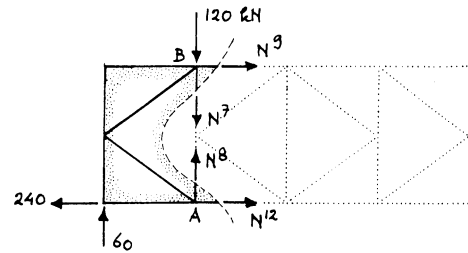


Figure 9.46 Section for determining the forces in members 9 and 12. They are found from the moment equilibrium about respectively A and B.

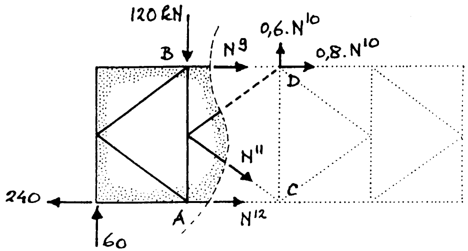


Figure 9.47 Section for determining the force in member 10. This force follows from the moment equilibrium about C. However, we do have to know N^9 first.

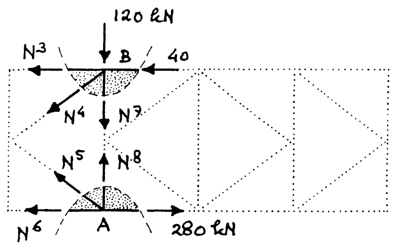


Figure 9.48 N^7 and N^8, the forces in the verticals, are found from the equilibrium of joints B and A, although we must first know the forces in members 3 or 4, respectively 5 or 6.

The forces in the diagonal members 10 and 11 are found using the section in Figure 9.47 over members 9, 10, 11 and 12. Since we already know N^9, we can find N^{10} from the moment equilibrium about C. This gives the following (with force N^{10} shifted along its line of action to point D, only the horizontal component of N^{10} is left in the equation for the moment equilibrium)

$$\sum T_z|\text{C} = -(60\text{ kN})(4\text{ m}) + (120\text{ kN})(2\text{ m}) - N^9 \times (3\text{ m}) + \\ - 0.8N^{10} \times (3\text{ m}) = 0.$$

With $N^9 = -40$ kN this gives

$$N^{10} = +50\text{ kN}.$$

In the same way, from the moment equilibrium about D we find

$$N^{11} = -50\text{ kN}.$$

Since we know both N^9 and N^{12}, we can find N^{10} and N^{11} from the two equations for the force equilibrium of the isolated part in Figure 9.47:

$$\sum F_x = N^9 + 0.8N^{10} + 0.8N^{11} + N^{12} - (240\text{ kN}) = 0,$$
$$\sum F_y = 0.6N^{10} - 0.6N^{11} + (60\text{ kN}) - (120\text{ kN}) = 0.$$

With $N^9 = -40$ kN and $N^{12} = +280$ kN these equations are now

$$0.8N^{10} + 0.8N^{11} = 0,$$
$$0.6N^{10} - 0.6N^{11} = +60\text{ kN}.$$

The solution is

$$N^{10} = +50 \text{ kN},$$

$$N^{11} = -50 \text{ kN}.$$

This is in agreement with earlier results.

The forces in the verticals 7 and 8 follow from the (force) equilibrium of joints B and A respectively, although we do first have to know the forces in one of members 3 and 4 and one of members 5 or 6 (see Figure 9.48).

The forces N^3 and N^6 can be found in the section in Figure 9.49 from the moment equilibrium about G and E respectively. In fact, with this section on the end of the truss, we isolate joints E and G. N^3 and N^6 can therefore also be found directly from the horizontal force equilibrium of joints E and G:

$$N^3 = 0,$$

$$N^6 = +240 \text{ kN}.$$

In Figure 9.50, joints A and B have been isolated, and all the known forces N^3, N^6, N^9 and N^{12} are shown as they act in reality on the joints.

At joint B, two forces are still unknown: N^4 and N^7. From the equilibrium for this joint we find

$$\sum F_x = -(40 \text{ kN}) - 0.8N^4 = 0,$$

$$\sum F_y = -(120 \text{ kN}) - 0.6N^4 - N^7 = 0$$

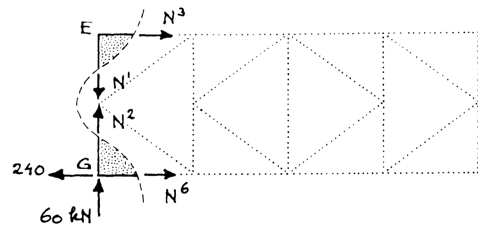


Figure 9.49 N^3 and N^6 follow from the force equilibrium of respectively joint E and G.

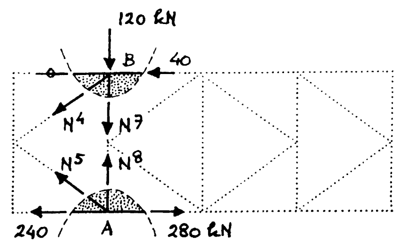


Figure 9.50 If $N^3 = 0$ and $N^6 = 240$ kN, then N^7 and N^8 are found from the force equilibrium of respectively joint B and A.

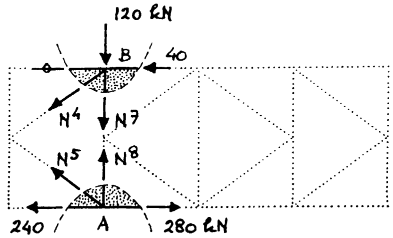


Figure 9.50 If $N^3 = 0$ and $N^6 = 240$ kN, then N^7 and N^8 are found from the force equilibrium of respectively joint B and A.

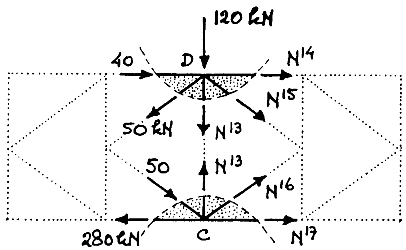


Figure 9.51 The force in member 13 is found from the force equilibrium of joint D or C, although we must first determine one of the member forces N^{14} and N^{15}, or one of N^{16} and N^{17}.

with the solution

$$N^4 = -50 \text{ kN},$$
$$N^7 = -90 \text{ kN}.$$

In the same way, we can determine N^5 and N^8 from the equilibrium of joint A:

$$\sum F_x = -(240 \text{ kN}) + (280 \text{ kN}) - 0.8N^5 = 0,$$
$$\sum F_y = 0.6N^5 + N^8 = 0$$

such that

$$N^5 = +50 \text{ kN},$$
$$N^8 = -30 \text{ kN}.$$

The force in member 13 is the most complicated one to determine. This force is found from the equilibrium of joint D or C. However, we first have to determine one of the member forces N^{14} and N^{15}, or one of N^{16} and N^{17} (see Figure 9.51).

With the section in Figure 9.52, N^{14} is found from the moment equilibrium about H:

$$\sum T_z|\text{H} = (180 \text{ kN})(2 \text{ m}) - (240 \text{ kN})(3 \text{ m}) + N^{14} \times (3 \text{ m}) = 0$$
$$\Rightarrow N^{14} = +120 \text{ kN}.$$

At joint D, N^{13} and N^{14} are now the only unknowns (see Figure 9.51). The two equations for the force equilibrium of the joint are:

$$\sum F_x = (40\text{ kN}) + N^{14} - 0.8 \times (50\text{ kN}) + 0.8 \times N^{15} = 0,$$

$$\sum F_y = -(120\text{ kN}) - N^{13} - 0.6 \times (50\text{ kN}) - 0.6 \times N^{15} = 0.$$

Here substitute $N^{14} = +120$ kN to find the solution:

$$N^{15} = -150\text{ kN},$$

$$N^{13} = -60\text{ kN}.$$

Table 9.3 provides a summary of all the member forces.

In the *method of sections*, member forces are determined from the equilibrium of a sectioned part of the truss. In the examples, the *sectioned part* sometimes degenerates into a *joint*. The following section looks at the *method of joints*. With this method, all the member forces are consistently derived from the equilibrium of the joints.

9.3.2 The method of joints

In the *method of joints*, all the joints are isolated, and we investigate the *force equilibrium* of the individual joints.

For the truss in Figure 9.53a, all the joints have been isolated in Figure 9.53b. On the isolated joints are acting

- loads (joints C and D);
- support reactions (joints A and B);
- member forces.

Here, the support reactions and member forces are the unknown forces.

Since only two equations for the force equilibrium are available per joint, we have to start the calculation at a joint where no more than two forces

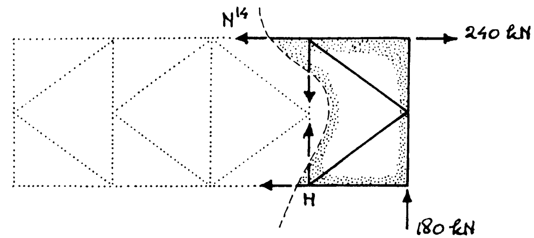


Figure 9.52 In this section, N^{14} is found from the moment equilibrium about H.

Table 9.3 Member forces Example 3.

Mem. no. *i*	N^i (kN)	Mem. no. *i*	N^i (kN)
1	0	14	+120
2	−60	15	−150
3	0	16	+150
4	−50	17	+120
5	+50	18	+90
6	+240	19	−90
7	−90	20	+240
8	−30	21	−150
9	−40	22	+150
10	+50	23	0
11	−50	24	0
12	+280	25	−180
13	−60		

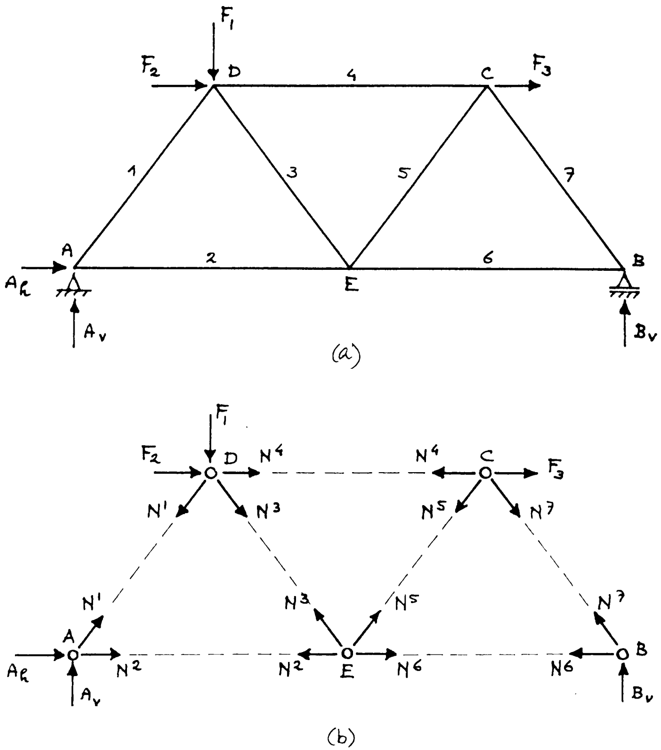


Figure 9.53 (a) Truss with support reactions and (b) all isolated joints of the truss with all the forces acting on them.

are unknown. These forces are determined from the joint equilibrium, after which we move to the next joint where, again, no more than two forces are unknown. In this way, we pass along each of the joints in the truss.

If there are k joints, it is not the intention to first generate all $2k$ equations for the force equilibrium, and then to solve them together as a system of equations. We will often encounter the problem in which we cannot start with a joint with only two unknowns, as in Figure 9.53. This can be avoided by previously determining the support reactions from the truss as a whole. In Figure 9.53b, we can now start the procedure at one of the joints A or B.

The method of joints is mostly used if one wants to find all the member forces in a truss. If you want only to calculate the member force somewhere in the middle of the truss, you will often have to work out the equilibrium for several joints. In that case, the method of sections is faster.

Calculating the two unknown forces per joint can be done either *analytically* or *graphically*. The graphical approach is preferable; it is not only faster but also gives a better insight in the force flow. The method of joints is illustrated using a number of examples.

Example 1
The truss crane in Figure 9.54 is loaded at A by means of a vertical force $4F$.

Question:
Determine all the member forces, with the correct sign for tension and compression.

Solution:
In this case, we do not have to determine the support reactions as we can start directly at joint A. Here, two forces are unknown: N^1 and N^2. These forces can be determined both *analytically* and *graphically*.

Analytical solution for the equilibrium of joint A:
In Figure 9.55a, all the forces acting on joint A are shown. In this figure, the member forces are again shown as tensile forces. For a tensile force, N is by convention positive.

For the angles α^1 and α^2 shown in the figure, the equilibrium equations are

$$\sum F_x = -N^1 \cos\alpha^1 - N^2 \cos\alpha^2 = 0,$$
$$\sum F_y = -N^1 \sin\alpha^1 - N^2 \sin\alpha^2 - 4F = 0.$$

From the slopes of the members 1 and 2 we find

$$\sin\alpha^1 = \cos\alpha^1 = \tfrac{1}{2}\sqrt{2},$$
$$\sin\alpha^2 = \tfrac{1}{5}\sqrt{5} \text{ and } \cos\alpha^2 = \tfrac{2}{5}\sqrt{5}.$$

Both equations in N^1 and N^2 now become

$$-N^1 \times \tfrac{1}{2}\sqrt{2} - N^2 \times \tfrac{2}{5}\sqrt{5} = 0,$$
$$-N^1 \times \tfrac{1}{2}\sqrt{2} - N^2 \times \tfrac{1}{5}\sqrt{5} = 4F$$

with solution:

$$N^1 = -8F\sqrt{2},$$
$$N^2 = +4F\sqrt{5}.$$

Member 1 is a *compression member* and exerts a compressive force on joint A. Member 2 is a *tension member*. Figure 9.55b shows the forces as they really act on both the joint and on the two members.

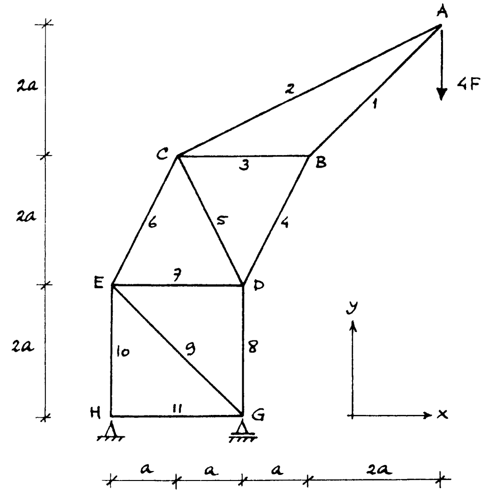


Figure 9.54 A truss crane. We can start the method of joints in A without having to first determine the support reactions.

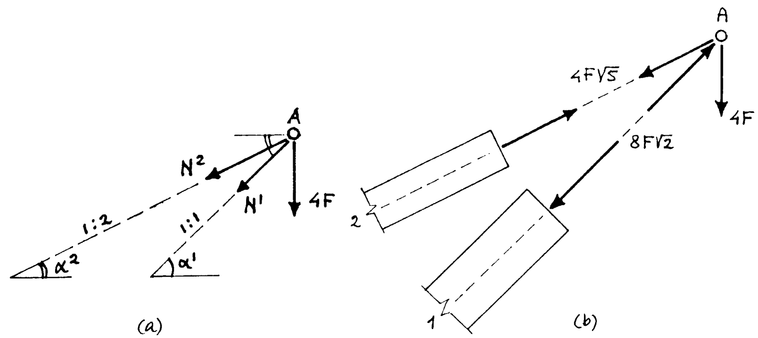


Figure 9.55 (a) The isolated joint A. The unknown forces N^1 and N^2 exerted by the members 1 and 2 on joint A are shown as tensile forces. (b) The interaction forces between joint A and members 1 and 2 as they really act. Member 1 is a compression member and member 2 is a tension member.

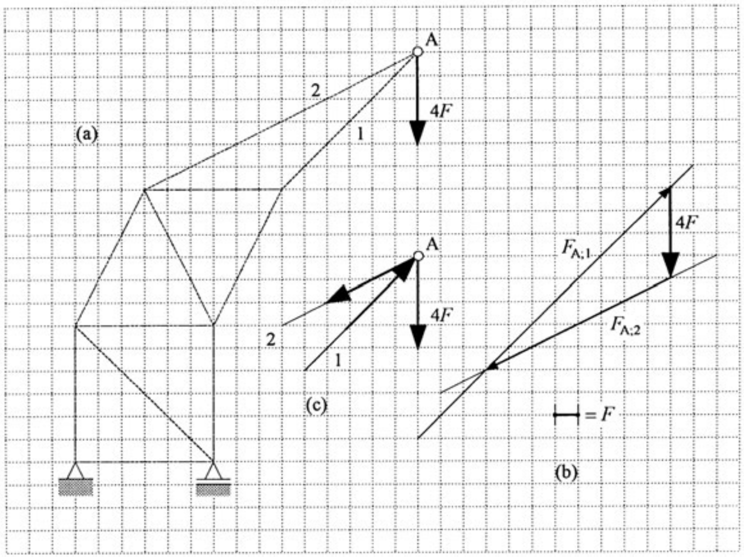


Figure 9.56 (a) The forces in members 1 and 2 follow from the equilibrium of joint A. (b) The closed force polygon for the equilibrium of joint A. $F_{A;1}$ and $F_{A;2}$ are the forces that members 1 and 2 exert on joint A. (c) Joint A with all the forces acting on it. From this figure we can see that N^1 is a compressive force and N^2 is a tensile force.

Graphical solution for the equilibrium of joint A:
$F_{A;1}$ and $F_{A;2}$ are the forces that members 1 and 2 exert on joint A. The forces $F_{A;1}$ and $F_{A;2}$ have their line of action along the members 1 and 2, but we do not know their magnitudes, nor their directions (see Figure 9.56a). Joint A is in equilibrium if all forces acting on joint A form a *closed force polygon*. Figure 9.56b shows the closed force polygon for the equilibrium of joint A. From here, we can read off the magnitude of $F_{A;1}$ and $F_{A;2}$ (or calculate it):

$$F_{A;1} = 8F\sqrt{2},$$

$$F_{A;2} = 4F\sqrt{5}.$$

From the force polygon, we can also find the directions of $F_{A;1}$ and $F_{A;2}$, but we cannot see whether they are tensile or compressive forces. To do so, we first have to draw the forces found as they act on joint A, see Figure 9.56c. Only then we can see that $F_{A;1}$ is a compressive force, and $F_{A;2}$ is a tensile force, so that

$$N^{(1)} = -F_{A;1} = -8F\sqrt{2},$$

$$N^{(2)} = +F_{A;2} = +4F\sqrt{5}.$$

Note that the forces in the force polygon have not been denoted as N. The force polygon provides information only on the magnitude of the member forces, and not on the sign for tension or compression.

The order in which one writes down the forces in a force polygon does not influence the result (vector addition is associative and commutative). Figure 9.57 shows two equivalent force polygons. The first force polygon is created by ranking the various forces acting on joint A in an order that is associated with an anti-clockwise rotation about joint A: $4F \Rightarrow F_{A;2} \Rightarrow F_{A;1}$.

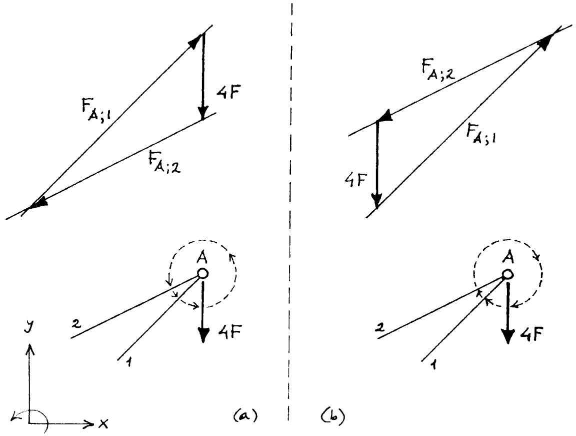


Figure 9.57 The order in which the forces in a force polygon are plotted does not influence the result (vector addition is associative and commutative).

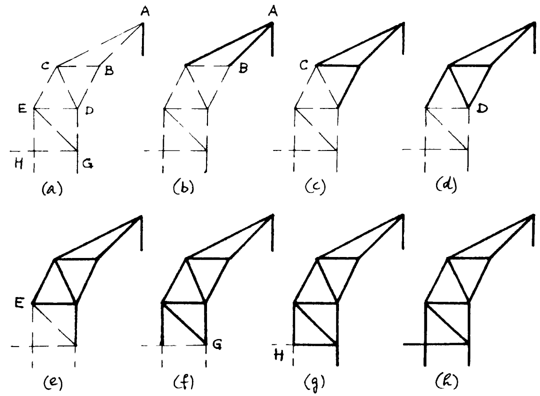


Figure 9.58 The order (a) to (h) shows how we can repeatedly determine two member forces per joint (and finally the support reactions at G and H). The members for which the normal force is known are shown in bold.

The second force polygon arises from ranking the forces in a clockwise order, so that $4F \Rightarrow F_{A;1} \Rightarrow F_{A;2}$.

In Figure 9.58, the order (a) to (h) shows how, per joint, we can consecutively calculate two member forces (and then the support reactions in G and H). The members for which the forces are known are shown in bold.

Figure 9.58a shows the initial situation. A is the only joint with two unknown member forces. Once we have calculated these, we get the situation shown in Figure 9.58b. Now B is the only joint with only two unknown member forces. Once these have been determined, we get the situation in Figure 9.58c, and so forth. The order in which the joint equilibrium is determined, with no more than two unknowns per joint, is

$$A \Rightarrow B \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow G \Rightarrow H.$$

For calculating the still unknown member forces, we now use the graphical method. After A, the next joint is B, where we can calculate the member forces. Joint B is subject to the forces $F_{\mathrm{B};1}$, $F_{B;3}$ and $F_{B;4}$, of which $F_{\mathrm{B};1}$ is known.

Earlier, we found that the force in member 1 is a compressive force: $N^1 = -8F\sqrt{2}$. Member 1 therefore exerts a compressive force on joint B of $8F\sqrt{2}$, so that $F_{\mathrm{B};1} = 8F\sqrt{2}$ (see Figure 9.59a).

The two unknowns $F_{\mathrm{B};3}$ and $F_{\mathrm{B};4}$ can be determined from the closed force polygon for the equilibrium of joint B (see Figure 9.59b):

$$F_{\mathrm{B};3} = 4F,$$
$$F_{\mathrm{B};4} = 4F\sqrt{5}.$$

In Figure 9.59c, the forces from the force polygon are shown as they act on joint B in reality. Here we see that $F_{\mathrm{B};3}$ and $F_{\mathrm{B};4}$ are both compressive forces. Converted into the normal forces in the members 3 and 4, with the correct sign for tension and compression, we therefore get

$$N^3 = -F_{\mathrm{B};3} = -4F,$$
$$N^4 = -F_{\mathrm{B};4} = -4F\sqrt{5}.$$

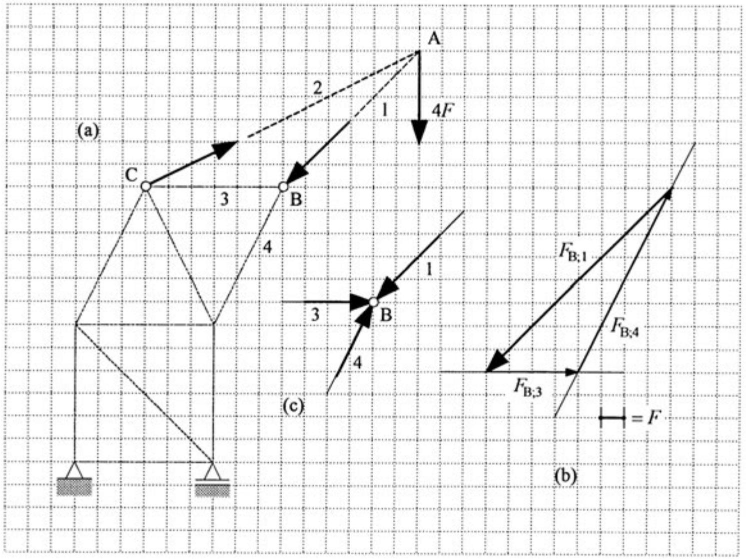


Figure 9.59 (a) The forces in members 3 and 4 follow from the equilibrium of joint B. (b) The closed force polygon for the equilibrium of joint B. $F_{\mathrm{B};1}$ is known. (c) Joint B with all the forces acting on it. From this figure we can see that N^3 and N^4 are compressive forces.

The following joint with only two unknowns is C. The forces that the members 2 and 3 exert on the joint are known (see Figure 9.60a):

$$F_{\mathrm{C};2} = 4F\sqrt{5},$$
$$F_{\mathrm{C};3} = 4F.$$

The unknown forces $F_{\mathrm{C};5}$ and $F_{\mathrm{C};6}$ follow from the force polygon in Figure 9.60b:

$$F_{C;5} = F\sqrt{5},$$
$$F_{C;6} = 3F\sqrt{5}.$$

In Figure 9.60c, all the forces are shown as they act on joint C in reality. Member 5 presses against the joint and is a compression member, member 6 pulls on the joint and is a tension member:

$$N^5 = -F\sqrt{5},$$
$$N^6 = +3F\sqrt{5}.$$

In Figures 9.61 to 9.64, the other member forces are calculated using the same method.

Table 9.4 provides a summary of all the member forces.

Table 9.4 Member forces Example 1.

Mem. no. *i*	**N^i (kN)**
1	$-8F\sqrt{2}$
2	$+4F\sqrt{5}$
3	$-4F$
4	$-4F\sqrt{5}$
5	$-F\sqrt{5}$
6	$+3F\sqrt{5}$
7	$-3F$
8	$-10F$
9	0
10	$+6F$
11	0

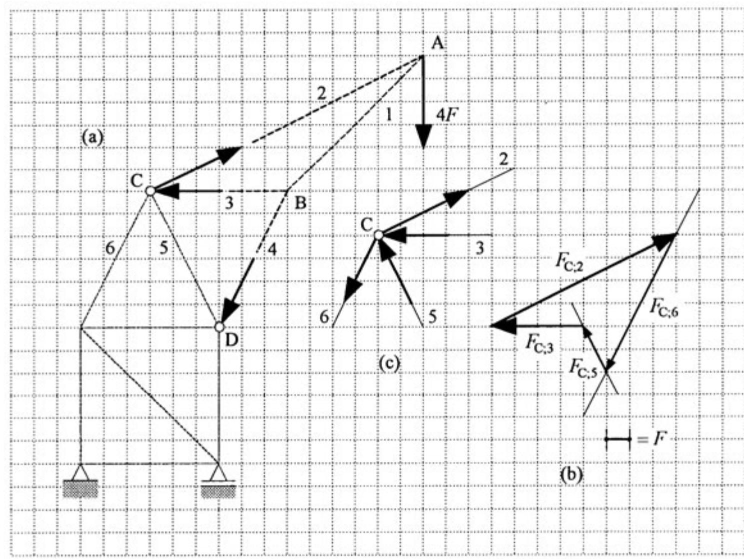


Figure 9.60 (a) The forces in members 5 and 6 follow from the equilibrium of joint C. (b) The closed force polygon for the equilibrium of joint C. $F_{C;2}$ and $F_{C;3}$ are known forces. (c) Joint C with all the forces acting on it. From this figure we can see that N^5 is compressive and N^6 is tensile.

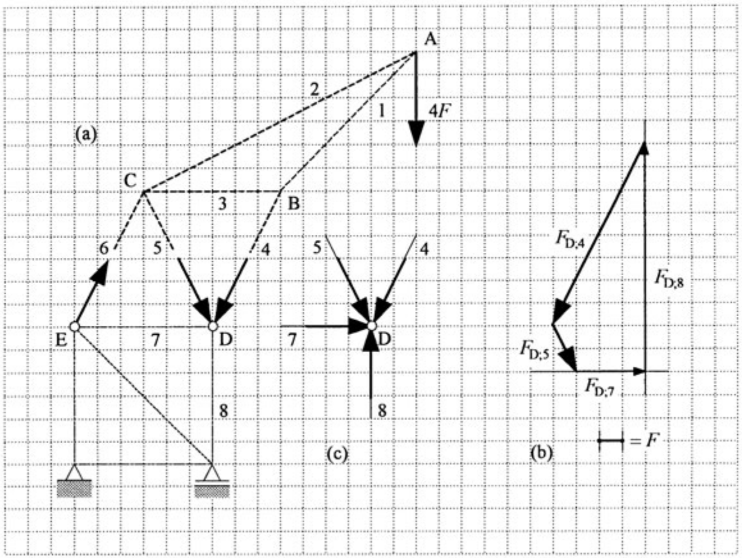


Figure 9.61 (a) The forces in members 7 and 8 follow from the equilibrium of joint D. (b) The closed force polygon for the equilibrium of joint D. $F_{D;4}$ and $F_{D;5}$ are known forces. (c) Joint D with all the forces acting on it. From this figure we can see that N^7 and N^8 are compressive.

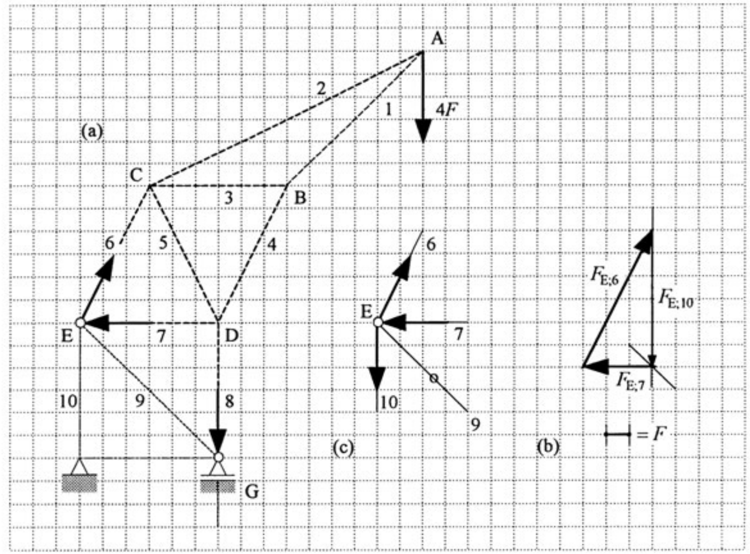


Figure 9.62 (a) The forces in members 9 and 10 follow from the equilibrium of joint E. (b) The closed force polygon for the equilibrium of joint E. $F_{E;6}$ and $F_{E;7}$ are known. (c) Joint E with all the forces acting on it. From this figure we can see that N^{10} is tensile. Member 9 is a zero-force member.

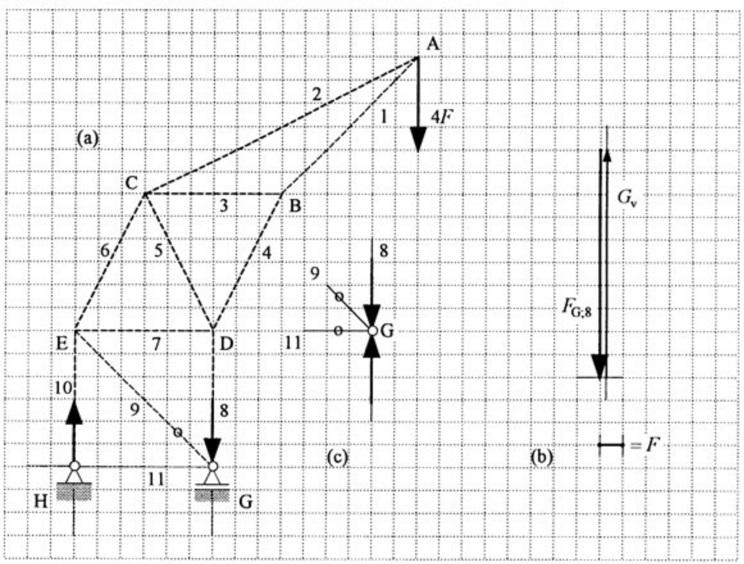


Figure 9.63 (a) The force in member 11 and the vertical support reaction at G is found from the equilibrium of joint G. (b) The closed force polygon for the equilibrium of joint G. Member 9 is a zero-force member and does not participate. $F_{G;8}$ is known. (c) Joint G with all the forces acting on it. Member 11 is a zero-force member. The vertical support reaction at G is a compressive force.

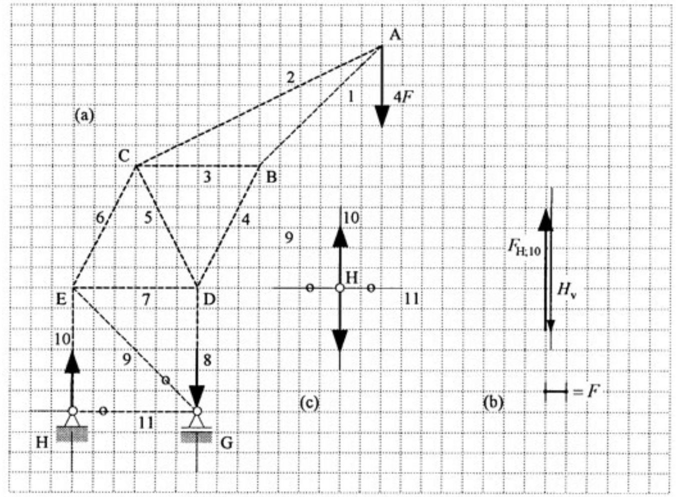


Figure 9.64 (a) The horizontal and vertical support reaction at H is found from the equilibrium of joint H. (b) The closed force polygon for the equilibrium of joint H. Member 11 is a zero-force member and does not participate. $F_{H;10}$ is known. (c) Joint H with all the forces acting on it. The horizontal support reaction at H is zero. The vertical support reaction in H is a tensile force.

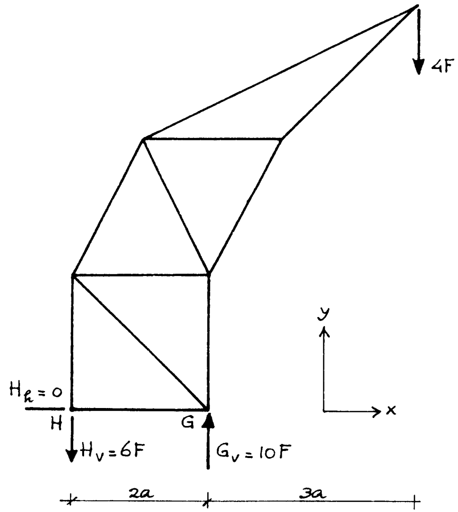


Figure 9.65 The truss crane with the support reactions as they are acting in reality.

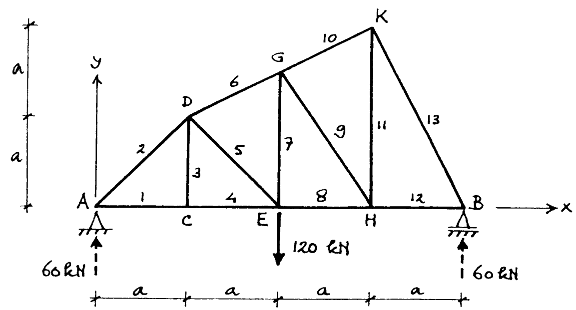


Figure 9.66 In this truss, we can apply the method of joints only when we know the support reactions.

In Figures 9.63 and 9.64 the support reactions in G and H have also been calculated:

$$G_v = 10F,$$

$$H_h = 0,$$

$$H_v = 6F.$$

In Figure 9.65, the support reactions are shown with the directions in which they are acting.

To check the calculation, we can look at the equilibrium of the truss as a whole:

$$\sum F_x = H_h = 0,$$

$$\sum F_y = G_v - H_v - 4F = 10F - 6F - 4F = 0,$$

$$\sum T_z|\mathrm{H} = G_v \times 2a - 4F \times 5a = 10F \times 2a - 4F \times 5a = 0.$$

The truss as a whole meets the equilibrium conditions.

Example 2

The truss in Figure 9.66 is loaded at joint E by a vertical force of 120 kN.

Question:
Calculate the member forces, with the correct sign for tension and compression.

Solution:
In this truss, we cannot find a joint with only two unknown forces. Before we can start the procedure for the joint equilibrium, we first have to determine the support reactions from the truss as a whole. Then we can start the calculation at joint A or B.

In Figure 9.67, the order (a) to (g) shows how, starting at A, we can consecutively determine two member forces per joint. The members for which we know the normal force are shown in bold. We will look at the joints in the following order:

$$A \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow G \Rightarrow H \Rightarrow K \text{ or } B.$$

The last two joints K and B both offer an opportunity to check the results: both force polygons have to be closed and give the same force in member 13.

Table 9.5 provides a summary of all the member forces.

Table 9.5 Member forces Example 2.

Mem. no. i	N^i (kN)
1	$+60$
2	$-60\sqrt{2}$
3	0
4	$+60$
5	$+20\sqrt{2}$
6	$-40\sqrt{5}$
7	$+100$
8	$+80$
9	$-25\sqrt{13}$
10	$-15\sqrt{5}$
11	$+75$
12	$+30$
13	$-30\sqrt{5}$

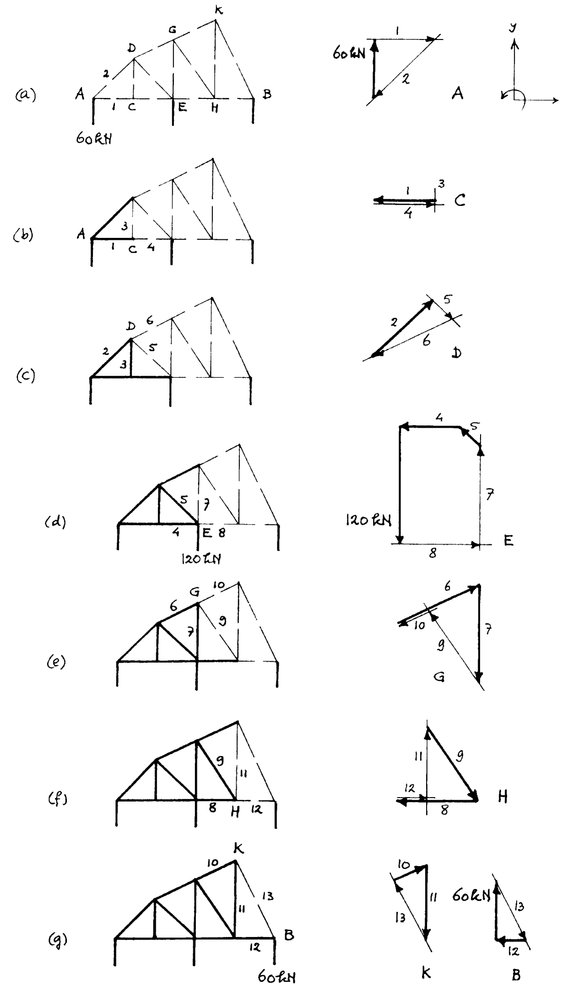


Figure 9.67 The order (a) to (g) shows how, starting at A, we can consecutively determine two member forces per joint. The members for which we know the normal force are shown in bold.

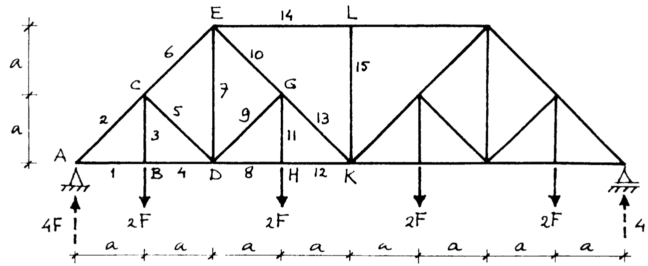


Figure 9.68 Truss with support reactions.

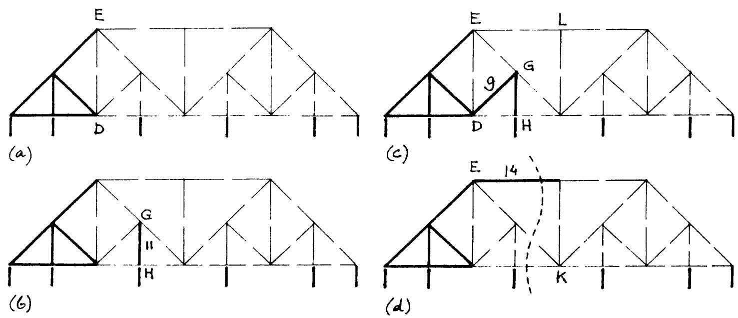


Figure 9.69 The members for which we know the normal force are shown in bold. (a) The method of joints gets stuck at joints E and D, as more than two member forces are unknown. (b) The force in member 11 follows from the vertical equilibrium of joint H, (c) after which we can find the force in member 9 from the force equilibrium of joint G. The method of joints can now continue at D. (d) We could also switch to the method of sections to calculate the force in member 14. The method of joints can then be resumed at E.

Example 3

You are given the (Baltimore) truss beam in Figure 9.68.

Question:
Determine the member forces N^1 to N^{15} using the method of joints. In which order should we handle the joint equilibrium?

Solution:
After first determining the support reactions from the equilibrium of the truss as a whole, we can determine the forces in members 1 to 6 from the equilibrium of joints A, B and C respectively. In the situation shown in Figure 9.69a we get stuck, as more than two member forces are unknown in both D and E.

Since members 8 and 12, and 10 and 13 are in a direct line with one another, we can determine the forces in the members 11 and 9 from the equilibrium of joints H and G.

The vertical equilibrium of joint H in Figure 9.70 gives

$$N^{11} = 2F.$$

We now have the situation as shown in Figure 9.69b. From the equilibrium in G in the direction normal to members 10 and 13 we find next (see Figure 9.70):

$$N^9 + \tfrac{1}{2}N^{11}\sqrt{2} = N^9 + \tfrac{1}{2} \times 2F \times \sqrt{2} = 0 \Rightarrow N^9 = -F\sqrt{2}.$$

Now that N^9 is known (see Figure 9.69c), we can find the remaining member forces by consecutively elaborating the equilibrium of joints D, E, G, H and L. The order in which we handle the joints is therefore

$$\text{A} \Rightarrow \text{B} \Rightarrow \text{C} \Rightarrow \text{H} \Rightarrow \text{G} \Rightarrow \text{D} \Rightarrow \text{E} \Rightarrow \text{G} \Rightarrow \text{H} \Rightarrow \text{L}.$$

Instead of determining N^9 and N^{11} from the equilibrium of joints H and G, it is far easier to revert to the method of sections. With the section shown in Figure 9.69d across members 12, 13 and 14, we can determine the force in member 14 from $\sum T_z|\text{K} = 0$. The other member forces are then found from the equilibrium for the successive joints E, D, G, H and L.

In certain cases, it can be useful to switch from one method to the other at the right moment.

Table 9.6 provides a summary of member forces N^1 to N^{15}.

Table 9.6 Member forces Example 3.

Mem. no. i	N^i (kN)
1	$+4F$
2	$-4F\sqrt{2}$
3	$+2F$
4	$+4F$
5	$-F\sqrt{2}$
6	$-3F\sqrt{2}$
7	$+2F$
8	$+4F$

Mem. no. i	N^i (kN)
9	$-F\sqrt{2}$
10	$+F\sqrt{2}$
11	$+2F$
12	$+4F$
13	0
14	$-4F$
15	0

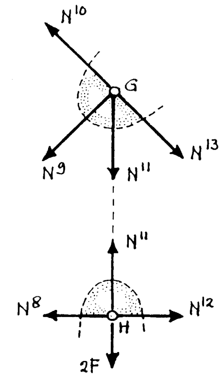


Figure 9.70 The isolated joints G and H.

9.3.3 Zero-force members and continuous members; simplifying the calculation

We can often shorten the calculation that needs to be done by first looking for *zero-force members* in a truss. Zero-force members are members in which no forces are acting ($N = 0$) due to the present loading.

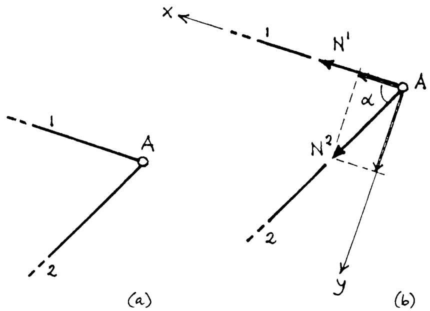


Figure 9.71 (a) If two members meet in an unloaded joint, both members are zero-force members. (b) The forces acting on isolated joint A.

There are three situations of frequent occurrence in which zero-force members can be easily recognised:

1. *If only two members meet in an unloaded joint, both are zero-force members (see Figure 9.71).*
2. *If three members meet in an unloaded joint of which two are in a direct line with one another, then the third is a zero-force member (see Figure 9.72).*
3. *If two members meet in an unloaded joint and the line of action of the load coincides with one of the members, the other member is a zero-force member (see Figure 9.73).*

These three rules are the direct consequence of the joint equilibrium, as shown below for each of the cases.

Rule 1. Two members meet in unloaded joint A in Figure 9.71. The force in one of the members has a component normal to the direction of the other member. If we write down the equilibrium of joint A in the given (local) xy coordinate system, we find

$$\sum F_x = N^1 + N^2 \cos\alpha = 0,$$
$$\sum F_y = N^2 \sin\alpha = 0$$

with the solution (because $\sin\alpha \neq 0$):[1]

$$N^1 = N^2 = 0.$$

Equilibrium is possible only if both member forces are zero.

1 In a kinematically determinate truss, members 1 and 2 cannot be an extension of one another, so that $\alpha \neq 0$ and $\alpha \neq 180°$.

Rule 2. In Figure 9.72, three members meet in joint B, of which members 1 and 3 are in a direct line with one another. The force in member 2 has a component normal to members 1 and 3. There can be equilibrium only if this component is zero, or in other words, if $N^2 = 0$. If we write down the equilibrium of joint B in the given (local) xy coordinate system, we find

$$\sum F_x = N^1 + N^2 \cos\alpha - N^3 = 0,$$
$$\sum F_y = N^2 \sin\alpha = 0$$

so that

$$N^2 = 0 \quad \text{and} \quad N^1 = N^3.$$

In addition to the fact that member 2 is a zero-force member, the normal forces in the continuous members 1 and 3, which are in a direct line with one another, are equal.

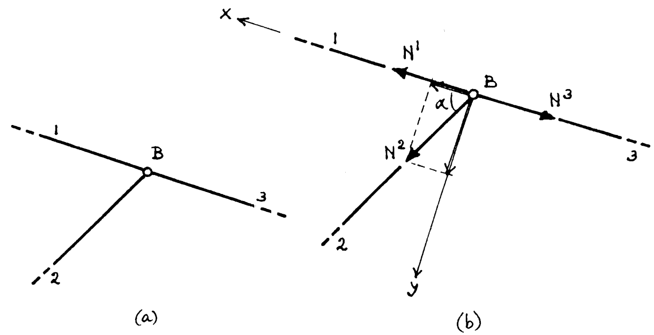


Figure 9.72 (a) If three members meet in an unloaded joint of which two are in a direct line with one another, the third member is a zero-force member. The normal forces in continuous members 1 and 3 are equal. (b) The forces acting on isolated joint B.

Rule 3. The situation in Figure 9.73 is clearly similar to that in Figure 9.71. The equations for the equilibrium of joint C are

$$\sum F_x = N^1 + N^2 \cos\alpha - F = 0,$$
$$\sum F_y = N^2 \sin\alpha = 0$$

so that

$$N^2 = 0 \quad \text{and} \quad N^2 = F.$$

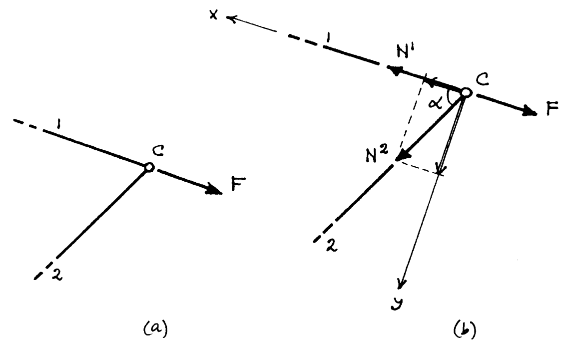


Figure 9.73 (a) If two members meet in an unloaded joint and the line of action of the load is in a direct line with one of the members, the other member is a zero-force member. (b) The forces acting on isolated joint C.

By using rules 1 to 3 to determine the zero-force members first, you can often shorten the required calculation. A fourth rule with which we can shorten the calculation relates to an unloaded joint, in which four members meet and in pairs are in a direct line with one another. This situation is

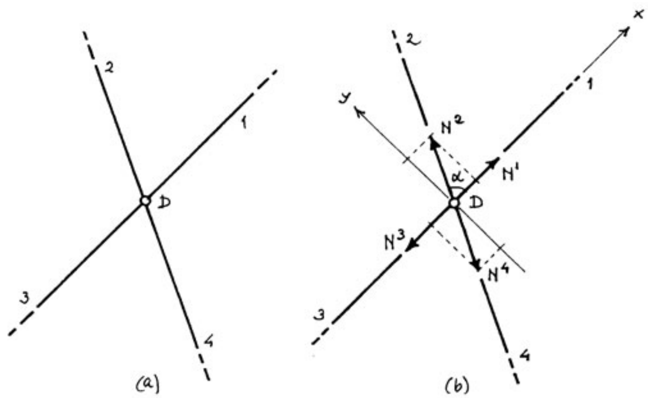


Figure 9.74 If four members meet in an unloaded joint that in pairs are in a direct line with one another, these members can be considered crossing members as far as the transfer of forces is concerned. (b) The forces acting on isolated joint D.

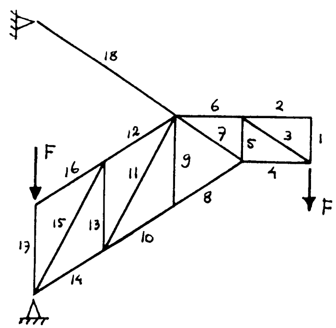


Figure 9.75 A truss.

shown in Figure 9.74. For the given xy coordinate system, the equilibrium of joint D gives

$$\sum F_x = N^1 + N^2 \cos\alpha - N^3 - N^4 \cos\alpha = 0,$$
$$\sum F_y = N^2 \sin\alpha - N^4 \sin\alpha = 0$$

so that

$$N^1 = N^3 \quad \text{and} \quad N^2 = N^4.$$

Conclusion:

Rule 4. If four members meet in an unloaded joint that in pairs are in a direct line with one another, these members can be considered crossing members *as far as the transfer of forces is concerned.*

The three examples below show how it is possible to simplify the calculation with these four rules.

Example 1
You are given the truss in Figure 9.75.

Question:
Which members are zero-force members for the given load?

Solution:
A is an unloaded joint in which two members meet (see Figure 9.76). Both members are zero-force members (rule 1), so that

$$N^1 = 0 \quad \text{and} \quad N^2 = 0.$$

B is an unloaded joint in which three members meet, and of which two are in a direct line with one another. The third member is therefore a zero-force member (rule 2), so that

$$N^9 = 0.$$

C is a loaded joint where two members meet, and where the line of action of the load coincides with member 17. Thus (rule 3)

$$N^{16} = 0.$$

The zero-force members are shown in Figure 9.76 with a “0” through the member axis.

Zero-force members do not participate in the force flow for the present load. When calculating the forces in the other members, you can leave out the zero-force members from the truss. If you leave out zero-force member 9 from the truss, you immediately notice that

$$N^8 = N^{10}.$$

If you leave out zero-force member 16, you see that

$$N^{17} = -F.$$

That this (imaginary) omission of zero-force members can significantly reduce the effort in calculating is further emphasised in the following two examples.

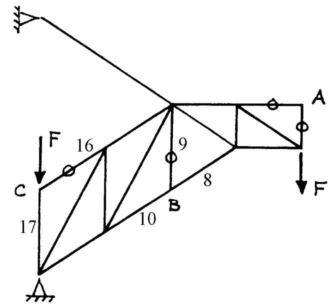


Figure 9.76 The zero-force members in the truss.

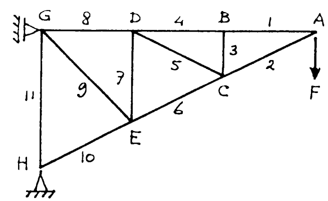


Figure 9.77 A truss.

Example 2
You are given the truss in Figure 9.77.

Question:
Determine all the zero-force members for the given load.

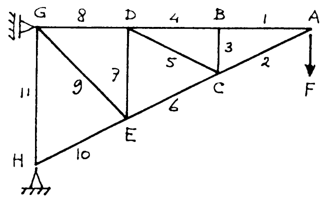


Figure 9.77 A truss.

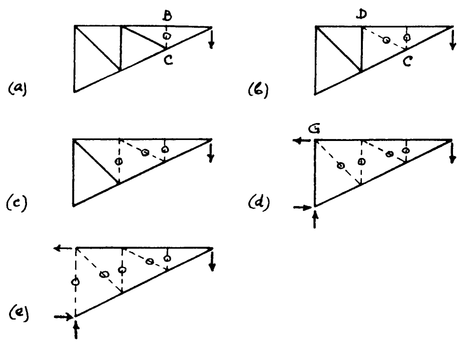


Figure 9.78 A zero-force member does not participate in the transfer of forces and can be omitted from the calculation. This is shown here by depicting the member with a dashed line. (a) to (e) shows the order in which one can find the zero-force members. (e) In the end, all bracing members turn out to be zero-force members, and the top and bottom chord members only transfer the load to the supports.

Solution:
From the equilibrium in joint B, it follows that member 3 is a zero-force member (rule 2). With the given load, this member does not participate in the transfer of forces, and could therefore be omitted. In Figure 9.78a, the member is now shown by means of a dashed line. The equilibrium of joint C means that member 5 is also a zero-force member (rule 2 again) (see Figure 9.78b). If we continue, we notice that members 7 and 9 are also zero-force members (see Figures 9.78c and 9.78d).

Since the support reaction in G is horizontal, member 11 is also a zero-force member (rule 3) (see Figure 9.78e).

All the verticals and diagonals are zero-force members. The load is therefore fully transferred by the bottom and top chord members. For the (continuous) top chord members we find

$$N^1 = N^4 = N^8$$

For the (continuous) bottom chord members we find

$$N^2 = N^6 = N^{10}.$$

When we talk about *omitting* zero-force members, this is done only to simplify the calculation. If the zero-force members are removed from the truss in reality, the truss becomes kinematically indeterminate.

Zero-force members therefore have a genuine function in the truss. On the one hand they ensure the truss retains its shape, while on the other they can prevent *buckling* (in the plane of the structure) of (long) compressed members, such as the bottom chord in Figure 9.77, or the top chord in Figure 9.79.

Example 3

You are given the truss in Figure 9.80. The diagonals are crossing members.

Question:

Determine all the zero-force members for the given load.

Solution:

In this truss, it is not possible to find a section across three members (that do not intersect in one point), nor is there a joint with less than two unknowns (member forces or support reactions). We therefore cannot determine the member forces with the method of sections, or with the method of joints, unless we first determine the support reactions.

For determining the zero-force members in the truss, it is enough to know that the support reaction at the point of the roller is vertical, so that $N^2 = 0$. This means that $N^7 = 0$, and so forth (see Figures 9.81a–9.81d). We subsequently discover that members 10, 12 and 13 are zero-force members.

Determining the other member forces is now a relatively simple task. Note that the force flow does not change when the crossing diagonals are joined at the point where they cross (rule 4).

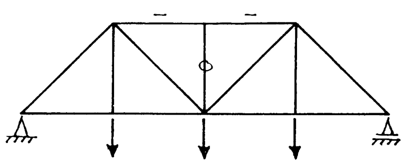


Figure 9.79 Zero-force members have a definite function in a truss. On the one hand, they ensure the truss retains its shape. On the other hand they can prevent buckling (in the plane of the structure) of (long) compressed members, such as the top chord in this truss.

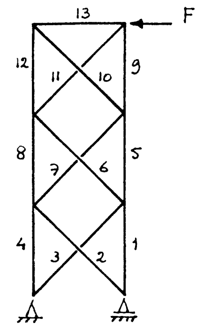


Figure 9.80 A truss in which the diagonals cross one another.

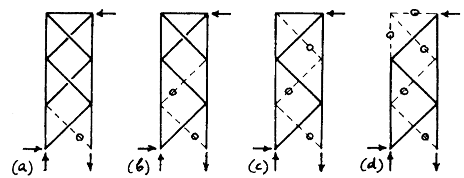
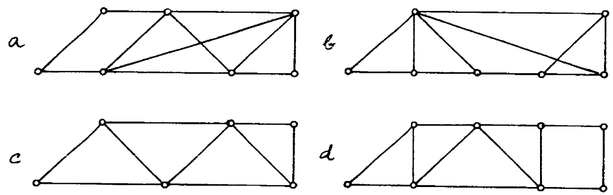


Figure 9.81 The zero-force members in the truss. (a) to (d) represent the order in which the zero-force members can be found.

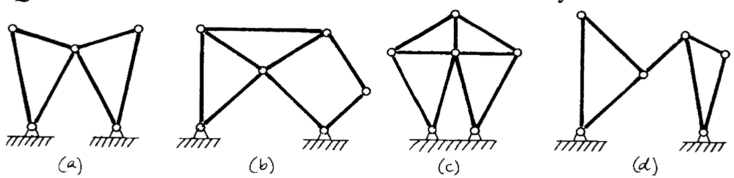
9.4 Problems

Kinematically/statically (in)determinate trusses (Section 9.2)

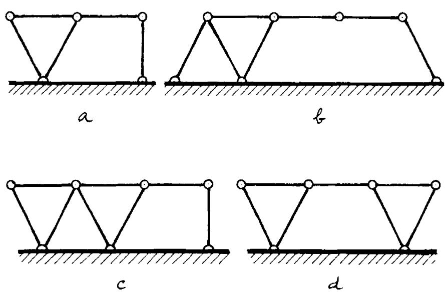
9.1 *Question*: Which of these structures retains its shape?



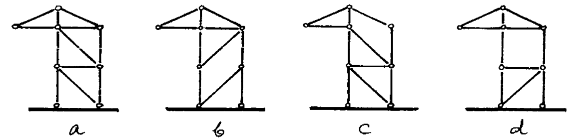
9.2 *Question*: Which of these structures is kinematically indeterminate?



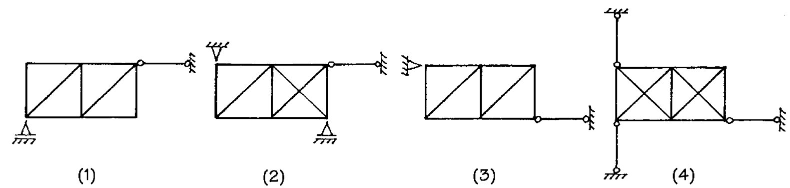
9.3 *Question*: Which structure is kinematically determinate?



9.4 *Question*: Which structure is kinematically determinate?



9.5: 1–4 You are given four simple or self-contained trusses that are supported in four different ways:



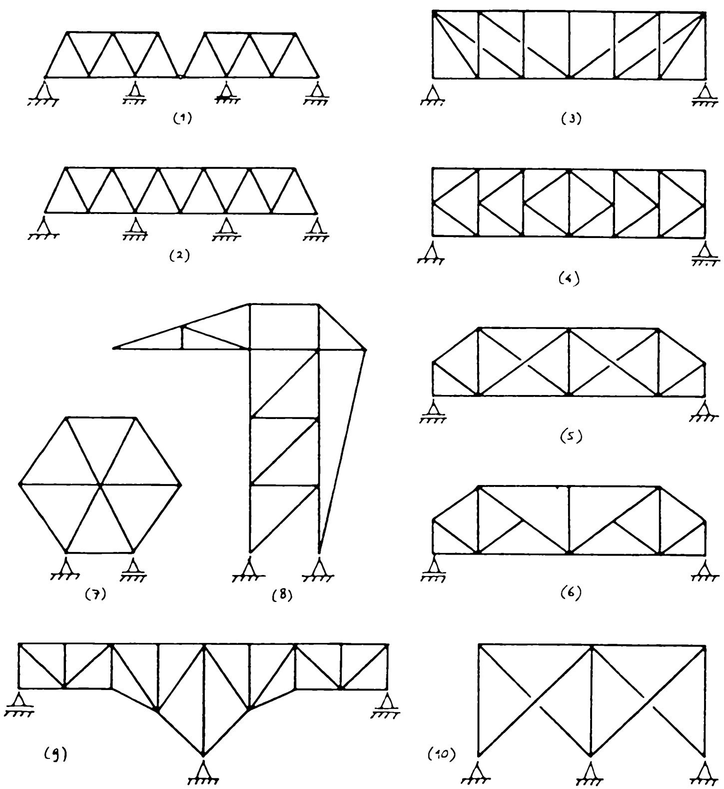
Questions:

a. What is the essential difference between a kinematically determinate and a kinematically indeterminate structure?
b. What is the essential difference between a statically determinate and a statically indeterminate structure?
c. Indicate whether the structure is
   – kinematically determinate (kd) or kinematically indeterminate (ki), and (if kinematically determinate) whether the structure is
   – statically determinate (sd) or statically indeterminate (si).

9.6: 1–10 The given trusses are kinematically determinate.

Questions (for each truss):

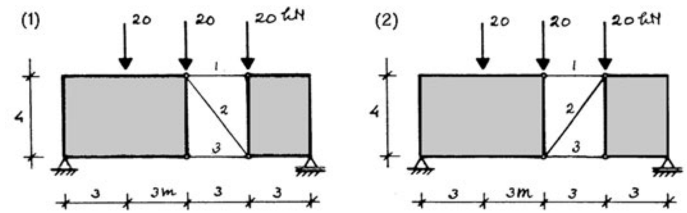
a. Is the truss statically determinate or statically indeterminate?
b. What is the degree of static indeterminacy if the truss is statically indeterminate?



Method of sections (Section 9.3.1)

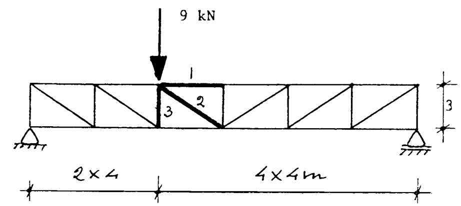
Note: Unless indicated otherwise, all structures in the problems are trusses.

9.7: 1–2 Two weightless blocks are connected by means of three bars. Structure (1) is different to structure (2) owing to the different placement of diagonal member 2.



Question: Determine the normal force in bars 1 to 3.

9.8 to 9.53 *Question*: Determine the normal force in the member(s) shown in bold.



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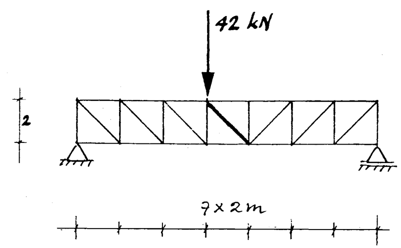

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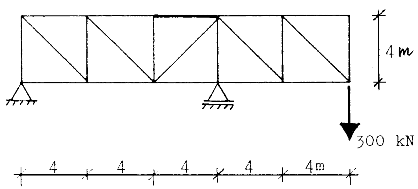

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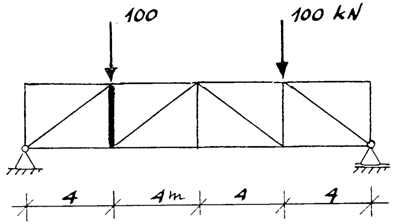

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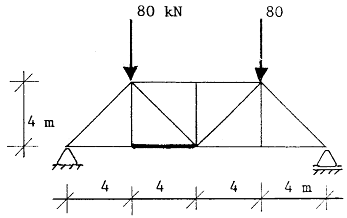

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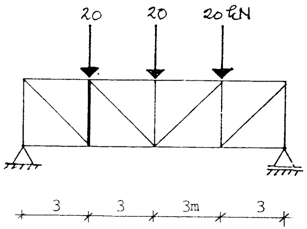

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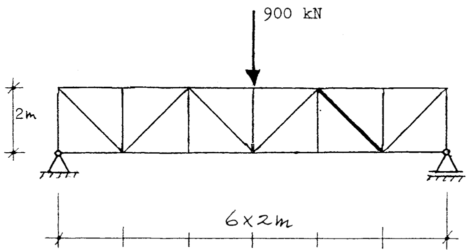

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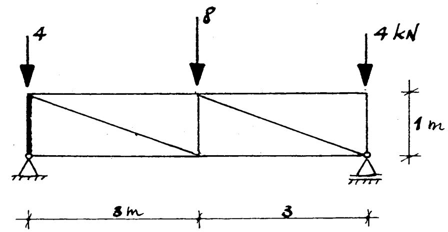

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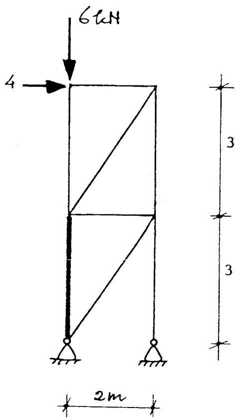

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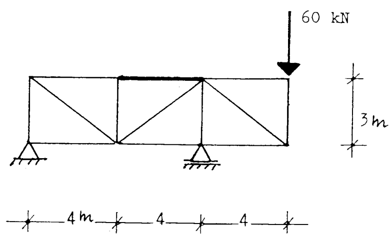

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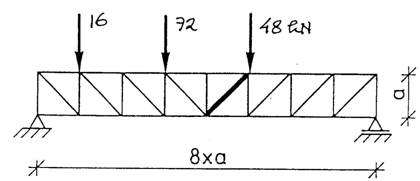

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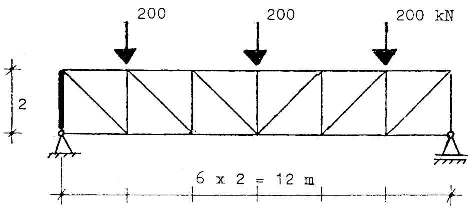

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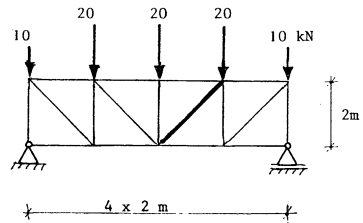

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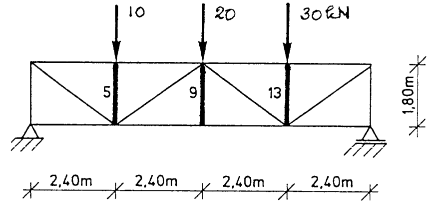

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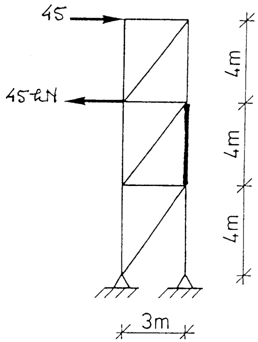

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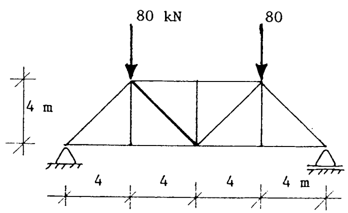

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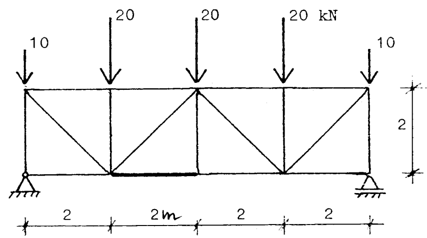

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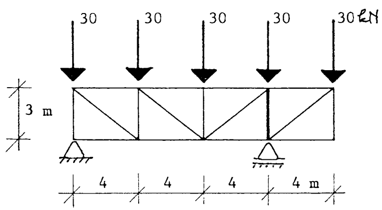

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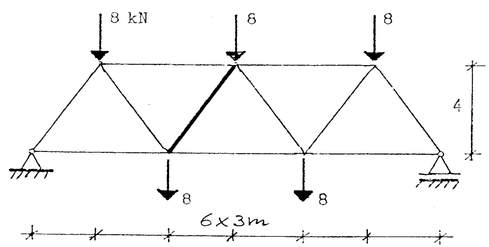

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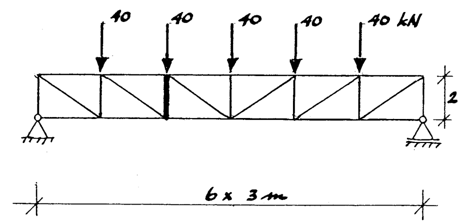

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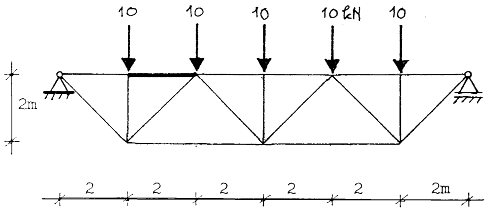

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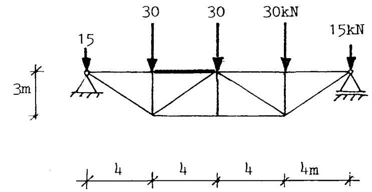

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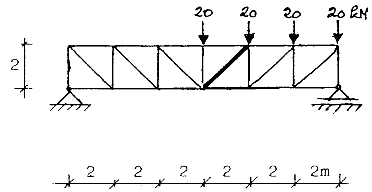

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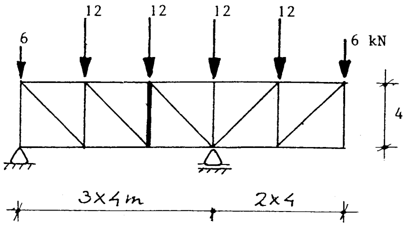

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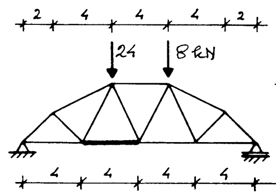

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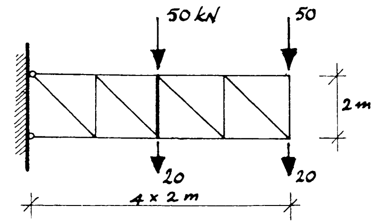

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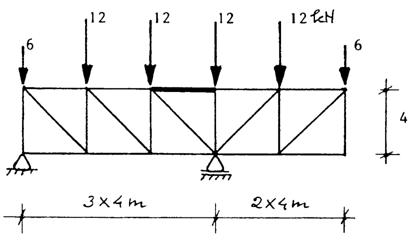

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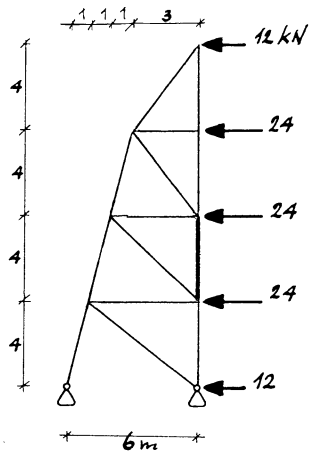

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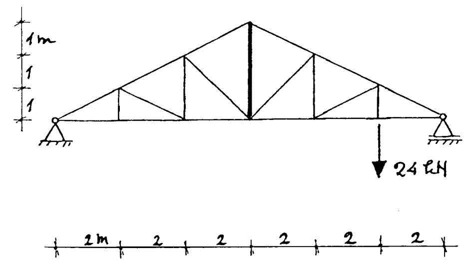

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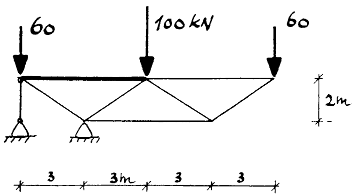

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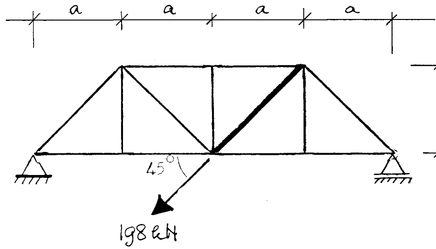

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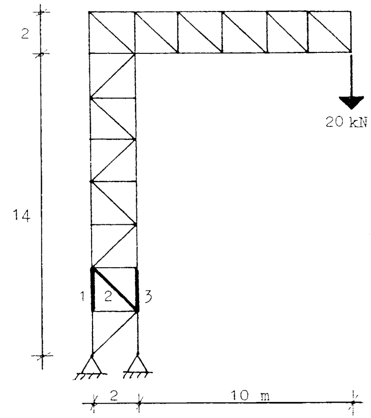

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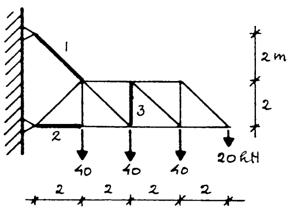

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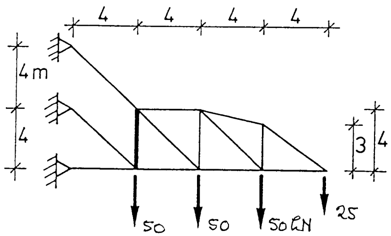

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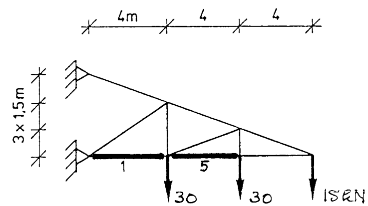

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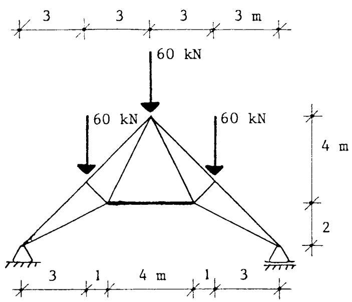

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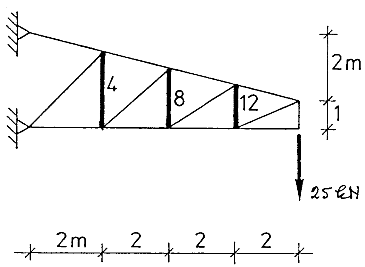

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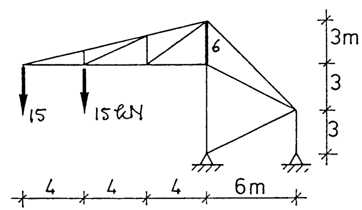

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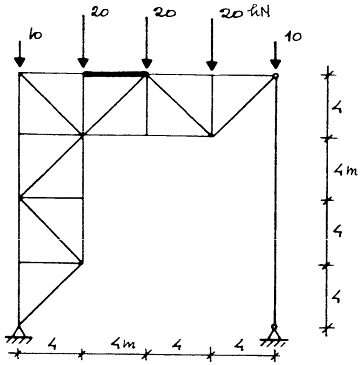

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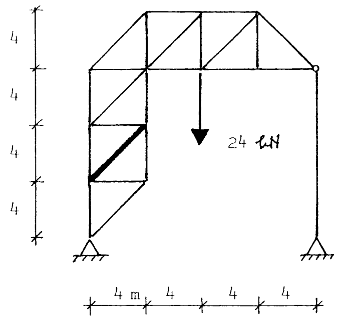

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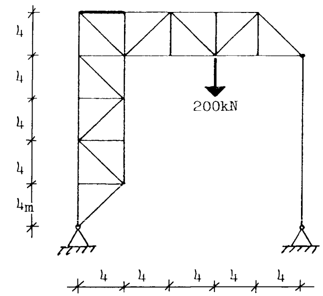
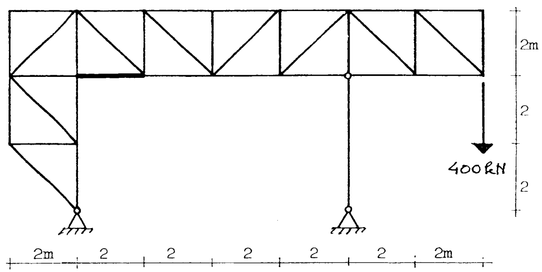
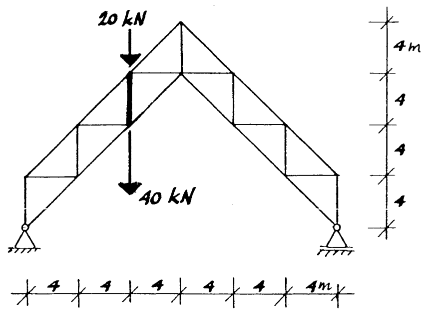


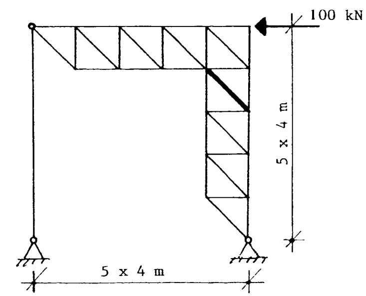
(9.48)



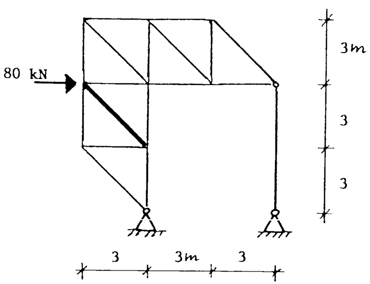
(9.49)



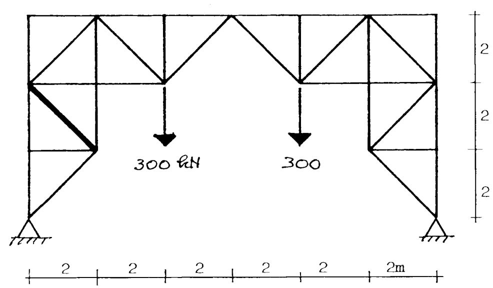
(9.50)



(9.51)



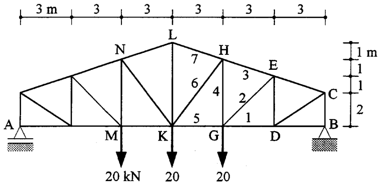
(9.52)



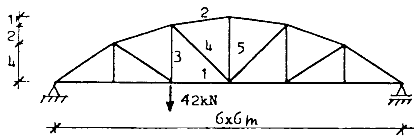
(9.53)

9.54 *Questions*:

a. Using the method of sections, determine the forces in members 1 to 7.
b. Draw the force polygon for the equilibrium of joint G. Plot the forces in the order 1, 2, 4, etc.

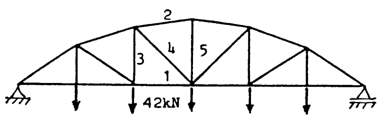


9.55 You are given a parabolic truss beam whose bottom chord is loaded by a single force of 42 kN.



Question: Using the method of sections, determine the normal force in the members 1 to 5.

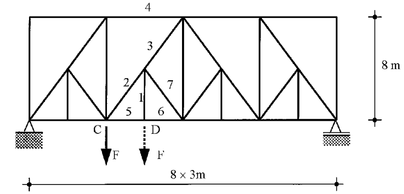
9.56 The truss from the previous problem is now loaded at the joints on the bottom chord by five equally large forces of 42 kN.



Question: Using the method of sections, determine the normal force in members 1 to 5.

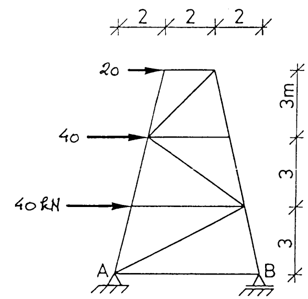
9.57 Use the method of sections to determine the normal force in members 1 to 7.

a. for $F = 160$ kN at joint C;
b. for $F = 160$ kN at joint D.

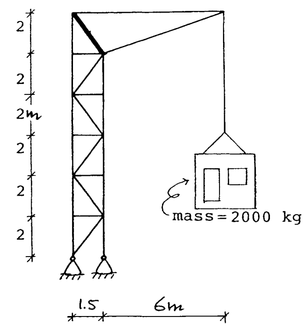


Method of joints (Section 9.3.2)

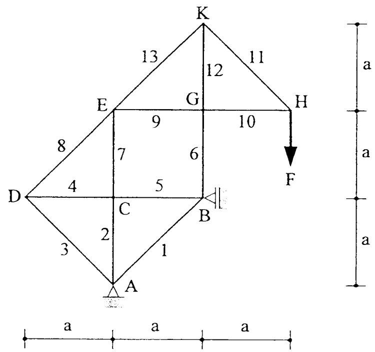
9.58 *Question*: Determine the normal force in member AB.



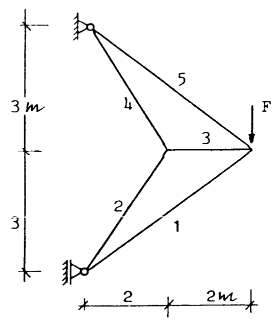
9.59 *Question*: Determine the normal force in the member shown in bold.



9.60 *Question*:
Using the method of joints, determine all the member forces.

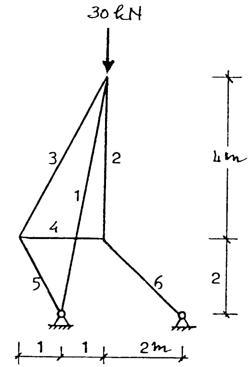


9.61 *Question*:
Using the method of joints, determine all the member forces due to $F = 6$ kN.



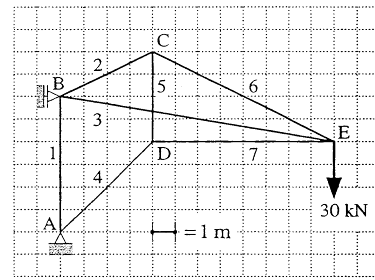
9.62 You are given a truss in which members 1 and 4 cross one another.

Question:
Using the method of joints, determine the normal forces in members 1 to 6 due to the vertical force of 30 kN in the top of the truss.

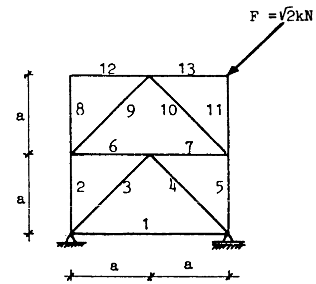


9.63 You are given a truss in which members 3 and 5 cross one another.

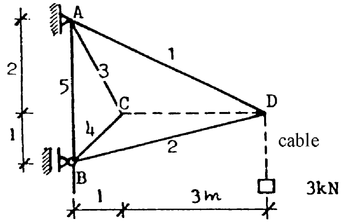
Question:
Using the method of joints, determine all the member forces. To do so, draw the force polygon for all the joints.



9.64 *Question*: Using the method of joints, determine all the member forces.

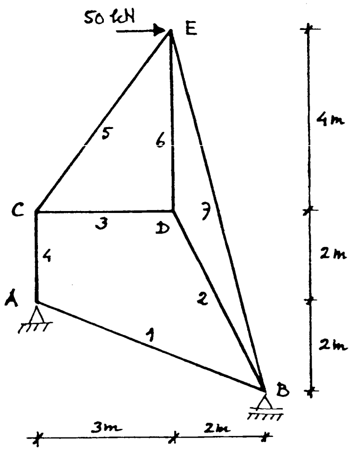


9.65 In the truss shown, the dashed line is a cable that is connected to the truss at C and runs over a pulley (without friction) at D. A load of 3 kN hangs from the cable.

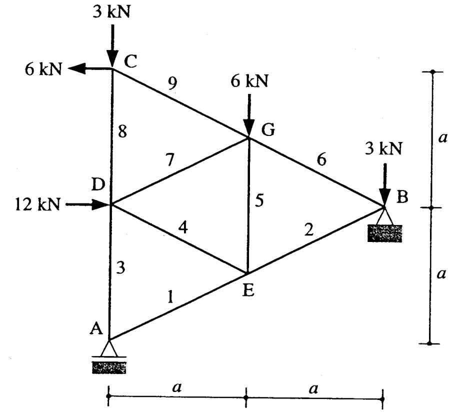


Question:
Using the method of joints, determine all the member forces.

9.66 *Questions*:
a. Using the method of joints, determine the forces in members 1 to 7.
b. Draw the force polygons for joints B and E.

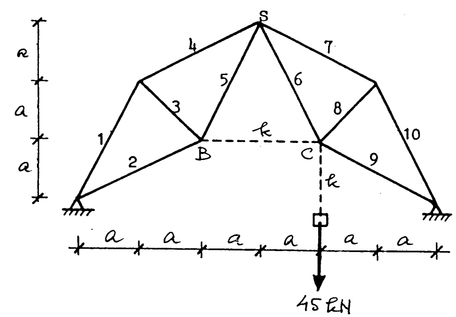


9.67 *Questions*:
a. Determine and draw the support reactions at A and B.
b. Determine all the member forces. To do so, draw the force polygon for the equilibrium for all the joints. Choose a scale of 5 mm ≡ 1 kN for the forces.



9.68 In the truss shown, the dashed line k is a cable that is joined to the truss at B and runs over a trolley (without friction) at C. The cable is loaded with a weight of 45 kN.

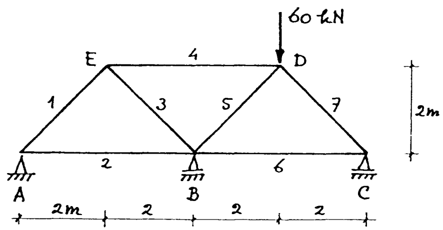
Questions:
a. Draw the forces that the cable exerts at B and C on the truss.
b. Using the method of joints, determine the forces in members 3 and 8.
c. Draw the force polygon for the equilibrium of joint C.



9.69 In the truss shown, there is a tensile force in member 4 of 20 kN: $N^4 = +20$ kN.

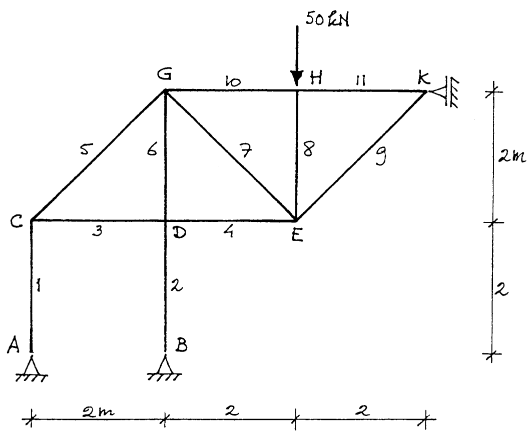
Questions:

a. Show that the truss is statically indeterminate to the first degree.
b. Using the method of joints, determine all the member forces.
c. Draw the support reactions in the direction in which they act, and give their values.
d. Draw the force polygon for joint B. Plot the forces in the order 2, 3, 5 and 6. Use 10 mm ≡ 10 kN as force scale.



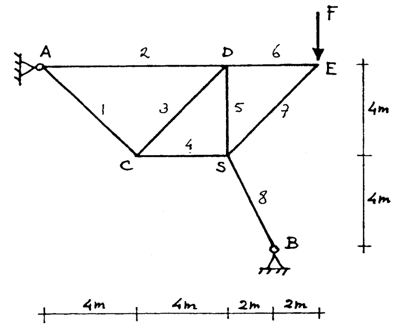
9.70 *Questions*:

a. Draw the support reactions as they act in reality on the structure and give their values.
b. Using the method of joints, determine the forces in members 1 to 11.



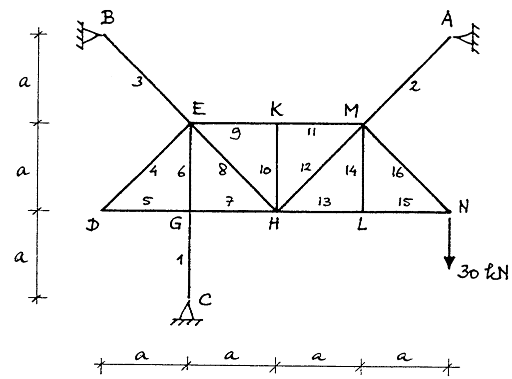
9.71 *Questions*:

a. Using the method of joints, determine all the member forces.
b. Draw the force polygon for the equilibrium in joint S.

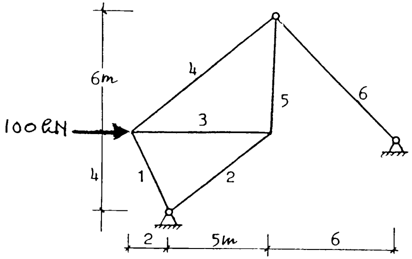


9.72 *Questions*:

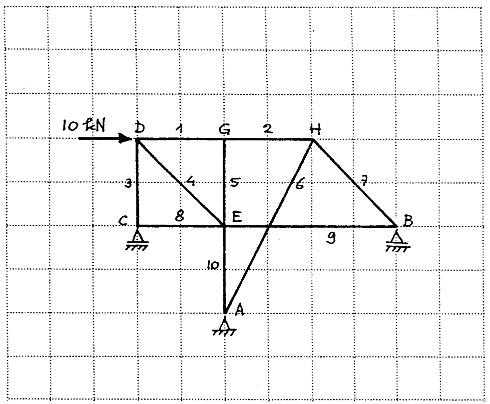
a. Determine the support reactions at A, B and C.
b. Using the method of joints, determine the forces in members 1 to 16.
c. Check the force equilibrium of joint E graphically.



9.73 *Question*: Using the method of joints, determine all the member forces.



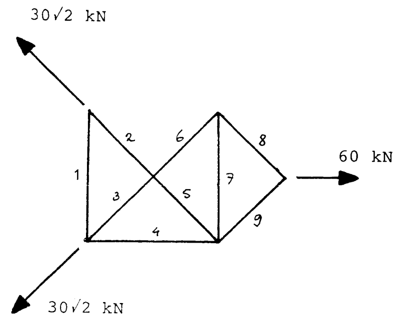
9.74 You are given a truss that is supported on a hinge at A and on rollers at B and C. The truss is loaded by means of a horizontal force of 10 kN at D. The members 6 and 9 cross one another.



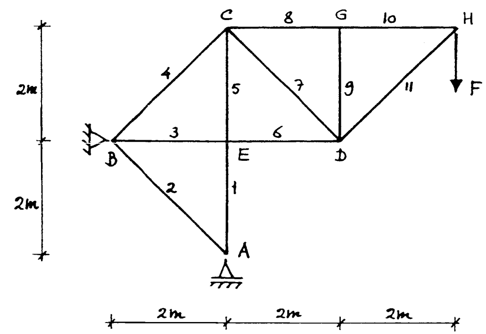
Questions:

a. First determine (as far as possible) the support reactions.
b. Determine the force in member 6.
c. Also determine all the other member forces.
d. Draw all the support reactions as they act in reality on the structure and give their values.

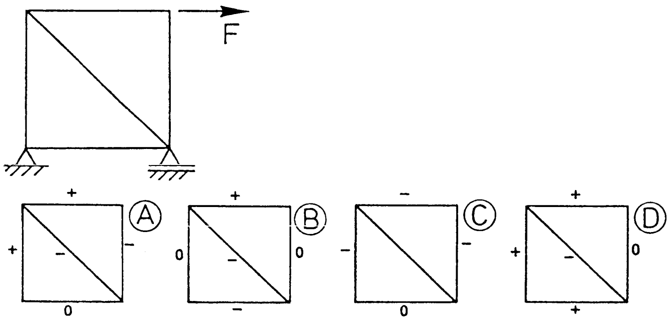
9.75 *Question*: How many tension members does this truss have due to the given load?



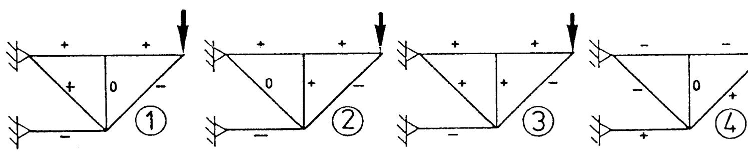
9.76 *Question*: For each of the members in the truss, indicate whether it is a zero-force member, a tension member, or a compression member. You do not have to calculate the member forces.



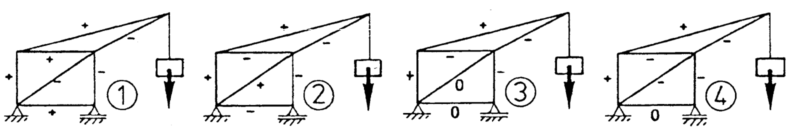
9.77 *Question*: In which figure are the correct signs for the member forces given?



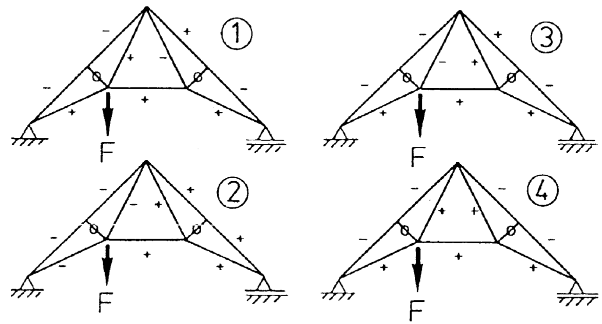
9.78 *Question*: In which figure are the correct signs for the member forces given?



9.79 *Question*: In which figure are the correct signs for the member forces given?

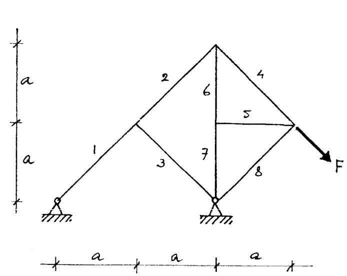


9.80 *Question*: In which figure are the correct signs for the member forces given?

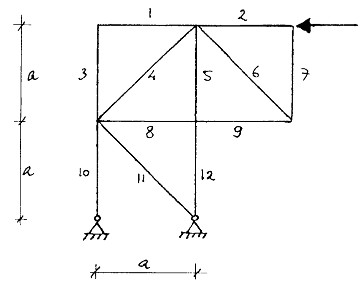


Zero-force members (Section 9.3.3)

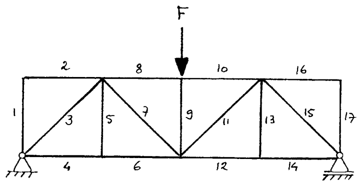
9.81 to 9.92 *Question*: Which of the members are zero-force members?



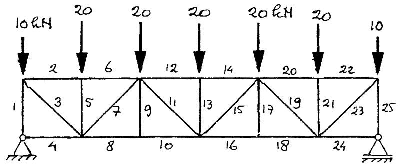
(9.81)



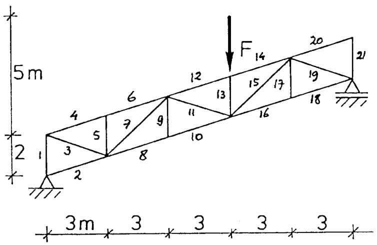
(9.82)



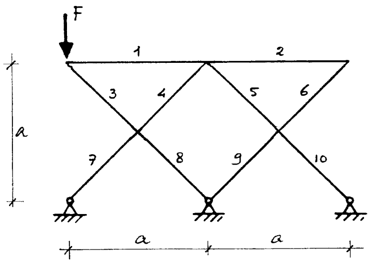
(9.83)



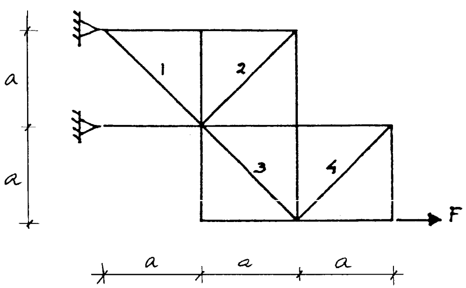
(9.84)



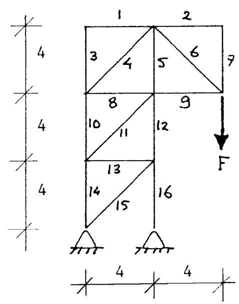
(9.85)



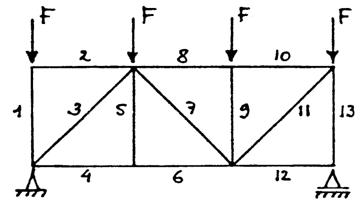
(9.86)



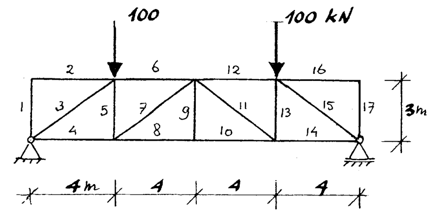
(9.87)



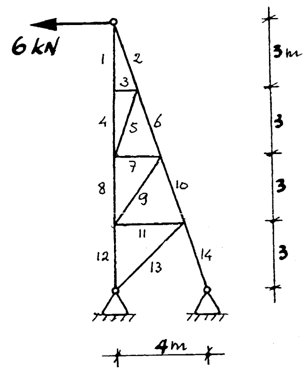
(9.88)



(9.89)



(9.90)



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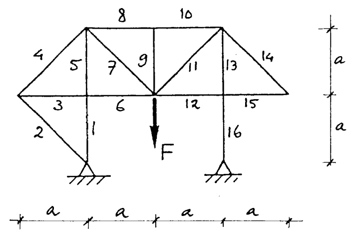
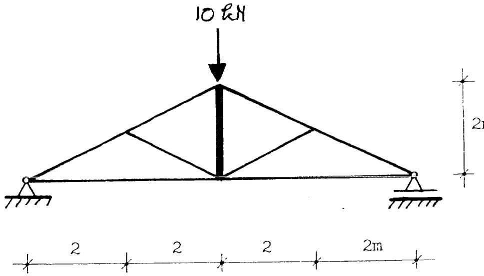


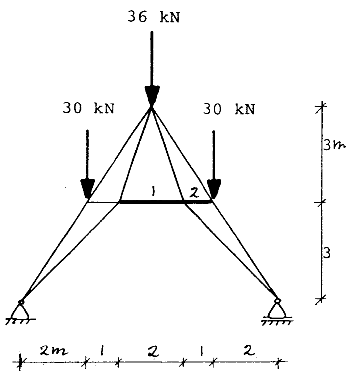
(9.92)

Mixed problems (Section 9.3)

9.93 *Question*: Determine the normal force in the vertical shown in bold.

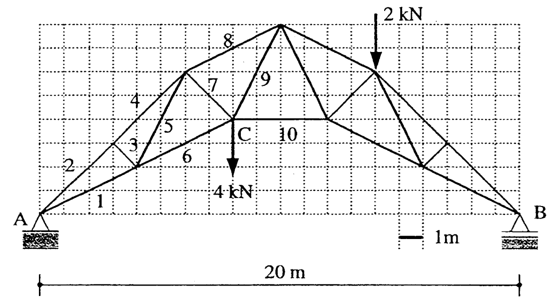


9.94 *Question*: Determine the normal force in members 1 and 2 shown in bold.



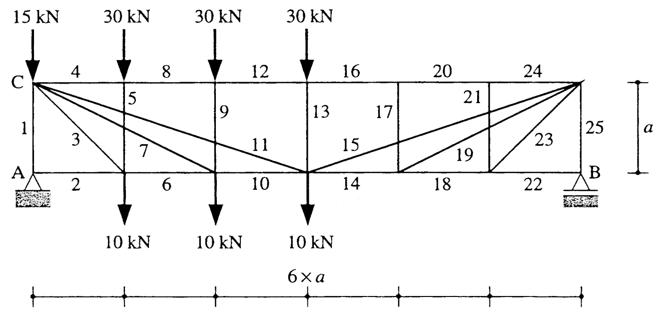
9.95 *Questions*:

a. Determine the member forces 1 to 10, with the correct sign for tension and compression.
b. Draw the force polygon for joint C in the order 6, 7, 9 and 10.



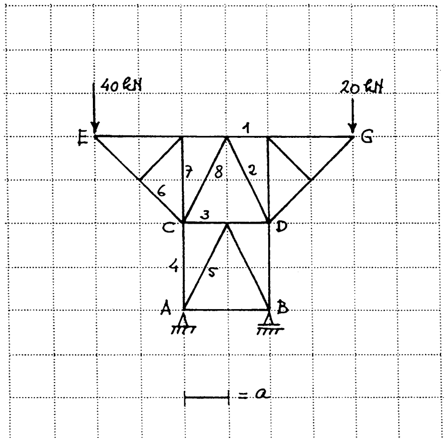
9.96 *Questions*:

a. Determine all the member forces.
b. Draw the force polygon for joint C.



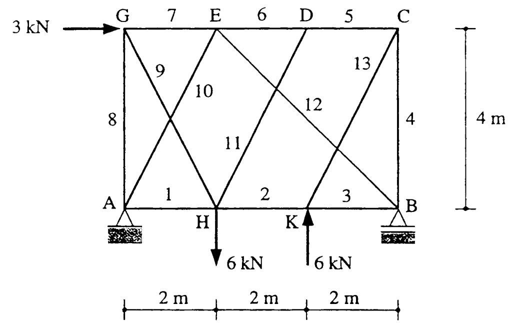
9.97 *Questions*:

a. Using the method of sections, determine the forces in members 1 to 4.
b. Using a method of your own choice, determine the forces in members 5 to 8.
c. Draw the force polygon for joint C. Plot the forces in the order 3, 4, 6, 7 and 8. Use a force scale of 10 mm ≡ 10 kN.
d. How many zero-force members are there in the truss? Indicate them (clearly) in the truss.



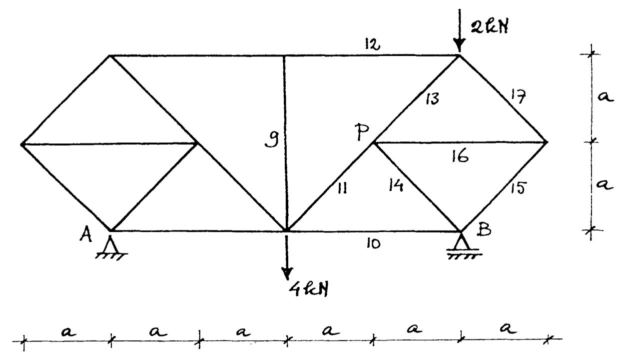
9.98 *Questions*:

a. The truss is kinematically determinate. What does that mean?
b. Show that the truss is statically determinate.
c. Determine all the member forces.
d. Draw the force polygon for the equilibrium of joint E, in the order 6, 7, 10 and 12. Use a scale of 5 mm ≡ 1 kN for the forces.



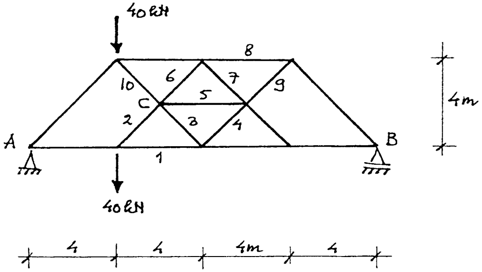
9.99 *Questions*:

a. Determine the forces in the members 9 to 17.
b. Draw the force polygon for the equilibrium at joint P.

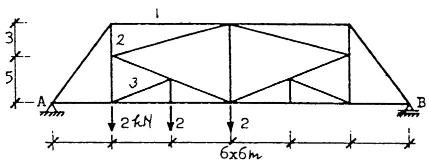


9.100 *Questions*:

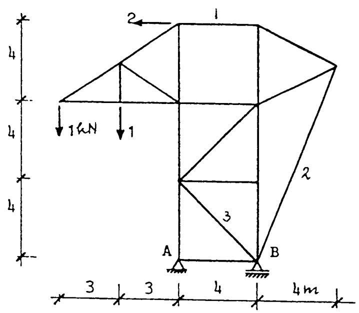
a. Determine the forces in the members 1 to 9.
b. Draw the force polygon for the equilibrium at joint C. Plot the member forces in the order 2, 3, 5, 6 and 10.



9.101: 1–2 *Question*: Determine the forces in the members 1 to 3.



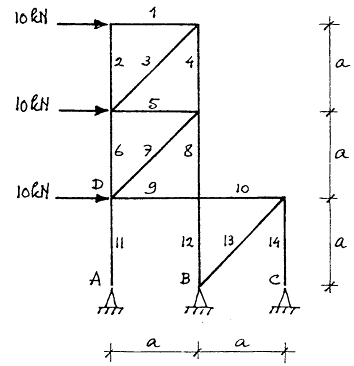
(1)



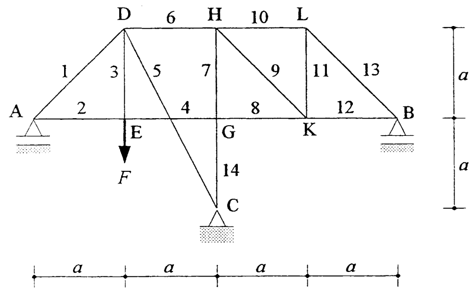
(2)

9.102 *Questions*:

a. Determine the support reactions at A, B and C. Draw them as they act in reality on the structure and write down their values alongside.
b. Determine the forces in the members 1 to 14.
c. Draw the force polygon for the equilibrium of joint D. Plot the forces in the order 11, 9, 7 and 6.



9.103 You are given a truss in which the members 4 and 5 cross one another.



Questions:

a. Determine the member forces.
b. Draw the support reactions as they are acting in reality.
c. Draw the force polygon for joint H. Plot the member forces in the order 6, 7, 9 and 10.

10 Section Forces

Section forces is the collective name for *interaction forces* or *joining forces* in a member axis. We make a distinction between *normal force*, *shear force*, *bending moment* and *torsional moment*. Section forces always occur in pairs and ensure *force transfer* in a member. This is addressed in further detail in Section 10.1.

The section forces can vary along the member axis. Here they are a function of the x coordinate, chosen along the member axis. Drawing these functions provides a graphic representation of the distribution of the section forces, known as diagrams.

In Section 10.2, we will determine the diagrams for the *normal force* (N), *shear force* (V) and *bending moment* (M) directly from the equilibrium.

For a correct interpretation of the signs in the M and V diagrams, we must always know the coordinate system in which we are working. For manual calculations, one often prefers the use of so-called *deformation symbols*. The deformation symbols are independent of the coordinate system. In Section 10.3, we will introduce the *bending symbol* for bending moments and the *shear symbol* for shear forces.

Section 10.4 summarises the sign conventions for N, V and M diagrams.

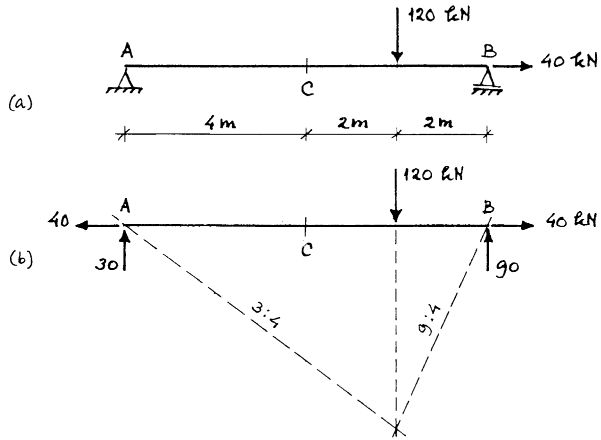


Figure 10.1 (a) A member modelled as a line element, loaded by two forces of 120 kN and 40 kN. (b) The isolated member with its support reactions.

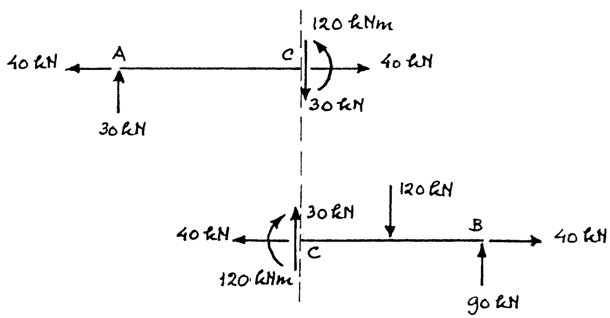


Figure 10.2 The section forces (interaction forces) that the member, modelled as a line, element has to transfer at C.

10.1 Force flow in a member

In mechanics, members in a frame are represented by means of lines. Each one-dimensional *line element* represents a three-dimensional member (see Section 4.3.2). All *member properties* are assigned to this single line. The *force flow* in the member is also assumed to occur along this line, which is known as the *member axis*.

Section forces is the collective name for *interaction forces* or *joining forces* in the member modelled as a line element. They always occur in pairs. An example is provided in Section 10.1.1.

In reality, the force transfer is not concentrated in the member axis, but is distributed over the *member cross-section*, and is the sum of a large number of small interactions between adjacent particles of matter. These interactions are described using the concept *stress*. We look at this in more detail in Section 10.1.2.

In Section 10.1.3, we discuss the *general definition* for the section forces, related to the *stresses in the cross-section*.

The *sign conventions* for section forces are closely related to those for stresses. They are summarised in Section 10.1.4.

10.1.1 Member axis and member cross-section; section forces

In Figure 10.1a, the load on beam AB in the one-dimensional model is transferred to the supports via the *member axis*. Figure 10.1b shows the support reactions. The lines of action of the resultant forces at A and B intersect in the line of action of the force of 120 kN (graphical check of the moment equilibrium for a body subjected to three forces, see Section 3.3.2).

Figure 10.2 shows the *interaction forces* that the member has to transfer at C. After introducing a section at C across the member, the interaction forces are found from the equilibrium of one of the isolated parts, to the right or left of C.

In reality, the member is not one-dimensional, but has cross-sectional dimensions. Figure 10.3 shows an arbitrary section across the member at C. The force transfer is not concentrated in the member axis, but varies over the section as the sum of a large number of very small interactions between adjacent particles of matter. Mathematically, we describe this phenomenon in the section by means of the concept *stress* (see Section 6.5).[1]

The distributions in magnitude and direction of the stresses in the section are as yet unknown. The equilibrium, however, shows that the stress resultant R, regardless of the shape of the section, must be 50 kN, and that the line of action of R must coincide with the line of action of the support reaction at A (see Figure 10.3).

We usually do not give the section across a member an arbitrary shape, but rather choose one that is straight and normal to the member axis, as shown in Figure 10.4. This type of section is called a *normal section* or simply *cross-section*. Hereafter, when we refer to a section, we always mean a normal section.

The intersection of the line of action of the stress resultant R and the cross-sectional plane is known as the *centre of force*.

The intersection of the member axis with the cross-sectional plane is the *normal force centre*, or *normal centre*, of the section.[2] The normal (force) centre is indicated by the two-character symbol NC (see Figure 10.4).

Consistent with the model used for a line element, it is usual to represent

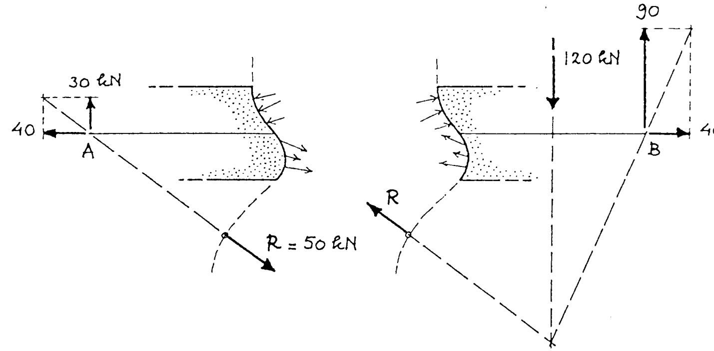


Figure 10.3 An arbitrarily shaped cross-section at C. The force transfer is not concentrated in the member axis but is distributed over the section as the sum of many very small interactions between adjacent particles of matter. The stress resultant R is the resultant force due to the stresses in the section.

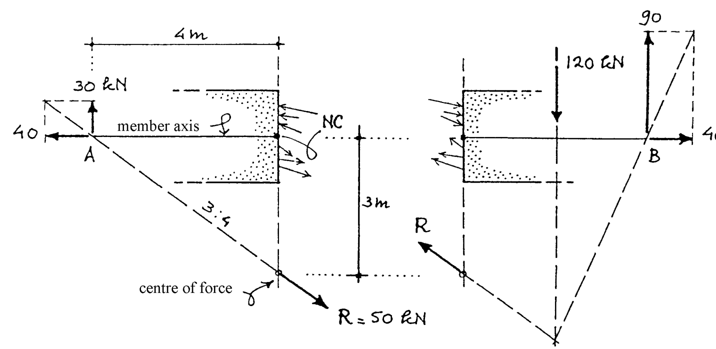


Figure 10.4 It is usual to choose the section across a member as a plane normal to the member axis. The centre of force is the intersection of the line of action of the stress resultant R with the plane of the cross-section.

---

1 In Section 10.1.2, the definition of the concept stress, as introduced in Section 6.5, is adapted to describe the interaction between the particles of matter.

2 The member axis is by definition chosen through the normal centre NC of the cross-section. The location of the normal centre is covered in Volume 2 *Stresses, Deformations, Displacements*. In so-called *homogeneous cross-sections* (the whole cross-section consists of the same material) the normal centre coincides with the centroid of the cross-section.

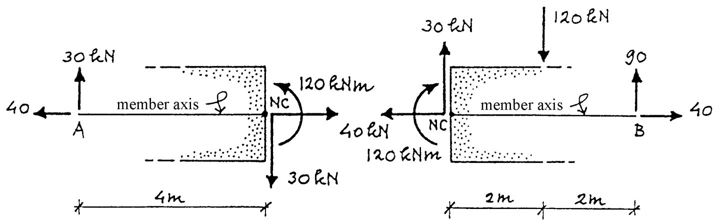


Figure 10.5 Linked up with the modelling as line element, the section forces (interaction forces) are said to act at the normal centre NC, the intersection of the member axis with the cross-sectional plane. Here there are three different section forces: a *normal force* of 40 kN, a *shear force* of 30 kN and a *bending moment* of 120 kNm.

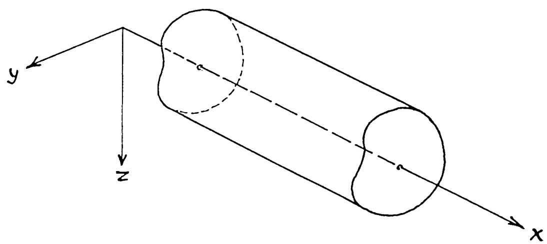


Figure 10.6 The sign of the section forces is related to a (local) coordinate system with the x axis along the member axis and the yz plane parallel to the cross-sections.

the forces in a section as acting in the member axis, or in other words, at the normal centre NC. The vertical component of the stress resultant R at section C is 30 kN and can be shifted directly along its line of action to the member axis. The horizontal component of R is 40 kN and has to be shifted 3 m in section C parallel to its line of action. This gives a moment of (40 kN)(3 m) = 120 kNm. The *section forces* in section C, acting at the member axis, are shown in Figure 10.5.

The section forces on the left- and right-hand sides of the section are equal and opposite. Section forces are *interaction forces* and always occur *in pairs*. You should always keep this in mind, even if you are drawing only one of the member segments to the right or left of the section.

In the case shown in Figure 10.5, we can distinguish between the following three section forces:

- A *normal force*: this is the pair of forces of 40 kN with their lines of action along the member axis; a normal force acts *normal to the cross-sectional plane*.
- A *shear force*: this is the pair of forces of 30 kN in the cross-sectional plane; a shear force acts *transverse to the member axis*.
- A *bending moment*: this is the pair of couples of 120 kNm in a plane normal to the cross-sectional plane.

For *normal force*, *shear force* and *bending moment*[1] we use the symbols N, V and M respectively.

Since section forces are interaction forces, their sign convention is somewhat more complicated than that for a force F or couple T. The sign of the section forces is related to a (*local*) *coordinate system* with the x axis along the member axis and the yz plane parallel to the member cross-sections (see Figure 10.6).

[1] These names used in practice in no sense reflect that we are talking about *interaction forces* (pair of forces).

After applying a section, there are two cross-sectional planes. To distinguish these from one another, we call the sectional plane *positive* where the x axis points outwards, and the sectional plane negative where the x axis points inwards. This is shown in Figure 10.7 where the sectional planes I are positive, and sectional planes II are negative.

More formally, we describe the position of a sectional plane using a so-called *unit normal vector* $\vec{n}$. This is a unit vector (a vector with length 1) pointing outwards from the matter and normal to the sectional plane that is considered. The position of a sectional plane in space is fully determined by the scalar components n_x; n_y; n_z of the unit normal vector $\vec{n}$. Since the cross-section is normal to the x axis as chosen along the member axis, $n_y = n_z = 0$. The sectional plane is now said to be *positive* if the unit normal vector $\vec{n}$ points in the positive x direction ($n_x = +1$), and *negative* if $\vec{n}$ is pointing in the negative x direction ($n_x = -1$). Again, see Figure 10.7.

Figure 10.8 shows the *positive directions* in the given xz coordinate system of the normal force N, the shear force V and the bending moment M. The *sign conventions* are as follows:

- A normal force N is positive when it acts on a positive sectional plane in the positive x direction and on a negative sectional plane in the negative x direction. To simplify: a normal force N is positive as a tensile force and negative as a compressive force. This sign convention has already been used in trusses (see Section 9.3).
- A shear force V is positive when it acts on a positive sectional plane in the positive z direction, and on a negative sectional plane in the negative z direction.
- A bending moment M is positive when it causes tension (tensile stresses) at the positive z side of the x axis, and causes compression (compressive stresses) at the negative z side.

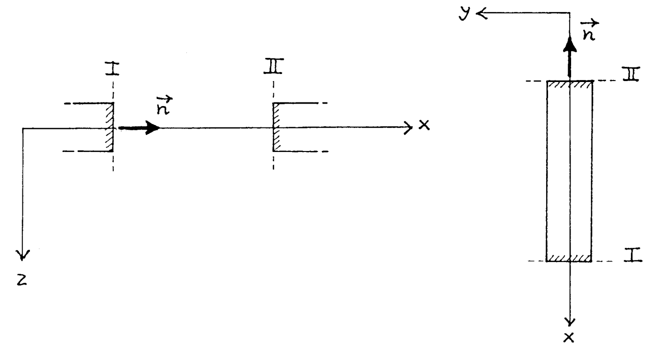


Figure 10.7 The sectional planes I are positive because the x axis points out of the matter and the unit normal vector $\vec{n}$ points in the positive x direction. The sectional planes II are negative because the x axis points into the matter and the unit normal vector $\vec{n}$ points in the negative x direction.

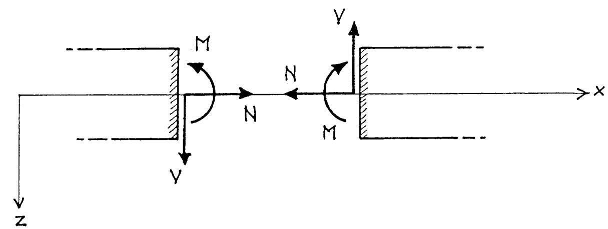


Figure 10.8 The positive directions of the normal force N, shear force V and the bending moment M in an xz coordinate system.

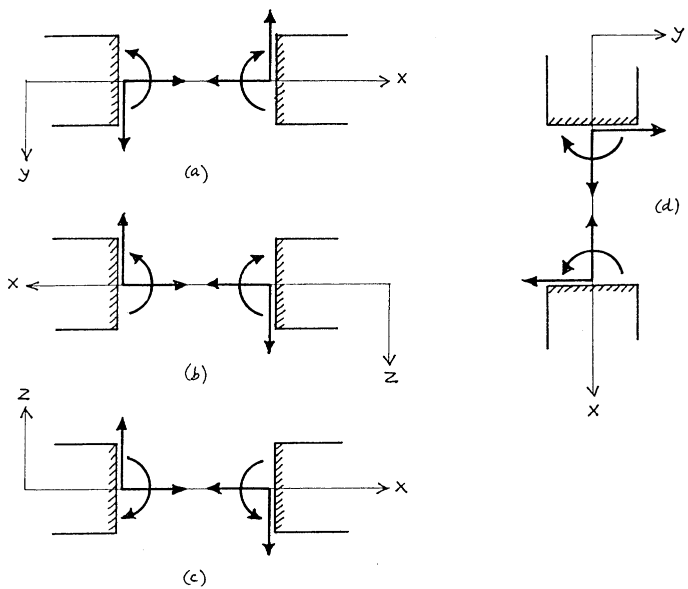


Figure 10.9 The positive directions of the section forces N, V and M in different coordinate systems.

In an xy coordinate system, the positive/negative section forces are defined in the same way. Figure 10.9 shows the positive directions of the section forces in various coordinate systems.[1]

The sign convention given here for the section forces N, V and M is associated with the sign convention for stresses in the cross-section. We look at this in more detail in Section 10.1.2.

10.1.2 Stresses in the cross-section

On a *positive sectional plane*, consider a small area ΔA. Let $\Delta\vec{F}$ be the resultant of all the small forces that are transferred by the matter via that small area. $\Delta\vec{F}$ is built up by the contributions of a large number of interactions between the particles of matter. Figure 10.10a shows the components $\Delta\vec{F}_x$; $\Delta\vec{F}_y$; $\Delta\vec{F}_z$ of the small force $\Delta\vec{F}$.

If ΔA is smaller, so is $\Delta\vec{F}$. It is assumed that the relationship between $\Delta\vec{F}$ and ΔA has a limit when ΔA approaches zero. This limit was defined in Section 6.5 as the stress vector $\vec{p}$:

$$\vec{p} = \lim_{\Delta A \to 0} \frac{\Delta\vec{F}}{\Delta A}.$$

The definition of the stress vector is based on the idealised model of continuous matter. Figure 10.10b shows the components p_x; p_y; p_z of the stress vector $\vec{p}$.

---

[1] Note: it is wrong to say that a bending moment is positive when the couple acts on the positive sectional plane in accordance with the positive sense of rotation and on the negative sectional plane in accordance with the negative sense of rotation. This is shown in Figures 10.9a and 10.9d.

If we look at the same small area ΔA on the *negative sectional plane*, there is an equal but opposite force, in accordance with the principle of action and reaction. The stress vectors $\vec{p}^{(\mathrm{I})}$ and $\vec{p}^{(\mathrm{II})}$ have the same magnitude at corresponding points ($\Delta A \to 0$) on the positive and negative sectional plane, but have opposite directions (see Figure 10.11):

$$\vec{p}^{(\mathrm{I})} = -\vec{p}^{(\mathrm{II})}.$$

The stress vector is defined in a particular point and for a particular sectional plane. If we want to indicate the *force transfer* (*interaction*) at a point of the cross-section, the stress vector $\vec{p}$ alone is not enough, as we also have to indicate the status of the sectional plane that is considered. This is done by means of the unit normal vector $\vec{n}$ on that plane.

To describe the action of the forces that the matter to the right of the section exerts on the matter to the left, and vice versa, we introduce the following quantities, which are known as *cross-sectional stresses* (see Figure 10.12):

$$\begin{aligned}
\sigma_{xx} &= \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A \cdot n_x},\\
\sigma_{xy} &= \lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A \cdot n_x},\\
\sigma_{xz} &= \lim_{\Delta A \to 0} \frac{\Delta F_z}{\Delta A \cdot n_x}.
\end{aligned}$$

Here, n_x is the x component of the unit normal vector $\vec{n}$ on the sectional plane that is considered.

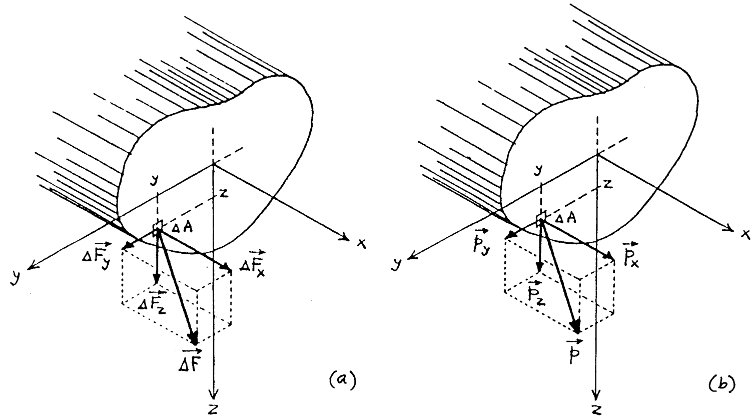


Figure 10.10 (a) The small force $\Delta\vec{F}$ is the resultant of all the small forces acting on a small but finite area ΔA. (b) The stress vector $\vec{p}$ is defined as the limit value of $\Delta\vec{F}/\Delta A$ for $\Delta A \to 0$.

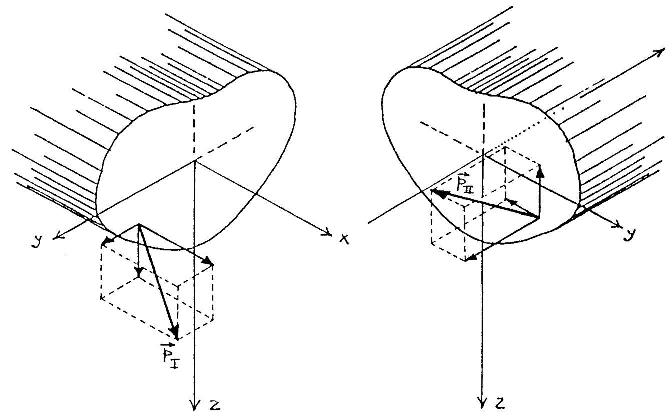


Figure 10.11 The stress vectors $\vec{p}^{(\mathrm{I})}$ and $\vec{p}^{(\mathrm{II})}$ in corresponding points on the positive and negative sectional plane are equal and opposite, so $\vec{p}^{(\mathrm{I})} = -\vec{p}^{(\mathrm{II})}$.

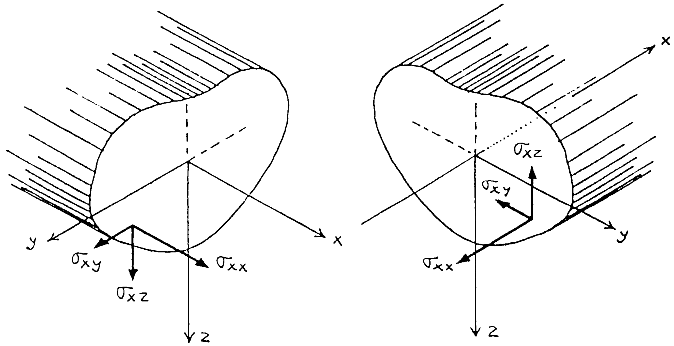


Figure 10.12 The stresses in the cross-section reflect the interaction through the area ΔA ($\Delta A \to 0$), of the right-hand part on the left-hand part, and vice versa. The normal stress σ_{xx} acts normal to the cross-sectional plane; the shear stresses σ_{xy} and σ_{xz} act in cross-sectional plane.

The kernel symbol σ for stress has two sub-indices. The first index relates to the *normal of the plane* on which the stress is acting; the second index relates to the *direction of the stress* (that is, the direction of the corresponding force component on that plane).

If we look at two corresponding equal areas ΔA to the right and to the left of the section, they are subject to two equal and opposite forces ΔF. Since the unit normal vectors also have opposite directions, the limit results for the negative sectional plane are the same as those for the positive sectional plane. The concept stress reflects the *interaction* through the small area ΔA ($\Delta A \to 0$), both for the right-hand part on the left-hand part, and vice versa.

The stress σ_{xx}, acting normal to the cross-sectional plane is known as the *normal stress*. The stresses σ_{xy} and σ_{xz}, that act in the cross-sectional plane are known as *shear stresses*.

The *sign convention for the stresses* results directly from their definition. The normal stress σ_{xx} is positive if n_x and ΔF_x are both positive or are both negative; the normal stress is negative if n_x and ΔF_x have different signs. In the same way, the shear stresses σ_{xy} and σ_{xz} are positive if n_x and ΔF_y, respectively n_x and ΔF_z are both positive or both negative; the shear stresses are negative if n_x and ΔF_y, respectively n_x and ΔF_z, have different signs.

The *sign convention* can be summarised as follows:

- A stress is positive when it acts on a positive plane in the positive direction or on a negative plane in the negative direction.
- A stress is negative when it acts on a positive plane in the negative direction or on a negative plane in the positive direction.

For more general cases, the *stress definition* can be summarised in short as follows:

$$\sigma_{ij} = \lim_{\Delta A \to 0} \frac{\Delta F_j}{\Delta A \cdot n_i} \quad (i, j = x, y, z)$$

in which both i and j can be replaced by x, y or z.[1]

Figure 10.13 shows the *positive stresses* acting on the sides of an (infinitesimally) small rectangular block. The block is bounded by six planes, of which three are positive and three are negative.

σ_{ij} is the stress

- on a small area with the unit normal vector parallel to the i axis (1st index),
- due to a force component parallel to the j axis (2nd index).

The stress σ_{ij} is a *normal stress* when the indices are the same ($i = j$) and a *shear stress* when the indices are different ($i \neq j$).

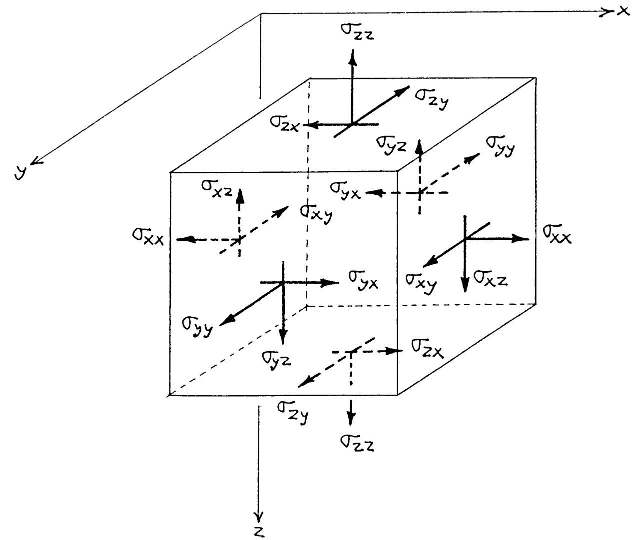


Figure 10.13 Positive stresses on the sides of a rectangular block. The kernel symbol σ for stress has two indices. The first index relates to the normal of the plane on which the stress is acting; the second index relates to the direction of the stress. The stress is a normal stress when both indices are equal, and a shear stress when both indices are different.

10.1.3 General definition of section forces

In a member cross-section, there are only normal stresses σ_{xx} and shear stresses σ_{xy} and σ_{xz} (see Figure 10.12). These stresses are as yet unknown functions of y and z, so that

$$\sigma_{xx} = \sigma_{xx}(y, z), \quad \sigma_{xy} = \sigma_{xy}(y, z) \quad \text{and} \quad \sigma_{xz} = \sigma_{xz}(y, z).$$

---

[1] The stresses σ_{ij} $(i, j = x, y, z)$ are the components of a quantity (the so-called *stress tensor*) that in a certain point for each arbitrary plane links the components of the *stress vector* $\vec{p}$ and the components of the *unit normal vector* $\vec{n}$.

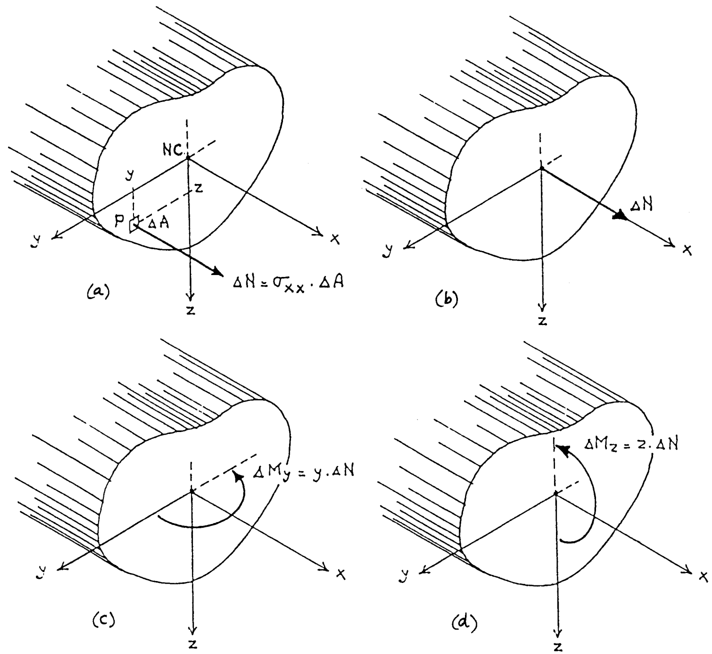


Figure 10.14 (a) The resultant of the normal stresses on a small area ΔA around a point P is a small force ΔN. This force in P is statically equivalent to (b) a small force ΔN in the normal force centre NC (the intersection of the member axis with the cross-sectional plane), together with (c) a small moment ΔM_y in the xy plane and (d) a small moment ΔM_z in the xz plane.

The resultant of the *normal stresses* on a small area ΔA around a point P is a small force ΔN:

$$\Delta N = \sigma_{xx} \Delta A.$$

This small force ΔN in P is statically equivalent to a small force ΔN in the member axis (the origin of the yz coordinate system), together with two small moments ΔM_y and ΔM_z, acting in the xy plane and the xz plane respectively (see Figure 10.14):

$$\Delta M_y = y \cdot \Delta N = y \cdot \sigma_{xx} \Delta A,$$
$$\Delta M_z = z \cdot \Delta N = z \cdot \sigma_{xx} \Delta A.$$

If we sum up the contributions of all the forces ΔN for the entire cross-section, this gives:

$$N = \int_A \sigma_{xx}\,\mathrm{d}A,$$
$$M_y = \int_A y\sigma_{xx}\,\mathrm{d}A,$$
$$M_z = \int_A z\sigma_{xx}\,\mathrm{d}A.$$

- N is the resulting force (or rather: the resulting pair of forces) due to the normal stresses in the cross-section, and is by definition known as *normal force* when it acts at the *normal centre* NC of the cross-section (the intersection of the member axis with the cross-sectional plane).
- M_y is a moment (or rather: a pair of moments) that acts in the xy plane. M_y is known as the *bending moment in the xy plane*.
- M_z is a moment (or rather: a pair of moments) that acts in the xz plane. M_z is known as the *bending moment in the xz plane*.

Note that indices y and z in M_y and M_z also occur under the integral symbol. This makes the formulas easy to memorise. In addition, y and z reoccur in the indication of the planes in which the bending moments act: M_y in the $x\underline{y}$ plane and M_z in the $x\underline{z}$ plane.

The *normal force* N is positive as a tensile force and negative as a compressive force.

The *bending moments* M_y and M_z are positive when a tensile stress ($\sigma_{xx} > 0$) on a small elemental area ΔA for $y > 0$ makes a positive contribution to M_y or for $z > 0$ makes a positive contribution to M_z.

Figure 10.15 shows the positive directions of N, M_y and M_z. These are the section forces that are transferred via *normal stresses* in the member cross-section.

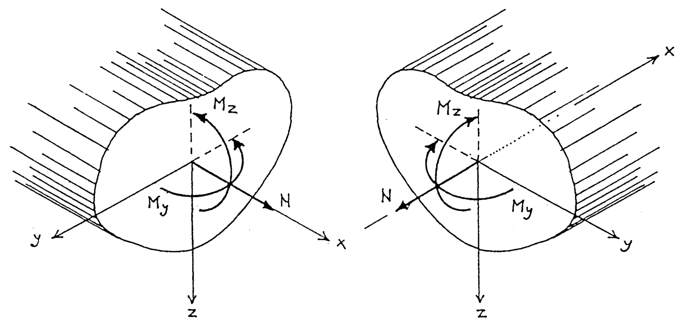


Figure 10.15 The positive directions of the section forces that are transferred via normal stresses. N is the normal force, M_y is the bending moment in the xy plane and M_z is the bending moment in the xz plane.

The resultant of the *shear stresses* on a small area ΔA around a point P is a small shear force ΔV, with components ΔV_y and ΔV_z:

$$\Delta V_y = \sigma_{xy} \Delta A,$$
$$\Delta V_z = \sigma_{xz} \Delta A.$$

When assuming these small forces act in the member axis (by shifting them to the origin of the yz coordinate system), we have to add a small moment ΔM_t *in the cross-sectional plane* (see Figure 10.16):

$$\Delta M_t = y \cdot \Delta V_z - z \cdot \Delta V_y = (y\sigma_{xz} - z\sigma_{xy})\Delta A.$$

Summation of the contributions of all the forces ΔV_y and ΔV_z for the entire cross-section results in

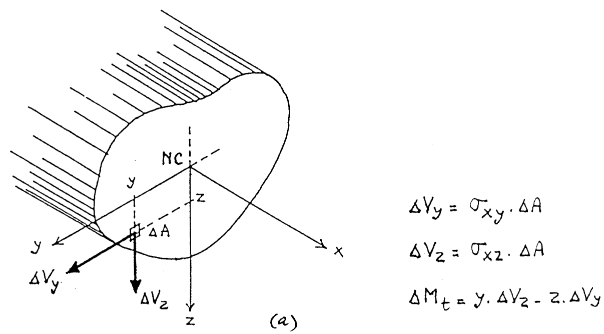


Figure 10.16 (a) The resultant of the shear stresses on a small area ΔA around a point P is a small shear force, with components ΔV_y and ΔV_z.

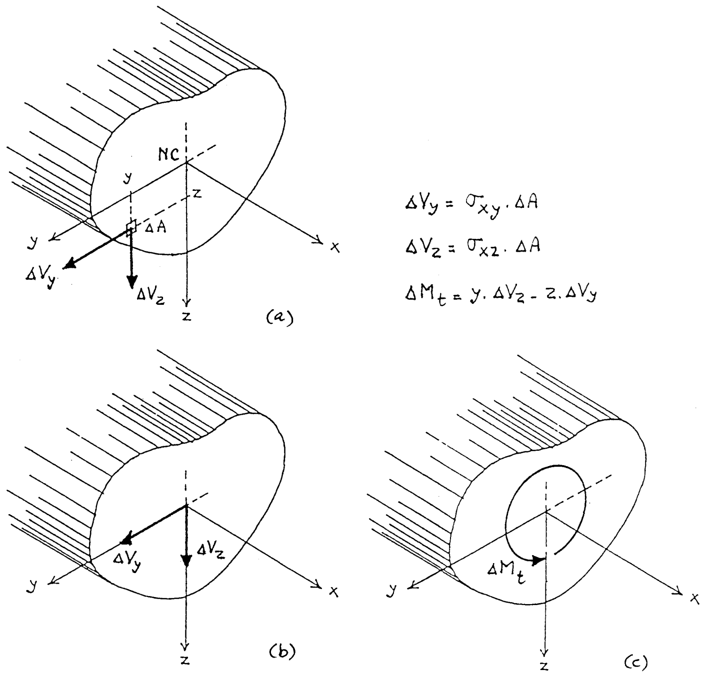


Figure 10.16 (a) The resultant of the shear stresses on a small area ΔA around a point P is a small shear force, with components ΔV_y and ΔV_z. The forces ΔV_y and ΔV_z in P are statically equivalent to (b) the small forces ΔV_y and ΔV_z in the normal force centre NC (the intersection of the member axis with the cross-sectional plane), together with (c) a small moment ΔM_t in the cross-sectional plane.

$$V_y = \int_A \sigma_{xy}\,\mathrm{d}A,$$

$$V_z = \int_A \sigma_{xz}\,\mathrm{d}A,$$

$$M_\text{t} = \int_A (y\sigma_{xz} - z\sigma_{xy})\,\mathrm{d}A.$$

- V_y and V_z are the components of the *shear force* V, that are the resultant forces (or rather: pair of forces) due to the shear stresses in the cross-section.
- M_t is a moment (or rather: pair of moments) that acts in the cross-sectional plane (the yz plane). M_t is known as a *torsional moment*.

The components V_y and V_z of the *shear force* V are (in accordance with the sign convention for the shear stresses σ_{xy} and σ_{xz}) positive when they act on a positive plane in the positive coordinate direction and on a negative plane in the negative coordinate direction.

The *torsional moment* M_t is positive when the couple acts on the positive sectional plane in the positive sense of rotation about the x axis and when the couple acts on the negative plane in the negative direction of rotation.

Figure 10.17 shows the positive directions of V_y, V_z and M_t. These are the section forces that are transferred in the cross-section via *shear stresses*.

Note: The expression given for the torsional moment is not universally applicable. Sometimes, to determine the torsional moment, we do not move the lines of action of the shear forces V_y and V_z to the *normal centre* NC (or the member axis, where we selected the origin of the yz coordinate system), but to another point in the cross-section that we refer to as the *shear force* *centre*, or *shear centre*, SC.[1] With (y_{SC}, z_{SC}) as the coordinates of the shear force centre, the expression for the torsional moment in that case is

$$M_t = \int_A [(y - y_{SC})\sigma_{xz} - (z - z_{SC})\sigma_{xy}]\,dA.$$

The expression given earlier,

$$M_t = \int_A (y\sigma_{xz} - z\sigma_{xy})\,dA,$$

applies only when $y_{SC} = 0$; $z_{SC} = 0$, or in other words, when the shear centre SC coincides with the normal centre NC. This occurs for cross-sections that have *rotational symmetry*.

A cross-section is said to have rotational symmetry when we rotate the cross-section n times $(n > 1)$ with an angle of $\alpha = 360°/n$ about the member axis, and the rotated cross-section coincides with the original, un-rotated cross-section.

Figure 10.18 gives a number of examples of cross-sections with rotational symmetry; the angle of rotation α is given for each of the cross-sectional shapes.

10.1.4 Summary of the sign conventions for stresses and section forces

We use a (*local*) *coordinate system* with the x axis along the member axis.

A *cross-section* is straight and normal to the member axis. The location of a cross-section is determined by the x coordinate.

[1] Volume 2, *Stresses, Deformations, Displacements*, addresses the location of the shear force centre SC in more detail.

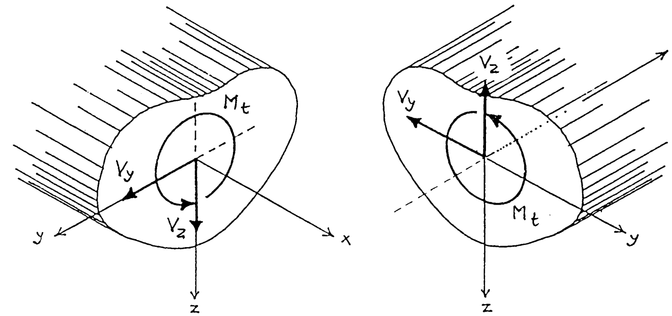


Figure 10.17 The positive directions of the section forces that are transferred via shear stresses. V_y and V_z are the components of the shear force V in respectively the y and z direction. M_t is the torsional moment and acts in the plane of the cross-section.

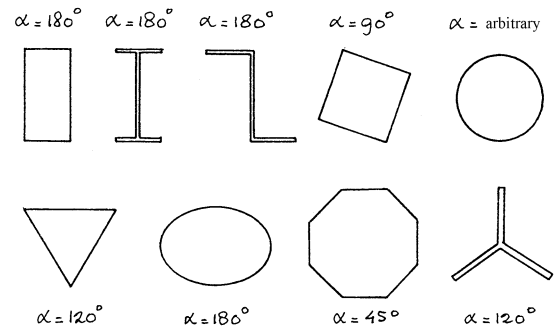


Figure 10.18 Examples of cross-sections with rotational symmetry. The angle of rotation α is mentioned for each cross-sectional shape.

The *unit normal vector* is a unit vector pointed outwards from matter, and normal to the sectional plane that is considered.

A *sectional plane* is
- positive when the unit normal vector is pointing in the positive coordinate direction;
- negative when the unit normal vector is pointing in the negative coordinate direction.

This can also be formulated as follows, without the unit normal vector.

A *sectional plane* is
- positive when the coordinate axis points out of the matter;
- negative when the coordinate axis points into the matter.

A *stress* is
- positive when it acts on a positive plane in the positive coordinate direction or on a negative plane in the negative coordinate direction.
- negative when it acts on a positive plane in the coordinate negative direction or on a negative plane in the positive coordinate direction.

In general, stress σ_{ij} acts
- on a plane with the unit normal vector parallel to the i axis (1st index),
- due to a force component, parallel to the j axis (2nd index).

The stress σ_{ij} is a *normal stress* when the indices are the same ($i = j$) and a *shear stress* when the indices are different ($i \neq j$).

The section forces transferred by *normal stresses* are
- the *normal force* N;
- the *bending moment* M_y, acting in the xy plane;
- the *bending moment* M_z, acting in the xz plane.

The section forces transferred by *shear stresses* are
- the *shear force* V_y in y direction;
- the *shear force* V_z in z direction;
- the *torsional moment* M_t, acting in the yz plane.

The following sign conventions apply for *section forces*:

- A *normal force* N is positive when it acts on a positive cross-sectional plane in the positive x direction. In other words, a normal force N is positive as a tensile force and negative as a compressive force.
- A *shear force* V_y (V_z) is positive when it acts on a positive cross-sectional plane in the positive y direction (z direction) and on a negative cross-sectional plane in the negative y direction (z direction).
- A *bending moment* M_y (M_z) is positive when it causes tensile stresses at the positive y side (z side) of the x axis and compressive stresses at the negative y side (z side).
- A *torsional moment* M_t is positive when the couple on the positive cross-sectional plane acts in accordance with the positive direction of rotation about the x axis and the couple on the negative cross-sectional plane acts in accordance with the negative direction of rotation.

10.2 Diagrams for the normal force, shear force and bending moment

The section forces in a member are in general not constant, but may vary along the member axis. They are then a function of the x coordinate chosen along the member axis. By drawing these functions, we get a graphical representation of the distribution of the section forces. These types of *diagrams* are extremely useful to see at a glance where the section forces change sign (direction) and where they are at largest.

In this section, we cover examples of diagrams for the *normal force*, *shear force* and *bending moment*. Section 10.2.1 covers members subject to concentrated forces and couples, while Sections 10.2.2 and 10.2.3 look at members subject to a uniformly distributed load.

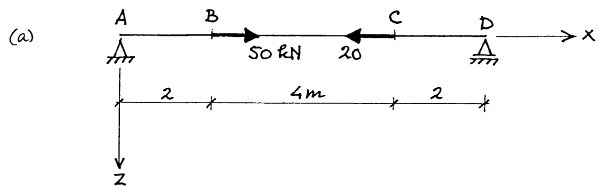


Figure 10.19 (a) A simply supported member loaded by two forces of which the lines of action coincide with the member axis.

10.2.1 Members subject to concentrated forces/couples

We look at three examples:

1. a simply supported member with forces in the direction of the member axis;
2. a simply supported member with forces normal to the member axis;
3. a simply supported member subject to a couple.

Example 1

The simply supported member AD in Figure 10.19a is loaded at B and C by two forces of respectively 50 and 20 kN, of which the lines of action coincide with the member axis.

Question:
Determine the diagrams for the section forces.

Solution:
In Figure 10.19b, the member has been isolated from its supports and the support reactions are shown; the vertical support reactions are zero.

The interaction forces in a section (the section forces) can be determined from the equilibrium of the isolated member segments to the left or to the right of the section. Figure 10.19c shows the member segment to the left of a section located between A and B. In the section, both segments are rigidly joined. The section must therefore be able to transfer a normal force N, shear force V and bending moment M. In Figure 10.19c, the unknown section forces are shown with their positive directions in the given xz axis system.[1]

Actually, the shear force and the bending moment in this xz coordinate system should be shown as respectively V_z and M_z. In obvious situations, the indices are generally omitted to simplify the writing.

---

[1] Remember that you should always include quantities shown as symbols to which a sign is linked positively in the calculation.

From the force and moment equilibrium of the segment to the left of the section it follows that

$$\sum F_x = -(30\ \text{kN}) + N = 0 \Rightarrow N = +30\ \text{kN},$$

$$\sum F_z = 0 \Rightarrow V = 0,$$

$$\sum T_y|\text{section} = 0 \Rightarrow M = 0.$$

The normal force N is a tensile force of 30 kN, while the shear force V and the bending moment M are zero. These values are independent of the location of the section between A and B and therefore apply for $(0\ \text{m}) \leq x < (2\ \text{m})$.

The shear force and bending moment are not only zero in AB, but also in the rest of the member. This follows from equilibrium of each member segment to the left or right of a (arbitrarily chosen) section. For this reason, we will look only at the distribution of the normal force.

Figure 10.19d shows the isolated member segment to the left of a section between B and C. The equation for the force equilibrium in the x direction now also includes the force of 50 kN at B:

$$\sum F_x = -(30\ \text{kN}) + (50\ \text{kN}) + N = 0 \Rightarrow N = -20\ \text{kN}.$$

This result, a compressive force of 20 kN, is independent of the location of the section between B and C and therefore applies for $(2\ \text{m}) < x \leq (6\ \text{m})$.

Of course, instead of the equilibrium for the part to the left of the section, we can also determine the equilibrium for the part to the right of the section (see Figure 10.19e):

$$\sum F_x = -N - (20\ \text{kN}) = 0 \Rightarrow N = -20\ \text{kN}.$$

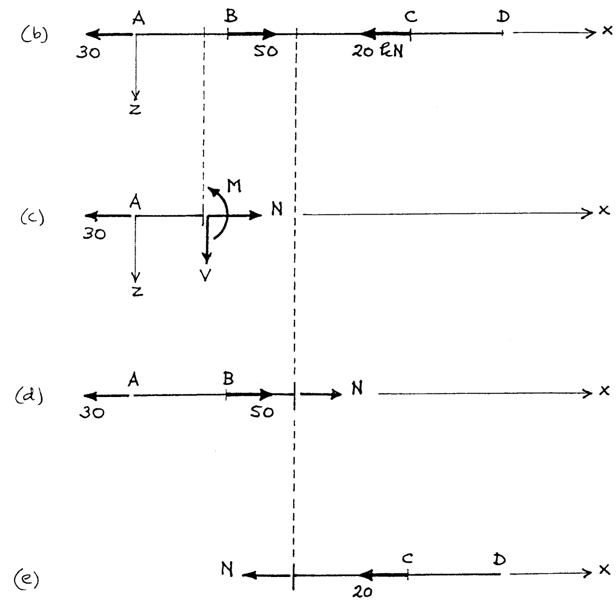


Figure 10.19 (b) The isolated member with its support reactions. (c) The isolated part of the member to the left of a section between A and B. The section can transfer a normal force N, shear force V and bending moment M. The unknown section forces are shown in accordance with their positive directions in the coordinate system. (d) The isolated part of the member to the left of a section between B and C. In the section, only the unknown normal force N is shown as it was determined earlier that the shear force V and the bending moment M are zero throughout the member. (e) The isolated part of the member to the right of a section between B and C, with the unknown normal force N.

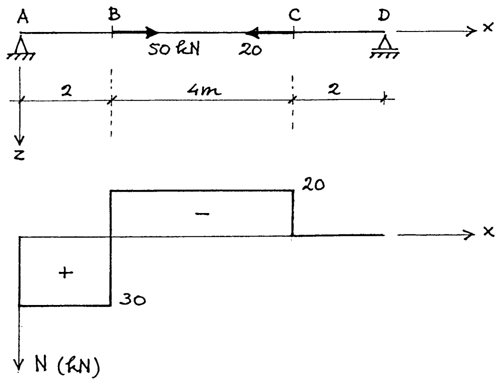


Figure 10.20 The normal force diagram (N diagram) for the simply supported member loaded by two forces of which the lines of action coincide with the member axis. Step changes occur in the N diagram at the location of the point loads at B and C.

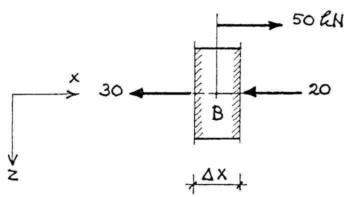


Figure 10.21 A step change in the N diagram can be found from the equilibrium of a small member segment. In this way, the normal forces on both sectional planes of a small member segment are in equilibrium with the load of 50 kN (shown eccentrically for the sake of clarity).

Note that the positive direction of N on a cross-sectional plane is by definition always that of a tensile force.

For a section between C and D, the equilibrium of the part to the right of the section gives

$$N = 0.$$

To summarise, for normal force N applies:

$$\begin{aligned} N &= +30\text{ kN} \quad \text{for} \quad (0\text{ m}) \le x < (2\text{ m}), \\ N &= -20\text{ kN} \quad \text{for} \quad (2\text{ m}) < x < (6\text{ m}), \\ N &= 0 \qquad\quad\ \ \text{for} \quad (6\text{ m}) < x \le (8\text{ m}). \end{aligned}$$

Figure 10.20 shows the distribution of the normal force N graphically in a diagram. This is called the *normal force diagram*, or *N diagram*. Positive values of N (tensile forces) are plotted at the positive side of the z axis and negative values (compressive forces) are plotted at the negative side of the z axis. We usually place the sign of N ("+" for tension and "−" for compression) *within the diagram* and write down the relevant values *without a sign*.

At $x = 2$ m and $x = 6$ m there is a *step change* in the N diagram equal to the forces acting there. In these sections, the value of N is undetermined. This is a result of modelling the load into concentrated forces (acting in a particular point).

The *step change in the normal force diagram* can be found from the equilibrium of a small member segment with length Δx ($\Delta x \to 0$), at the point load. Figure 10.21 shows "*joint*" B between the member segments AB and

BC. From the *N diagram* we can read off that there is a tensile force of 30 kN directly to the left of B and a compressive force of 20 kN directly to the right of B. Both forces are in equilibrium with the 50 kN load (which is shown eccentrically for clarity).

Example 2

The simply supported member AD in Figure 10.22a is loaded at B normal to the member axis by a force of 60 kN.

Question:
Determine the distribution of the section forces.

Solution:
The units used are m and kN. To simplify the writing, the units have been omitted from the calculation.

In Figure 10.22b, the member has been isolated and the support reactions are shown; the horizontal support reaction at A is zero.

Figure 10.22c shows the member segment to the left of a section between A and B, with all the forces acting on it. The section forces N (normal force), V (shear force) and M (bending moment) follow from the equilibrium.[1]

For a length x of the isolated member segment it holds that

$$\sum F_x = 0 \Rightarrow N = 0,$$

$$\sum F_z = -45 + V = 0 \Rightarrow V = +45 \text{ kN},$$

$$\sum T_y|\text{section} = -45x + M = 0 \Rightarrow M = 45x \text{ kNm}.$$

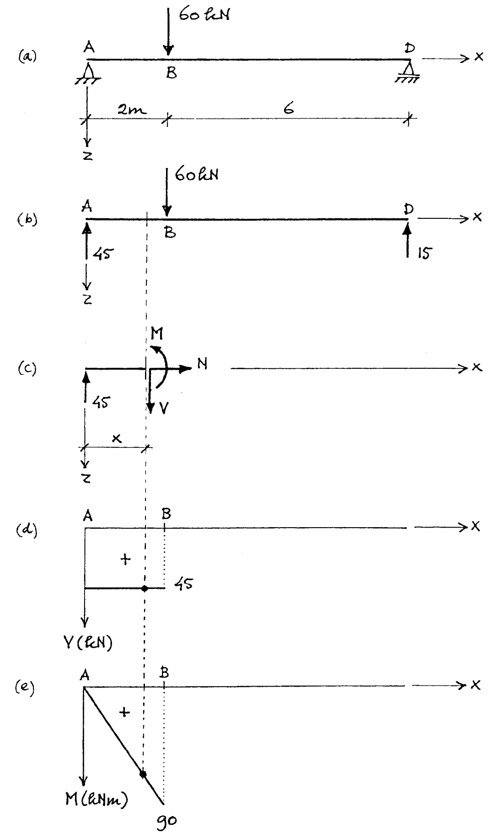


Figure 10.22 (a) A simply supported member that is loaded in B normal to the member axis by a force of 60 kN. (b) The isolated member with its support reactions. (c) The isolated part of the member to the left of a section between A and B. The section can transfer a normal force N, shear force V and a bending moment M. The unknown section forces are shown in accordance with their positive directions in the coordinate system. (d) The shear force diagram (V diagram) for AB. (e) The moment diagram (M diagram) for AB.

[1] The sub-index z is again omitted from the symbols for shear force (V_z) and bending moment (M_z); see also Example 1.

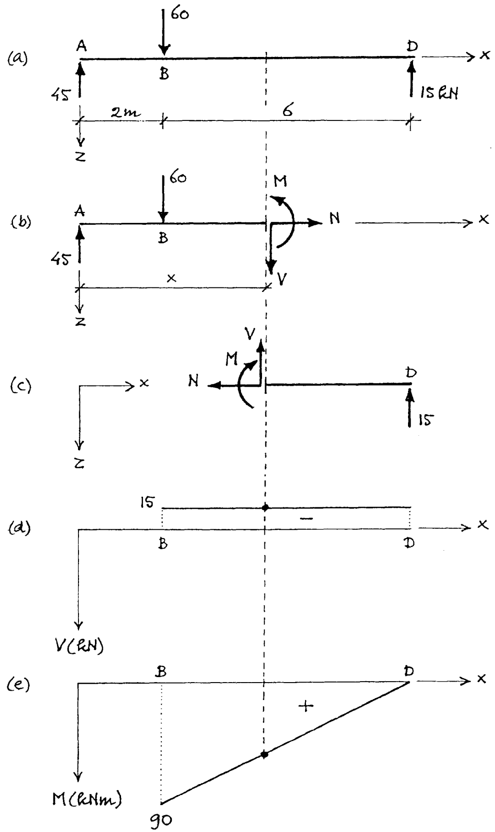


Figure 10.23 (a) The isolated member with its support reactions. (b) The isolated part of the member to the left of a section between B and D. The section can transfer a normal force N, shear force V and a bending moment M. The unknown shear forces are shown in accordance with their positive directions in the coordinate system. (c) The isolated part of the member to the right of a section between B and D. (d) The shear force diagram (V diagram) for BD. (e) The moment diagram (M diagram) for BD.

The normal force N is not only zero in AB, but also in the rest of the member. This follows from the force equilibrium in the x direction of each member segment to the left or right of a (arbitrarily chosen) section. We will therefore only look at the shear force and the bending moment.

The shear force is constant between A and B: $V = +45$ kN. The bending moment M varies linearly, from 0 at A ($x = 0$ m) to $+90$ kNm at B ($x = 2$ m).

Figures 10.22d and 10.22e show the variation for AB of the shear force and the bending moment with a so-called *shear force diagram* (V *diagram*), respectively a *bending moment diagram* (M *diagram*).

Positive values of V and M are plotted at the positive side of the z axis, and negative values are plotted at the negative side. The sign is shown *within the diagram*; relevant values are written down *without a sign*.

In Figure 10.23b, the member segment to the left of a section located between B and D has been isolated. The equations for the force equilibrium in the z direction and the moment equilibrium now includes the load of 60 kN:

$$\sum F_z = -45 + 60 + V = 0 \rightarrow V = -15 \text{ kN},$$

$$\sum T_y|\text{section} = -45x + 60 \times (x - 2) + M = 0$$

$$\Rightarrow M = (-15x + 120) \text{ kNm}.$$

These values can also be found from the equilibrium of the member segment to the right of the section, as shown in Figure 10.23c. It should be noted that the sectional plane in the coordinate system shown is negative, and that the positive directions of N, V and M are therefore opposite to those on a positive sectional plane.

$$\sum F_z = -V - 15 = 0 \Rightarrow V = -15 \text{ kN},$$

$$\sum T_y|\text{section} = 15 \times (8 - x) - M = 0 \Rightarrow M = (-15x + 120) \text{ kNm}.$$

The shear force between B and D is constant: $V = -15$ kN. The bending moment M decreases linearly, from $+90$ kNm at B ($x = 2$ m) to 0 at D ($x = 8$ m).

Figures 10.23d and 10.23e show the distribution for BD of respectively the shear force V and the bending moment M.

The *shear force diagram* (V diagram) and *bending moment diagram* (M diagram) for the entire member AD are shown in Figure 10.24.

At B, the point of application of the concentrated force of 60 kN, there is an *abrupt change in slope of the bending moment diagram*. Here the bending moment is at its largest.

The shear force in B is undetermined; this is the result of modelling the load as a point load. This finds expression in the shear force diagram as a *step change*: the shear force is $+45$ kN directly to the left of B and -15 kN directly to the right of B. The magnitude of the step change equals the magnitude of the point load at B.

The *step change in the shear force diagram* can be found from the force equilibrium in z direction of a small member segment at B, with length Δx ($\Delta x \to 0$) (see Figure 10.25). The 60 kN point load is kept in equilibrium by both shear forces in the sectional planes (the bending moments are not shown).

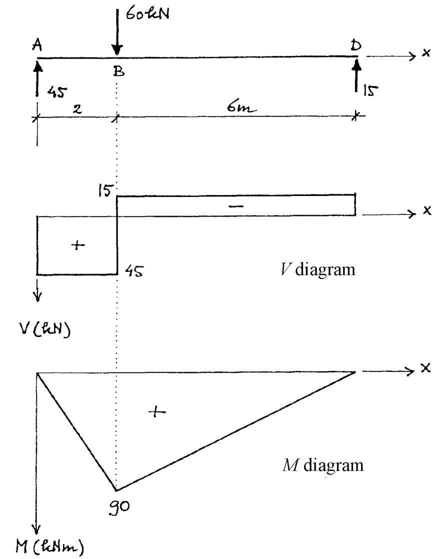


Figure 10.24 A simply supported member that is loaded at B normal to the member axis by a force of 60 kN, with its shear force diagram (V diagram) and bending moment diagram (M diagram). A step change occurs at the location of the point load in the V diagram and an abrupt change in slope occurs in the M diagram.

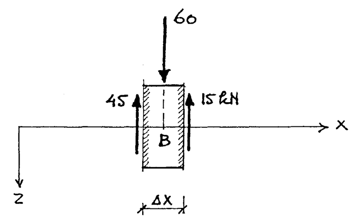


Figure 10.25 A step change in the V diagram can be found from the equilibrium of a small member segment. In this way, the shear forces on both sectional planes at B are in equilibrium with the load of 60 kN.

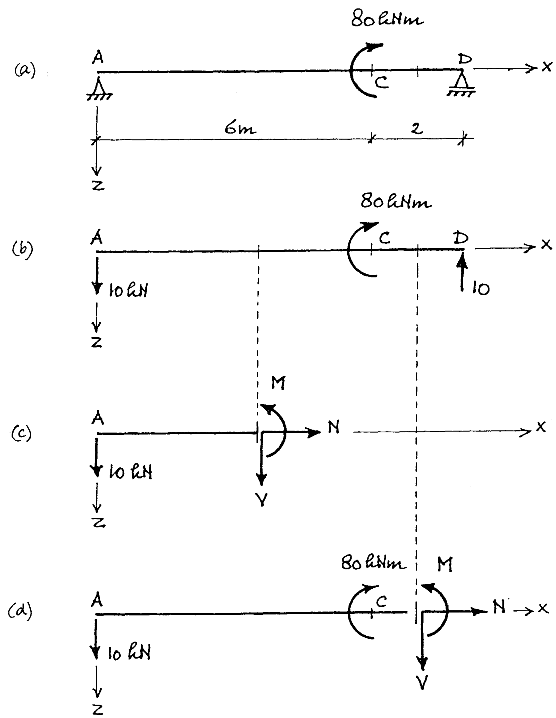


Figure 10.26 (a) A simply supported member AD, which is loaded in C by a couple of 80 kNm. (b) The isolated member with its support reactions. (c) The isolated part of the member to the left of a section between A and C. The section can transfer a normal force N, shear force V and a bending moment M. The unknown shear forces are shown in accordance with their positive directions in the coordinate system. (d) The isolated part of the member to the left of a section between C and D.

Example 3

The simply supported member in Figure 10.26a is loaded at C by a couple of 80 kNm.

Question:
Determine the distribution of the section forces.

Solution:
The units used are m and kN.

In Figure 10.26b, the member has been isolated and the support reactions are shown.

Figure 10.26c shows the isolated member segment to the left of a section between A and C, with all the forces acting on it. From the equilibrium we find:

$$\sum F_x = N = 0 \quad \Rightarrow N = 0,$$

$$\sum F_z = 10 + V = 0 \quad \Rightarrow V = -10\text{ kN},$$

$$\sum T_y|\text{section} = 10x + M = 0 \quad \Rightarrow M = -10x\text{ kNm}.$$

Figure 10.26d shows the member segment to the left of a section between C and D, with all the forces acting on it. The equation for the moment equilibrium now includes the load from the couple of 80 kNm:

$$\sum F_x = N = 0 \quad \Rightarrow N = 0,$$

$$\sum F_z = 10 + V = 0 \quad \Rightarrow V = -10\text{ kN},$$

$$\sum T_y|\text{section} = 10x - 80 + M = 0 \quad \Rightarrow M = (-10x + 80)\text{ kNm}.$$

Figures 10.26e and 10.26f show the *shear force diagram* (V diagram) and the *bending moment diagram* (M diagram) for member AD. Since the normal force is zero everywhere, the normal force diagram has been omitted. The shear force is constant across the entire length of the member: $V = -10$ kN. The bending moment varies linearly, from 0 at A ($x = 0$ m) to -60 kNm directly to the left of C ($x = 6$ m) and from $+20$ kNm directly to the right of C ($x = 6$ m) to 0 at D ($x = 8$ m).

In C, where the couple acts, the bending moment is undetermined. This is a result of modelling the load as a couple that is concentrated in a single point. This finds expression in the bending moment diagram as a *step change* equal to the magnitude of the couple.

The *step change in the bending moment diagram* can be found from the moment equilibrium of a small member segment at C, with a length of Δx ($\Delta x \to 0$) (see Figure 10.27; the shear forces are not shown). The bending moments on both sectional planes are in equilibrium with the couple of 80 kNm.

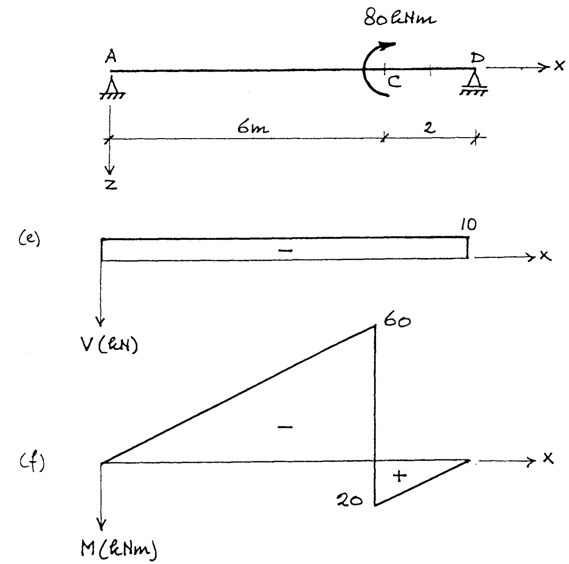


Figure 10.26 (e) The shear force diagram (V diagram) for AD. (f) The bending moment diagram (M diagram) for AD. The bending moment varies linearly and makes a step change at C, the point of application of the couple. The magnitude of the step change is equal to the magnitude of the couple.

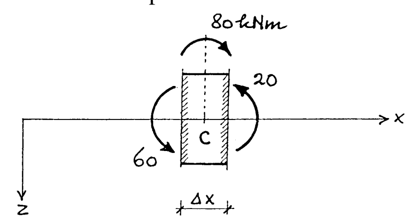


Figure 10.27 A step change in the M diagram can be found from the equilibrium of a small member segment. In this way, the bending moments on both sectional planes of a small member segment are in equilibrium with the couple of 80 kNm.

10.2.2 Members with a uniformly distributed load in the direction of the member axis

In straight members a (distributed) longitudinal load does not produce bending moments or shear forces. In these cases, there are only normal forces. The variation of the normal force is elaborated for two examples:

1. a column subject to its dead weight;
2. a simply supported member subject to a uniformly distributed axial load over three-quarters of its length.

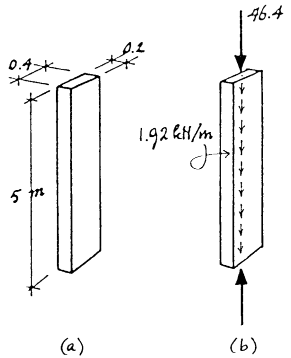


Figure 10.28 (a) Dimensions of a concrete column of the building used in Section 6.4. (b) The column is loaded on top by a force of 46.4 kN. The dead weight is to be considered as a uniformly distributed (line)load along the member axis of 1.92 kN/m.

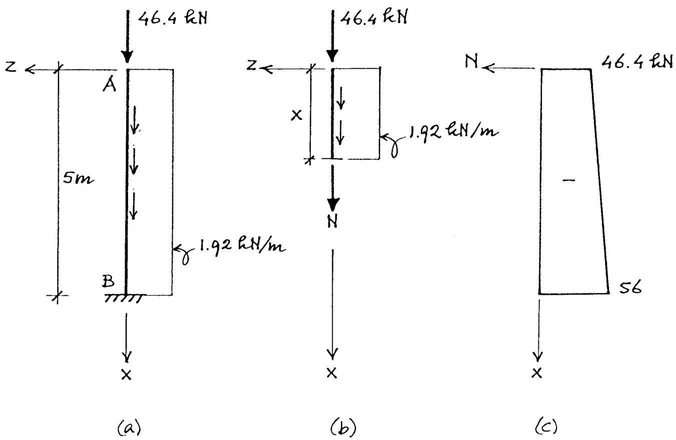


Figure 10.29 (a) The model for the column and the load. (b) To determine the normal force N we look at the equilibrium of the part above the section. (c) The normal force diagram.

Example 1

For the example in Figure 10.28, we will use the concrete column from the building that we looked at in Section 6.4.

Question:
Determine the N diagram.

Solution:
The column is loaded on top by a force of 46.4 kN (see Figure 10.28b). The dead weight is a uniformly distributed (line) load along the member axis. With a specific weight of concrete of 24 kN/m^3, and the cross-sectional dimensions given in Figure 10.28a, the dead weight is

$$(0.4 \text{ m})(0.2\text{m})(24\text{kN/m}^3) = 1.2\text{kN/m}.$$

The model for the column and load is shown in Figure 10.29a.

In Figure 10.29b, a segment with length x has been isolated at the top of the column. In the section, the as yet unknown normal force N is shown according to its positive direction (that of a tensile force). For this segment, the equation for the force equilibrium in the x direction is

$$\sum F_x = (46.4 \text{ kN}) + (1.92 \text{ kN/m})(x \text{ m}) + N = 0$$

from which it follows that (x expressed in m)

$$N = (-46.4 - 1.92x) \text{ kN}.$$

In Figure 10.29c, the normal force N is shown as a function of x. The normal force is a compressive force everywhere and varies linearly, from 46.4 kN at A ($x = 0$ m) to 56 kN at B ($x = 5$ m).

As expected, the compressive force increases downwards due to the column's dead weight. The compressive force of 56 kN at B is in conformity with the previously determined support reaction (in Section 6.3) from the equilibrium of the column as a whole.

Example 2
In Figure 10.30a, the simply supported member AC is subject to a uniformly distributed axial load q along segment BC.

Question:
Determine the normal force distribution.

Solution:
There is only one support reaction not equal to zero, namely the horizontal support reaction at A. Figure 10.30b shows the isolated member, with all the forces acting on it.

The variation of the normal force can be determined from the force equilibrium in the x direction for the member segment to the left of a section at a distance x from A. Here, we have to distinguish between two parts, or *fields*:

- AB ($0 < x < a$),
- BC ($a < x < 3a$).

For $0 < x < a$ (the section is within AB) the equation for the force equilibrium of the left-hand member segment is (see Figure 10.30c)

$$\sum F_x = -2qa + N = 0$$

from which it follows that

$$N = 2qa.$$

The normal force in field AB is a constant tensile force.

For $a < x < 3a$ (the section is within BC) the equilibrium equation for the left-hand member segment is (see Figure 10.30d)

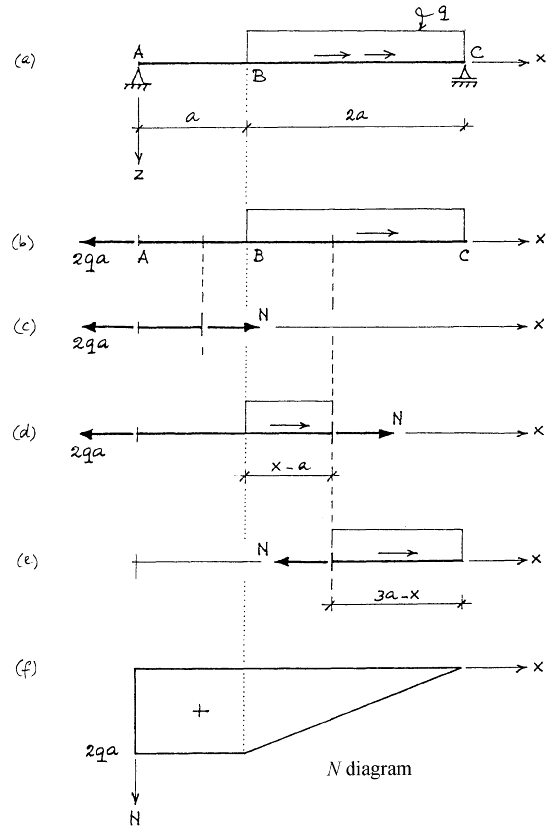


Figure 10.30 (a) A simply supported member AC, with a uniformly distributed load q over part BC along the member axis. (b) The isolated member with its support reactions. (c) The isolated part to the left of a section between A and B. (d) The isolated part to the left of a section between B and C. (e) The isolated part to the right of a section between B and C. (f) The normal force diagram. It has an abrupt change in slope at the joining of fields AB and BC.

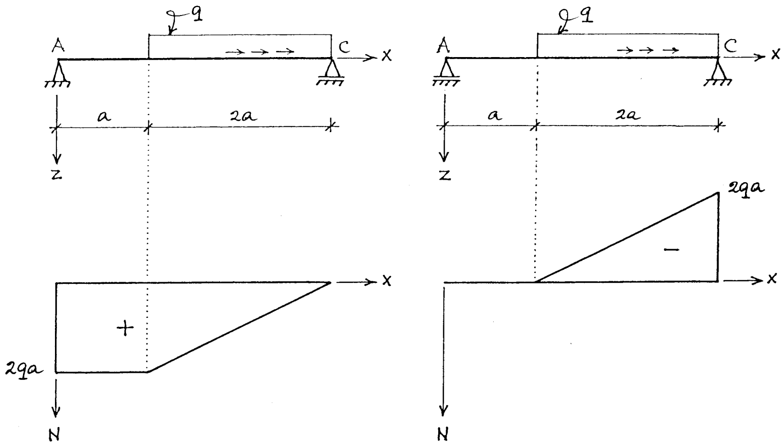


Figure 10.31 The simply supported member AC is subject to a uniformly distributed axial load q along part BC. If we switch the hinged support and roller support, the N diagram changes.

$$\sum F_x = -2qa + q(x-a) + N = 0.$$

Here $q(x-a)$ is the resultant of the distributed load on the isolated left-hand segment. This leads to

$$N = q(-x+3a).$$

Of course we find the same result if we look at the segment to the right of the section (see Figure 10.30e).

In field BC the normal force is a tensile force that decreases linearly, from $2qa$ at $x = a$ to zero at $x = 3a$.

Figure 10.30f shows the entire *normal force diagram*. This gives a bend (an abrupt change of slope) at the joining of the fields AB and BC.

It should be noted that the normal force variation changes if you swap the hinged and roller support at A and C (see Figure 10.31). It is up to you to check this.

10.2.3 Members with a uniformly distributed load normal to the member axis

This section looks at two examples:

1. a simply supported member;
2. a member fixed at one side and free at the other.

Example 1
The simply supported beam AB in Figure 10.32a carries a uniformly distributed load q over its entire length ℓ.

Question:
Determine the distribution of all the section forces.

Solution:
In Figure 10.32b, the beam has been isolated and the support reactions are shown. To determine the variation of the section forces, we will look at the equilibrium of the part to the left of the section (see Figure 10.32c):

$$\sum F_x = N = 0,$$

$$\sum F_z = -\tfrac{1}{2}q\ell + qx + V = 0,$$

$$\sum T_y|\text{section} = -\tfrac{1}{2}q\ell \cdot x + qx \cdot \tfrac{1}{2}x + M = 0$$

so that

$$N = 0$$

$$V = -qx + \tfrac{1}{2}q\ell, \qquad \text{(a)}$$

$$M = -\tfrac{1}{2}qx^2 + \tfrac{1}{2}q\ell x = \tfrac{1}{2}qx(\ell - x). \qquad \text{(b)}$$

The normal force is zero everywhere, and therefore not interesting.

The shear force varies linearly from $+\frac{1}{2}q\ell$ in A ($x = 0$) to $-\frac{1}{2}q\ell$ in B ($x = \ell$). The *shear force diagram* is shown in Figure 10.32d.

The bending moment varies quadratically in x and is positive everywhere. The bending moment diagram is shown in Figure 10.32e and is shaped like a (second degree) *parabola*. In A and B, the tangents of the parabola are also shown; both tangents intersect at the middle of AB.[1]

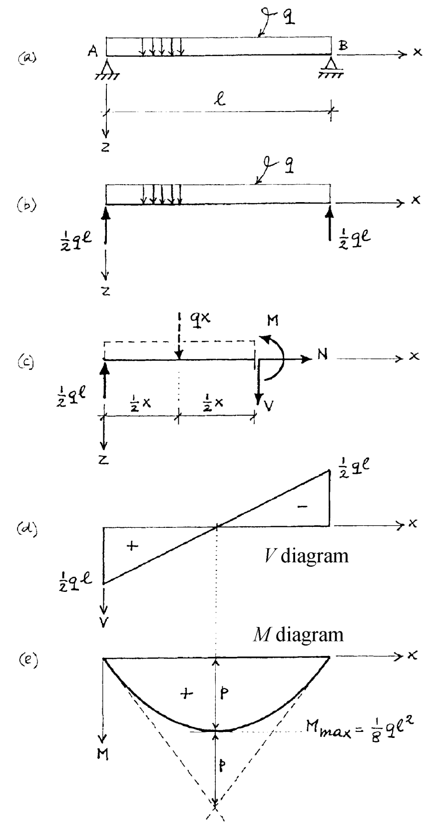


Figure 10.32 (a) A simply supported beam AB is bearing a uniformly distributed load q over its entire length ℓ. (b) The isolated member with its support reactions. (c) The isolated part of the member to the left of a section. The section can transfer a normal force N, shear force V and a bending moment M. The unknown section forces are shown in accordance with their positive directions in the coordinate system. (d) The shear force diagram for AB. (e) The bending moment diagram for AB, with the tangents in A and B.

[1] It is assumed that the reader is familiar with plotting graphical functions, drawing tangents, and calculating extreme values.

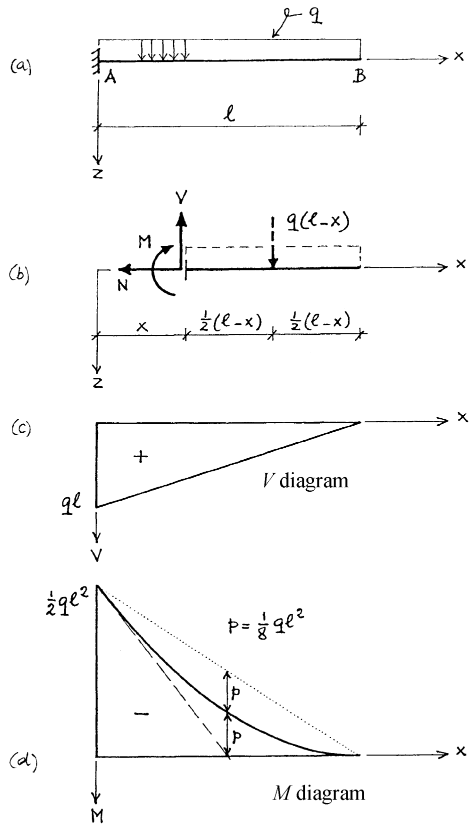


Figure 10.33 (a) A member AB fixed at A carries a uniformly distributed load q over its entire length ℓ. (b) To determine the section forces, we look at the equilibrium of the part to the right of the section. (c) The shear force diagram for AB. (d) The bending moment diagram for AB, with the tangents at A and B.

The bending moment is an *extreme* (*maximum* or *minimum*) when the derivative of M with respect to x is zero:

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -qx + \tfrac{1}{2}q\ell = 0 \Rightarrow x = \tfrac{1}{2}\ell \text{ and } M_{\max} = \tfrac{1}{8}q\ell^2.$$

If we differentiate expression (b) for the bending moment with respect to x we find the expression (a) for the shear force. The derivative of the bending moment M is therefore equal to the shear force V:

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V.$$

Consequently: *the gradient of the moment diagram is equal to the shear force*. In Chapter 11 we will demonstrate that this property is generally applicable. It is up to you to check the property for Examples 2 and 3 in Section 10.2.1.

Example 2

In Figure 10.33a, the member AB is fixed at A and carries a uniformly distributed load q over its entire length ℓ.

Question:
Determine the variation of the section forces.

Solution:
To determine the section forces, we will look at the equilibrium of the segment to the right of the section. In this case, it is not necessary to previously determine the support reactions at A (see Figure 10.33b):

$$\sum F_x = N = 0 \Rightarrow N = 0,$$

$$\sum F_z = -V + q(\ell - x) = 0 \Rightarrow V = q(\ell - x),$$

$$\sum T_y|\text{section} = -M - q(\ell - x)\cdot\tfrac{1}{2}(\ell - x) = 0 \Rightarrow M = -\tfrac{1}{2}q(\ell - x)^2.$$

The normal force is zero everywhere, and therefore not of interest.

The shear force is positive everywhere and varies linearly, from $q\ell$ at A ($x = 0$) to zero at B ($x = \ell$). The *shear force diagram* is shown in Figure 10.33c.

The bending moment is quadratic in x and negative everywhere. It varies from $-\frac{1}{2}q\ell^2$ at the fixed end A ($x = 0$) to zero at the free end B ($x = \ell$).

Figure 10.33d shows the *bending moment diagram*: a parabola with its apex at B. The tangents at A and B are also shown. The tangent at B is horizontal. Both tangents intersect at the middle of AB. The values p are equal to $\frac{1}{8}q\ell^2$.

The *maximum*[1] bending moment occurs at the fixed support in A:

$$|M|_{\max} = \tfrac{1}{2}q\ell^2.$$

Note that at A: $\mathrm{d}M/\mathrm{d}x = V \neq 0$; here it concerns a maximum at a field boundary.

The support reactions at A can be derived according to magnitude and direction from the shear force diagram and the bending moment diagram:

$$\begin{aligned} V_{\mathrm{A}} &= +q\ell, \\ M_{\mathrm{A}} &= -\tfrac{1}{2}q\ell^2. \end{aligned}$$

Since the support reactions at A act on a negative sectional plane, they have the directions shown in Figure 10.34. Whether this is correct can easily be checked by looking at the equilibrium of the structure as a whole.

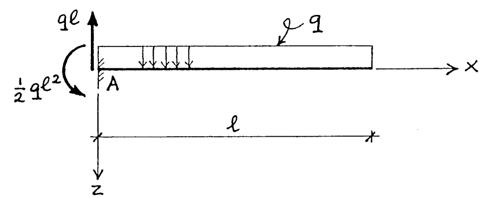


Figure 10.34 The magnitude and direction of the support reactions in A can be derived from the shear force diagram and the bending moment diagram.

[1] With "*maximum*" we often mean "the largest value in an *absolute sense*"; we call this the global maximum.

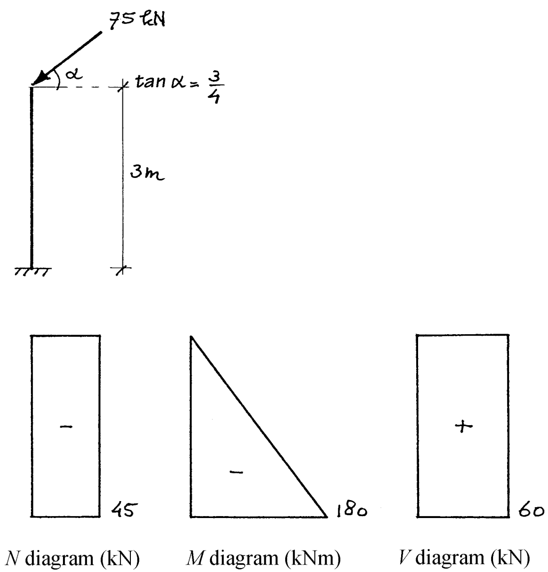


Figure 10.35 A fixed member, loaded at its free end by a force of 75 kN, with its N diagram, M diagram and V diagram. The magnitude of the section forces follows directly from these diagrams. The direction is determined by the sign. To do so, we have to know the coordinate system in which we are working. Since that is not given, the signs in the M and V diagrams have here lost their meaning.

10.3 Deformation symbols for shear forces and bending moments

Figure 10.35 shows a fixed member that is loaded at its free end by a force of 75 kN. The same figure also shows the N diagram, M diagram and the V diagram. The magnitude of the section forces can be read directly from these diagrams. The direction is determined by the plus or minus sign.

The normal force N can be read directly from the N diagram without coordinate system. The direction of the normal force follows directly from the plus or minus sign. The N diagram shows that the normal force is negative and therefore a compressive force.

Other than for the normal force N, we have to know the coordinate system in which we are working to interpret the signs in the M and V *diagrams*. In Figure 10.35, without the coordinate system, the signs in the M and V *diagrams* have lost their meaning.

Assume we were working in a xz coordinate system with, of course, the x axis along the member axis. In order to determine the direction of the bending moment M from the sign, we have to know the direction of the z axis.[1] In order to determine direction of the shear force V from the sign, we have to know the direction of the z axis and also the x axis.[2]

---

1 A bending moment is positive if it causes tension at the positive z side of the member axis and compression at the negative z side. The direction of the x axis is not important here.

2 A shear force is positive if it acts in the positive z direction on a positive sectional plane, and in the negative direction on a negative sectional plane. Now you also have to know the direction of the x axis to determine whether a sectional plane is positive or negative.

The signs in the M and V diagrams are in accordance only with the correct directions for the bending moment M and the shear force V for the coordinate system given in Figure 10.36.

To interpret the signs in the M and V diagrams correctly, we therefore have to know the coordinate system. So far, in all the examples showing the M and V *diagrams*, the structure has consisted of a single straight member, and the coordinate system was always shown. When you have to deal with bent members or structures consisting of several members, you have to introduce a local coordinate system along each straight member segment if you want to indicate the directions of M and V using the plus and minus signs. For the simple structure in Figure 10.37, this already leads to three local coordinate systems: one for AB, one for BC and one for CD.

This soon becomes cumbersome and cluttered. In manual calculations, we will therefore use *deformation symbols*: the *bending symbol* for bending moments, and the *shear symbol* for shear forces.

The bending symbol and shear symbol symbolise the deformation of the member axis due to a bending moment and a shear force respectively. These deformation symbols can be used to set the direction of the section forces unequivocally, regardless of a coordinate system.

We always use the plus and minus sign for normal forces.

The bending symbol and shear symbol will be explained in more detail below.

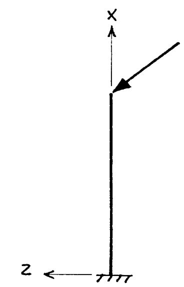


Figure 10.36 The fixed member with the coordinate system used.

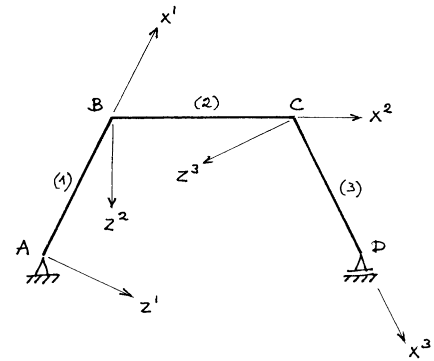


Figure 10.37 When you have to deal with bent members or structures consisting of several members, you have to introduce a local coordinate system along each straight member segment if you want to indicate the directions of M and V using the plus and minus signs. This soon becomes cumbersome and cluttered in manual calculations.

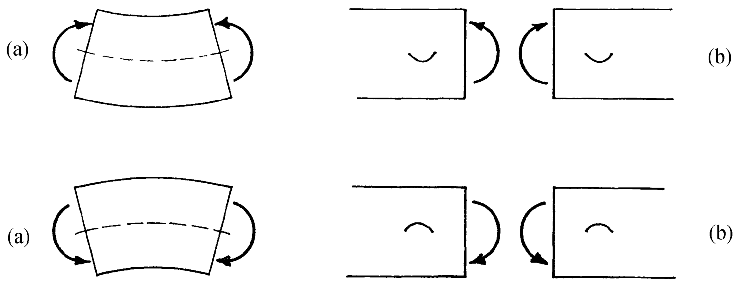


Figure 10.38 Bending symbols.

Figure 10.38 (a) A small member segment subject to bending moments will lengthen at the tension side and shorten at the compression side. The member segment will bend. (b) The small arc that represents the deformation is used as deformation symbol for the bending moment and is known as the bending symbol.

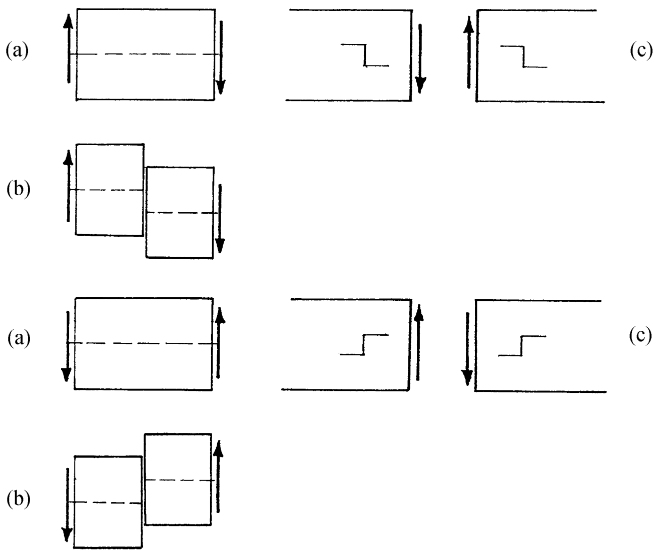


Figure 10.39 Shear symbols.

Figure 10.39 (a) In a member segment subject to shear forces, one sectional plane will try to shift with respect to the other. (b) This effect can be visualised by introducing an (imaginary) slide joint in the segment, so that both sectional planes can move with respect to one another. (c) The step formed by the moved member axes is used as the deformation symbol for shear forces and is known as the shear symbol.

- *Bending symbol* (deformation symbol for *bending moments*)
  Figure 10.38a shows a small member segment subject to bending moments. The member segment will lengthen at the side being pulled, and shorten at the side being compressed. The member segment will bend. Since it is possible to determine the bending moment from the bent shape of the member axis, we use the small arc as deformation symbol for the bending moment (see Figure 10.38b).
- *Shear symbol* (deformation symbol for *shear forces*)
  Figure 10.39a again shows a small member segment, but now with shear forces. When subject to shear forces, one sectional plane will try to shift with respect to the other. This effect can be visualised by applying an (imaginary) slide joint within the segment, so that both parts can move with respect to one another (see Figure 10.39b). The step change formed by the moved member axes is used as the deformation symbol for shear forces (see Figure 10.39c).

Figure 10.40 shows the V diagram and the M diagram with the deformation symbols for a simply supported beam, loaded by a single point load.

Since there is no coordinate system, we are in principle free to choose on which side of the member axis we plot the bending moment and the shear force, as long as we use the correct deformation symbols. See the shear force diagrams in Figures 10.40b and 10.40c, which are both correct.

We make an exception for the bending moment. It is agreed that the M values are plotted at the side where the bending moment causes tension, so at the convex side of the member axis (see Figure 10.40d). The open side of the deformation symbol is therefore always faced towards the member axis.[1]

The advantage of this is that the gradient of the M diagram ($\Delta M/\Delta x$), shown in Figure 10.40d as a "step", now corresponds directly with the shear symbol for the shear force. This allows us to easily and directly check the relationship between the moment diagram and the shear force diagram.

In Figure 10.40e, the M diagram has been plotted at the wrong side of the member axis. Although you will come across this often in books, we strongly recommend that you do not draw the moment diagram in this way, as the relationship with the deformation symbol for the shear force (the "step" in the M diagram) is lost.

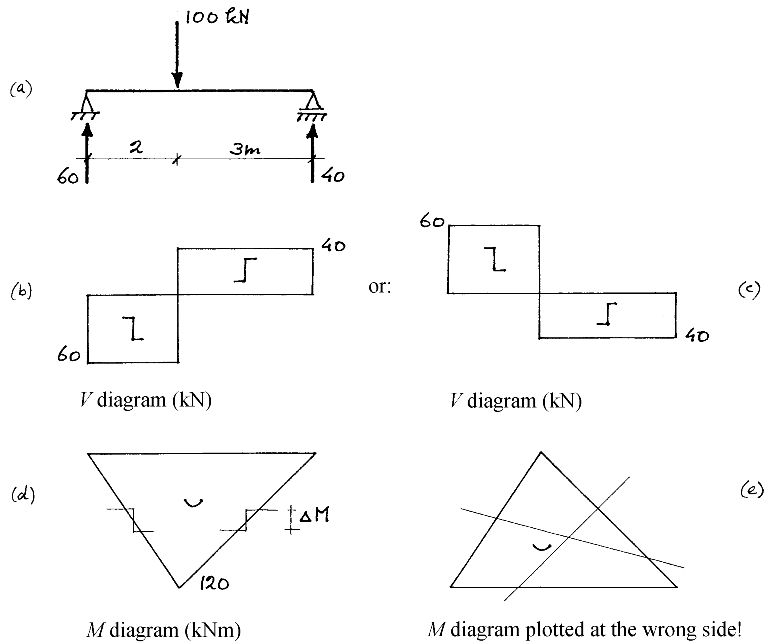


Figure 10.40 (a) A simply supported beam, loaded by a single point load. (b) and (c) When using deformation symbols, we are free to choose at which side of the member axis the shear force is plotted. (d) The bending moment is plotted at the side where the bending moment causes tension, so at the convex side of the member axis. The open side of the bending symbol therefore always faces towards the member axis. The gradient of the M diagram ($\Delta M/\Delta x$), shown as a "step", now corresponds directly to the shear symbol for the shear force. (e) If however the M diagram is plotted at the wrong side of the member axis, the relationship with the deformation symbol for the shear force (the "step" in the M diagram) is lost.

[1] Thanks to this agreement, the deformation symbol in the M diagram is actually unnecessary. The deformation symbol is nevertheless always shown for clarity.

10.4 Summary sign convention for the *N*, *V* and *M* diagrams

When drawing the N, V and M diagrams, you can use:

- plus and minus signs, related to a (local) coordinate system with the x axis along the member axis;
- deformation symbols (only for the V diagram and the M diagram).

If working with plus and minus signs, in a xz coordinate system:

- positive section forces are plotted at the positive side of the z axis, and negative values at the negative side of the z axis;
- the sign is placed *within the diagram*; relevant values are included *without their sign*.

If working with deformation symbols:

- You use plus and minus signs for the normal force N, and you use the bending symbol for the bending moment M and the shear symbol for the shear force V.
- The bending moment is plotted at the side where the bending moment causes tension, this is at the convex side of the member axis. The open side of the deformation symbol therefore always faces the member axis. The gradient of the M diagram, shown as a “step”, now corresponds directly to the shear symbol in the V diagram.
- It does not matter at which side of the member axis you plot the values for the normal force and the shear force.
- Plus and minus signs for the normal force and deformation symbols for the bending moment and shear force are placed within the diagram; relevant values in the diagram are written without sign.

10.5 Problems

Member axis and member cross-section; section forces (Section 10.1.1)

10.1 The sign of the section forces N, V and M can be related to a coordinate system with the x axis along the member axis and the yz plane parallel to the member cross-sections.

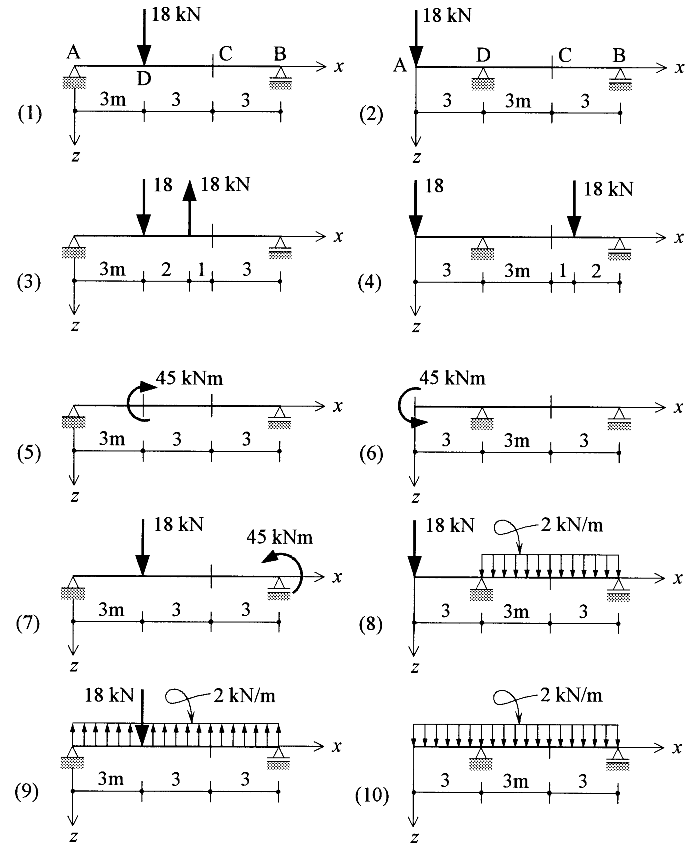
Questions:

a. When is a section plane positive, and when is it negative?
b. When is a normal force positive, and when is it negative?
c. When is a shear force positive, and when is it negative?
d. When is a bending moment positive, and when is it negative?

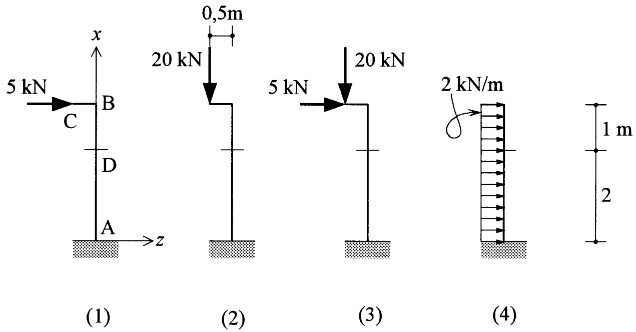
10.2: 1–10 You are given two beams loaded in five different ways.

Question:
Determine the bending moment and the shear force, with the correct sign in the given coordinate system, in the following cross-sections:

a. directly to the right of A.
b. directly to the left of B.
c. in C.
d. directly to the left of D.
e. directly to the right of D.



10.3: 1–4 You are given a column AB fixed at A, with console BC, loaded in four different ways.



Question:
Determine the sections forces, with the correct sign in the given coordinate system, in the following cross-sections:

a. directly below the console at B.
b. at D, one meter below B.
c. directly above fixed support A.

Stresses (Section 10.1.2)

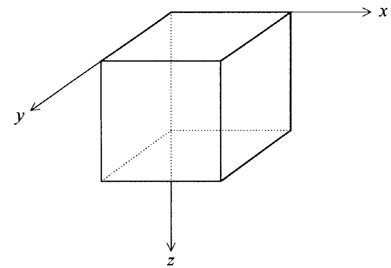
10.4: 1–4 You are given a block with the following four states of stress:

(1) $\sigma_{xx} = -25 \text{ N/mm}^2$
$\sigma_{xy} = \sigma_{yx} = -30 \text{ N/mm}^2$
$\sigma_{yz} = \sigma_{zy} = +15 \text{ N/mm}^2$
$\sigma_{zz} = +10 \text{ N/mm}^2$

(2) $\sigma_{xy} = \sigma_{yx} = +15 \text{ N/mm}^2$
$\sigma_{xz} = \sigma_{zx} = +10 \text{ N/mm}^2$
$\sigma_{yy} = -5 \text{ N/mm}^2$
$\sigma_{zz} = +20 \text{ N/mm}^2$

(3) $\sigma_{xx} = +30 \text{ N/mm}^2$
$\sigma_{xz} = \sigma_{zx} = +15 \text{ N/mm}^2$
$\sigma_{yz} = \sigma_{zy} = -25 \text{ N/mm}^2$
$\sigma_{zz} = -5 \text{ N/mm}^2$

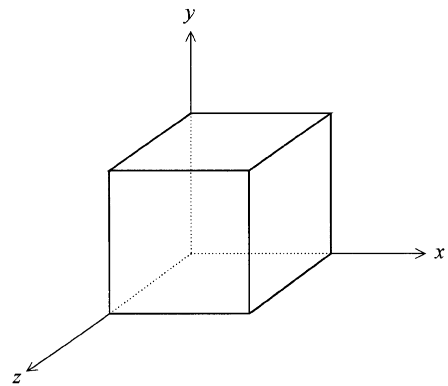
(4) $\sigma_{xx} = -6 \text{ N/mm}^2$
$\sigma_{xy} = \sigma_{yx} = -8 \text{ N/mm}^2$
$\sigma_{yy} = +5 \text{ N/mm}^2$
$\sigma_{yz} = \sigma_{zy} = +12 \text{ N/mm}^2$



Question:
Draw (for each case) the stresses on the block in the directions in which they act and include their values:

a. for the planes shown.
b. for the planes not shown.

10.5: 1–4 You are given a block with the following four states of stress:



(1) $\sigma_{xx} = +10 \text{ N/mm}^2$
$\sigma_{xy} = \sigma_{yx} = +15 \text{ N/mm}^2$
$\sigma_{xz} = \sigma_{zx} = -30 \text{ N/mm}^2$
$\sigma_{yy} = -5 \text{ N/mm}^2$

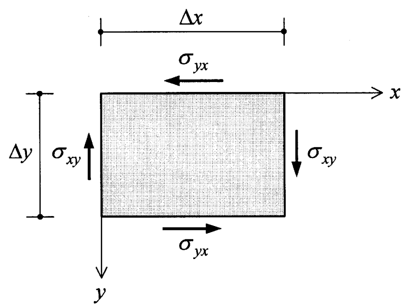
(2) $\sigma_{xy} = \sigma_{yx} = +10 \text{ N/mm}^2$
$\sigma_{yy} = -3 \text{ N/mm}^2$
$\sigma_{yz} = \sigma_{zy} = -8 \text{ N/mm}^2$
$\sigma_{zz} = +14 \text{ N/mm}^2$

(3) $\sigma_{xx} = -12\ \text{N/mm}^2$
$\sigma_{xz} = \sigma_{zx} = +5\ \text{N/mm}^2$
$\sigma_{yz} = \sigma_{zy} = -8\ \text{N/mm}^2$
$\sigma_{zz} = -15\ \text{N/mm}^2$

(4) $\sigma_{xx} = +4\ \text{N/mm}^2$
$\sigma_{xz} = \sigma_{zx} = +10\ \text{N/mm}^2$
$\sigma_{yy} = +7\ \text{N/mm}^2$
$\sigma_{yz} = -16\ \text{N/mm}^2$

Question:
Draw (for each case) the stresses on the block in the directions in which they act and include their values:
a. for the planes shown.
b. for the planes not shown.

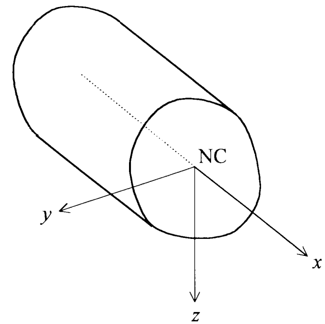
10.6 The dimensions of a rectangular block are Δx; Δy; Δz and are so small that three stresses on opposite planes are equal. The figure shows the top view of the block with only the shear stresses σ_{xy} and σ_{yx} on two planes. The other stresses are not shown.



Questions:
a. Draw all the other stresses that act parallel to the xy plane. Show that, from the moment equilibrium of the block in the xy plane, it follows that $\sigma_{xy} = \sigma_{yx}$.
b. Also show that $\sigma_{xz} = \sigma_{zx}$.
c. Show that $\sigma_{yz} = \sigma_{zy}$.

General definition section forces (Section 10.1.3)

10.7 A normal force N and the bending moments M_y and M_z act in a cross-section. If $\sigma = \sigma(x, y)$ is the normal stress in the cross-section, then the normal force is: $N = \int_A \sigma \, \mathrm{d}A$.



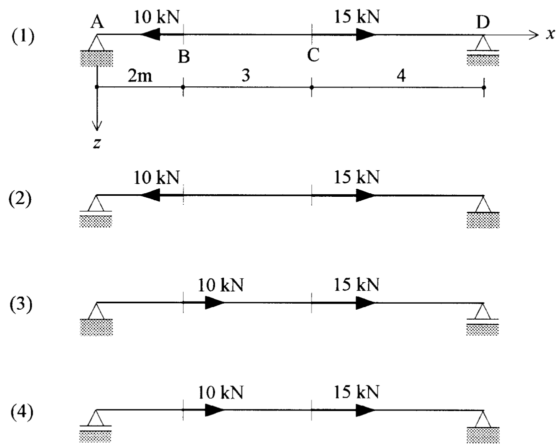
Questions:
a. Draw the (positive) normal force N in the cross-section.
b. Draw the (positive) bending moments M_y and M_z in the cross-section.
c. How can we express the bending moments M_y and M_z in the normal stress σ?

10.8 The shear stresses $\sigma_{xy} = \sigma_{xy}(x, y)$ and $\sigma_{xz} = \sigma_{xz}(x, y)$ in a cross-section lead to the shear forces V_y and V_z and a torsional moment M_t.

Questions:
a. Draw the (positive) shear forces V_y and V_z in the cross-section.
b. How can the shear forces V_y and V_z be expressed in the shear stresses σ_{xy} and σ_{xz}?
c. Draw the (positive) torsional moment M_t in the cross-section.
d. How can the torsional moment M_t be expressed in the shear stresses σ_{xy} and σ_{xz}?

Diagrams for the normal force, shear force and bending moment (Section 10.2)

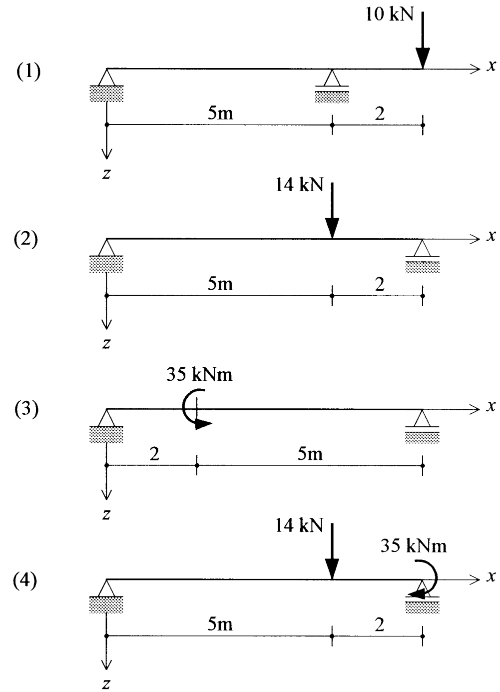
10.9: 1–4 Member AD is supported in two different ways and is loaded at B and C by forces of respectively 10 and 15 kN.



Questions:

a. From the equilibrium determine the normal force N as a function of x.
b. Draw the N diagram.

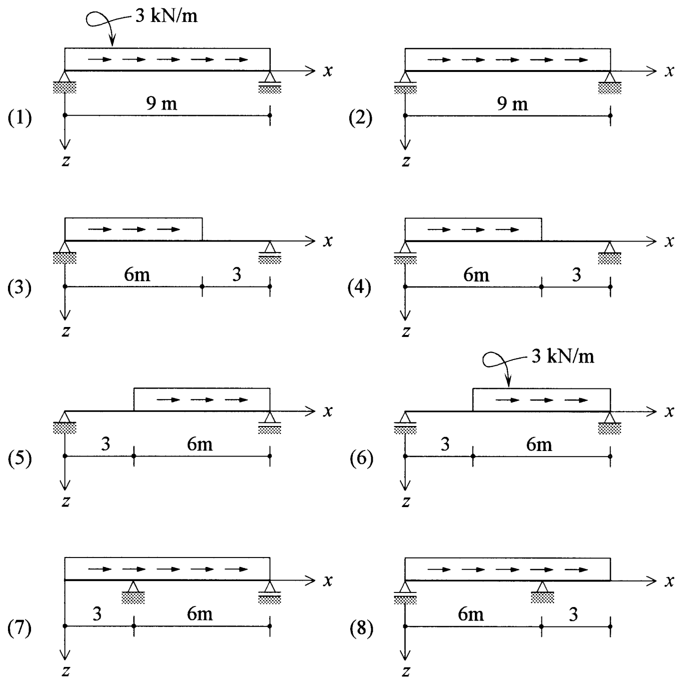
10.10: 1–4 You are given four different loaded beams.



Questions:

a. From the equilibrium, determine the shear force V as a function of x.
b. Draw the shear force diagram.
c. From the equilibrium, determine the bending moment M as a function of x.
d. Draw the bending moment diagram.

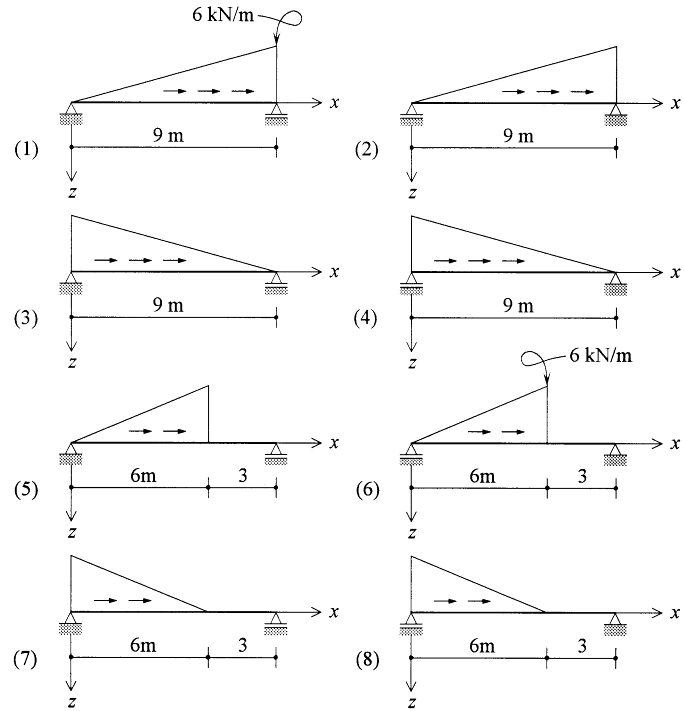
10.11: 1–8 A number of members are loaded for extension by a uniformly distributed load of 3 kN/m.



Questions:

a. From the equilibrium, determine the normal force as a function of x.
b. Draw the normal force diagram.

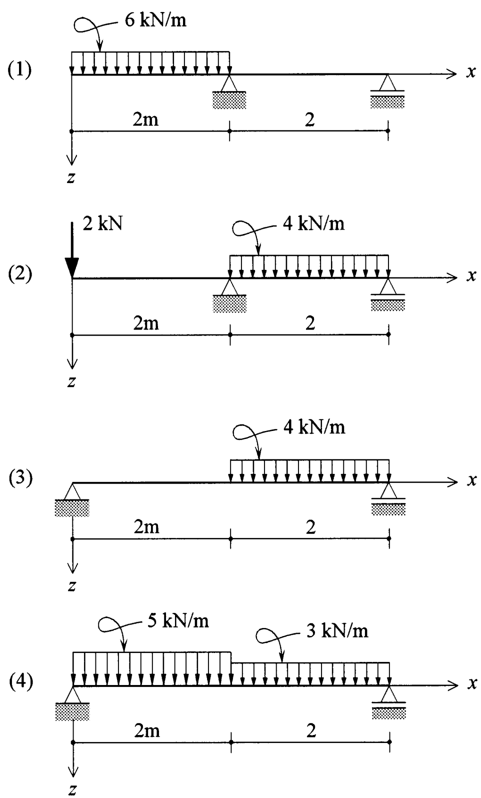
10.12: 1–8 The same member is loaded in two different ways and is loaded for extension in four different ways by a linearly distributed load. In all cases, the top value of the distributed load is 6 kN/m.



Questions:

a. From the equilibrium, determine the normal force as a function of x.
b. Draw the normal force diagram.

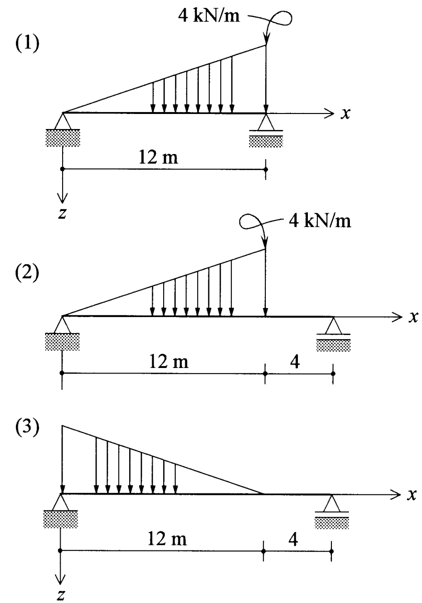
10.13: 1–4 You are given four differently loaded beams.



Questions:

a. From the equilibrium, determine the shear force V as a function of x.
b. Draw the shear force diagram.
c. From the equilibrium determine the bending moment M as a function of x.
d. Draw the bending moment diagram.

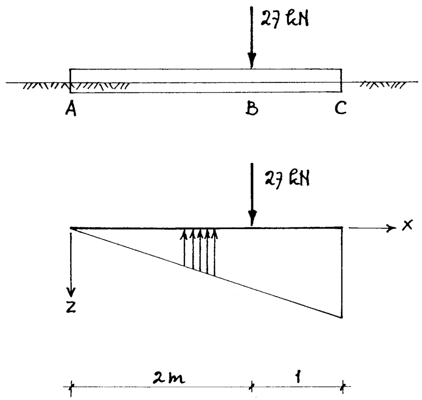
10.14: 1–3 You are given three beams with a linearly distributed load normal to the member axis. The top value for the distributed load in all three cases is 4 kN/m.



Questions:

a. From the equilibrium, determine the shear force V as a function of x.
b. Draw the V diagram.
c. Form the equilibrium, determine the bending moment M as a function of x.
d. Draw the M diagram.

10.15 The figure shows a foundation beam on sand and loaded by a force of 27 kN. For this load, the earth pressure is linearly distributed, as shown in the load diagram. The dead weight of the beam is ignored.



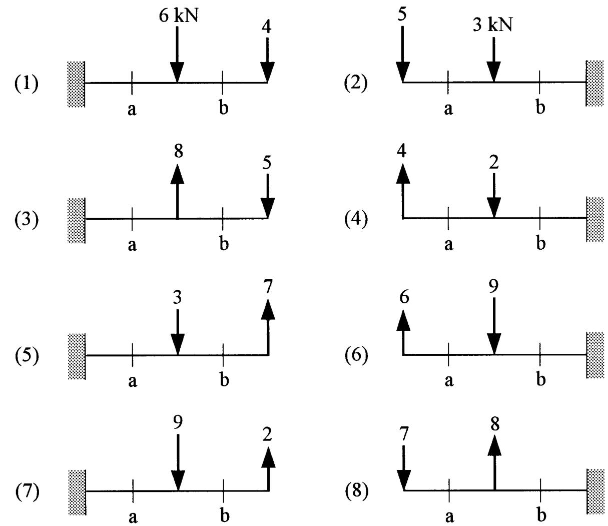
Questions:

a. Determine the top value of the earth pressure.
b. From the equilibrium, determine the shear force and the bending moment for AB as a function of x ($0 \leq x < 2$ m).
c. From the equilibrium, determine the shear force and the bending moment for BC as a function of x ($2 \text{ m} < x \leq 4$ m).
d. For ABC, draw the shear force diagram and the bending moment diagram.

Deformation symbols for shear force and bending moment (Section 10.3)

10.16 *Question*: Explain the *shape* of the deformation symbols that are used for shear force and bending moment.

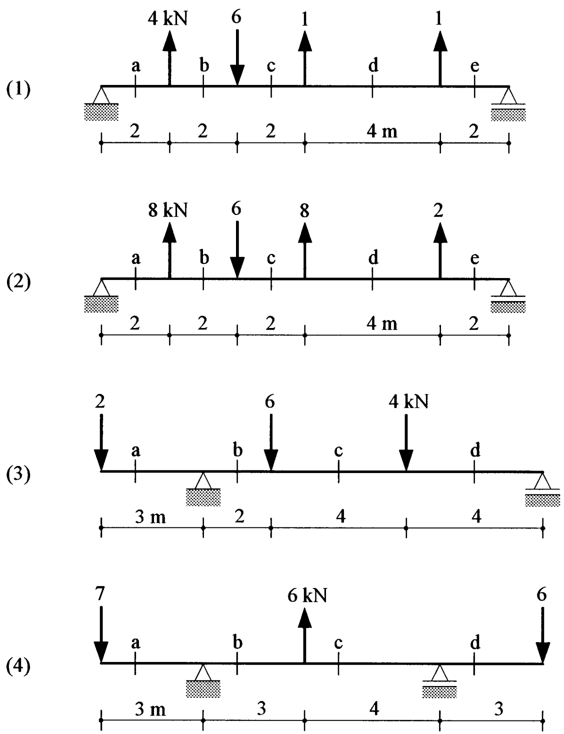
10.17: 1–8 A fixed beam is loaded in various ways.



Question:
Which deformation symbol belongs to the shear force in cross-sections a and b respectively?

a.	‾|_
b.	_|‾
c.	0

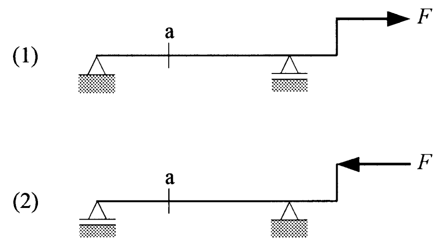
10.18: 1–4 You are given four different beams.



Question:
Which deformation symbol belongs to the shear force in cross-sections a to e respectively?

a.	¯|_
b.	_|¯
c.	0

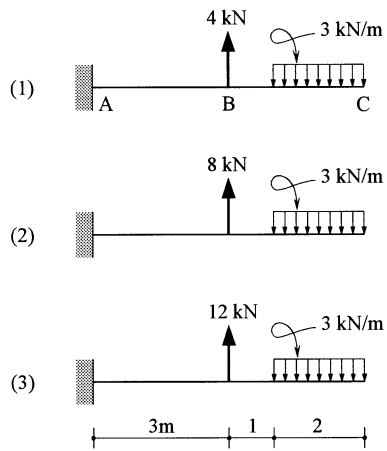
10.19: 1–2 Two beams are loaded by an eccentrically applied normal force.



Question:
Which combination of deformation symbols belongs to the shear force and the bending moment in cross-section a?

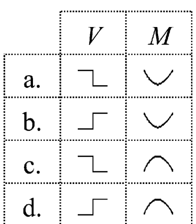
	V	M
a.	¯|_	∪
b.	_|¯	∪
c.	¯|_	∩
d.	_|¯	∩

10.20: 1–3 You are given the same cantilever beam with three different loads.

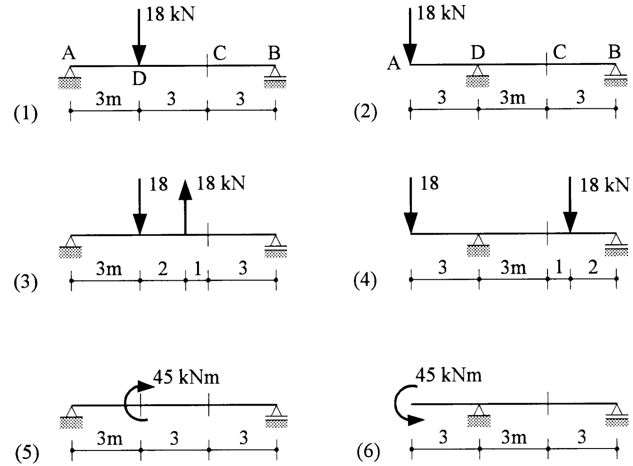


Question:
Determine the combination of deformation symbols that belong to the shear force and the bending moment:

a. in the cross-section directly to the right of B.
b. in the cross-section directly to the left of B.
c. in the cross-section directly next to the fixed support A.



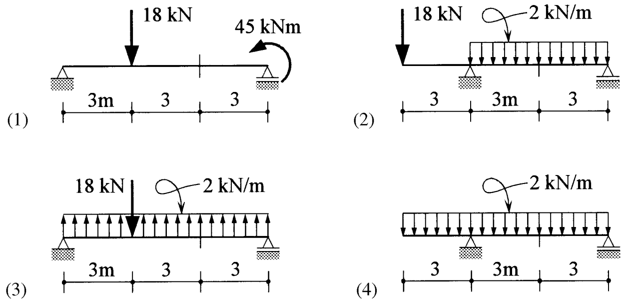
10.21: 1–6 You are given two beams loaded in three different ways.



Question:
Determine the bending moment and the shear force, with the correct deformation symbol, in the following cross-sections:

a. directly to the right of A.
b. directly to the left of B.
c. at C.
d. directly to the left of D.
e. directly to the right of D.

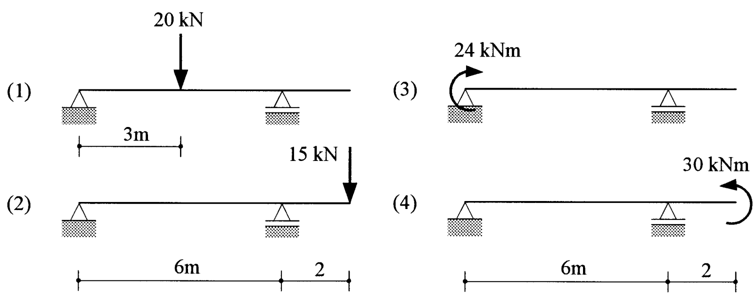
10.22: 1–4 You are given two beams loaded in different ways.



Question:
Determine the bending moment and the shear force, with the correct deformation symbol, in the following cross-sections:

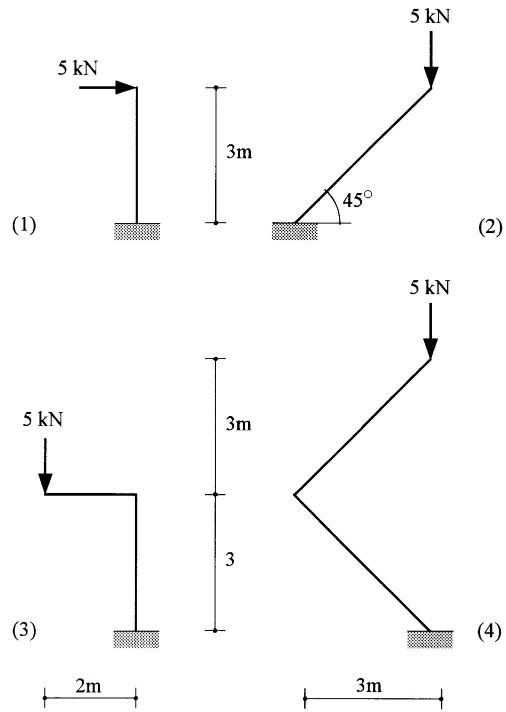
a. directly to the right of A.
b. directly to the left of B.
c. at C.
d. directly to the left of D.
e. directly to the right of D.

10.23: 1–4 You are given a cantilever beam and four different loads.



Question:
For the entire beam, draw the shear force diagram and the bending moment diagram, with the deformation symbols. Include their values at relevant points.

10.24: 1–4 You are given four different structures.



Question:
For the entire structure, draw the V diagram and the M diagram, with the deformation symbols. Include the values.

11 Mathematical Description of the Relationship between Section Forces and Loading

In the previous chapter, a direct approach was used to determine the variation of section forces. Section forces were determined from the equilibrium of the isolated member part on the one or other side of the section. Usually the support reactions have to be determined first.

This chapter introduces a more *mathematical approach* based on the equilibrium of a small member segment with length Δx that approaches zero ($\Delta x \to 0$).

In Section 11.1, we derive the *differential equations for the equilibrium* of such an infinitesimal member segment.

Using examples, Sections 11.2 and 11.3 show how to determine the variation of the section forces. The examples in Section 11.2 relate to *extension* (relationship between N and q_x); those in Section 11.3 relate to *bending* (relationship between M_z, V_z and q_z).

Since no misunderstanding is possible, we will omit the index z in M_z and V_z to simplify the writing.

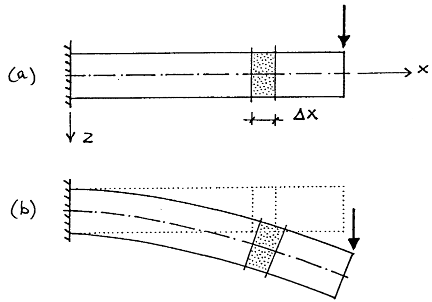


Figure 11.1 Member with (a) non-deformed geometry and (b) deformed geometry.

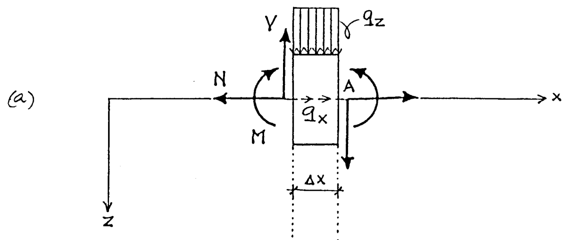


Figure 11.2 (a) Section forces N, V and M acting on the left-hand sectional plane of a small member segment with length Δx ($\Delta x \to 0$).

11.1 Differential equations for the equilibrium

The differential equations for equilibrium are derived from the equilibrium of a small member segment with a length Δx that approaches zero ($\Delta x \to 0$). We assume that the displacements due to the deformation of the member are negligible small. Therefore the equilibrium, including that of a small member element, can be related to the *non-deformed geometry* (see Figure 11.1).

In Figure 11.2a, a small segment with length Δx has been isolated from a member and greatly magnified. The member segment is subjected to q_x and q_z. The loads act on the member axis (this has not been drawn as such for q_z for the sake of clarity).

If length Δx of the member segment is sufficiently small ($\Delta x \to 0$), the distributed loads q_x and q_z can be considered *uniformly distributed*.

The (unknown) section forces on the left and right-hand sectional planes are shown in their positive direction. The section forces are a function of x, the location of the cross-section, and are generally different in both sectional planes.

In Figure 11.2a, it is assumed that the forces on the left-hand section are N, V and M. If the section forces increase over a distance Δx in the x direction by amounts ΔN, ΔV and ΔM, respectively (see Figures 11.2b to 11.2d), the forces on the right-hand sectional plane are then $N + \Delta N$, $V + \Delta V$ and $M + \Delta M$ (see Figure 11.2e).

From the *force equilibrium* of the small member segment it follows that

$$\sum F_x = -N + (N + \Delta N) + q_x \Delta x = 0, \tag{a}$$

$$\sum F_z = -V + (V + \Delta V) + q_z \Delta x = 0. \tag{b}$$

From the *moment equilibrium* it follows that (we have selected the moment sum about point A on the right-hand sectional plane)

$$\sum T_y|\text{A} = -M - V\Delta x + (M + \Delta M) + q_z\Delta x \cdot \tfrac{1}{2}\Delta x = 0. \qquad \text{(c)}$$

With the three equilibrium conditions (a) to (c) this gives

$$\Delta N + q_x\Delta x = 0,$$
$$\Delta V + q_z\Delta x = 0,$$
$$\Delta M - V\Delta x + \tfrac{1}{2}q_z(\Delta x)^2 = 0.$$

After dividing by Δx we find

$$\frac{\Delta N}{\Delta x} + q_x = 0,$$
$$\frac{\Delta V}{\Delta x} + q_z = 0,$$
$$\frac{\Delta M}{\Delta x} - V = -\tfrac{1}{2}q_z\Delta x.$$

$\Delta N/\Delta x$ is the increase in the normal force per length in the x direction (see Figure 11.2b). In the limit $\Delta x \to 0$ this is known as the derivative from N with respect to x and is written $\mathrm{d}N/\mathrm{d}x$:

$$\lim_{\Delta x\to 0}\frac{\Delta N}{\Delta x} = \frac{\mathrm{d}N}{\mathrm{d}x}.$$

In the same way:

$$\lim_{\Delta x\to 0}\frac{\Delta V}{\Delta x} = \frac{\mathrm{d}V}{\mathrm{d}x},$$

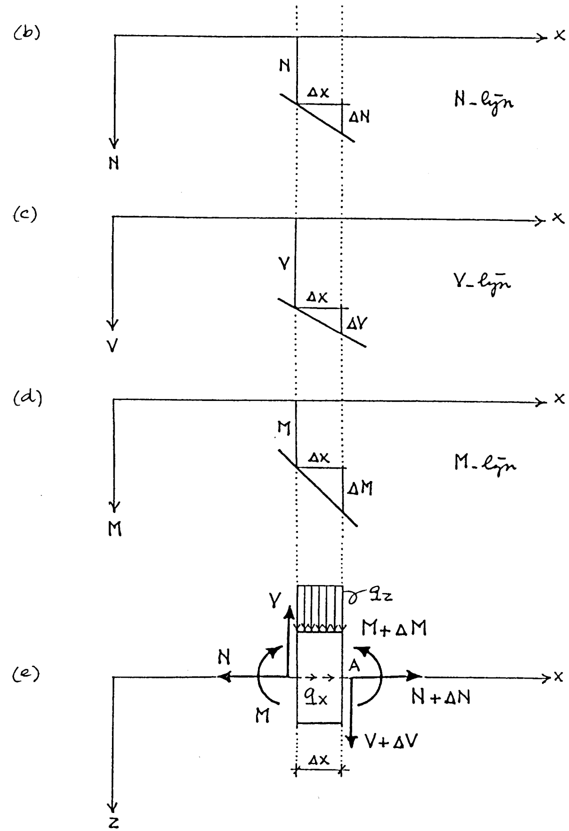


Figure 11.2 (b) to (d) Over a distance Δx in the x direction the section forces increase by amounts ΔN, ΔV and ΔM respectively. (e) The section forces on the right-hand sectional plane are then $N + \Delta N$, $V + \Delta V$ and $M + \Delta M$.

$$\lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} = \frac{\mathrm{d}M}{\mathrm{d}x}.$$

The three equations for the equilibrium of an elementary member segment with length Δx in the limit $\Delta x \to 0$ are

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0,$$
$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0,$$
$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0.$$

In the last equation, with $\Delta x \to 0$, the term $\frac{1}{2}q_z\Delta x$ has disappeared. This is justified since the contribution by q_z in the equation for the moment equilibrium is *one order smaller* than the contribution of the other terms.

The formulas derived give important *general information* about the variation of N, V and M in member segments (*fields*) where no concentrated forces and/or couples are acting. In Sections 11.2 and 11.3 this general information is translated into rules that allow us to easily draw N, V and M diagrams.

The first-order differential equation equation

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0 \quad (\textit{extension}) \tag{a}$$

provides a direct relationship between the (distributed) load q_x acting in the direction of the member axis, and the normal force N. This is known as the *equilibrium equation for extension*.

The equations

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0, \tag{b}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0 \tag{c}$$

indicate the relationship between the (distributed) load q_z acting normal to the member axis, the shear force V and the bending moment M.

The shear force V can be eliminated by differentiating (c) to x and adding it to (b). This gives the second-order differential equation

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q_z = 0 \quad (\textit{bending}). \tag{d}$$

This equation provides the direct relationship between the (distributed) load q_z normal to the member axis and the bending moment M. This is known as the *equilibrium equation for bending*.

The variation of the normal force N depends only on the load in the direction of the member axis, q_x. The variation of the bending moment M and the shear force V depends only on the load q_z normal to the member axis. For a member, this means that the equilibrium equations for *extension* (only normal forces due to axial loads) and *bending* (only bending moments and shear forces due to loads normal to the member axis) can be treated separately.[1]

Comment 1: In Sections 10.2.1 to 10.2.3 we discussed the fact that axial loads give only normal forces (*extension*) and that loads normal to the mem-

1 The loads have to be applied on the member axis.

ber axis (including couples) give only shear forces and bending moments (*bending*).

Comment 2: The derivation is not applicable if a concentrated force or a concentrated couple acts on the member segment. In that case, there is a "step change" or "abrupt change in slope" in the N, V and/or M diagram; see the examples in Section 10.2.1. N, V and/or M are, as functions of x, no longer continuous and/or continuously differentiable. In such a case, the member can be split into a number of *fields*, so that the differential equation is applicable for each individual field (see Section 11.2, Example 2 and Section 11.3, Example 4).

11.2 Mathematical elaboration of the relationship between N and q_x (extension)

We derived for the relationship between the normal force N and the distributed axial load q_x

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0$$

or in other words

$$\frac{\mathrm{d}N}{\mathrm{d}x} = -q_x.$$

By integrating once, we find the variation of the normal force N:

$$N = -\int q_x \,\mathrm{d}x.$$

With the exception of a constant, we have determined the *indefinite integral* (or *primitive function*) of q_x, and therefore the variation of the normal force N.

The unknown integration constant is found using a known (prescribed) value of N at one of the member *ends*. This is referred to as an *end condition*.

It is sometimes necessary to divide the member into a number of segments (*fields*), as in Figure 11.3. In that case, we also have to formulate conditions for the *joining* from one field to another. These conditions are referred to as *joining conditions*.

Purely mathematically both the end conditions and joining conditions can be regarded as boundary conditions for a specific field. They can be derived from the equilibrium of a small member segment with length Δx ($\Delta x \to 0$) at the boundaries of the field (an end and/or a joining).

For a statically determinate member, there are always sufficient boundary conditions to find the normal force variation *without previously determining the support reactions*.

We will illustrate this by means of two examples previously covered in Section 10.2.3:

- A column subject to its dead weight.
- A simply supported member that is loaded over two-thirds of its length by a uniformly distributed axial load along the member axis.

Example 1
Figure 11.4a gives the model of a column and its load.

Question:
Determine the variation of the normal force (the N diagram) from the differential equations for the equilibrium.

Solution:
The units used are m and kN; they are omitted hereafter from the calculation.

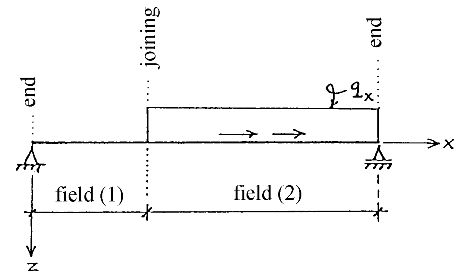


Figure 11.3 Since the distributed load is not acting along the entire length, the member has to be divided into two fields.

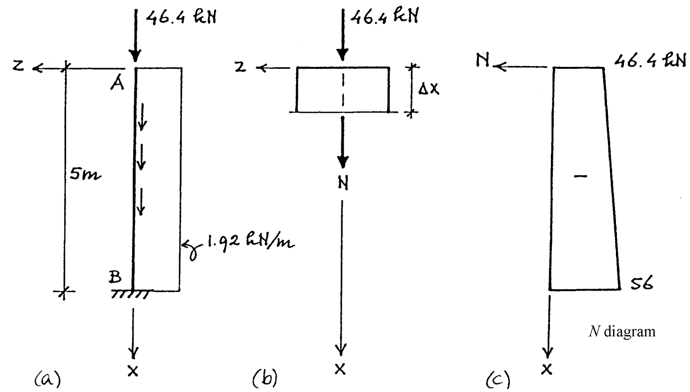


Figure 11.4 (a) Model of a column and its load. (b) The boundary condition (end condition) $N = -46.4$ kN follows from the force equilibrium in the x direction of the small end segment in A (with $\Delta x \to 0$). (c) N diagram.

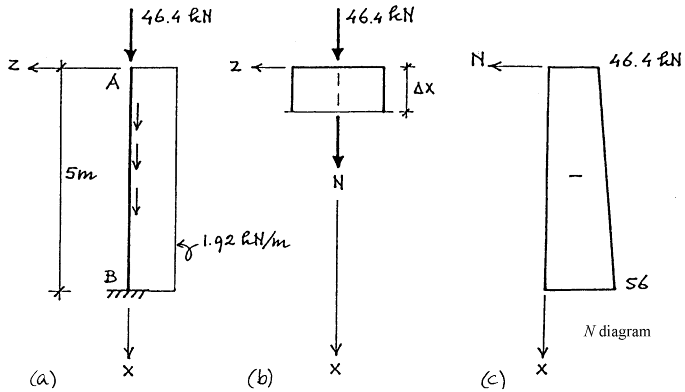


Figure 11.4 (a) Model of a column and its load. (b) The boundary condition (end condition) $N = -46.4$ kN follows from the force equilibrium in the x direction of the small end segment in A (with $\Delta x \to 0$). (c) N diagram.

In the given coordinate system:

$$q_x = 1.92 \text{ kN/m}$$

so that

$$N = -\int q_x \,\mathrm{d}x = -\int 1.92 \,\mathrm{d}x = (-1.92x + C) \text{ kN.} \tag{a}$$

The *integration constant* C is found from the fact that there is a compressive force of 46.4 kN at the top of the column. The boundary condition (end condition) is therefore

$$x = 0 : N = -46.6 \text{ kN.} \tag{b}$$

This boundary condition can also be derived more formally from the force equilibrium in the x direction of a small member segment with length Δx ($\Delta x \to 0$) at the top of the column (see Figure 11.4b):

$$\sum F_x = 46.4 + N = 0 \to N = -46.4 \text{ kN.}$$

With $\Delta x \to 0$ the contribution of $q_x \Delta x$, due to the distributed load, disappears.

Substitute the values of x and N from (b) in (a) and we find

$$C = -46.6 \text{ kN.}$$

This gives the variation of the normal force N:

$$N = (-1.92x - 46.4) \text{ kN.}$$

The normal force diagram is shown in Figure 11.4c. The results agree with what we found earlier in Section 10.2.3, Example 1.

Example 2

In Figure 11.5a, a uniformly distributed axial load q is acting on segment BC of member ABC which is simply supported at A and C.

Question:

Determine the variation of the normal force (the N diagram) from the differential equations for the equilibrium.

Solution:

Since the uniformly distributed load q acts only on part of the member, we have to distinguish between two segments:

- segment AB ($0 < x < a$), hereafter known as field (1).
- segment BC ($a < x < 3a$), hereafter known as field (2).

The normal force variation is determined per field. The field number is used as upper index for units that are field-dependent.

Field (1):

$$\frac{\mathrm{d}N^{(1)}}{\mathrm{d}x} = -q_x^{(1)} = 0 \rightarrow N^{(1)} = C^{(1)}.$$

In an unloaded field, the normal force is constant.

Field (2):

$$\frac{\mathrm{d}N^{(2)}}{\mathrm{d}x} = -q_x^{(2)} = -q \rightarrow N^{(2)} = -qx + C^{(2)}.$$

In a field with a uniformly distributed load, the normal force is linear.

Per field, there is one unknown integration constant; with two fields there is a total of two integration constants, $C^{(1)}$ and $C^{(2)}$. There are two boundary conditions available to solve these constants: a *joining condition* at B ($x = a$) and an *end condition* at C ($x = 3a$).

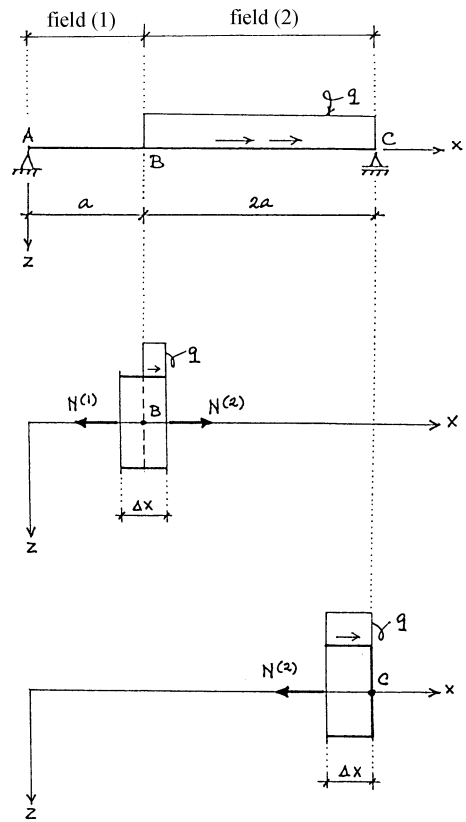


Figure 11.5 (a) A simply supported member with a uniformly distributed axial load on section BC. (b) The boundary condition (joining condition) at B, $N^{(1)} = N^{(2)}$, follows from the force equilibrium in the x direction of a small member segment at the joining at B (with $\Delta x \rightarrow 0$). (c) The boundary condition (end condition) at C, $N^{(2)} = 0$, follows from the force equilibrium in the x direction of a small end segment at C (with $\Delta x \rightarrow 0$).

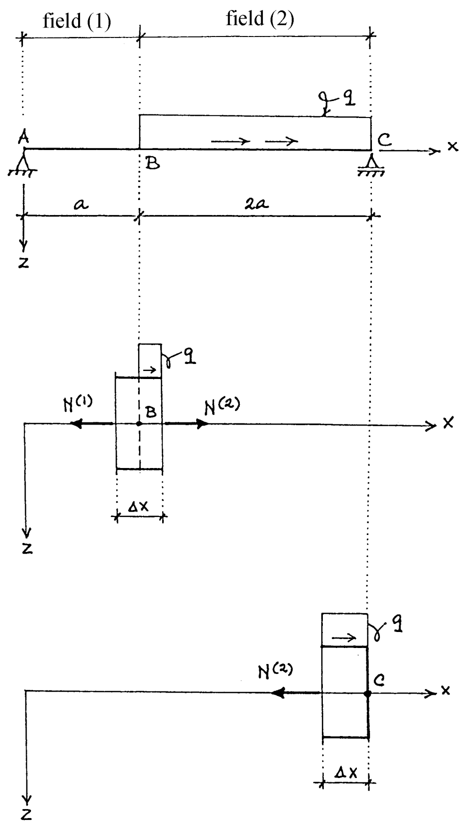


Figure 11.5 (a) A simply supported member with a uniformly distributed axial load on section BC. (b) The boundary condition (joining condition) at B, $N^{(1)} = N^{(2)}$, follows from the force equilibrium in the x direction of a small member segment at the joining at B (with $\Delta x \to 0$). (c) The boundary condition (end condition) at C, $N^{(2)} = 0$, follows from the force equilibrium in the x direction of a small end segment at C (with $\Delta x \to 0$).

- The *joining condition* at B ($x = a$)
  At B, the normal force in field (1) is equal to the normal force in field (2):
  $$x = a : N^{(1)} = N^{(2)}. \tag{a}$$
  This boundary condition can be derived directly from the force equilibrium in the x direction of the small member segment with length Δx ($\Delta x \to 0$) at the joining in B (see Figure 11.5b). The contribution of q disappears from the equilibrium equation as $\Delta x \to 0$.

- The *end condition* at C ($x = 3a$)
  At a roller support C the horizontal support reaction is zero, as is therefore the normal force:
  $$x = 3a : N^{(2)} = 0. \tag{b}$$
  This boundary condition can also be derived from the equilibrium of a small member segment at the end of the member (see Figure 11.5c). Here also, the contribution of q disappears in the equilibrium equation as $\Delta x \to 0$.

Elaboration of the conditions (a) and (b) leads to two equations with two unknowns:

$$C^{(1)} - C^{(2)} = -qa,$$
$$C^{(2)} = 3qa.$$

The solution is

$$C^{(1)} = 2qa,$$
$$C^{(2)} = 3qa.$$

This results in the normal force variation for both fields:

Field (1):

$$N^{(1)} = 2qa \quad (0 \leq x < a).$$

Field (2):

$$N^{(2)} = -qx + 3qa \quad (a < x \leq 3a).$$

Figure 11.6 shows the N diagram. The results agree with those found previously in Section 10.2.2, Example 2.

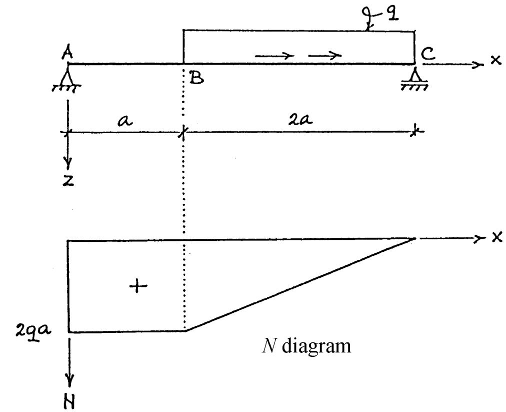


Figure 11.6 The loaded member with its N diagram.

11.3 Mathematical elaboration of the relationship between M, V and q_z (bending)

We derived the following for the relationship between M, V and a distributed load q_z normal to the member axis:

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0,$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0.$$

Eliminating the shear force leads to a direct relationship between the bending moment M and the distributed load q_z:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q_z = 0$$

or in other words:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -q_z.$$

On the basis of this latter equation, we find the shear force V after integrating once:

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V = -\int q_z \, \mathrm{d}x$$

and after integrating again we find the variation of the bending moment M:

$$M = \int V \, \mathrm{d}x = -\int \left(\int q_z \, \mathrm{d}x \right) \mathrm{d}x.$$

With each integration, an integration constant appears. This means that the expression for the shear force V contains one unknown (C_1), and that for the bending moment M contains two (C_1 and C_1).

The two constants C_1 and C_2 follow from *end conditions and/or joining conditions* relating to V and M. They can be derived from the equilibrium of a small member segment with length Δx ($\Delta x \to 0$) on the boundaries (end or joining) of the field.

For statically-determinate members, there are always sufficient end and/or joining conditions to find the variation of the shear force and bending moment *without previously determining the support reactions*. This is illustrated using the following four examples:

1. A fixed beam, loaded at its free end by a concentrated load.
2. A simply supported beam and a beam fixed at one of its ends, both with a uniformly distributed load along its entire length.
3. A simply supported beam with a triangular load.
4. A simply supported beam with overhang (cantilever beam) and a uniformly distributed load along its entire length.

Example 1
Figure 11.7a shows a beam AB fixed at A and of length ℓ. At its free end B the beam is loaded normal to its axis by a force F.

Question:
Determine the V and M diagrams using the differential equations for equilibrium.

Solution:
For $0 < x < \ell$ it holds that

$$\frac{d^2M}{dx^2} = -q_z = 0.$$

Repeated integration gives

$$\frac{dM}{dx} = V = C_1, \tag{a}$$

$$M = C_1 x + C_2. \tag{b}$$

In an unloaded field the shear force is constant and the bending moment is linear.

The integration constants C_1 and C_2 are found from the boundary conditions at the free end B. Here both V and M have a prescribed value: the shear force is equal to F (pay attention to the sign), and the bending moment is zero:

$$x = \ell : V = +F, \tag{c}$$

$$x = \ell : M = 0. \tag{d}$$

The boundary conditions can also be derived from the force and moment equilibrium of a small member segment with length Δx ($\Delta x \to 0$) at the

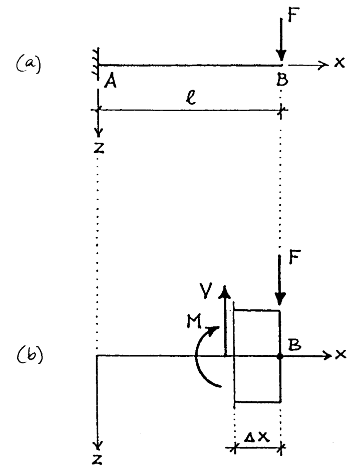


Figure 11.7 (a) A beam fixed at A and loaded at its free end B by a force F normal to the beam axis. (b) The boundary conditions $V = +F$ and $M = 0$ are found from the force and moment equilibrium of a small boundary segment at B (with $\Delta x \to 0$).

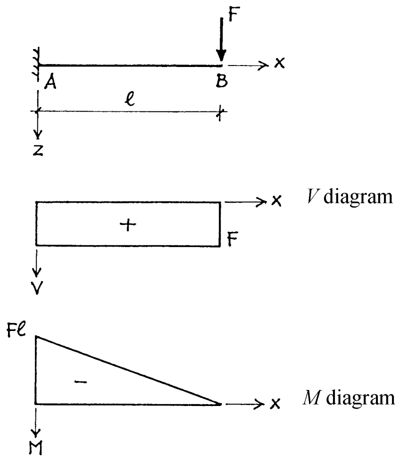


Figure 11.8 The loaded beam and its V and M diagrams.

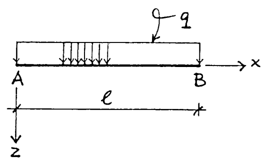


Figure 11.9 A beam for which the type of support at A and/or B is still unknown, with a uniformly distributed load normal to the member axis.

free end B (see Figure 11.7b). The section forces V and M, which here are acting on a negative section plane, have to be drawn in accordance with their positive directions. The equilibrium of the member segment gives

$$\sum F_z = -V + F = 0 \Rightarrow V = +F,$$

$$\sum T_z|\mathrm{B} = -M - V\Delta x = 0 \text{ (with } \Delta x \to 0) \;\Rightarrow M = 0.$$

Substitute (c) in (a) and (d) in (b); elaboration of the boundary conditions leads to

$$C_1 = +F,$$

$$C_2 = -F\ell.$$

This gives the variation of shear force V and bending moment M for beam AB:

$$V = F,$$

$$M = F_x - F\ell = -F(\ell - x).$$

The V and M diagrams are shown in Figure 11.8. The shear force V is constant. The bending moment M is negative everywhere and is linear. The bending moment (in the absolute sense) has its maximum at the fixed end:

$$|M|_{\max} = F\ell.$$

Example 2
In Figure 11.9, a uniformly distributed load q is acting normal to the beam axis over the entire length ℓ of beam AB. The method of support in A and/or B is given below.

Question:
For the following three cases, determine the V and M diagrams using the differential equations for equilibrium (see Figure 11.10):

(a) The beam is simply supported at A and B;
(b) The beam has a fixed support at A and a roller support at B;
(c) The beam has a fixed support at B and a roller support at A.

Solution:
In all three cases, with $q_z = q$ the following applies:

$$\frac{d^2M}{dx^2} = -q,$$

$$\frac{dM}{dx} = V = -\int q\,dx = -qx + C_1,$$

$$M = \int V\,dx = \int (-qx + C_1)\,dx = -\tfrac{1}{2}qx^2 + C_1x + C_2.$$

Due to a uniformly distributed load, the shear force is linear, and the bending moment is quadratic (parabolic).

The constants C_1 and C_2 are determined by the boundary conditions (associated with the type of support) at A and/or B.

The boundary conditions are

case (a)	case (b)	case (c)
$x = 0 : M = 0$	$x = \ell : V = 0$	$x = 0 : V = 0$
$x = \ell : M = 0$	$x = \ell : M = 0$	$x = 0 : M = 0$

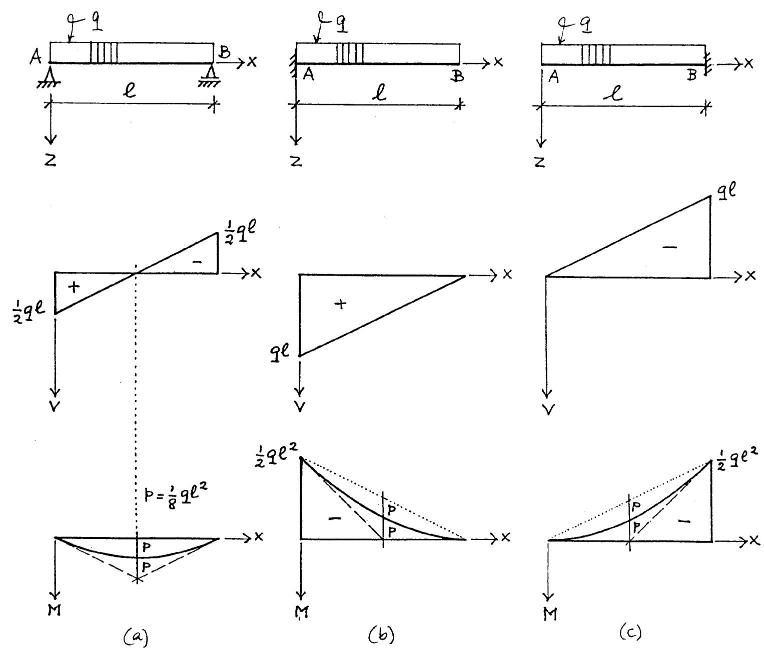


Figure 11.10 The same beam supported in three different ways, with the associated V and M diagrams: (a) simply supported at A and B, (b) fixed at A and (c) fixed at B.

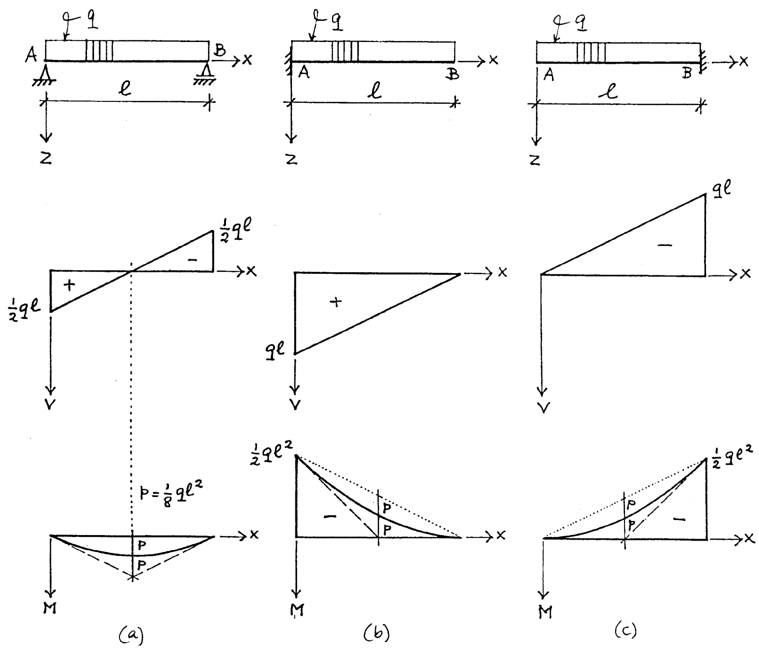


Figure 11.10 The same beam supported in three different ways, with the associated V and M diagrams: (a) simply supported at A and B, (b) fixed at A and (c) fixed at B.

- Elaboration of the boundary conditions in case (a):

$$x = 0 : M = C_2 = 0 \Rightarrow C_2 = 0,$$

$$x = \ell : M = -\tfrac{1}{2}q\ell^2 + C_1\ell + C_2 = 0 \Rightarrow C_1 = \tfrac{1}{2}q\ell$$

from which it follows that

$$V = -qx + \tfrac{1}{2}q\ell = \tfrac{1}{2}q(\ell - 2x),$$

$$M = -\tfrac{1}{2}qx^2 + \tfrac{1}{2}q\ell x = \tfrac{1}{2}qx(\ell - x).$$

- Elaboration of the boundary conditions in case (b):

$$x = \ell : V = -q\ell + C_1 = 0 \Rightarrow C_1 = q\ell,$$

$$x = \ell : M = -\tfrac{1}{2}q\ell^2 + C_1\ell + C_2 = 0 \Rightarrow C_2 = -\tfrac{1}{2}q\ell^2$$

from which it follows that

$$V = -qx + q\ell = q(\ell - x),$$

$$M = -\tfrac{1}{2}qx^2 + q\ell x - \tfrac{1}{2}q\ell^2 = -\tfrac{1}{2}q(\ell - x)^2.$$

- Elaboration of the boundary conditions in case (c):

$$x = 0 : V = C_1 = 0 \Rightarrow C_1 = 0,$$

$$x = 0 : M = C_2 = 0 \Rightarrow C_2 = 0$$

from which it follows that

$$V = -qx,$$

$$M = -\tfrac{1}{2}qx^2.$$

Figure 11.10 shows the V and M diagrams for all three cases. The tangents to the M diagram are also shown at A and B. These intersect in $x = \frac{1}{2}\ell$, at mid-span. In the figure, an important variable p is shown: $p = \frac{1}{8}q\ell^2$. We will make use of p in Chapter 12.

Below we again show how, for two cases, the boundary conditions (end conditions) can be derived from the equilibrium of a small element with length Δx ($\Delta x \to 0$) at the beam ends.

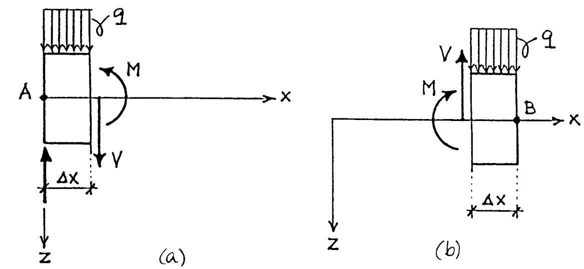


Figure 11.11 (a) Small end segment at hinged support A (see Figure 11.10a). (b) Small end segment at free end B (see Figure 11.10b).

Boundary condition at the hinged support A in case (a)

In Figure 11.11a, a member segment with length Δx ($\Delta x \to 0$) has been isolated at the hinged support at A. The figure shows all the forces acting on it, including the unknown vertical support reaction at A. Moment equilibrium about A requires

$$\sum T_y|\mathrm{A} = M - V\Delta x - q\Delta x \cdot \tfrac{1}{2}\Delta x = 0.$$

For $\Delta x \to 0$ the terms with Δx disappear and we find the boundary condition at A:

$$M = 0.$$

Boundary conditions at the free member end B in case (b)

In Figure 11.11b, the small "last" member segment at the free end B is shown, with all the forces acting on it. The element has a length Δx.

The equations for the equilibrium are

$$\sum F_z = -V + q\Delta x = 0,$$

$$\sum T_y|\mathrm{B} = -M - V\Delta x + q\Delta x \cdot \tfrac{1}{2}\Delta x = 0.$$

For $\Delta x \to 0$ the terms with Δx disappear in both equations and we find the boundary conditions at the free member end B:

$$V = 0,$$

$$M = 0.$$

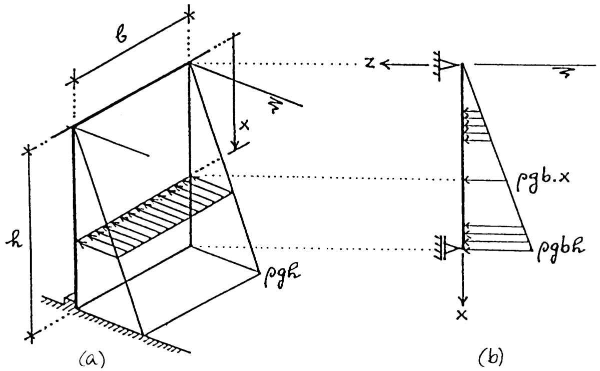


Figure 11.12 (a) The water pressure on a water-retaining slide (b) modelled as a line load on a line element.

Example 3

Figure 11.12a shows a water-retaining slide of width b and height h, which is supported by a hinge at the top and supported against a sill below. The mass density of water is ρ.

Question:
Model the slide as a line element with a line load, and use the differential equations for equilibrium to find the variation of the shear force and the bending moment.

Solution:
The water pressure on the slide at a depth x is

$$p = \rho g x,$$

in which g is the gravitational field strength. The water pressure increases linearly with the depth.

In Figure 11.12b the slide with width b is modelled as a line element (beam). The support at the base is considered a roller support. At a depth x the load on the slide is

$$q_z = pb = \rho g b x.$$

It holds

$$\frac{d^2 M}{dx^2} = -q_z = -\rho g b x,$$

$$V = \frac{dM}{dx} = -\int q_z \, dx = -\int \rho g b x \, dx = -\tfrac{1}{2}\rho g b x^2 + C_1,$$

$$M = \int V \, dx = \int \left(-\tfrac{1}{2}\rho g b x^2 + C_1\right) dx = -\tfrac{1}{6}\rho g b x^3 + C_1 x + C_2.$$

Due to a linear distributed load, the shear force is a quadratic (parabolic) function in x, and the bending moment is a third degree (cubic) function in x.

The constants C_1 and C_2 follow from the boundary conditions that the bending moment at both the top and the base is zero:

$$x = 0 : M = 0,$$

$$x = h : M = 0.$$

Elaboration of the boundary conditions leads to

$$C_1 = \tfrac{1}{6}\rho g b h^2,$$

$$C_2 = 0.$$

The expressions for the shear force and the bending moment are therefore

$$V = \tfrac{1}{6}\rho g b(h^2 - 3x^2), \tag{a}$$

$$M = \tfrac{1}{6}\rho g b(h^2 x - x^3). \tag{b}$$

The V diagram is a second degree curve (parabola); the M diagram is a third degree curve (cubic).

Both diagrams are shown in Figure 11.13. At A and B tangents to the V and M diagrams are also shown. Note that the tangents to the M diagram intersect at $x = \frac{2}{3}h$, the location where the resultant R of the *triangular load* acts. We will make use of this in the next chapter.

The bending moment is extreme when

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V = 0.$$

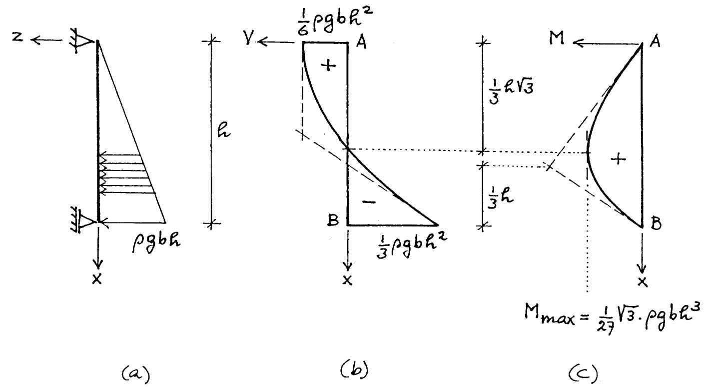


Figure 11.13 (a) The water-retaining slide modelled as a beam with its (b) shear force diagram and (c) bending moment diagram.

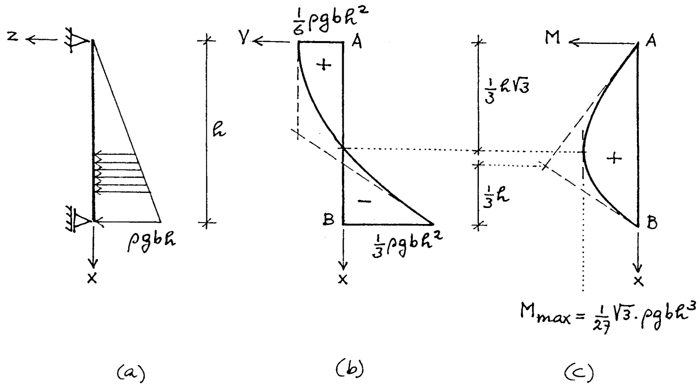


Figure 11.13 (a) The water-retaining slide modelled as a beam with its (b) shear force diagram and (c) bending moment diagram.

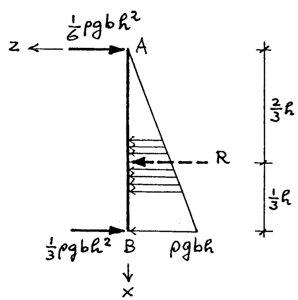


Figure 11.14 The support reactions are found from the V diagram as the shear forces at the member ends.

In other words: the bending moment is extreme where the shear force is zero. This location can be found from (a):

$$V = \tfrac{1}{6}\rho g b(h^2 - 3x^2) = 0 \Rightarrow x = \tfrac{1}{3}h\sqrt{3}.$$

Substituting this value of x in the expression for M gives the maximum bending moment:

$$\begin{aligned} M_{\max} &= M_{(x=\frac{1}{3}h\sqrt{3})} = \tfrac{1}{6}\rho g b\left\{h^2\left(\tfrac{1}{3}h\sqrt{3}\right) - \left(\tfrac{1}{3}h\sqrt{3}\right)^3\right\} \\ &= \frac{\sqrt{3}}{27}\rho g b h^3 = 0.064\rho g b h^3. \end{aligned}$$

The support reactions can be found from the V diagram as the shear forces on the beam ends:

$$x = 0 : V = +\tfrac{1}{6}\rho g b h^2,$$
$$x = h : V = -\tfrac{1}{3}\rho g b h^2.$$

These shear forces on the boundaries of the beam are shown in Figure 11.14. As a check, one can be examine whether the beam as a whole is in equilibrium. The resultant $R = \frac{1}{2}\rho g b h^2$ of the distributed load acts at $x = \frac{2}{3}h$. This gives

$$\sum F_z = R - \tfrac{1}{6}\rho g b h^2 - \tfrac{1}{3}\rho g b h^2 = 0,$$
$$\sum T_y|\mathrm{B} = R \cdot \tfrac{1}{3}h - \tfrac{1}{6}\rho g b h^2 \cdot h = 0.$$

Force and moment equilibrium therefore are satisfied.

Example 4

Cantilever beam ABC in Figure 11.15a is simply supported at A and B, and has an overhang BC at B. The beam is carrying a uniformly distributed load of 40 kN/m over its entire length. The dimensions are given in the figure.

Question:

Using the differential equations for equilibrium, determine the variation of the shear force and the bending moment.

Solution:

The as yet unknown support reaction at B gives a discontinuity in the distributed load on the isolated member. At this point, the differential equations for the equilibrium are not valid (see Section 11.1). The beam therefore has to be split into two parts or *fields*:

- part AB with (0 m) < x < (5 m), hereafter known as field (1).
- part BC with (5 m) < x < (7 m), hereafter known as field (2).

The differential equations for the equilibrium are elaborated per field. For the units that are field-dependent, the field number is used as upper index.

All units are expressed in m and kN. The units are hereafter omitted from the calculation.

For field (1) with (0 m) < x < (5 m):

$$q_z = 40\ \text{kN/m},$$

$$\frac{d^2M^{(1)}}{dx^2} = -q_z = -40\ \text{kN/m},$$

$$V^{(1)} = \frac{dM^{(1)}}{dx} = -\int 40\,dx = (-40x + C_1^{(1)})\ \text{kN},$$

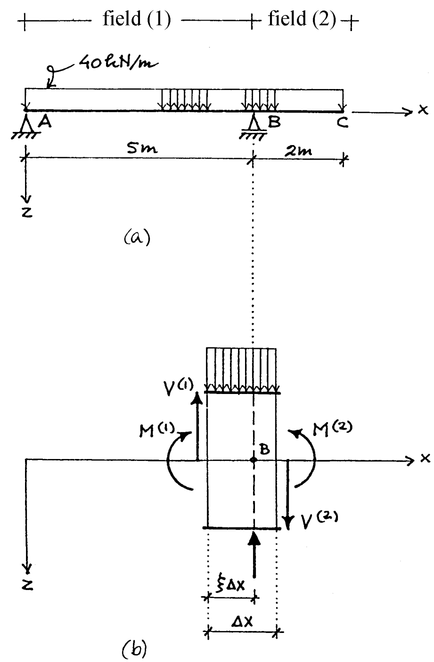


Figure 11.15 (a) For the cantilever beam with a uniformly distributed load along the entire length we have to distinguish two fields. (b) The joining condition, $M^{(1)} = M^{(2)}$, at support B is found from the moment equilibrium of a small beam segment with length $\Delta x \to 0$.

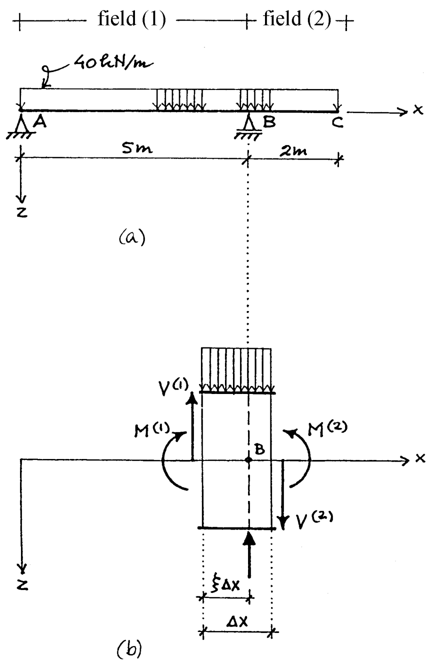


Figure 11.15 (a) For the cantilever beam with a uniformly distributed load along the entire length we have to distinguish two fields. (b) The joining condition, $M^{(1)} = M^{(2)}$, at support B is found from the moment equilibrium of a small beam segment with length $\Delta x \to 0$.

$$M^{(1)} = \int V^{(1)}\,\mathrm{d}x = \int (-40x + C_1^{(1)})\,\mathrm{d}x$$
$$= (-20x^2 + C_1^{(1)}x + C_2^{(1)})\ \text{kNm}.$$

For field (2) with (5 m) $< x <$ (7 m):

$$q_z = 40\ \text{kN/m},$$
$$\frac{\mathrm{d}^2 M^{(2)}}{\mathrm{d}x^2} = -q_z = -40\ \text{kN/m},$$
$$V^{(2)} = \frac{\mathrm{d}M^{(2)}}{\mathrm{d}x} = -\int 40\,\mathrm{d}x = (-40x + C_1^{(2)})\ \text{kN},$$
$$M^{(2)} = \int V^{(2)}\,\mathrm{d}x = \int (-40x + C_1^{(2)})\,\mathrm{d}x$$
$$= (-20x^2 + C_1^{(2)}x + C_2^{(2)})\ \text{kNm}.$$

There are four boundary conditions available for solving the total of four unknown integration constants $C_1^{(1)}$, $C_2^{(1)}$, $C_1^{(2)}$ and $C_2^{(2)}$:

1. end condition at A: $x = 0;\ M^{(1)} = 0.$
2. joining condition at B: $x = 5;\ M^{(1)} = M^{(2)}.$
3. end condition at C: $x = 7;\ V^{(2)} = 0.$
4. end condition at C: $x = 7;\ M^{(2)} = 0.$

For the joining condition at B, we will show below how this can be derived from the equilibrium of a member segment with length Δx ($\Delta x \to 0$) at the joining of the two fields.

Figure 11.15b shows the small member segment with the four section forces acting on it and the unknown support reaction at B. If B is not located in the middle of the element, but at a distance $\xi\Delta x$ from the left-hand section plane, respectively $(1-\xi)\Delta x$ from the right-hand section plane $(0 < \xi < 1)$, then the equation for the moment equilibrium about B is

$$\sum T_y|\mathrm{B} =$$
$$-M^{(1)} + M^{(2)} - V^{(1)}\xi\Delta x - V^{(2)}(1-\xi)\Delta x + q_z\Delta x\left(\xi - \tfrac{1}{2}\right)\Delta x = 0.$$

As $\Delta x \to 0$ all terms with Δx disappear and we are left with

$$M^{(1)} = M^{(2)}.$$

This is the joining condition we are looking for.

The derivation is considerably simpler if, as is standard, the beam element is chosen such that B is in the middle. In that case $\xi = \frac{1}{2}$, and

$$\sum T_y|\mathrm{B} = -M^{(1)} + M^{(2)} - V^{(1)}\tfrac{1}{2}\Delta x - V^{(2)}\tfrac{1}{2}\Delta x = 0.$$

As $\Delta x \to 0$ this again gives the joining condition we are looking for.

Elaboration of the end conditions and the joining condition leads to a set of four equations and four unknowns:

1. end condition at A:
$$C_2^{(1)} = 0.$$
2. joining condition at B:
$$-20 \times 5^2 + C_1^{(1)} \times 5 + C_2^{(1)} = -20 \times 5^2 + C_1^{(2)} \times 5 + C_2^{(2)}.$$
3. end condition at C:
$$-40 \times 7 + C_1^{(2)} = 0.$$
4. end condition at C:
$$-20 \times 7^2 + C_1^{(2)} \times 7 + C_2^{(2)} = 0.$$

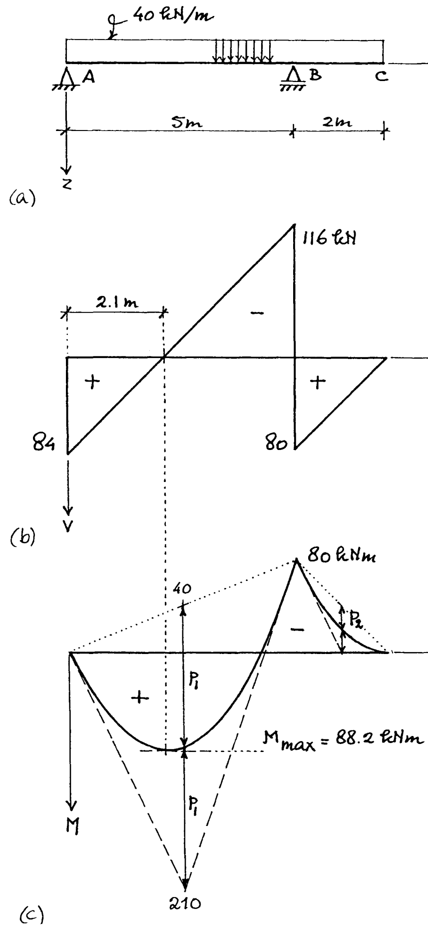


Figure 11.16 (a) A cantilever beam with a uniformly distributed load, and the associated (b) V diagram and (c) M diagram. The bending moment M is extreme where the shear force V is zero or changes sign.

Or more neatly put

$$\begin{aligned} C_2^{(1)} &= 0, \\ 5C_1^{(1)} + C_2^{(1)} - 5C_1^{(2)} - C_2^{(2)} &= 0, \\ C_1^{(2)} &= 280, \\ 7C_1^{(2)} + C_2^{(2)} &= 980. \end{aligned}$$

The solution to the set is

$$C_1^{(1)} = 84 \text{ kN},$$

$$C_2^{(1)} = 0,$$

$$C_1^{(2)} = 280 \text{ kN},$$

$$C_2^{(2)} = -980 \text{ kNm}.$$

From this it follows for field (1) with $(0 \text{ m}) < x < (5 \text{ m})$ that

$$V^{(1)} = (-40x + 84) \text{ kN},$$

$$M^{(1)} = (-20x^2 + 84x) \text{ kNm},$$

and for field (2) with $(5 \text{ m}) < x < (7 \text{ m})$ that

$$V^{(2)} = (-40x + 280) \text{ kN},$$

$$M^{(2)} = (-20x^2 + 280x - 980) \text{ kNm},$$

Figure 11.16 shows the V and M diagrams. At A, B and C, the tangents to the M diagram are also shown. These intersect at the middle of each field.

Note that $p_1 = \frac{1}{8} \times 40 \times 5^2 = 125$ kNm and $p_2 = \frac{1}{8} \times 40 \times 2^2 = 20$ kNm, or in other words, for each field: "$p = \frac{1}{8}q\ell^2$".

The bending moment in field (1) is a maximum where the tangent to the M diagram is horizontal, or where

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V = -40x + 84 = 0 \Rightarrow x = 2.1 \text{ m}.$$

The maximum bending moment therefore occurs to the left of the middle of AB. Substituting $x = 2.1$ in the expression for $M^{(1)}$ gives the value of the maximum bending moment:

$$M_{\max} = -20 \times 2.1^2 + 84 \times 2.1 = 88.2 \text{ kNm}.$$

The support reactions at A and B are shown in Figure 11.17a. Their magnitude and direction can be found directly from the shear force diagram (see Figure 11.17b). This is shown below for the support reaction at B.

Figure 11.17c shows (only) the shear forces directly to the left and right of joint B. The vertical force equilibrium of joint B gives

$$B_\mathrm{v} = 116 + 80 = 196 \text{ kN}.$$

The support at B is carrying 116 kN from the left-hand field and 80 kN from the right-hand field. *The support reaction at B is exactly the same magnitude as the "step change" in the shear force diagram.*

The support reactions derived from the shear force diagram can be checked using the equilibrium of the beam as a whole.

With $R = 7 \times 40 = 280$ kN (see Figure 11.18)

$$\sum F_z = 280 - 84 - 196 = 0,$$

$$\sum T_y|\mathrm{A} = -280 \times 3.5 + 196 \times 5 = 0.$$

The beam as a whole therefore satisfies force and moment equilibrium.

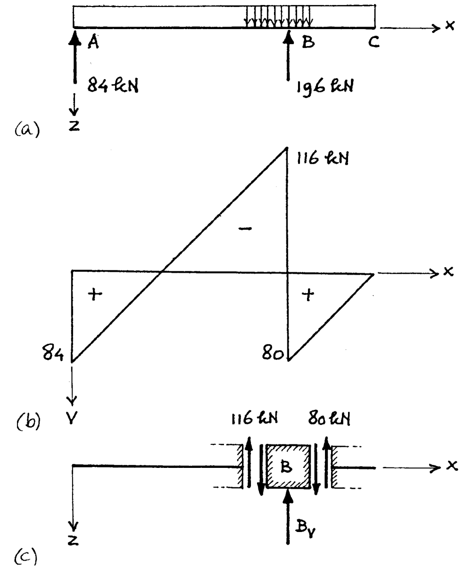


Figure 11.17 (a) The magnitude and direction of the support reactions at A and B follow from (b) the shear force diagram. (c) The shear forces directly to the left and right of joint B. The support reaction at B is the same magnitude as the step change in the shear force diagram.

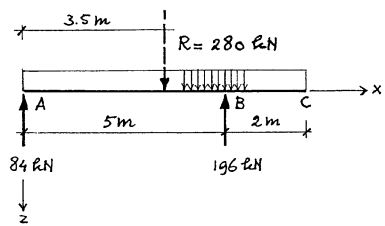


Figure 11.18 The isolated beam with all the forces acting on it.

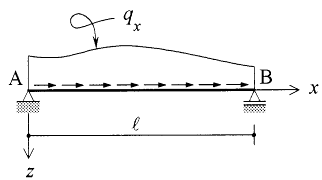
11.4 Problems

Differential equations for the equilibrium (Section 11.1)

11.1 Member AB with length ℓ is subjected to extension by a distributed axial load $q_x = q_x(x)$.

Questions:

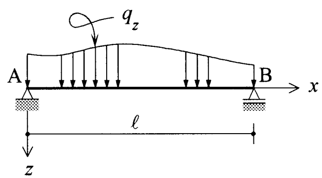
a. Isolate a small segment of length Δx ($\Delta x \to 0$) from the member and draw all the forces acting on it.
b. From the equilibrium of the member segment, derive the relationship between the normal force in the member and the distributed load.



11.2 Beam AB with length ℓ is subjected to bending by a distributed load $q_z = q_z(x)$, normal to the beam axis.

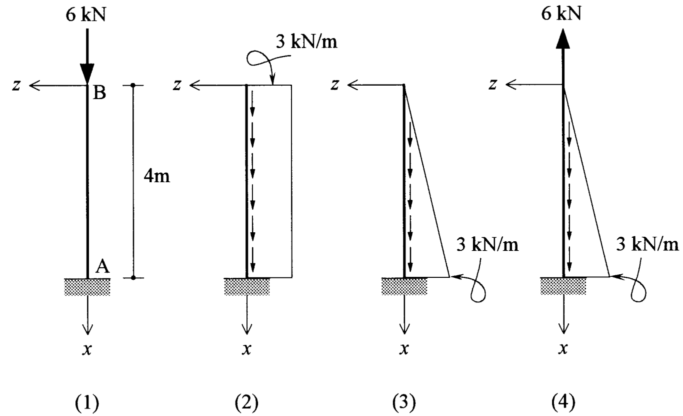
Questions:

a. Isolate a small segment of length Δx ($\Delta x \to 0$) from the member and draw all the forces acting on it.
b. From the equilibrium of the member segment, derive the relationship between the bending moment and the shear force.
c. From the equilibrium of the member segment, derive the relationship between the shear force and the distributed load.
d. From the equilibrium of the member segment, derive the relationship between the bending moment and the distributed load.



Mathematical elaboration of the relationship between N and q_x (extension) (Section 11.2)

11.3: 1–4 A four-metre high column AB is subjected to extension by four different axial loads.



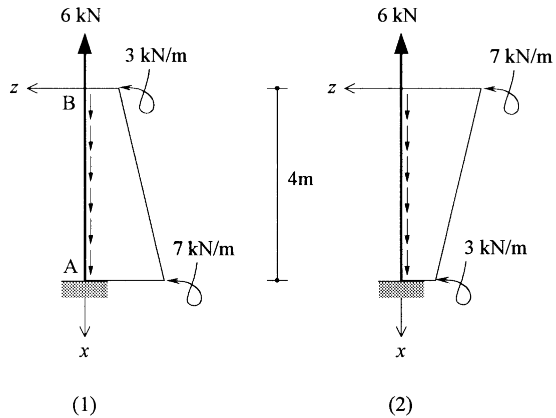
Questions:

a. By integrating the differential equations for the equilibrium, determine the normal force as a function of x, without previously calculating the vertical support reaction at A.
b. Draw the normal force diagram.
c. Calculate the vertical support reaction at A from the equilibrium of the column as a whole and check whether this agrees with the normal force diagram found.

11.4: 1–2 Column AB, 4 m high, is subjected to extension by two different axial loads.

Questions:

a. Write down the distributed load as a function of x.



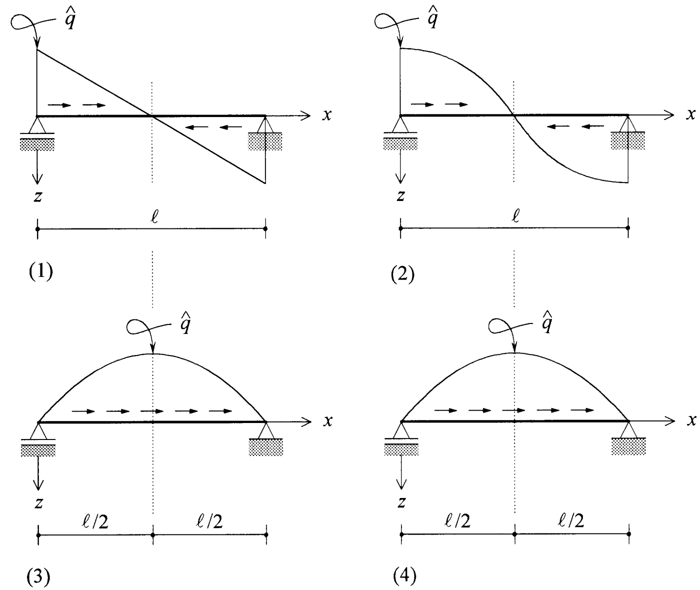
b. Determine the variation of N as a function of x by integration of the differential equation for the equilibrium (without previously calculating the vertical support reaction at A).
c. Draw the N diagram.
d. At which height is the normal force in the column zero?
e. Calculate the vertical support reaction at A from the equilibrium of the column as a whole and check whether this is in agreement with the N diagram found.

11.5: 1–4 A simply supported member with length ℓ is subjected to extension by four different distributed loads $q(x)$ with top value $\hat{q}$:

(1) $q(x) = \hat{q} \cdot \left(1 - 2\dfrac{x}{\ell}\right)$, (2) $q(x) = \hat{q} \cos \dfrac{\pi x}{\ell}$,

(3) $q(x) = 4\hat{q} \cdot \left(\dfrac{x}{\ell} - \dfrac{x^2}{\ell^2}\right)$, (4) $q(x) = \hat{q} \sin \dfrac{\pi x}{\ell}$.

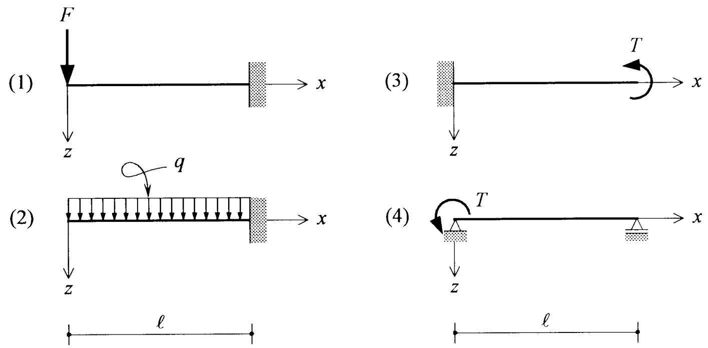
In the calculation use $\ell = 5$ m and $\hat{q} = 2.4$ kN/m.



Questions:
a. Using the differential equation for the equilibrium, determine the variation of N as a function of x.
b. Draw the N diagram. Include the numerical values.
c. Where is N extreme, and what is this extreme value?
d. Determine the support reactions, and draw them as they are really acting on the member.

Mathematical elaboration of the relationship between M, V and q_z ***(bending)*** (Section 11.3)

11.6: 1–4 Four beams subjected to bending.



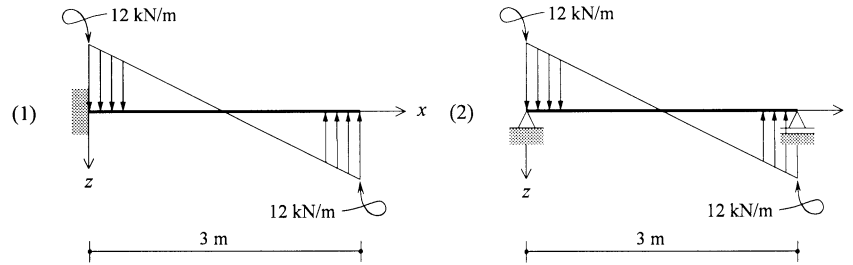
Questions:

a. By integrating the differential equations for the equilibrium, determine the variation of the shear force V and the bending moment M as a function of x, without previously determining the support reactions.
b. Draw the V and M diagrams.
c. Use the V and M diagrams to determine the magnitude and direction of the support reactions. Draw them as they act on the beam and check their values on the basis of the equilibrium of the beam as a whole.

11.7: 1–2 A beam with a linearly distributed load is supported in two different ways.

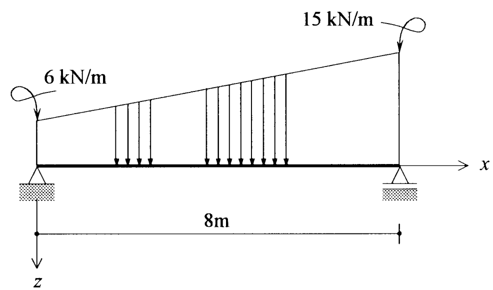
Questions:

a. Write down the distributed load as a function of x.
b. Without previously calculating the support reactions, use the differential equations for the equilibrium to determine the variation of V and M as a function of x.



c. Draw the V and M diagram and include their values and signs.
d. In which cross-sections are V and M extreme, and what are their extreme values?
e. Using the V and M diagrams, determine the magnitude and direction of the support reactions. Draw them as they act on the beam, and check their values on the basis of the equilibrium of the beam as a whole.

11.8 A beam subjected to bending by a trapezoidal load.



Questions:

a. Write down the distributed load as a function of x.
b. Without previously calculating the support reactions, use the differential equations for the equilibrium to determine V and M as a function of x.
c. Draw the V and M diagrams and include their values and signs.

d. In which cross-section is M extreme, and what is its value?
e. Using the V diagram determine the magnitude and direction of the support reactions. Draw them as they act on the beam and check their values on the basis of the equilibrium of the beam as a whole.

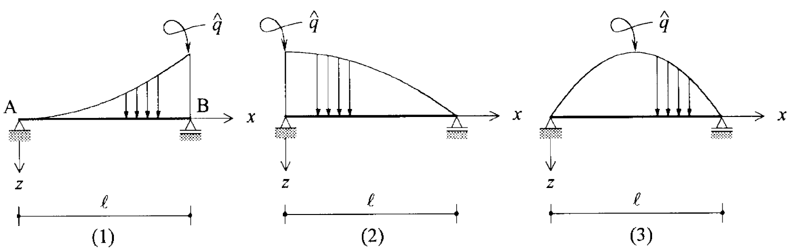
11.9: 1–3 A simply supported beam AB with length ℓ is subjected to bending by three different parabolic distributed loads with the same top value $\hat{q}$:

$$(1)\quad q(x) = \hat{q}\frac{x^2}{\ell^2},$$

$$(2)\quad q(x) = \hat{q}\cdot\left(1 - \frac{x^2}{\ell^2}\right),$$

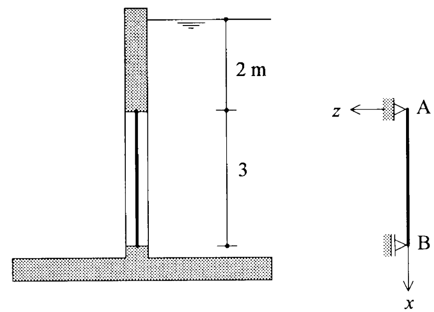
$$(3)\quad q(x) = 4\hat{q}\cdot\left(\frac{x}{\ell} - \frac{x^2}{\ell^2}\right).$$

In the calculation use $\ell = 4$ m and $\hat{q} = 30$ kN/m.



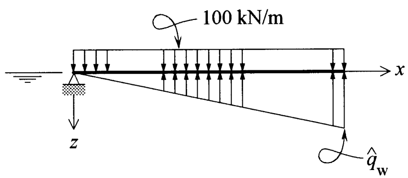
Questions:
a. Determine M and V as a function of x.
b. Draw the M and V diagrams. Include the values and signs.
c. Determine the location and magnitude of the maximum bending moment.
d. Using the V diagram, determine the support reactions at A and B, and draw them as they actually act on the beam.

11.10 An opening in a dam is closed by means of a 3 metre high slide. The top of the slide is two metres below water level. A one metre wide strip from the slide is modelled as the simply supported beam AB. The specific weight of water is 10 kN/m^3.



Questions:
a. Write down the distributed load on AB due to the water pressure as a function of x.
b. Without previously calculating the support reactions, use the differential equations for the equilibrium to determine V and M as a function of x.
c. Draw the V and M diagram and include their values and signs.
d. In which cross-section is M extreme, and what is its value?
e. Using the V diagram, determine the magnitude and direction of the support reactions. Draw them as they act on the beam and check their values on the basis of the equilibrium of beam AB as a whole.

11.11 A 30-m long ship is stranded on rock just below the water level. The figure shows a rough model of the situation. The ship is modelled as a line element with a weight of 100 kN/m. The rock is acting as a hinged support. The upward water pressure on the ship is modelled as line load and varies linearly from zero at the rock to the top value $\hat{q}_w$ at the free-floating end, where the ship is deepest.



Questions:

a. Using the equilibrium of the ship as a whole, determine the value of $\hat{q}_w$.
b. Write down the total distributed load on the ship as a function of x.
c. Use the differential equations for the equilibrium to determine V and M as a function of x.
d. Draw the V and M diagrams, and include the values and signs.
e. In which cross-sections are V and M extreme, and what are their values?
f. Give an assessment of the reality of this model.

12 Bending Moment, Shear Force and Normal Force Diagrams

In this chapter, we will look at how to calculate and draw M, V and N diagrams, with deformation symbols.

The formal approach using differential equations will often become far too laborious. Therefore, we will first present a number of rules in Sections 12.1 and 12.2 on how to draw the M, V and N diagrams more quickly. The rules follow directly from the differential equations for the equilibrium derived in Section 11.1.

In Section 12.3, we present a number of examples in which we calculate and draw the M, V and N diagrams for bent and compound bar structures loaded by concentrated forces and couples. We calculated the support reactions and/or interaction forces for these structures earlier in Chapter 5.

When calculating and drawing M, V and N diagrams, the influences of the various load contributions can be added. The individual contributions can often be found again from the shape of the M, V and N diagrams. We address this *principle of superposition* in Section 12.4.

Concentrated loads, couples and uniformly distributed loads are *models* of the actual load. We will be looking at the consequences of such modelling in Section 12.5. We also have a closer look at the shear force at a support, and investigate the effect of *eccentrically applied axial forces*.

12.1 Rules for drawing V and M diagrams more quickly

The formal approach for determining the variation of shear forces by solving differential equations is rather laborious, certainly if several fields within a member have to be distinguished.

We are often not so much interested in the precise variation of the section forces as a function of x, but rather in the extreme values and the locations where these occur, or the locations where the section forces change direction.

If we want to calculate the section forces in only a few locations, the direct method, whereby we look at the equilibrium of an isolated segment (*method of sections*) is far quicker.

The mathematical approach based on the differential equations for the equilibrium of a small member segment has the advantage that it leads to a number of generally-applicable rules for the relationship between load, shear force and bending moment that can be translated into properties of the V and M diagrams.[1]

In combination with the direct method, these rules allow us to sketch the V and M diagrams, and determine the relevant values quickly. Since the ability to sketch the V and M diagrams and indicate relevant locations and values is extremely important in engineering practice, the direct method is of great practical relevance.

We will look at the rules relating to the V and M diagrams below, using a number of examples. The rules often highlight various sides of one and the same property. In the examples, we will be using deformation symbols.

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[1] The mathematical approach in Chapter 11 is also important at a later stage, when we further develop the theory to be able to determine deformations and displacements.

The calculations are not always performed in their entirety, and the reader is left to complete certain parts.

At the end of this section, we will also look at the properties of a *parabola*, the shape of the M diagram due to a uniformly distributed load.

12.1.1 Relationship between the variation of the distributed load q_z and the shape of V and M diagrams

In Section 11.3 we showed how to find the shear force V from the distributed load q_z by integration:

$$V = -\int q_z \,\mathrm{d}x$$

and the bending moment M by integrating again:

$$M = \int V \,\mathrm{d}x.$$

With a simple variation of the distributed load q_z we can show directly what the *variation* of the shear force and the bending moment will be, and which *shape* the V and M diagrams will have. This leads to the following three rules for an unloaded field ($q_z = 0$), a field with a uniformly distributed load (q_z constant and $\neq 0$) and a field with a linearly distributed load respectively:

• ***Rule 1***
In an unloaded field, the shear force V is constant and the bending moment M is linear. If the shear force is zero, the bending moment is constant.

$$q_z = 0 \Rightarrow V \text{ constant;} \quad \begin{aligned} V &= 0 \Rightarrow M \text{ constant,} \\ V &\neq 0 \Rightarrow M \text{ linear.} \end{aligned}$$

Table 12.1 Relationship between the variation of the distributed load q_z normal to the member axis and the variation of the shear force V and the bending moment M.

Variation q_z ⇒	**variation V ⇒**	**Variation M**
constant $= 0$	constant $= 0$	constant
constant $= 0$	constant $\neq 0$	linear
constant $\neq 0$	linear	quadratic
linear	quadratic	cubic

• ***Rule 2***
In a field with a uniformly distributed load q_z, the shear force V varies linearly, and the bending moment M varies quadratically (parabolic).

$$q_z \text{ constant } (\neq 0) \Rightarrow V \text{ linear } \Rightarrow M \text{ quadratic.}$$

• ***Rule 3***
In a field with a linearly distributed load q_z, the shear force V varies quadratically, and the bending moment M is a cubic function.

$$q_z \text{ linear } \Rightarrow V \text{ quadratic } \Rightarrow M \text{ cubic.}$$

The three rules are summarised in Table 12.1. The correctness of the rules can be verified in Section 11.3, Examples 1 to 3.

Using these rules, it is often possible to draw the V and M diagrams more quickly by determining the values in a limited number of points and then to sketch the path between these points.

The rules are illustrated below by means of two examples.

Example 1
The simply supported beam AD in Figure 12.1a is loaded at B and C by two equal forces of 30 kN.

Question:
Determine the V and M diagrams.

Solution:
First the support reactions at A and D are calculated. Both support reactions turn out to be the same: upward forces of 30 kN (see Figure 12.1b).

V diagram
As far as the V diagram is concerned, it can be said that for each of the unloaded fields AB, BC and CD, the shear force is constant between the

concentrated loads (rule 1). To fully plot the shear force diagram, we have to calculate only three shear forces, namely V^{AB}, V^{BC} and V^{CD}, the shear forces in the fields AB, BC and CD,[1] respectively. These shear forces can be calculated as shown in Section 10.2.1, Example 2, etc.

Shear force V^{BC} is thus found by introducing a section at an arbitrary location in field BC. The magnitude and direction of V^{BC} then follows from the vertical force equilibrium of the isolated part to the right or to the left of the section. In this case the calculation leads to $V^{BC} = 0$. The calculation is left to the reader.

The shear force in the end fields AB and CD is calculated in the same way. The shear forces V^{AB} and V^{CD} in the end fields are of equal magnitude to the support reactions at A and D respectively. Only their directions are different. See Figure 12.1b, which shows the actual directions and the associated deformation symbols. V^{AB} and V^{CD} are therefore plotted in the V diagram on different sides of the member axis.

In the V diagram in Figure 12.1c, the three values calculated are shown by means of dots. Since the shear force is constant, the V diagram can now be completed by drawing a horizontal line in each field through these points.

M diagram

As far as the M diagram is concerned, we know that the bending moment is zero at the supports A and D, and that the bending moment in each field varies linearly (rule 1). To be able to plot the M diagram it is therefore only necessary to calculate the bending moments at the field joinings at B and C after which straight lines can be drawn between the values at A to D.

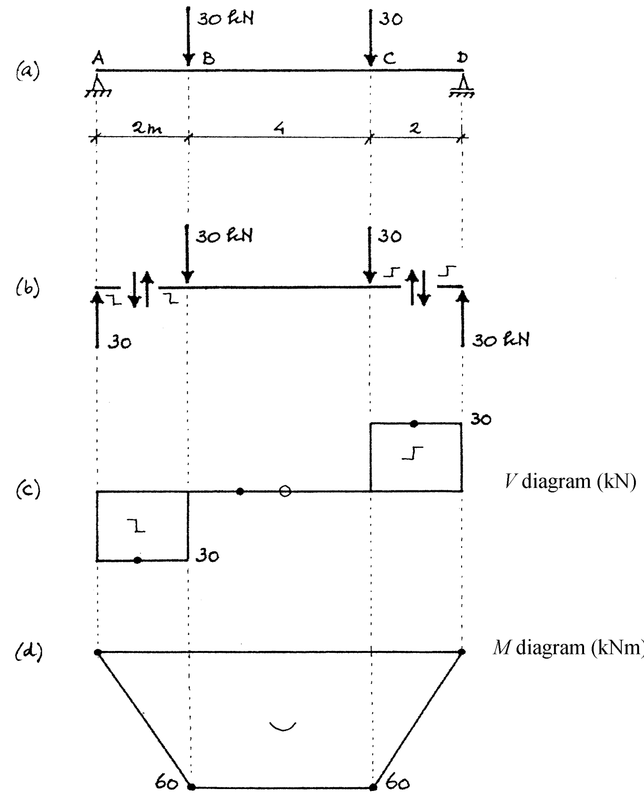


Figure 12.1 (a) Simply supported beam loaded symmetrically by two forces. (b) The support reactions at A and B, and the shear forces in fields AB and CD. (c) Shear force diagram. The shear force is constant in each field. (d) Bending moment diagram. This is fully determined by the values at the boundaries (ends and joinings of the fields).

[1] The upper index refers to the *segment* in which shear force V is acting.

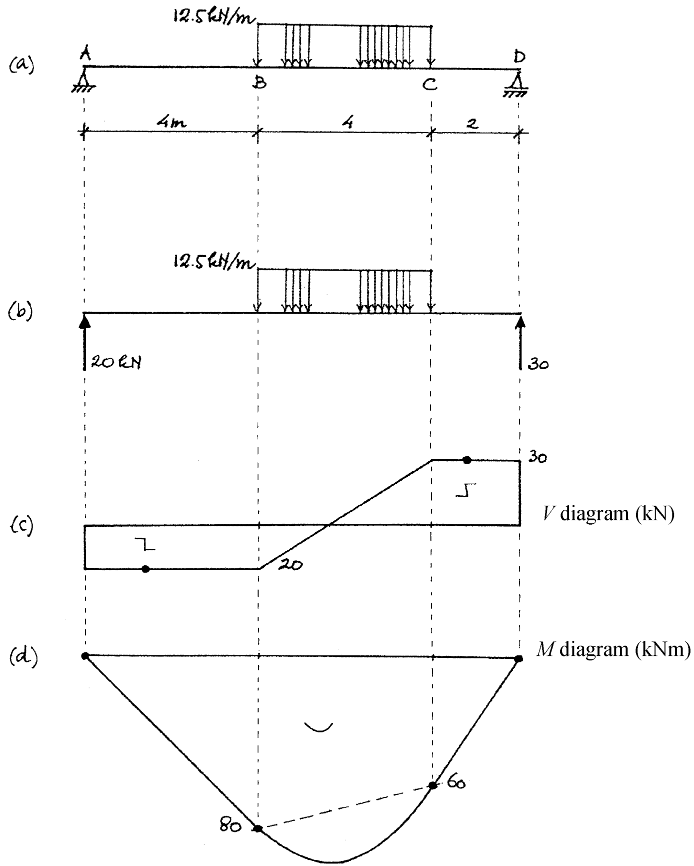


Figure 12.2 (a) Simply supported beam with a uniformly distributed load in field BC. (b) Support reactions. (c) Shear force diagram. The variation of the shear forces is linear under the uniformly distributed load. (d) Bending moment diagram. The bending moment is parabolic under the uniformly distributed load.

The bending moment M_B at B[1] is found from the moment equilibrium (about B) of the isolated part to the left or right of B. This bending moment is 60 kNm; the calculation is left to the reader. As this moment causes tension at the lower side of the beam, this value in the M diagram has to be plotted at the underside of the member axis. See the conventions discussed earlier in Sections 10.2.4 and 10.2.5 and see also Figure 10.40.

In the same way, the bending moment found at C is 60 kNm, also with tension at the lower side of the beam.

In the M diagram in Figure 12.1d, the two known and the two calculated values are shown by means of dots. The M diagram is completed by drawing straight lines between these values.

The M diagram shows that in field BC, where the shear force is zero, the bending moment is indeed constant (according to rule 1).

Example 2

In Figure 12.2a, the simply supported beam AD is subject to a uniformly distributed load of 12.5 kN/m in field BC.

Question:
Determine the V and M diagrams.

Solution:
The support reactions at A and D are 20 and 30 kN respectively, both upwards (see Figure 12.2b).

V diagram
We know that the shear force is constant in field AB (rule 1), varies linearly in field BC (rule 2) and is constant in field CD (rule 1). To draw the shear force diagram we have to calculate only the shear forces V^{AB} and V^{CD} in the end fields. Their magnitudes are the same as the support reactions at A

1 The lower index refers to the *section* in which the bending moment M is acting.

and D respectively.

In the V diagram in Figure 12.2.c, the two calculated shear forces are shown by means of dots. Since the shear forces have a different shear symbol, they are plotted at different sides of the member axis. The V diagram is now completed by means of the horizontal lines in the fields AB and CD, after which the linear path in field BC can be drawn.

M diagram
The bending moment in the beam varies linearly in AB (rule 1), quadratically in BC (rule 2) and linearly in CD (rule 1). At A and D, the bending moment is zero. For the field joinings at B and C, we can calculate that the bending moment is 80 and 60 kNm respectively, both with tension at the lower side of the beam. In the M diagram in Figure 12.2d, these values are plotted downwards of the member axis.

On the M diagram, the four known values are shown by means of dots. The M diagram can now be completed by drawing straight lines between the values at A and B, respectively C and D (linear variation) and by drawing a parabola between the values at B and C (quadratic variation).

More detailed information is required to draw the parabola between B and C somewhat accurately. This is provided in the following subsections, in particular Section 12.1.6.

12.1.2 Slope of the V diagram and M diagram and extreme values of V and M

In Section 11.1, the differential equations for the equilibrium of an infinitesimally small member segment loaded normal to its axis were derived:

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0,$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0.$$

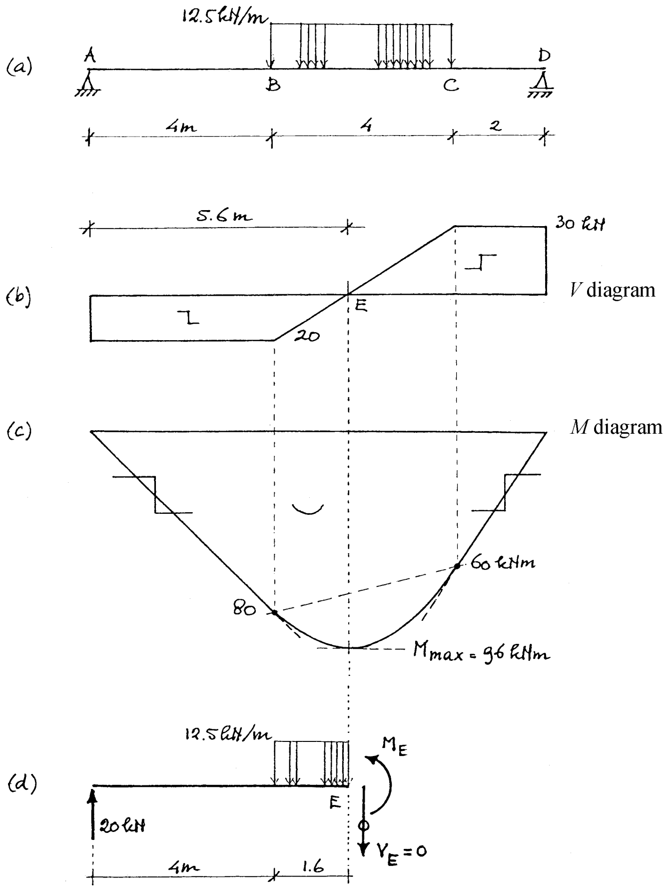


Figure 12.3 (a) Simply supported beam with a uniformly distributed load in field BC. (b) Shear force diagram. (c) Bending moment diagram. The steps in the M diagram agree with the deformation symbols in the V diagram. The bending moment is an extreme where the shear force is zero. (d) The isolated part AE for calculating the maximum bending moment at E.

These equations can also be written as

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q_z, \tag{a}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V. \tag{b}$$

Expressing the differential equations (a) and (b) in words gives the following two rules:

- ***Rule 4***

The slope of the V diagram ($\mathrm{d}V/\mathrm{d}x$) is equal to the distributed load q_z (but with an opposite sign).

- ***Rule 5***

The slope of the M diagram ($\mathrm{d}M/\mathrm{d}x$) is equal to the shear force V.

The correctness of these rules can be verified directly using the two examples in the previous section.

If working with deformation symbols, the direction of the slope of the V or M diagram can no longer be shown by means of plus and minus. We therefore have to work with the absolute values $|\mathrm{d}V/\mathrm{d}x|$ and $|\mathrm{d}M/\mathrm{d}x|$. The directions are deduced from the V and M diagrams respectively.

Example 1
You are given the beam in Figure 12.3a with its V and M diagram in respectively Figures 12.3b and 12.3c.

Questions:
a. Examine whether the M and V diagrams comply with rules 4 and 5.
b. Where is the bending moment an extreme, and how large is that moment?

Solution:
a. In the end fields AB and CD, where the distributed load is zero, the slope

of the V diagram is also zero. This is in line with rule 4.

In the middle field BC the slope of the V diagram is

$$\left|\frac{\Delta V}{\Delta x}\right| = \frac{(20\ \text{kN}) + (30\ \text{kN})}{4\ \text{m}} = 12.5\ \text{kN/m}.$$

This also agrees with rule 4, as it is exactly the value of the uniformly distributed load in field BC.

The slopes of the M diagram in the end fields AB and CD are

$$\left|\frac{\Delta M}{\Delta x}\right|^{(\text{AB})} = \frac{80\ \text{kNm}}{4\ \text{m}} = 20\ \text{kN} = V^{(\text{AB})},$$

$$\left|\frac{\Delta M}{\Delta x}\right|^{(\text{CD})} = \frac{60\ \text{kNm}}{2\ \text{m}} = 30\ \text{kN} = V^{(\text{CD})}.$$

In line with rule 5, the slopes of the M diagram are equal to the shear forces.

As a check for the directions of the shear force, we notice that each "*step*" in the M diagram (which stands for $|\Delta M/\Delta x|$) corresponds to the deformation symbol for the shear force. This check is possible only if one plots the M diagram in accordance with the convention in Section 10.2.4 which requires that the concave side of the bending symbol is faced to the member axis.

The shear forces directly to the left and right of B are equal:

$$V_\text{B}^{(\text{AB})} = V_\text{B}^{(\text{BC})} = 20\ \text{kN}.$$

So, in accordance with rule 5, the slopes of the M diagram directly to the left and right of B are equal. In other words, the straight M path in field AB is the tangent at B to the parabola in field BC. In the same way, the straight M path in field CD is the tangent to the parabola at C.

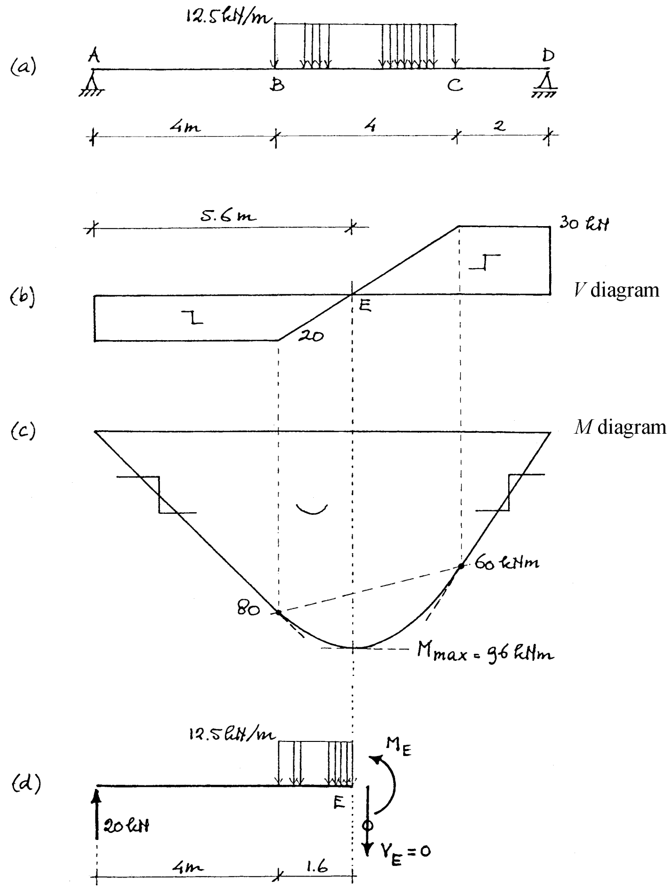


Figure 12.3 (a) Simply supported beam with a uniformly distributed load in field BC. (b) Shear force diagram. (c) Bending moment diagram. The steps in the M diagram agree with the deformation symbols in the V diagram. The bending moment is an extreme where the shear force is zero. (d) The isolated part AE for calculating the maximum bending moment at E.

b. At E, the shear force is zero and the parabola has a horizontal tangent. Here the bending moment is an extreme. The bending moment at E can be calculated from the moment equilibrium of the isolated segment AE or EB. Figure 12.3d shows segment AE. From the bending symbol in the M diagram, we know that the bending moment causes tension at the lower side of the beam; M_E in Figure 12.3d therefore has been plotted at the underside of the beam axis. For AE it holds that

$$\begin{aligned}\sum T|\mathrm{E} \circlearrowleft &= -(20\ \mathrm{kN})(5.6\ \mathrm{m}) + (1.6\ \mathrm{m})(12.5\ \mathrm{kN/m})(0.8\ \mathrm{m}) + M_E \\ &= 0 \end{aligned} \tag{a}$$

so that

$$M_E = (112\ \mathrm{kNm}) - (16\ \mathrm{kNm}) = 96\ \mathrm{kNm}.$$

Since no coordinate system is shown in Figure 12.3d, the positive direction of rotation of the moment about E in expression (a) has been depicted by means of a bent arrow. The positive direction of rotation chosen here is anti-clockwise.

The value of the maximum bending moment at E and the tangents at B, C and E are aids for sketching the parabolic M diagram in field BC (see also Section 12.1.6).

In general, the shear force V has an extreme value where $\mathrm{d}V/\mathrm{d}x = 0$. Also, the bending moment M is an extreme where $\mathrm{d}M/\mathrm{d}x = 0$. This leads to the following two rules.

• ***Rule 6***
The shear force V is an extreme where the distributed load q_z is zero (or changes sign). Per field, we have to take into account that the occurrence of values at the boundaries (e.g. at concentrated loads and supports).

• ***Rule 7***
The bending moment is an extreme where the shear force is zero (or changes sign). Per field, we have to take into account the occurrence of extreme values at the boundaries (e.g. where concentrated loads and/or couples act or at supports).

Example 2
The simply supported beam AB in Figure 12.4a carries a linearly distributed load of 30 kN/m upwards at A to 60 kN/m downwards at B.

Questions:
a. Determine the support reactions.
b. Make a good sketch of the V and M diagrams.
c. Determine the extreme values of V and M.

Solution:
a. To calculate the support reactions, the beam is divided into the fields AC and CB. Per field, the resultant of the triangle load is (see Figure 12.4b)

$$R^{(\mathrm{AC})} = \tfrac{1}{2}(3\ \mathrm{m})(30\ \mathrm{kN/m}) = 45\ \mathrm{kN},$$

$$R^{(\mathrm{CB})} = \tfrac{1}{2}(6\ \mathrm{m})(60\ \mathrm{kN/m}) = 180\ \mathrm{kN}.$$

The vertical support reaction at A ($A_\mathrm{v}\uparrow$) follows from the moment equilibrium of AB about B:

$$\sum T|\mathrm{B}\circlearrowleft\ = -A_\mathrm{v} \times (9\ \mathrm{m}) - 45\ \mathrm{kN})(8\ \mathrm{m}) + (180\ \mathrm{kN})(2\ \mathrm{m}) = 0$$

$$\Rightarrow A_\mathrm{v} = 0.$$

The vertical support reaction in B ($B_\mathrm{v}\uparrow$) follows from the vertical force equilibrium of AB:

$$\sum F_{\mathrm{vert}}\uparrow = (45\ \mathrm{kN}) - (180\ \mathrm{kN}) + B_\mathrm{v} = 0 \Rightarrow B_\mathrm{v} = 135\ \mathrm{kN}(\uparrow).$$

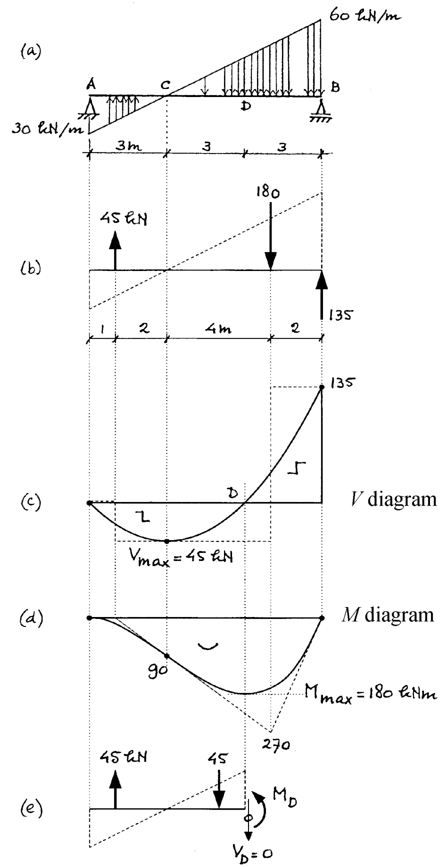


Figure 12.4 (a) Simply supported beam with linear distributed load. (b) The isolated beam with the distributed loads in field AC and CB replaced by their resultant. The support reaction at A is zero and at B is 135 kN. (c) Shear force diagram. (d) Bending moment diagram. (e) The isolated part AD for calculating the maximum bending moment at D.

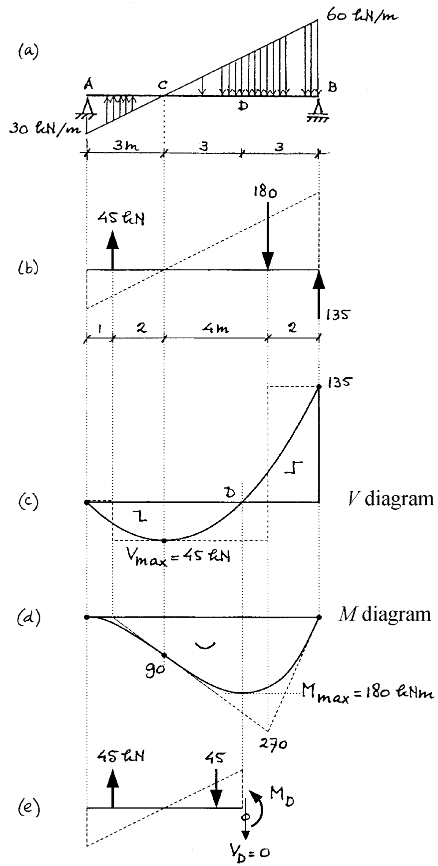


Figure 12.4 (a) Simply supported beam with linear distributed load. (b) The isolated beam with the distributed loads in field AC and CB replaced by their resultant. The support reaction at A is zero and at B is 135 kN. (c) Shear force diagram. (d) Bending moment diagram. (e) The isolated part AD for calculating the maximum bending moment at D.

b. For the beam in Figure 12.4b, loaded only by concentrated loads, calculating and drawing the V and M diagrams is relatively simple using rule 1 from the previous section. The calculation is left to the reader. The result is shown in Figures 12.4c and 12.4d by means of a dashed line.

The dashed lines do not give the correct V and M diagrams, but the values of V and M at the field boundaries are correct! They are shown in Figures 12.4c and 12.4d by means of dots. If we isolate a field to calculate the shear force and the bending moment in the field boundaries, it does not matter in the equilibrium equations whether we consider the actual load in the field or its resultant.

The actual V and M diagrams therefore pass through the dots at the field boundaries.

Since the shear forces at the field boundaries A, C and B in the dashed case have the correct values, the slopes of the M diagram in the dashed case are also correct. This means that the dashed M diagram in Figure 12.4d at A, C and B is tangent to the actual M diagram.

According to rule 3, the shear force in Figure 12.4c varies parabolically. Since at C the distributed load is zero, the tangent to the V diagram is horizontal there. The V diagram is now easy to draw.

According to rule 3, the M diagram in Figure 12.4d is cubic. It passes through the three black dots and has the dashed lines at the dots as *tangents*. After this it is not difficult to sketch the M diagram.

c. The shear force is an extreme where the distributed load is zero (rule 6). Isolating section AC or CB the vertical force equilibrium gives

$$V_{\max} = 45 \text{ kN}.$$

However, the largest shear force occurs at support B and is 135 kN (with opposite sign). This is an example of a *maximum at a field boundary*.

The maximum bending moment occurs where the shear force is zero, in D. The V diagram varies parabolically. The top of the parabola is at C. Since distances AC and CD are equal, the distance from D to A is therefore 6 m.

The maximum bending moment at D can be calculated from the moment equilibrium of isolated parts AD or DB. In Figure 12.4e, part AD has been isolated, and the two triangular loads have been replaced by their resultants. From the deformation symbol in the M diagram we observe tension at the underside of the beam due to the bending moment M_D at D. In Figure 12.4e, M_D is shown in accordance with this direction. For AD it holds that

$$\sum T|\mathrm{D} \circlearrowleft = -(45\text{ kN})(5\text{ m}) + (45\text{ kN})(1\text{ m}) + M_\mathrm{D} = 0$$

so that

$$M_\mathrm{D} = M_\mathrm{max} = 180\text{ kNm}.$$

It is up to the reader to check that the same maximum bending moment is found from the equilibrium of part DB.

12.1.3 Tangents to the M diagram

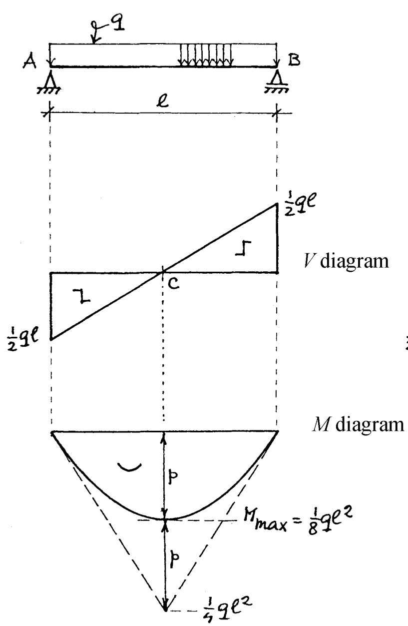
When drawing the V and M diagrams in Example 2 from the previous section, two new rules occur:

- ***Rule 8***

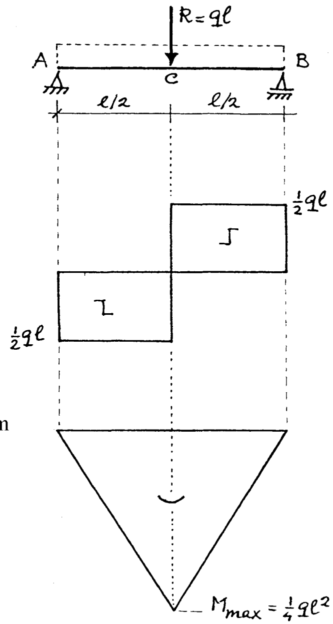
The tangents to the M diagram at the boundaries of a field intersect on the line of action of the load resultant in that field (for a distributed load this is at the centroid of the load diagram).

- ***Rule 9***

If we replace the load per field by its resultant and we draw the bending moment diagram due to these resultants, this bending moment diagram is tangent to the actual bending moment at the field boundaries.



(12.5)



(12.6)

Figure 12.5 Simply supported beam with V and M diagrams due to a uniformly distributed full load.

Figure 12.6 Simply supported beam with the V and M diagrams due to the resultant of a uniformly distributed full load.

We will illustrate this with three examples.

Example 1

The first example relates to a simply supported beam AB with length ℓ. In Figure 12.5, the beam is over its entire length subjected to a uniformly distributed load q. In Figure 12.6, the beam is subjected to a concentrated force $R = q\ell$ at midspan C, to be seen as the resultant of the previously mentioned uniformly distributed load q.

Question:
Verify the rules 8 and 9.

Solution:
In the case of Figure 12.5, the shear force varies linearly and the bending moment varies parabolically. At midspan C, it applies that

$$V = \frac{\mathrm{d}M}{\mathrm{d}x} = 0.$$

Here the M diagram has a horizontal tangent. This means that the bending moment M at C is a maximum (rule 7):

$$M_{\mathrm{max}} = \tfrac{1}{8}q\ell^2.$$

In the case in Figure 12.6, the shear force at C is not zero (as is often wrongly said), but rather changes sign. Here too the bending moment is a maximum (rule 7):

$$M_{\mathrm{max}} = \tfrac{1}{4}R\ell = \tfrac{1}{4}q\ell^2.$$

This maximum bending moment, twice as large as in the case of the uniformly distributed load, is a *boundary extreme* at the joining of fields AC and BC.

If the distributed load q in Figure 12.5 is replaced by its resultant R in Figure 12.6, the V and M diagrams change. The values at A and B do not change. Since the shear forces at A in both cases are equal, the slopes of the M diagram at A are also equal in both cases. The same holds for B.

It is now clear that the tangents at A and B for the distributed load can be found by drawing the M diagram due to the resultant R. Both tangents intersect, in accordance with rule 8, at the middle of AB, on the line of action R.

For the value p indicated in the M diagram,

$$p = \tfrac{1}{8} q \ell^2.$$

Note: This expression for p holds only if the load is *uniformly distributed*.

Example 2

Draw the V and M diagrams for the simply supported beam in Figure 12.7a, with the deformation symbols. See also Section 12.1.1, Example 2.

Solution:

If the distributed load $q = 12.5$ kN/m is replaced by its resultant $R = 50$ kN at the middle G of BC, then the V and M diagrams change only between B and C. The change over BC of V and M is shown by means of the dashed lines in Figures 12.7b and 12.7c.

The actual V diagram in field BC varies linearly (rule 2). This is shown in Figure 12.7b.

The resultant R gives the same shear force (20 kN) and bending moment (80 kNm) at B as the uniformly distributed load q. Since the shear force can be interpreted as the slope of the M diagram, this means that at B the M diagrams due to R and q have the same slope. The same applies at C.

The actual M diagram in field BC varies parabolically. At B and C the parabola is tangent to the dashed M diagram due to the resultant R (rule 9).

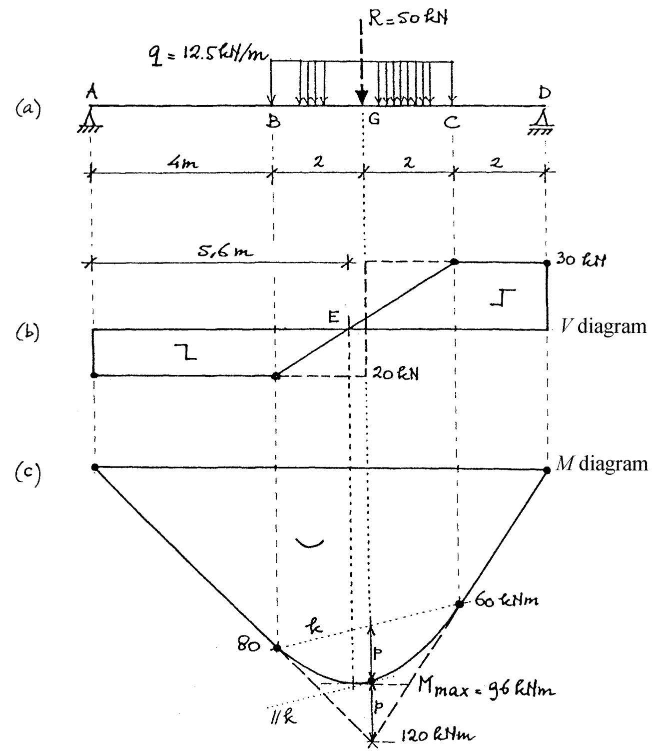


Figure 12.7 (a) Simply supported beam with uniformly distributed load in field BC. (b) Shear force diagram. The dashed V diagram in field BC corresponds with the resultant R of the distributed load. (c) Bending moment diagram. The dashed M diagram in field BC corresponds with the resultant R of the distributed load.

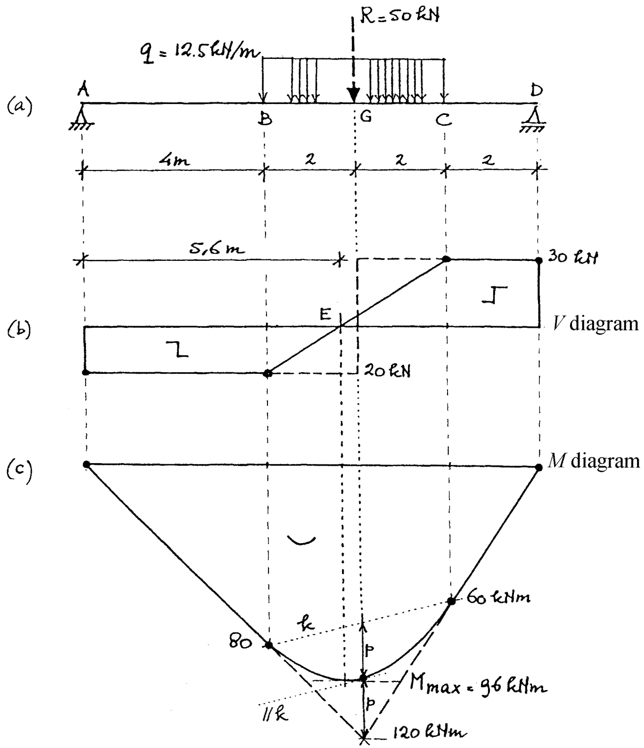


Figure 12.7 (a) Simply supported beam with uniformly distributed load in field BC. (b) Shear force diagram. The dashed V diagram in field BC corresponds with the resultant R of the distributed load. (c) Bending moment diagram. The dashed M diagram in field BC corresponds with the resultant R of the distributed load.

The tangents at B and C intersect on the line of action of resultant R, in the middle of BC (rule 8).

The parabola in field BC is the same parabola as in Example 1, except that it is now "*obliquely suspended*" between the values of 80 and 60 kNm at B and C respectively. The tangent midway of the parabola is parallel to chord k.

As in Example 1, in the middle of field BC, the value of p in the M diagram is

$$p = \tfrac{1}{8}q\ell^2 = \tfrac{1}{8}(12.5\text{ kN/m})(4\text{ m})^2 = 25\text{ kNm}.$$

Here ℓ represents the length of field BC.

It is up to the reader to check whether the distance p from the bend in the dashed M diagram to the actual M diagram, and from there to the dotted chord between the values at B and C is indeed 25 kNm.

It should be noted that the bending moment is not a maximum in the middle of BC, $x = 6$ m. The maximum occurs somewhat to the left of the middle ($x = 5.6$ m), where the shear force is zero (rule 7).

Example 3

A uniformly distributed load of 50 kN/m is acting on the simply supported beam AC in Figure 12.8a, downwards in field AB and upwards in field BC.

Questions:

a. Draw a good sketch of the V diagram, with the deformation symbols. Where is the shear force an extreme?

b. Draw a good sketch of the M diagram, with the deformation symbols. At relevant points, also draw the tangents to the M diagram. Where is the bending moment an extreme?

Solution:

a. Before drawing the shear force diagram and the bending moment diagram, we first have to determine the support reactions. The support reactions at A and C are respectively 170 kN (↑) and 70 kN (↓) (see Figure 12.8b).

The shear force varies linearly in both field AB and field BC (rule 2). To draw the V diagram we therefore have to calculate only the shear forces at A, B and C. The shear forces at A and C are equal to the support reactions at A and C. Beware of the deformation symbols! The shear force at B is found from the vertical force equilibrium of AB or BC.

Figure 12.8c shows the V diagram. The values calculated are shown by means of a dot.

At B, the distributed load changes sign, and the shear force is an extreme (rule 6). However, the maximum shear force occurs at the location of support A (a boundary maximum).

b. In order to draw the M diagram with its tangents, the distributed loads in fields AB and BC are replaced by their resultants (see Figure 12.8b). The bent M diagram due to these resultants is shown by means of the dashed path in Figure 12.8d. At A, B and C, the dashed M diagram is tangent to the actual M diagram (rule 9). Per field, the actual M diagram varies parabolically (rule 2).

If, in the middle of a field, one halves the distance between the maximum value of the dashed M diagram to the chord of the parabola, this gives an additional point on the parabola. At that point, the tangent to the parabola is parallel to the chord. Using this information, it is possible to make a very accurate free-hand sketch of the M diagram. The result is shown in Figure 12.8d.[1]

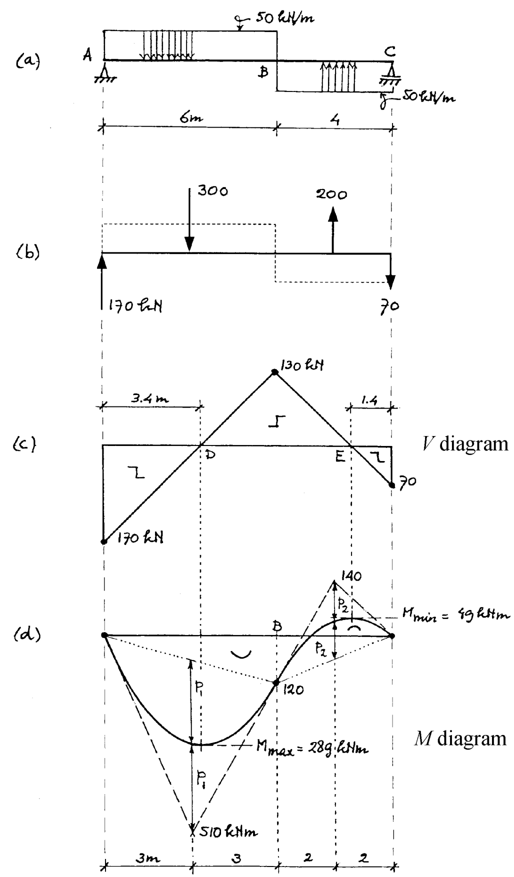


Figure 12.8 (a) Simply supported beam with an abruptly changing uniformly distributed load at B. (b) The isolated beam with its support reactions and the distributed loads in field AB and BC replaced by their resultants. (c) Shear force diagram. (d) Bending moment diagram. At the field boundaries A, B and C, this diagram is tangent to the dashed M diagram due to the load resultants.

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1 The tangents in the middle of the fields (parallel to the chords) are not shown, to ensure the figure remains somewhat legible.

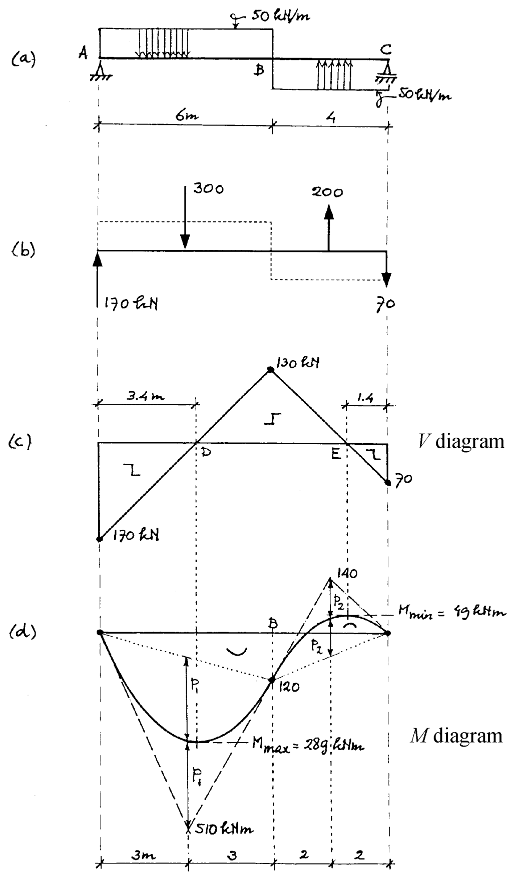


Figure 12.8 (a) Simply supported beam with an abruptly changing uniformly distributed load at B. (b) The isolated beam with its support reactions and the distributed loads in field AB and BC replaced by their resultants. (c) Shear force diagram. (d) Bending moment diagram. At the field boundaries A, B and C, this diagram is tangent to the dashed M diagram due to the load resultants.

The correctness of the M diagram can be checked using the values of p_1 and p_2:

$$p_1 = \tfrac{1}{8}(50\ \text{kN/m})(6\ \text{m})^2 = 225\ \text{kNm},$$

$$p_2 = \tfrac{1}{8}(50\ \text{kN/m})(4\ \text{m})^2 = 100\ \text{kNm}.$$

It is up to the reader to perform this check.

The bending moment is an extreme where the shear force is zero (rule 7), that is at D and E. These extreme values can be calculated from the moment equilibrium of the isolated part to the left or right of a section at respectively D and E.

In Section 12.1.4 we present an alternative method for calculating the maximum shear force and the maximum bending moment.

12.1.4 Interpreting the area of the load diagram and V diagram

The differential equations for the equilibrium of an infinitesimally small beam segment are

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q_z, \tag{a}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V. \tag{b}$$

This can also be written as

$$\mathrm{d}V = -q_z\,\mathrm{d}x, \tag{c}$$

$$dM = V\,dx. \tag{d}$$

Integrating expression (c) over the interval between x_1 and x_2 gives

$$\int_{x_1}^{x_2} dV = -\int_{x_1}^{x_2} q_z\,dx$$

so that

$$\Delta V = V(x_2) - V(x_1) = -\int_{x_1}^{x_2} q_z\,dx.$$

Expressed in words and ignoring the signs, this leads to rule 10:

• ***Rule 10***
Without concentrated forces,[1] the change in the shear force V over a certain length is equal to the area of the load diagram over that length.

Integrating expression (d) over the interval between x_1 and x_2 gives

$$\int_{x_1}^{x_2} dM = \int_{x_1}^{x_2} V\,dx$$

so that

$$\Delta M = M(x_2) - M(x_1) = \int_{x_1}^{x_2} V\,dx.$$

Expressed in words and ignoring the signs, this leads to rule 11:

[1] If there are acting concentrated couples, the differential equation (a) is no longer valid.

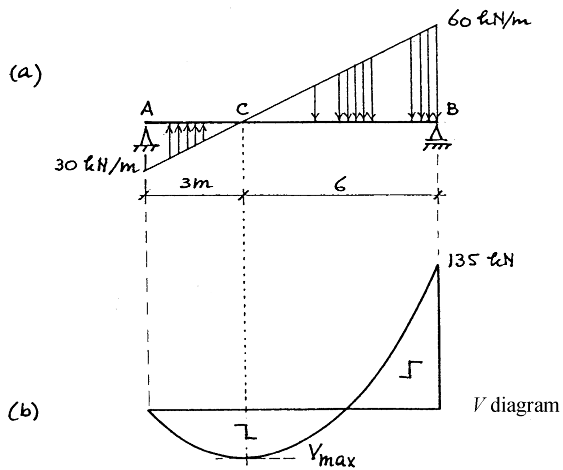


Figure 12.9 (a) Simply supported beam with linear distributed load and (b) the associated shear force diagram. The shear force is an extreme where the distributed load is zero – not taking into account the extremes at the field boundaries.

• ***Rule 11***

Without concentrated couples,[1] the change in the bending moment over a certain length equals the area of the V diagram over that length.

Rules 10 and 11 are demonstrated using two examples.

Example 1

The V diagram in Figure 12.9b for the beam in Figure 12.9a was previously calculated in Section 12.1.2, Example 2. The maximum shear force – boundary extremes not considered – occurs at C and can be calculated directly from the force equilibrium of AC or BC.

Question:
Determine the maximum shear force at C from the area of the load diagram.

Solution:
From the V diagram we can read that

$$\Delta V^{(\mathrm{AC})} = V_\mathrm{C} - V_\mathrm{A} = V_{\max}.$$

$\Delta V^{(\mathrm{AC})}$ is equal to the area of the load diagram over AC:

$$V_{\max} = \Delta V^{(\mathrm{AC})} = \tfrac{1}{2}(3\text{ m})(30\text{ kN/m}) = 45\text{ kN}.$$

Example 2

For the beam in Figure 12.10a, the V and M diagrams in Figures 12.10b and 12.10c were calculated earlier in Section 12.1.3, Example 3.

Question:
Calculate the extreme bending moments from the area of the shear force diagram.

---

[1] If there are acting concentrated couples, the differential equation (b) is no longer true.

Solution:
The bending moments are extreme where the shear force is zero, that is at D and E.

The *maximum* bending moment occurs at D. From the M diagram, with $M_{\mathrm{D}} = M_{\max}$ and $M_{\mathrm{A}} = 0$, we can read that

$$\Delta M^{(\mathrm{AD})} = M_{\mathrm{D}} - M_{\mathrm{A}} = M_{\max}.$$

$\Delta M^{(\mathrm{AD})}$ is equal to the area of the shear force diagram over AD:

$$M_{\max} = \Delta M^{(\mathrm{AD})} = \tfrac{1}{2}(3.4\ \mathrm{m})(170\ \mathrm{kN}) = 289\ \mathrm{kNm}.$$

The *minimum* bending moment occurs at E, and is found in the same way from the area of the shear force diagram over EC:

$$M_{\min} = \Delta M^{(\mathrm{EC})} = \tfrac{1}{2}(1.4\ \mathrm{m})(70\ \mathrm{kN}) = 49\ \mathrm{kNm}.$$

Check:

$$\Delta M^{(\mathrm{DE})} = M_{\min} + M_{\max} = 338\ \mathrm{kNm}.$$

This must be equal to the area of the V diagram over DE:

$$\Delta M^{(\mathrm{DE})} = \tfrac{1}{2}(5.2\ \mathrm{m})(130\ \mathrm{kN}) = 338\ \mathrm{kNm},$$

which is indeed the case.

From the above, and taking into account rule 11, we discover another property:

• ***Rule 12***

a. For a beam without concentrated couples, the total area of the V diagram is zero. More generally

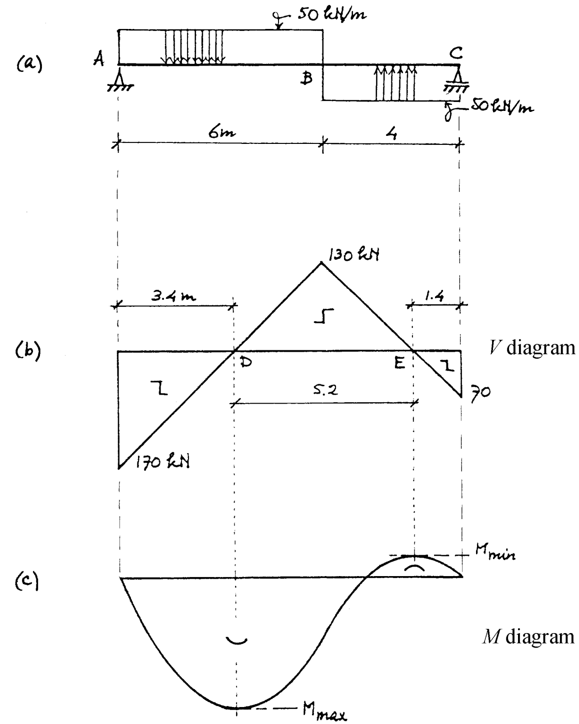


Figure 12.10 (a) Simply supported beam with a step change in the uniformly distributed load at B. (b) Shear force diagram. The shear force is an extreme where the distributed load changes sign. (c) Bending moment diagram. The bending moments are extreme where the shear force is zero.

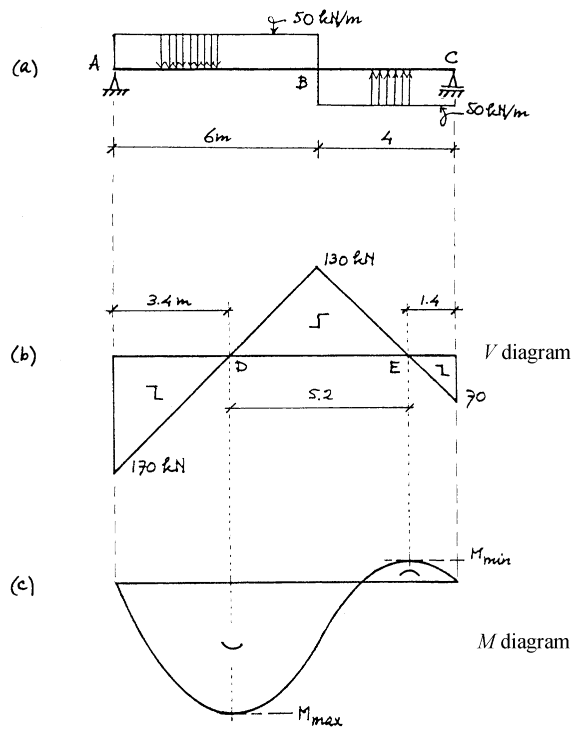


Figure 12.10 (a) Simply supported beam with a step change in the uniformly distributed load at B. (b) Shear force diagram. The shear force is an extreme where the distributed load changes sign. (c) Bending moment diagram. The bending moments are extreme where the shear force is zero.

b. The total area of the V diagram is equal to the sum of moments of all concentrated couples that act on the beam.

Example 3
Check rule 12a for the beam in Figure 12.10.

Solution:
For the beam simply supported at A and C, without concentrated couples, it applies that

$$\Delta M^{(\mathrm{AC})} = M_{\mathrm{C}} - M_{\mathrm{A}} = 0.$$

$\Delta M^{(\mathrm{AC})}$ is equal to the total area of the V diagram:

$$\tfrac{1}{2}(3.4\text{ m})(170\text{ kN}) - \tfrac{1}{2}(5.2\text{ m})(130\text{ kN}) + \tfrac{1}{2}(1.4\text{ m})(70\text{ kN}) = 0 \quad \text{(b)}$$

which is indeed zero.

Comment: When we calculate the area of the V diagram, the minus sign in expression (b) indicates that the deformation symbol for the shear force in field DE is opposite to that in the rest of the beam. The plus and minus signs alone here are arbitrary: they could just as well be interchanged one another. For this reason, we generally look at the *absolute value* of the area of the V diagram.

If there are (concentrated) couples acting on the beam, the (absolute value of the) total area of the V diagram is equal to the sum of the moments of the (concentrated) couples. This rule is closely bound up with the modelling of the couple as discussed in Section 12.5.3.

12.1.5 Step changes and bends in the V and M diagrams

• ***Rule 13***
A step change in the distributed load q_z gives a bend in the V diagram (and a point of inflection in the M diagram).

For rule 13, please refer to Figure 12.10. In both fields, the slope of the V diagram is 50 kN/m. The slopes are opposite as the distributed loads in the fields are opposite (rule 4). This causes a bend in the V diagram.

- ***Rule 14***

A (concentrated) force F normal to the member axis generates a step change in the V diagram, with magnitude F, and a bend in the M diagram.

- ***Rule 15***

A (concentrated) couple T gives a step change in the M diagram of magnitude T. The V diagram reveals no information about the point of application of the couple.

Two examples are given below to illustrate this.

Example 1

The simply supported beam in Figure 12.11a is loaded by a force of 40 kN and a (concentrated) couple of 80 kNm.

Questions:

a. Calculate and draw the V and M diagrams, with the deformation symbols.
b. Explain the step change in the V and M diagrams from the equilibrium of joints B and C respectively.
c. To what extent can rules 11 and 12 be applied here?

Solution:

a. The support reactions at A and B are both 20 kN (↑), see Figure 12.11b.

In principle, when drawing the V and M diagrams, we have to distinguish three fields: AB, BC and CD. There is no distributed load, so that the shear force is constant, and the bending moment varies linearly in all three fields (rule 1).

In the end fields AB and CD, the shear force is equal to the support reactions at A and D respectively. Beware of the deformation symbols!

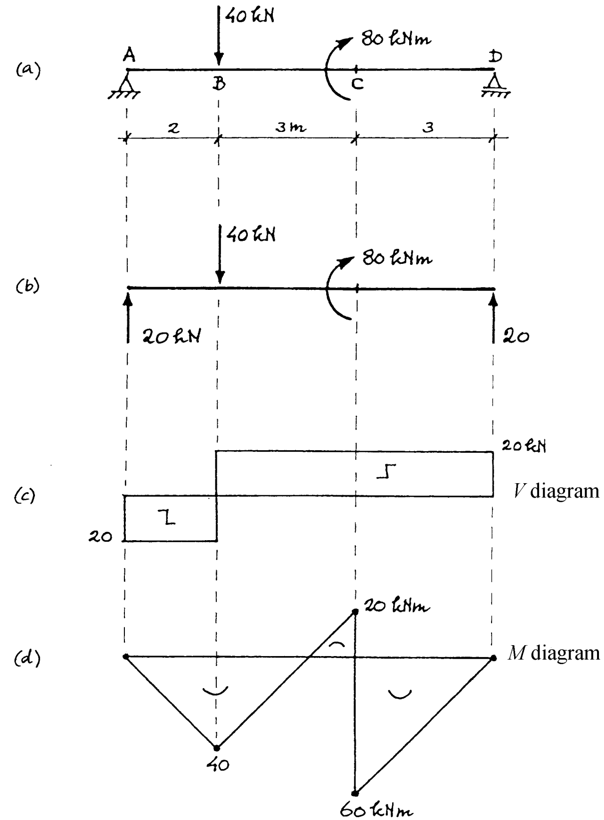


Figure 12.11 (a) Simply supported beam loaded by a concentrated force and a couple. (b) The isolated beam and its support reactions. (c) Shear force diagram. A step change occurs at the point of application of the concentrated force. The point of application of the couple cannot be derived from the V diagram. (d) Bending moment diagram. A bend occurs at the point of application of the concentrated force, and a step change occurs where the couple is applied.

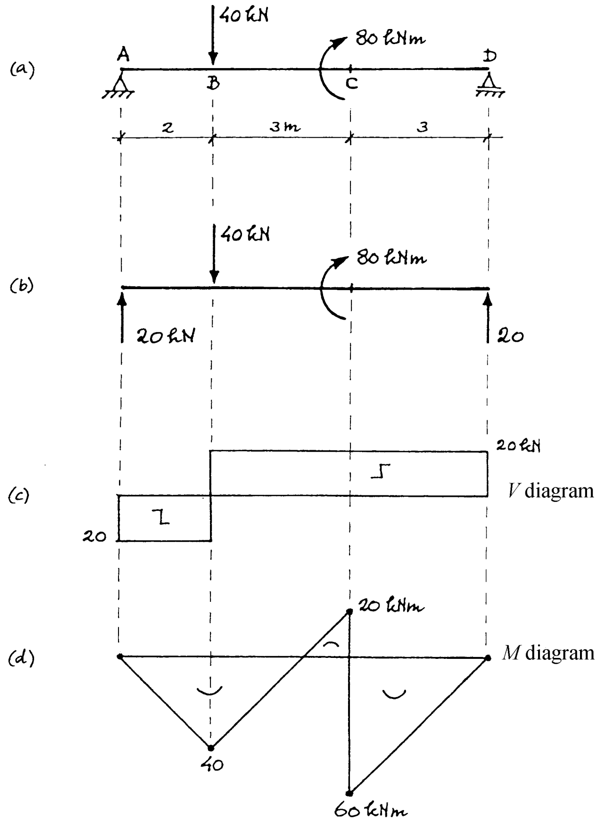


Figure 12.11 (a) Simply supported beam loaded by a concentrated force and a couple. (b) The isolated beam and its support reactions. (c) Shear force diagram. A step change occurs at the point of application of the concentrated force. The point of application of the couple cannot be derived from the V diagram. (d) Bending moment diagram. A bend occurs at the point of application of the concentrated force, and a step change occurs where the couple is applied.

To find the shear force in field BC, we have to introduce a section somewhere in field BC and investigate the vertical force equilibrium of the isolated part to the left or to the right of the section. If we write down the expression for the force equilibrium of the right-hand part, the couple of 80 kNm plays no part in this expression! The shear force in BC is therefore equal to the shear force in CD and is 20 kN.

Figure 12.11c shows the V diagram. The V diagram reveals no information about the point of application of the couple (rule 15). At B, at the location of the 40 kN load, a step change occurs in the V diagram. The magnitude of the step change is equal to the magnitude of the concentrated load.

To draw the M diagram, we have to know the values at the field boundaries A, B, C and D. Between them, the bending moment varies linearly. At A and D the bending moment is zero.

The bending moment at B is calculated from the moment equilibrium of the isolated part AB or BD. The result is 40 kNm with tension at the underside.

In the same way, the bending moment at C is calculated from the moment equilibrium of the isolated part AC or CD. But beware: due to the 80 kNm couple it makes a difference whether we take the section directly to the left or directly to the right of C. In the section directly to the left of C, we find 20 kNm with tension at the upper side, and in the section directly to the right of C we find 60 kNm with tension at the underside.

Figure 12.11d shows the M diagram. The values calculated at the field boundaries are shown by means of dots. Between these values, the moment varies linearly. At C, the point of application of the 80 kNm couple, there is a step change in the M diagram. The magnitude of the step change is equal to the magnitude of the (concentrated) couple.

Note that because the shear force over BCD is constant, the M diagram to the left and to the right of C have the same slope (rule 5).

At B, the shear force changes sign, and the M diagram is an extreme (rule

7). We cannot however see from the V diagram that the M diagram also has extreme values directly to the left and to the right of C, the point where the couple is applied. The maximum bending moment in an absolute sense occurs in the section directly to the right of C and is 60 kNm.

b. In Figure 12.12a, joint B has been isolated. In the sections, only the shear forces are shown. The shear forces directly to the left and right of B have to be in equilibrium with the vertical force on the joint. From this, it follows that the step change in the V diagram must be equal to the force on the joint.

In Figure 12.12b, joint C has been isolated. In the sections, only the bending moments are shown. From the moment equilibrium of joint C it follows that the bending moments directly to the left and to the right of C have to be in equilibrium with the couple at C. The magnitude of the step change in the M diagram must therefore be equal to the magnitude of the couple.

c. Rule 11, states that the change ΔM of the bending moment M is equal to the area of the V diagram. This rule is still valid as long as no couples are acting in the field that is considered.

If one looks at part AC of the beam, to the left of the couple, we could, for example, determine M_{B} and $M_{\text{C;left}}$ from the area of the V diagram. With $M_{\text{A}} = 0$ we find

$$M_{\text{B}} = \Delta M^{(\text{AB})} = (2\ \text{m})(20\ \text{kN}) = 40\ \text{kNm},$$

$$M_{\text{C;left}} = \Delta M^{(\text{AC})} = (2\ \text{m})(20\ \text{kN}) - (3\ \text{m})(20\ \text{kN}) = -20\ \text{kNm}.$$

In the latter case, because the V diagrams over AB and BC have different deformation symbols, the area of the V diagram is equal to the difference in the areas over AB and BC. Here, the opposite signs of M_{B} and $M_{\text{C;left}}$ indicate that the deformation symbols of M_{B} and $M_{\text{C;left}}$ are opposite. The plus and minus signs alone are however arbitrary and can be interchanged. For this reason, we generally work with absolute values and use for ΔM

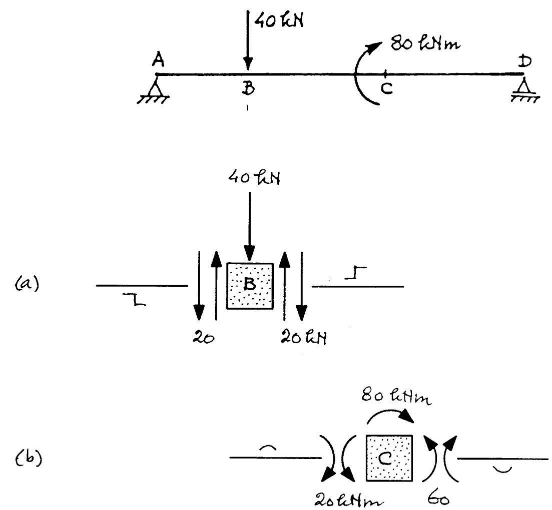


Figure 12.12 (a) The shear forces directly to the left and right of joint B are in equilibrium with the point load of 40 kN on joint B. (b) The bending moments directly to the left and right of joint C are in equilibrium with the load due to the concentrated couple of 80 kNm on joint C.

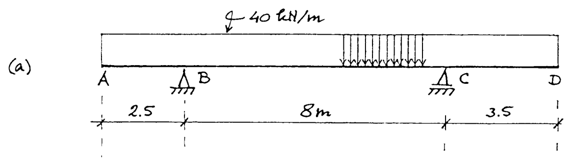


Figure 12.13 (a) Simply supported cantilever beam with unequal overhangs and a uniformly distributed full load.

the *absolute value* of the area of the V diagram.

Since a couple is acting on the beam, the area of the V diagram between A and D is not zero but rather

$$\left|(2\text{ m})(20\text{ kN}) - (6\text{ m})(20\text{ kN})\right| = 80\text{ kNm}.$$

In accordance with rule 12, this value is exactly equal to the magnitude of the couple at C.

Example 2

The simply supported cantilever beam in Figure 12.13a has two overhangs and is carrying a uniformly distributed load of 40 kN/m over its entire length.

Questions:

a. Draw the M diagram.
b. Derive the V diagram from the M diagram.
c. Verify the step changes in the V diagram.
d. Determine the maximum bending moments in field BC and at the supports.

Solution:

The support reactions at B and C are respectively 245 kN and 315 kN, both aimed upwards (see Figure 12.13b). Three fields, AB, BC and CD, are distinguished for drawing the M and V diagrams. In each of these fields, the shear force varies linearly and the bending moment varies parabolically (rule 2).

a. Per field, we replace the distributed load by its resultant, and draw the moment due to these resultants. We find the bent M diagram shown by the dashed lines in Figure 12.13c. At the field boundaries A, B, C and D, the actual parabolic M diagram is tangent to the dashed M diagram. Extra points on the parabolic M diagram are found in the middle of each field by halving the distance between the chord and the bend in the dashed M diagram. At these points, the tangent to the M diagram is parallel to the chord.[1] With three values and three tangents per field, it is now easy to make a free-hand sketch of the M diagram, see the solid line in Figure 12.13c. To keep the image legible, the tangents in the field middles are not shown.

The correctness of the M diagram can be checked using the values $p = \frac{1}{8}q\ell^2$, where ℓ stands for field length:

$$p_1 = \tfrac{1}{8}(40\ \text{kN/m})(2.5\ \text{m})^2 = 31.25\ \text{kNm},$$

$$p_2 = \tfrac{1}{8}(40\ \text{kN/m})(8\ \text{m})^2 = 320\ \text{kNm},$$

$$p_3 = \tfrac{1}{8}(40\ \text{kN/m})(3.5\ \text{m})^2 = 61.25\ \text{kNm}.$$

The value $2p$ is the distance between the chord and the bend in the dashed M diagram. This distance can also be derived from the M diagram. It is up to the reader to check whether the calculated values of p fit on the M diagram shown.

b. The V diagram varies linearly with step changes at B and C where the support reactions act. At the field boundaries, the shear forces can be derived from the slope of the dashed M diagram (rule 5). This gives

$$V_\text{A} = 0,$$

$$V_\text{B;left} = \frac{125\ \text{kNm}}{1.25\ \text{m}} = 100\ \text{kN},$$

$$V_\text{B;right} = \frac{(125 + 455)\ \text{kNm}}{4\ \text{m}} = 145\ \text{kN},$$

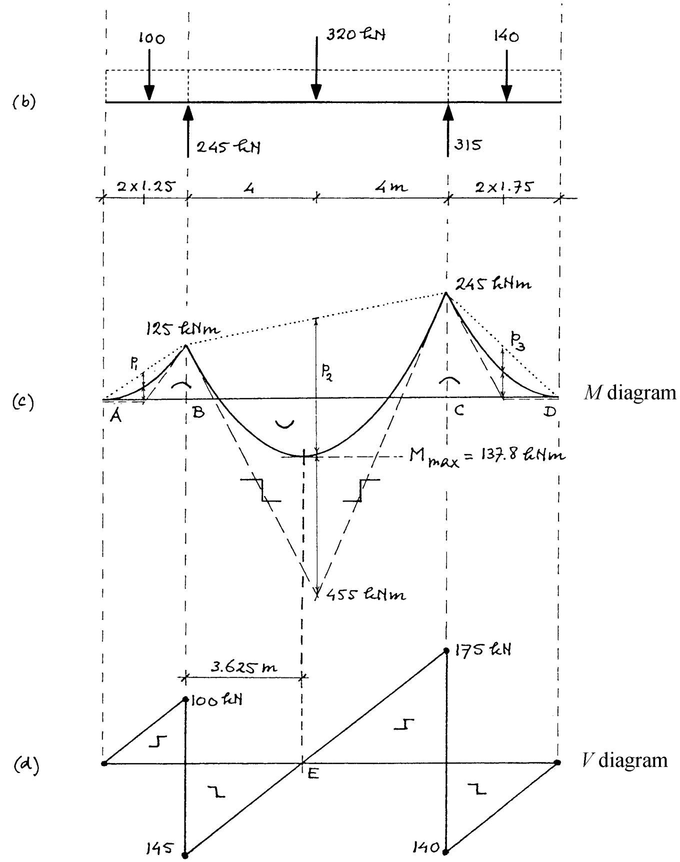


Figure 12.13 (b) The isolated beam with the resultants of the distributed loads in the fields AB, BC and CD, and the support reactions. (c) Bending moment diagram. At the field boundaries A to D, this diagram is tangent to the dashed bending moment diagram due to the load resultants. (d) Shear force diagram. The bending moments are extreme where the shear force is zero or changes sign.

[1] See Section 12.1.6, "Properties of parabolic M diagrams".

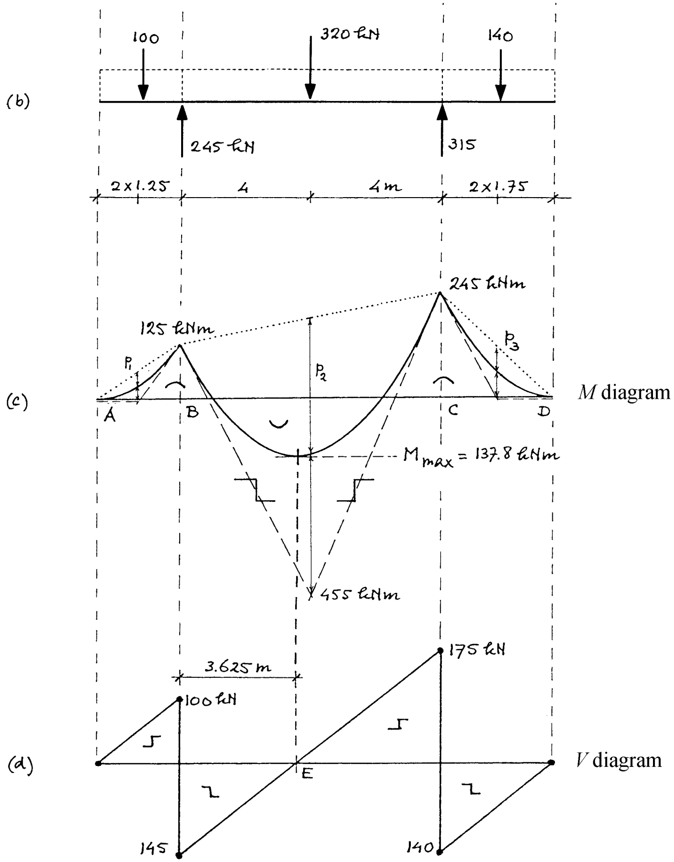


Figure 12.13 (b) The isolated beam with the resultants of the distributed loads in the fields AB, BC and CD, and the support reactions. (c) Bending moment diagram. At the field boundaries A to D, this diagram is tangent to the dashed bending moment diagram due to the load resultants. (d) Shear force diagram. The bending moments are extreme where the shear force is zero or changes sign.

$$V_{\mathrm{C;left}} = \frac{(245+455)\ \mathrm{kNm}}{4\ \mathrm{m}} = 175\ \mathrm{kN},$$

$$V_{\mathrm{C;right}} = \frac{245\ \mathrm{kNm}}{1.75\ \mathrm{m}} = 140\ \mathrm{kN},$$

$$V_{\mathrm{D}} = 0.$$

The deformation symbols also follow from the slope of the M diagram; they are equal to the "steps" in the M diagram. These "steps" are shown explicitly only for $V_{\mathrm{B;right}}$ and $V_{\mathrm{C;left}}$ in the M diagram in Figure 12.13c.

Figure 12.13d shows the V diagram. The V values derived from the M diagram are shown by means of dots. Between these values the shear force varies linearly. Since the same uniformly distributed load acts over the entire length of the beam, the slopes of the V diagram are equal in all the fields, and are 40 kN/m (rule 4).

c. The step changes in the V diagram at the supports at B and C are 245 kN and 315 kN respectively, and are equal in magnitude to the support reactions (rule 14).

d. The bending moments at the supports at B and C are referred to as *support moments*.[1] The maximum support moment occurs at C.

The largest moment in field BC is known as the maximum *field moment*. This occurs at E, where the shear force is zero (rule 7). Here the tangent is horizontal. The magnitude can be derived from the moment equilibrium of the isolated part AE or ED, but also from the area of the V diagram (rule 11). With $M_{\mathrm{E}} = M_{\max}$ and $M_{\mathrm{A}} = 0$,

$$\Delta M^{(\mathrm{AE})} = M_{\mathrm{E}} - M_{\mathrm{A}} = M_{\max}.$$

1 For a fixed-end, the *support moment* is called a *fixed-end moment*.

$\Delta M^{(\mathrm{AE})}$ is equal to the (absolute value of the) area of the V diagram over AE:

$$\begin{aligned} M_{\max} &= \Delta M^{(\mathrm{AE})} \\ &= \left| \tfrac{1}{2}(2.5\ \mathrm{m})(100\ \mathrm{kN}) - \tfrac{1}{2}(3.625\ \mathrm{m})(145\ \mathrm{kN}) \right| = 137.8\ \mathrm{kNm}. \end{aligned}$$

Because the V diagrams over AB and BE have different deformation symbols, the total area is equal to the difference in the areas over AB and BE. And because the sign is not so important, we look at the absolute value.

Of course we can also look at the right-hand part ED:

$$\Delta M^{(\mathrm{ED})} = M_{\mathrm{E}} - M_{\mathrm{D}} = M_{\max}$$

so that

$$\begin{aligned} M_{\max} &= \Delta M^{(\mathrm{ED})} \\ &= \left| \tfrac{1}{2}(4.375\ \mathrm{m})(175\ \mathrm{kN}) - \tfrac{1}{2}(3.5\ \mathrm{m})(140\ \mathrm{kN}) \right| = 137.8\ \mathrm{kNm}. \end{aligned}$$

Note that the total area of the V diagram is zero (rule 12).

If in this example (as well as in other examples) we further investigate the M diagram, we notice a correlation between the shape of the M diagram and the shape that a cable or cord assumes under the same load. This leads to the following statement:

• ***Rule 16***
If a beam is exclusively loaded by forces normal to its axis, the M diagram has the same shape as a cable (or cord) on which one lets the same forces act.

In addition to the shape of the M diagram, this rule also allows us to easily check whether the M diagram has been drawn at the correct side and there-

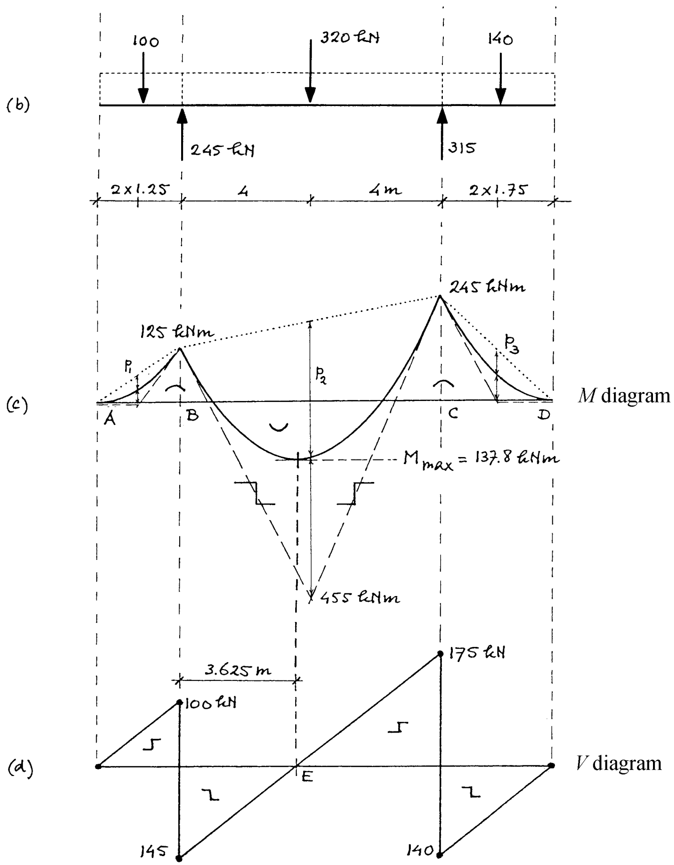


Figure 12.13 (b) The isolated beam with the resultants of the distributed loads in the fields AB, BC and CD, and the support reactions. (c) Bending moment diagram. At the field boundaries A to D, this diagram is tangent to the dashed bending moment diagram due to the load resultants. (d) Shear force diagram. The bending moments are extreme where the shear force is zero or changes sign.

fore with the correct deformation symbol. A downward force generates a downward bend in a cable, and also in the M diagram. An upward force lifts the cable, and therefore also lifts the M diagram.

In the M diagram in Figure 12.13c, we can recognise a cable AD hanging (as a parabola) under the influence of the uniformly distributed full load, and pushed upwards by the support reactions at B and C.

The general validity of rule 16 is explained in Chapter 14.

12.1.6 Properties of parabolic M diagrams

Due to a uniformly distributed load, the M diagram has the shape of a parabola. Since uniformly distributed loads occur frequently in practice we will discuss a number of the striking properties of parabolas below. They can be used to sketch a parabolic M diagram quickly.

We will use the isolated beam segment in Figure 12.14a as starting point, with length ℓ and a uniformly distributed load q over the full length. In addition to shear forces, the section planes at A and B are subject to bending moments M_A and M_B, both causing tension at the underside. Figure 12.14b shows the associated M diagram.

Properties of the parabolic M diagram include the following:

- The tangents to the M diagram at A and B are found by drawing the M diagram for the resultant of the distributed load. The tangents at A and B intersect in the middle C of AB.
- The vertically measured distance p between chord k and the parabola is

$$p = \tfrac{1}{2}qab \quad \text{(in which } a + b = \ell\text{)}.$$

- In the middle C ($a = b = \tfrac{1}{2}\ell$) this distance is

$$p_C = \tfrac{1}{8}q\ell^2.$$

- The intersection of the tangents at A and B is at a distance $2p_C$ under the chord k.

- In the example, with bending moments M_A and M_B, both causing tension at the underside, the bending moment in the middle C is:

$$M_C = \tfrac{1}{2}(M_A + M_B) + \tfrac{1}{8}q\ell^2.$$

- The tangent in the middle C is parallel to chord k.
- The field moment in AB is an extreme where the tangent is horizontal. This is generally not in the middle of AB.

12.1.7 Summary of all the rules relating to M and V diagrams

This section includes a summary of all the rules discussed in Sections 12.1.1 to 12.1.5.

• *Rule 1*

In an unloaded field, the shear force V is constant, and the bending moment M varies linearly. If the shear force is zero, the bending moment is constant.

$$q_z = 0 \Rightarrow V \text{ constant;} \quad V = 0 \Rightarrow M \text{ constant}$$
$$V \neq 0 \Rightarrow M \text{ linear.}$$

• *Rule 2*

In a field with a uniformly distributed load q_z, the shear force V varies linearly, and the bending moment M varies quadratically (parabolic).

$$q_z \text{ constant } (\neq 0) \Rightarrow V \text{ linear } \Rightarrow M \text{ quadratic.}$$

• *Rule 3*

In a field with a linearly distributed load q_z, the shear force V varies quadratically, and the bending moment M is a cubic function.

$$q_z \text{ linear } \Rightarrow V \text{ quadratic } \Rightarrow M \text{ cubic.}$$

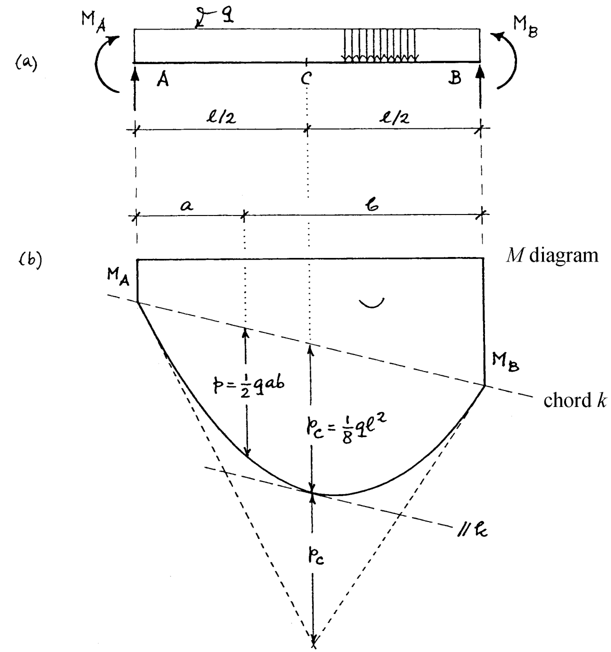


Figure 12.14 (a) An isolated beam segment with uniformly distributed full load and (b) the associated parabolic bending moment diagram.

- ***Rule 4***
The slope of the V diagram ($\mathrm{d}V/\mathrm{d}x$) is equal to the distributed load q_z (but with an opposite sign).

- ***Rule 5***
The slope of the M diagram ($\mathrm{d}M/\mathrm{d}x$) is equal to the shear force V.

- ***Rule 6***
The shear force V is an extreme where the distributed load q_z is zero (or changes sign). Per field, we have to take into account the occurrence of extreme values at the boundaries (e.g. at concentrated loads and supports).

- ***Rule 7***
The bending moment is an extreme where the shear force is zero (or changes sign). Per field, we have to take into account the occurrence of extreme values at the boundaries (e.g. where concentrated loads and/or couples are applied or at supports).

- ***Rule 8***
The tangents to the M diagram at the boundaries of a field intersect on the line of action of the load resultant in that field (for a distributed load this is at the centroid of the load diagram).

- ***Rule 9***
If we replace the load per field by its resultant, and draw the bending moment diagram due to these resultants, this bending moment diagram is tangent to the actual bending moment diagram at the field boundaries.

- ***Rule 10***
Without concentrated forces, the change of the shear force V over a certain length is equal to the area of the load diagram over that length.

- ***Rule 11***
Without concentrated couples, the change in the bending moment over a certain length is equal to the area of the V diagram over that length.

• ***Rule 12***
The total area of the V diagram is equal to the sum of moments of all concentrated couples that act on the beam. For a beam without concentrated couples, the total area of the V diagram is zero.

• ***Rule 13***
A step change in the distributed load q_z gives a bend in the V diagram and a point of inflection in the M diagram.

• ***Rule 14***
A (concentrated) force F normal to the member axis generates a step change in the V diagram, of magnitude F, and a bend in the M diagram.

• ***Rule 15***
A (concentrated) couple T gives a step change in the M diagram of magnitude T. The V diagram reveals no information about the point of application of the couple.

• ***Rule 16***
If a beam is exclusively loaded by forces normal to its axis, the M diagram has the same shape as a cable (chord) on which one lets the same forces act.

12.2 Rules for drawing the N diagram more quickly

The differential equations for the force equilibrium of an infinitesimal member segment in Section 11.1 are

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0 \quad \text{(extension)}, \tag{a}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0 \quad \text{(bending)}. \tag{b}$$

Table 12.2 Relationship between the distributed axial load q_x and the variation of the normal force N.

Variation q_x	Variation N
constant $= 0$	constant
constant $\neq 0$	linear
linear	quadratic

Based on the analogy of the differential equations (a) and (b), we can say that, for the relationship between N and q_x, the same rules apply as derived in the previous section for the relationship between V and q_z. Without any further commentary, and set down in the same order, the rules for the relationship between N and q_x are presented.

• ***Rule 1***
In an unloaded field, the normal force N is constant:

$$q_x = 0 \Rightarrow N \text{ constant.}$$

• ***Rule 2***
In a field with a uniformly distributed load q_x, the normal force N varies linearly:

$$q_x \text{ constant } (\neq 0) \Rightarrow N \text{ linear.}$$

• ***Rule 3***
In field with a linearly distributed load q_x, the normal force N varies quadratically (parabolic).

$$q_x \text{ linear } \Rightarrow N \text{ quadratic.}$$

Rules 1 to 3 are summarised in Table 12.2.

• ***Rule 4***
The slope of the N diagram ($\mathrm{d}N/\mathrm{d}x$) is equal to the distributed load q_x (but with an opposite sign).

• ***Rule 6***
The normal force N is an extreme where the distributed load q_x is zero (or changes sign). Per field, we have to take into account the occurrence of extreme values at the boundaries (e.g. at concentrated loads and supports).

• ***Rule 10***
Without concentrated forces, the change in the normal force N over a certain length is equal to the area of the load diagram over that length.

• ***Rule 13***
A step change in the distributed load q_x gives a bend in the N diagram.

• ***Rule 14***
A (concentrated) axial member force F generates a step change in the N diagram of magnitude F.

12.3 Bent and compound bar type structures

With bent and compound bar type structures, the force flow can be found by dividing the structure into all its (straight) members and by calculating all the support reactions and joining forces as shown in Chapter 5. It is then possible to draw the M, V and N diagrams for each separate member. These diagrams are then linked together to form the M, V and N diagrams for the structure as a whole. Only when the diagram becomes illegible should you draw part of the structure with its M, V and N diagrams separately.

Below we determine and draw the M, V and N diagrams for a number of structures for which we previously calculated the support reactions and/or joining forces in Chapter 5. The load consists of concentrated forces and couples. Distributed loads are covered in detail in Chapter 13.

Example 1
The support reactions were calculated for the lighting mast in Figure 12.15a in Section 5.1, Example 1.

Question:
Determine the M, V and N diagrams.

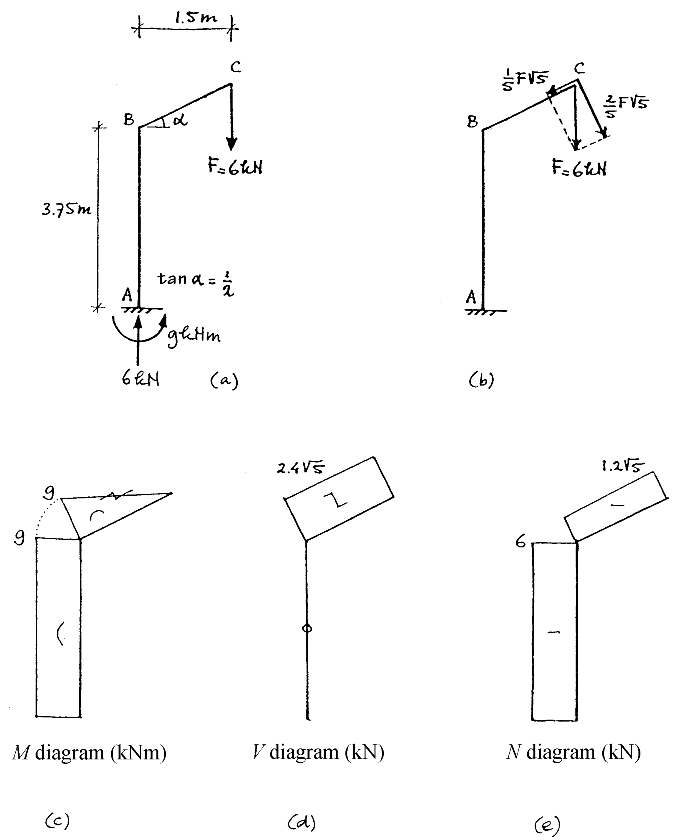


Figure 12.15 (a) The support reactions at A of a lighting mast loaded by a vertical force at C. (b) The force at C resolved into components normal to and parallel to member axis BC. (c) Bending moment diagram. The bending moment "*goes round the corner*" at B. (d) Shear force diagram. (e) Normal force diagram.

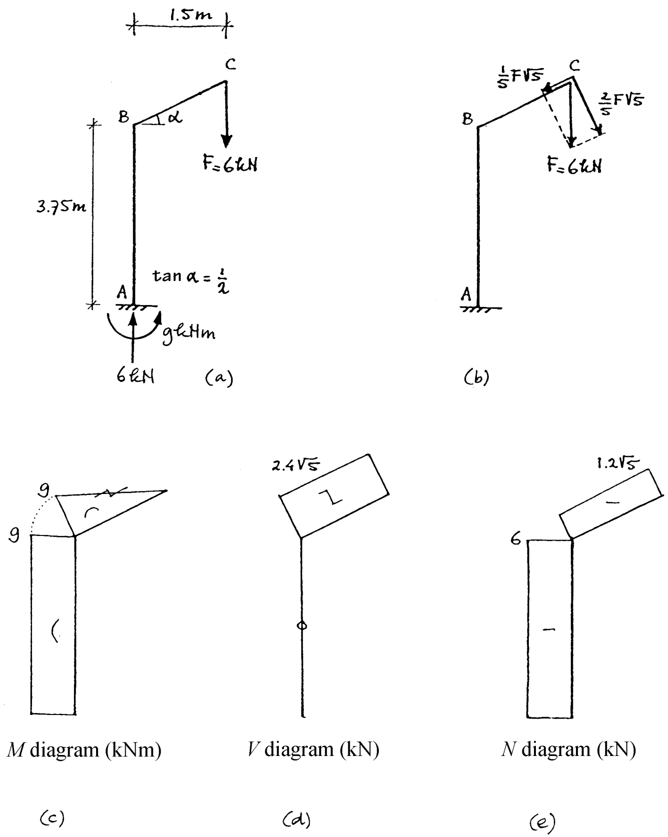


Figure 12.15 (a) The support reactions at A of a lighting mast loaded by a vertical force at C. (b) The force at C resolved into components normal to and parallel to member axis BC. (c) Bending moment diagram. The bending moment "*goes round the corner*" at B. (d) Shear force diagram. (e) Normal force diagram.

Solution:
Neither of the segments AB or BC is subject to a distributed load, so that the shear force in each segment is constant, and the bending moment varies linearly (rule 1).

M diagram
We can draw the M diagram as soon as we know the bending moments at A, B and C. The fixed-end moment at A and the bending moment at the free end C are known:

$$M_A = 6 \text{ kNm},$$

$$M_C = 0.$$

We now have to determine only the bending moment at B. Without resolving F, we can calculate M_B from the moment equilibrium of the isolated segment BC:

$$M_B = (6 \text{ kN})(1.5 \text{ m}) = 9 \text{ kNm}.$$

Of course it is also possible to determine M_B from the equilibrium of the isolated segment AB.

The M diagram is shown in Figure 12.15c. All values are plotted normal to the member axis.

The bending moment at joint B is the same magnitude on both sides of the joint and is also plotted at the same side. This follows directly from the moment equilibrium of joint B (see Figure 12.16, which shows only the bending moments). It is said that the bending moment at B "*goes round the corner*", which is further emphasised in Figure 12.15c by the dotted arc (normally not drawn).

V diagram
The shear forces can be calculated directly from the slopes of the M diagram (rule 5):

$$V^{(\mathrm{AB})} = \frac{\Delta M^{(\mathrm{AB})}}{\ell^{(\mathrm{AB})}} = \frac{0 \text{ kNm}}{3.75 \text{ m}} = 0,$$

$$V^{(\mathrm{BC})} = \frac{\Delta M^{(\mathrm{BC})}}{\ell^{(\mathrm{BC})}} = \frac{9 \text{ kNm}}{0.75\sqrt{5} \text{ m}} = 2.4\sqrt{5} \text{ kN}.$$

Since we are concerned here with the magnitude (and not the direction) of the shear force, we use the absolute value of ΔM in the calculation.

The V diagram is shown in Figure 12.15d. The deformation symbol for the (direction of the) shear force is found from the "*steps*" in the M diagram.

The shear force in AB is zero; this is in agreement with the horizontal support reaction at A.

In Figure 12.15b, the force at C has been resolved into components parallel to and normal to the member axis. The component normal to the member axis corresponds with the magnitude and direction of the shear force in BC as calculated earlier from the M diagram.

Comment: It is often useful, particularly for *oblique members*, to draw the bending moment diagram first and then use it to calculate the shear forces. In that case it is not necessary to resolve forces into their components.

N diagram
For determining the normal force in BC, we cannot escape from resolving the 6 kN force at C into components parallel to and normal to BC (see Figure 12.15b). The N diagram is shown in Figure 12.15e. The normal force is a constant compressive force in both members.

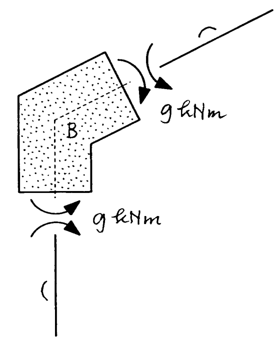


Figure 12.16 From the moment equilibrium of joint B it follows that the bending moment in B "*goes round the corner*".

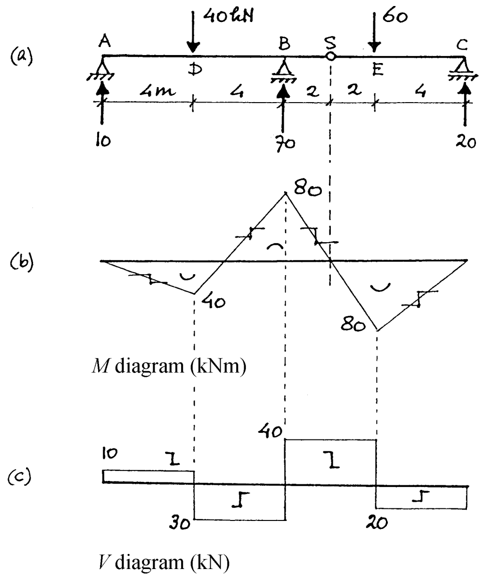


Figure 12.17 (a) Hinged beam with load and support reactions. (b) Bending moment diagram (M diagram) with the step changes for the deformation symbols in the shear force diagram (V diagram). (c) Shear force diagram.

Example 2

The support reactions and joining forces for the hinged beam in Figure 12.17a were calculated in Section 5.2, Example 1.

Question:
Determine the M and V diagrams.

Solution:
M diagram
The M diagram is shown in Figure 12.17b. The bending moment is zero at A, S and C; furthermore, the M diagram is linear with bends at the points of application of the forces at D, B and E. To be able to draw the M diagram, we have to calculate only the bending moments at D and E. The value at B is found by drawing a straight line through the values at E and S.

V diagram
The shear forces can be found from the slopes of the M diagram. The V diagram is shown in Figure 12.17c. The deformation symbols must correspond to the "*steps*" in the M diagram. The step changes in the V diagram must correspond with the forces on the beam (including the support reactions).

For a straight (continuous) beam, the shape of the V diagram is also found easily by plotting the successive step changes due to the concentrated loads one behind the other. These step changes are shown from left to right in Figure 12.18a. The values of the step changes are included in the figure. To draw the V diagram, the deformation symbols need to be included, as do the values of the shear forces. The values of the step changes are usually not included. Here this has been done only to illustrate the method.

Instead of going from left to right, we can plot the successive step changes from right to left (see Figure 12.18b). The result is the same figure again (Figure 12.18a), but now in reverse. If we include the correct deformation symbols, this V diagram is also correct. If we use the deformation symbols, it does not make a difference for the V diagram at which side of the member axis we plot the values.

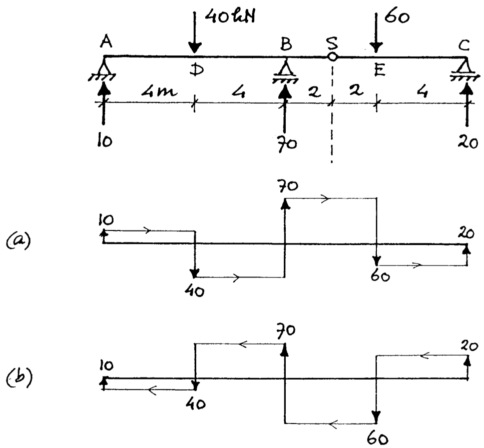


Figure 12.18 The shape of the V diagram can also be found by plotting the successive step changes due to the point loads at the beam: (a) from left to right or (b) from right to left.

Note that, between the two successive zero moments, in AS and SC, ΔM is zero, and therefore the corresponding area of the V diagram is also zero (rule 12). It is left to the reader to check this.

Example 3

The support reactions and the interaction forces at joint C for the structure in Figure 12.19a were calculated in Section 5.1, Example 5 (see Figure 12.20).

Question:

Determine the M, V and N diagrams.

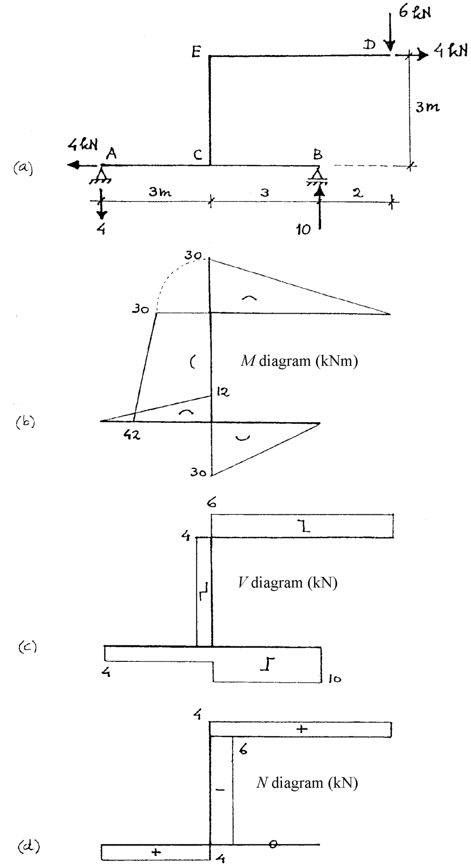


Figure 12.19 (a) A structure of which parts AC, BC and DC are rigidly joined at C. (b) Bending moment diagram. (c) Shear force diagram. (d) Normal force diagram.

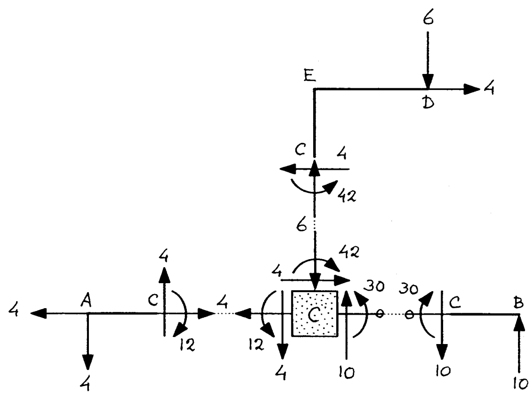


Figure 12.20 The interaction forces between joint C and the isolated parts AC, BC and DC.

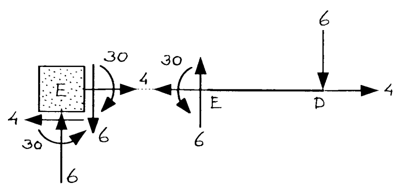


Figure 12.21 Joint E and member ED isolated.

Solution:
If we determine the interaction forces at joint E, it is possible to plot the M, V and N diagrams for the entire structure. As there are no distributed loads, the bending moment varies linearly along all members, and the shear force and normal force are constant in all members (rule 1).

M diagram
The M diagram is shown in Figure 12.19b. Since the variation of M along all members is linear, it is sufficient to determine the bending moments at the member ends to get the M diagram.

The bending moments $M_{\mathrm{C}}^{(\mathrm{CA})}$, $M_{\mathrm{C}}^{(\mathrm{CB})}$ and $M_{\mathrm{C}}^{(\mathrm{CE})}$ at joint C have already been calculated[1] (see Figure 12.20).

$M_{\mathrm{E}}^{(\mathrm{ED})}$ follows from the moment equilibrium of the isolated part ED (see Figure 12.21):

$$M_{\mathrm{E}}^{(\mathrm{CE})} = M_{\mathrm{E}}^{(\mathrm{ED})}$$

with tension on the upper side of ED. This value in the M diagram is therefore plotted at the upper side.

The moment equilibrium of joint E in Figure 12.21 gives

$$M_{\mathrm{E}}^{(\mathrm{CE})} = M_{\mathrm{E}}^{(\mathrm{ED})}.$$

The bending moment "*goes round the corner*". This is emphasised in the M diagram in Figure 12.19b by means of a dotted arc at joint E.

---

1 The upper index refers to the member in which the bending moment acts and the lower index refers to the location.

V diagram

The magnitude of the shear force and the associated deformation symbol follow from the slope of the M diagram.[1] The V diagram is shown in Figure 12.19c.

N diagram

The N diagram is shown in Figure 12.19d.

Example 4

We are given the trussed beam in Figure 12.22a. All the forces acting on the isolated hinged beam ASB shown in Figure 12.22b were calculated in Section 5.6.

The N, V and M diagrams are shown in Figures 12.22c to 12.22e.

Determining and drawing the V and M diagrams is done in the same way as for the hinged beam in Example 2. To draw the M diagram, we have to calculate only the bending moments at C and D. The bending moment varies linearly between D and E, so that the M diagram must pass through S where the bending moment is zero. This also fixes the value at E. The V diagram can subsequently be calculated from the M diagram.

We can change the order: first draw the V diagram and calculate the values at C, E and D from the areas of the V diagram.

Note: The shear forces at A and B are not equal to the support reactions at A and B! Why not?

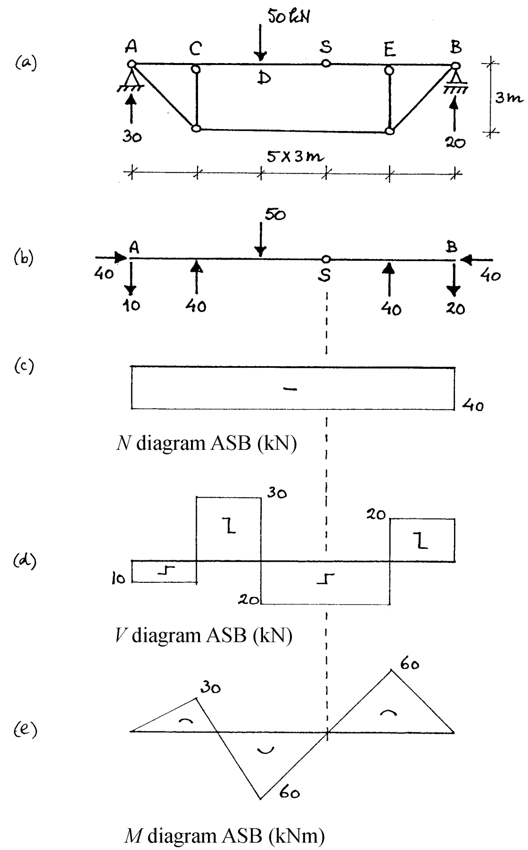


Figure 12.22 (a) Trussed beam with load and support reactions. (b) Isolated beam ASB. (c) Normal force diagram. (d) Shear force diagram. The shear forces at A and B are not equal to the support reactions at A and B. (e) Bending moment diagram.

[1] In Figure 12.19b, the "steps" in the M diagrams are no longer shown.

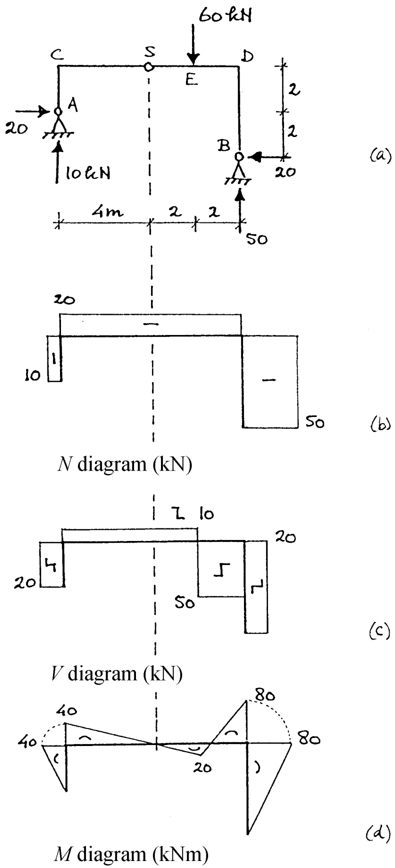


Figure 12.23 (a) Three-hinged portal frame with load and support reactions. (b) Normal force diagram. (c) Shear force diagram. (d) Bending moment diagram.

Example 5
The support reactions for the three-hinged portal frame in Figure 12.23a were calculated in Section 5.3, Example 1.

As there are no distributed loads, the normal forces and shear forces in each field are constant, and the bending moment varies linearly (rule 1).

N and *V diagrams*
To draw the *N* and *V* diagrams, we have to investigate the force equilibrium of the separate parts. The necessary calculations are left to the reader. The result is shown in Figures 12.23b and 12.23c.

Due to the concentrated load at E, a step change of 60 kN occurs in the *V* diagram (rule 14).

M diagram
To draw the *M* diagram, we have to know only the bending moments at C and D.

The bending moment at C follows from the moment equilibrium of the isolated part AC:

$$M_{\mathrm{C}}^{(\mathrm{AC})} = 40 \text{ kNm}$$

with tension at the "*outside*" of the frame.

From the moment equilibrium of joint C, where the two members AC and CS are rigidly joined to one another, it follows that the bending moments $M_{\mathrm{C}}^{(\mathrm{AC})}$ in column AC and $M_{\mathrm{C}}^{(\mathrm{CS})}$ in beam CS are of equal magnitude, and that both cause tension at the "*outside*" of the frame (see Figure 12.24, which shows only the bending moments). Both moments are plotted "*outside*". In the *M* diagram, we can see the bending moment in C "*going round the corner*".

For the bending moment at D we find

$$M_{\mathrm{D}}^{(\mathrm{BD})} = M_{\mathrm{D}}^{\mathrm{d}(DS)} = 80 \text{ kNm}$$

also with tension at the "outside" of the frame.

The M diagram for AC and BD varies linearly, from 0 to 40 and 80 kNm respectively. The M diagram for CSD consists of two straight lines that have a bend at the concentrated load at E. In addition, the M diagram passes through hinge S where the bending moment is zero. We therefore have to draw a straight line from the value of 40 kNm at C, through S, up to 20 kN on the opposite side at E. From there, we continue with a straight line to the value of 80 kNm at D.

The M diagram is shown for the entire three-hinged portal frame in Figure 12.23d.

Check 1:
We can read from the M diagram that the bending moment at the position of the point load is 20 kNm, with tension at the underside of the beam. This can be checked using the moment equilibrium of the isolated part BDE.

Check 2:
Note that the magnitude of the shear forces and the deformation symbols agree with the slopes of the M diagram. This relationship between the M and V diagram represents a simple and fast way of checking their correctness.

Example 6
All the forces on the isolated members ACE and BDE for the structure in Figure 12.25a were calculated in Section 5.5, Example 2.

M diagram
To draw the M diagram we have to calculate only the bending moment at the three points C, D and G. To do so, it is sufficient to know the support reactions and the normal force in member CD (see Figure 12.25b):

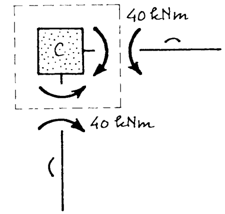


Figure 12.24 From the moment equilibrium of joint C it follows that the bending moment "*goes round the corner*" at C (see Figure 12.23).

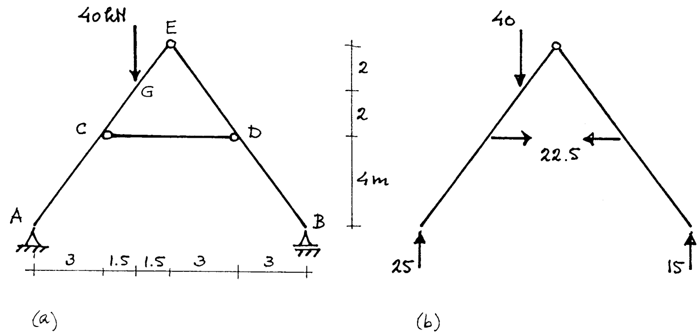


Figure 12.25 (a) A structure loaded by a vertical force on the left frame leg. (b) The isolated member AEB. (c) Bending moment diagram.

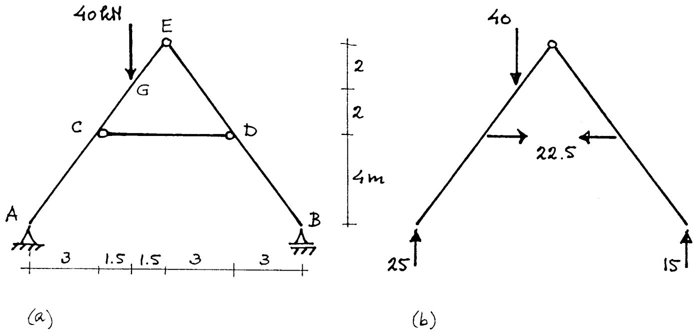


Figure 12.25 (a) A structure loaded by a vertical force on the left frame leg. (b) The isolated member AEB. (c) Bending moment diagram.

$$M_\mathrm{C} = (25 \text{ kN})(3 \text{ m}) = 75 \text{ kNm},$$

$$M_\mathrm{G} = (25 \text{ kN})(4.5 \text{ m}) - (22.5 \text{ kN})(2 \text{ m}) = 67.5 \text{ kNm},$$

$$M_\mathrm{D} = (15 \text{ kN})(3 \text{ m}) = 45 \text{ kNm}.$$

The M diagram is shown in Figure 12.25c. Since there is only a normal force in the two-force member CD, this member has been omitted to simplify the figure.

V diagram

The shear forces can be determined directly from the slopes of the M diagram. For example:

$$V^{(\mathrm{AC})} = \frac{\Delta M^{(\mathrm{AC})}}{\ell^{(\mathrm{AC})}} = \frac{75 \text{ kNm}}{5 \text{ m}} = 15 \text{ kN},$$

$$V^{(\mathrm{CG})} = \frac{\Delta M^{(\mathrm{CG})}}{\ell^{(\mathrm{CG})}} = \frac{(75 \text{ kNm}) - (67.5 \text{ kNm})}{2.5 \text{ m}} = 3 \text{ kN},$$

$$V^{(\mathrm{GE})} = \frac{\Delta M^{(\mathrm{GE})}}{\ell^{(\mathrm{GE})}} = \frac{67.5 \text{ kNm}}{2.5 \text{ m}} = 27 \text{ kN}.$$

The associated deformation symbols follow from the "*steps*" in the M diagram (they are not shown here). The complete V diagram is shown in Figure 12.25d. Here too the two-force member CD has been omitted.

N diagram

The N diagram is shown in Figure 12.25e. Determining the N diagram is relatively laborious as we have to resolve all the forces on the members into components normal to the member axis (step changes in the V diagram) and components parallel to the member axis (step changes in the N diagram).

For example, for the horizontal force of 22.5 kN at C, the component normal to the axis of member ACE is

$\frac{4}{5} \times (22.5 \text{ kN}) = 18 \text{ kN}$

and the component parallel to the member axis is

$\frac{3}{5} \times (22.5 \text{ kN}) = 13.5 \text{ kN}.$

At C we observe a step change of 18 kN in the V diagram, and a step change of 13.5 kN in the N diagram (rule 14).

In the same way, the component normal to ACE of the vertical force of 40 kN in G is

$\frac{3}{5} \times (40 \text{ kN}) = 24 \text{ kN}$

and the component parallel to ACE is

$\frac{4}{5} \times (40 \text{ kN}) = 32 \text{ kN}.$

At G we observe a step change of 24 kN in the V diagram, and a step change of 32 kN in the N diagram (rule 14).

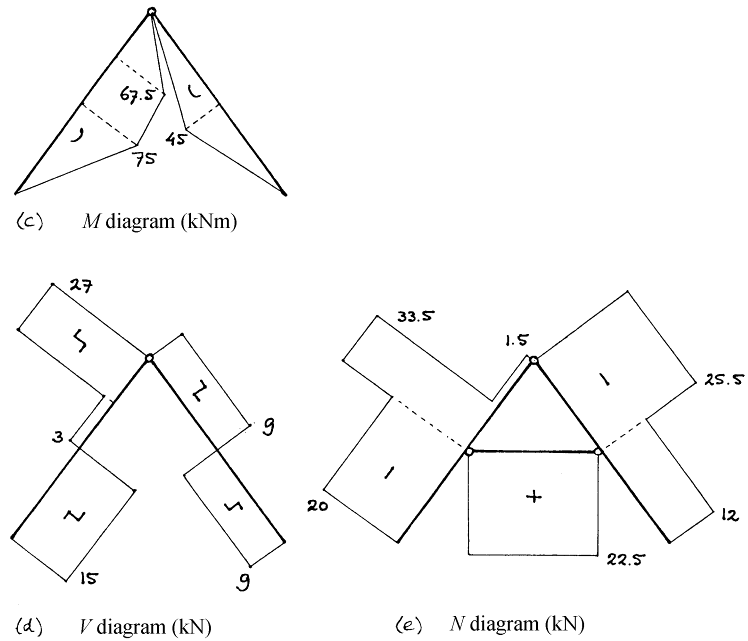


Figure 12.25 (c) Bending moment diagram. (d) Shear force diagram. (e) Normal force diagram.

12.4 Principle of superposition

If several loads are acting on a structure, the separate influences of the various loads on the support reactions and section forces can be added together.[1]

The validity of this so-called *principle of superposition* is a result of the linear relationships between the loads, section forces and support reactions.

[1] In Section 6.3.1, we showed that distributed loads can be split and that the individual influences on the support reactions can be added together.

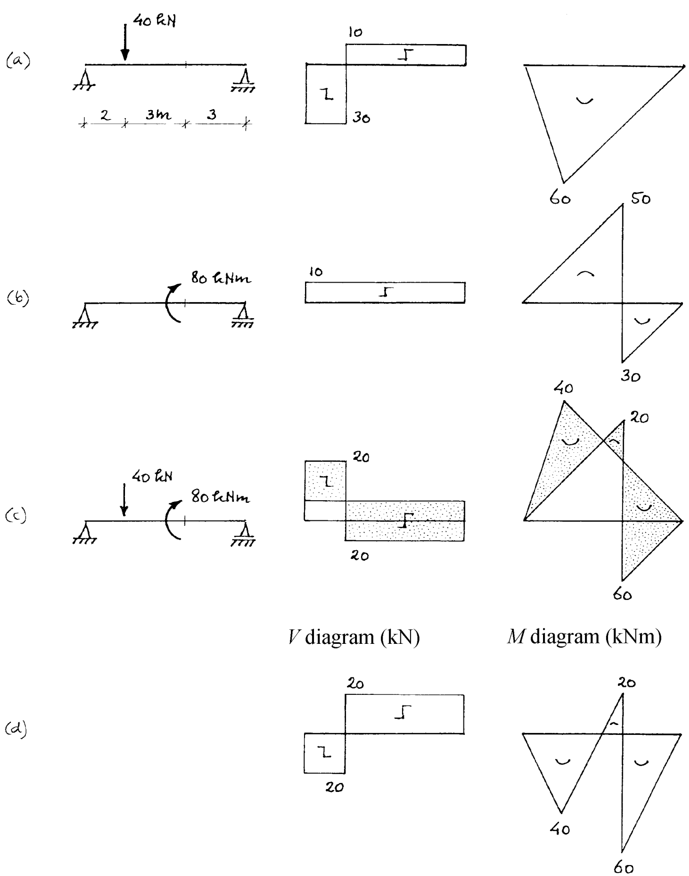


Figure 12.26 Simply supported beam with the V and M diagrams due to (a) the force and (b) the couple. (c) The V and M diagrams due to the force and the couple together, found by superposing the V and M diagrams from (a) and (b). (d) The V and M diagrams from (c) transferred to a horizontal axis.

Example 1

In Figure 12.26, the principle of superposition has been applied to determine the V and M diagrams for an 8-metre beam that is loaded by a force of 40 kN and a couple of 80 kNm.

In Figure 12.26a, the V and M diagrams have been calculated due to the force only. In Figure 12.26b, the V and M diagrams have been calculated due to the couple only. The final V and M diagrams with concurrent loading by the force and the couple is shown in Figure 12.26c. To draw these V and M diagrams, one of the two diagrams to be superposed has been reflected with respect to the horizontal axis to simplify the graphics. In areas with opposite deformation symbols that overlap one another, the combined contribution to the section force is zero. The remaining areas have been filled and form the final V and M diagrams.

In Figure 12.26d, these diagrams have been transferred to a horizontal axis, but this is generally not necessary.

Of course the superposition can also be performed by determining the ordinates at a number of points and adding them together.

Example 2

The second example relating to the principle of superposition concerns further analysis of the force flow in segment BC of beam AD in Figure 12.27, with a uniformly distributed load over BC. The V and M diagrams for this beam were calculated in Section 12.1.3, Example 2.

For BC, we find the same V and M diagrams if we isolate segment BC from beam AD, support it simply at its ends B and C, and there load it by couples of 80 and 60 kNm respectively (see Figure 12.28).

We can distinguish three loads on beam BC in Figure 12.28:

(a) a couple of 80 kNm at end B;
(b) a couple of 60 kNm at end C;
(c) a uniformly distributed full load of 12.5 kN/m.

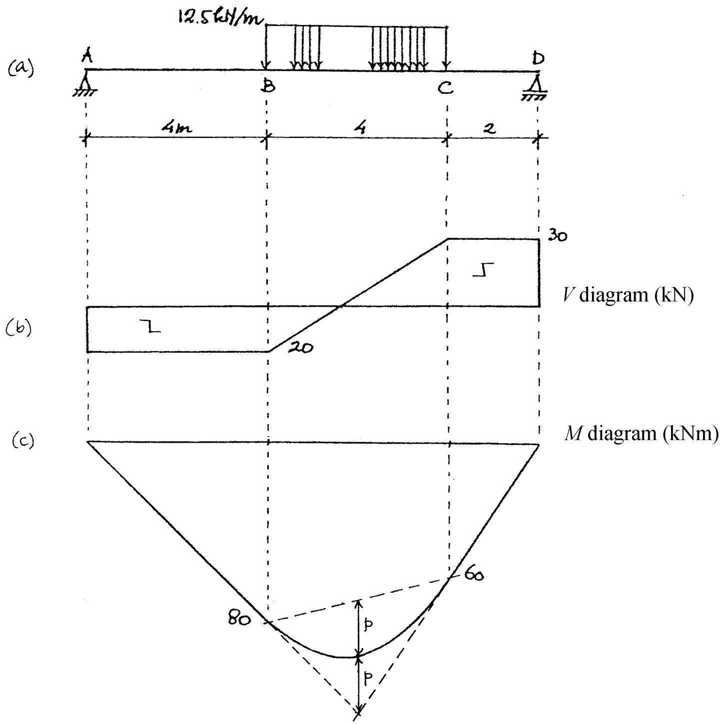


Figure 12.27 (a) Simply supported beam with a uniformly distributed load in field BC. (b) Shear force diagram. (c) Bending moment diagram.

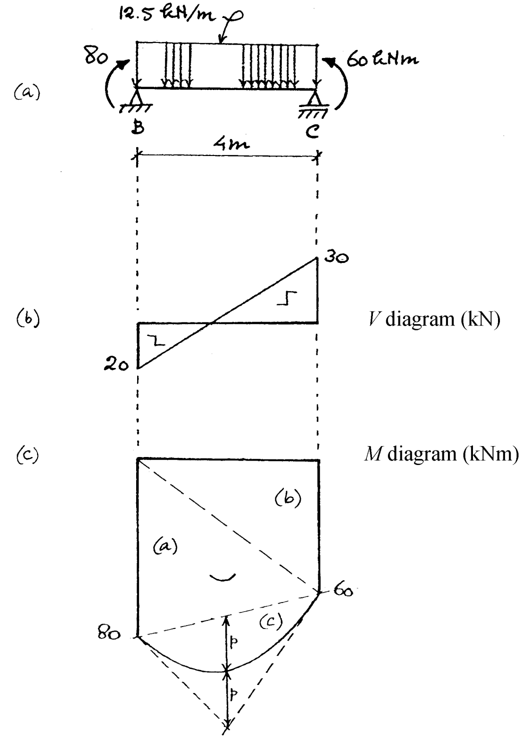


Figure 12.28 For BC in Figure 12.27, the same V and M diagrams are found if BC is isolated from AD, simply supported at its ends B and C, and loaded there by couples of 80 and 60 kNm respectively.

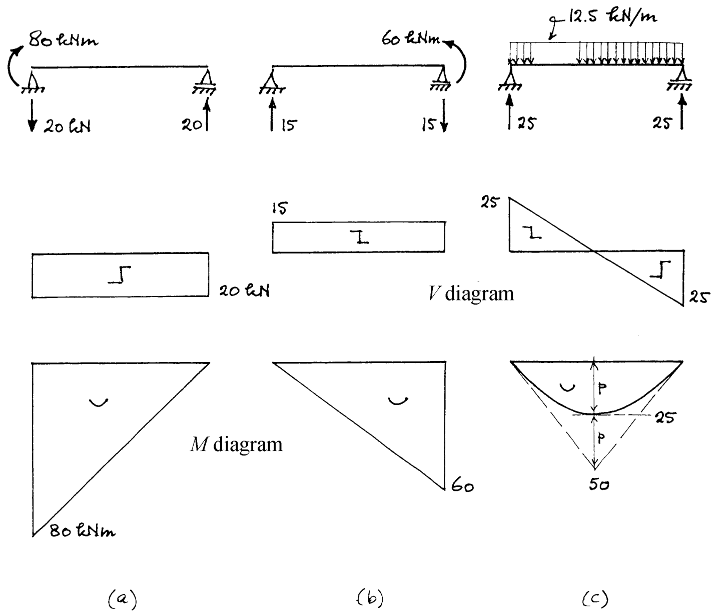


Figure 12.29 The V and M diagrams in Figure 12.28 can be found by superposing the V and M diagrams due to (a) a couple of 80 kNm in B, (b) a couple of 60 kNm in C and (c) a uniformly distributed full load of 12.5 kN/m.

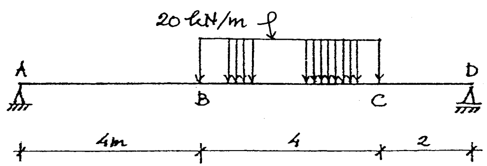


Figure 12.30 Simply supported beam with a uniformly distributed load in field BC.

In Figure 12.29, the V and M diagrams are shown for each of these loads.

The V and M diagrams in Figure 12.28 can be found by superposing the V and M diagrams from Figure 12.29. In the M diagram in Figure 12.28c we can clearly recognise the contributions (a) to (c) of the separate loads.

The principle of superposition can be used also to compare the effects of similar loads of different magnitudes. Since the entire system behaves linearly, we can say that if a load leads to certain values for the section forces, a similar load that is n times as large causes section forces that are in turn n times as large.

Example 3
If the maximum bending moment in Figure 12.27c is 96 kNm, how large is the maximum bending moment for the beam in Figure 12.30?

Solution:
The load in Figure 12.30 is similar to that in Figure 12.27a and is $20/12.5 = 1.6$ times as large. In both cases, the V and M diagrams have the same shape, except that the values for the beam in Figure 12.30 are now 1.6 times as large as those for the beam in Figure 12.27. This applies also for the maximum bending moment. Therefore, for the beam in Figure 12.30,

$$M_{\text{max}} = 1.6 \times (96\text{ kNm}) = 153.6\text{ kNm}.$$

12.5 Schematisations and reality

Forces, couples and uniformly distributed loads are schematisations of the real loads. In this section we will look at the consequences of these schematisations. We will also look at the shear forces at a support and the influence of eccentric axial forces.

12.5.1 Point load

The simply supported beam in Figure 12.31a is loaded at the middle of span ℓ by a force F. Figures 12.31b and 12.31c show the associated V and M diagrams.

The force F, modelled as a point load, is a load concentrated at one single point, or in other words, a load that applies over a length zero. This does not exist in reality. In fact, F is the resultant of a distributed load over a small yet finite length $\xi\ell$. Figure 12.32 shows the V and M diagrams for the case in which load F is uniformly distributed over length $\xi\ell$:

$$q = \frac{F}{\xi\ell}.$$

For a point load, ξ approaches zero and at the same time the force intensity q increases such that $q \cdot \xi\ell = F$ remains constant.

Over the (small) area $\xi\ell$ in Figure 12.32, the shear force varies (steeply) from $+\frac{1}{2}F$ to $-\frac{1}{2}F$ and the M diagram is parabolic. The maximum bending moment occurs at midspan:

$$M_{\max} = \tfrac{1}{4}F\ell - p = \tfrac{1}{4}F\ell - \tfrac{1}{8}\frac{F}{\xi\ell}(\xi\ell)^2 = \tfrac{1}{4}F\ell\left(1 - \tfrac{1}{2}\xi\right).$$

For calculating this moment, we used the property that

$$p = \text{“}\tfrac{1}{8}q\ell^2\text{”} = \tfrac{1}{8}\frac{F}{\xi\ell}(\xi\ell)^2 = \tfrac{1}{8}F\xi\ell.$$

If we allow ξ to approach zero, the slope of the V diagram gets increasingly steep and eventually changes into a step change of magnitude F. The shear force, and therefore the slope of the M diagram, changes increasingly rapidly as length $\xi\ell$ gets smaller. In other words, the parabola curves more and more. In the limiting case $\xi \to 0$ the area of the parabola is reduced to a point: the M diagram gets bent under the concentrated load.

For uniformly distributed loads over a finite length $\xi\ell$, the maximum bend-

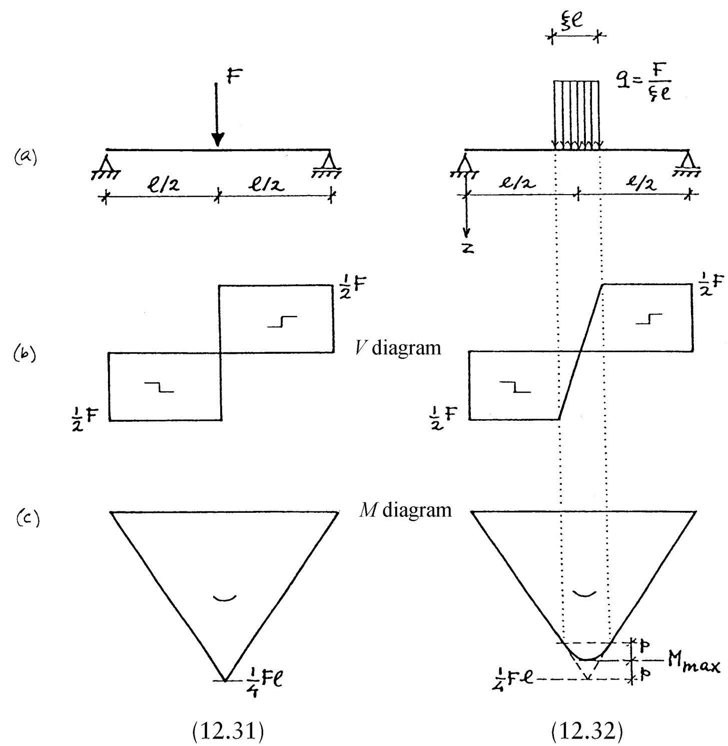


Figure 12.31 The V and M diagrams due to a point load F at midspan.

Figure 12.32 The V and M diagrams in the case that load F is uniformly distributed over a small length $\xi\ell$.

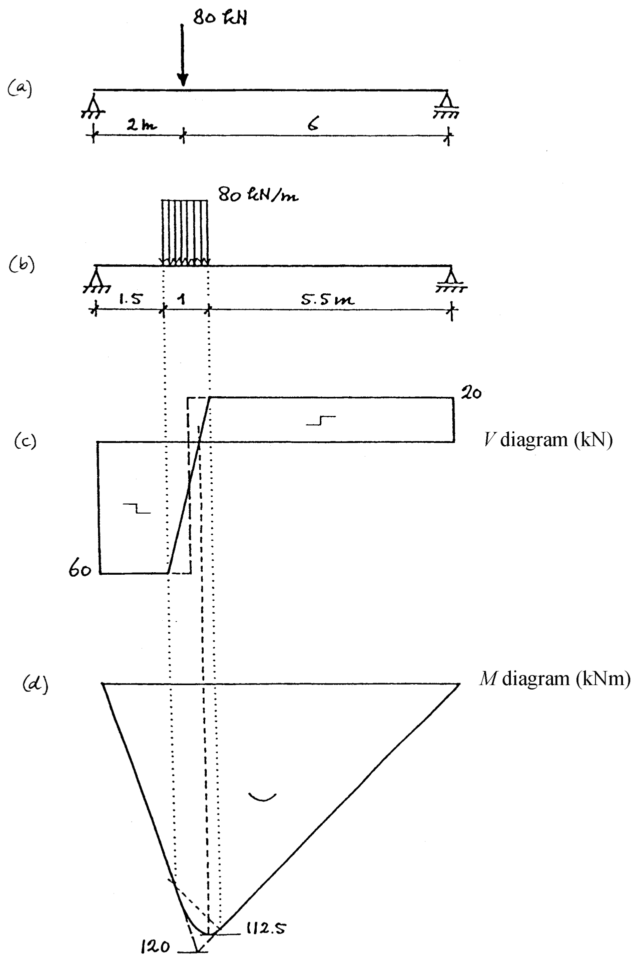


Figure 12.33 The V and M diagrams due to a point load (dashed line) to replace a uniformly distributed load over a small length (solid line).

ing moment is smaller than when the load is concentrated at a single point.

The difference is

$$\frac{p}{\frac{1}{4}F\ell} = \frac{\frac{1}{8}F\xi\ell}{\frac{1}{4}F\ell} \times 100\% = \tfrac{1}{2}\xi \times 100\%.$$

We can show that this value applies also when the load is not acting at midspan.

To illustrate, we compare the two loads in Figures 12.33a and 12.33b. With

$$\xi = \tfrac{1}{8}$$

we find

$$\tfrac{1}{2}\xi \times 100\% = \tfrac{1}{2} \times \tfrac{1}{8} \times 100\% = 6.25\%.$$

The maximum bending moment due to the distributed load will be 6.25% smaller than the maximum bending moment due to the point load.

A calculation (which is left to the reader) shows that the maximum bending moment due to the concentrated load is 120 kNm, and due to the distributed load is 112.5 kNm, which indeed is 6.25% less than before.

In Figures 12.33c and 12.33d, the V and M diagrams are shown for both loads. The V and M diagrams due to the concentrated loads are shown by dashed lines, insofar they deviate.

Note: The maximum values of 120 and 112.5 kNm do not occur in the same cross-section!

12.5.2 Uniformly distributed load

If a large number of almost equal point loads are acting on a beam at regular distances, such as a bridge with a traffic jam or a train crossing, the point loads can often be replaced by a uniformly distributed load to simplify the calculation. The uniformly distributed load is then a schematisation of the actual load.

To gain a picture of the consequences of this type of modelling, we will look at the simply supported beam in Figure 12.34, which is subject to a system of equal point loads at mutually equal distances. With n point loads F at mutually equal distances a on a span of length ℓ it applies that

$$\ell = n \cdot a.$$

The substitute uniformly distributed load is

$$q = \frac{F}{a} = \frac{nF}{\ell}.$$

The point loads generate a stepped shear force diagram and a bent bending moment diagram. The uniformly distributed load causes a linear shear force diagram and a parabolic bending moment diagram. Both V and M diagrams have the same value midway between two successive point loads. Because the shear force V is equal to the slope of the M diagram, the parabola is tangent to the bent M diagram there.

With an even number of point loads, both moment diagrams have the same maximum bending moment at midspan, see the M diagram in Figure 12.34b:

$$M_{\max} = \tfrac{1}{8} q \ell^2 = \tfrac{1}{8} \frac{nF}{\ell} \ell^2 = \tfrac{1}{8} nF\ell.$$

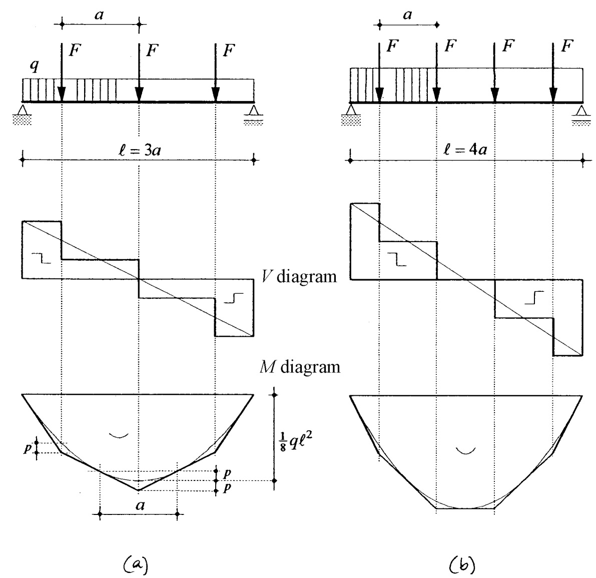


Figure 12.34 The V and M diagrams due to a number of point loads, respectively a substitute uniformly distributed load. (a) With an odd number of point loads the substitute distributed load gives a somewhat smaller maximum bending moment at midspan. (b) With an even number of point loads, both bending moment diagrams have the same maximum bending moment.

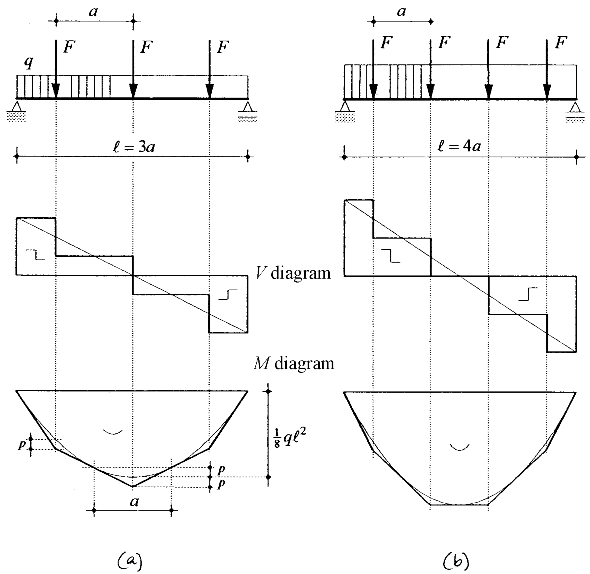


Figure 12.34 The V and M diagrams due to a number of point loads, respectively a substitute uniformly distributed load. (a) With an odd number of point loads the substitute distributed load gives a somewhat smaller maximum bending moment at midspan. (b) With an even number of point loads, both bending moment diagrams have the same maximum bending moment.

With an odd number of point loads, the maximum bending moment at midspan due to the point loads is a value of p larger than the bending moment due to the substitute uniformly distributed load, see the M diagram in Figure 12.34a:

$$M_{\max} = \tfrac{1}{8}q\ell^2 + p$$

in which

$$p = \tfrac{1}{8}qa^2.$$

The actual maximum bending moment is therefore larger. The difference is

$$\frac{p}{\frac{1}{8}q\ell^2} \cdot 100\% = \frac{a^2}{\ell^2} \cdot 100\% = \frac{100}{n^2}\%.$$

With one concentrated load ($n = 1$) the difference is 100%,[1] but this reduces rapidly for more loads. With $n = 3$ the difference is 11% and for $n = 5$ we are already down to a difference of 4%. It now does not make much difference whether we use in calculations point loads or a substitute uniformly distributed load.

12.5.3 Couple

The load on the beam in Figure 12.35a consists of two parallel and opposite forces F. If the distance a between these forces is small as compared to the length of the beam, the load can also be modelled as a concentrated couple $T = F \cdot a$ (see Figure 12.35b).

[1] See Section 12.1.3, Example 1, with Figures 12.5 and 12.6.

Figure 12.35 shows the M and V diagrams for

$$F = 120\ \text{kN},$$

$$a = 0.5\ \text{m}.$$

Thus

$$T = F \cdot a = 60\ \text{kNm}.$$

The differences in the M diagrams are minor: the maximum bending moments due to the forces are a fraction smaller than those due to the substitute couple.

On the other hand, the differences in the V diagrams are far larger. The shear force is equal to the slope of the M diagram. At the concentrated couple the slope becomes infinitessimally large over an infinitely small length. Since infinitessimally small and infinitessimally large do not exist in physical reality, the dashed part of the V diagram in Figure 12.35b is omitted.

For the simply supported beam in Figure 12.35a, the total area of the V diagram is equal to zero (rule 12):

$$\Delta M^{(\text{AB})} = M_{\text{B}} - M_{\text{A}} = \int_0^{\ell} V\,\mathrm{d}x = 0.$$

It is clear that the area of the V diagram is no longer zero when a concentrated couple acts on the beam. By omitting the dashed part in the V diagram in Figure 12.35b at the concentrated couple (an infinitely large value over an infinitessimally small length, but with a finite area), the total area of the V diagram changes. This is no longer zero, but is now equal to the magnitude of the couple.

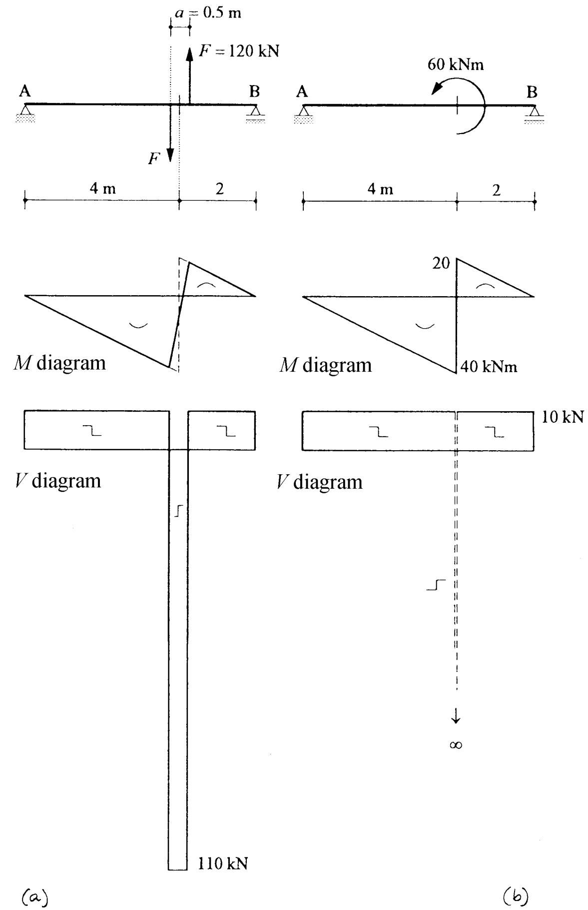


Figure 12.35 The M and V diagrams (a) due to a couple of 60 kNm formed by two parallel opposite forces and (b) due to a concentrated couple of 60 kNm.

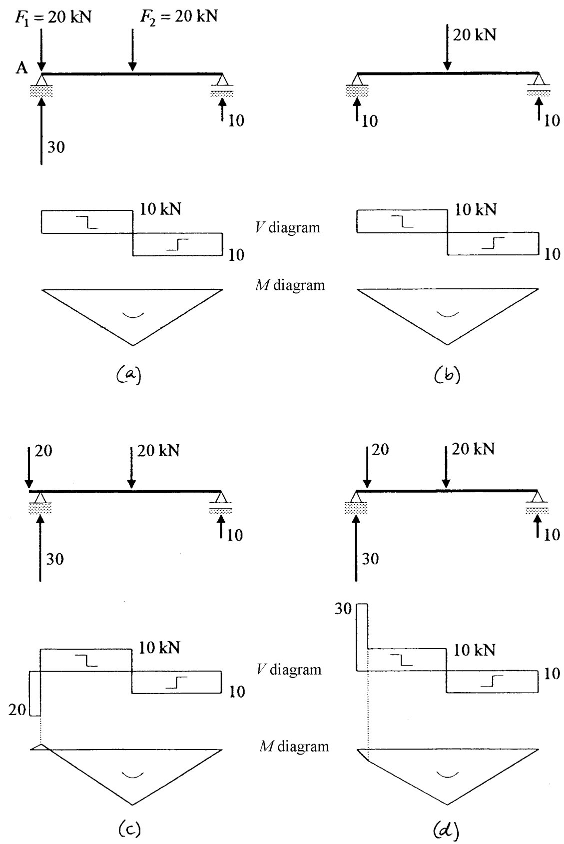


Figure 12.36 Various loads with associated V and M diagrams. In cases (a) and (b) the V and M diagrams are the same, but the support reactions differ due to the load at A. If this load is not directly above the support, but slightly to the left or to the right, the situations in respectively (c) and (d) occur.

12.5.4 Shear forces at a support

The simply supported beam in Figure 12.36a is loaded by two forces F_1 and F_2. The force F_1 at support A is directly transferred to the foundation and does not influence the force flow in the beam. The V and M diagrams are equal to those for the beam in Figure 12.36b, without force F_1. Only the support reactions differ.

Note: In Figure 12.36a, the shear force directly to the right of A is not equal to the support reaction at A!

The fact that force F_1 is exactly above support A is theoretically possible. However, it is more likely that F_1 acts slightly to the left (Figure 12.36c) or slightly to the right (Figure 12.36d) of A. In both cases the maximum bending moment changes only by a small amount, but there are major differences in the V diagram at A.

The differences noted in the V diagram may be less serious than sketched here, for concentrated forces do not exist in reality. Also, members idealised as line elements (the member axis) in reality have cross-sectional dimensions in which the shear force is a model for the transfer of forces normal to the member axis. One should always keep in mind that there are differences between an idealised and the real situation.

12.5.5 Eccentric axial forces

Line elements are structural elements for which the cross-sectional dimensions are considerably smaller than the length. Through simplifying assumptions in the smaller directions (those of the cross-section) the properties of such a structural element can be ascribed to a single line. This line, the so-called member axis, is a one-dimensional model of a structural element that in reality is three-dimensional. In mechanical diagrams, we usually represent a line element by its axis, and draw it without its cross-sectional dimensions.

We have not yet discussed the position of the member axis within the cross-section. This location is not important as long as all the forces are acting normal to the member axis. It does make a difference if there are also forces (with components) parallel to the member axis.

As an example, we use the simply supported beam in Figure 12.37a, for which we assume that the member axis is halfway down the depth of the beam.

The support reactions follow directly from the equilibrium of the beam. If we model the beam as a line element, none of the forces on the beam (including the support reactions) have their point of application on the member axis (see Figure 12.37b).

Forces (force components) normal to the member axis can be shifted along their line of action to the member axis.

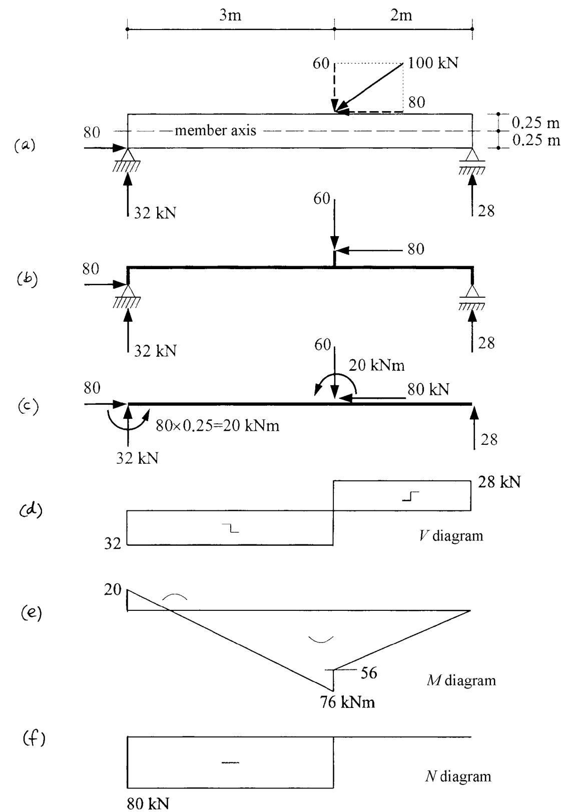
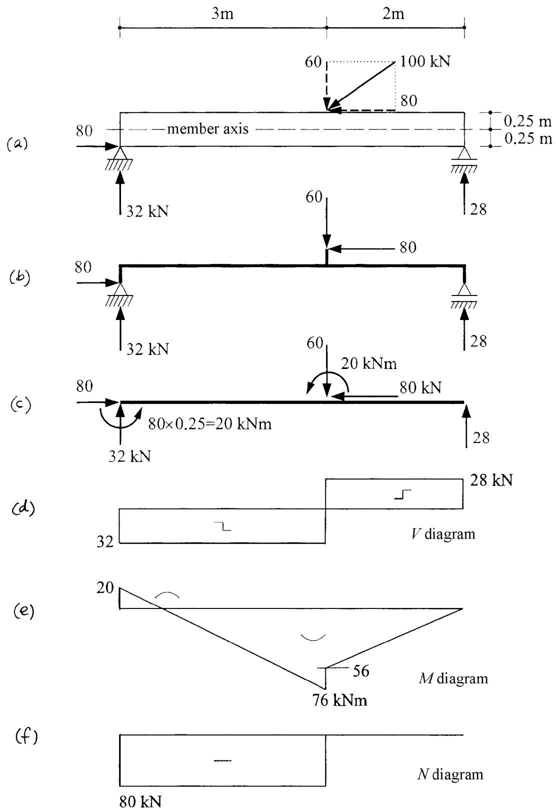


Figure 12.37 (a) Simply supported beam with load and support reactions. The member axis is at half the depth of the beam. (b) The beam modelled as a line element. None of the forces have their point of application on the member axis. (c) Forces (force components) normal to the member axis can be shifted along their line of action to the member axis. Forces (force components) acting eccentrically and parallel to the member axis can be shifted provided that a couple is added concurrently. The magnitude of the couple is equal to the product of force and eccentricity. (d) Shear force diagram. (e) Bending moment diagram. Note that the bending moment at the hinged support is not zero. (f) Normal force diagram.



Forces (force components) acting eccentrically and parallel to the member axis can be shifted to the member axis as long as we concurrently add couples that are equal in magnitude to the product of force and eccentricity (see Section 3.1.5). In this example, the two couples are:

$$(80\ \text{kN})(0.25\ \text{m}) = 20\ \text{kNm}.$$

In Figure 12.37c, all the loads are acting on the member axis. Figures 12.37d to 12.37f show the associated V, M and N diagrams.

Eccentric axial forces exert moments on the beam modelled as a line element and cause step changes in the M diagram.

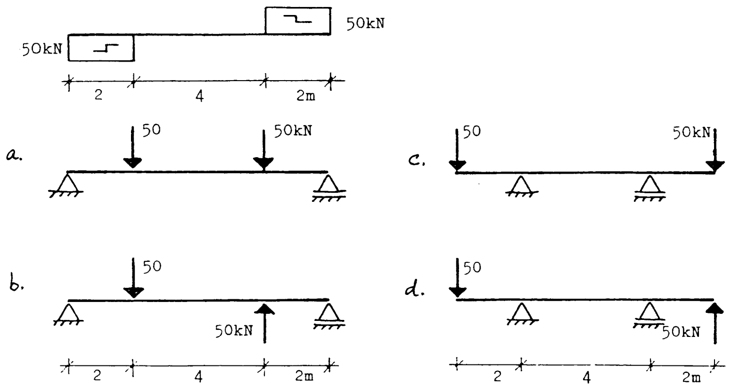
Note that the bending moment at the hinged support is not zero. Also note that the slope of the M diagram agrees with the shear force, and that the total area of the V diagram is no longer zero, but is equal to the sum of the concentrated couples on the line element.

Figure 12.37 (a) Simply supported beam with load and support reactions. The member axis is at half the depth of the beam. (b) The beam modelled as a line element. None of the forces have their point of application on the member axis. (c) Forces (force components) normal to the member axis can be shifted along their line of action to the member axis. Forces (force components) acting eccentrically and parallel to the member axis can be shifted provided that a couple is added concurrently. The magnitude of the couple is equal to the product of force and eccentricity. (d) Shear force diagram. (e) Bending moment diagram. Note that the bending moment at the hinged support is not zero. (f) Normal force diagram.

12.6 Problems

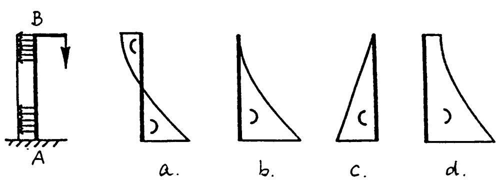
Shape of the V and M diagrams (Section 12.1.1)

12.1 Given a shear force diagram and four loaded beams.



Question :
Which beam matches the shear force diagram?

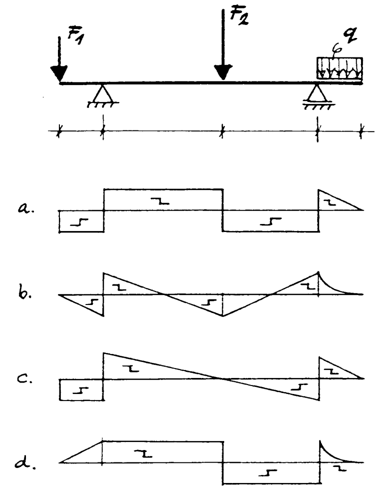
12.2 Four bending moment diagrams are given for column AB.



Question :
Which bending moment diagram matches the given load?

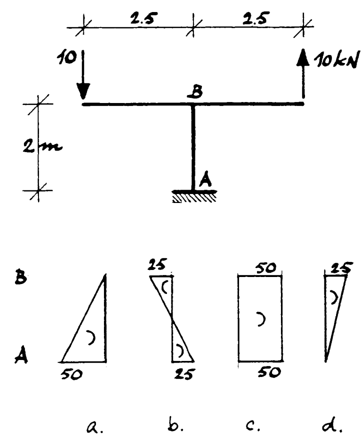
12.3 Given a loaded beam and four shear force diagrams.

Question :
Which shear force diagram matches the loaded beam?



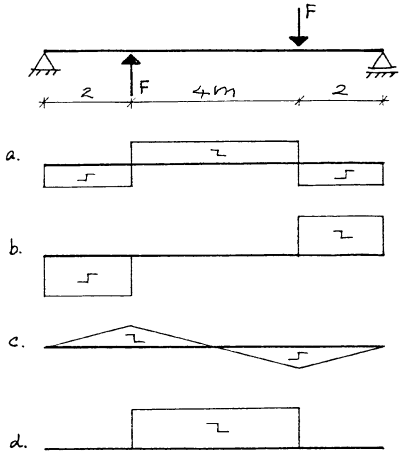
12.4 Four bending moment diagrams are given for column AB (values in kNm).

Question :
Which bending moment diagram matches the given load?



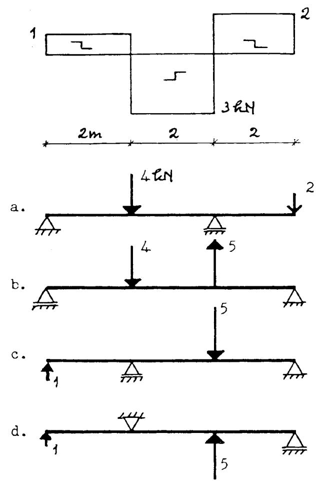
12.5 A loaded beam and four shear force diagrams are given.

Question :
Which shear force diagram matches the loaded beam?



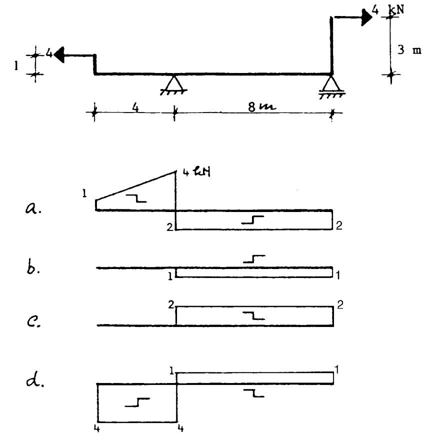
12.6 A shear force diagram and four loaded beams are given.

Question :
Which loaded beam matches the shear force diagram?



12.7 Four shear force diagrams are shown for beam ABC, loaded by two eccentric tensile forces.

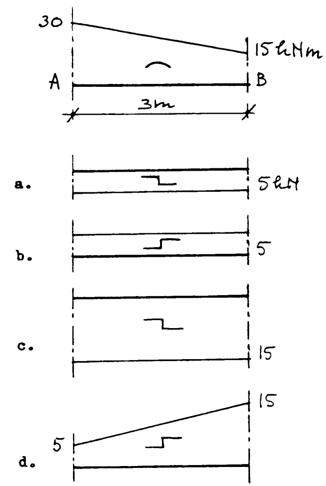
Question :
Which shear force diagram is correct?



Slope of the V and M diagrams and extreme values (Section 12.1.2)

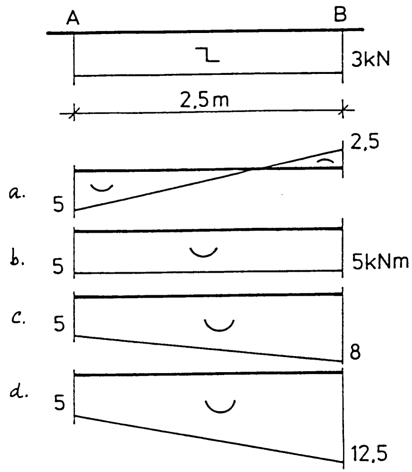
12.8 Given the bending moment diagram for beam segment AB and four shear force diagrams.

Question:
Which shear force diagram is correct?



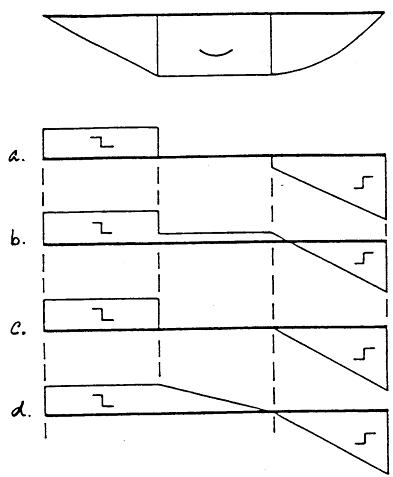
12.9 The shear force diagram for beam segment AB and four bending moment diagrams are given.

Question:
Which bending moment diagram may be correct?



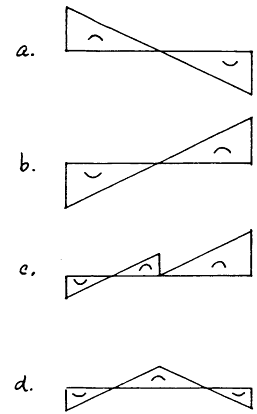
12.10 A bending moment diagram and four shear force diagrams are given.

Question:
Which shear force diagram matches the bending moment diagram?



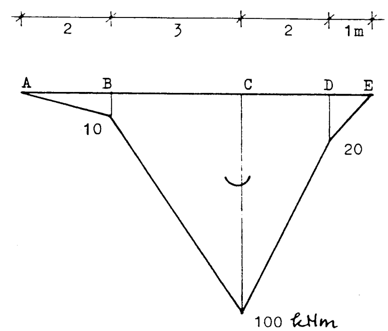
12.11 Four bending moment diagrams are drawn to the same scale.

Question:
Which two bending moment diagrams match the same shear force diagram?

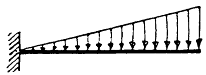


12.12 The bending moment diagram for beam AE is given.

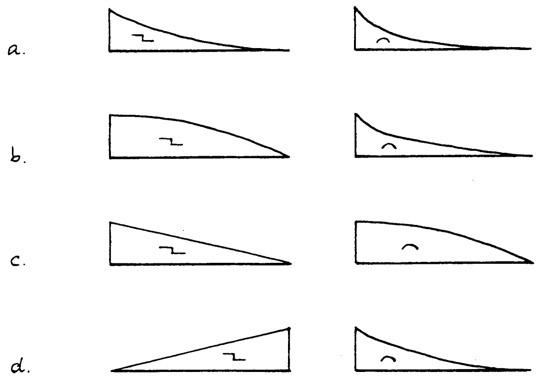
Question:
Where in the beam is the shear force an extreme?



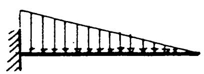
12.13 A cantilever beam with a linear distributed load is given.



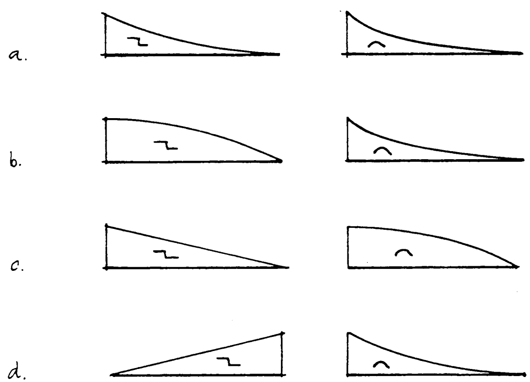
Question:
Which combination of V and M diagrams matches this loading case?



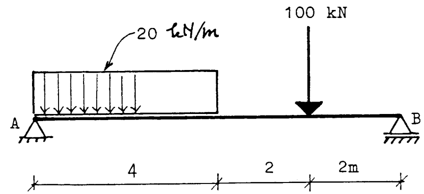
12.14 A cantilever beam with a linear distributed load is given.



Question:
Which combination of V and M diagrams matches this loading case?

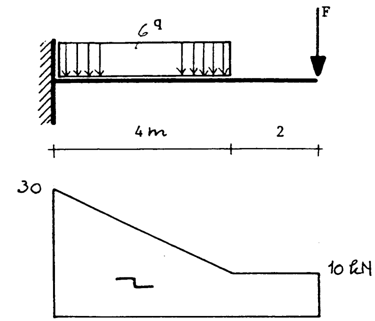


12.15 A beam with its load is given.



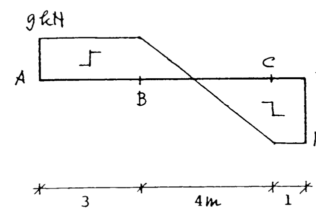
Question:
Determine at A the slope of the tangent to the M diagram (in kNm/m).

12.16 A rigidly supported beam with its shear force diagram is given.



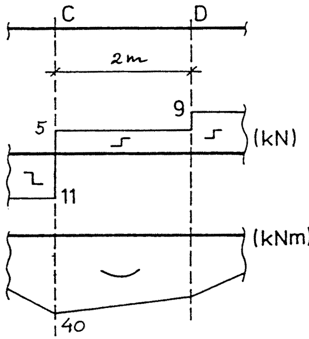
Question:
Determine the magnitude of the uniformly distributed load q.

12.17 The shear force diagram for beam AD is given.



Question:
Determine the magnitude and direction of the uniformly distributed load in field BC.

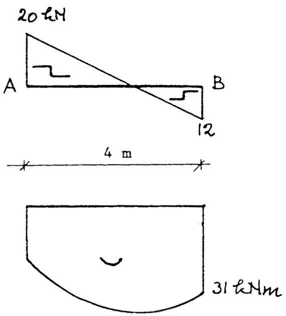
12.18 Given the shear force diagram for beam segment CD and the bending moment at C.



Questions:

a. Isolate segment CD and draw all the (section) forces acting on it.
b. From the equilibrium of CD determine the bending moment at D.
c. Check for CD that the slope of the M diagram is equal to the shear force.

12.19 Given the shear force diagram for beam segment AB and a sketch of the bending moment diagram. The bending moment at B is 31 kNm.



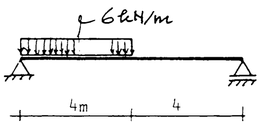
Questions:

a. Isolate beam segment AB and draw all the forces acting on it.
b. From the equilibrium of AB determine the bending moment at A.
c. Where is the bending moment an extreme?
d. Determine this moment.

12.20 Given a simply supported beam with a uniformly distributed load on the left-hand half.

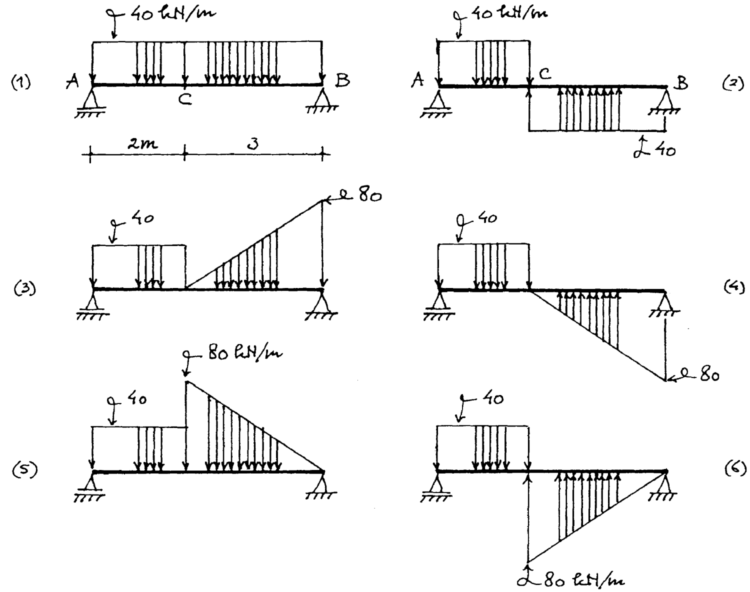
Questions:

a. Draw the shear force diagram.
b. Where is the bending moment an extreme?
c. Determine this bending moment.



Tangents to the M diagram (Section 12.1.3)

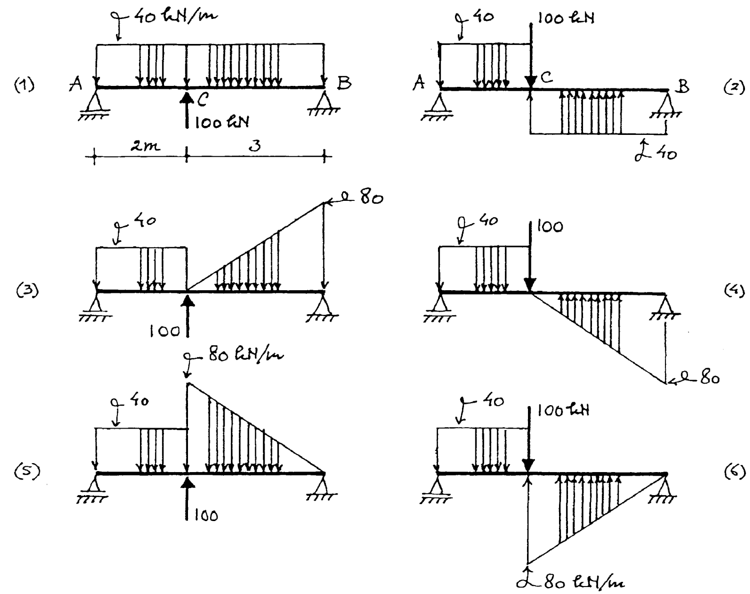
12.21: 1–6 The same beam ACB is loaded in six different ways.



Questions:

a. Draw the bending moment diagram due to the load resultants in the fields AC and CB.
b. Use this bending moment diagram to determine the bending moment and the shear force at C, the join of the fields.
c. Sketch the bending moment diagram due to the distributed load.

12.22: 1–6 The same beam ACB is loaded in six different ways. The difference to the previous problem is an additional force of 100 kN at C.



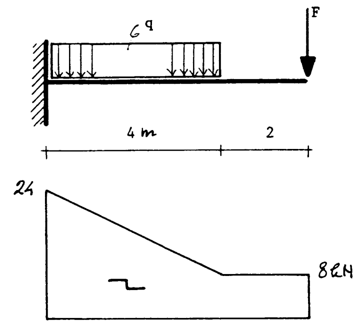
Questions:

a. Draw the bending moment diagram due to the load resultants in the fields AC and CB and the force of 100 kN at C.
b. Sketch the bending moment diagram due to the load actually present.

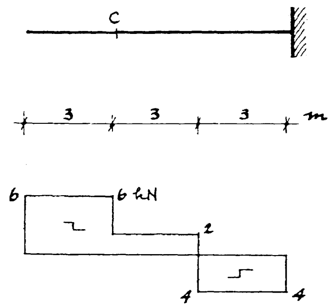
Interpreting the area of the load diagram and V diagram (Section 12.1.4)

12.23 Given a fixed beam and its shear force diagram.

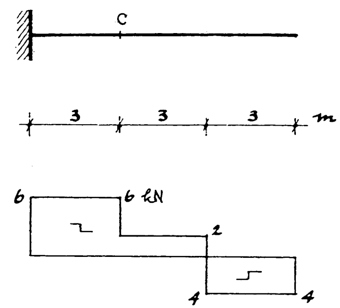
Question:
Determine the magnitude of the fixed-end moment.



12.24: 1–2 Given the shear force diagram for a fixed beam, loaded by three point loads.



(1)

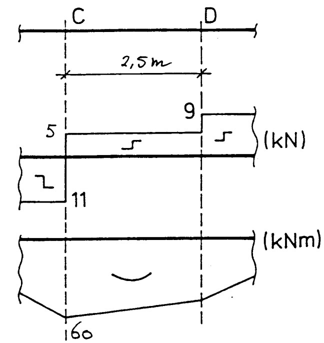


(2)

Question:
How large is the bending moment at C?

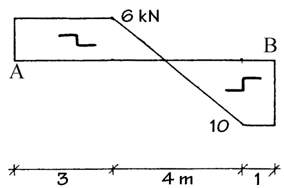
12.25 Given the shear force diagram for a beam segment CD and a sketch of the bending moment diagram. The bending moment at C is 60 kNm.

Question:
Determine the bending moment at D.



12.26 Given the shear force diagram for a simply supported beam AB. No (concentrated) couples are acting on the beam.

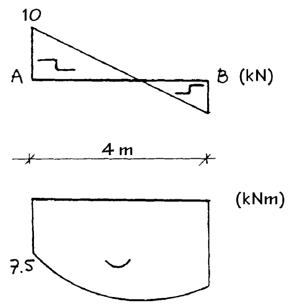
Question:
Determine the maximum bending moment.



12.27 Given the shear force diagram for beam segment AB and a sketch of the bending moment diagram. The bending moment at A is 7.5 kNm.

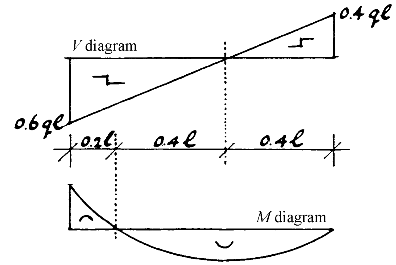
Questions:

a. Determine the maximum bending moment.
b. Determine the bending moment at B.



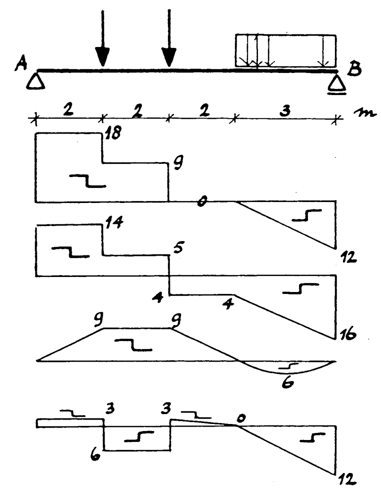
12.28 Given a shear force diagram and the associated bending moment diagram.

Question:
Determine the extreme values of the bending moment.

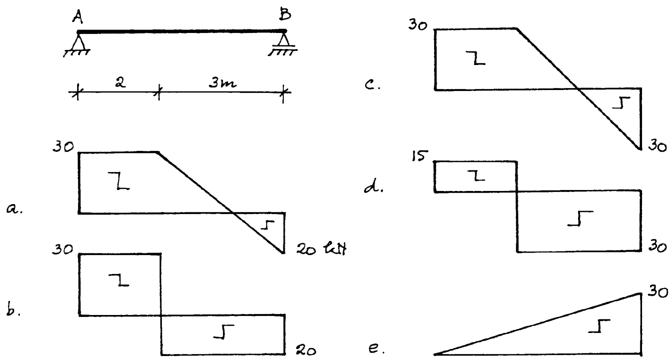


12.29 Given a loaded beam and four shear force diagrams.

Question:
Which shear force diagram could be the correct one?



12.30 Given beam AB with five shear force diagrams. In addition to (distributed) forces normal to the beam axis, the beam may also be subject to a (concentrated) couple.

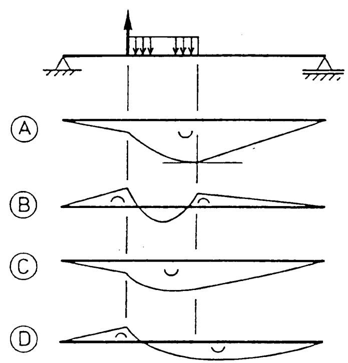


Question:
In which cases is a (concentrated) couple acting on the beam? In those cases determine the magnitude of the couple and the direction in which it is acting.

Step changes and bends in the V and M diagrams (Section 12.1.5)

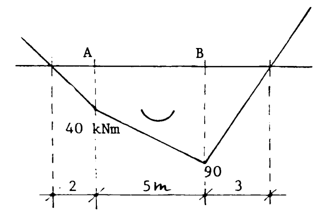
12.31 Given a loaded beam and four bending moment diagrams.

Question:
Which bending moment diagram has the right shape?



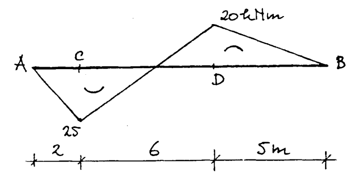
12.32 Given the bending moment diagram for a beam segment subject to forces at A and B.

Question:
The magnitude and direction of these forces.



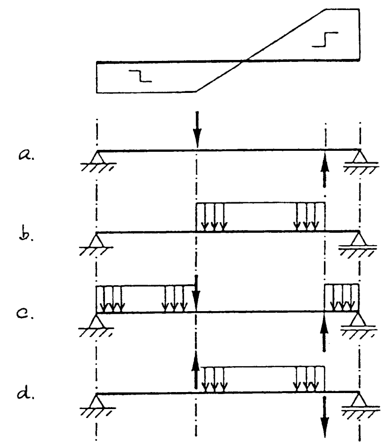
12.33 Given the bending moment diagram for beam AB.

Question:
The magnitude and direction of the forces at C and D.

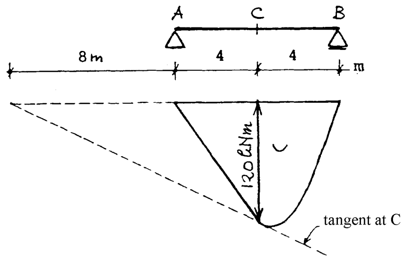


12.34 Given a shear force diagram for four different loaded beams.

Question:
Which load matches the shear force diagram?

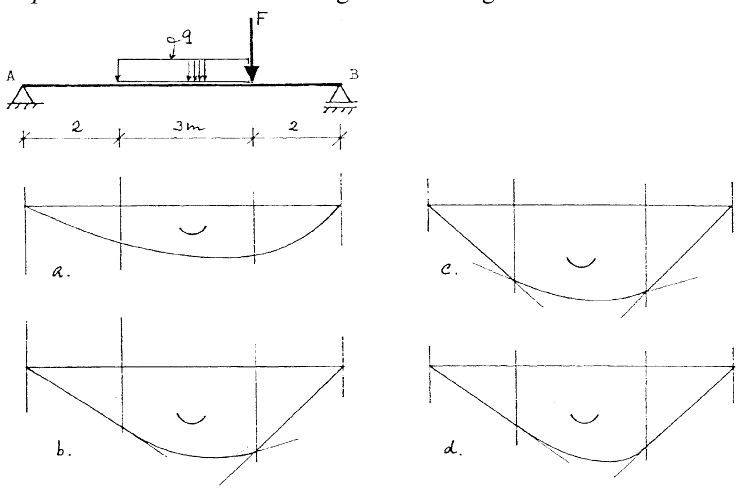


12.35 The bending moment diagram of beam AB varies linearly along the left-hand side and is curved on the right-hand side. The tangent to the curved part of the M diagram in the middle C is also shown.



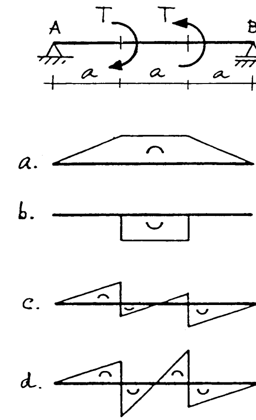
Question:
Determine the magnitude and direction of the point load at C.

12.36 The simply supported beam AB is loaded by a uniformly distributed load q and a force F. Four bending moment diagrams are shown.



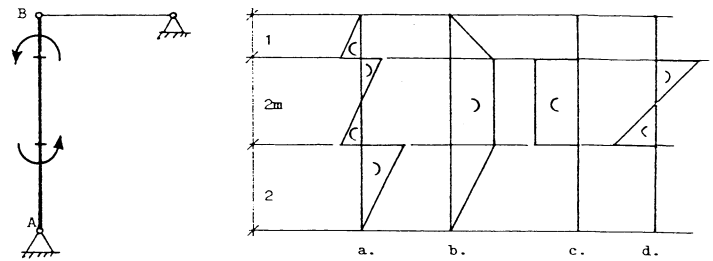
Question:
Which bending moment diagram matches the loaded beam?

12.37 Given four bending moment diagrams for beam AB.



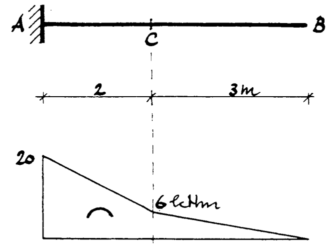
Question:
Which bending moment diagram matches the given load?

12.38 Given four bending moment diagrams for post AB.



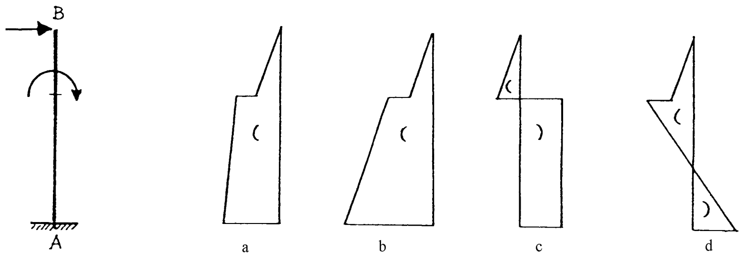
Question:
Which bending moment diagram could match the given load?

12.39 Given the bending moment diagram for the cantilever beam AB.



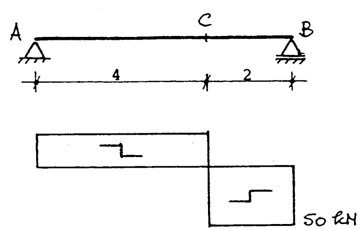
Question:
Determine the magnitude and direction of the force at C.

12.40 Given four bending moment diagrams for column AB.



Question:
Which bending moment diagram could match the given load?

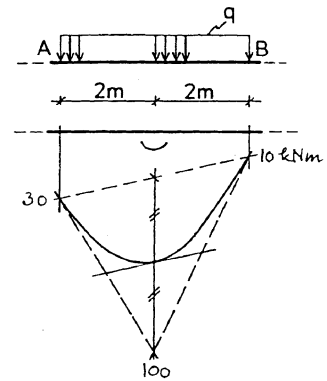
12.41 A sketch is given of the shear force diagram of the simply supported beam ACB, loaded by a force F at C. The magnitude and direction of the shear force in field CB are also given.



Question:
Determine the magnitude and direction of F.

Properties of parabolic M diagrams (Section 12.1.6)

12.42 Given the bending moment diagram for an isolated beam segment AB with a uniformly distributed load q. At A and B the tangents to the M diagram are shown.



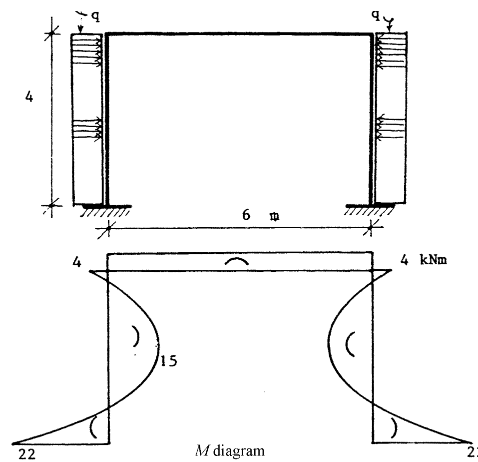
Questions:

a. The magnitude of the uniformly distributed load q.
b. The shear force in the middle of AB.
c. The shear forces at A and B.
d. Isolate beam segment AB, draw all the forces acting on it and check the equilibrium.

12.43 Given a statically indeterminate portal frame with its bending moment diagram due to the given load. The bending moment half-way up the column is 15 kNm.

Questions:

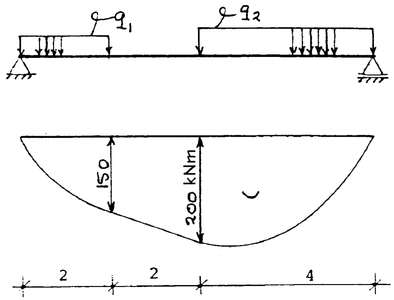
a. The magnitude of the uniformly distributed load q on column AB.
b. The shear force diagram for column AB.
c. The location and magnitude of the maximum bending moment in column AB.



12.44 Given a sketch of the bending moment diagram of a simply supported beam with uniformly distributed loads q_1 and q_2.

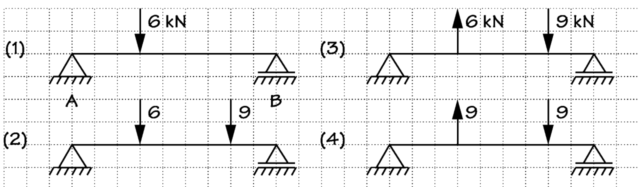
Questions:

a. From the shape of the M diagram determine the magnitude of the distributed loads.
b. Draw the shear force diagram for the entire beam.
c. Determine the location and magnitude of the maximum bending moment in the beam.



Mixed problems (Section 12.1)

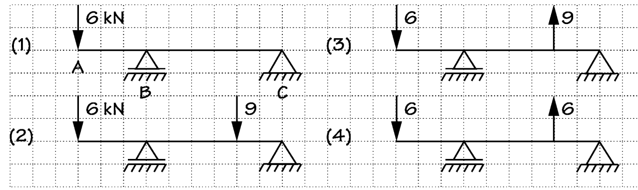
12.45: 1–4 The same simply supported beam is loaded in various ways. Length scale: 1 square ≡ 1 m; forces in kN.



Questions:

a. Determine the bending moment diagram.
b. Determine the shear force diagram.

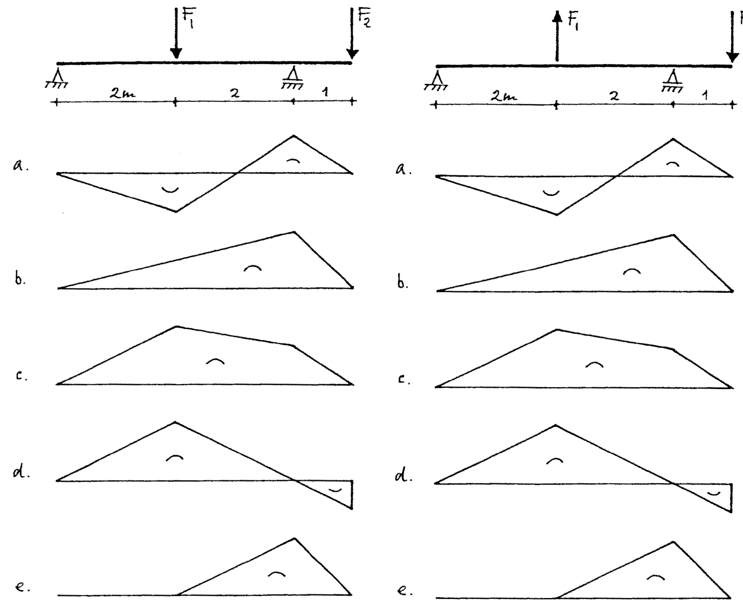
12.46: 1–4 The same beam with overhang is loaded in various ways. Length scale: 1 square ≡ 1 m; forces in kN.



Questions:

a. Determine the bending moment diagram.
b. Determine the shear force diagram.

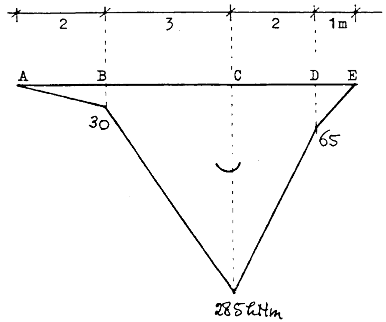
12.47: 1–2 Two loading cases and five bending moment diagrams are given.



(1) (2)

Question:
Which bending moment diagram(s) in no way matches (match) the loading case?

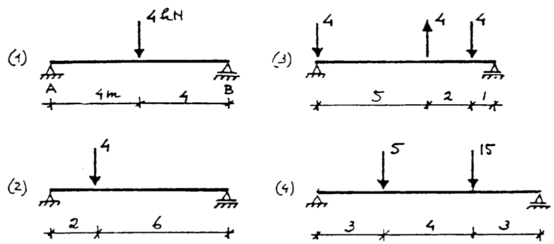
12.48 Given the bending moment diagram for beam AE. Five forces are acting on the isolated beam.



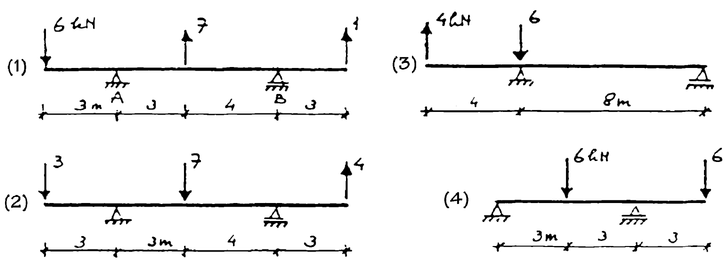
Question:
Determine the magnitude and direction of these forces.

12.49: 1–4 Given a number of beams loaded in various ways.

Questions:
a. Determine the bending moment diagram.
b. Determine the shear force diagram.

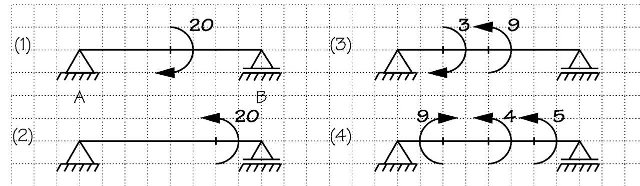


12.50: 1–4 Given a number of cantilever beams are loaded in various ways.



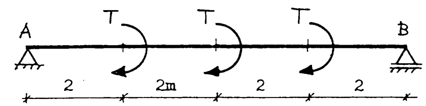
Questions:
a. Determine the bending moment diagram.
b. Determine the shear force diagram.

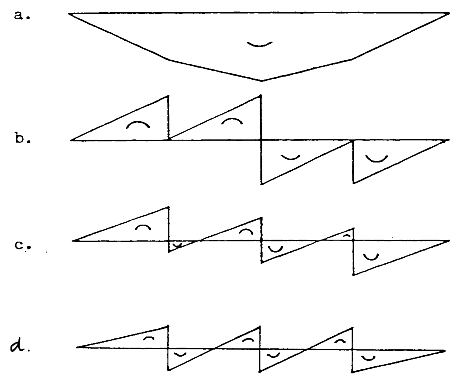
12.51: 1–4 The same simply supported beam is loaded in various ways by only couples. Length scale: 1 square ≡ 1 m; couples in kNm.



Questions:
a. Determine the bending moment diagram.
b. Determine the shear force diagram.

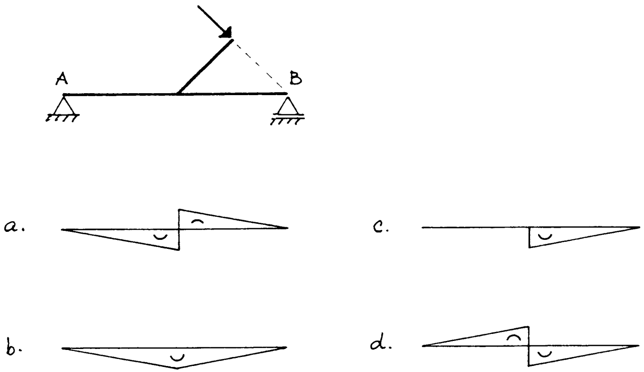
12.52 Given four bending moment diagrams for beam AB.





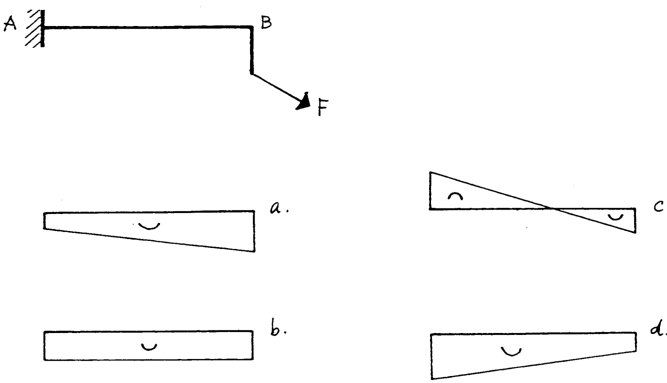
Question:
Which bending moment diagram matches the given load?

12.53 Given four bending moment diagrams for beam AB.



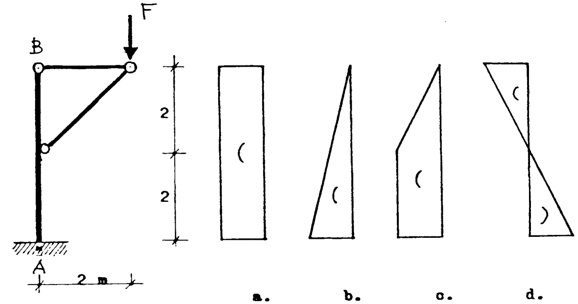
Question:
Which bending moment diagram matches the given load?

12.54 Given four bending moment diagrams for beam AB.



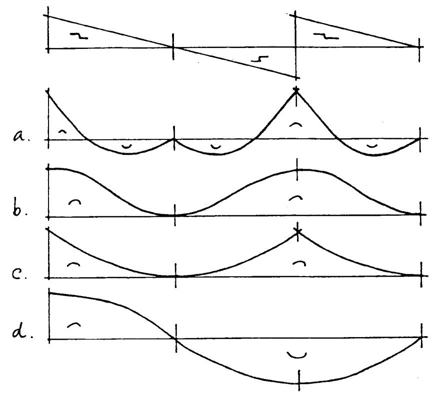
Question:
Which bending moment diagram matches the given load?

12.55 Four bending moment diagrams for column AB are given.



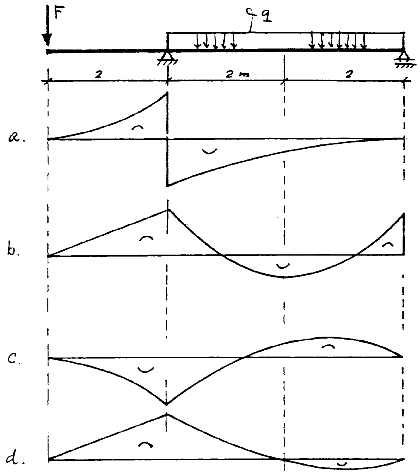
Question:
Which bending moment diagram is correct?

12.56 A shear force diagram and four bending moment diagrams are given.



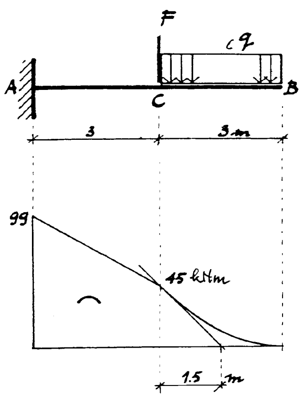
Question:
Which bending moment diagram matches the shear force diagram?

12.57 Given a loaded beam and four bending moment diagrams.



Question:
Which bending moment diagram matches the loaded beam?

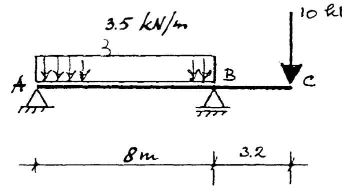
12.58 Given the bending moment diagram for beam AB due to a force F at C and a uniformly distributed load q along field CB. The direction of F is not given.



Questions:

a. Determine the magnitude of q.
b. Determine the magnitude and direction of F.

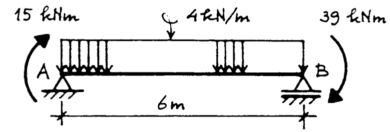
12.59 A beam with overhang is loaded as shown.



Questions:

a. Determine the bending moment diagram with the tangents at A and B.
b. Determine the shear force diagram.
c. Determine the maximum field moment in AB.

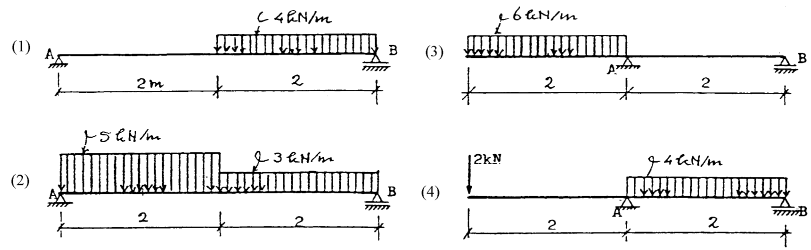
12.60 Given the simply supported beam AB loaded as shown.



Questions:

a. Determine the bending moment diagram with the tangents at A and B.
b. Determine the shear force diagram.
c. Determine the maximum field moment.

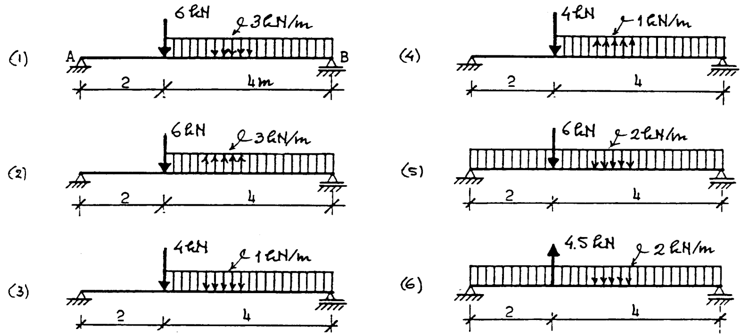
12.61: 1–4 Given four different beams with a uniformly distributed load.



Questions:

a. Determine the M diagram with its tangents at the field boundaries.
b. Determine the V diagram.
c. Determine the location and magnitude of the extreme bending moments.

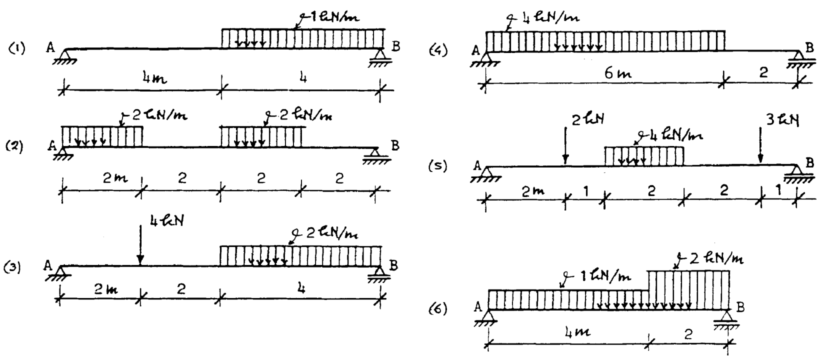
12.62: 1–6 The same beam AB is loaded in six different ways.



Questions:

a. Determine the *M* diagram with the tangents at relevant points.
b. Determine the *V* diagram.
c. Determine the location and magnitude of the maximum/minimum bending moment.

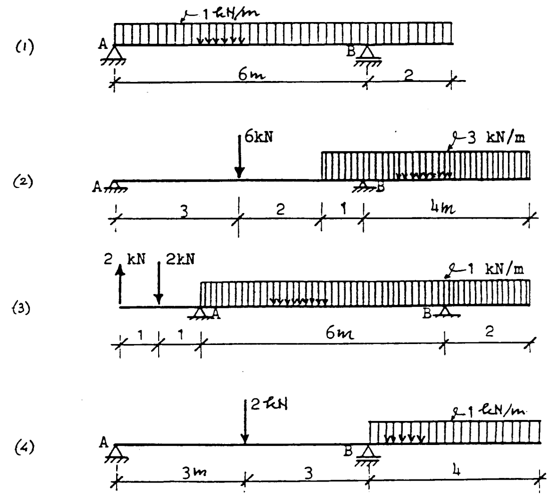
12.63: 1–6 Six different loaded beams are given.



Questions:

a. Determine the *M* diagram with the tangents at relevant points.
b. Determine the *V* diagram.
c. Determine the location and magnitude of the maximum/minimum bending moment.

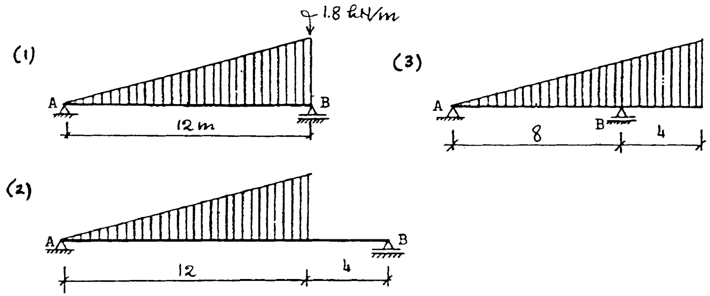
12.64: 1–4 Four different loaded beams with overhang are given.



Questions:

a. Determine the *M* diagram with the tangents at relevant points.
b. Determine the *V* diagram.
c. Determine the location and magnitude of the maximum/minimum bending moment.

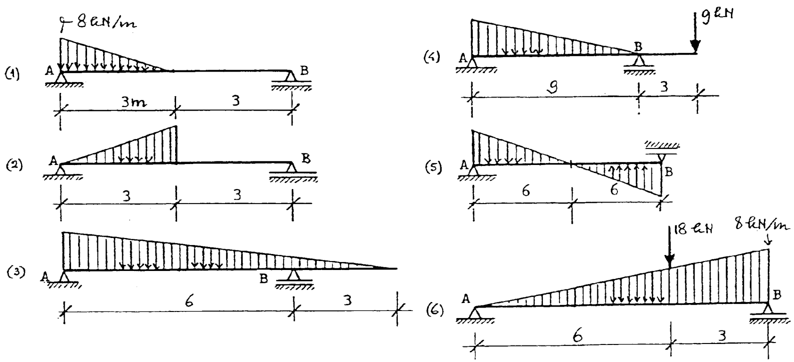
12.65: 1–3 Three beams with a linear distributed load are given. The top value of the distributed load is in all cases 1.8 kN/m.



Questions:

a. Sketch the M diagram with its tangents at the field boundaries.
b. Sketch the V diagram.

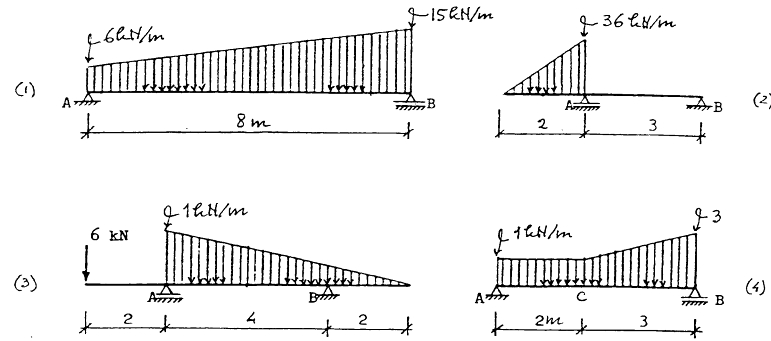
12.66: 1–6 A number of beams with linear distributed load and in two cases also a point load are given. The figures are not all drawn to the same scale. The top value of the linear distributed load is in all cases 8 kN/m. The magnitude of the two point loads can be read off from the figure.



Questions:

a. Sketch the M diagram with its tangents at the field boundaries.
b. Sketch the V diagram.

12.67: 1–4 Four different loaded beams are given.



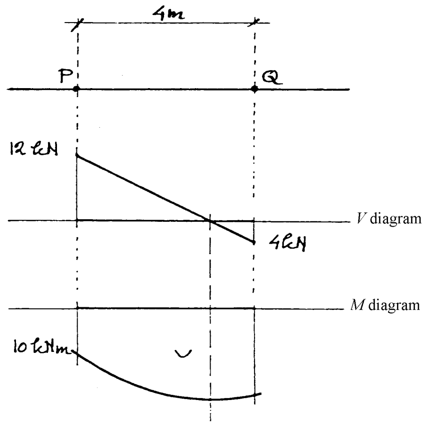
Question:

Sketch the bending moment diagram with the tangents at A, B and C.

12.68 For the segment PQ of a beam the shear force diagram (without deformation symbols), the bending moment at P and a sketch of the bending moment diagram are given.

Questions:

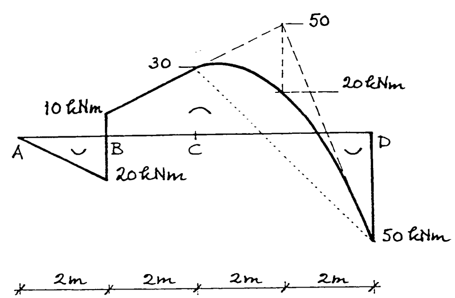
a. Place the deformation symbols in the V diagram.
b. Which load is acting on PQ? Draw the load.
c. Determine the bending moment at Q.
d. Determine the maximum bending moment at PQ.
e. Draw PQ with all the forces (loads and section forces at the edges) acting on it and include the values.



12.69 Member ABCD is loaded normal to its axis. The bending moment diagram is given with the tangents in C and D. The bending moment between C and D is a parabola.

Questions:

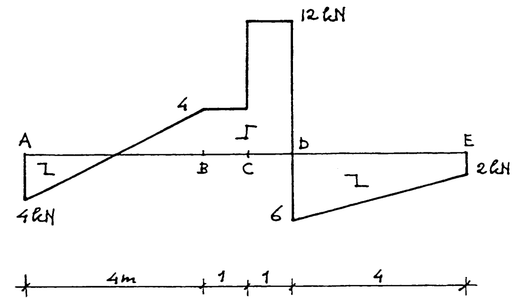
a. Draw the associated shear force diagram with the deformation symbols. Include the relevant values.
b. Draw all the forces (couples) acting on the isolated member. Include the relevant values.
c. Check the equilibrium of member ABCD.



12.70 The V diagram of beam ABCDE is given. There are no couples acting on the beam.

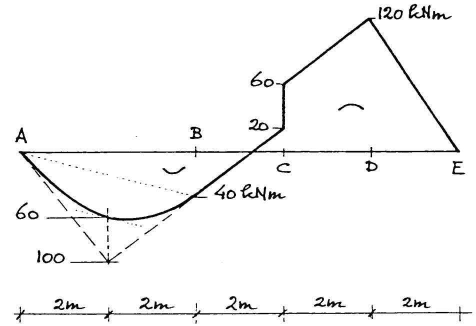
Questions:

a. Draw all the (distributed) forces associated with this V diagram acting on beam ABCDE.



b. For the entire beam, draw the M diagram with the deformation symbols. Include the relevant values. Draw in relevant points the tangents to the M diagram.

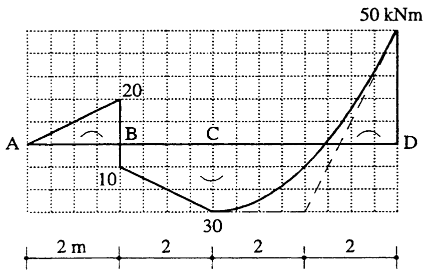
12.71 Member ABCDE is loaded normal to its member axis. The bending moment diagram is given, as are the tangents in A and B. The bending moment between A and B is a second-degree curve (parabola).



Questions:

a. Draw the associated shear force diagram with deformation symbols. Include the relevant values.
b. Draw all the forces (couples) acting on the isolated member. Include the relevant values.
c. Check the equilibrium of the member.

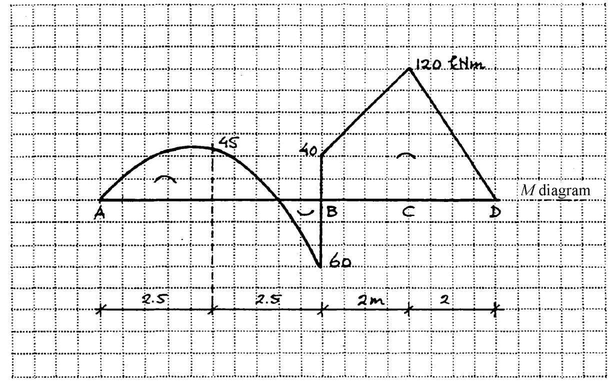
12.72 Member ABCD is loaded normal to its axis. The bending moment diagram is given, as are the tangents at C and D. The bending moment between C and D is a second-degree curve (parabola).



Questions:

a. Draw the associated shear force diagram with deformation symbols. Include the relevant values.
b. Draw all the forces (couples) acting on the isolated member. Include the relevant values.
c. Check the equilibrium of the member. Clearly indicate how you performed this check.

12.73 The bending moment diagram is given for member ABCD. The bending moment between A and B is a parabola (second-degree curve).

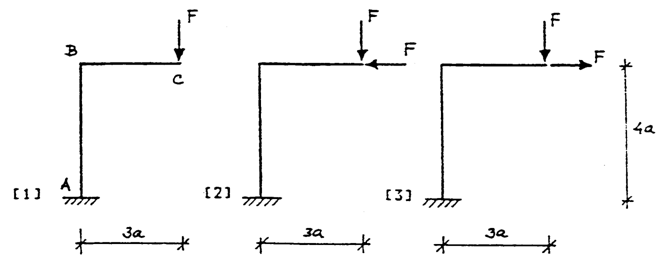


Questions:

a. Draw the associated shear force diagram with deformation symbols. Include the relevant values.
b. Draw all the forces (couples) acting on the isolated member. Include the relevant values.
c. Check the equilibrium of the member.

Bent and compound bar type structures (Section 12.3)

12.74: 1–3 The same bent cantilever is loaded in three different loading ways.

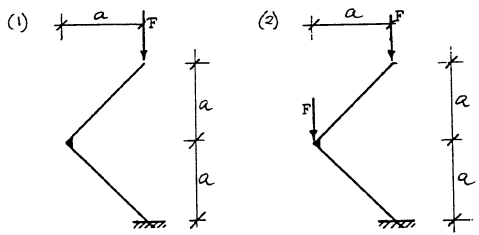


Question:

For the entire structure determine:

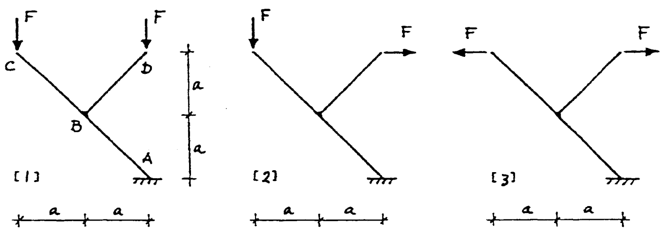
a. the M diagram.
b. the V diagram.
c. the N diagram.

12.75: 1–2 The same bent cantilever is loaded in two different ways.



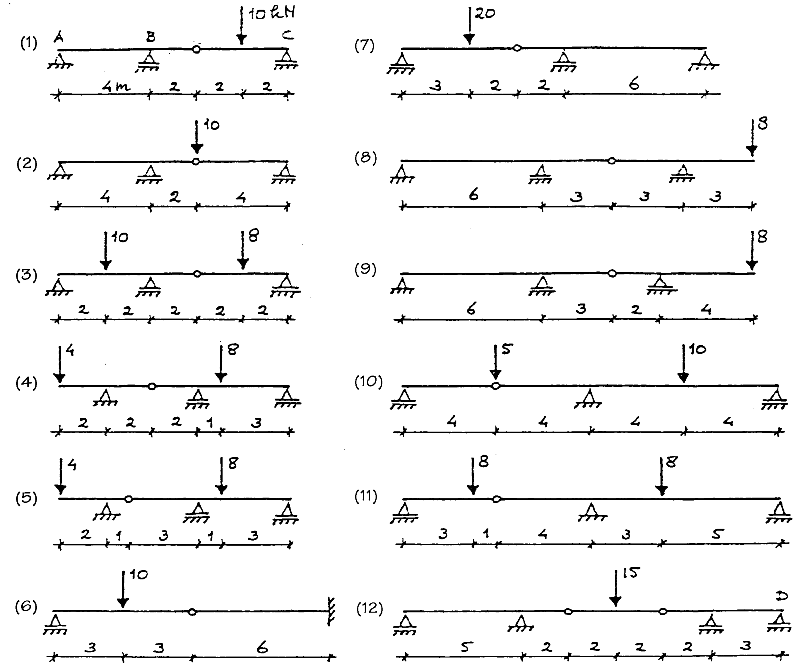
Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.
c. the N diagram.

12.76: 1–3 The same structure is loaded in three different ways.



Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.
c. the N diagram.

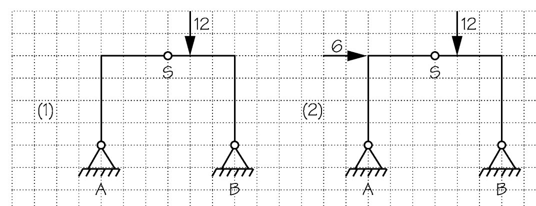
12.77: 1–12 A number of hinged beams with load are given. Dimensions are in m; forces in kN.



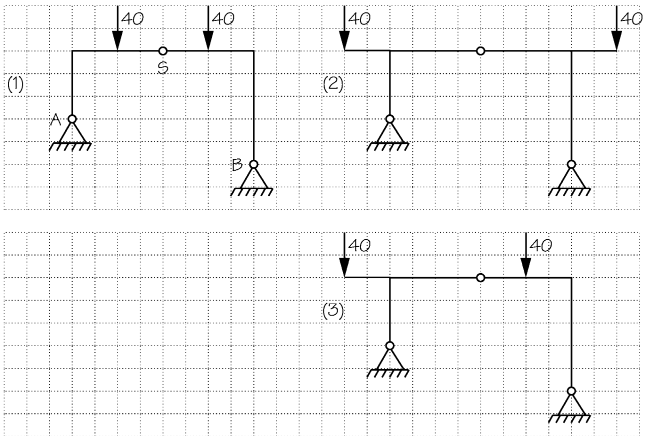
Question:
Determine the bending moment diagram and shear force diagram.

12.78: 1–2 The same three-hinged portal frame is loaded in two ways. Length scale: 1 square ≡ 1 m; forces in kN.

Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.
c. the N diagram.

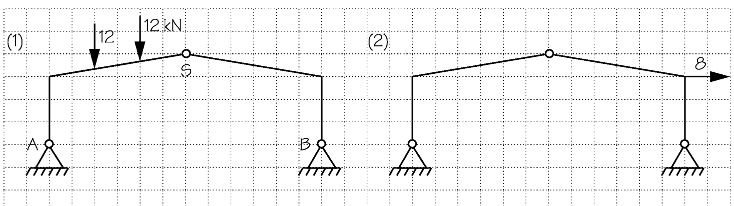


12.79: 1–3 Given three three-hinged portal frames with their load. Length scale: 1 square ≡ 1 m; forces in kN.



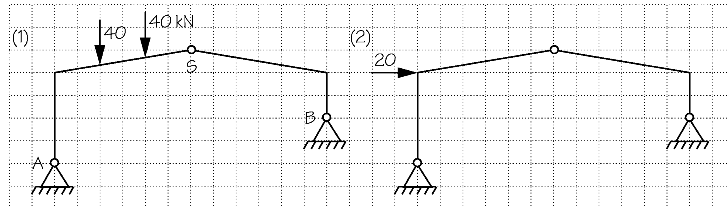
Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.
c. the N diagram.

12.80: 1–2 The same pitched roof portal frame is loaded in two different ways. Length scale: 1 square ≡ 1 m; forces in kN.



Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.

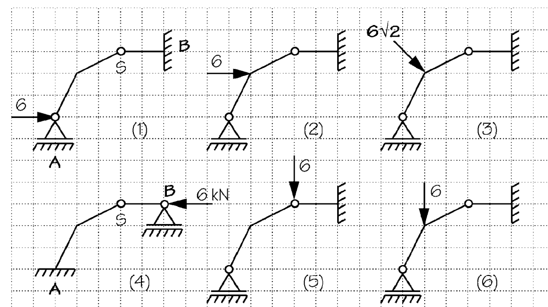
12.81: 1–2 The same three-hinged portal frame is loaded in two different ways. Length scale: 1 square ≡ 1 m; forces in kN.



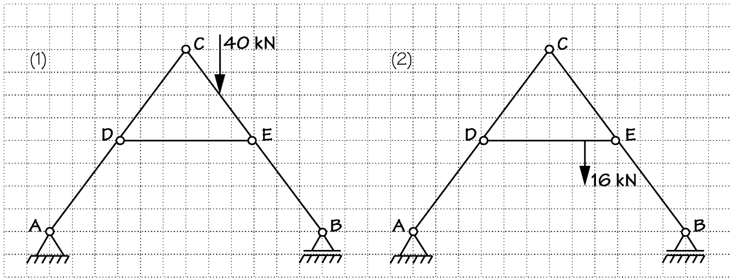
Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.

12.82: 1–6 Given six bent structures. Length scale: 1 square ≡ 1 m; forces in kN.

Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.
c. the N diagram.

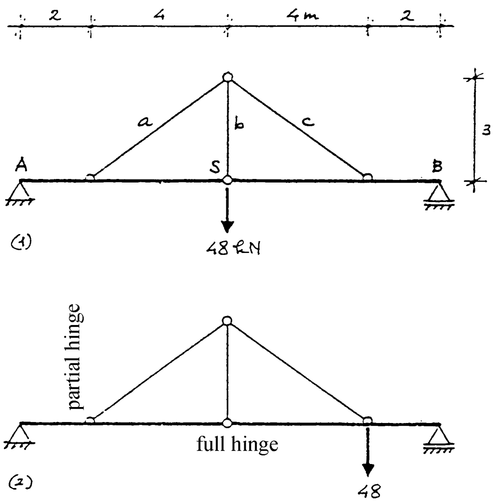


12.83: 1–2 The same structure is loaded in two different ways.



Question:
For the entire structure determine:
a. the M diagram.
b. the V diagram.
c. the N diagram.

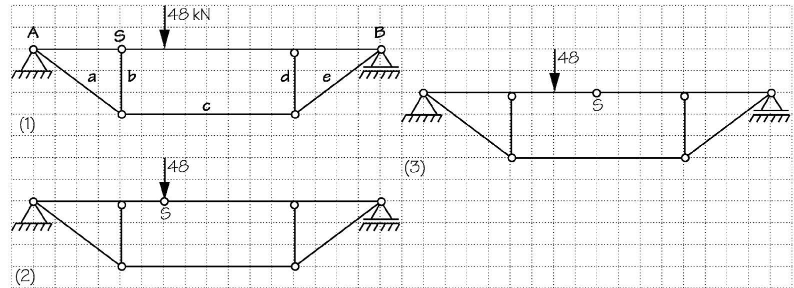
12.84: 1–2 A trussed beam ASB is loaded in two different ways.



Questions:
a. Determine the M and V diagram for ASB.
b. Determine the N diagram for ASB.

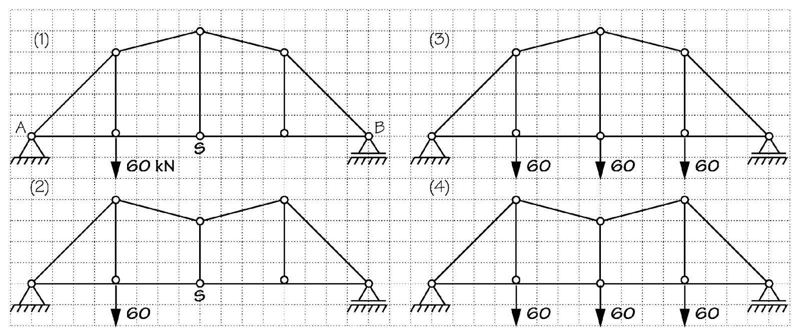
Mind the difference between full hinges and partial hinges.

12.85: 1–5 Hinge S in trussed beam ASB is located in different places. Length scale: 1 square ≡ 1 m; forces in kN.



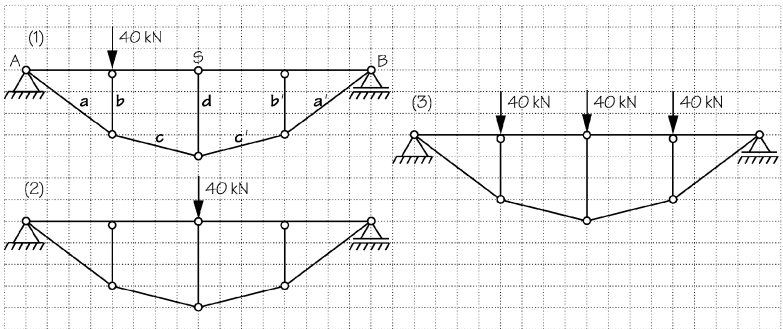
Questions:
a. Determine the M and V diagram for ASB.
b. Determine the N diagram for ASB.

12.86: 1–4 Two trussed beams ASB are loaded in two different ways. Length scale: 1 square ≡ 1 m; forces in kN.



Questions:
a. Determine the M and V diagrams for ASB.
b. Determine the N diagram for ASB.

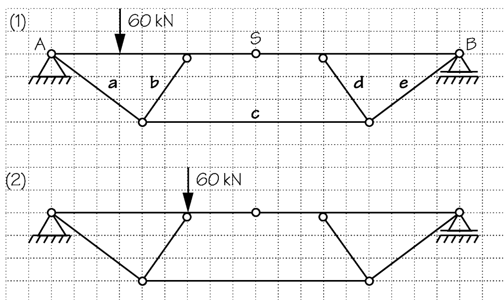
12.87: 1–3 The same trussed beam ASB is loaded in three different ways. Length scale: 1 square ≡ 1 m; forces in kN.



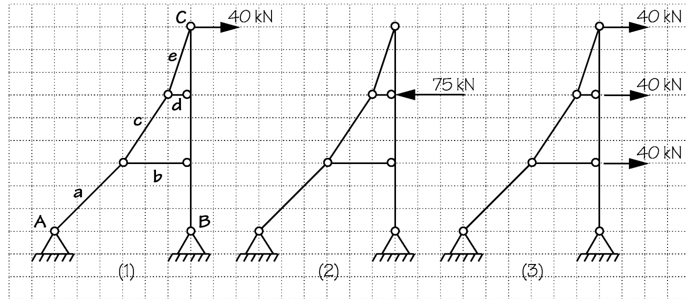
Questions:
a. Determine the M and V diagrams for ASB.
b. Determine the N diagram for ASB.

12.88: 1–2 The same trussed beam ASB is loaded in two different ways. Length scale: 1 square ≡ 1 m; forces in kN.

Questions:
a. Determine the M diagram for ASB.
b. Determine the V diagram for ASB.
c. Determine the N diagram for ASB.

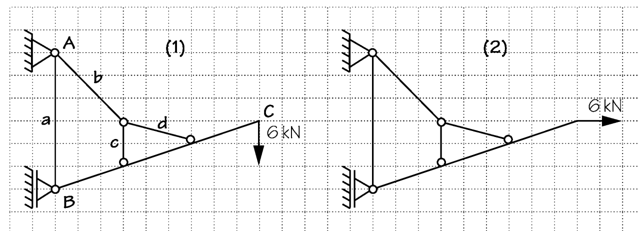


12.89: 1–3 Given a trussed beam with three different loads. Length scale: 1 square ≡ 1 m; forces in kN.



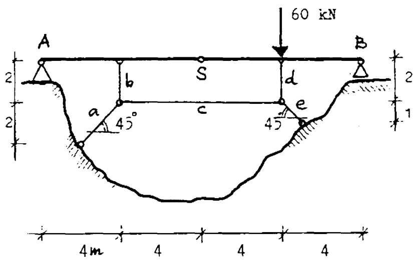
Questions:
a. Determine the M and V diagrams for BC.
b. Determine the N diagram for BC.

12.90: 1–2 Given a trussed beam with two different loads. Length scale: 1 square ≡ 1 m; forces in kN.



Questions:
a. Determine the M and V diagrams for BC.
b. Determine the N diagram for BC.

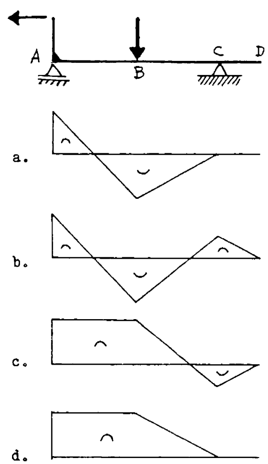
12.91 Given a queen post truss.



Question:
Determine the M and V diagrams for ASB.

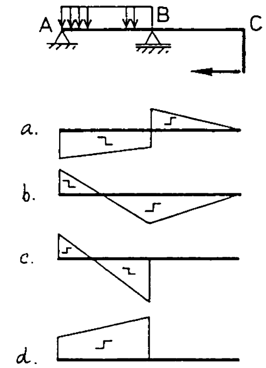
Principle of superposition (Section 12.4)

12.92 Given four bending moment diagrams for beam ABCD.



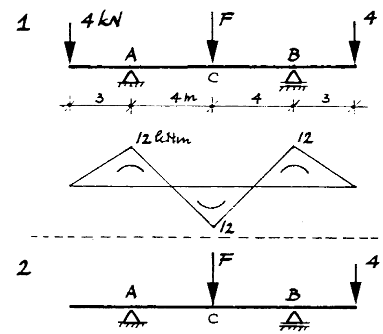
Question:
Which bending moment diagram matches the given load?

12.93 Given four shear force diagrams for beam ABC.



Question:
Which shear force diagram matches the given load?

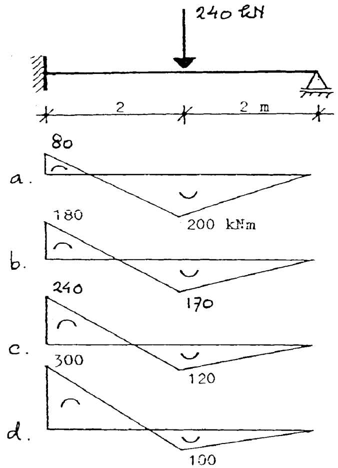
12.94 Given loading case 1 with the associated bending moment diagram and loading case 2 without bending moment diagram. In loading case 2 there is no force on the left-hand overhang.



Question:
Determine the bending moment at C for loading case 2.

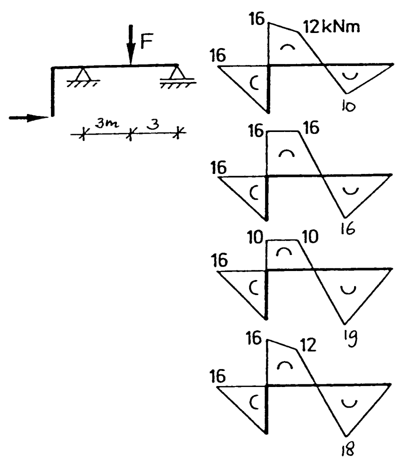
12.95 Given a statically indeterminate beam with four bending moment diagrams.

Question:
Which bending moment diagram(s) may be correct?



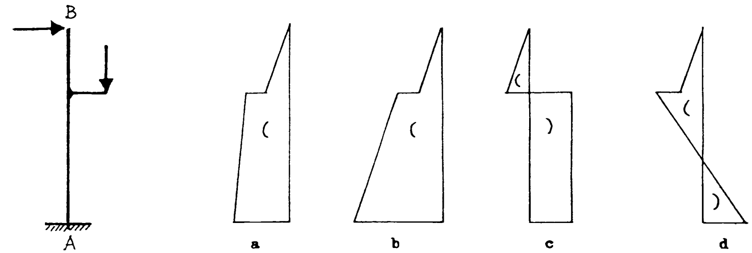
12.96 Four bending moment diagrams are shown for the loaded structure, of which only one is correct.

Question:
Using the correct bending moment diagram determine the magnitude of the force F.



Eccentric axial forces (Section 12.5.5)

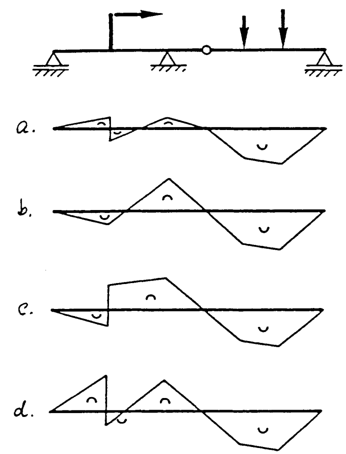
12.97 Given four bending moment diagrams for column AB.



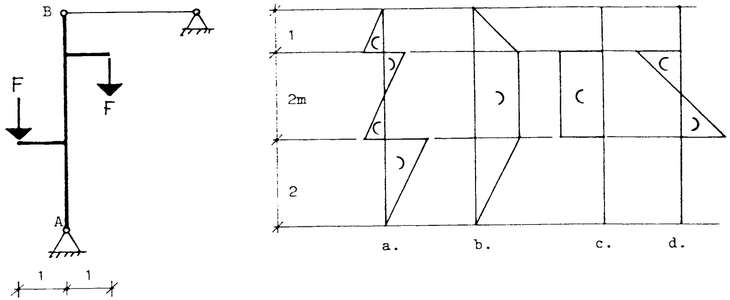
Question:
Which bending moment diagram, with the given load, could be correct?

12.98 Given a loaded beam with four bending moment diagrams.

Question:
Which bending moment diagram has the right shape?

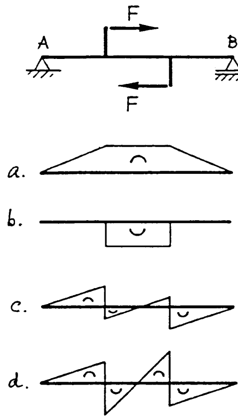


12.99 Given four bending moment diagrams for post AB.



Question:
Which bending moment diagram matches the given load?

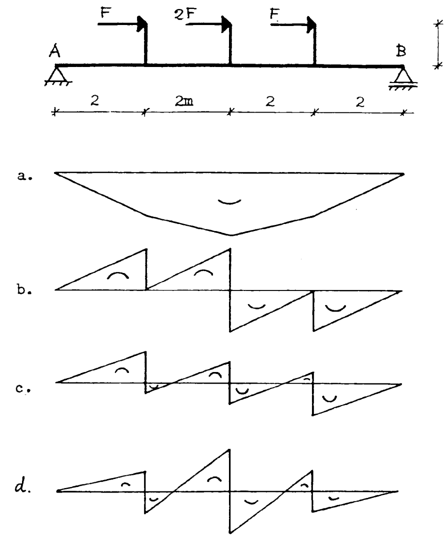
12.100 Given four bending moment diagrams for beam AB.



Question:
Which bending moment diagram matches the given load?

12.101 Given four bending moment diagrams for beam AB.

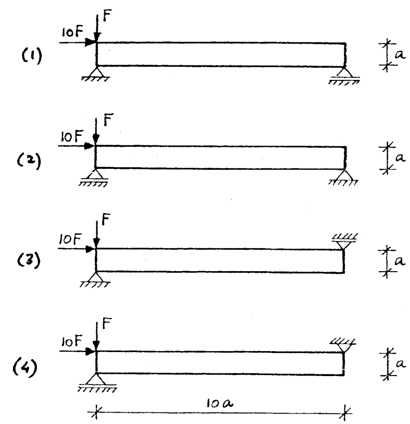
Question:
Which bending moment diagram matches the given load?



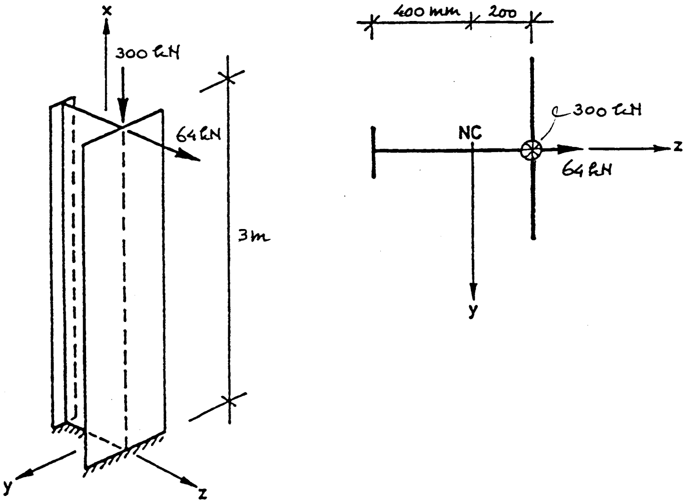
12.102: 1–4 A beam with rectangular cross-section is supported in four different ways. The beam axis is half-way up.

Questions:

a. Indicate how the load is acting on the beam modelled as a line element.
b. Draw the N, V and M diagrams.

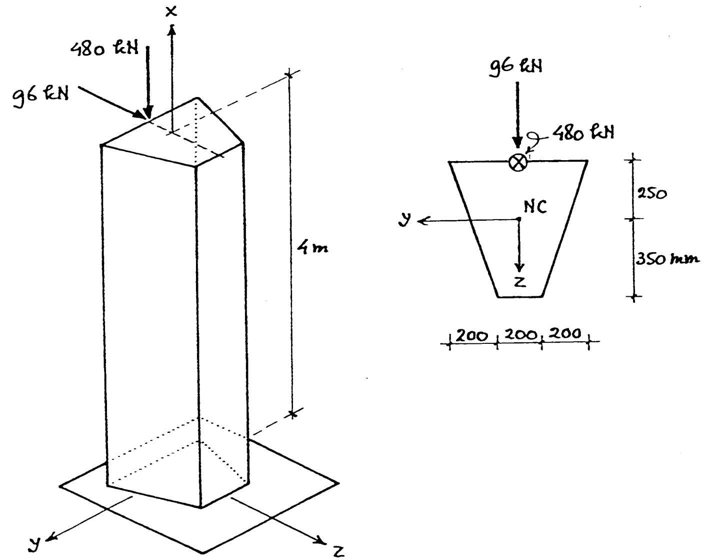


12.103 A fixed column is loaded at its free end as shown. The column axis passes through the normal centre NC and coincides with the x axis shown.



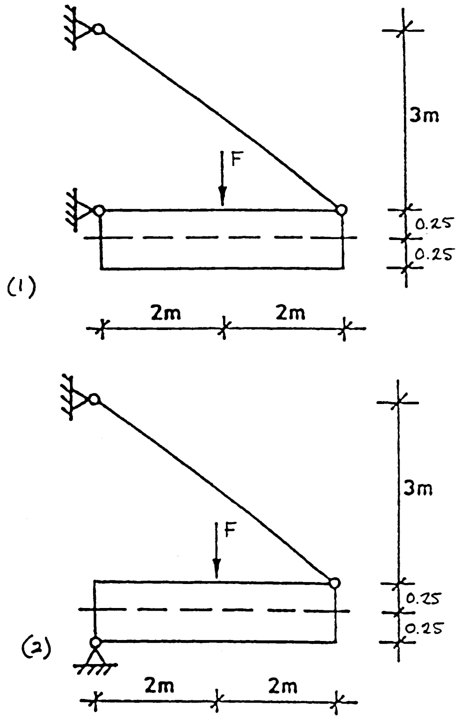
Question:
Draw the N, V and M diagrams for the column modelled as a line element.

12.104 A fixed column fixed is loaded at its free end as shown. The column axis passes through the normal centre NC and coincides with the x axis shown.



Question:
Draw the N, V and M diagrams for the column modelled as a line element.

12.105: 1–2 In the figure, the beam axis is shown by means of a dashed line.



Question:
Schematise the beam as a line element and draw the N, V and M diagrams due to the force $F = 84$ kN.

Calculating M, V and N Diagrams

13

In this chapter, we will look at a number of examples for calculating M, V and N diagrams. In the presentation we distinguish between the self-contained structures in Section 13.1, the somewhat more complex compound and associated structures in Section 13.2 and the statically indeterminate structures in Section 13.3. For some of the calculations we will, to prevent repetition, make only a start, and leave it to the reader to work out the answer further. Should you decide to work out the questions yourself, you will notice that there are several ways to arrive at the answer.

13.1 Self-contained structures

In this section we will be determining the M, V, and sometimes the N diagrams for self-contained structures subject to distributed loads.

13.1.1 Beam with triangular load (lintel)

The lintel in Figure 13.1a is supporting the part of the brickwork shown above a door opening. The load on the lintel modelled as a line element is the triangular load in Figure 13.1b, with top value $\hat{q}$. The brick wall is $d = 240$ mm thick. The mass density of the brickwork is $\rho = 1800$ kg/m^3.

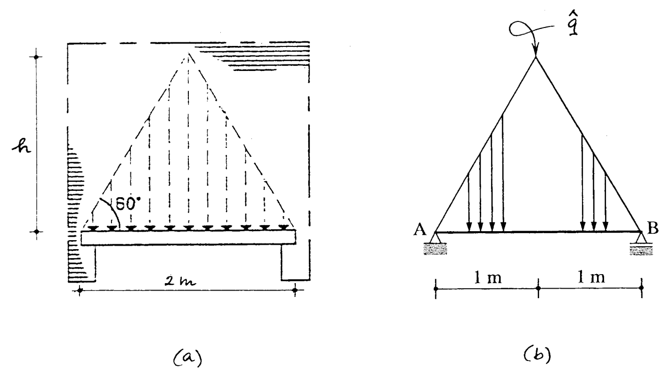


Figure 13.1 (a) A lintel carrying the triangular part of the brickwork. (b) Modelling of lintel and load as a beam with triangular load.

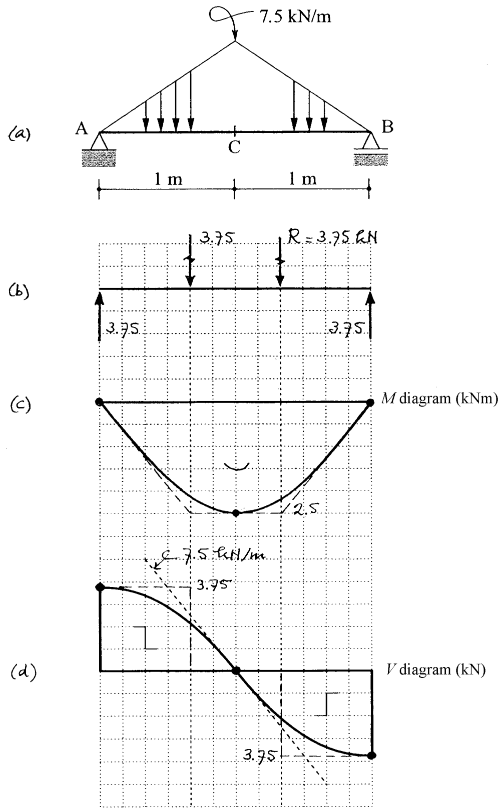


Figure 13.2 (a) Beam with triangular load. (b) The isolated beam with the field loads on AC and BC replaced by their resultants, and the associated support reactions. (c) Bending moment diagram and (d) shear force diagram.

Questions:

a. Determine the top value $\hat{q}$ of the triangular load.

b. For AB, determine and draw the M and V diagrams. At A and B also draw the tangents to the M diagram. How large is the maximum bending moment?

Solution:

a. The height h of the brickwork in Figure 13.1a is

$$h = (1 \text{ m}) \times \tan 60^\circ = 1.732 \text{ m}.$$

With a gravitational field strength of $g = 10$ N/kg, the top value $\hat{q}$ of the triangular load on the lintel is

$$\hat{q} = \rho g h d = (1800 \text{ kg/m}^3)(10 \text{ N/kg})(1.732 \text{ m})(0.240 \text{ m}) \approx 7.5 \text{ kN/m}$$

(see Figure 13.2a).

b. In Figure 13.2b, the distributed loads on AC and BC have been replaced by their resultants R:

$$R = \tfrac{1}{2} \times (1 \text{ m})(7.5 \text{ kN/m}) = 3.75 \text{ kN}.$$

The support reactions are also shown.

In Figures 13.2c and 13.2d, the M and V diagrams due to these resultants are shown by means of dashed lines. This way, we can find the correct values for M and V at A, B and C (shown by means of dots) and the correct slopes of the M diagram.

We can now draw the actual M diagram, a cubic, see the solid line in Figure 13.2c. The maximum bending moment occurs at midspan and is 2.5 kNm.

The actual V diagram is parabolic (see the solid line in Figure 13.2d). The slope of the V diagram is equal to the magnitude of the distributed load. At A and B, the distributed load is zero, and the V diagram has horizontal tangents. At C, the distributed load is largest, and the slope of the V diagram is steepest. The slope is 7.5 kN/m, and is shown separately in Figure 13.2d.

13.1.2 Beam with parabolic distributed load

Beam ABC in Figure 13.3 is supported by a hinge at A and on a roller at B. The beam is loaded by a parabolic distributed load in field AB and a point load of 25 kN at end C of cantilever BC. The longitudinal dimensions of the beam are shown in the figure. The parabolic distributed load can be represented with

$$q(x) = -30\left(\frac{x}{\ell}\right)^2 + 30\left(\frac{x}{\ell}\right) \text{ kN/m.}$$

Here, $\ell = 10$ m is the length of AB. The dead weight of the beam is not considered in the calculation.

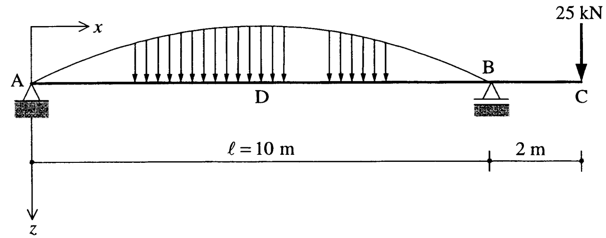


Figure 13.3 Parabolic load over AB on beam ABC.

Questions:

a. Replace the distributed load over AB by its resultant, and draw the M and V diagrams for the entire beam ABC.
b. Draw a (rough) sketch of the actual M and V diagrams for AB. In addition to the deformation symbols in the M and V diagrams, also include the plus and minus signs in the given xz coordinate system.
c. For AB, through consecutive integration, determine the shear force V and the bending moment M as a function of x. Determine the values of V and M at A and B and at the middle D of field AB. At D draw the tangent to the M diagram.
d. Where in AB is the field moment a maximum? It is enough to give a rough indication of the location. Using the M diagram estimate the

value of the maximum field moment. This value need not be determined accurately.

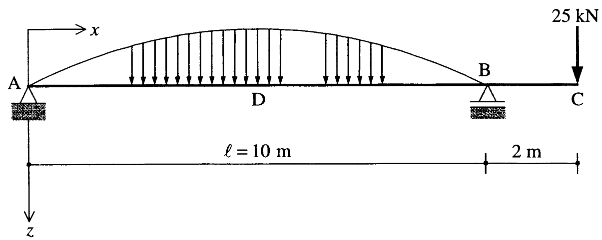


Figure 13.3 Parabolic load over AB on beam ABC.

Solution (units kN and m):

a. With $\ell = 10$ m, for the parabolically distributed load on AB it applies that

$$q(x) = (-0.3x^2 + 3x) \text{ kN/m}$$

if x is expressed in metres. The top of the parabola is at the middle of AB. This is derived from

$$\frac{\mathrm{d}q(x)}{\mathrm{d}x} = -0.6x + 3 = 0 \Rightarrow x = 5 \text{ m}.$$

On the basis of symmetry, the resultant R of the distributed load is acting here. The magnitude of R is equal to the area of the load diagram, and is found by integrating the distributed load:

$$R = \int_0^{10} q(x)\,\mathrm{d}x = \int_0^{10} (-0.3x^2 + 3x)\,\mathrm{d}x = (-0.1x^3 + 1.5x^2)\Big|_0^{10}$$
$$= 50 \text{ kN}.$$

Figure 13.4a shows the resultant R, together with the support reactions at A and B. In Figures 13.4b and 13.4c, the M and V diagrams due to this (concentrated) force R are shown (with dashed lines for AB).

b. The M and V diagrams are correct for the cantilever BC. In field AB, only the values at A and B (shown by means of dots) are correct. In addition, at A and B the dashed M diagram B is tangent to the actual M diagram. There are no other handholds to sketch the M diagram, but we can now certainly make a rough sketch (see the solid line in Figure 13.4b).

The actual V diagram has horizontal tangents at A and B because the distributed load is zero there. This allows us to make a pretty good sketch of the V diagram (see the solid line in Figure 13.4c).

c. With

$$q(x) = -0.3x^2 + 3x$$

integrating gives

$$V = -\int q(x)\,\mathrm{d}x = +0.1x^3 - 1.5x^2 + C_1.$$

Beware of the signs!

After integrating again we find

$$M = \int V\,\mathrm{d}x = +0.025x^4 - 0.5x^3 + C_1x + C_2.$$

The integration constants C_1 and C_2 follow from the boundary conditions. Because the M and V diagrams are roughly known, we have a free choice here. Below we have selected the boundary conditions relating to the bending moments at A and B:

$$x = 0;\ \ M = 0 \Rightarrow C_2 = 0,$$

$$x = 10;\ \ M = -50 \Rightarrow C_1 = +20\ \mathrm{kN}.$$

For the variation of the shear force and the bending moment we find

$$V = (0.1x^3 - 1.5x^2 + 20)\ \mathrm{kN}, \tag{a}$$

$$M = (+0.025x^4 - 0.5x^3 + 20x)\ \mathrm{kNm}. \tag{b}$$

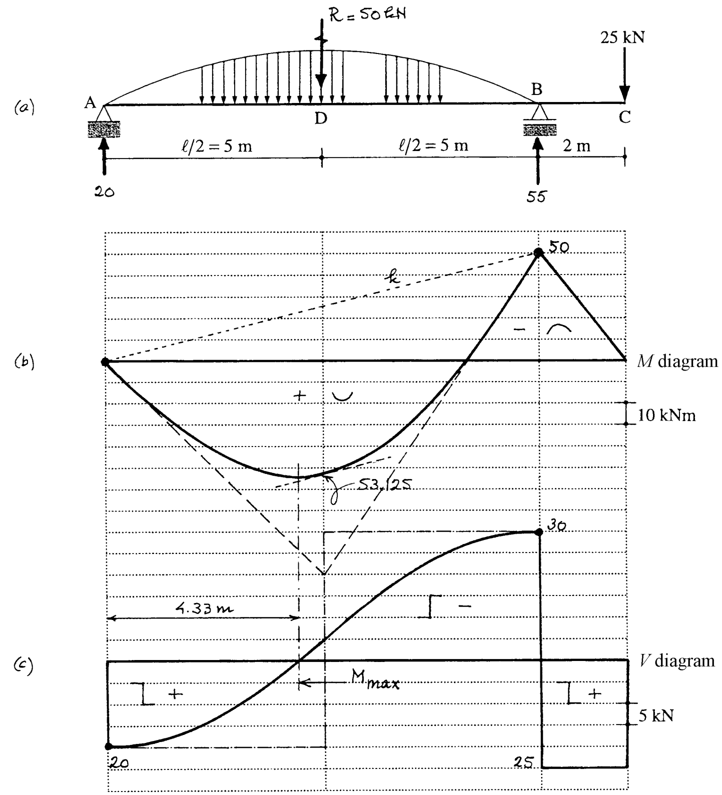


Figure 13.4 (a) Support reactions, (b) bending moment diagram and (c) shear force diagram.

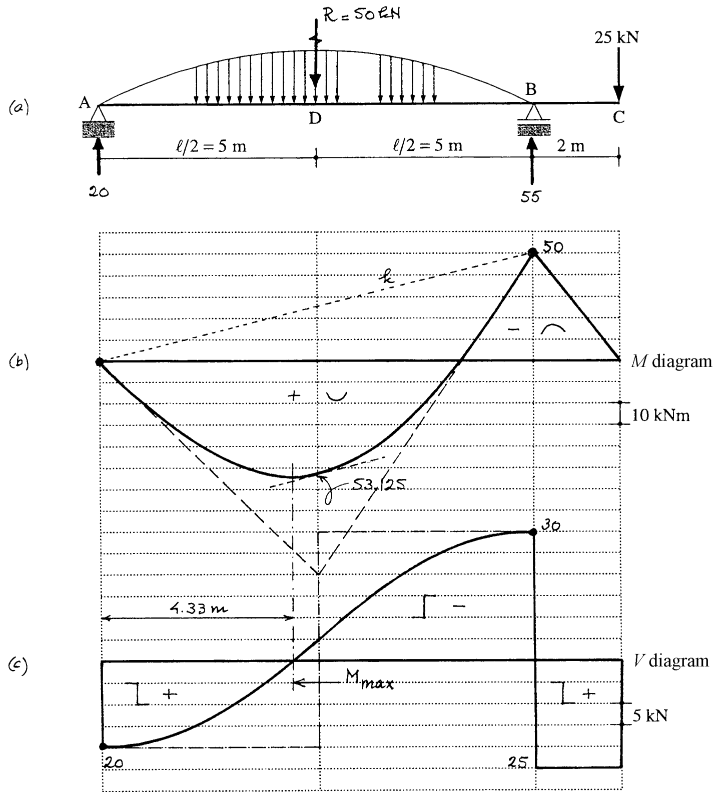


Figure 13.4 (a) Support reactions, (b) bending moment diagram and (c) shear force diagram.

Check: With $x = 0$ and $x = 10$ expression (a) must give the shear force at A, and that to the left of B, respectively:

$$x = 0; \; V = +20 \text{ kN (correct)},$$
$$x = 10; \; V = +100 - 150 + 20 = -30 \text{ kN (correct)}.$$

At D ($x = 5$):

$$V = +0.1 \times 5^3 - 1.5 \times 5^2 + 20 = -5 \text{ kN},$$
$$M = +0.025 \times 5^4 - 0.5 \times 5^3 + 20 \times 5 = +53.125 \text{ kNm}.$$

At D, the middle of span AB, the tangent to the M diagram is parallel to the chord k (see Figure 13.4.b).

d. The maximum bending moment in AB will occur slightly to the left of the middle D. Looking at the M diagram in Figure 13.4b, we can estimate the magnitude of that moment as approximately 55 kNm.

Accurate determination:
If we are looking for the root of the V diagram, (a) gives

$$x = 4.33 \text{ m}.$$

Substituting this value in (b) leads to an accurate value of the maximum bending moment:

$$M_{\max} = 54.8 \text{ kNm } (\smile).$$

13.1.3 Beam on three bar supports with a uniformly distributed load

The structure in Figure 13.5 consists of a beam supported by three bars. Dimensions and loads are given in the figure.

Questions:

a. Determine the support reactions at P, Q and R. Draw them as they act in reality, and include their values.
b. For ABCD, draw the V and M diagrams, with the deformation symbols. Include relevant values. At A, B and E also draw the tangents to the M diagram.
c. Determine the location and magnitude of the maximum field moment in BC.

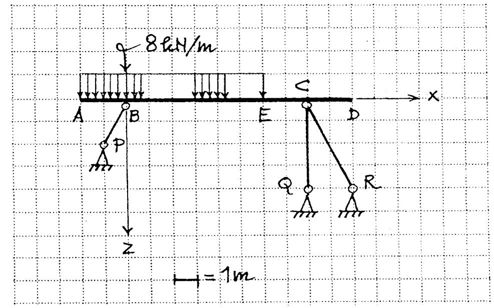


Figure 13.5 Beam on three bar supports.

Solution:

a. In Figure 13.6, the distributed load over AE has been replaced by its resultant of (8 kN/m)(8 m) = 64 kN. This simplifies the calculation for the support reactions. From the moment equilibrium about S we can find the vertical support reaction at Q:

$$\sum T_y|\mathrm{S} = 0 \Rightarrow Q_\mathrm{v} = 32\ \mathrm{kN}\ (\downarrow).$$

From the moment equilibrium about T we can find the vertical support reaction at R:

$$\sum T_y|\mathrm{T} = 0 \Rightarrow R_\mathrm{v} = 48\ \mathrm{kN}\ (\uparrow).$$

From the slope of bar support RC we find

$$R_\mathrm{h} = \tfrac{1}{2} R_\mathrm{v} = 24\ \mathrm{kN}\ (\leftarrow).$$

Finally, the horizontal and vertical force equilibrium gives

$$\sum F_z = 0 \Rightarrow P_\mathrm{v} = 48\ \mathrm{kN}\ (\uparrow),$$

$$\sum F_x = 0 \Rightarrow P_\mathrm{h} = 24\ \mathrm{kN}\ (\rightarrow).$$

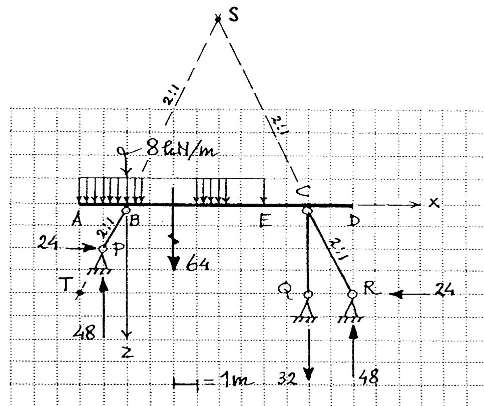


Figure 13.6 Support reactions.

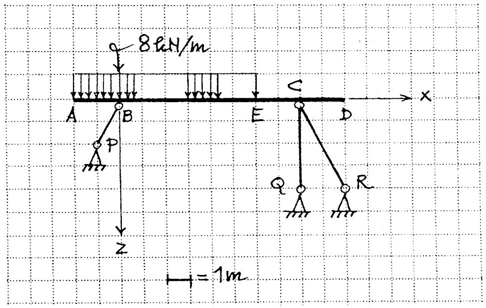


Figure 13.5 Beam on three bar supports.

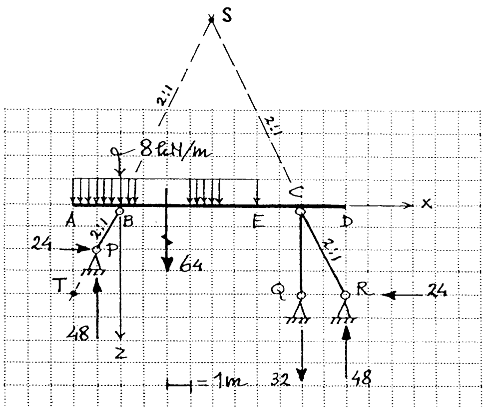


Figure 13.6 Support reactions.

Optional solution question a:
With $P_\mathrm{h} = \frac{1}{2} P_\mathrm{v}$ the moment equilibrium about C gives

$$P_\mathrm{v} = 48 \text{ kN } (\uparrow),$$

$$P_\mathrm{h} = 24 \text{ kN } (\rightarrow).$$

The horizontal force equilibrium gives

$$R_\mathrm{h} = 24 \text{ kN } (\leftarrow).$$

The slope of the bar support RC gives

$$R_\mathrm{v} = 2R_\mathrm{h} = 48 \text{ kN } (\uparrow).$$

Finally, the vertical force equilibrium gives:

$$Q_\mathrm{v} = 32 \text{ kN } (\downarrow).$$

b. Figure 13.7a shows the isolated beam AD with all the forces acting on it. To simplify the calculation and drawing of the V and M diagrams for the fields AB and BC, the resultants of the distributed loads are also shown.

In Figures 13.7b and 13.7c, the dashed line shows the V and M diagrams due to the concentrated forces. These diagrams have to be adjusted in fields AB and BE. Here the shear force is linear and the bending moment is parabolic. The parabolic bending moment diagram "hangs" between the values at A, B and E. The definitive V and M diagrams are shown as solid lines.

Checking the M diagram for field BE:
In the middle of the field, the parabola bisects the distance between the chord (8 kNm) and the top value due to the load resultant (80 kNm). From

Figure 13.7c we can deduce:

$$p = \frac{(80 - 8)\ \text{kNm}}{2} = 36\ \text{kNm}.$$

This value of p, the rise of the parabola, must be equal to $\frac{1}{8}q\ell^2$, in which ℓ is the length of field BE:

$$p = \tfrac{1}{8}(8\ \text{kN/m})(6\ \text{m})^2 = 36\ \text{kNm}.$$

This is the case.

c. From the V diagram we can deduce that the shear force in field BC is zero at G, 4 m to the right of B. Here the maximum field moment occurs. We can determine the magnitude from the moment equilibrium of beam segments AG or GD, or from the area of the V diagram for beam segments AG or GD. From the area of the V diagram for beam segment AG we find

$$M_{\text{max}} = \left|\tfrac{1}{2}(2\ \text{m})(16\ \text{kN}) - \tfrac{1}{2}(4\ \text{m})(32\ \text{kN})\right| = 48\ \text{kNm}\ (\smile).$$

If we look at beam segment GD this must off course give the same value:

$$M_{\text{max}} = \left|\tfrac{1}{2}(2\ \text{m})(16\ \text{kN}) + (2\ \text{m})(16\ \text{kN})\right| = 48\ \text{kNm}\ (\smile).$$

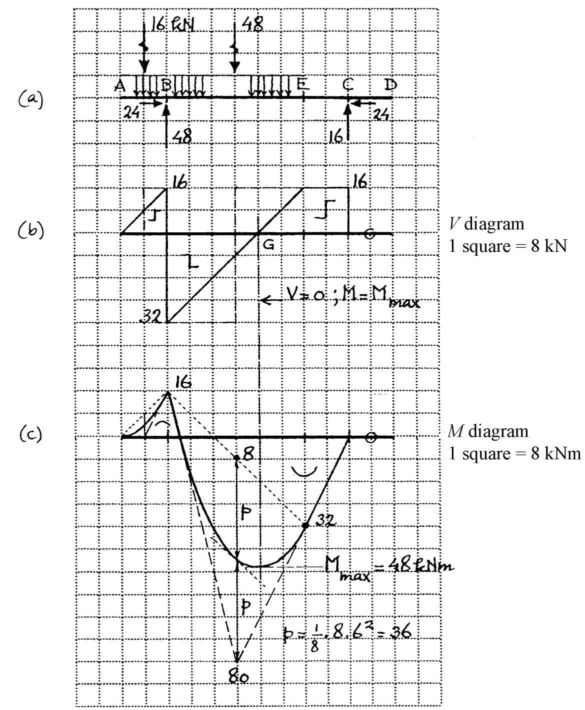


Figure 13.7 (a) The isolated beam AD with its (b) shear force diagram and (c) bending moment diagram.

13.1.4 Pile (cantilever beam)

A concrete pile with length $\ell = 20.5$ m and square cross-section of $0.35 \times 0.35\ \text{m}^2$ is supported as shown in Figure 13.8. The mass density ρ of concrete is assumed $\rho = 2500\ \text{kg/m}^3$.

Questions:

a. Determine and draw the M and V diagrams. At A, B and C also draw the tangents to the M diagram.

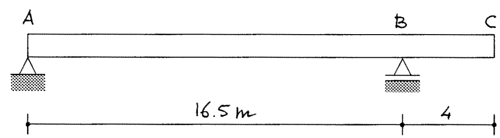


Figure 13.8 A pile that is picked up to be driven, can be seen as a simply supported beam with overhang.

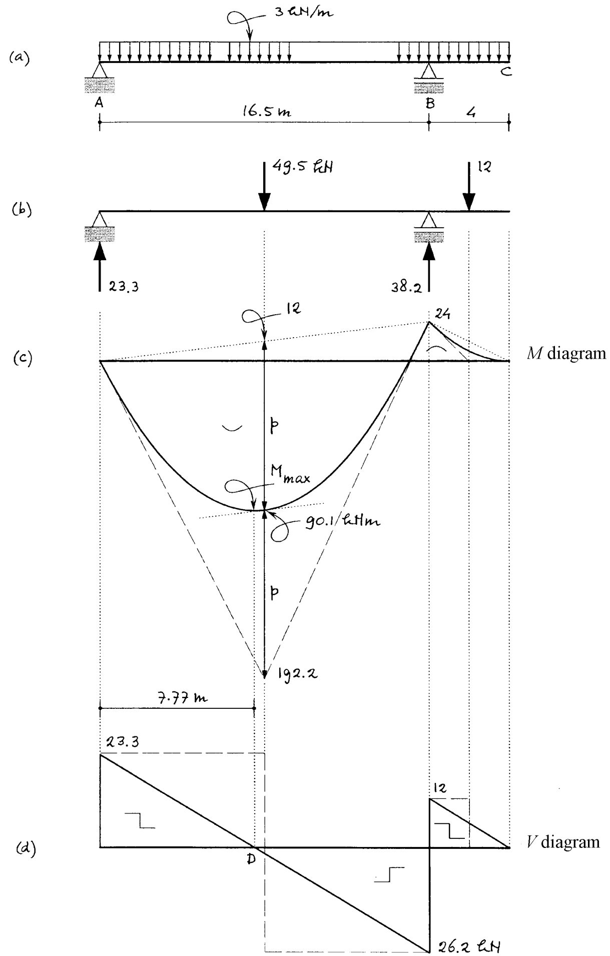


Figure 13.9 (a) Model for the pile subject to its dead weight. (b) The load resultants on AB and BC and the associated support reactions. (c) Bending moment diagram and (d) shear force diagram.

b. Determine the extreme value(s) of the bending moment.

c. Where should the support B be placed to minimise the bending moment? Draw the associated M and V diagrams.

Solution:

a. The dead weight of the pile is $q = \rho g A$, in which $g = 10$ N/kg is the gravitational field strength and A is the cross-sectional area of the pile:

$$q = \rho g A = (2500 \text{ kg/m}^3)(10 \text{ N/kg})(0.35 \text{ m})^2 = 3062.5 \text{ N/m}.$$

Hereafter, assume $q = 3$ kN/m.

Figure 13.9a shows the model for the pile. In Figure 13.9b, the distributed loads in fields AB and BC have been replaced by their resultants, and the support reactions are shown. In Figure 13.9c, the M diagram due to the load resultants is shown by means of dashed lines. At A, B and C the dashed diagram gives the correct values for the actual M diagram and the correct tangents. The actual M diagram is shown by means of a solid line.

Checking the M diagram in field AB (see Figure 13.9c):

$$p = \tfrac{1}{8} q \ell^2 = \tfrac{1}{8}(3 \text{ kN/m})(16.5 \text{ m})^2 = 102.1 \text{ kNm} = (12 + 192.2)/2 \text{ kNm}.$$

In Figure 13.9d, the V diagram due to the resultants is shown by dashed lines. This V diagram gives the correct values in A, B and C. The actual V diagram is linear, and is shown by means of a solid line.

Checking the V diagram:

The slope of the V diagram is equal to the distributed load, and is the same in both fields.

b. From the V diagram in Figure 13.9d we find that the shear force in field AB is zero at D. This is where the bending moment in the field is an extreme. The distance from D to A is

$$\ell^{\mathrm{AD}} = \frac{23.3}{23.3 + 26.2} \times (16.5 \text{ m}) = 7.77 \text{ m}.$$

The bending moment at D can be found from the moment equilibrium of the isolated segment AD or, as shown below, from the area of the V diagram for AD:

$$M_{\max} = \tfrac{1}{2}(7.77 \text{ m})(23.3 \text{ kN}) = 90.5 \text{ kNm}.$$

Another extreme bending moment is the *support moment*[1] at B. Note that this moment can be found from the area of the V diagram for segment BC (see Figure 13.9d):

$$M_{\min} = \tfrac{1}{2}(4 \text{ m})(12 \text{ kN}) = 24 \text{ kNm}.$$

c. Let the total length of the pile be ℓ and the length of the overhang be a (see Figure 13.10a). Figure 13.10b shows a sketch of the V diagram. The shear force to the right of B is equal to qa. The slope of the V diagram is the same everywhere. The extreme bending moments occur at E and B. The bending moment at B is equal to the hatched area of the V diagram between B and C:

$$M_{\mathrm{B}} = \tfrac{1}{2} q a^2.$$

The bending moment at E is equal to the hatched area of the V diagram between A and E. The bending moment in the pile is least when the extreme bending moments at E and B are equal:

$$M_{\mathrm{E}} = M_{\mathrm{B}} = \tfrac{1}{2} q a^2.$$

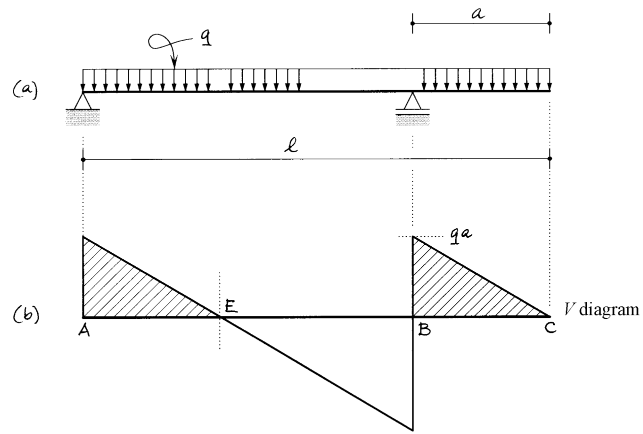


Figure 13.10 (a) Simply supported beam with total length ℓ and overhang of length a. (b) If the maximum bending moment at E is equal to the bending moment at B, the hatched areas in the V diagram are also equal.

[1] A support moment is the bending moment in the beam at a support.

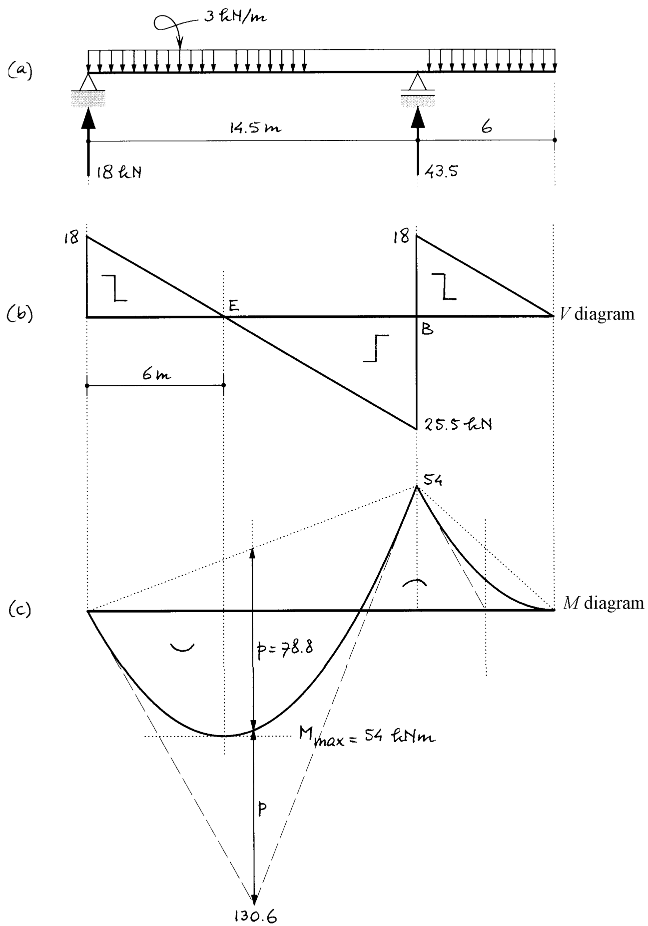


Figure 13.11 (a) The pile supported in such a way that the maximum field moment in AB and the support moment at B are of equal magnitude. (b) Associated shear force diagram and (c) bending moment diagram.

In that case, the shear force diagrams for AE and BD must be equal. From this it follows that the shear force at A is equal to qa, and the length of AE is equal to a. From the linear variation of the shear force along AB, it follows that the shear force to the left of B is equal to $q(\ell - 2a)$. The total area of the M diagram is zero as there are no concentrated couples acting. The hatched area of the V diagram must therefore be equal to the non-hatched area:

$$2 \times \tfrac{1}{2} q a^2 = \tfrac{1}{2} q (\ell - 2a)^2.$$

This leads to the following quadratic equation in a:

$$2a^2 - 4\ell a + \ell^2 = 0.$$

The solution is

$$a = \frac{-(-4\ell) \pm \sqrt{(-4\ell)^2 - 4 \times 2 \times \ell^2}}{2 \times 2} = \left(1 \pm \tfrac{1}{2}\sqrt{2}\right)\ell.$$

Since $a < \ell$ the solution with the plus sign is invalid, so that

$$a = \left(1 - \tfrac{1}{2}\sqrt{2}\right)\ell = 0.293\ell.$$

With $\ell = 20.5$ m this gives

$$a = 0.293 \times (20.5 \text{ m}) = 6 \text{ m}.$$

Figure 13.11 shows the associated M and V diagrams. The extreme bending moments are

$$M_{\mathrm{E}} = M_{\mathrm{B}} = \tfrac{1}{2}(3 \text{ kNm})(6 \text{ m})^2 = 54 \text{ kNm}.$$

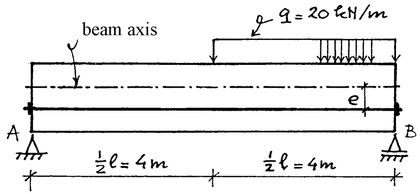


Figure 13.12 Simply supported prestressed beam with a uniformly distributed load on the right-hand side. The straight single bar tendon has eccentricity e.

13.1.5 Prestressed beam

The simply supported beam AB in Figure 13.12 has a length $\ell = 8$ m and is prestressed with a straight single bar tendon. The tendon is at a distance $e = 0.2$ m under the beam axis. The prestressing force is $F_\mathrm{p} = 200$ kN. The right-hand half of the beam is loaded by a uniformly distributed load $q = 20$ kN/m.

Questions:

a. Model beam AB as a line element and draw all the forces acting on it.
b. Draw the V and M diagrams with the deformation symbols. Include relevant values.
c. Determine the location and magnitude of the maximum bending moment in the beam.

Solution:

a. Via the anchorages the tensile force $F_\mathrm{p} = 200$ kN in the tendon exerts equal compressive forces $F_\mathrm{p} = 200$ kN on the beam ends (see Figure 13.13a). In Figure 13.13b, the beam has been modelled as a line element. By definition, the line element coincides with the beam axis. So the force flow is assumed to occur via the beam axis. All the forces are therefore shifted to the beam axis. The eccentric compressive forces on the

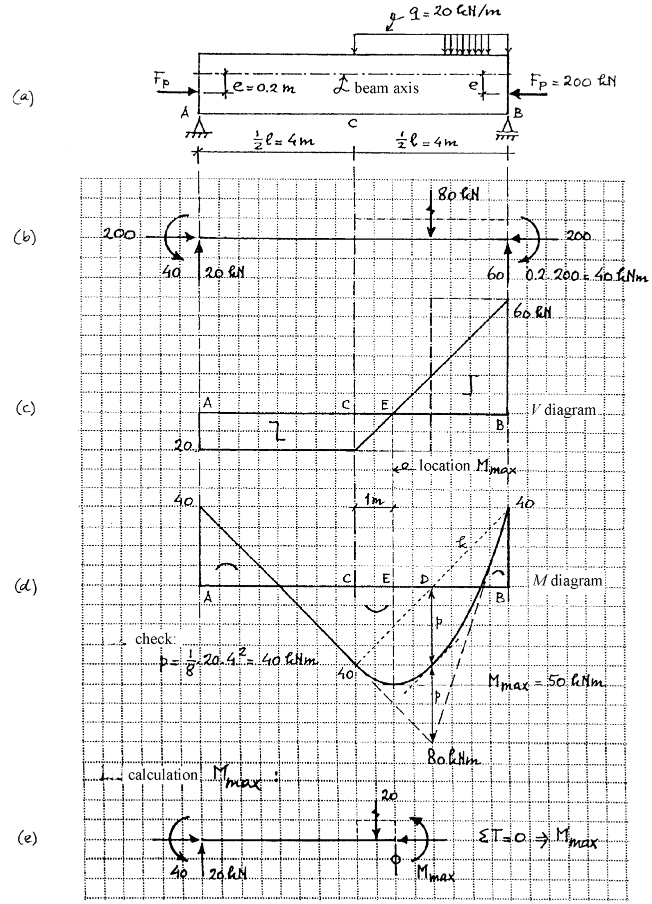


Figure 13.13 (a) Due to the prestressing, the eccentric compressive forces F_p are exerted on the ends of the beam. (b) The beam modelled as a line element. The eccentric compressive forces on the beam ends cause couples. (c) Shear force diagram and (d) bending moment diagram. (e) The maximum bending moment at the shear force zero E can be found from the moment equilibrium of AE.

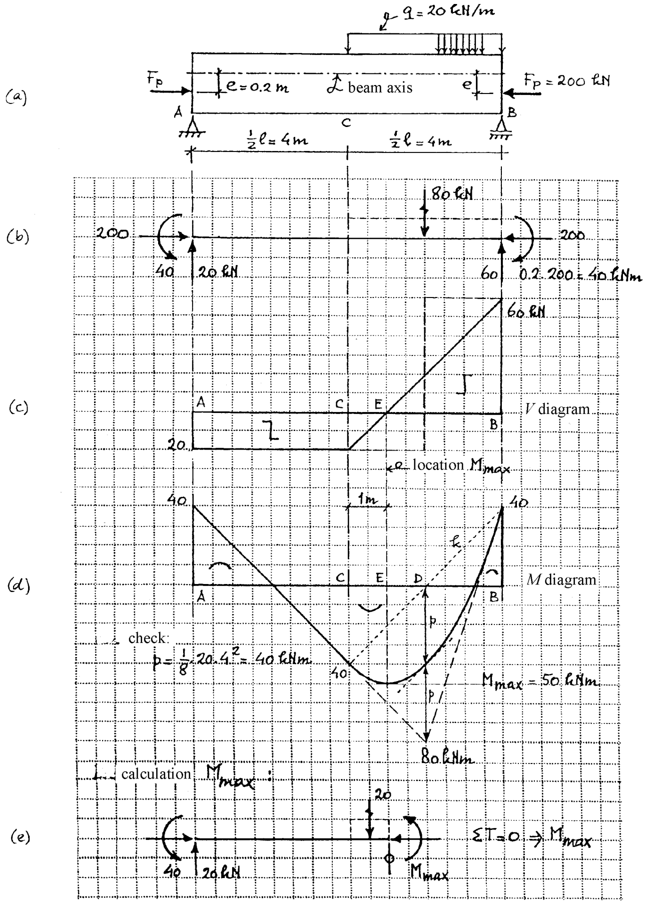


Figure 13.13 (a) Due to the prestressing, the eccentric compressive forces F_p are exerted on the ends of the beam. (b) The beam modelled as a line element. The eccentric compressive forces on the beam ends cause couples. (c) Shear force diagram and (d) bending moment diagram. (e) The maximum bending moment at the shear force zero E can be found from the moment equilibrium of AE.

beam ends are statically equivalent with centric compressive forces (forces acting in the beam axis) $F_p = 200$ kN and additional couples T:

$$T = F_p e = (200 \text{ kN})(0.2 \text{ m}) = 40 \text{ kNm}.$$

To simplify the calculation, the uniformly distributed load over BC has been replaced by its resultant R in Figure 13.13b:

$$R = q \times \tfrac{1}{2}\ell = (20 \text{ kN/m})(4 \text{ m}) = 80 \text{ kN}.$$

The support reactions follow from the moment equilibrium about supports A and B.

b. Figure 13.13c shows the V diagram. We can first draw the V diagram for all the concentrated forces (dashed line), and then adapt them for field CB by drawing a linear path between the values at C and B.

Figure 13.13d shows the M diagram. We first draw the M diagram due to the concentrated forces (dashed line) and then adjust the variation for field CB by sketching a parabola between the values at C and B, where it is tangent to the M diagram due to the resultant R of the distributed load. At the middle D of field CB the distance p between chord k and the parabola is

$$p = \tfrac{1}{8} q \left(\tfrac{1}{2}\ell\right)^2 = \tfrac{1}{8}(20 \text{ kN/m})(4 \text{ m})^2 = 40 \text{ kNm}.$$

Here the tangent is parallel to chord k.

For the bending moment at the middle D of field CB, we can read from the M diagram in Figure 13.13d:

$$M_D = p = 40 \text{ kNm } (\smile)$$

Check: The tangents at C and B intersect at D at a distance $2p$ under chord k.

c. The maximum bending moment in field AB occurs where the shear force $V = \mathrm{d}M/\mathrm{d}x$ is zero. This is at E, 1 m to the right of the middle C of beam AB. The magnitude of this maximum can, for example, be determined from the moment equilibrium about E of beam segment AE (see Figure 13.13e). This gives

$$M_{\max} = 50 \text{ kNm } (\smile).$$

Note that this maximum moment is not equal to the area of the V diagram for beam segment AE or BE, while the total area of the V diagram certainly is zero. It is left to the reader to explain this.

13.1.6 Slice from a long floating barge

A transverse slice has been isolated from the long floating concrete trough in Figure 13.14a, and has been modelled as the line element in Figure 13.14b. The dead weight of the slice is uniformly distributed over walls and bottom and is 12 kN/m. The dimensions and depth can be read from the figure. Note: The width of the slice is unknown.

Questions:

a. From the equilibrium of the slice modelled as a line element, determine the water pressure on the bottom AB. Draw the water pressure on both the bottom and the walls. Include the values.
b. Isolate bottom AB, and draw all the forces acting on it. Include the values.
c. For the entire slice, draw the M, V and N diagrams, with the deformation symbols. Include relevant values.
d. Determine the maximum bending moment. Where does it occur?

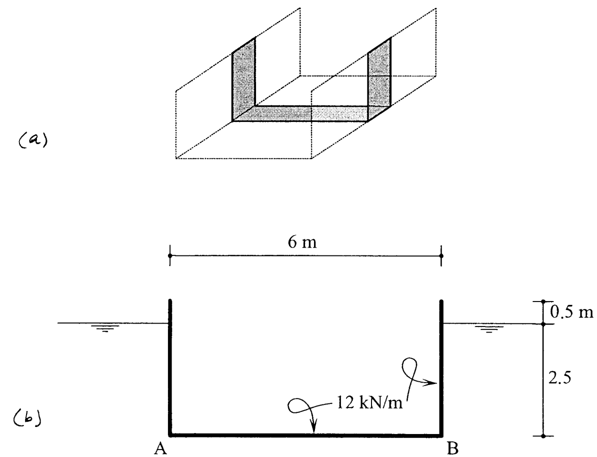


Figure 13.14 (a) A slice from a long floating concrete barge. (b) The slice modelled as a line element.

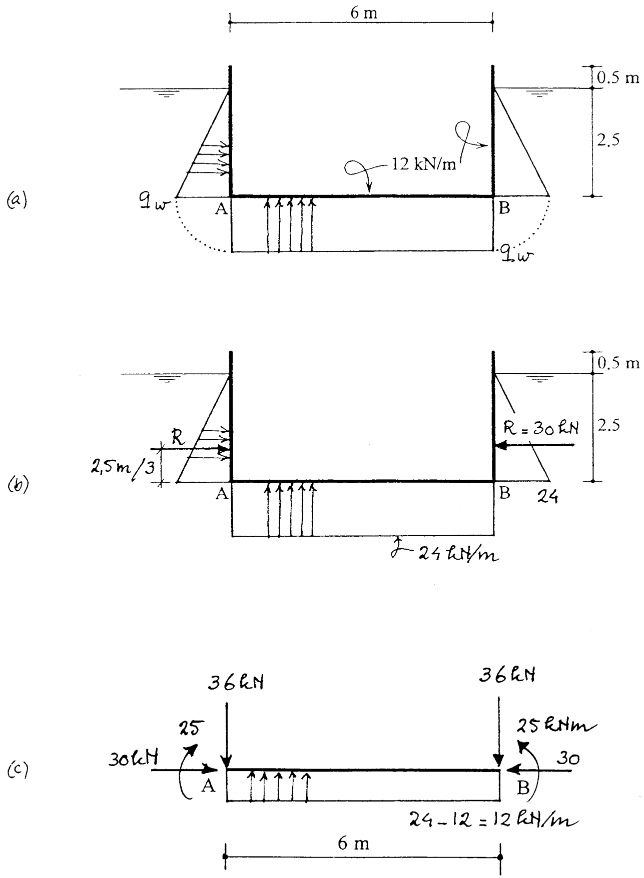


Figure 13.15 (a) The distribution of the water pressure and (b) the resulting water pressure on the walls. (c) The isolated bottom with all the forces and couples acting on it.

Solution:

a. Figure 13.15a shows the water pressure on the slice modelled as a line element. The water pressure on the bottom is constant. Let the water pressure there be q_w. The water pressure on the walls varies linearly from zero at the water level to q_w at the bottom. The upward water pressure q_w on the bottom must be in equilibrium with the dead weight of the bottom and walls of the strip:

$$\begin{aligned}(6\text{ m}) \times q_w\ (\uparrow) &= \{(6\text{ m}) + 2 \times (3\text{ m})\} \times (12\text{ kN/m})\ (\downarrow)\\ &= 144\text{ kN}\ (\downarrow).\end{aligned}$$

This gives $q_w = 24$ kN/m.

b. The resulting water pressure R on the walls is (see Figure 13.15b):

$$R = \tfrac{1}{2}(24\text{ kN/m})(2.5\text{ m}) = 30\text{ kN}.$$

The forces R, which pass through the centroid of the load diagram and therefore act (2.5 m)/3 above bottom AB, exert horizontal forces of 30 kN on AB and couples of (30 kN)(2.5 m)/3 = 25 kNm. The bottom AB can be seen as an eccentrically compressed beam. In addition, at A and B, the vertical forces due to the dead weight of the walls are (3 m)(12 kN/m) = 36 kN. In Figure 13.15c, the base AB has been isolated, and all the forces are shown. The resulting (uniformly) distributed load q on base AB is equal to the difference between the upward water pressure $q_w = 24$ kN/m (↑) and the dead weight $q_w = 12$ kN/m (↓):

$$q = q_w - q_{dw} = (24\text{ kN/m}) - (12\text{ kN/m}) = 12\text{ kN/m})\ (\uparrow).$$

c. Figures 13.16a to 13.16c shows the M, V and N diagrams for the entire structure.

Walls: Due to the linearly distributed water pressure, the M diagram is a cubic, and the V diagram is a parabola. In order to find the tangents to the M diagram at A and B, the M diagram due to resultant R has been shown by means of dashed lines. The slope of the V diagram is zero, where the water pressure is zero and increases downwards. Due to the uniformly distributed dead weight, the normal force in the wall is linear.

Bottom: Due to the uniformly distributed (upward) load, the bending moment is parabolic and the shear force is linear. The moments at A and B "go round the corner". Between A and B a parabola is "hanging" with a rise p in the middle:

$$p = \tfrac{1}{8}q\ell^2 = \tfrac{1}{8} \times (12 \text{ kN/m})(6 \text{ m})^2 = 54 \text{ kNm}.$$

Because the distributed load is acting upwards, the parabolic M diagram is turned upwards. The normal force in the bottom is a constant compressive force of 30 kN.

d. In the middle of field AB, the shear force is zero and the bending moment is an extreme. The maximum field moment is an upward bending moment and can be determined from the M diagram in Figure 13.16b:

$$M_{\max} = (54 \text{ kNm}) - (25 \text{ kNm}) = 29 \text{ kNm } (\frown).$$

In addition, there are boundary extremes at A and B of 25 kNm ($\smile$).

13.1.7 Floating tunnel segment

A tunnel segment is afloat, waiting to be towed to the location where it will be sunk. The tunnel segment can be seen as a rigid beam and has a freeboard of 0.09 m (see Figure 13.17). The length ℓ, width b and height h of the tunnel segment are respectively 80, 9.5 and 6 m.

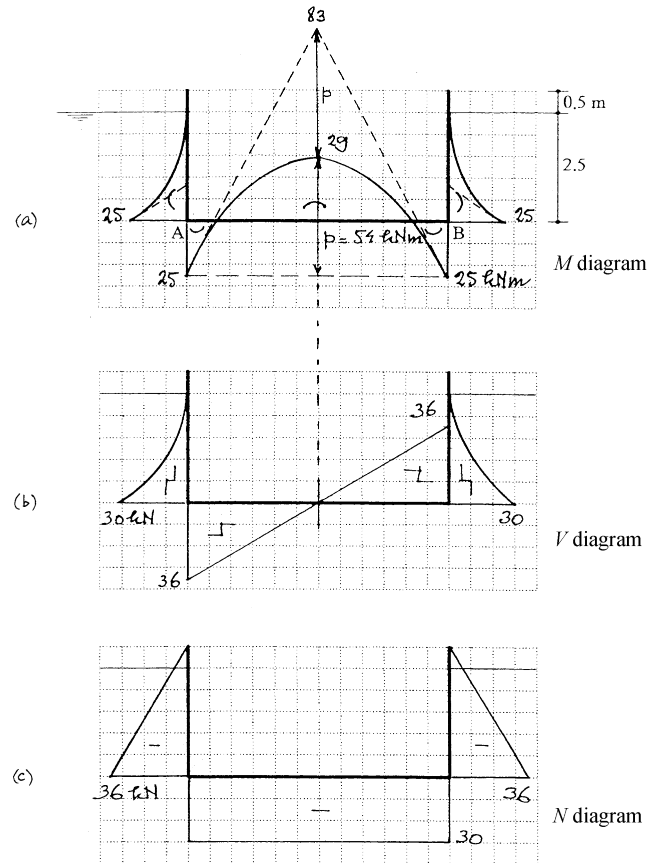


Figure 13.16 (a) Bending moment diagram, (b) shear force diagram and (c) normal force diagram for the slice from the floating concrete barge modelled as a line element.

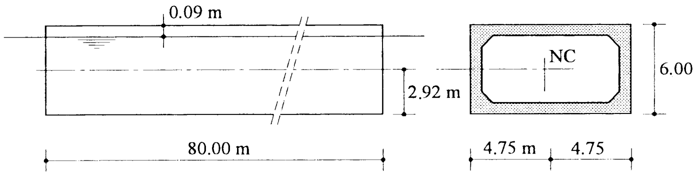


Figure 13.17 Longitudinal view and cross-section of a floating tunnel segment.

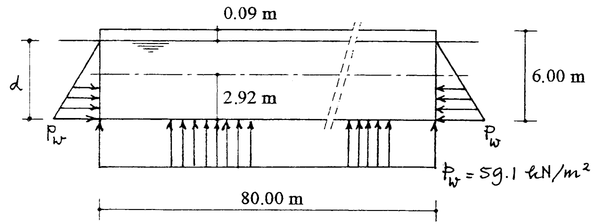


Figure 13.18 Distribution of the water pressure on the tunnel segment.

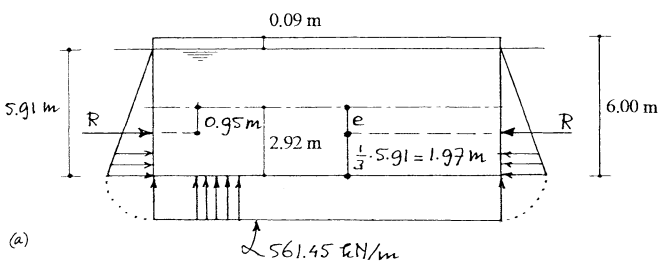


Figure 13.19 (a) Water pressure on the tunnel segment modelled as a plane element.

Figure 13.17 also shows the place of the normal force centre NC of the tunnel segment. The dead weight of the tunnel is 554 kN/m. The two temporary bulkheads both have a dead weight of 298 kN. The mass density of water is $\gamma_w = 10$ kN/m^3.

Questions:

a. Determine and draw the water pressure on the base of the tunnel. Write down the units.
b. Draw the variation of the water pressure on a bulkhead. Write down the units.
c. Model the tunnel segment as a line element. Draw all the forces (distributed or not) (and/of couples) acting on it. Include the values.
d. Draw the N diagram, V diagram and M diagram for the tunnel segment including the deformation symbols. Include relevant values. Determine the maximum bending moment in the tunnel segment.

Solution:

a. The water pressure p_w on the base of the tunnel is

$$p_w = \gamma_w d$$

in which $d = (6.00 \text{ m}) - (0.09 \text{ m}) = 5.91$ m is the depth of water at the base of the tunnel (see Figure 13.18), so that

$$p_w = (10 \text{ kN/m}^3)(5.91 \text{ m}) = 59.1 \text{ kN/m}^2.$$

b. The horizontal water pressure on the bulkheads varies linearly over the height (see Figure 13.18).

c. In Figure 13.19a, the tunnel has been modelled as a plane element. The water pressure on the base is

$$q_w = bp_w = (9.5 \text{ m})(59.1 \text{ kN/m}^2) = 561.45 \text{ kN/m}.$$

The resulting water pressures R on the bulkheads are

$$R = \tfrac{1}{2} q_w d = \tfrac{1}{2} \times (561.45\ \text{kN/m})(5.91\ \text{m}) = 1659\ \text{kN}.$$

The forces R act at a distance $d/3 = 5.91\ \text{m})/3 = 1.97$ m from the base of the tunnel segment. The eccentricity e with respect to the tunnel axis (through the normal force centre NC) is

$$e = (2.92\ \text{m}) - (1.97\ \text{m}) = 0.95\ \text{m}.$$

In Figure 13.19b, the tunnel segment has been modelled as a line element. The force flow is assumed to take place along the tunnel axis, through the normal centre NC. All the forces are therefore shifted to the tunnel axis. By shifting the eccentric water pressures R on the bulkheads to the tunnel axis, couples T are generated at the ends of the line element:

$$T = Re = (1659\ \text{kN})(0.95\ \text{m}) = 1576\ \text{kNm}.$$

In addition to the upward water pressure $q_w = 561.45$ kN/m (↑), there is also the dead weight of the tunnel segment $q_{dw} = 554$ kN/m (↓). The resulting distributed load is an upward load q:

$$q = q_w - q_{dw} = (561.45\ \text{kN/m}) - (554\ \text{kN/m}) = 7.45\ \text{kN/m})\ (\uparrow).$$

Check: The resulting upward load q, shown in Figure 13.20a, must be in equilibrium with the dead weight of the bulk heads:

$$\sum F_{\text{vert}}\ (\downarrow) = 2 \times (298\ \text{kN}) - (7.45\ \text{kN/m})(80\ \text{m}) = 0.$$

d. Figure 13.20b to d shows the N, V and M diagrams. The normal force is a constant compressive force. The shear force varies linearly. The bending moment varies parabolically. Since the distributed load is acting upwards the parabolic M diagram is also aimed upwards. The maximum bending

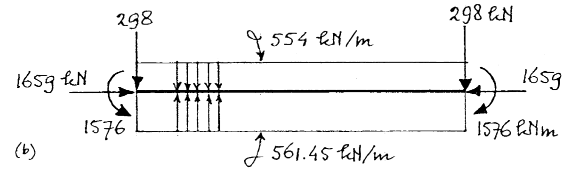


Figure 13.19 (b) Forces (and couples) on the tunnel section modelled as a line element.

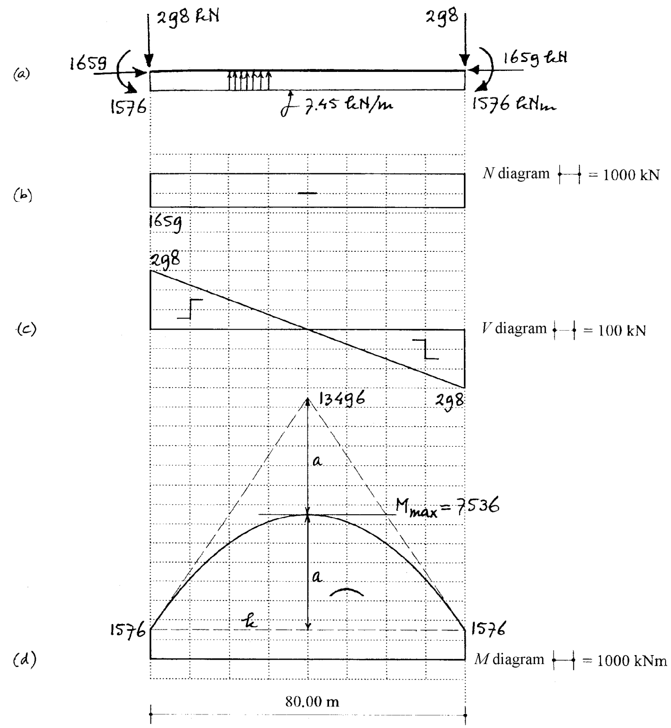


Figure 13.20 (a) The tunnel segment modelled as a line element with its (b) normal force diagram (c) shear force diagram and (d) bending moment diagram.

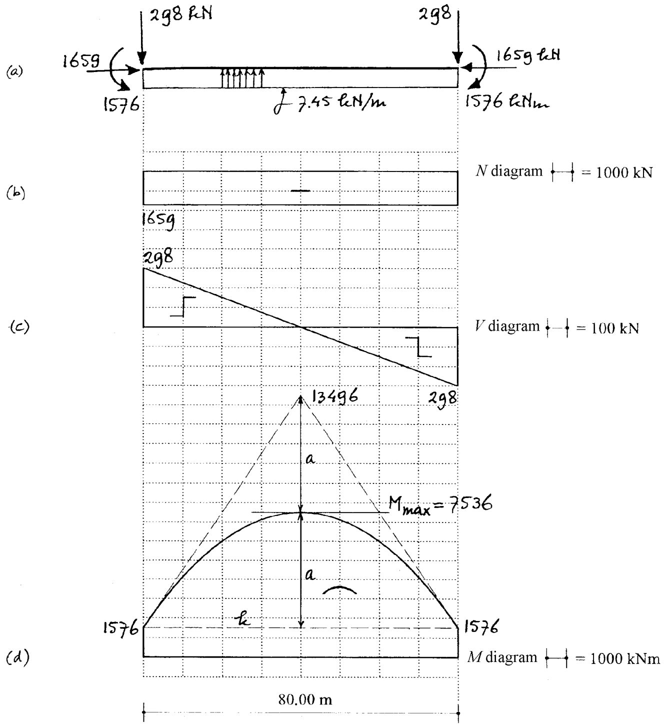


Figure 13.20 (a) The tunnel segment modelled as a line element with its (b) normal force diagram (c) shear force diagram and (d) bending moment diagram.

moment occurs in the middle of the tunnel segment. This can be determined from the moment equilibrium of half a tunnel segment or directly from the M diagram. In the middle of the tunnel segment, the distance a from the chord k to the parabola is

$$a = \tfrac{1}{8}q\ell^2 = \tfrac{1}{8} \times (7.45\,\text{kN/m})(80\,\text{m})^2 = 5960\,\text{kNm}$$

with which we can find the maximum bending moment:

$$M_{\max} = (1576\,\text{kNm}) + (5960\,\text{kNm}) = 7536\,\text{kNm}\ (\frown).$$

Figure 13.20d shows the end tangents to the M diagram.

13.1.8 Oblique roof beam on bar supports with triangular load

The structure in Figure 13.21 is subject to a linear distributed load normal to ABC, varying from from 18 kN/m at A to zero at C.

Questions:

a. Determine the support reactions at A, E and D. Draw them as they act in reality and include their values.
b. Isolate beam ABC, and draw all the forces acting on it.
c. For ABC draw a clear sketch of the V and M diagrams, with the deformation symbols and the plus and minus signs in the given (local) xz coordinate system. Include relevant values, and at A, B and C draw the tangents to the M diagram.
d. For AB, determine the shear force V and the bending moment M as a function of x. Use the given xz coordinate system.

Solution:

a. The support reactions are shown in Figure 13.22. To determine the support reactions, the triangular load on ABC is replaced by its resultant

R^{ABC}:

$$R^{\text{ABC}} = \tfrac{1}{2} \times (18 \text{ kN/m})(6\sqrt{2} \text{ m}) = 54\sqrt{2} \text{ m}.$$

The vertical support reaction $A_{\text{v}}\,(\uparrow)$ at A is found from the moment equilibrium about G, the intersection of the two-force members BE and CD:

$$\sum \text{T}|\text{G} \circlearrowleft \; = (4\sqrt{2} \text{ m})(54\sqrt{2} \text{ kN}) - (4 \text{ m}) \times A_{\text{v}}\,(\uparrow) = 0$$

$$\Rightarrow A_{\text{v}} = 108 \text{ kN } (\uparrow).$$

The vertical support reaction $E_{\text{v}}\,(\uparrow)$ in E is then found from the moment equilibrium about C:

$$\sum \text{T}|\text{C} \circlearrowleft \; = (4\sqrt{2} \text{ m})(54\sqrt{2} \text{ kN}) - (6 \text{ m})(108 \text{ kN}) - (2 \text{ m}) \times E_{\text{v}}\,(\uparrow) = 0$$

so that

$$E_{\text{v}}\,(\uparrow) = -108 \text{ kN}$$

or in other words

$$E_{\text{v}} = 108 \text{ kN } (\downarrow).$$

Finally, the support reactions at D follow from the horizontal and vertical force equilibrium of the structure:

$$D_{\text{h}} = 54 \text{ kN } (\leftarrow),$$

$$D_{\text{v}} = 54 \text{ kN } (\uparrow).$$

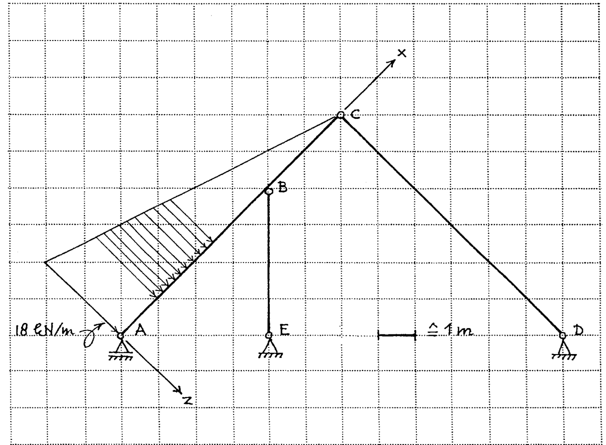


Figure 13.21 Oblique roof beam with triangular load.

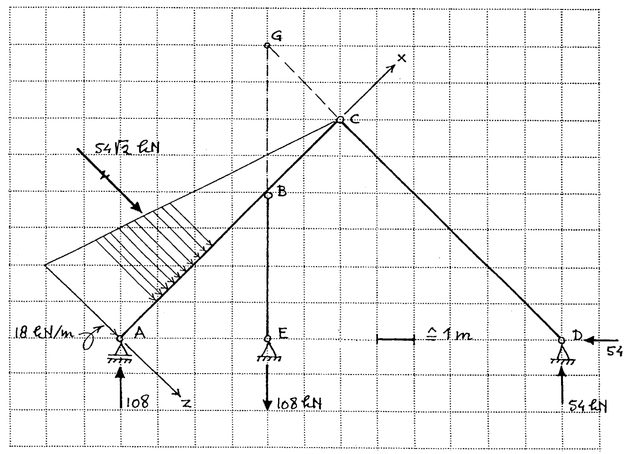


Figure 13.22 Support reactions.

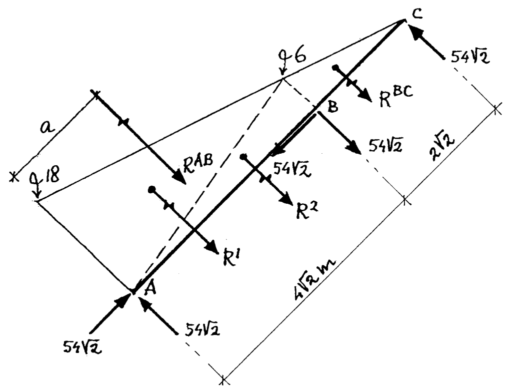


Figure 13.23 Isolated beam ABC with the load resultants for the fields AB and BC.

This final calculation is left to the reader.

b. In Figure 13.23, the beam ABC has been isolated, and all forces acting on it at A, B and C are shown. To draw the M and V diagrams, the distributed loads in fields AB and BC have been replaced by their resultants R^{AB} and R^{BC}. For the triangular load on BC

$$R^{\mathrm{BC}} = \tfrac{1}{2} \times (6\ \mathrm{kN/m})(2\sqrt{2}\ \mathrm{m}) = 6\sqrt{2}\ \mathrm{kN}.$$

The trapezoidal load on AB is divided into triangular loads (1) and (2); their resultants can be calculated easier:

$$R^{(1)} = \tfrac{1}{2} \times (18\ \mathrm{kN/m})(4\sqrt{2}\ \mathrm{m}) = 36\sqrt{2}\ \mathrm{kN},$$

$$R^{(2)} = \tfrac{1}{2} \times (6\ \mathrm{kN/m})(4\sqrt{2}\ \mathrm{m}) = 12\sqrt{2}\ \mathrm{kN},$$

$$R^{\mathrm{AB}} = R^{(1)} + R^{(2)} = 48\sqrt{2}\ \mathrm{kN}.$$

The location of the line of action of R^{AB} is found from the moment about A:

$$a R^{\mathrm{AB}} = \tfrac{1}{3} \times 4\sqrt{2} \times R^{(1)} + \tfrac{2}{3} \times 4\sqrt{2} \times R^{(2)} = 160\ \mathrm{kNm}$$

so that

$$a = \frac{160\ \mathrm{kNm}}{48\sqrt{2}\ \mathrm{kN}} = \tfrac{5}{3}\sqrt{2}\ \mathrm{m}.$$

c. Figure 13.24a shows all the forces acting normal to the beam axis. They generate shear forces and bending moments in the beam. The distributed loads on AB and BC have been replaced by their load resultants. In Figures 13.24b and 13.24c, the V and M diagrams due to the load resultants are shown by means of dashed lines. The solid lines are the actual V and M diagrams.

The actual V diagram has a parabolic variation with a step change at B. At C, the distributed load is zero, and the V diagram has a "horizontal" tangent. To the left and to the right of B, the V diagram has the same slope. In other words: the slope of the V diagram is continuous at B.

The actual M diagram is a cubic, with a bend at B. The M diagram due to the load resultants gives the tangents to the actual M diagram at A, B and C.

d. The load q on ABC varies linearly:

$$q = c_1 x + c_2.$$

The coefficients c_1 and c_2 follow from the values $q = +18$ for $x = 0$ and $q = 0$ for $x = 6\sqrt{2}$:

$$q = -\tfrac{3}{2}x\sqrt{2} + 18.$$

The units used are kN and m, and are omitted in this part of the answer.

By integrating, we can find the variation of the shear force and the bending moment from the distributed load:

$$V = -\int q\,\mathrm{d}x = \tfrac{3}{4}x^2\sqrt{2} - 18x + C_1,$$

$$M = \int V\,\mathrm{d}x = \tfrac{1}{4}x^3\sqrt{2} - 9x^2 + C_1 x + C_2.$$

The integration constants C_1 and C_2 can be found from the boundary conditions at A:

$$x = 0 : V = +54\sqrt{2} \Rightarrow C_1 = +54\sqrt{2},$$

$$x = 0 : M = 0 \qquad \Rightarrow C_2 = 0.$$

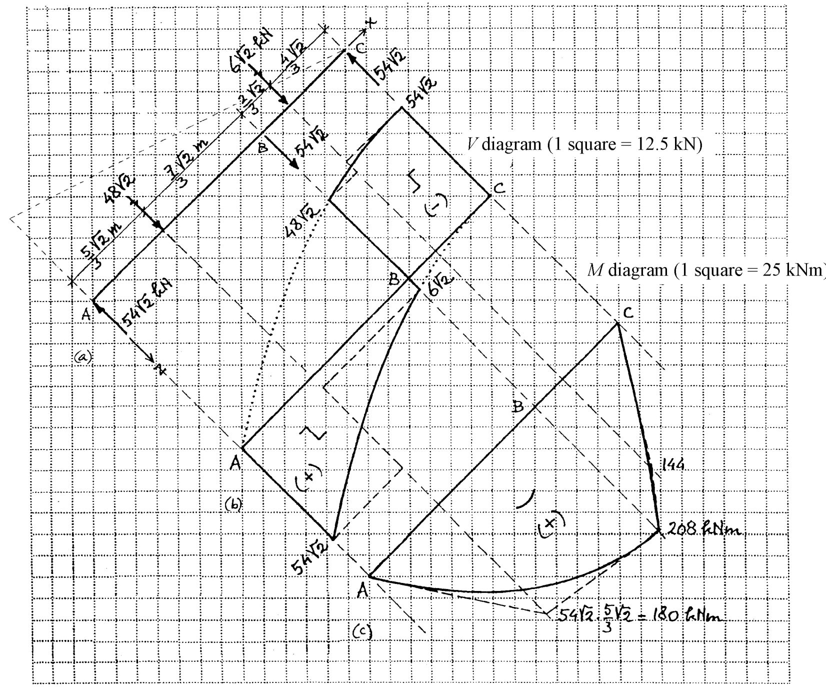


Figure 13.24 (a) Isolated beam ABC with (b) shear force diagram and (c) bending moment diagram.

The functional forms of the shear force V and the bending moment M in AB are

$$V = +\tfrac{3}{4}x^2\sqrt{2} - 18x + 54\sqrt{2},$$
$$M = +\tfrac{1}{4}x^3\sqrt{2} - 9x^2 + 54\sqrt{2}.$$

These expressions can be verified by substituting $x = 4\sqrt{2}$ m to obtain the previously determined values of V and M at B:

$$x = 4\sqrt{2}\ \text{m} : V = +6\sqrt{2}\ \text{kN}\ \ \text{(correct)}$$
$$x = 4\sqrt{2}\ \text{m} : M = +208\ \text{kNm}\ \ \text{(correct)}.$$

13.1.9 Railway sleeper in a ballast bed

Figure 13.25a shows a railway sleeper in a ballast bed. In Figure 13.25b the sleeper is modelled as a line element. The sleeper is loaded across the width of the rail by a uniformly distributed load $q = 15$ kNm. It is assumed that the ballast bed exerts a uniformly distributed counter-pressure p (kN/m) on the entire length of the sleeper. The dimensions are shown in the figure.

Questions:

a. Determine the counter-pressure p exerted by the ballast bed.
b. Draw the V diagram with the deformation symbols, and include relevant values.
c. Draw the M diagram with the deformation symbols and the tangents at A, B and C.
d. Determine the extreme bending moments in the railway sleeper. Where do they occur?

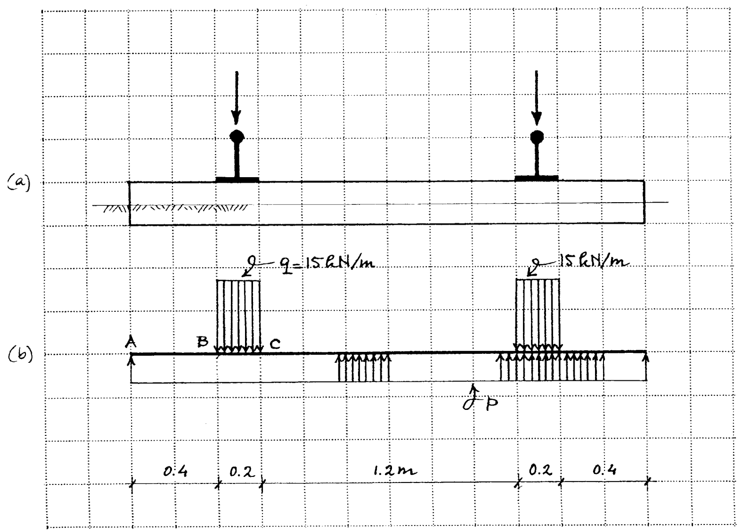


Figure 13.25 (a) Railway sleeper in a ballast bed with (b) the schematic representation.

Solution:

a. The magnitude of p follows from the vertical force equilibrium of the railway sleeper (see Figure 13.25b):

$$\sum F_{\text{vert}}\ (\downarrow) = 2 \times (0.2\ \text{m})(15\ \text{kN/m}) - (2.4\ \text{m}) \times p = 0$$

so that

$$p = 2.5\ \text{kN/m}.$$

b. In Figure 13.26a, the railway sleeper is split into the fields AB, BC, etc. Figure 13.26b shows the resultants of the field loads. In Figure 13.26c, the V diagram due to the load resultants is shown by means of dashed lines. The values denoted by a dot at the field boundaries A, B, C, and so forth, are the correct values. The actual (solid) V diagram varies linearly per field between the values indicated by means of dots.

c. Figure 13.26d shows the M diagram due to the load resultants by means of dashed lines. The values on the field boundaries A, B, C, and so forth, indicated by a dot, are the correct values. They can be determined directly from the dashed M diagram. The dashed M diagram also gives the tangents to the actual (solid) M diagram at A, B, C and so forth. In each field, the actual M diagram varies parabolically. In the fields AB, CD and EG, the parabola is turned upwards (the distributed load is acting upwards), in the fields BC and DE the parabola is turned downwards (the resulting distributed load is acting downwards).

Per field, the parabolic variation can be drawn in the standard manner. For field CD it holds that

$$p_1 = \tfrac{1}{8}q\ell^2 = \tfrac{1}{8} \times (2.5\ \text{kN/m})(1.2\ \text{m})^2 = 0.45\ \text{kNm}$$

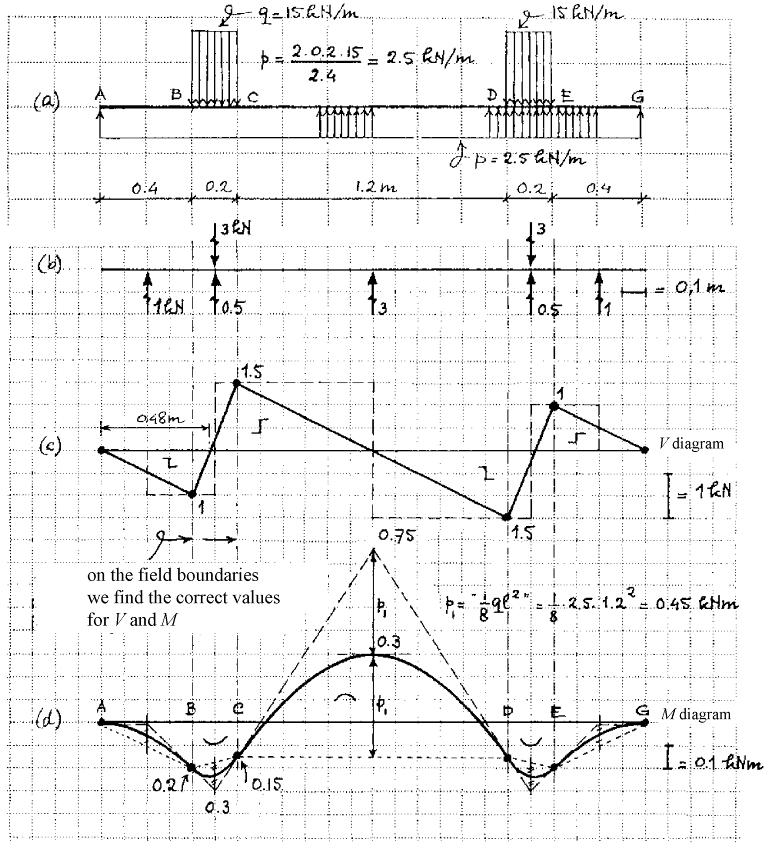


Figure 13.26 (a) The distributed loads on the railway sleeper replaced by (b) their resultants. (c) Shear force diagram and (d) bending moment diagram.

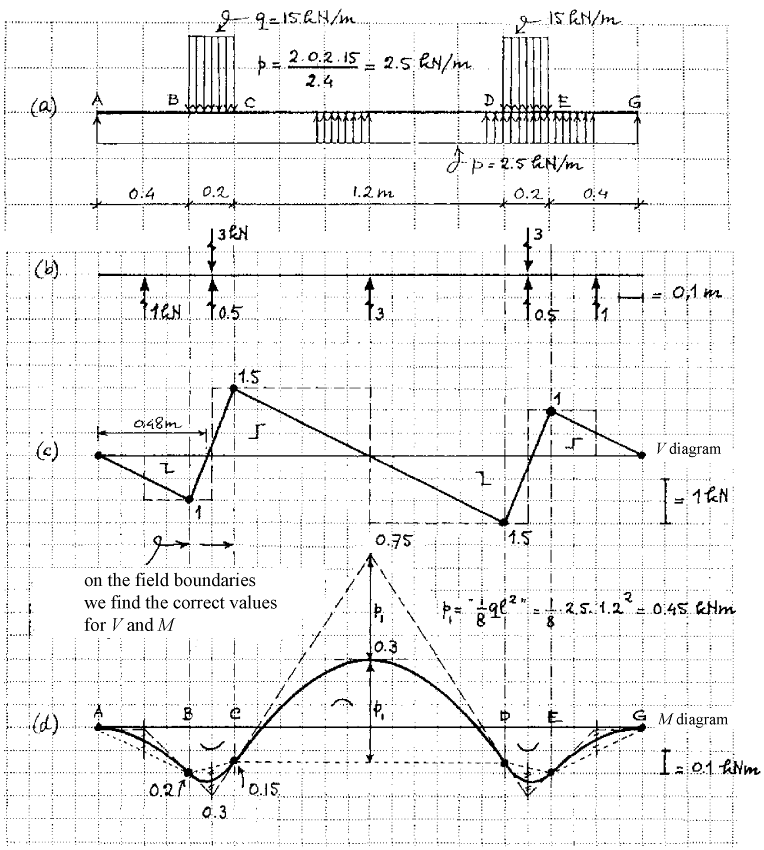


Figure 13.26 (a) The distributed loads on the railway sleeper replaced by (b) their resultants. (c) Shear force diagram and (d) bending moment diagram.

so that

$$M_1 = p_1 - (0.15 \text{ kNm}) = 0.3 \text{ kNm} (\frown).$$

In the middle of the fields BC and DE it applies that

$$p_2 = \tfrac{1}{8} q \ell^2 = \tfrac{1}{8} \times (12.5 \text{ kN/m})(0.2 \text{ m})^2 = 0.0625 \text{ kNm}$$

so that

$$M_2 = (0.3 \text{ kNm}) - p_2 = 0.2375 \text{ kNm} (\smile).$$

The values p_2 and M_2 are not shown in Figure 13.26d.

d. The M diagram has three extreme values, as the V diagram has three zeros (not including the end zeros). The largest extreme value $M_{\max}$ occurs in the middle of the railways sleeper. This maximum was determined in question c, and can be read directly from the M diagram:

$$M_{\max} = M_1 = 0.3\ kNm\ (\frown).$$

The two other extreme values are equal, and occur in the fields BC and DE. Here we will determine the extreme value $M_{\min}$ for field BC. In the shear force diagram the distance from A to the zero in BC is (see Figure 13.26c)

$$(0.4 \text{ m}) + \frac{1 \text{ kN}}{(1 \text{ kN}) + (1.5 \text{ kN})} \times (0.2 \text{ m}) = 0.48 \text{ m}.$$

The magnitude of $M_{\min}$ is now found most easily from the area of the V diagram:

$$M_{\min} = \tfrac{1}{2} \times (0.48 \text{ m})(1 \text{ kN}) = 0.24 \text{ kNm} (\smile).$$

13.1.10 Beam on the ground

Figure 13.27 shows a beam AB lying on the ground, of which the dead weight can be ignored. A uniformly distributed load q is acting on the right-hand side of the beam over a length a. Due to this load, the earth pressure on the underside of the beam varies linearly, from 0 at A to 48 kN/m at B.

Questions:

a. From the equilibrium of the beam, determine length a and load q.
b. Make a good sketch of the V diagram and the M diagram for the beam.
c. At which cross-section is the shear force an extreme? Write down the extreme values for the V diagram. For these cross-sections, also include the tangents to the M diagram.
d. At which cross-section is the bending moment an extreme? Determine this value, and include it with the M diagram.

Solution:

a. The resultant of the earth pressure and the resultant of the q load must have the same line of action (moment equilibrium of a body subject to two forces). The distance from B to the line of action of both resultants is (see Figure 13.27)

$$\tfrac{1}{2}a = \tfrac{1}{3} \times (1.80\text{ m}) \Rightarrow a = 1.20\text{ m}.$$

On the basis of the vertical force equilibrium, both resultants must be of equal magnitude:

$$qa = \tfrac{1}{2} \times (1.80\text{ m})(48\text{ kN/m}) = 43.2\text{ kN}$$

so that

$$q = \frac{43.2\text{ kN}}{1.20\text{ m}} = 36\text{ kN/m}.$$

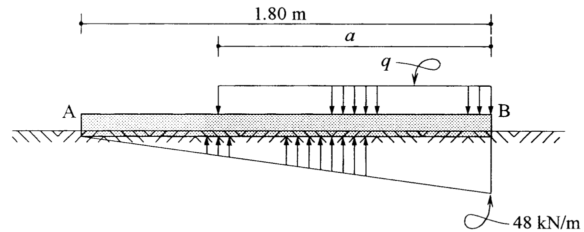


Figure 13.27 Beam on the ground. Due to the uniformly distributed load q, the ground pressure is distributed linearly.

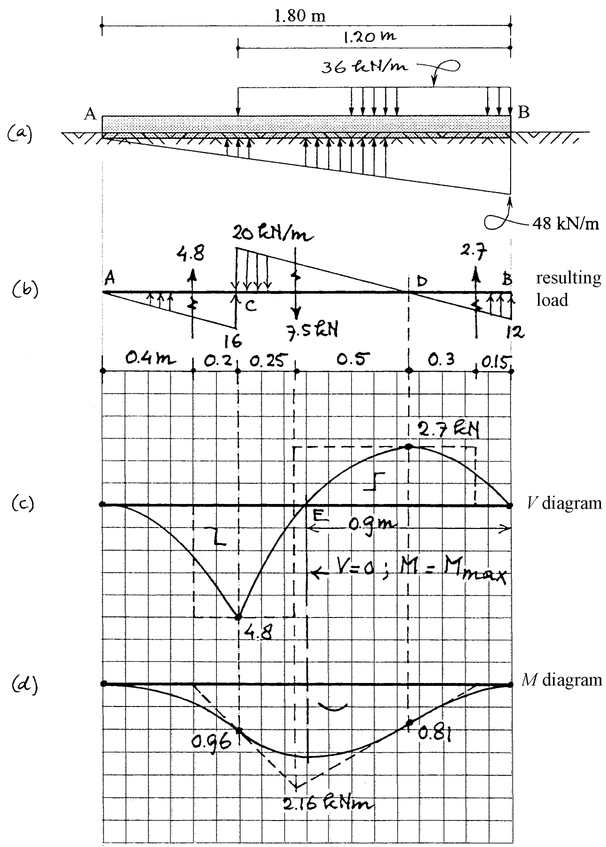


Figure 13.28 (a) Beam with load and earth pressure. (b) Resulting distributed load on the beam and the resultants per field. (c) Shear force diagram and (d) bending moment diagram.

b. In Figure 13.28a, all the loads on beam AB are shown. In Figure 13.28b, the beam has been modelled as a line element, and the resulting load is shown. Three fields with a triangular load can be distinguished. The figure also shows the resultants of these triangular loads.

In Figures 13.28c and 13.28d, the V and M diagrams due to the three load resultants are shown by means of dashed lines. They give the correct values on the field boundaries, in both figures shown by dots. In the field boundaries, the dashed M diagram also gives the tangents to the actual M diagram. The actual V and M diagrams are shown by means of solid lines.

c. A linearly distributed load produces a parabolic V diagram. The shear force is an extreme where the (resulting) distributed load is zero or changes sign, so at A, C and D. At A and D, the V diagram has a horizontal tangent (the distributed load is zero here) and the parabolas have their top. At C, the step change in the distributed load gives a bend in the V diagram. The extreme values can be read off directly from the V diagram.

d. The bending moment is a cubic and is relatively simple to draw using the tangents at the field boundaries. The maximum bending moment $M_{\max}$ occurs where the shear force is zero. This is at E, 0.9 m to the left of B, see the V diagram in Figure 13.28c. The magnitude of $M_{\max}$ can be determined from the moment equilibrium of the isolated part EB in Figure 13.29:

$$M_{\max} = (2.7 \text{ kN})(0.6 \text{ m}) = 1.62 \text{ kNm } (\smile).$$

$M_{\max}$ can also be determined from the area of the parabolic V diagram for EB. To do so we have to know that the area of the parabola is equal to two-thirds of the area of the rectangle with a width of 0.9 m and a height of 2.7 kN. This then gives the same value:

$$M_{\max} = \tfrac{2}{3} \times (0.9 \text{ m})(2.7 \text{ kN}) = 1.62 \text{ kNm } (\smile).$$

13.1.11 Lean-to subject to dead weight, wind and snow loads

In Figure 13.30, the lean-to ACD is modelled as a line element. We want to determine the N diagram and the extreme values of the bending moment due to the three uniformly distributed loads:

a. A wind pressure of $q_{\mathrm{w}} = 5$ kN/m (force per length measured along ACD).
b. A dead weight of $q_{\mathrm{dw}} = 5$ kN/m (force per length measured along ACD).
c. A snow load of $q_{\mathrm{sn}} = 5$ kN/m (force per length measured along the projection of ACD on the horizontal ground plane).

Solution:
Since the dead weight and the snow load have components transverse to the beam axis (q_{tr}) and parallel to the beam axis (q_{pa}), the N, V and M diagrams are first determined due to the separate loads $q_{\mathrm{tr}} = 1$ kN/m (Figure 13.31a) and $q_{\mathrm{pa}} = 1$ kN/m (Figure 13.31b). By means of superposition, we then determine the final N diagram for each of the given loads and the extreme values of the bending moments.

In preparation, the dimensions given in Figure 13.30 are first used to determine the angles α, β and γ and the lengths ℓ^{AC}, ℓ^{CD} and ℓ^{ACD} of AC, CD and ACD respectively. The angles α, β and γ we find from

$$\tan\alpha = 2/5 \Rightarrow \alpha = 21.8°,$$

$$\tan\beta = 4/5 \Rightarrow \beta = 38.7°,$$

$$\gamma = \alpha + \beta = 60.5°.$$

The lengths of AC, CD and ACD are

$$\ell^{\mathrm{AC}} = (5\text{ m})/\cos\alpha = 5.385\text{ m},$$

$$\ell^{\mathrm{CD}} = (2\text{ m})/\cos\alpha = 2.154\text{ m},$$

$$\ell^{\mathrm{ACD}} = \ell^{\mathrm{AC}} + \ell^{\mathrm{CD}} = 7.539\text{ m}.$$

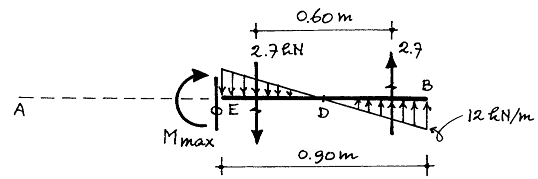


Figure 13.29 The maximum bending moment at E can be derived from the moment equilibrium of EB.

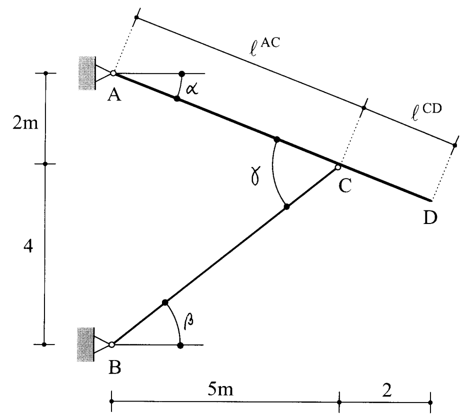


Figure 13.30 Lean-to.

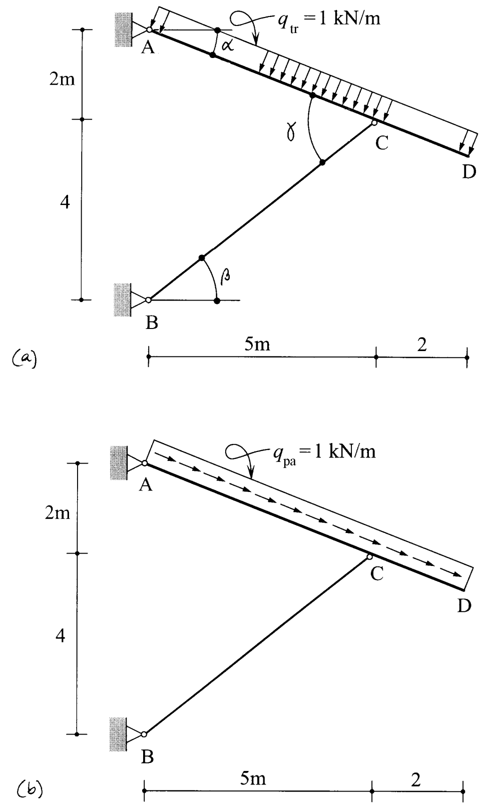


Figure 13.31 Lean-to loaded by a uniformly distributed load of 1 kN/m (a) normal to and (b) parallel to roof plane ACD.

• ***N, V and M diagrams due to q_{tr} = 1 kN/m*** (Figure 13.31a)
In Figure 13.32, ACD has been isolated. The distributed load $q_{tr} = 1$ kN/m has been replaced by its resultant R_{tr}. In addition, the joining forces acting at A and C on ACD are also shown. The indices "*pa*" and "*tr*" point to the directions "*pa*rallel to the beam axis" respectively "*tr*ansverse to the beam axis".

Since BC is a two-force member, the resultant of C_{tr} and C_{pa} must be along BC:

$$C_{pa} = C_{tr}/\tan\gamma.$$

The resultant of the uniformly distributed load is

$$R_{tr} = q_{tr}\ell^{ACD} = (1\ \text{kN/m})(7.539\ \text{m}) = 7.539\ \text{kN}.$$

The moment equilibrium about A gives C_{tr}:

$$\begin{aligned}\sum T|A \circlearrowright &= R_{tr} \times \tfrac{1}{2}\ell^{ACD} - C_{tr} \times \ell^{AC}\\ &= (7.539\ \text{kN}) \times \tfrac{1}{2} \times (7.539\ \text{m}) - C_{tr} \times (5.385\ \text{m}) = 0\end{aligned}$$

so that

$$C_{tr} = 5.277\ \text{kN}.$$

C_{pa} is found from the direction of two-force member BC:

$$C_{pa} = C_{tr}/\tan\gamma = \frac{5.277\ \text{kN}}{\tan 60.5^\circ} = 2.986\ \text{kN}.$$

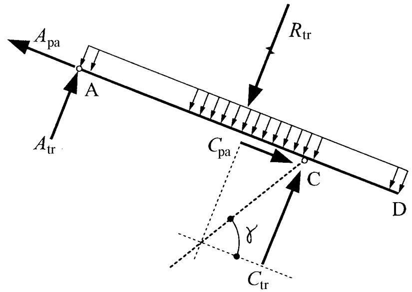


Figure 13.32 Isolated beam ACD with the support reactions at A and C due to a uniformly distributed load normal to the roof plane.

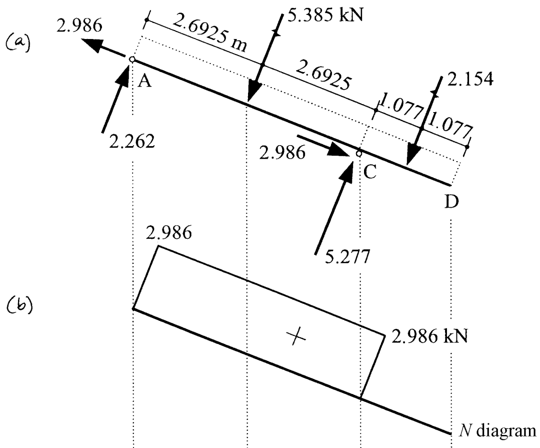


Figure 13.33 (a) Isolated beam ACD with a uniformly distributed load of 1 kN/m normal to the roof plane and associated (b) normal force diagram.

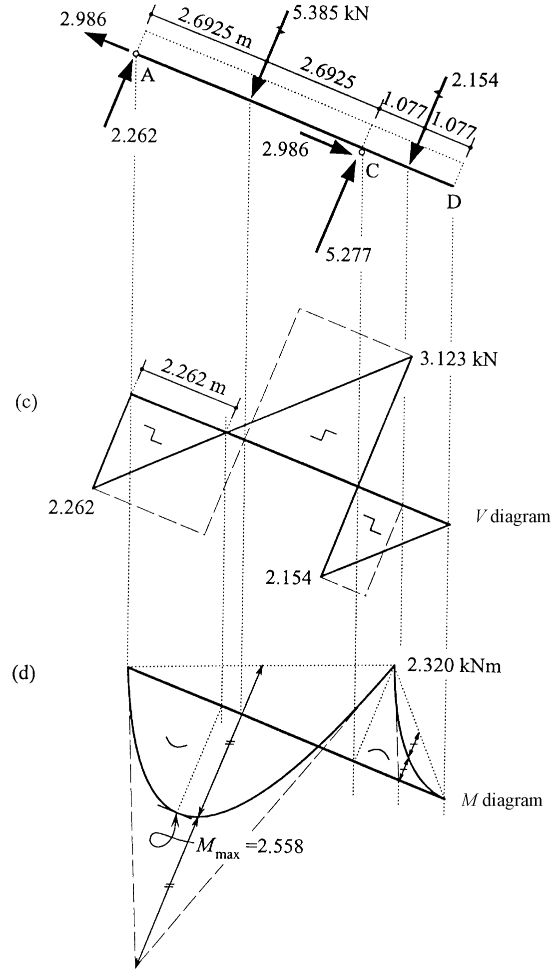


Figure 13.33 (c) Shear force diagram and (d) bending moment diagram due to a uniformly distributed load of 1 kN/m normal to the roof plane.

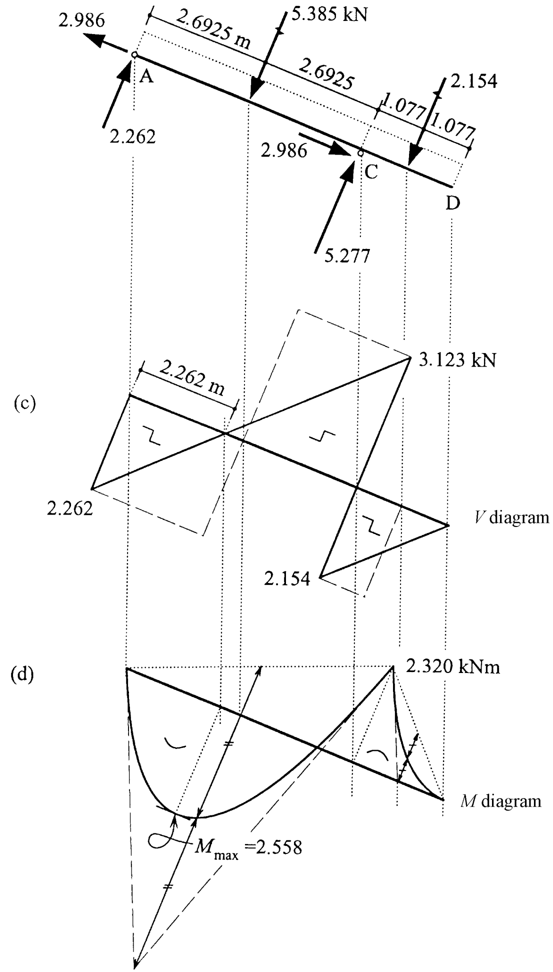


Figure 13.33 (c) Shear force diagram and (d) bending moment diagram due to a uniformly distributed load of 1 kN/m normal to the roof plane.

The force equilibrium in the longitudinal direction of ACD gives

$$A_{pa} = C_{pa} = 2.986 \text{ kN}$$

and the force equilibrium normal to ACD

$$A_{tr} = R_{tr} - C_{tr} = (7.539 \text{ kN}) - (5.277 \text{ kN}) = 2.262 \text{ kN}.$$

Figure 13.33a again shows ACD with the forces determined at A and C, and the resultants of the load on fields AC and CD. In Figures 13.33b to 13.33d, the N, V and M diagrams are shown. It is assumed that the reader is familiar with the necessary calculation. The V and M diagrams due to the load resultants are shown by means of dashed lines.

The M diagram has two extreme values: the (in an absolute sense) smallest moment M_{min} is the bending moment at C and the (in an absolute sense) largest moment M_{max} is the bending moment in field AC:

$$M_{min} = 2.320 \text{ kNm } (\frown).$$

The maximum moment in field AC is found 2.262 m from A. The magnitude can be determined from the area of the V diagram:

$$M_{max} = \tfrac{1}{2}(2.262 \text{ m})(2.262 \text{ kN}) = 2.558 \text{ kNm } (\smile).$$

• ***N, V and M diagrams due to*** $\boldsymbol{q_{pa} = 1}$ ***kN/m*** (Figure 13.31b)
In Figure 13.34, ACD has been isolated and the joining forces acting on ACD are shown. The distributed load $q_{pa} = 1$ kN/m has been replaced by its resultant R_{pa}:

$$R_{pa} = q_{pa}\ell^{ACD} = (1\ \text{kN/m})(7.539\ \text{m}) = 7.539\ \text{kN}.$$

The moment equilibrium about A gives $C_{tr} = 0$ and so $C_{pa} = 0$. The force equilibrium of ACD gives $A_{tr} = 0$ and $A_{pa} = R_{pa} = 7.539$ kN.

In Figure 13.35a, ACD is shown again with the forces determined. In Figure 13.35b, the associated N diagram is shown: due to a uniformly distributed load the normal force is linear. With this load there are no bending moments and shear forces.

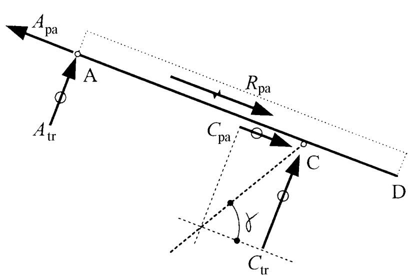


Figure 13.34 Isolated beam ACD with the support reactions in A and C due to a uniformly distributed load parallel to the roof plane.

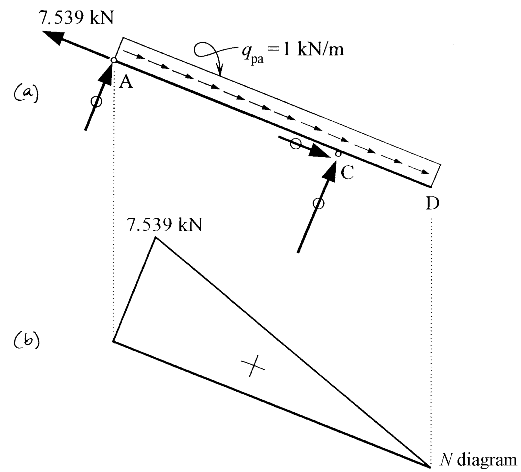


Figure 13.35 (a) Isolated beam ACD with a uniformly distributed load of 1 kN/m normal to the roof plane and associated (b) normal force diagram.

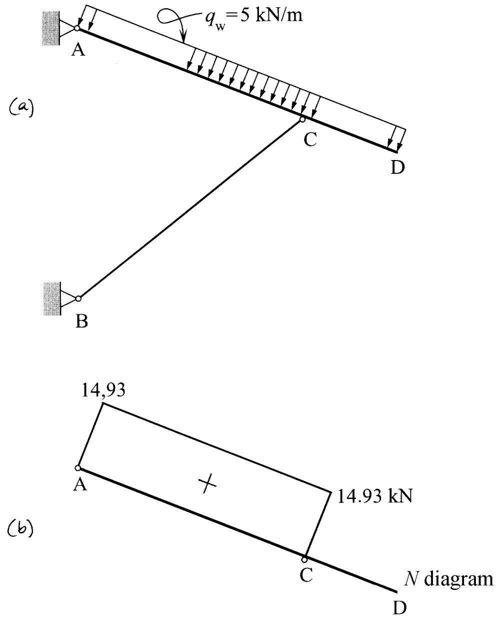


Figure 13.36 (a) Lean-to with wind load of 5 kN/m and (b) associated normal force diagram.

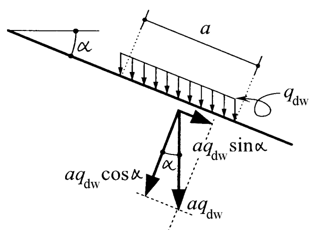


Figure 13.37 The components of the dead weigh q_{dw} of a member segment with length a.

a. *Wind load*

The wind load $q_w = 5$ kN/m in Figure 13.36a is normal to the roof plane ACD. The associated N, V and M diagrams are equal to those in Figure 13.33b to d, but with values that are five times as large:

$$M_{w;max} = (2.558 \text{ kNm}) \times 5 = 12.79 \text{ kNm},$$

$$M_{w;min} = (2.320 \text{ kNm}) \times 5 = 11.60 \text{ kNm},$$

$$N^{AC} = (2.986 \text{ kNm}) \times 5 = 14.93 \text{ kNm}.$$

The N diagram for ACD is shown in Figure 13.36b.

b. *Dead weight*

In Figure 13.37 we take a closer look at a member segment with length a. The dead weight of this member segment is aq_{dw} with components $aq_{dw} \cos\alpha$ and $aq_{dw} \sin\alpha$ respectively normal to and parallel to the beam axis. For the components of the *distributed load* normal to and parallel to the beam axis we find

$$q_{dw;tr} = \frac{aq_{dw} \cos\alpha}{a} = q_{dw} \cos\alpha = (5 \text{ kN/m}) \cos 21.8° = 4.642 \text{ kN/m},$$

$$q_{dw;pa} = \frac{aq_{dw} \sin\alpha}{a} = q_{dw} \sin\alpha = (5 \text{ kN/m}) \sin 21.8° = 1.857 \text{ kN/m}.$$

The distributed load $q_{dw} = 5$ kN/m due to the dead weight (Figure 13.38a) has components of 4.642 kN/m normal to the beam axis (Figure 13.38b) and 1.857 kN/m parallel to the beam axis (Figure 13.38c).

The bending moment in ACD is caused by the load of 4.642 kN/m normal to the beam axis. The M diagram is equal to that in Figure 13.33d, but 4.462 as large, so that the extreme values of the bending moments are

$$M_{dw;max} = (2.558 \text{ kNm}) \times 4.642 = 11.87 \text{ kNm},$$

$$M_{dw;min} = (2.320 \text{ kNm}) \times 4.642 = 10.77 \text{ kNm}.$$

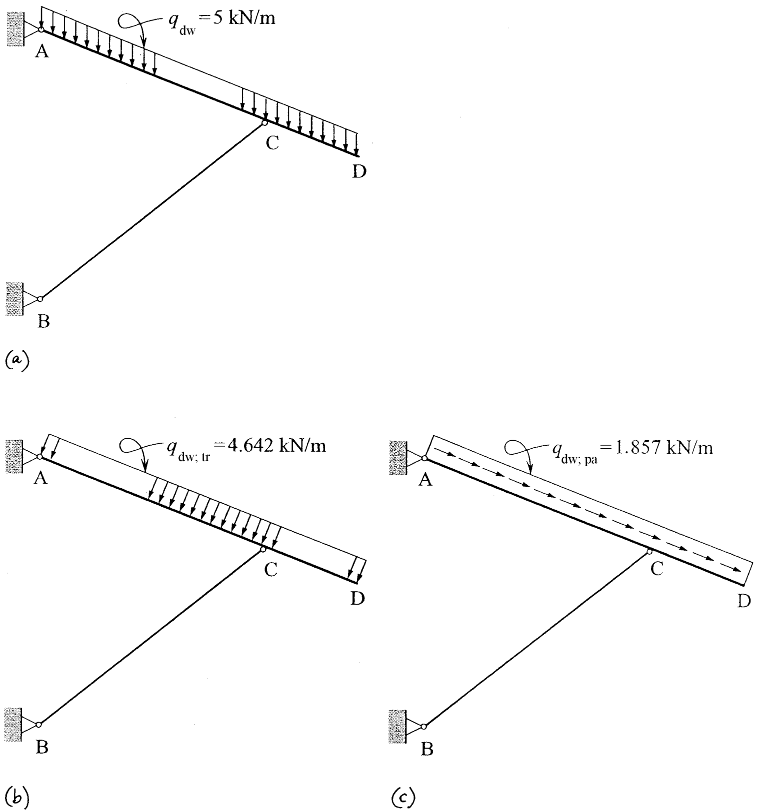


Figure 13.38 (a) The dead weight of beam ACD of 5 kN/m, resolved into components (b) normal to the roof plane and (c) parallel to the roof plane.

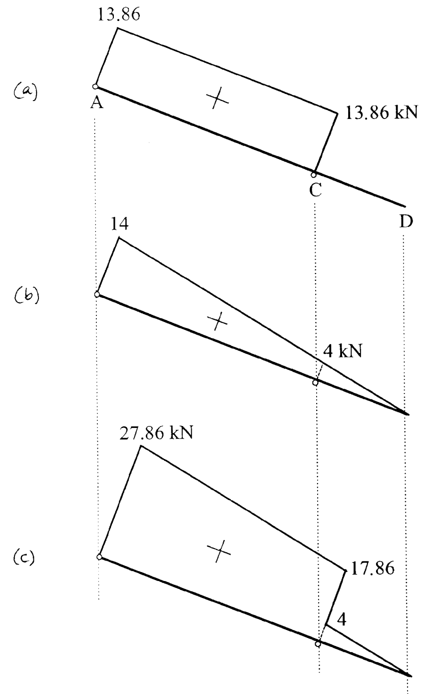


Figure 13.39 (a) The *N* diagram due to the component normal to the roof plane, superposed on (b) the *N* diagram due to the component parallel to the roof plane gives (c) the requested *N* diagram due to the dead weight.

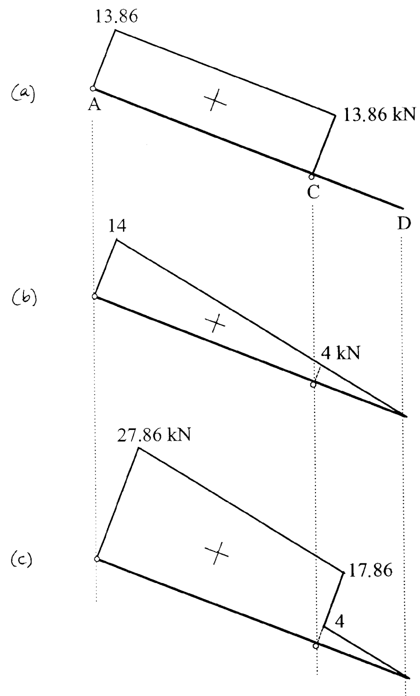


Figure 13.39 (a) The N diagram due to the component normal to the roof plane, superposed on (b) the N diagram due to the component parallel to the roof plane gives (c) the requested N diagram due to the dead weight.

The N diagram in Figure 13.39a due to the 4.642 kN/m load normal to the beam axis is equal to the N diagram in Figure 13.33b, but with values that are 4.642 times as large. The N diagram in Figure 13.39b due to the load of 1.857 kN/m parallel to the beam axis is equal to the N diagram in Figure 13.35b, but then with values that are 1.857 times as large. Superposing the N diagrams in Figures 13.39a and 13.39b gives the N diagram in Figure 13.39c. This is the requested N diagram due to the dead weight $q_{\text{dw}} = 5$ kN/m.

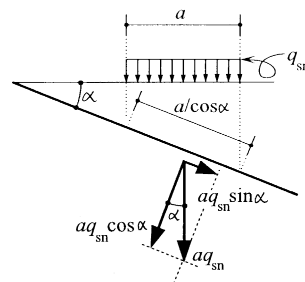


Figure 13.40 The components of the snow load q_{sn} on a member segment with length a measured horizontally.

c. *Snow load*

Over a length a measured horizontally, the resultant of the snow load is aq_{sn} (see Figure 13.40). The components of this force normal to and parallel to the axis are respectively $aq_{\text{sn}} \cos\alpha$ and $aq_{\text{sn}} \sin\alpha$. They act on a member segment with length $a/\cos\alpha$. For the components of the distributed load normal to and parallel to the beam axis we now find

$$q_{\text{sn;tr}} = \frac{aq_{\text{sn}} \cos\alpha}{a/\cos\alpha} = q_{\text{sn}} \cos^2\alpha,$$

$$= (5\ \text{kN/m}) \cos^2 21.8° = 4.310\ \text{kN/m},$$

$$q_{\text{sn;pa}} = \frac{aq_{\text{sn}} \sin\alpha}{a/\cos\alpha} = q_{\text{sn}} \sin\alpha \cos\alpha,$$

$$= (5\ \text{kN/m}) \sin 21.8° \cos 21.8° = 1.724\ \text{kN/m}.$$

The distributed load $q_{\text{sn}} = 5$ kN/m due to the snow (Figure 13.41a) has components of 4.310 kN/m normal to the beam axis (Figure 13.41b) and 1.724 kN/m parallel to the beam axis (Figure 13.41c).

The bending moment in ACD is caused by the load of 4.310 kN/m normal to the beam axis. The M diagram is equal to that in Figure 13.33d, but now 4.310 times as large, so that the extreme values of the bending moment are:

$$M_{\text{sn;max}} = (2.558\ \text{kNm}) \times 4.310 = 11.02\ \text{kNm}\ (\smile),$$

$$M_{\text{sn;min}} = (2.320\ \text{kNm}) \times 4.310 = 10\ \text{kNm}\ (\frown).$$

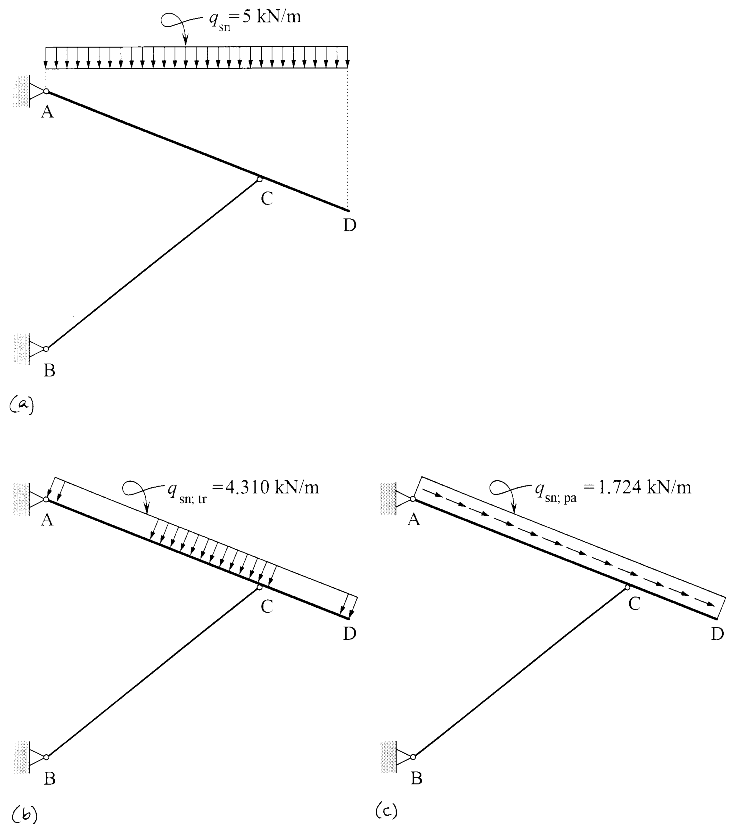


Figure 13.41 (a) The snow load of 5 kN/m on the lean-to, resolved into components (b) normal to the roof plane and (c) parallel to the roof plane.

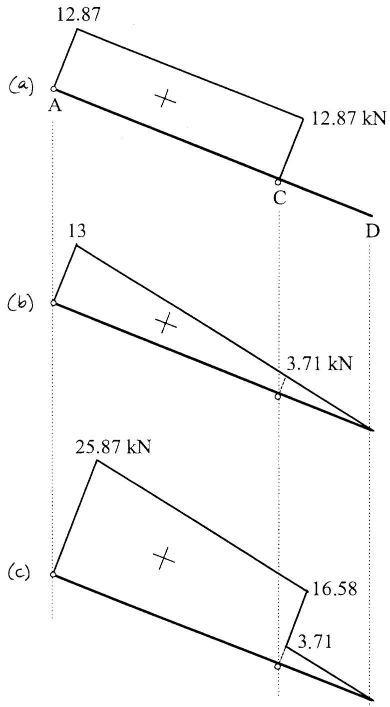


Figure 13.42 (a) The N diagram due to the component normal to the roof plane, superpositioned on (b) the N diagram due to the component parallel to the roof plane gives (c) the requested N diagram resulting from the snow load.

The N diagram in Figure 13.42a due to the load of 4.310 kN/m normal to the beam axis is equal to the N diagram in Figure 13.33b, but with values that are 4.310 times as large. The N diagram in Figure 13.42b due to the load of 1.724 kN/m parallel to the beam axis is equal to the N diagram in Figure 13.35b, but with values that are 1.724 times as large. By superposing the N diagrams in Figures 13.42a and 13.42b on one another we get the N diagram in Figure 13.42c. This is the requested N diagram due to the snow loading $q_{\text{sn}} = 5$ kN/m.

13.1.12 Indirectly loaded beam

With indirectly loaded beams, the load does not act on the beam directly, but is rather transferred to the beam by means of a system of stringer beams and cross beams.

Figure 13.43 shows a schematic representation of a bridge constructed as an indirectly loaded beam. *Main beam* (mb) AB is carrying *cross beams* (cb) at regular distances which in turn are carrying *stringer beams* (sb).

The main beam is divided into a number of fields by the cross beams, five in Figure 13.43. It is assumed that the lengths of the stringer beams are equal to the field lengths, and that the stringer beams are simply supported at the cross beams.

Since the main beam is loaded only by forces exerted by the cross beams, the shear force in each field is constant, and the bending moment in each field is linear (excluding the dead weight of the main beam).

For the indirectly loaded beam in Figure 13.43, the M and V diagrams are determined for the following two loading cases:

1. a concentrated load,
2. a uniformly distributed full load.

Example 1
In Figure 13.44a, the main beam AB is indirectly loaded by a point load of 60 kN in field CD. The dimensions can be read from the figure.

Question:
Determine the M and V diagrams for the indirectly loaded main beam and for the stringer beams.

Solution:
The support reactions at A and B are 40 kN and 20 kN respectively (see Figure 13.44a).

In Figure 13.44b, the M diagram is shown for the *directly loaded* beam. The dashed line indicates how this deviates from the requested M diagram for the *indirectly loaded* beam.

The force of 60 kN in field CD exerts forces on the main beam of 20 kN and 40 kN via the cross beams in C and D respectively (see Figure 13.44f). The other cross beams do not exert any forces on the main beam. The M diagram due to these forces of 20 and 40 kN at C and D is equal to the M diagram due to the (resulting) force of 60 kN (this is the M diagram for the directly loaded beam), with the exception of field CD. In field CD, the M diagram varies linearly between the values of 120 kNm at C and 180 kNm at D. The M diagram of the indirectly loaded beam can therefore be found by snipping the M diagram of the directly loaded beam over field CD.

The snipped part of the M diagram is equal to the M diagram of the simply supported stringer beam CD (see Figure 13.44c).

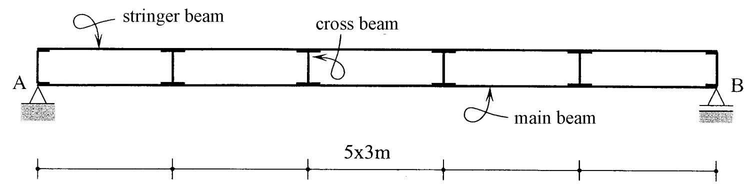


Figure 13.43 A bridge constructed as an indirectly loaded beam. The load on the stringer beams is transferred to the main beams via crossbeams.

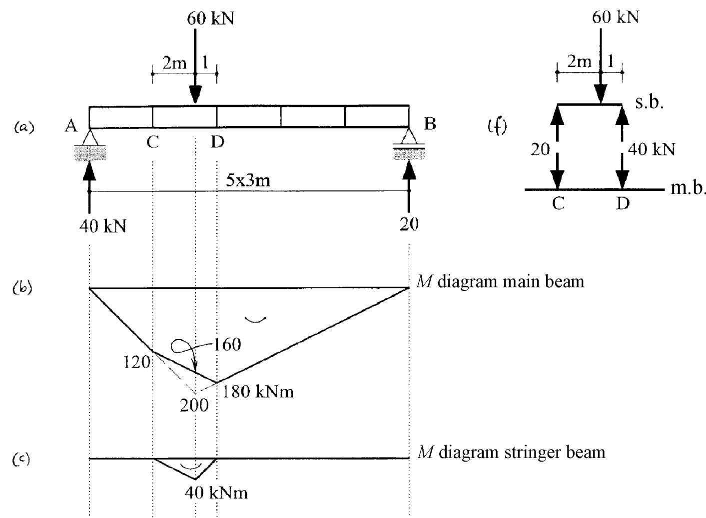


Figure 13.44 (a) Indirectly loaded beam AB, loaded in field CD by a point load. (b) The bending moment diagram of the indirectly loaded beam can be found from the dashed bending moment diagram of the directly loaded beam by snipping it between C and D. (c) The bending moment diagram of stringer beam CD is equal to the difference between the bending moment diagrams for the directly and indirectly loaded beam.

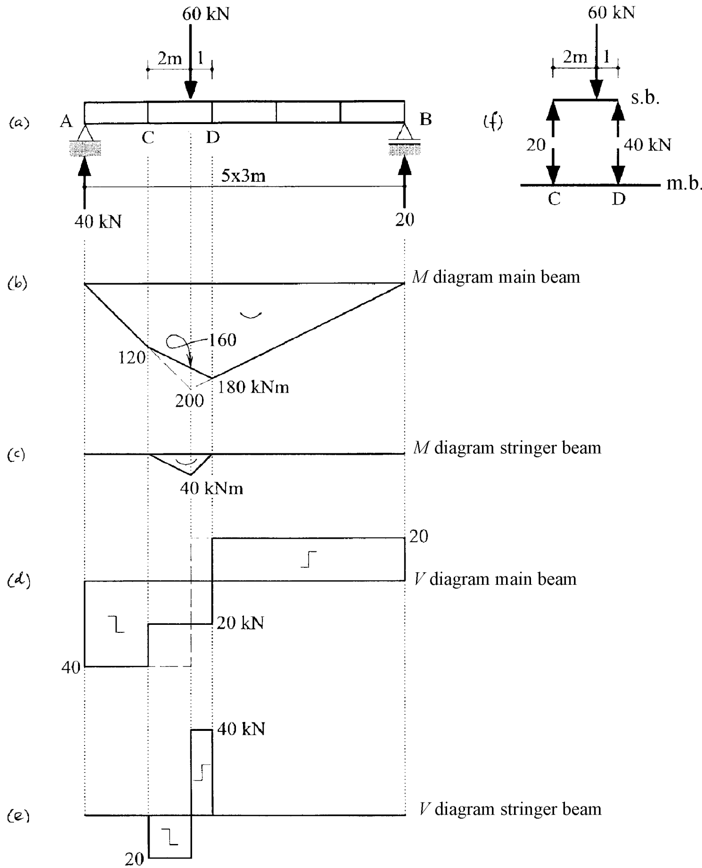


Figure 13.44

The V diagram is found from the slope of the M diagram (see Figure 13.44d). The V diagram for the *indirectly loaded* beam deviates from the dashed V diagram for the *directly loaded* beam only in the field CD.

The area enclosed in field CD between dashed and solid lines (the V diagrams for the directly and indirectly loaded beam respectively) is exactly the same as the V diagram for the simply supported stringer beam CD (see Figure 13.44e). The V diagram for the indirectly loaded beam can therefore be found by reducing the shear force of the directly loaded beam in field CD by the shear force in the stringer beam.

Example 2

In Figure 13.45a the indirectly loaded beam AB is carrying a uniformly distributed load of 16 kN/m. The dimensions are found in the figure.

Question:
Determine the M and V diagrams for the indirectly loaded main beam and for the stringer beams.

Figure 13.44 (a) Indirectly loaded beam AB, loaded in field CD by a point load. (b) The bending moment diagram of the indirectly loaded beam can be found from the dashed bending moment diagram of the directly loaded beam by snipping it between C and D. (c) The bending moment diagram of stringer beam CD is equal to the difference between the bending moment diagrams for the directly and indirectly loaded beam. (d) The shear force diagram of the indirectly loaded beam can be found from the slopes of the associated bending moment diagram. (e) The shear force diagram of the stringer beams can be found from the slopes of the associated bending moment diagram and is equal to the difference between the shear force diagrams for the directly and indirectly loaded beam. (f) The forces exerted via the cross beams in C and D on the main beam are found from the equilibrium of the stringer beam CD.

Solution:
The support reactions at A and B are 120 kN (see Figure 13.45a).

In Figure 13.45b, the M diagram for the *directly loaded* beam is shown. The values at the cross beams can be determined directly from the equilibrium or (because the M diagram is parabolic) by means of the formula $M = \frac{1}{2}qab$, in which a is the distance to A and b is the distance to B (see Section 12.1.6). By *snipping* the M diagram for the directly loaded beam over the fields we can find the M diagram for the *indirectly loaded* beam.

The snipped part of the M diagram is equal to the M diagram of the simply supported stringer beams (see Figure 13.45c).

The V diagram is found from the slope of the M diagram (see Figure 13.45d). The V diagram for the directly loaded beam is shown by means of a dashed line. The difference between both V diagrams is equal to the V diagram of the simply supported stringer beams (see Figure 13.45e).

Note that the shear forces at the end fields are not equal to the support reactions. Half of the load on the end fields is not carried by the main beam but is transferred by the end cross beams directly to the supports.

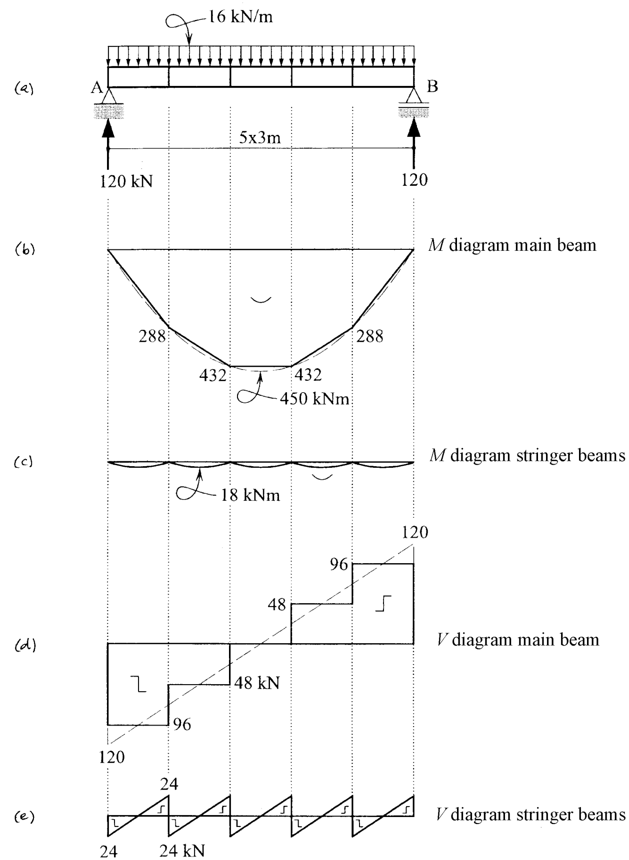


Figure 13.45

Figure 13.45 (a) Indirectly loaded beam AB with a uniformly distributed load. (b) The bending moment diagram of the indirectly loaded beam is found by snipping the dashed (parabolic) bending moment diagram of the directly loaded cross beams. (c) The bending moment diagram of the stringer beams is equal to the difference between the bending moment diagrams for the directly and indirectly loaded beam. (d) The shear force diagram of the indirectly loaded main beam can be found from the slopes of the associated bending moment diagram. (e) The shear force diagram of the stringer beams can be found from the slopes of the associated bending moment diagram and is equal to the difference between the shear force diagrams for the directly and indirectly loaded beam.

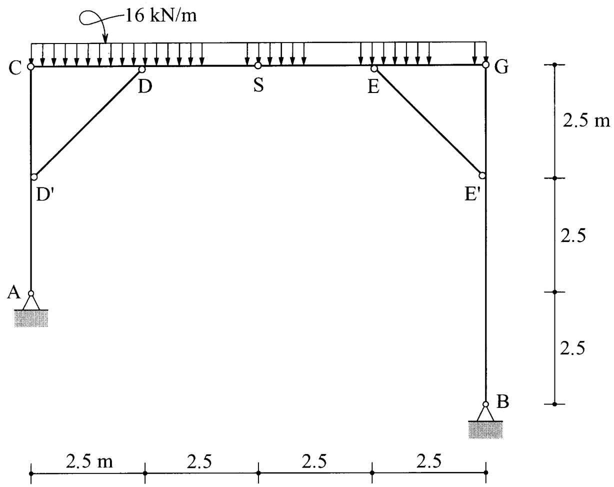


Figure 13.46 Three-hinged shored frame.

13.2 Compound and associated structures

To be able to draw the M, V and N diagrams for compound and associated structures, it is first necessary to determine the support reactions and the joining forces between the compound sections. Subsequently, the M, V and N diagrams can be determined and drawn for the constituent parts, in the same way as for the self-contained structures in Section 13.1. By then adding together the M, V and N diagrams of the constituent parts we can determine the requested M, V and N diagrams for the entire structure.

13.2.1 Three-hinged shored frame

The three-hinged shored frame ASB in Figure 13.46 is loaded over CDSEG by a uniformly distributed load of 16 kN/m. The dimensions are shown in the figure.

Questions:

a. Determine the support reactions at A and B.
b. Determine the forces in shores DD′ and EE′, with the correct signs for tension and compression.
c. Isolate CDSEG, and draw all the forces acting on it.
d. For CDSEG, draw the N, V and M diagrams with the deformation symbols and the tangents at C, D, S, E and G to the M diagram. Include relevant values.

Solution (units kN and m):

a. Determining the support reactions is left to the reader (see Section 5.3, Example 1).

b. In Figure 13.47, all parts of the frame have been isolated, and all the joining forces are shown. The support reactions at A and B are also shown. Both shores DD′ and EE′ are at 45°:

$$D_h = D_v = \tfrac{1}{2} N^{DD'} \sqrt{2},$$

$$E_h = E_v = \tfrac{1}{2} N^{EE'} \sqrt{2}.$$

D_h can be determined from the moment equilibrium of post AD′C about C:

$$\sum T|C \circlearrowleft \; = +32 \times 5 + D_h \times 2.5 = 0 \Rightarrow D_h = D_v = -64 \text{ kN}$$

so that

$$N^{DD'} = -64\sqrt{2} \text{ kN (a compressive force).}$$

In the same way, E_h can be determined from the moment equilibrium of post BE′G about G:

$$\sum T|G \circlearrowleft \; = -32 \times 7.5 - E_h \times 2.5 = 0 \Rightarrow E_h = E_v = -96 \text{ kN}$$

so that

$$N^{EE'} = -96\sqrt{2} \text{ kN (a compressive force).}$$

The forces in C and G follow from the force equilibrium of AC and BG:

$$C_h = -32 - D_h = +32 \text{ kN},$$

$$C_v = +72 + D_v = +8 \text{ kN},$$

$$G_h = -32 - E_h = +64 \text{ kN},$$

$$G_v = +88 + E_v = -8 \text{ kN}.$$

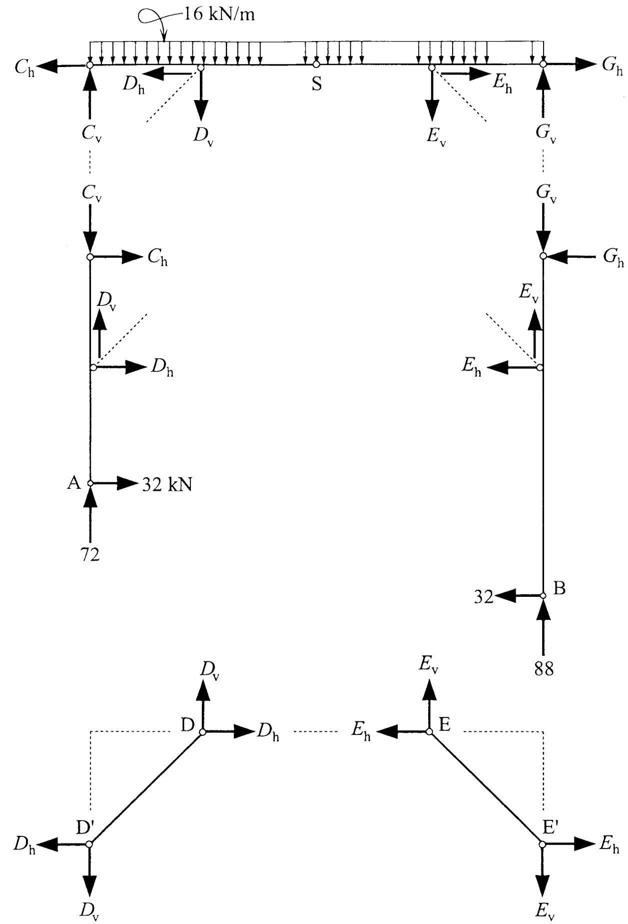


Figure 13.47 Girder, posts and shores isolated from the three-hinged shored frame, with joining forces and support reactions.

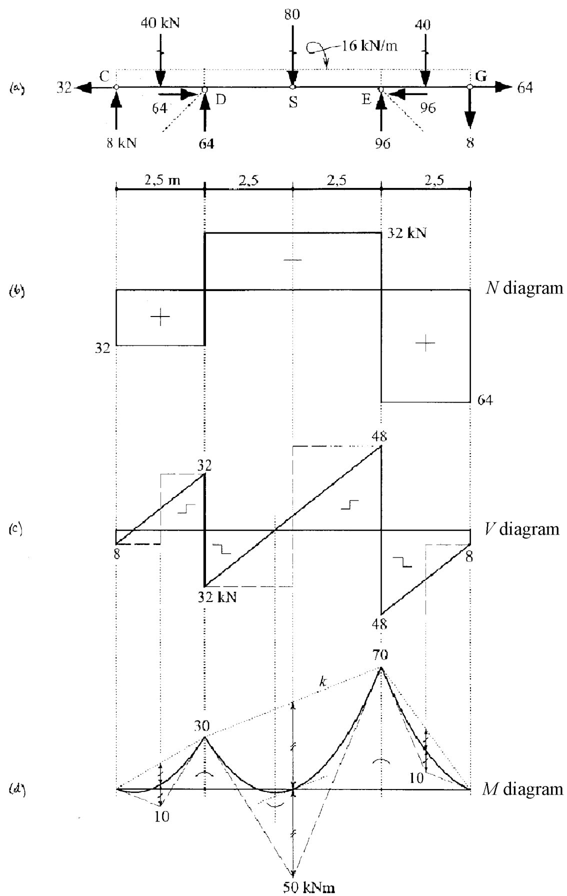


Figure 13.48 (a) Isolated girder with the associated (b) normal force diagram, (c) shear force diagram and (d) bending moment diagram.

c. In Figure 13.48a, CDSEG has been isolated, and all the forces are shown. The force and moment equilibrium of CDSEG can be used to check the correctness of the forces determined above.

d. In Figures 13.48b to d the N, V and M diagrams are shown for CDSEG. Three fields are distinguished: CD, DSE and EG.

In each field the normal force is constant. In CD and EG, the normal force is a tensile force, while in DSE it is a compressive force.

The shear force is linear in each field, and the bending moment is parabolic. The V and M diagrams due to the resultants of the field loads are shown in Figure 13.48c and d by means of dashed lines.

The V diagram has the same slope in all fields, equal to the distributed load of 16 kN/m.

The dashed M diagram due to the load resultants gives the tangents at A, D, E and G. The parabola in field DE passes through hinge S, since $M = 0$. Here, in the middle of field DE, the tangent is parallel to the chord k between the M values at D and E.

Note that the M diagram at C and G has no horizontal tangents as the shear force is not zero.

Check M diagram:
Per field, $p = \frac{1}{8} q \ell^2$ applies for the rise p of the parabolic M diagram.

13.2.2 Three-hinged frame with tie rod

Figure 13.49 shows a three-hinged frame ASB with tie rod AB. Girder CSD is carrying a uniformly distributed full load of 25 kN/m. The dimensions are shown in the figure.

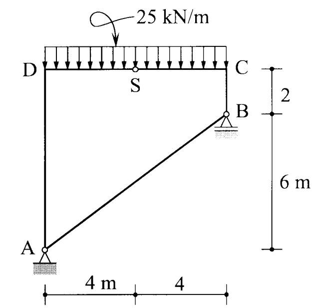


Figure 13.49 Three-hinged frame with a tie rod.

Questions:

a. Determine the support reactions at A and B and the force in tie rod AB.
b. Isolate frame ASB, and draw all the forces acting on it at A and B.
c. For ASB, draw the M, V and N diagrams, with the deformation symbols. Include relevant values. At D, S and C, draw the tangents to the M diagram.
d. Determine the maximum bending moment in field CSD.

Solution (units kN and m):

a. The support reactions at A and B are forces of 100 kN aimed upwards. The calculation is left to the reader. In Figure 13.50, frame ASB has been isolated at A and B. In addition to the support reactions of 100 kN, there are also joining forces exerted by the tie rod AB. With a tensile force N in the tie rod, the horizontal forces exerted on ASB at A and B are $\frac{4}{5}N$, and the vertical forces are $\frac{3}{5}N$, as shown in Figure 13.50. N can be found from the moment equilibrium about S of ADS or BCS. The moment equilibrium of BCS about S gives:

$$\sum T|S \circlearrowright = +100 \times 2 + \tfrac{4}{5}N \times 2 + \tfrac{3}{5}N \times 4 - 100 \times 4 = 0$$

$$\Rightarrow N = +50 \text{ kN}.$$

In bar AB, there is therefore a tensile force of 50 kN.

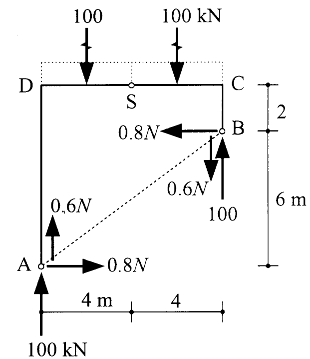


Figure 13.50 Three-hinged frame ASB with the support reactions and the forces exerted by tie rod AB at A and B.

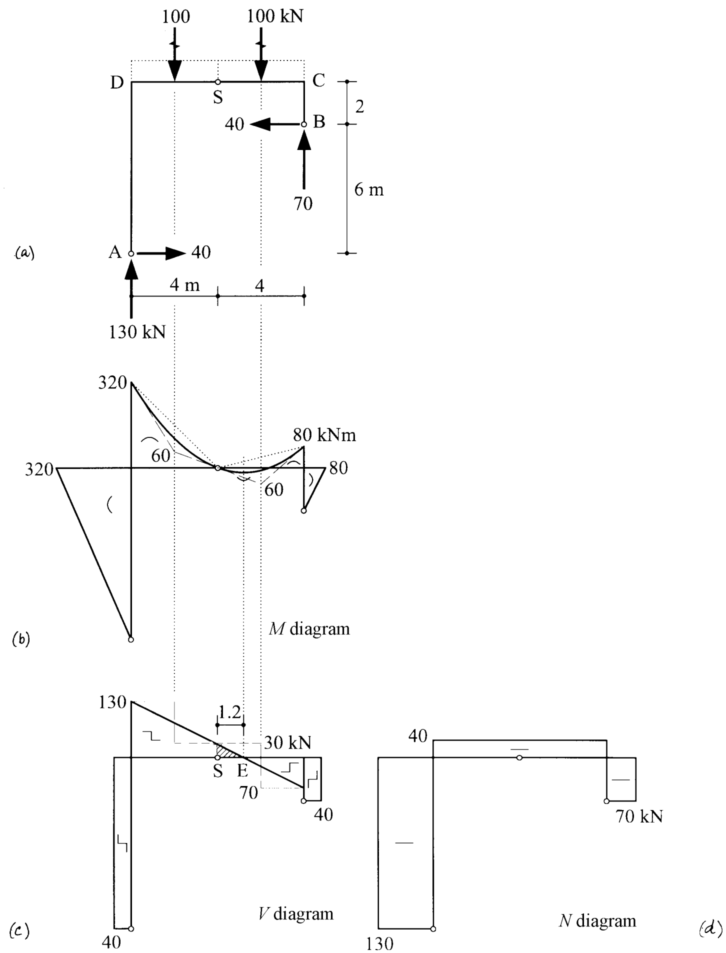


Figure 13.51 (a) Isolated three-hinged frame ASB with the resulting forces at A and B and the associated (b) bending moment diagram, (c) shear force diagram and (d) normal force diagram.

b. In Figure 13.51a, the resulting forces on the isolated frame ASB are shown, and the associated M, V and N diagrams are shown in Figures 13.51b to 13.51d.

c. To draw the M, V and N diagrams due to the forces at A and B, and the resultants of the field loads on CS and DS please refer to Section 5.3, Example 5.

The M and V diagrams due to the resultants of the field loads on CS and DS are not correct for girder CSD. They are therefore shown by means of dashed lines in Figures 13.51b and 13.51c. For the uniformly distributed load, the shear force over DSC is linear and the bending moment is parabolic. The dashed M and V diagrams give the correct values of V and M at C, S and D, and the correct tangents to the M diagram.

d. The zero shear force in CSD is found at E, $\frac{30}{30+70} \times 4 = 1.2$ m to the right of S (see Figure 13.51c). Here the bending moment is an extreme. Because the bending moment at S is zero, the bending moment at E is equal to the (hatched) area of the V diagram for SE:

$$M_{\rm E} = \tfrac{1}{2} \times 1.2 \times 30 = 18 \text{ kNm } (\smile).$$

This maximum field moment, which is significantly smaller than the boundary moments at C and D, can of course also be found from the moment equilibrium of the frame part ADSE to the left of E, or ECB to the right of E.

13.2.3 Trussed beam

The trussed beam ABSC in Figure 13.52 is carrying a uniformly distributed load of 12 kN/m. The dimensions of the structure are shown in the figure.

Questions:

a. Determine the forces in the members AD, BD and CD with the correct signs for tension and compression. Include the force polygon for joint D.
b. Isolate ABSC, and draw all the forces acting on it.
c. For ABSC, draw the M, V and N diagrams, with the deformation symbols. At A, B, S and C draw the tangents to the M diagram.
d. Determine the maximum bending moment in ABSC.

Solution (units kN and m):

a. In Figure 13.52 the support reactions are already shown. The calculation is left to the reader. In order to determine the normal forces in the two-force members AD, BD and CD, part SCD has been isolated in Figure 13.53. At D, the (normal) force N^{CD} has been resolved into its components. The force N^{CD} is found from the moment equilibrium of the isolated part about S:

$$\sum T|S \circlearrowright = +72 \times 3 - 96 \times 6 + \tfrac{2}{5}\sqrt{5} \times N^{\mathrm{CD}} \times 3 = 0$$

so that:

$$N^{\mathrm{CD}} = +60\sqrt{5}\ \mathrm{kN}\ (= +134.2\ \mathrm{kN}).$$

The normal forces in the members AD and BD can now be determined from the force equilibrium of joint D. To do so, we have created the closed force polygon in Figure 13.54a. We first set down the tensile force $F_{\mathrm{D}}^{\mathrm{CD}} = N^{\mathrm{CD}}$, which is the force member CD exerts on joint D. Next, the force polygon is closed with two lines parallel to AD and BD.

The magnitude of the forces $F_{\mathrm{D}}^{\mathrm{AD}}$ and $F_{\mathrm{D}}^{\mathrm{BD}}$ can be found from the force polygon. In order to interpret them as normal forces N, with the correct signs (positive as tensile force and negative as compressive force), we first have to check whether the forces $F_{\mathrm{D}}^{\mathrm{AD}}$ and $F_{\mathrm{D}}^{\mathrm{BD}}$ from the force polygon

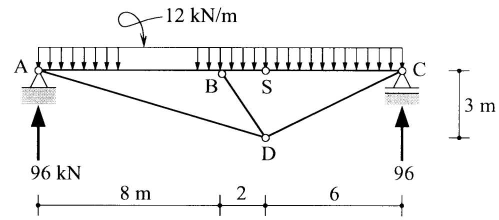


Figure 13.52 Trussed beam ABC with uniformly distributed load.

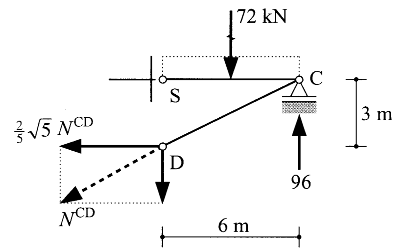


Figure 13.53 Normal force N^{CD} in two-force member CD is found from the moment equilibrium of part SCD about S.

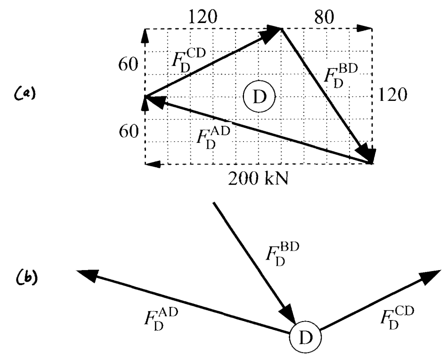


Figure 13.54 (a) Force polygon for the equilibrium of joint D. (b) Joint D with the forces exerted on it by members AD, BD and CD.

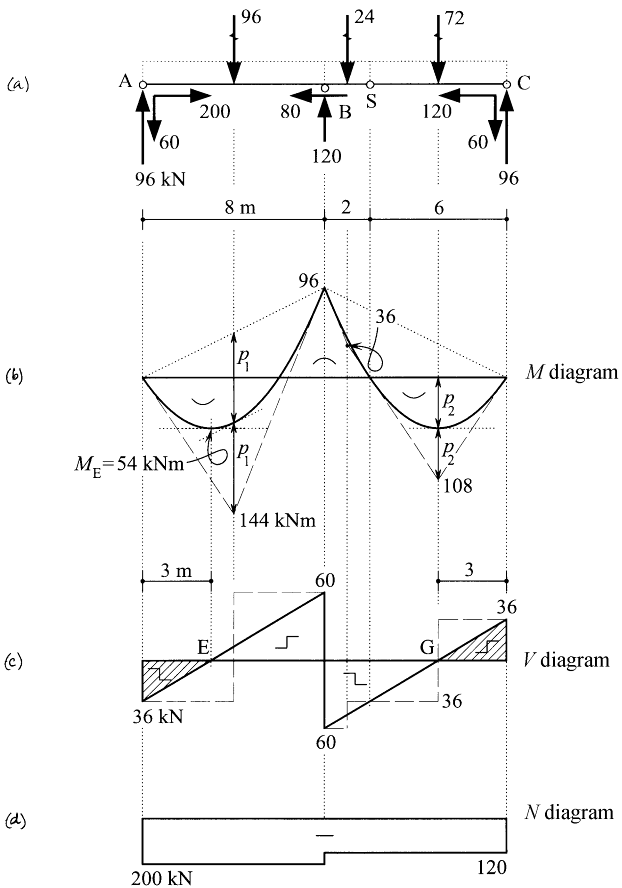


Figure 13.55 (a) Isolated beam ASC with (b) bending moment diagram, (c) shear force diagram and (d) normal force diagram.

exert tension or compression on joint D (see Figure 13.54b). We find

$$N^{AD} = +F_D^{AD} = +20\sqrt{109}\text{ kN } (= +208.8\text{ kN}),$$

$$N^{BD} = -F_D^{BD} = -40\sqrt{13}\text{ kN } (= -144.2\text{ kN}).$$

There is therefore tension in AD and CD and compression in BD.

b. In Figure 13.55a, the beam ABSC has been isolated. At A and C, in addition to the support reactions, the components of the tensile forces in AD and CD are also active. At B the components of the compressive force in BD are acting. In the figure, the distributed loads in the fields AB, BS and SC have been replaced by their resultants.

c. In Figures 13.55b to 13.55d the M, V and N diagrams are shown. The M and V diagrams due to the resultants of the field loads are shown by means of dashed lines. They give the correct values in the field boundaries. Here the dashed M diagram also gives the tangents.

The final M diagram shown in Figure 13.55b with a solid line, can be checked using the rise p of the parabolas for both fields:

$$p_1 = \tfrac{1}{8} \times 12 \times 8^2 = 96\text{ kNm},$$

$$p_2 = \tfrac{1}{8} \times 12 \times 6^2 = 54\text{ kNm}.$$

These values of p fit in the M diagram shown.

Note that in Figure 13.55c the shear forces at the supports A and C are not equal to the support reactions there. This is caused by the vertical components of the member forces in AD and CD.

Also note that the shear force in all fields has the same slope, equal to the distributed load of 12 kN/m.

There is a compressive force over the entire length of beam ABSC (see Figure 13.55d). At B, a step change in the N diagram occurs due to the horizontal component of the member force in BD.

d. The largest bending moment in an absolute sense is the *support moment* at B:

$$M_B = 96 \text{ kNm } (\frown).$$

In addition, there are extreme *field moments* at E and G three metres from the supports, where the shear force is zero (see Figure 13.55c). The easiest way to find their magnitudes is from the hatched area of the V diagram:

$$M_E = M_G = \tfrac{1}{2} \times 3 \times 36 = 54 \text{ kNm } (\smile).$$

M_G is also equal to the maximum bending moment in the simply supported beam SC with uniformly distributed full load:

$$M_G = \tfrac{1}{8} \times 12 \times 6^2 = 54 \text{ kNm } (\smile).$$

Note that the M diagram has mirror symmetry about B.

13.2.4 Sideways-supported mast

The mast ABC in Figure 13.56a is supported sideways by a number of bars. Dimensions and load are shown in the figure.

Questions:

a. Determine the support reactions at A and D.
b. Determine the forces in the bars 1 to 3, with the correct signs for tension and compression.
c. Isolate beam ABC, and draw all the forces acting on it.
d. For beam ABC draw the M, V and N diagram, with the deformation

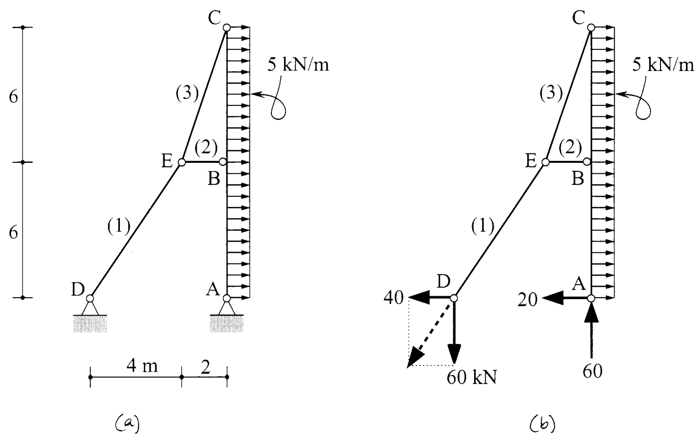


Figure 13.56 (a) Sideways-supported mast ABC with (b) its support reactions.

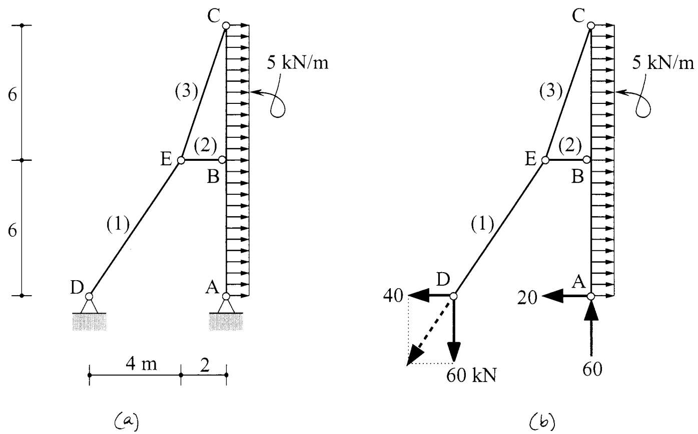


Figure 13.56 (a) Sideways-supported mast ABC with (b) its support reactions.

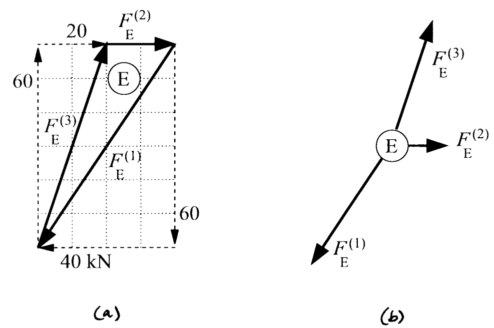


Figure 13.57 (a) Force polygon for the equilibrium of joint E. (b) Joint E with the forces exerted on it by bars (1), (2) and (3).

symbols. At A, B and C, also draw the tangents to the M diagram.

e. Determine the extreme moments in ABC.

Solution (units kN and m):

a. The vertical support reaction D_v (↓) at D follows from the moment equilibrium of the entire structure about A:

$$\sum T|A \circlearrowleft \; = -(12 \times 5) \times 6 + D_v \times 6 = 0 \Rightarrow D_v = 60 \text{ kN } (\downarrow).$$

Bar (1) is a two-force member so that the line of action of the support reaction at D coincides with DE. The horizontal component D_h is therefore

$$D_h = \tfrac{4}{6} \times D_v = 40 \text{ kN } (\leftarrow).$$

The support reactions at A follow from the force equilibrium of the structure as a whole:

$$A_h = 20 \text{ kN } (\leftarrow),$$

$$A_v = 60 \text{ kN } (\uparrow).$$

The support reactions are shown in Figure 13.56b.

b. The support reactions at D show that there is a tensile force in bar (1):

$$N^{(1)} = +\sqrt{40^2 + 60^2} = +20\sqrt{13} \text{ kN } (= +72.11 \text{ kN}).$$

The (normal) forces in bars (2) and (3) can now be determined from the force equilibrium of joint E. To do so we have to draw the force polygon for joint E (see Figure 13.57a). The force $F_E^{(1)} = N^{(1)} = 20\sqrt{13}$ kN, which bar (1) exerts on joint E, is known. We close the force polygon with the forces $F_E^{(2)}$ and $F_E^{(3)}$, parallel to the two-force members (2) and (3). Figure 13.57b shows that all the forces in the force polygon are tensile forces:

$$N^{(2)} = +F_E^{(2)} = +20 \text{ kN},$$

$$N^{(3)} = +F_E^{(3)} = +20\sqrt{10} \text{ kN } (= +63.24 \text{ kN}).$$

c. In Figure 13.58a, ABC has been isolated and all the forces acting on it have been shown. For the fields AB and BC, the resultants of the distributed load are also shown.

d. In Figure 13.58b to d the M, V and N diagrams are shown. The dashed M and V diagrams, which are determined first, are an important tool for drawing the actual M and V diagrams. The answer is left to the reader. The value p can be used to check the M diagram shown:

$$p = \tfrac{1}{8} \times 5 \times 6^2 = 22.5 \text{ kNm}.$$

Note that the M diagram has mirror symmetry about B.

e. The bending moment is an extreme at G and H, where the shear force is zero, and at B, where the shear force changes sign (see Figure 13.58c).

The M diagram in Figure 13.58b gives

$$M_B = 30 \text{ kNm } (\rangle).$$

The actual maximum bending moment occurs at G and H, 4 m from the ends A and B, and is most easily determined from the area of the V diagram:

$$M_G = M_H = M_{max} = \tfrac{1}{2} \times 4 \times 20 = 40 \text{ kNm } (\rangle).$$

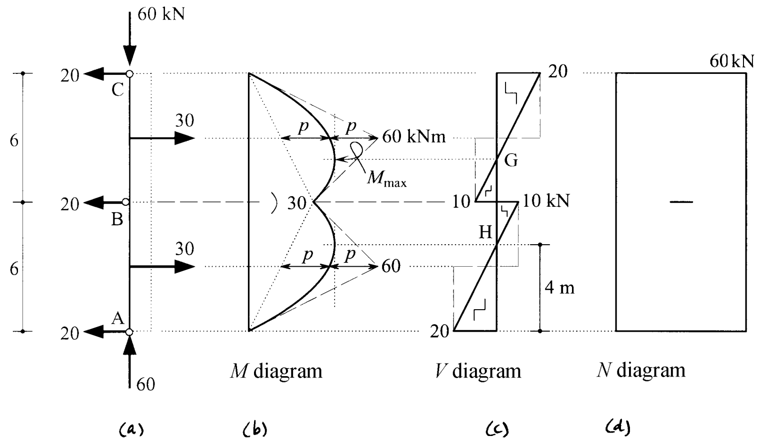


Figure 13.58 (a) Isolated mast ABC with (b) bending moment diagram, (c) shear force diagram and (d) normal force diagram.

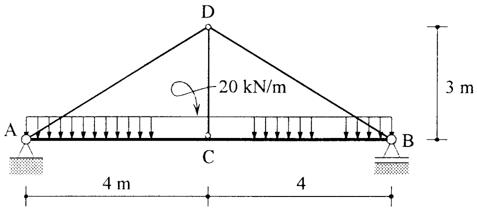


Figure 13.59 Statically indeterminate trussed beam with uniformly distributed load.

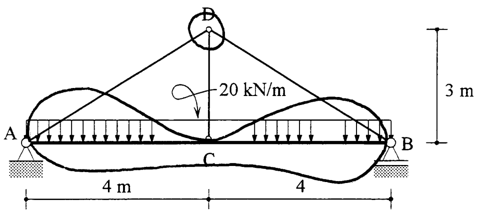


Figure 13.60 Three joining forces are acting between joint D and beam ACB, namely the normal forces in the three two-force members.

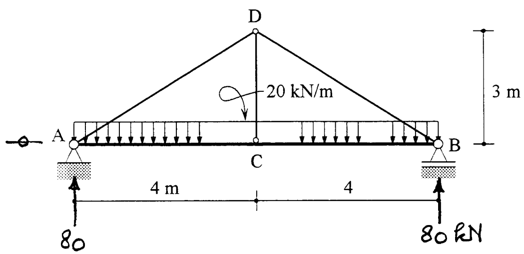


Figure 13.61 Support reactions.

13.3 Statically indeterminate structures

With statically indeterminate structures, it is not possible to determine all the support reactions and joining forces directly from the equilibrium, as there are too few equilibrium equations. In this section, sufficient support reactions and/or joining forces are given in magnitude and direction for a statically indeterminate structure so that all the other support reactions and joining forces can be determined with the available number of equilibrium equations. Thereafter, it is possible to determine and draw the M, V and N diagrams for the entire structure.

13.3.1 Trussed beam with a given normal force

The dimensions and load for the trussed beam ACB can be found in Figure 13.59. For the given load, there is a tensile force of 60 kN in member CD.

Questions:

a. Determine the degree of static indeterminacy of the structure.
b. Isolate beam ACB, and draw all the forces acting on it.
c. For ACB, draw the N, V and M diagrams, with the deformation symbols. At A, B and C also draw the tangents to the M diagram.

Solution:

a. In Figure 13.60, beam ACB and joint D have been isolated from one another. There are $v = 3$ unknown joining forces acting between beam ACB and joint D: the normal forces N^{AD}, N^{BD} and N^{CD}. In addition, there are $r = 3$ support reactions, namely A_{h}, A_{v} and B_{v}. That makes a total of $r + v = 6$ unknowns. Beam ACB provides three equilibrium equations (force equilibrium and moment equilibrium); joint D provides two (force equilibrium). In total, there are therefore $e = 5$ equilibrium equations available. The degree of static indeterminacy n is equal to the difference between the number of unknowns and the number of available equilibrium equations:

$$n = (r + v) - e = 6 - 5 = 1.$$

The structure is therefore statically indeterminate to the first degree.

b. In Figure 13.61 the support reactions are shown; they follow from the equilibrium of the structure as a whole.

If it is known that a tensile force of 60 kN is acting in CD, the normal forces in two-force members AD and BD can be determined from the equilibrium of joint D. In Figure 13.62a the closed force polygon is shown for the equilibrium of the forces acting on joint D. Figure 13.62b shows how these forces are acting on the joint. This figure also shows whether the forces are tensile or compressive. The normal forces are

$$N^{\mathrm{AD}} = -F_{\mathrm{D}}^{\mathrm{AD}} = -50 \text{ kN},$$
$$N^{\mathrm{BD}} = -F_{\mathrm{D}}^{\mathrm{BD}} = -50 \text{ kN},$$
$$N^{\mathrm{CD}} = +F_{\mathrm{D}}^{\mathrm{CD}} = +60 \text{ kN (given)}.$$

Figure 13.63a shows the isolated beam ACB. At A and B there are not only support reactions, but also (the components of) the compressive forces exerted by the members AD and BD.

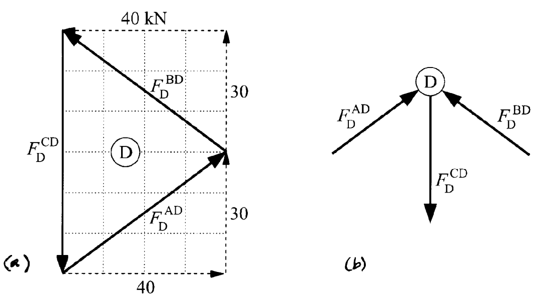


Figure 13.62 (a) Force polygon for the equilibrium of joint D. (b) Joint D with the forces exerted on it by members AD, BD and CD.

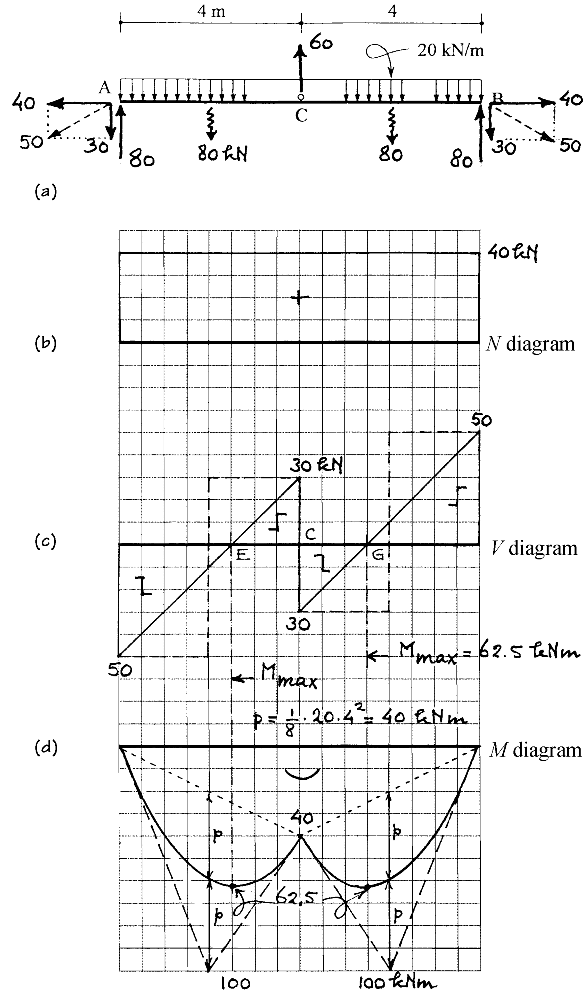


Figure 13.63 (a) Isolated beam ACB with (b) normal force diagram, (c) shear force diagram and (d) bending moment diagram.

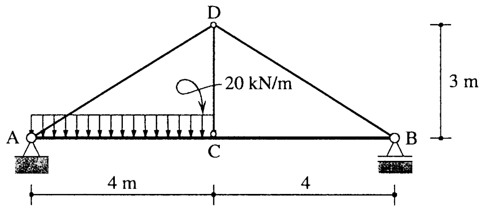


Figure 13.64 Statically indeterminate trussed beam with a uniformly distributed load on the left-hand side. We are given a zero bending moment at C.

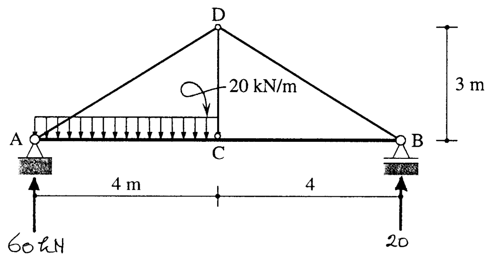


Figure 13.65 Support reactions.

c. The N, V and M diagrams are shown in Figures 13.63b to 13.63d. In beam ACD there is a tensile force of 40 kN. When drawing the V and M diagrams, we used the dashed V and M diagram associated with the load resultants of 80 kN in the fields AC and BC. The bending moment is an extreme at E and G, where the shear force is zero, and at C where the shear force changes sign:

$$M_{\text{min}} = M_{\text{C}} = 40 \text{ kNm } (\smile),$$

$$M_{\text{max}} = M_{\text{E}} = M_{\text{G}} = \tfrac{1}{2}(2.5 \text{ m})(50 \text{ kN}) = 62.5 \text{ kNm } (\smile).$$

13.3.2 Trussed beam with a given bending moment

The trussed beam ACB in Figure 13.64 is the same as that in the previous section, except it now has a different load. Further, we are given a zero bending moment at C.

Question:
For ACB draw the N, V and M diagrams, with the deformation symbols. At A, B and C also draw the tangents to the M diagram.

Solution (units kN and m):
The support reactions follow from the equilibrium of the structure as a whole, and are shown in Figure 13.65.

In the unloaded field BC, the bending moment (dependent on the shear force) is constant or linear. Since the bending moment is zero at both B and C (given), the bending moment must be zero throughout field BC.

Due to the uniformly distributed load, the bending moment in field AC is parabolic. In addition, the bending moment is zero at both A and C. This allows us to directly draw the M diagram for AB (see Figure 13.66d). At the middle of AC:

$$M = M_{\max} = p = \tfrac{1}{8} \times 20 \times 4^2 = 40 \text{ kNm}.$$

The V diagram can be determined from the M diagram. In field BC, the shear force is zero, in field AC it varies linearly. The shear forces at A and to the left of C are

$$\frac{2p}{\frac{1}{2}\ell^{AC}} = \frac{2 \times 40}{\frac{1}{2} \times 4} = 40 \text{ kN}.$$

Their deformation symbols follow from the slope of the M diagram (see Figure 13.66c).

Check: The shear forces found must agree with the support reactions of the simply supported beam AC.

The vertical force equilibrium of joint C gives (see Figure 13.67)

$$N^{CD} = +40 \text{ kN}.$$

Using the equilibrium of joint D, we can now find the normal forces in AD and BD. They turn out to be compressive forces:

$$N^{AD} = N^{BD} = -100/3 \text{ kN}.$$

The calculation is left to the reader.

In Figure 13.66a, beam ACB has been isolated, and all forces acting on it are shown. At A and B, there are acting support reactions and (components of the) compressive forces exerted by members AD and BD.

Check: By reducing the support reactions (pointed upwards) at A and B by the vertical component (pointed downwards) of these member forces we

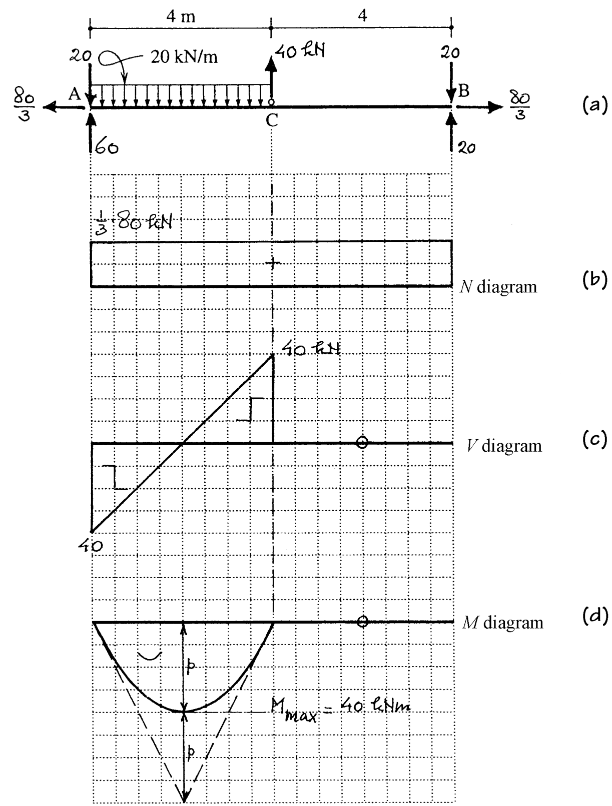


Figure 13.66 (a) Isolated beam ACB with (b) normal force diagram, (c) shear force diagram and (d) bending moment diagram.

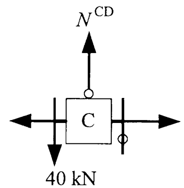


Figure 13.67 The normal force N^{CD} in member CD follows from the vertical force equilibrium of joint C.

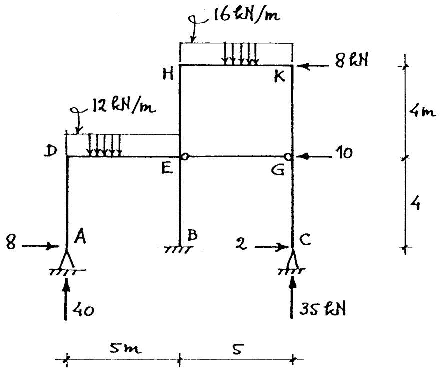


Figure 13.68 Statically indeterminate portal structure with the support reactions at A and C.

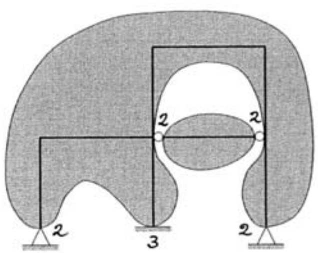


Figure 13.69 The portal structure consists of two singly-cohesive sub-structures.

find the same shear forces as from the V diagram.

The horizontal component of the compressive member forces at A and B results in a tensile force of 80/3 kN in ACB, see the N diagram in Figure 13.66b.

13.3.3 Portal structure with a number of given support reactions

With the load given, the support reactions at A and C for the structure are given in Figure 13.68.

Questions:

a. Determine the degree of static indeterminacy of the structure.
b. Determine the support reactions at B.
c. Isolate ADEBH, and draw all the forces acting on it, with the additional information of a compressive force of 6 kN acting in EG.
d. For ADE and BH, draw the M, V and N diagrams with the deformation symbols. At D and E, draw the tangents to the M diagram.

Solution (units kN and m):
a. The structure consists of two singly-cohesive sub-structures (see Figure 13.69). There are $v = 2 + 2 = 4$ unknown joining forces acting in the hinged joints between both sub-structures. In addition, there are $r = 2 + 3 + 2 = 7$ unknown support reactions. In total, that makes $r + v = 11$ unknowns. Each sub-structure offers three equilibrium equations, making a total of $e = 2 \times 3 = 6$ equilibrium equations available. The degree of static indeterminacy n is equal to the number of unknown joining forces and support reactions minus the number of available equilibrium equations:

$$n = (r + v) - e = 11 - 6 = 5.$$

The structure is therefore statically indeterminate to the fifth degree.

b. The support reactions at B follow from the equilibrium of the structure as a whole. For the assumed directions of $B_{\rm h}$, $B_{\rm v}$ and $B_{\rm m}$ in Figure 13.70a, we find

$$\sum F_x = 8 + B_{\rm h} + 2 - 10 - 8 = 0 \Rightarrow B_{\rm h} = 8\ \text{kN},$$

$$\sum F_y = 40 + B_{\rm v} + 35 - 80 - 60 = 0 \Rightarrow B_{\rm v} = 65\ \text{kN},$$

$$\sum T_z|\text{B} = -40 \times 5 + B_{\rm m} + 35 \times 5 + 10 \times 4$$

$$+ 8 \times 8 - 80 \times 2.5 + 60 \times 2.5 = 0 \Rightarrow B_{\rm m} = -29\ \text{kN}.$$

The fixed-end moment reaction $B_{\rm m}$ is acting opposite to the direction assumed in Figure 13.70a. In Figure 13.70b, the support reactions at B are shown as they act in reality.

c. In Figure 13.71a, ADEBH has been isolated and all forces acting on it are shown. Additional information given is that member EG exerts a horizontal compressive force of 6 kN on joint E. The three joining forces at H can now be determined from the equilibrium of ADEBH, or (less laboriously) from the equilibrium of CGKH. The calculation is left to the reader.

d. In Figures 13.71b to 13.71d, the N, M and V diagrams for ADEBH are shown. When determining and drawing these lines, it is best to work from the member ends A, B and H to joint E. To verify the calculation, the force and moment equilibrium of joint E can be investigated. One could also first isolate all the members and determine all the joining forces at D and E. See for example Section 5.3, Example 3.

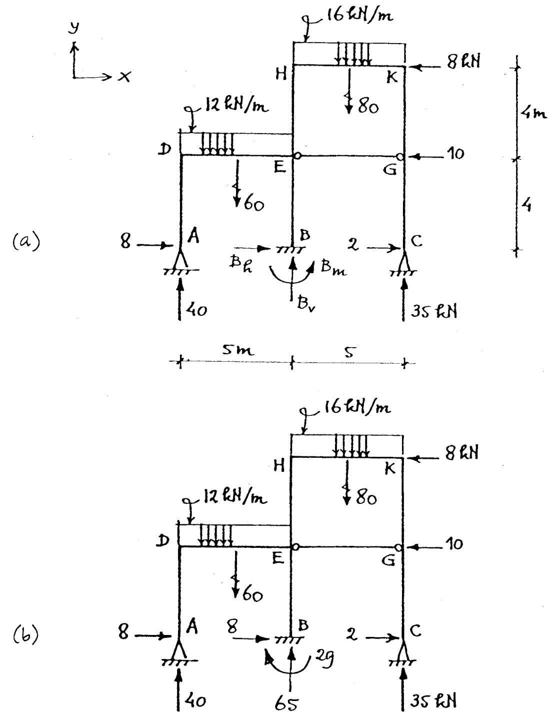


Figure 13.70 (a) Assumed and (b) calculated support reactions at B.

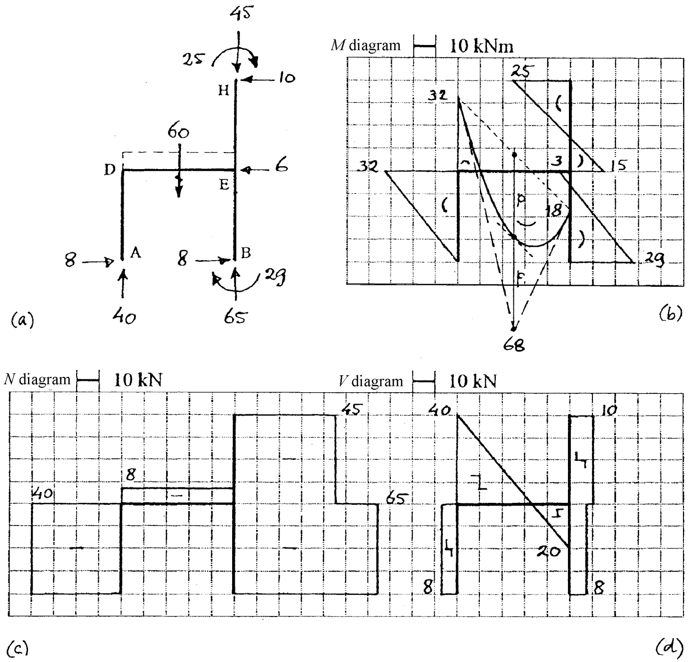


Figure 13.71 (a) Isolated part ADEBH with (b) bending moment diagram, (c) normal force diagram and (d) shear force diagram.

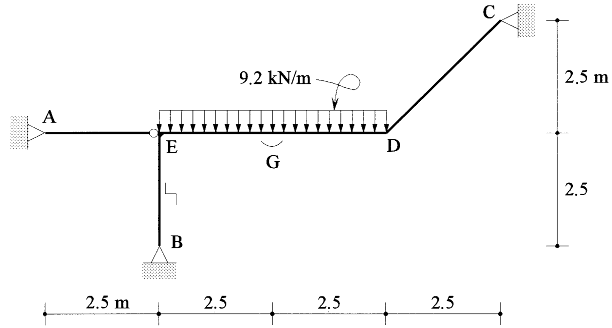


Figure 13.72 Statically indeterminate frame for which the bending moment at G, and the shear force in BE are given.

13.3.4 Frame with given shear force and bending moment

For the statically indeterminate structure in Figure 13.72 we are given the following:

- the bending moment at the middle G of DE: $M_G = 12.5$ kNm, and
- the shear force in BE: $V^{BE} = 5$ kN.

The associated deformation symbols are given in the figure, as are the measurements and the load.

Questions:

a. Determine the degree of static indeterminacy of the structure.
b. Determine the support reactions. Draw them as they are acting in reality on the structure.
c. Determine the normal force in CD.
d. Determine the N, V, and M diagrams for the entire structure, with the deformation symbols. At D, G and E draw the tangents to the M diagram.

Solution (units kN and m):

a. The structure BEGD is statically indeterminate to the second degree. The structure has five unknown support reactions while there are only three equilibrium equations.

b. The horizontal support reaction at B follows from the shear force in BE.

$$B_h = 5 \text{ kN } (\rightarrow).$$

Bar AE is a two-force member so that

$$A_v = 0.$$

Introduce a cut at G, and investigate the moment equilibrium of ABEG about G (see Figure 13.73):

$$\sum T|G \circlearrowright = +12.5 + 23 \times 1,25 + 5 \times 2.5 - B_v \times 2.5 = 0$$

$$\Rightarrow B_v = 21.5 \text{ kN } (\uparrow).$$

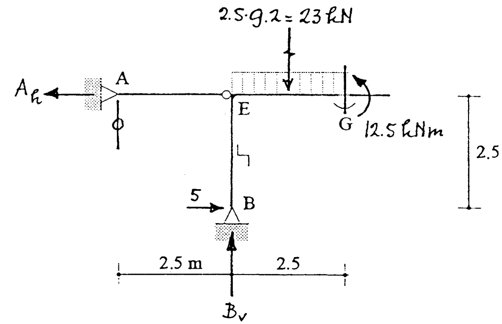


Figure 13.73 The vertical support reaction at B follows from the moment equilibrium of ABEG about G.

The support reactions at A and C are found from the equilibrium of the structure as a whole:

$$\sum F_{\text{vert}} = 0 \Rightarrow C_v = 24.5 \text{ kN } (\uparrow),$$

$$\sum T|A = 0 \Rightarrow C_h = 32.5 \text{ kN } (\rightarrow),$$

$$\sum F_{\text{hor}} = 0 \Rightarrow A_h = 37.5 \text{ kN } (\leftarrow).$$

In Figure 13.74, the support reactions are shown as they act in reality.

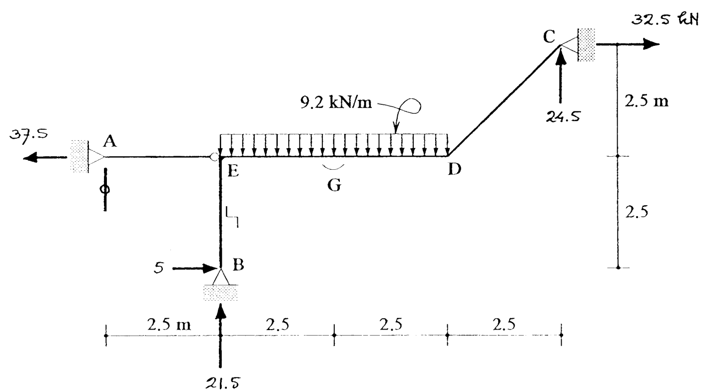


Figure 13.74 Support reactions.

c. By resolving the horizontal and vertical support reaction at C into components parallel to and normal to CD we find the normal force N^{CD} and the

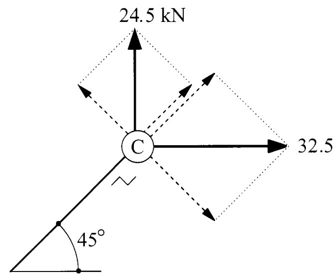


Figure 13.75 To determine the normal force and shear force in CD, the support reactions at C can be resolved into components normal to and parallel to member CD.

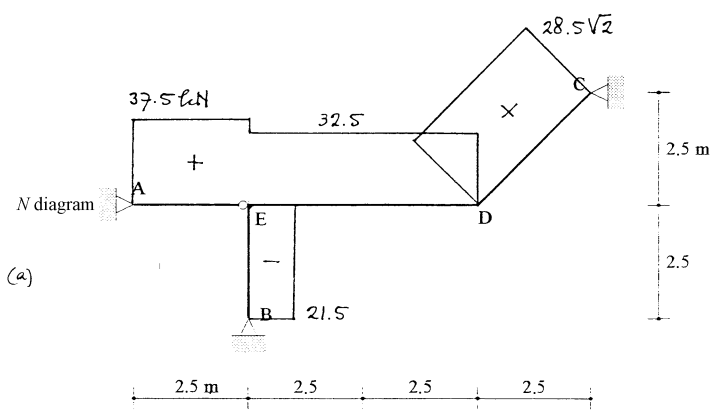


Figure 13.76 (a) Normal force diagram.

shear force V^{CD} (see Figure 13.75):

$$N^{\text{CD}} = \tfrac{1}{2}\sqrt{2} \times (32.5 + 24.5) = 28.5\sqrt{2}\,\text{kN},$$

$$V^{\text{CD}} = \tfrac{1}{2}\sqrt{2} \times (32.5 - 24.5) = 4\sqrt{2}\,\text{kN}.$$

The normal force is a tensile force; the deformation symbol for the shear force is given in Figure 13.75.

d. In Figures 13.76a to 13.76c, the N, V and M diagrams are shown. We provide a number of comments about the M and V diagrams below. At E and D, the bending moment "goes round the corner". At G, the tangent to the M diagram is parallel to the chord k of the parabola. The tangents at E and D are formed by the dashed M diagram due to the resultant of the distributed load on DE. The slope of this dashed M diagram gives the magnitude and the deformation symbol for the shear forces at E and D. The shear force in DE varies linearly between the values at D and E. The slope of the V diagram can be used as a check: it is equal to the distributed load. The maximum bending moment in DE is slightly to the left of the middle G of ED, and will be only marginally larger than M_{G}. From the area under the V diagram we find

$$M_{\max} = \tfrac{1}{2} \times 21.5 \times \tfrac{21.5}{21.5+24.5} \times 5 - 12.5 = 12.62\,\text{kNm}\ (\smile).$$

13.3.5 Frame with two given shear forces

The statically indeterminate structure in Figure 13.77 has a hinged joint at S. All other joints are rigid. Dimensions and loads are given in the figure. The shear forces directly next to joint C are given:

$$V_{\text{C}}^{\text{BC}} = 2.5\sqrt{2}\,\text{kN},$$

$$V_{\text{C}}^{\text{CS}} = 10\,\text{kN}.$$

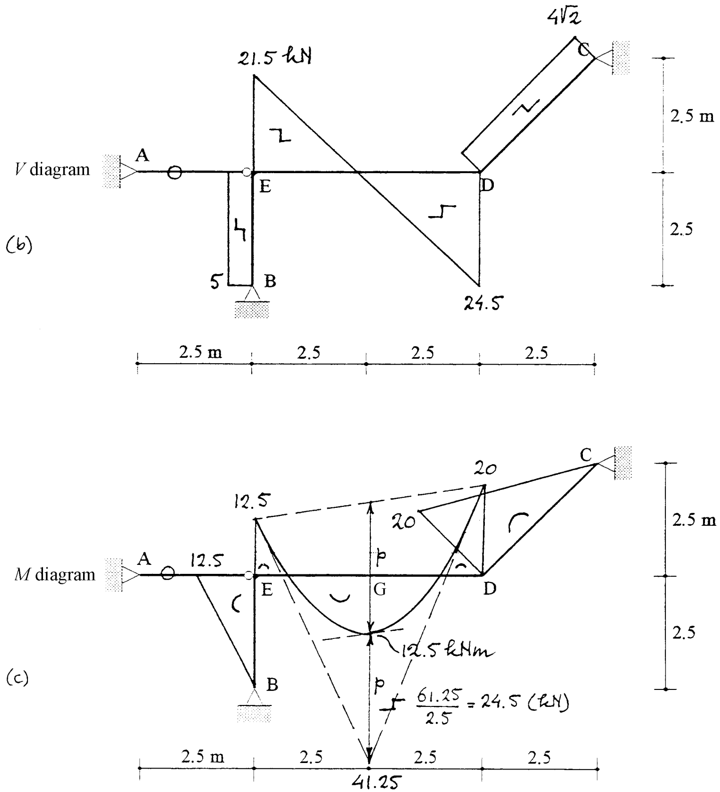


Figure 13.76 (b) Shear force diagram and (c) bending moment diagram.

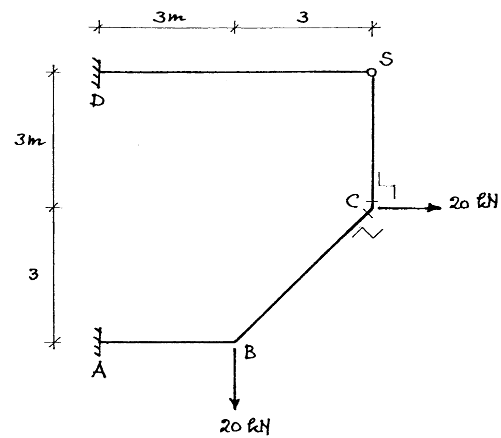


Figure 13.77 Statically indeterminate frame with two given shear forces directly next to joint C.

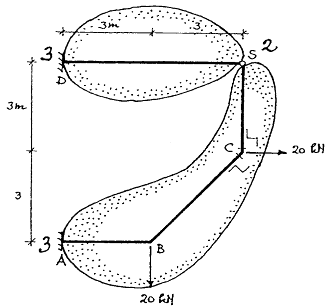


Figure 13.78 The frame consists of the two singly-cohesive sub-structures ABCS and DS.

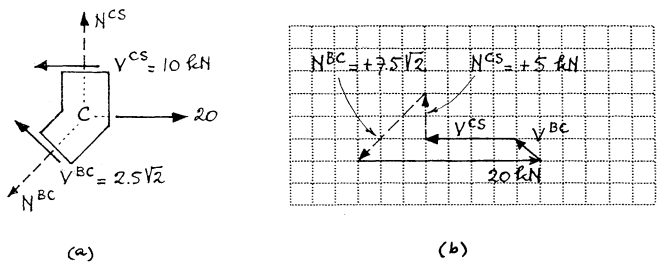


Figure 13.79 (a) The forces on joint C and (b) the closed force polygon for the force equilibrium (scale: 1 square = 2.5 kN).

The directions follow from the deformation symbols given in the figure.

Questions:

a. Determine the degree of static indeterminacy of the structure.
b. Draw the force polygon for the force equilibrium of joint C.
c. Determine the support reactions at A and D.
d. Determine the M, V and N diagrams for the entire structure, with the deformation symbols.

Solution (units kN and m):
a. The two parts ABCS and DS provide $e = 2 \times 3 = 6$ equilibrium equations (see Figure 13.78). The number of unknown support reactions at A and D is $r = 3 + 3 = 6$. The number of unknown joining forces at S is $v = 2$. The degree of static indeterminacy is:

$$n = r + v - e = 6 + 2 - 6 = 2.$$

The structure is therefore statically indeterminate to the second degree.

b. In Figure 13.79a, joint C has been isolated and all forces acting on it are shown. The bending moments acting on the joint are not shown! Figure 13.79b shows the closed force polygon for the force equilibrium of the joint (scale: 1 square = 2.5 kN). The force polygon gives

$$N^{\mathrm{BC}} = +7.5\sqrt{2}\ \mathrm{kN},$$
$$N^{\mathrm{CS}} = +5\ \mathrm{kN}.$$

c. With $N^{\mathrm{CS}} = +5$ kN the vertical equilibrium of CSD gives

$$D_{\mathrm{v}} = 5\ \mathrm{kN}\ (\uparrow).$$

With $V^{\mathrm{CS}} = 10$ kN the horizontal equilibrium of ABC gives

$$A_{\mathrm{h}} = 10 \text{ kN } (\leftarrow).$$

From the force equilibrium of the structure as a whole follows

$$D_{\mathrm{h}} = 10 \text{ kN } (\leftarrow),$$

$$A_{\mathrm{v}} = 15 \text{ kN } (\uparrow).$$

Finally, the fixed-end moment reactions at A and D follow from the moment equilibrium about S of DS and ABCS respectively. In Figure 13.80, the support reactions are shown as they act in reality.

d. In Figures 13.81a to 13.81c the M, V and N diagrams are shown. At B and C, the bending moment "*goes round the corner*". The slopes of the M diagram are in line with the magnitudes and the deformations symbols of the shear forces.

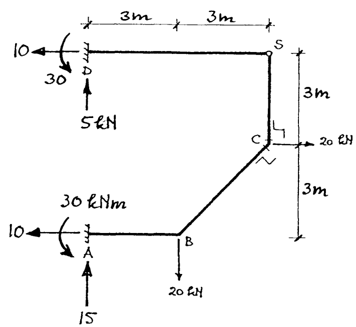


Figure 13.80 Support reactions.

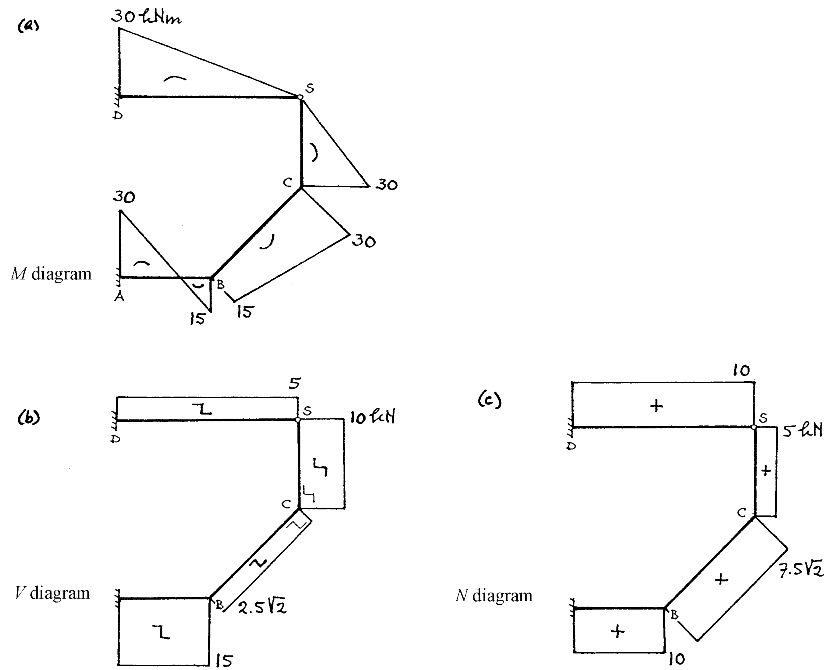


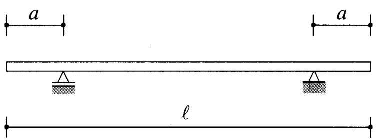
Figure 13.81 (a) Bending moment diagram, (b) shear force diagram and (c) normal force diagram.

13.4 Problems

General comment: When asked to draw an M, V or N diagram, please draw the diagrams including the (deformation) symbols (or plus and minus signs) and the values at relevant points.

Self-contained structures (Section 13.1)

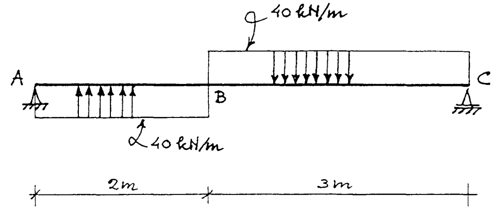
13.1 A beam with length $\ell = 16.90$ m is supported as shown. The dead weight of the beam is uniformly distributed and is 4 kN/m.



Questions:
a. How do you choose distance a to minimise the bending moment in the beam due to the dead weight (in an absolute sense)?
b. How large is this bending moment?
c. Draw the M and V diagrams.

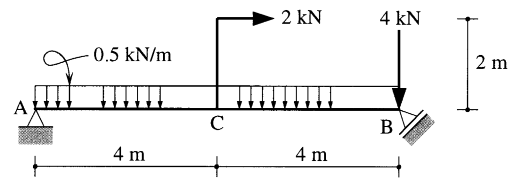
13.2 Beam ABC is simply supported at A and B. A uniformly distributed load of 40 kN/m acts upwards over AB, and downwards over BC.

Questions:
a. Determine the support reactions and draw them as they act on the beam.
b. For ABC, draw the bending moment diagram with the tangents at A, B and C. Clearly show where these tangents intersect.
c. For ABC, draw the shear force diagram.



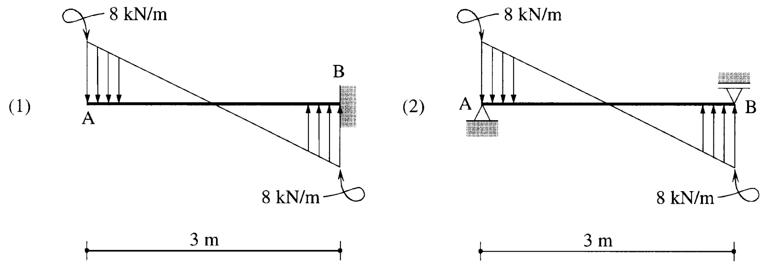
d. Determine the maximum and minimum bending moment in the beam and indicate where these moments occur.

13.3 Beam ACB is supported by a hinge at A, and on a roller at B. The roller track at B is on a slope of 45°. A uniformly distributed load of 0.5 kN/m acts over the entire length ACB. At B, the beam is loaded by a vertical force of 4 kN. At C there is an eccentric axial force of 2 kN.



Questions:
a. Determine and draw the support reactions at A and B.
b. For ACB draw the N diagram.
c. For ACB draw the V diagram.
d. For ACB draw the M diagram. At A, C and B, draw the tangents to the M diagram, and clearly indicate where they intersect.

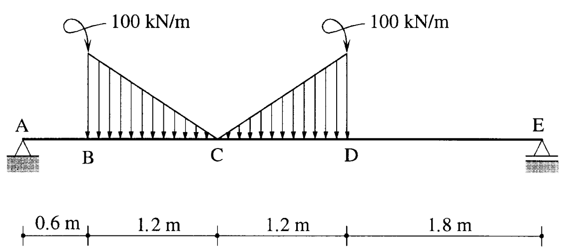
13.4: 1–2 Beam AB is supported in two different ways and carries a linearly distributed load.



Questions:

a. Determine and draw the support reactions.
b. Make a clear sketch of the *V* and *M* diagrams. At A, B and the middle of AB, also draw the tangents to the *M* diagram.
c. Where are *V* and *M* an extreme and how large are these extreme values?

13.5 The simply supported beam AE is loaded in the fields BC and CD by two equally large triangular loads. The top value of the distributed load is 100 kN/m.



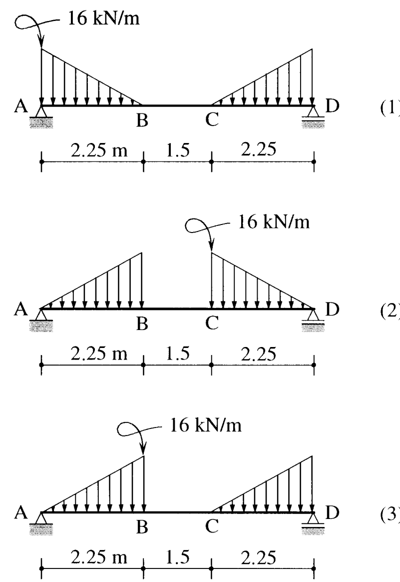
Questions:

a. Draw the *M* and *V* diagrams. At B, C and D draw the tangents to the *M* and *V* diagrams, and clearly show where they intersect.
b. Where is the bending moment an extreme? Using the bending moment diagram drawn, estimate the value of this moment.
c. Make an accurate calculation of the maximum bending moment.

13.6: 1–3 The simply supported beam AD is loaded in three different ways by triangular loads with a top value of 16 kN/m.

Questions:

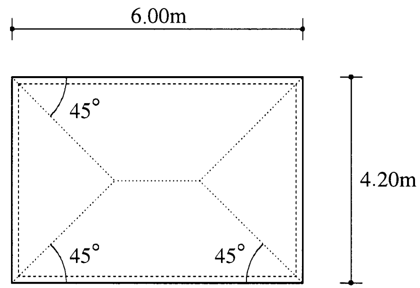
a. Draw the *M* and *V* diagrams. At A to D, also draw the tangents to the *M* and *V* diagrams, and clearly show where they intersect.
b. Where is the bending moment an extreme? Determine this moment.



13.7 A rectangular slab rests on four edge beams, of which it is assumed that they are simply supported on columns at the corners of the slab. A uniform full load on the slab of 4 kN/m^2 is transferred in accordance with the envelope pattern shown to the edge beams.

Questions:

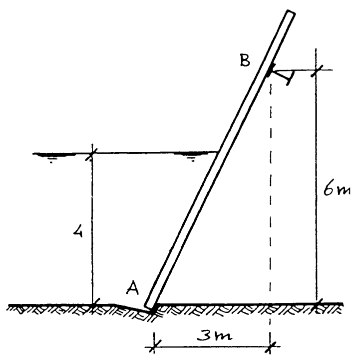
a. For each of the edge beams, draw the loading diagram.
b. Drawn the M and V diagrams for the short edge beam. How large is the maximum bending moment?
c. Draw the M and V diagrams for the long edge beam. How large is the maximum bending moment?



13.8 A barrage is composed of 1.5-metre-wide bulkheads that on the underside rest in a groove at A and on top rest against an I-beam. The I-beam is supported by the barrage walls.

Questions:

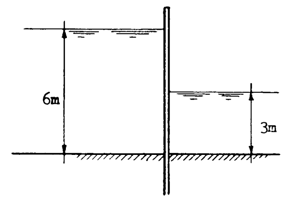
a. Draw the distribution of the water pressure on the bulkheads.
b. Draw a model of a bulkhead with a width of 1.5 m as a line element, and determine the support reactions at A and B.
c. Draw the M and V diagrams for bulkhead AB.
d. How large is the maximum bending moment, and where does it occur?



13.9 A steel sheet-pile wall is fixed in a concrete floor with 6 metres of water on one side, and 3 metres on the other side. The mass density of water is 1000 kg/m^3.

Questions:

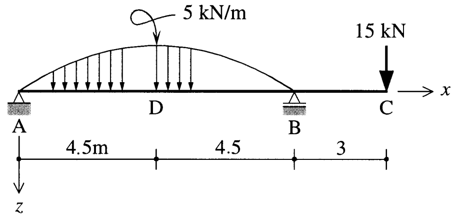
a. Schematize a 1-metre wide vertical strip from the sheet-pile wall as a line element, and draw the load diagram.
b. Determine the support reactions for the strip.
c. Draw the M and V diagrams for the strip. In a number of places, draw the tangents to the M and V diagrams.



13.10 Beam ABC is supported by a hinge at A and on a roller at B. In field AB the beam carries a parabolically distributed load, and at the end C of the overhang BC a point load of 15 kN. In the given coordinate system, the parabolically distributed load is represented by

$$q(x) = -20\left(\frac{x}{\ell}\right)^2 + 20\left(\frac{x}{\ell}\right) \text{ kN/m.}$$

Here $\ell = 9$ m is the length of AB. The dead weight of the beam is not considered here.



Questions:

a. Substitute the distributed load over AB by its resultant, and for the entire beam ABC draw the M and V diagrams.
b. Now give a (rough) sketch of the actual M and V diagrams for AB, with the deformation symbols, and also the plus and minus signs in the given xz coordinate system.
c. For AB, through successive integration, determine the shear force V and the bending moment M as a function of x. Determine the values of V and M at A and B and in the middle D of field AB. At D draw the tangent to the M diagram.
d. Where in AB is the field moment an extreme? It is enough to indicate the location of this maximum roughly. On the basis of the M diagram, estimate the value of the maximum field moment. This value need not be calculated accurately.

13.11: 1–3 A simply supported beam AB with length ℓ is loaded for bending by three different distributed loads with the same top value $\hat{q}$:

(1) $q(x) = \hat{q} \cdot \left(\frac{x^2}{\ell^2} - 2\frac{x}{\ell} + 1\right)$,

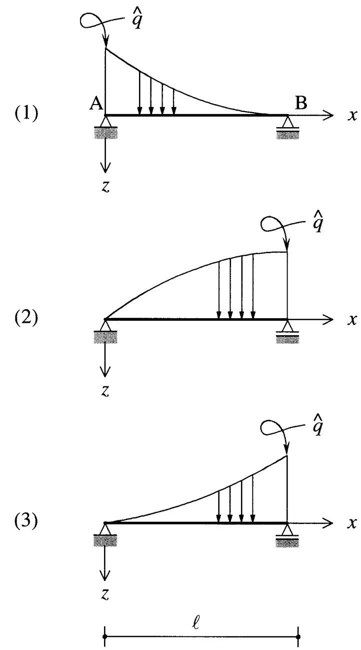
(2) $q(x) = \hat{q} \cdot \left(-\frac{x^2}{\ell^2} + 2\frac{x}{\ell}\right)$,

(3) $q(x) = \frac{1}{2}\hat{q} \cdot \left(\frac{x^2}{\ell^2} + \frac{x}{\ell}\right)$.

For the numerical calculation, assume $\ell = 4$ m and $\hat{q} = 48$ kN/m.

Questions:

a. Determine M and V as a function of x.
b. Draw the M and V diagrams with the *deformation symbols*.
c. Determine the location and magnitude of the maximum bending moment.
d. Determine the support reactions at A and B and draw them as they actually act on the beam.

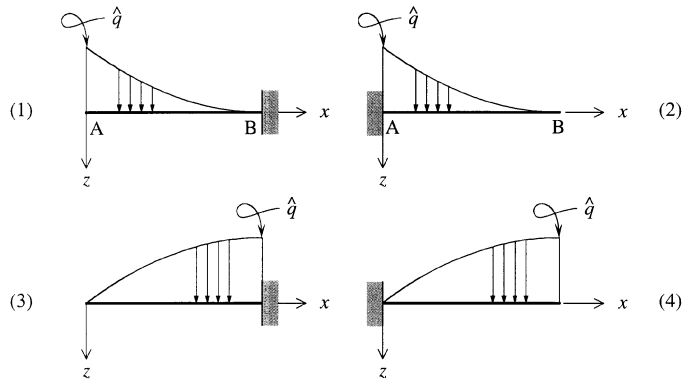


13.12: 1–4 Two cantilever beams AB with length ℓ are subject to bending by two different distributed loads with the same top value $\hat{q}$:

$$\text{(1) and (2)} \quad q(x) = \hat{q} \cdot \left(\frac{x^2}{\ell^2} - 2\frac{x}{\ell} + 1 \right),$$

$$\text{(3) and (4)} \quad q(x) = \hat{q} \cdot \left(-\frac{x^2}{\ell^2} + 2\frac{x}{\ell} \right).$$

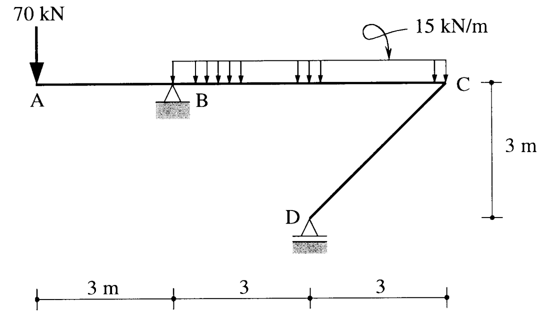
For the numerical calculation, assume $\ell = 4$ m and $\hat{q} = 48$ kN/m.



Questions:

a. Determine M and V as a function of x.
b. Draw the M and V diagrams with the *deformation symbols*.
c. Determine the support reactions as they act on the beam.

13.13 The bent beam ABCD is supported by a hinge at A and on a roller at D. The structure is loaded by a uniformly distributed load in field BC and a point load at A.



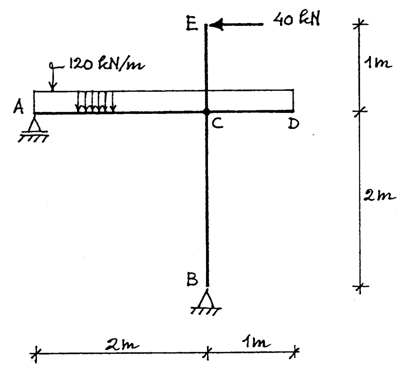
Questions:

a. Determine the support reactions. Draw them in the directions in which they act.
b. For the entire construction, draw the M, V and N diagrams with the deformation symbols. At B and C, draw the tangents to the M diagram.
c. Indicate in which cross-section of BC the field moment is an extreme. Determine this extreme value.

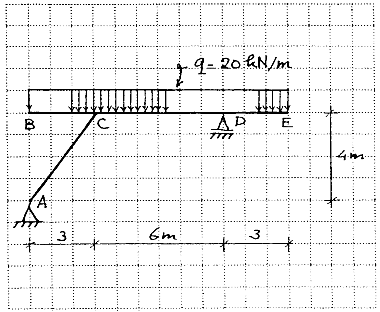
13.14 The structure consists of the members ACD and BCE that are rigidly joined to one another at C.

Questions:

a. Determine the support reactions and draw them as they act in reality.
b. Isolate ACD, and draw all the forces acting on it.
c. For ACD, draw the M and V diagrams. At A, C and D draw the tangents to the M diagram.
d. Determine the maximum bending moment in field AC. In which cross-section does this occur?
e. Draw the M and V diagrams for BCE.



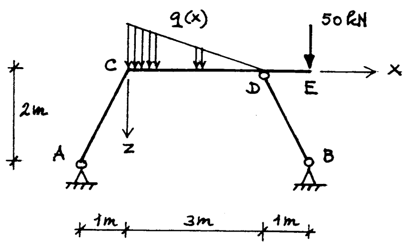
13.15 The structure in Figure 13.21 consists of the members AC and BCDE that are rigidly joined to one another at C.



Questions:

a. Determine the support reactions at A and D.
b. For the entire structure draw the M and V diagrams. At B to E, draw the tangents to the M diagram, and clearly indicate where they intersect.
c. Where in field CD is the bending moment an extreme? Determine this moment.
d. Draw the N diagram for the entire structure.

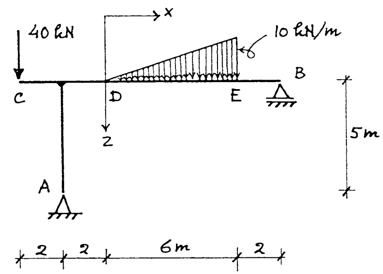
13.16 The structure is subject to a force of 50 kN at E, and a linearly distributed load $q(x)$ in field CD. The following applies in the given xz coordinate system: $q(x) = (-10x + 30)$ kN/m with x expressed in metres.



Questions:

a. Determine and draw the support reactions at A and B.
b. Isolate member CDE, and draw all the forces acting on it.
c. Write down the shear force in CD as a function of x. Verify the function values at C and D.
d. Write down the bending moment in CD as a function of x. Check the function values at C and D.
e. For CDE, draw the V and M diagrams with the deformation symbols. At C and D, draw the tangents to the V and M diagrams.
f. Where in field CD is the bending moment an extreme, and how large is this moment?

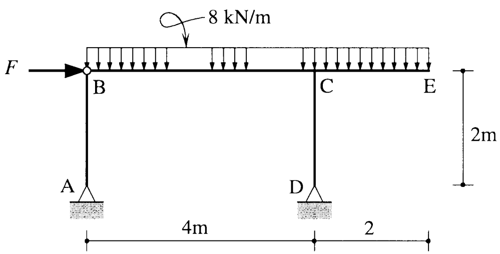
13.17 The structure is supported by a hinge at A and on a roller at B. At C, the overhang is subject to a vertical force of 40 kN. A triangular load acts between D and E, with a top value of 10 kN/m at E.



Questions:

a. Determine and draw the support reactions.
b. Draw the M, V and N diagrams for the entire structure with the deformation symbols. At D and E draw the tangents to the M and V diagrams.
c. Determine M and V in field DE as a function of x. Use the given xz coordinate system. Check the values (including the signs) of M and V at both D and E.
d. Determine the location and magnitude of the maximum bending moment in field DE.

13.18 The structure is subject to a uniformly distributed vertical load of 8 kN/m and a horizontal force $F = 3$ kN at B. The joint at B is hinged, the joint at C is rigid.



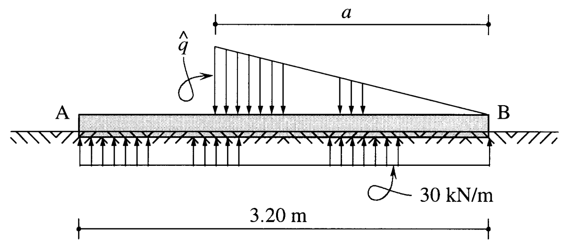
Questions:

a. Determine the support reactions at A and D.
b. Draw the M, V and N diagrams for the entire structure.
c. Determine the location and magnitude of the maximum field moment in BC.
d. Find the value of F for which the normal force in post AB is zero (with the given uniformly distributed load of 8 kN/m on BCE).

13.19 A structure, modelled as beam AB, is lying on the ground. Its dead weight can be ignored. On the right-hand side, the beam is subject to a linearly distributed load over length a with a top value $\hat{q}$. Due to this load, the earth pressure on the underside of the beam is constant, and is 30 kN/m.

Questions:

a. From the equilibrium of the beam determine length a and the top value $\hat{q}$.
b. Draw the resulting distributed load on the beam (the load diagram).
c. For the beam, draw good sketches of the V diagram and the M diagram (with their tangents at relevant points).
d. In which cross-section(s) is the shear force an extreme? At these cross-sections also draw the tangents to the M diagram.

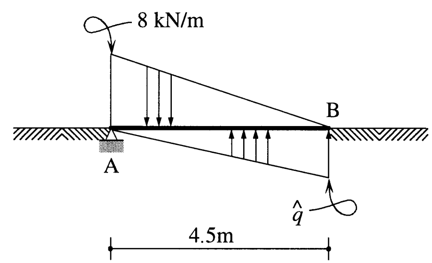


e. In which cross-section is the bending moment an extreme? Determine this value.

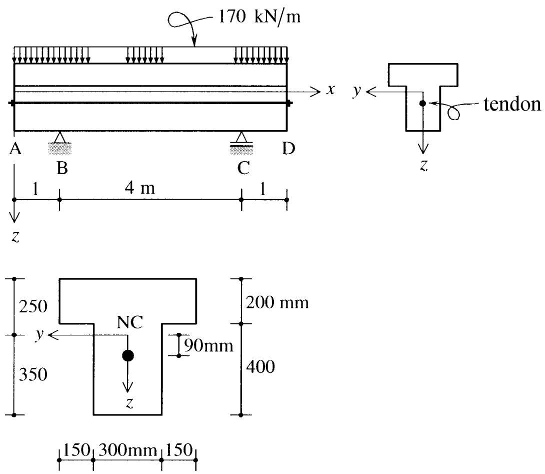
13.20 A weightless rigid beam AB is resting on a hinge at A, while the remainder is resting on the ground, which provides a linearly distributed counter-pressure with top value $\hat{q}$ at B. The load on the beam consists of a triangular load with a top value of 8 kN/m at A.

Questions:

a. From the equilibrium of the beam, determine the top value $\hat{q}$ of the earth pressure.
b. Draw the resulting load on the beam (the resultant load diagram).
c. For the beam, draw good sketches of the V diagram and the M diagram (with their tangents at relevant points).
d. At which cross-section(s) is the shear force an extreme? Draw the tangents to the M diagram at these cross-sections.
e. At which cross-section is the bending moment an extreme? Determine this value.



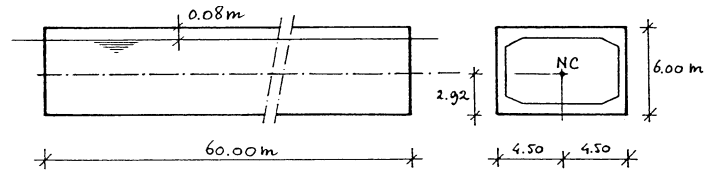
13.21 Given an eccentrically prestressed T-beam with overhangs and a uniformly distributed full load. The straight single bar tendon is 90 mm under the beam axis. The prestressing force is 1200 kN.



Questions:

a. Determine the support reactions.
b. Determine the N, V and M diagrams, with the deformation symbols. At A to D, draw the tangents to the M diagram.
c. In which cross-section(s) is the bending moment an extreme? Determine this/these extreme value(s).

13.22 A tunnel segment is afloat, ready to be moved to its final location where it will be sunk. The tunnel segment, which can be seen as a rigid body, has a freeboard of 0.08 m. The dead weight of the tunnel is 525 kN/m. The weight of each of the two temporary bulkheads is 234 kN. The specific weight of water is 10 kN/m^3. The dimensions of the tunnel segments are shown in the figure. The figure also shows the location of the normal centre NC in the cross-section.

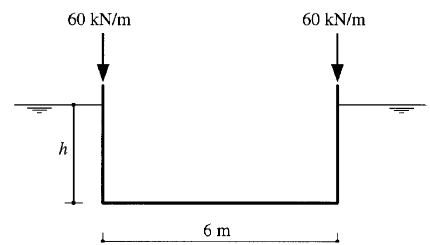


Questions:

a. How large is the water pressure on the underside of the tunnel?
b. Draw the distribution of the water pressure on a bulkhead, and determine the magnitude and location of the resultant.
c. Model the tunnel segment as a line element, and draw all (distributed and non-distributed) forces (and/or couples) acting on it.
d. For the tunnel segment, draw the M, V and N diagrams, with the deformation symbols. How large is the maximum bending moment?

13.23 A long weightless barge is loaded on its walls by a distributed load of 60 kN/m.

Questions:

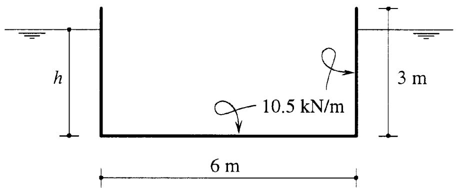


a. Determine the draught h of the barge.
b. Determine the distribution of the water pressure on the walls and bottom of the barge.
c. Draw the M, V and N diagrams for a 1-metre wide strip from the wall.
d. Isolate a 1-metre strip from the bottom of the barge and draw all the forces acting on it.
e. Draw the M, V and N diagrams for this 1-metre strip out of the bottom.

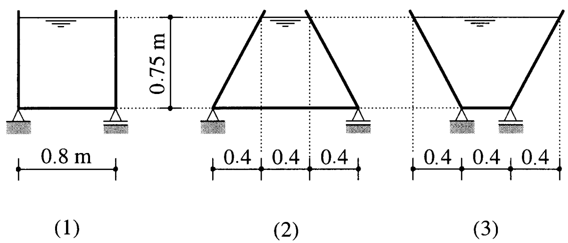
13.24 A 1-metre strip has been isolated from a long barge and is modelled as a bent line element. The dead weight of line element (walls and bottom) is 10.5 kN/m. The width of the strip is not given.

Questions:

a. Determine the draught h of the barge.
b. Determine the distribution of the water pressure on the walls and bottom.
c. Draw the M, V and N diagrams for the wall.
d. Isolate the bottom of the barge and draw all the forces acting on it.
e. Draw the M, V and N diagrams for the bottom.



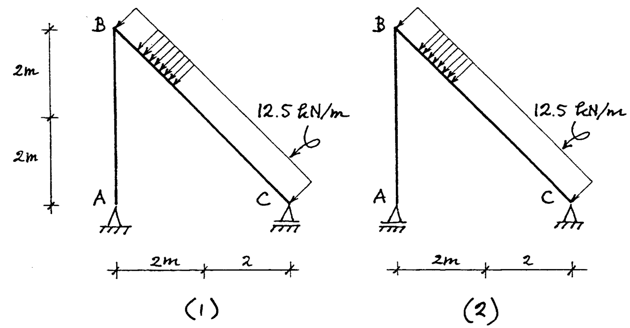
13.25: 1–3 A 1-metre strip has been isolated from a long trough filled with water and is modelled as a line element. There are three different trough shapes.



Questions:

a. Determine the support reactions.
b. Draw the distribution of the water pressure on the walls and the bottom.
c. Isolate the bottom and draw all the forces acting on it.
d. Draw the M, V and N diagrams for the bottom.
e. Determine the maximum field moment in the bottom.

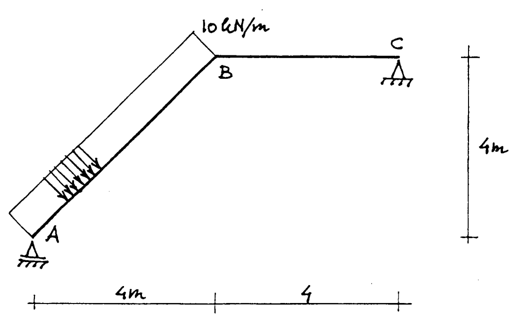
13.26: 1–2 The two structures shown differ only in their method of support.



Question:

Draw the M, V and N diagrams for the entire structure. At B and C, draw the tangents to the M diagram, and clearly indicate where they intersect.

13.27: 1–2 The two structures given differ only in their method of support.



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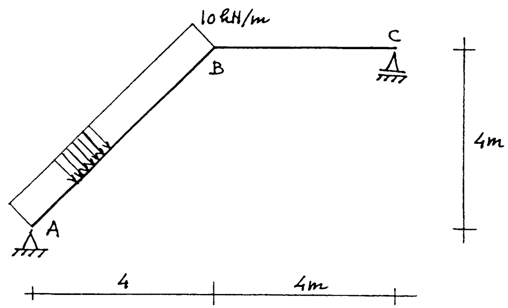
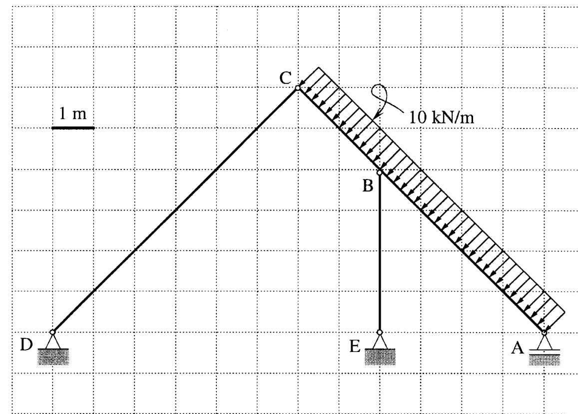


(2)

Question:

Draw the M, V and N diagrams for the entire structure. At A and B, draw the tangents to the M diagram, and clearly indicate where they intersect.

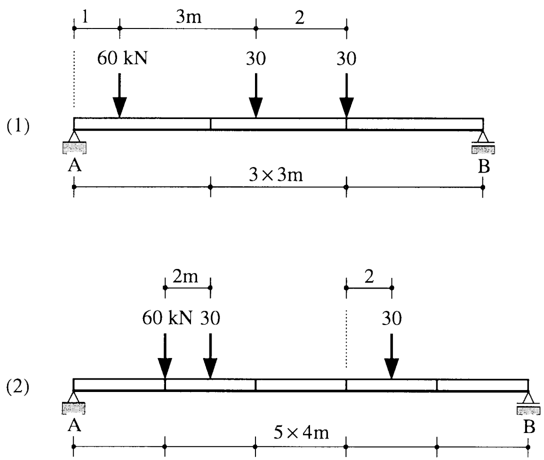
13.28 The structure consists of the members ABC, BE and CD that are joined together by hinges. A uniformly distributed load of 10 kN/m acts normal to ABC.



Questions:

a. Determine the support reactions at A, D and E. Draw them as they act on the structure.
b. Isolate ABC, and draw all the forces acting on it.
c. For ABC determine and draw the M and V diagram, with the deformation symbols. At A, B and C, draw the tangents to the M diagram, and clearly indicate where they intersect.

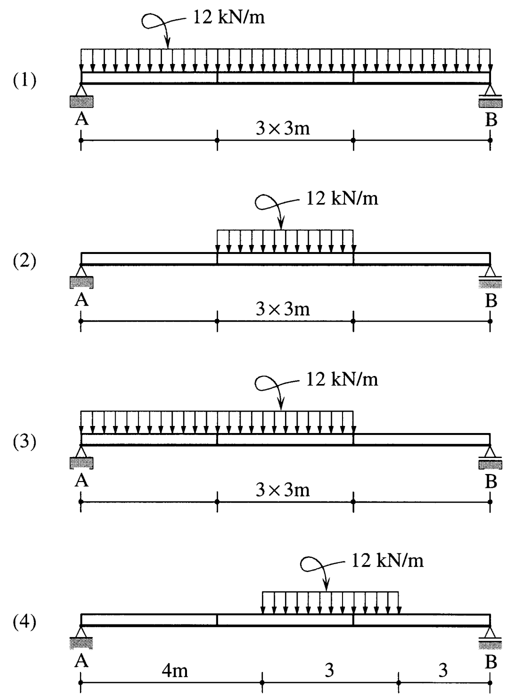
13.29: 1–2 Two different beams AB are indirectly loaded by a number of point loads.



Questions:

a. Determine the support reactions at A and B.
b. Determine the M and V diagrams for the indirectly loaded (main) beam.
c. Determine the M and V diagrams for the directly loaded (stringer) beams.
d. Explain any difference in magnitude between the support reactions at A and B, and the shear force in the main beam at those places.

13.30: 1–4 Four different distributed loads act on the same indirectly loaded beam AB.

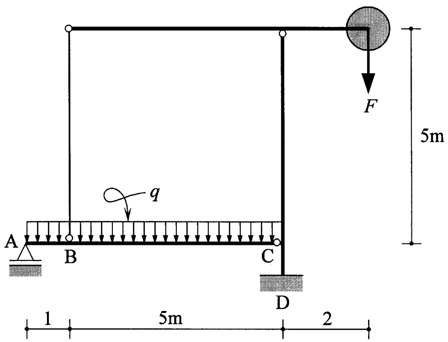


Questions:

a. Determine the support reactions at A and B.
b. Determine the M and V diagrams for the indirectly loaded (main) beam.
c. Determine the M and V diagrams for the directly loaded (stringer) beams.
d. Explain any difference in magnitude between the support reactions at A and B, and the shear force in the main beam at those places.

Compound and associated structures (Section 13.2)

13.31 The scheme of a (weightless) draw bridge is given. A uniformly distributed load q acts on the bridge deck ABC. The weight of the balance is F. Assume that $q = 12$ kN and $F = 90$ kN.



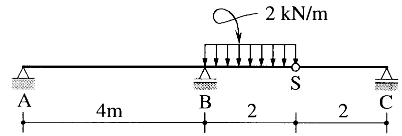
Questions:

a. Isolate ABC, and draw all the forces acting on it.
b. Draw the M and V diagram for ABC. At A, B and C, draw the tangents to the M diagram.
c. How large are the support reactions at D?
d. Calculate F in order to obtain a zero support reaction at A due to the given load.

13.32 As problem 13.31, but now with $q = 12$ kN and $F = 60$ kN.

13.33: 1–4 A number of hinged beams are given.

(1)



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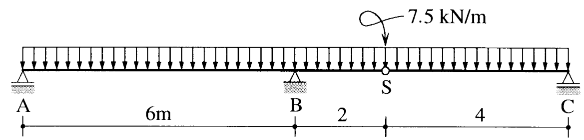

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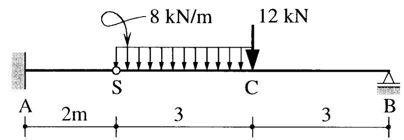

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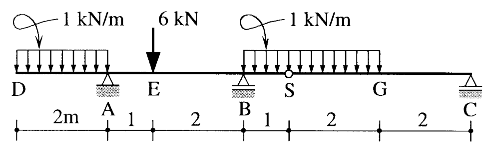
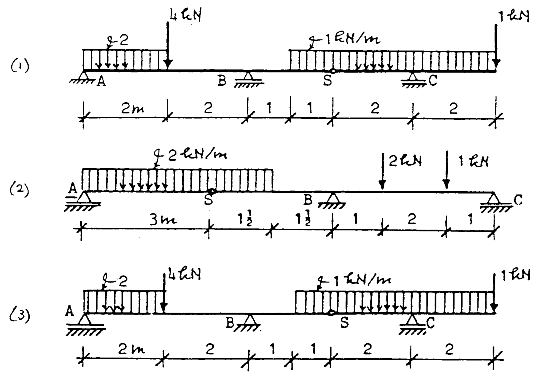


Questions:

a. Determine the support reactions.
b. Determine the V diagram.
c. Determine the M diagram, with the tangents at a number of points.
d. Determine the location and magnitude of the extreme bending moments.

13.34: 1–5 A number of hinged beams are given.



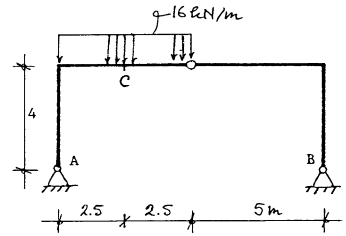
Questions:

a. Determine the support reactions.
b. Determine the V diagram.
c. Determine the M diagram, with the tangents at a number of points.
d. Determine the location and magnitude of the extreme bending moments.

13.35 A three-hinged portal frame with a uniformly distributed vertical load of 16 kN/m on the left-hand side of the girder is given.

Questions:

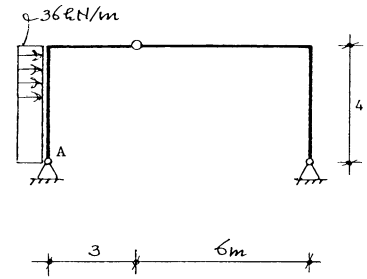
a. Determine the support reactions.
b. Determine the M diagram for the entire structure with the tangents at a number of points.
c. Determine the V diagram for the entire structure.
d. Determine the location and magnitude of the maximum field moment in the girder.
e. Determine the N diagram for the entire structure.



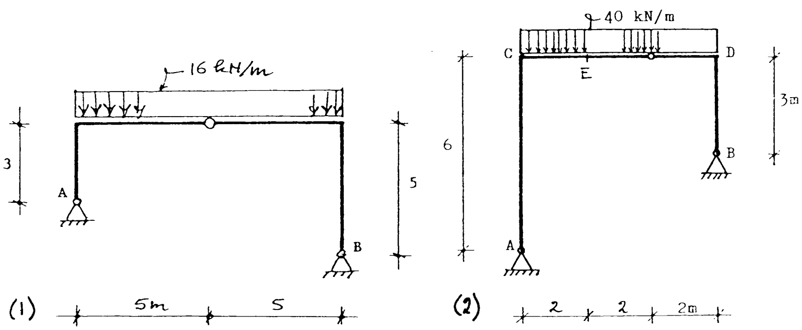
13.36 A uniformly horizontal distributed load of 36 kN/m acts on the left-hand column of a three-hinged portal frame.

Questions:

a. Determine the support reactions.
b. Determine the M diagram for the entire structure with the tangents at a number of points.
c. Determine the V diagram for the entire structure.
d. Determine the location and magnitude of the maximum field moment in the left-hand column.
e. Determine the N diagram for the entire structure.



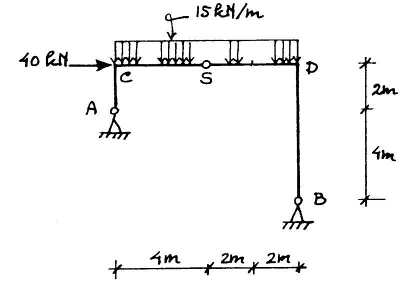
13.37: 1–2 Two three-hinged frames with unequal column lengths and a uniformly distributed full load on the beam are given.



Questions:
a. Determine the support reactions.
b. Determine the *M* diagram for the entire structure with the tangents at a number of points.
c. Determine the *V* diagram for the entire structure.
d. Determine the location and magnitude of the maximum field moment in the beam.
e. Determine the *N* diagram for the entire structure.

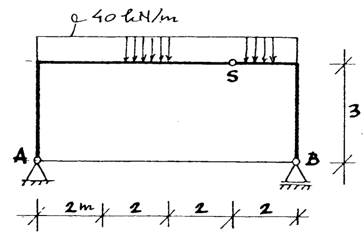
13.38 The girder of a three-hinged frame with unequal posts is loaded by a uniformly distributed vertical load and a horizontal force.

Questions:
a. Determine the support reactions.
b. Determine the *M* diagram for the entire structure with the tangents at a number of points.
c. Determine the *V* diagram for the entire structure.
d. Determine the location and magnitude of the maximum field moment in the girder.
e. Determine the *N* diagram for the entire structure.



13.39 A three-hinged frame with a tie rod is carrying a uniformly distributed load of 40 kN/m.

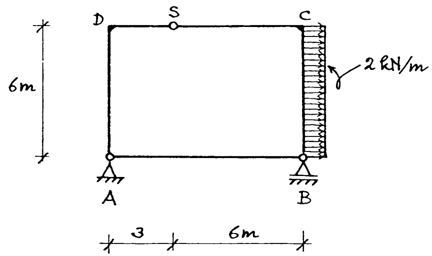
Questions:
a. Determine the support reactions.
b. Determine the force in the tie rod.
c. Determine the *M* diagram for the entire structure with the tangents at a number of points.
d. Determine the *V* diagram for the entire structure.
e. Determine the location and magnitude of the maximum field moment in the girder.
f. Determine the *N* diagram for the entire structure.



13.40 A uniformly horizontal distributed load of 2 kN/m acts on the right-hand post of a three-hinged frame with tie rod.

Questions:

a. Determine the support reactions.
b. Determine the force in the tie rod.
c. Determine the *M* diagram for the entire structure with the tangents at a number of points.
d. Determine the *V* diagram for the entire structure.
e. Determine the location and magnitude of the maximum field moment of the loaded post.
f. Determine the *N* diagram for the entire structure.

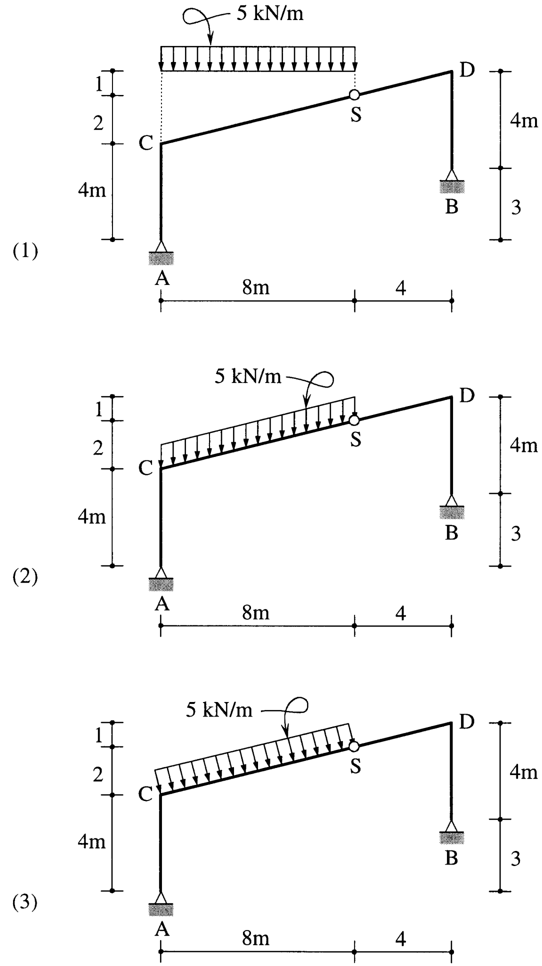


13.41 As problem 13.40, but now with the distributed load acting on the left-hand post.

13.42: 1–3 The same three-hinged frame is loaded in three different ways by a uniformly distributed load on CS.

Questions:

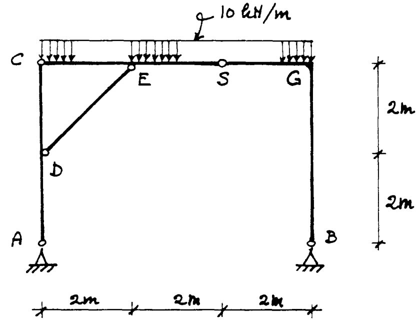
a. Determine the support reactions.
b. For the entire frame, draw the *M* diagram with the tangents at C and S.
c. Draw the *V* diagram for the entire frame.
d. Draw the *N* diagram for the entire frame.
e. Determine the location and magnitude of the maximum bending moment in field CS.



13.43 A shored three-hinged frame with a uniformly distributed full load of 10 kN/m on the girder is given.

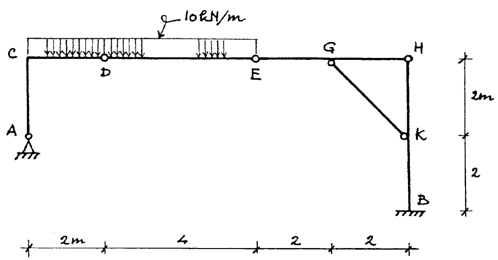
Questions:

a. Determine the support reactions.
b. Determine the force in shore DE, with the correct sign.
c. Isolate parts ADC, CES and SGB and draw all the forces acting on them.
d. Draw the *N* diagram for the entire structure.



e. Draw the *V* diagram for the entire structure.
f. Draw the *M* diagram for the entire structure, with the tangents at C, E and G.
g. Determine the location and magnitude of the extreme moments in CESG.

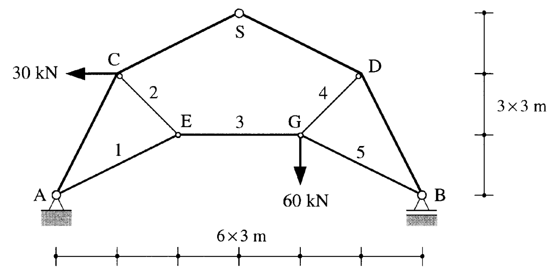
13.44 The structure is supported by a hinge at A and is fixed at B. GK is a shore with hinged connections at G and K. The structure also has hinged connections at D, E and H. A uniformly distributed load of 10 kN/m acts on CDE.



Questions:

a. For ACDEGH draw the *M* diagram, with the tangents at C, D and E.
b. For ACDEGH draw the *V* diagram.
c. Determine the support reactions.
d. Determine the force in shore GK, with the correct sign.
e. For HKB draw the *M* and *V* diagram.
f. Draw the *N* diagram for the entire structure.

13.45 The structure shown consists of the bent members ACS and BDS and the straight members 1 to 5, all joined by hinges. The structure is supported by a hinge at A, and on a roller at B. The load consists of a horizontal force of 30 kN at C and a vertical force of 60 kN at G.



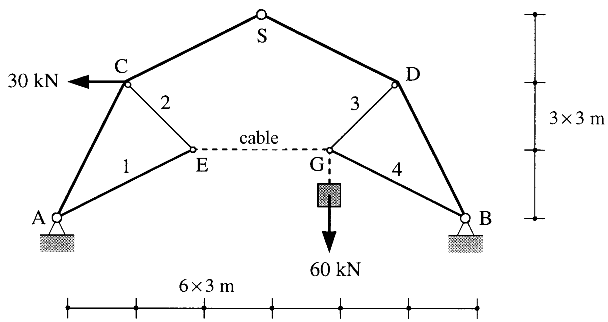
Questions:

a. Determine the support reactions at A and B.
b. Determine the forces in members 1 to 5 (with the correct sign). Draw the force polygons for joints E and G.
c. Isolate part ACS and draw all the forces acting on it.
d. For ACS draw the *M* and *V* diagram.
e. For ACS draw the *N* diagram.
f. Isolate part BDS and draw all the forces acting on it.
g. For BDS draw the *M* and *V* diagram.
h. For BDS draw the *N* diagram.

13.46 As problem 13.45, but now without the horizontal force at C.

13.47 As problem 13.45, but now without the vertical force at G.

13.48 The structure shown consists of the bent members ACS and BDS and the straight bars 1 to 4, all joined at hinges. The structure is supported by hinges at A and B. At E, a cable is fixed that at G runs through a frictionless pulley. A weight of 60 kN is attached to the cable. At C there is a horizontal force of 30 kN.

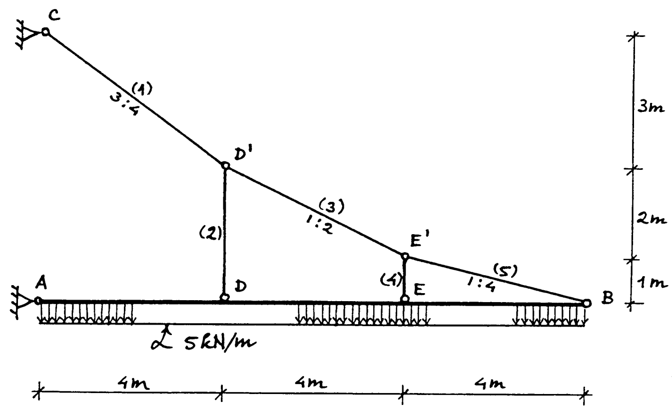


Questions:
a. Determine the support reactions at A and B.
b. Determine the forces in members 1 to 4 (with the correct sign). Draw the force polygons for joints E and G.
c. Isolate part ACS and draw all the forces acting on it.
d. For ACS draw the M and V diagram.
e. For ACS draw the N diagram.
f. Isolate part BDS and draw all the forces acting on it.
g. For BDS draw the M and V diagram.
h. For BDS draw the N diagram.

13.49 As problem 13.48, but now without the horizontal force at C.

13.50 As problem 13.48, but now without the weight of 60 kN at G.

13.51 The cantilever beam AB, with a uniformly distributed full load of 5 kN/m, is supported by means of a cable structure.

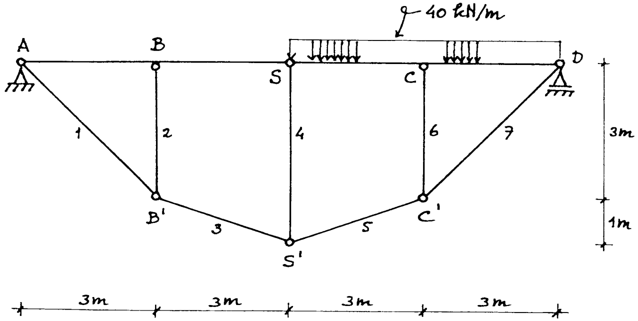


Questions:
a. Determine the support reactions.
b. Determine the forces in cables 1 to 5.
c. Isolate beam AB and draw all the forces acting on it.
d. Draw the M, V and N diagrams for AB. At A, D, E and B, draw the tangents to the M diagram.
e. Determine the magnitude and location of the extreme bending moments in AB.

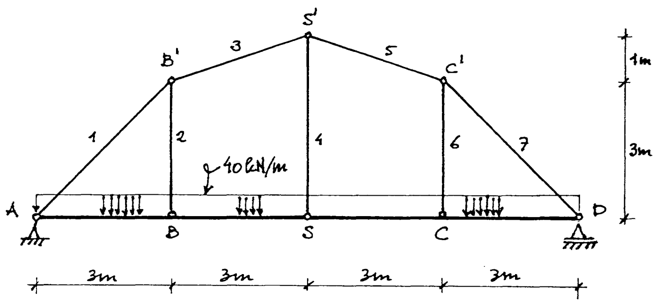
13.52 The trussed beam ASD is carrying a uniformly distributed load of 40 kN/m over SD.

Questions:
a. Determine the support reactions.
b. Determine the forces in members 1 to 7.
c. Isolate beam ASD and draw all the forces acting on it.

d. Draw the M, V and N diagrams for ASD. In S, C and D also draw the tangents to the M diagram.
e. Determine the location and magnitude of the extreme bending moments in ASD.

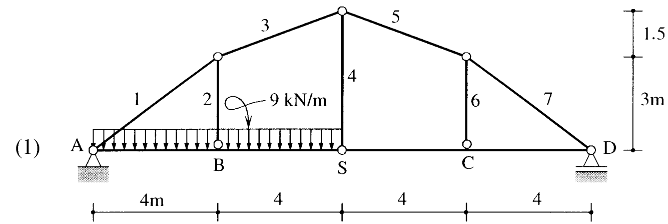


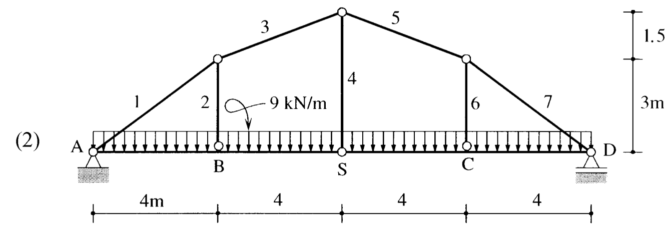
13.53 The trussed beam ASD is carrying a uniformly distributed full load of 40 kN/m.



Questions:
a. Determine the support reactions.
b. Determine the forces in members 1 to 7.
c. Isolate beam ASD and draw all the forces acting on it.
d. Draw the M, V and N diagrams for ASD. At A, B, S, C and D, draw the tangents to the M diagram.
e. Determine the location and magnitude of the extreme bending moments in ASD.

13.54: 1–2 The same trussed beam ASD is loaded in two different ways.

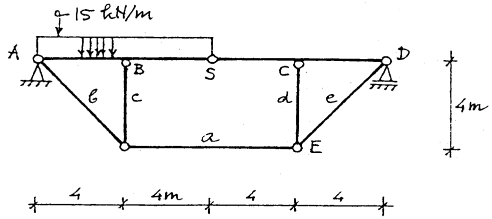




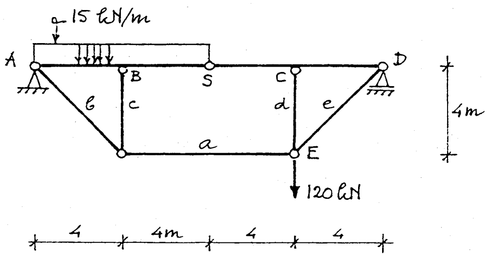
Questions:
a. Determine the support reactions.
b. Determine the forces in members 1 to 7.
c. Isolate beam ASD and draw all the forces acting on it.
d. Draw the M, V and N diagrams for ASD. Also draw at relevant points the tangents to the M diagram.
e. Determine the location and magnitude of the extreme bending moments in ASD.

13.55: 1–2 The same trussed beam ASD is loaded in two different ways.

(1)



(2)



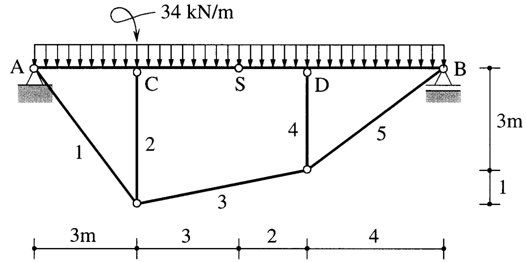
Questions:

a. Determine the support reactions.
b. Determine the forces in members a to e.
c. Isolate beam ASD, and draw all the forces acting on it.
d. Draw the M, V and N diagrams for ASD. At A, B and S, draw the tangents to the M diagram.

13.56 The trussed beam ASD carries a uniformly distributed full load of 34 kN/m.

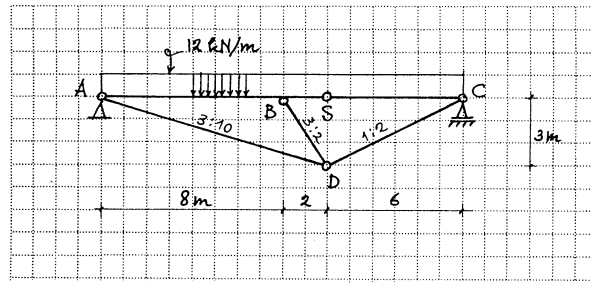
Questions:

a. Determine the forces in members 1 to 5, with the correct sign.
b. Isolate beam ASB and draw all the forces acting on it.
c. For ASB draw the N diagram.
d. For ASB draw the V diagram.
e. For ASB draw the M diagram with the tangents at A, C, D and B.



f. Determine the location and magnitude of the extreme bending moments in beam ASB.

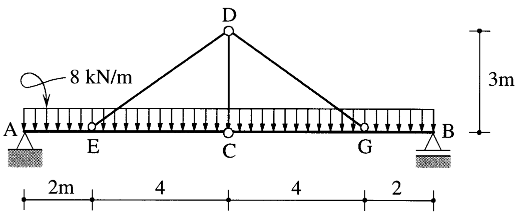
13.57 The trussed beam ASC carries a uniformly distributed full load of 12 kN/m.



Questions:

a. Determine the forces in members AD, BD and CD. Draw the force polygon for joint D. Use a force scale of 1 cm ≡ 40 kN.
b. Isolate beam ASC, and draw all the forces acting on it.
c. For ASC draw the N diagram.
d. For ASC draw the V diagram.
e. For ASC draw the M diagram, with the tangents at A, B and C.
f. Determine the location and magnitude of the extreme bending moments in beam ASC.

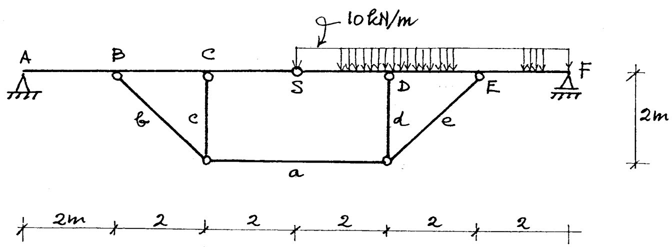
13.58 The trussed beam ACB carries a uniformly distributed full load of 8 kN/m.



Questions:

a. Determine the forces in members DC, DE and DG.
b. Isolate beam ASC and draw all the forces acting on it.
c. For ACB draw the *N* diagram.
d. For ACB draw the *V* diagram.
e. For ACB draw the *M* diagram, with the tangents at A, E and C.
f. Determine the location and magnitude of the extreme bending moments in beam ACB.

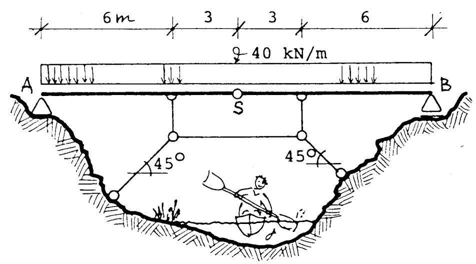
13.59 The trussed beam ASF carries a uniformly distributed load of 10 kN/m over SF.



Questions:

a. Isolate beam ASF, and draw all the forces acting on it.
b. For ASF draw the *N* diagram.
c. For ASF draw the *V* diagram.
d. For ASF draw the *M* diagram with the tangents at S, D, E and F.
e. Determine the location and magnitude of the extreme bending moments in beam ASF.

13.60 A queen post truss with a uniformly distributed full load of 40 kN/m is given.

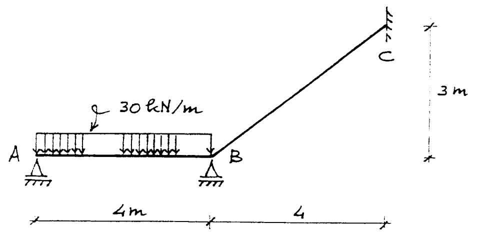


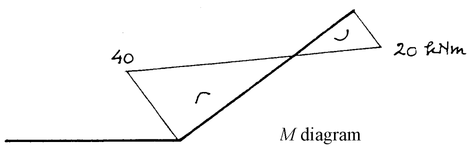
Questions:

a. Determine the *M* and *V* diagrams for beam ASB. Also draw at relevant points the tangents to the *M* diagram.
b. Determine the location and magnitude of the extreme bending moments in beam ASB.

Statically indeterminate structures (Section 13.3)

13.61 The *M* diagram for BC is given for the structure shown.



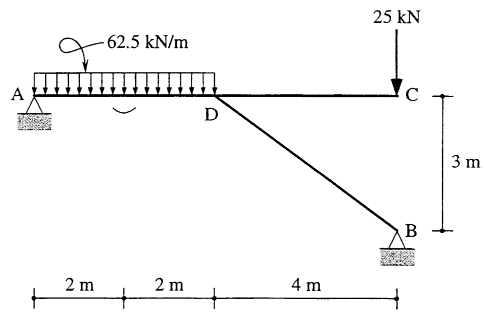


M diagram

Questions:

a. To what degree is the structure statically indeterminate? Substantiate your answer.
b. Draw the *M* diagram for AB, with the tangents at A and B.
c. Draw the *V* diagram for the entire structure.
d. How large is the maximum field moment in AB, and where does it occur?
e. Draw the *N* diagram for the entire structure.
f. Determine the support reactions at A, B and C. Draw them as they act on the structure.

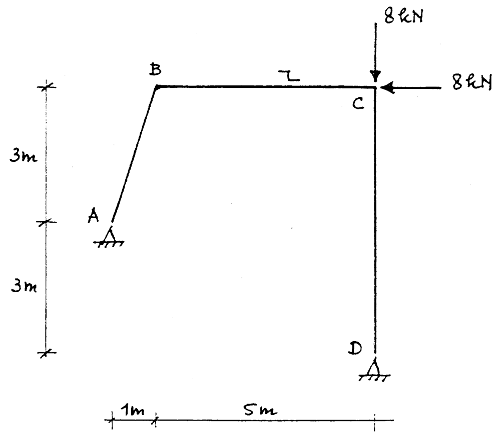
13.62 The structure is supported by hinges at A and B. At joint D all the members are rigidly joined to one another. With the given load, the bending moment in the middle of field AD is 25 kNm. The associated deformation symbol is given in the figure.



Questions:

a. To what degree is the structure statically indeterminate? Substantiate your answer.
b. Determine the support reactions, and draw them as they act on the structure.
c. For the entire structure, determine and draw the *M* and *V* diagrams. At A and D and the middle of field AD, draw the tangents to the *M* diagram.
d. Determine and draw the *N* diagram for the entire structure.

13.63 A two-hinged frame is loaded at C by the forces of 8 kN as shown. With this load, the shear force in girder BC is 6 kN. The associated deformation symbol is shown in the figure.



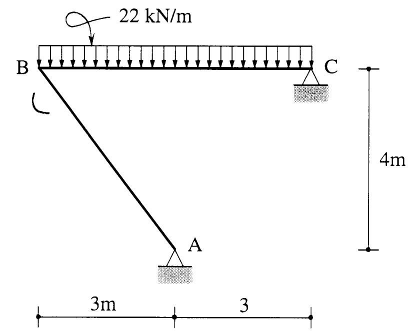
Questions:

a. Determine the support reactions. Draw them as they act in reality.
b. Draw the bending moment diagram for the entire structure.
c. Draw the shear force diagram for the entire structure.
d. Draw the normal force diagram for the entire structure.

13.64 The bend beam is supported by hinges at A and C. The joint at B is entirely rigid. With the given load the bending moment in member BA, directly under joint B, is 60 kNm. The deformation symbol is given in the figure.

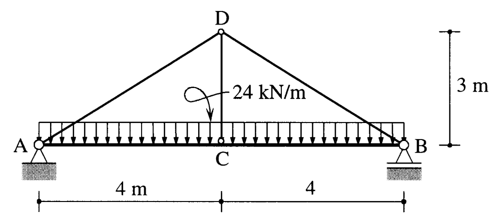
Questions:

a. To what degree is the structure statically indeterminate? Substantiate your answer.
b. Determine the support reactions and draw them as they act on the structure.



c. Determine and draw the *M* and *V* diagrams for the entire structure. At B and C draw the tangents to the *M* diagram.
d. Determine and draw the *N* diagram for the entire structure.

13.65 In the trussed beam ACB the bending moment at C is zero for the given load.



Questions:

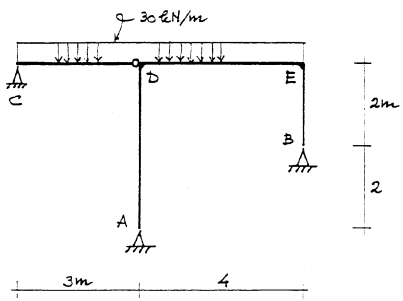
a. Determine the degree of static indeterminacy for the structure.
b. For ACB draw the *M* and *V* diagrams. At A, B and C also draw the tangents to the *M* diagram.
c. Determine the normal forces in bars AD, BD and CD.
d. Draw the *N* diagram for ACB.

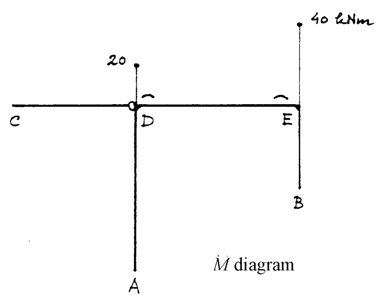
13.66 In the trussed beam ACB from problem 13.65 the bending moment at C with the given load is 36 kNm.

Questions:

a. Draw the M and V diagram for ACB (there are two possibilities). At A, B and C, draw the tangents to the M diagram.
b. Determine the normal forces in bars AD, BD and CD.
c. Draw the N diagram for ACB.

13.67 The structure consists of a two-hinged frame ADEB that is supported horizontally by member CD. The link between member CD and frame ADEB is a hinge. With the given uniformly distributed load of 30 kN/m on CDE, the bending moments with deformation symbols at D and E are given for DE.





M diagram

Questions:

a. Draw the M diagram for the entire structure. At D and E, draw the tangents to the M diagram.
b. Draw the V diagram for the entire structure.
c. Draw the N diagram for the entire structure.
d. Draw all the support reactions in the directions in which they act.

14

Cables, Lines of Force and Structural Shapes

A special type of tension-loaded line element is the entirely *flexible cable*. Cables have no “natural” shape, and adapt to the load.

In Section 14.1.1, we look at the behaviour of cables subject to a system of parallel forces.

In Section 14.1.2, we show that the shape of a cable with respect to its chord, due to a number of parallel forces, is similar to the bending moment diagram of a simply supported beam with the same span and the same load.

After deriving the *cable equation* from the equilibrium of a small cable element in Section 14.1.3, we apply this in Section 14.1.4 to a cable with a uniformly distributed full load (*force per horizontally measured length*). In this case, the cable is a *parabola*.

Next, the cable equation in Section 14.1.5 is applied to a cable loaded exclusively by its dead weight (*force per length measured along the cable*). The associated cable shape is a *catenary*.

In Section 14.2 we come back to the concept of *centre of force*, the point of application of the resultant of all normal stresses in the cross-section, or in other words, the point of application of the resultant of N and M (see Section 10.1.1). The centres of force in all consecutive cross-sections together form the *line of force*. If the line of force coincides with the member axis, the bending moments (and shear forces) are zero and the force flow occurs via normal forces.

In bending, the material in the cross-section is used less efficiently than in extension. To ensure maximum efficient use of material, the structural shape (member axis) should preferably be chosen in such a way that there is no bending and the force flow occurs via normal forces. In Section 14.3, on the basis of the cable shape and line of force, we look for structural shapes in which the force flow through bending remains limited.

14.1 Cables

Cables are line elements in which the resistance to bending is so small that it can be ignored. A fully flexible cable cannot transfer bending moments nor transverse forces. The force flow occurs entirely via normal forces, namely tensile forces.[1]

Cables are often used in structures with large spans such as suspension bridges and suspended roofs, but also in high-voltage cables, cableways, and the mooring of high structures such as radio and TV masts.

Cables do not have their own shape – they adapt to the load. Here, we assume that the axial stiffness of the cable is infinite. Therefore the cable has the same length before and after loading. The shape of the cable and the cable forces can then be deduced directly from the equilibrium equations.

In Section 14.1.1, we deduce the shape of the cable and cable forces directly from the equilibrium for a cable loaded by a number of parallel point loads.

---

[1] If there are compressive forces in the cable, the equilibrium is unstable (unreliable). In order to restore the equilibrium following a minor disruption in the cable shape under the given load, bending moments have to develop in the cable. Since this is not possible in an entirely flexible cable, the equilibrium is lost after a minor disruption in the cable shape.

In Section 14.1.2, we show that the *cable shape* is similar to the shape of the bending moment diagram of a simply supported beam with the same span and load.

We follow with a mathematical description of the relationship between cable force, cable shape and load in Section 14.1.3. We derive the so-called *cable equation* from the equilibrium equations for a small cable element.

Using the cable equation as basis, we calculate the cable shape in Section 14.1.4 due to a uniformly distributed load. The associated cable shape is a *parabola*.

At that point, the distributed load is a *force per horizontally measured length*. In Section 14.1.5, we calculate the cable shape due to its *dead weight*. The dead weight is a *force measured along the length of the cable*. The cable shape resulting from the dead weight is a *catenary*.

If the (vertical) sag of the cable with respect to its chord is small compared with the (horizontal) span, then the catenary can be approximated by the simpler parabola.

Finally, in Section 14.1.6, we present a number of examples.

14.1.1 Cables with point loads

Calculating the cable shape and cable forces from the equilibrium is illustrated using the cable in Figure 14.1a, supported at the fixed points A and B, and loaded by the vertical forces $F_C = 75$ kN and $F_D = 30$ kN. The cable has a (horizontal) span $\ell = 60$ m and a difference in elevation between supports A and B of $h = 9$ m. The distances between the supports and the lines of action of the forces are shown in the figure. The z coordinate of the cable of point E is also given: $z_E = 22$ m.

The dead weight of the cable is so small compared to the load that it can be ignored.

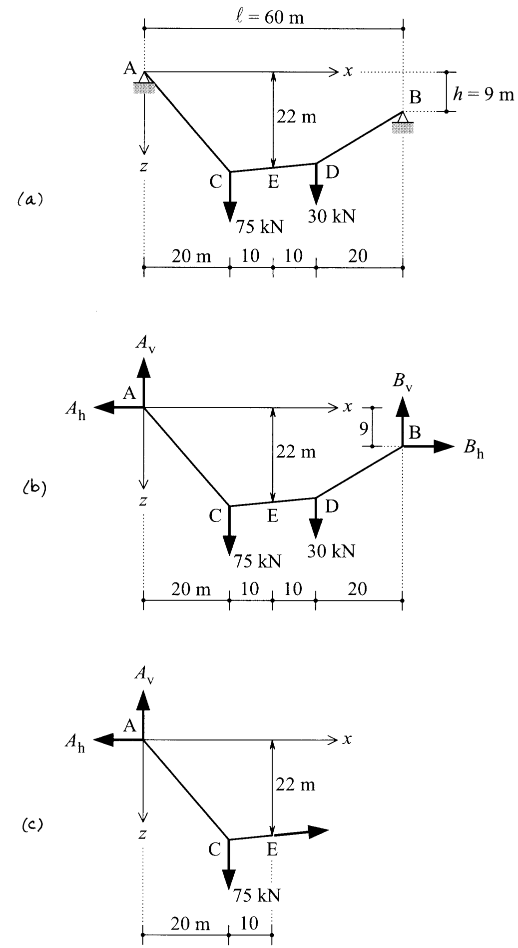


Figure 14.1 (a) Cable AB loaded by two vertical forces. (b) The isolated cable AB. (c) The isolated cable part AE.

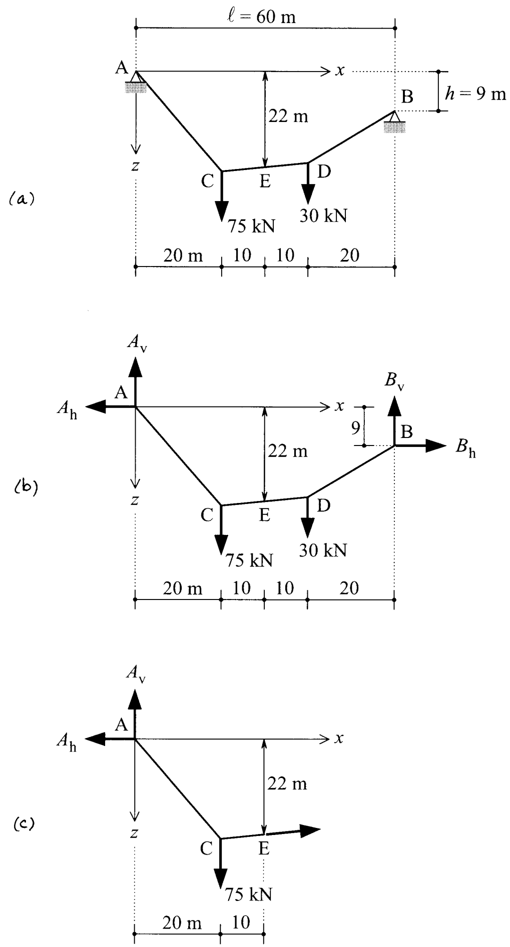


Figure 14.1 (a) Cable AB loaded by two vertical forces. (b) The isolated cable AB. (c) The isolated cable part AE.

Questions:

a. Determine the cable shape, or in other words, the z coordinates of C and D where kinks occur in the cable.
b. Determine the maximum and minimum cable force.

Solution:

a. With fully flexible cables, no bending moments can be transferred, and the cable remains straight between the places where forces are applied. Each straight part of the cable can be seen as a line element subject to a tensile force N, the cable force.

In Figure 14.1b the cable has been isolated. There are four unknown support reactions: A_h, A_v, B_h and B_v. There are three equilibrium equations:

$$\sum F_x = -A_h + B_h = 0, \tag{1}$$

$$\sum F_z = -A_v - B_v + (75\ \text{kN}) + (30\ \text{kN}) = 0, \tag{2}$$

$$\begin{aligned}\sum T_y|\text{B} &= +A_h(9\ \text{m}) - A_v(60\ \text{m}) \\ &\quad + (75\ \text{kN})(40\ \text{m}) + (30\ \text{kN})(20\ \text{m}) = 0.\end{aligned} \tag{3}$$

For a unique solution to these three equations with four unknowns, we need a fourth equation. This is found from the moment equilibrium of the part of the cable to the right or left of E, the point where the z coordinate of the cable is given. Here, we select the part to the left of E, as this equilibrium equation contains only the unknowns A_h and A_v and in combination with Equation (3) leads to the quicker result (see Figure 14.1c):

$$\sum T_y|\text{E} = +A_h(22\ \text{m}) - A_v(30\ \text{m}) + (75\ \text{kN})(10\ \text{m}) = 0. \tag{4}$$

From (3) and (4) we find

$$A_h = 60\ \text{kN},$$

$$A_v = 69\ \text{kN}.$$

From (1) and (2) we then find

$$B_h = 60\ \text{kN},$$

$$B_v = 36\ \text{kN}.$$

If the z coordinate of E (or of another point on the cable) is not given, the result remains undetermined, and the cable can assume various shapes, such as the two dotted shapes in Figure 14.2. The final shape of the cable is determined by the length of the cable. The approach via a given cable length is considerably more complicated than that with a locally given z coordinate of the cable, such as that at point E.

Hereafter we assume that the axial stiffness of the cable is infinite, so that the cable has the same length before and after loading. In that case, the cable shape and cable forces can be derived directly from the equilibrium. Since the cable cannot stretch, an increase in the load with a certain factor does not cause a change in the shape of the cable.

The z coordinates of respectively C and D are found from the moment equilibrium about C and D of the left-hand or right-hand part of the cable.

The following holds for the part to the left of C (see Figure 14.3a):

$$\sum T_y|\text{C} = +(60\ \text{kN}) \times z_\text{C} - (69\ \text{kN})(20\ \text{m}) = 0$$

so that

$$z_\text{C} = 23\ \text{m}.$$

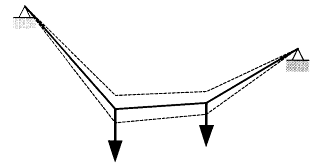


Figure 14.2 The equilibrium conditions are satisfied by an infinite number of cable shapes. The final shape is determined by the (developed) length of the cable.

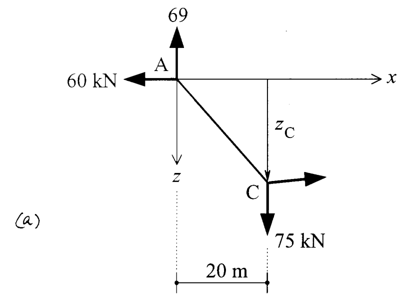
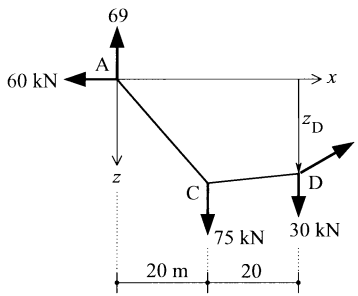


Figure 14.3 (a) The z coordinate follows from the moment equilibrium about C of AC.

(b)



(c)

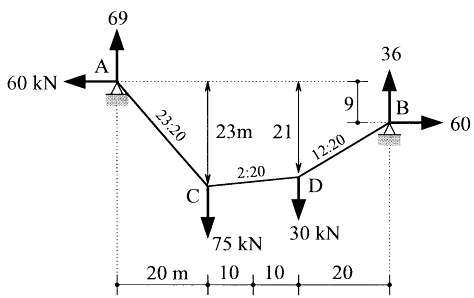


Figure 14.3 (b) The z coordinate follows from the moment equilibrium of ACD about D. (c) Support reactions and cable shape.

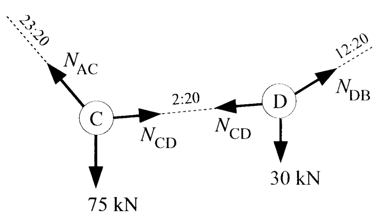


Figure 14.4 The cable forces can be determined using the force equilibrium for joints C and D.

For the part to the left of D applies (see Figure 14.3b)

$$\sum T_y|\text{D} = +(60\ \text{kN}) \times z_\text{D} - (69\ \text{kN})(40\ \text{m}) + (75\ \text{kN})(20\ \text{m}) = 0$$

so that

$$z_\text{D} = 21\ \text{m}.$$

The z coordinates of C and D fix the cable shape (see Figure 14.3c).

b. Cable forces N (in the straight parts) can now be calculated from the force equilibrium of joints C and D (see Figure 14.4).

Since the cable is loaded exclusively by vertical forces, it is easier to use the fact that the tensile force N in the cable has a horizontal component H that is constant over the entire length of the cable. This follows directly from the horizontal force equilibrium of an arbitrary part of the cable:

$$H = A_\text{h} = B_\text{h} = 60\ \text{kN}.$$

Assuming α is the angle that the cable makes with the horizontal, then (see Figure 14.5)

$$N = \frac{H}{\cos\alpha} = H\sqrt{1 + (\tan\alpha)^2}. \tag{5}$$

The maximum force in the cable occurs where $\tan\alpha$ is a maximum, that is where the slope of the cable is largest.

The geometry of the deformed cable gives

$$N^\text{AC} = H\sqrt{1 + (\tan\alpha^\text{AC})^2} = (60\ \text{kN})\sqrt{1 + (23/20)^2} = 91.44\ \text{kN},$$

$$N^\text{CD} = H\sqrt{1 + (\tan\alpha^\text{CD})^2} = (60\ kN)\sqrt{1 + (2/20)^2} = 60.30\ \text{kN},$$

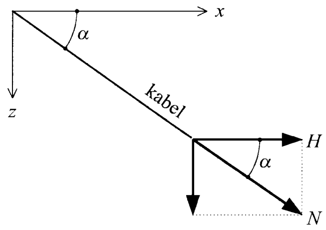


Figure 14.5 For cable force N it holds that $N = H/\cos\alpha = H\sqrt{1 + (\tan\alpha)^2}$.

$$N^{\mathrm{DB}} = H\sqrt{1 + (\tan\alpha^{\mathrm{DB}})^2} = (60\ \text{kN})\sqrt{1 + (12/20)^2} = 69.97\ \text{kN}.$$

Check: The cable forces in AC and DB can also be found directly from the support reactions at A and B respectively:

$$N^{\mathrm{AC}} = \sqrt{A_\mathrm{h}^2 + A_\mathrm{v}^2} = \sqrt{(60\ \text{kN})^2 + (69\ \text{kN})^2} = 91.44\ \text{kN},$$

$$N^{\mathrm{DB}} = \sqrt{B_\mathrm{h}^2 + B_\mathrm{v}^2} = \sqrt{(60\ \text{kN})^2 + (36\ \text{kN})^2} = 69.97\ \text{kN}.$$

14.1.2 Relationship between cable shape and bending moment diagram

Figure 14.6a shows a simply supported cable with compression bar, loaded by n vertical point loads $F_1, F_2, \ldots, F_n$. The cable has a (horizontal) span ℓ with a difference h between the support elevations at A and B.

The place of the roller support B is fixed by the *compression bar* AB so that the cable shape can be determined as if A and B were immovable supports.

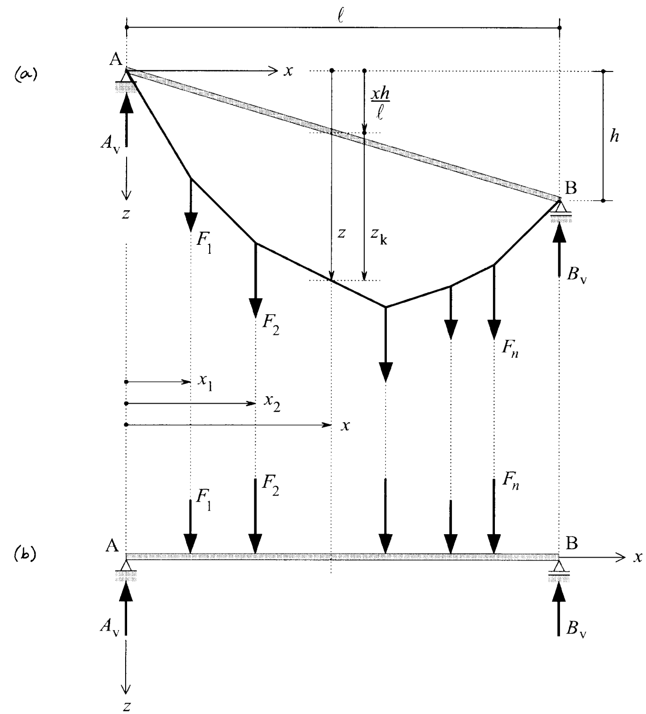


Figure 14.6 (a) A simply supported cable with compression bar loaded by a number of parallel forces. (b) A simply supported beam with the same span and load.

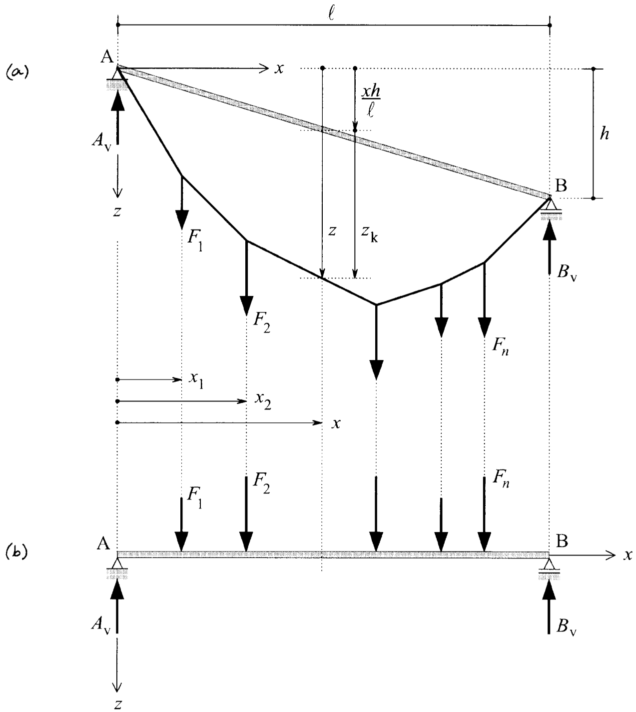


Figure 14.6 (a) A simply supported cable with compression bar loaded by a number of parallel forces. (b) A simply supported beam with the same span and load.

For the cable shape applies

$$z = z(x)$$

in which z is the distance from the x axis to the cable. The distance from the chord (compression bar) AB to the cable is hereafter indicated by

$$z_k = z_k(x).$$

From Figure 14.6a we can deduce that

$$z_k = z - \frac{x}{\ell}h.$$

Figure 14.6b shows a simply supported beam AB with the same span ℓ and the same load.

The cable with compression bar, and the simply supported beam have the same support reactions A_v and B_v at A and B respectively. There are no horizontal support reactions. That the support reactions are equal for cable and beam can easily be checked by calculation. In this way, the vertical support reaction A_v follows in both cases from the moment equilibrium about B of the structure as a whole:

$$\sum T_y|B = -A_v\ell + \sum_{i=1}^{n} F_i(\ell - x_i) = 0$$

so that

$$A_v = \frac{\sum_{i=1}^{n} F_i(\ell - x_i)}{\ell}.$$

In both cases, the vertical force equilibrium about B gives the following result for the vertical support reaction B_v:

$$B_v = \frac{\sum_{i=1}^{n} F_i x_i}{\ell}.$$

In Figure 14.7, the part to the left of an arbitrary (vertical) section x has been isolated for both the cable with compression bar and the simply supported beam.

Since the cable is loaded exclusively by vertical forces, the tensile forces in the cable have a constant horizontal component H (see Section 14.1.1). From the horizontal force equilibrium of the isolated part in Figure 14.7a we find that the compressive force in bar AB has the same horizontal component H.

In addition to the horizontal forces H, there are also the vertical forces V and Hh/ℓ in the section, components of the tensile force in the cable and the compressive force in the bar (a two-force member) respectively. On the basis of the vertical equilibrium of the isolated section, it holds that

$$\sum F_z = -A_v + \sum_{i=1}^{2} F_i + V - \frac{Hh}{\ell} = 0.$$

The vertical forces in the section are therefore

$$V - \frac{Hh}{\ell} = A_v - \sum_{i=1}^{2} F_i. \tag{6}$$

In Figure 14.7b, there is a bending moment M and a shear force V^{beam} at the cross-section of the beam. The vertical equilibrium of the isolated part of the beam gives the shear force:

$$V^{\text{beam}} = A_v - \sum_{i=1}^{2} F_i. \tag{7}$$

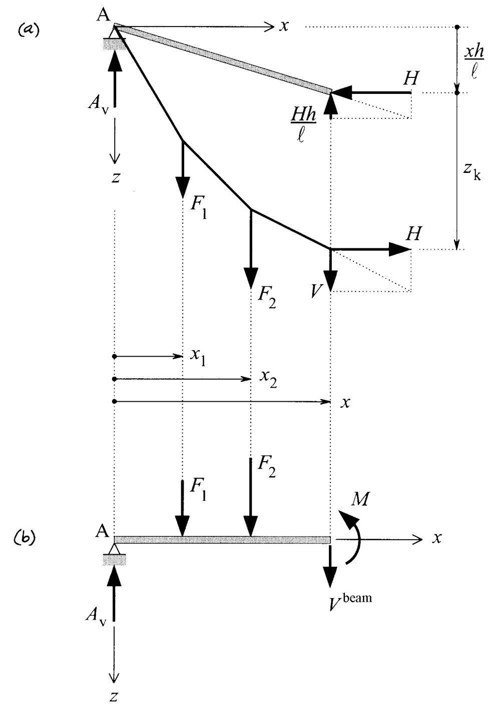


Figure 14.7 The isolated part to the left of section x of (a) the cable with a compression bar and (b) the beam.

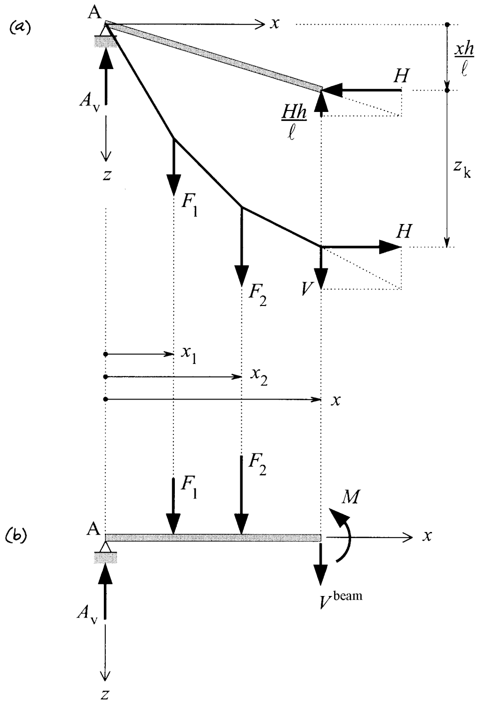


Figure 14.7 The isolated part to the left of section x of (a) the cable with a compression bar and (b) the beam.

From (6) and (7), we find the following relationship between the vertical component of the cable force and the shear force in the beam:

$$V - \frac{Hh}{\ell} = V^{\text{beam}}. \tag{8}$$

The term Hh/ℓ in (8) disappears when the supports of the cable are at equal elevations. The vertical component of the cable force is then equal to the shear force in the beam.

Conclusion: *The vertical component of the cable force is equal to the shear force in the beam (which can be read from the shear force diagram), only if the support reactions of the cable are at equal elevations.*

The moment equilibrium of the isolated part of the cable with compression bar about an arbitrary point in the section gives (see Figure 14.7a)

$$\sum T_y|\text{section} = -A_\text{v}x + \sum_{i=1}^{2} F_i(x - x_i) + Hz_\text{k} = 0$$

so that

$$Hz_\text{k} = A_\text{v}x - \sum_{i=1}^{2} F_i(x - x_i). \tag{9}$$

The moment equilibrium of the isolated part of the beam in Figure 14.7b gives

$$\sum T_y|\text{section} = -A_\text{v}x + \sum_{i=1}^{2} F_i(x - x_i) + M = 0$$

so that

$$M = A_\text{v}x - \sum_{i=1}^{2} F_i(x - x_i). \tag{10}$$

If we compare the equations (9) and (10) we find

$$H z_{\mathrm{k}} = M. \tag{11}$$

Conclusion: *The product of the horizontal component H of the tensile force in the cable and the distance z_{k} from the chord (compression bar) AB to the cable is equal to the bending moment M in a simply supported beam with the same span and the same load.*

The horizontal component H of the tensile force in the cable is constant and therefore independent from x. In contrast, the cable shape $z_{\mathrm{k}} = z_{\mathrm{k}}(x)$, under the chord, and the bending moment $M = M(x)$ are functions of x. The equation

$$H z_{\mathrm{k}}(x) = M(x)$$

shows that the cable shape under the chord (compression bar) AB has the same shape as the bending moment diagram. The force H can be seen as a scale factor.[1]

In the section in Figure 14.7, the left-hand part is subject to only two forces F_1 and F_2, and only these forces appear in the calculation. With an arbitrary alternative section, the number of forces on the left-hand side can be larger or smaller. The conclusions remain the same, however.

In Figure 14.8a, the compression bar AB has been isolated from the cable. In A and B, the compression bar is subject to horizontal forces H and vertical forces Hh/ℓ. In Figure 14.8b, equal but opposite forces act on the ends of the isolated cable, together with the forces A_{v} and B_{v}, which are equal to the support reactions of the beam AB in Figure 14.8c, with the same span and load.

1 As a help for drawing bending moment diagrams, rule 16 in Section 12.1.6 already pointed out the relationship between cable shape and bending moment diagram.

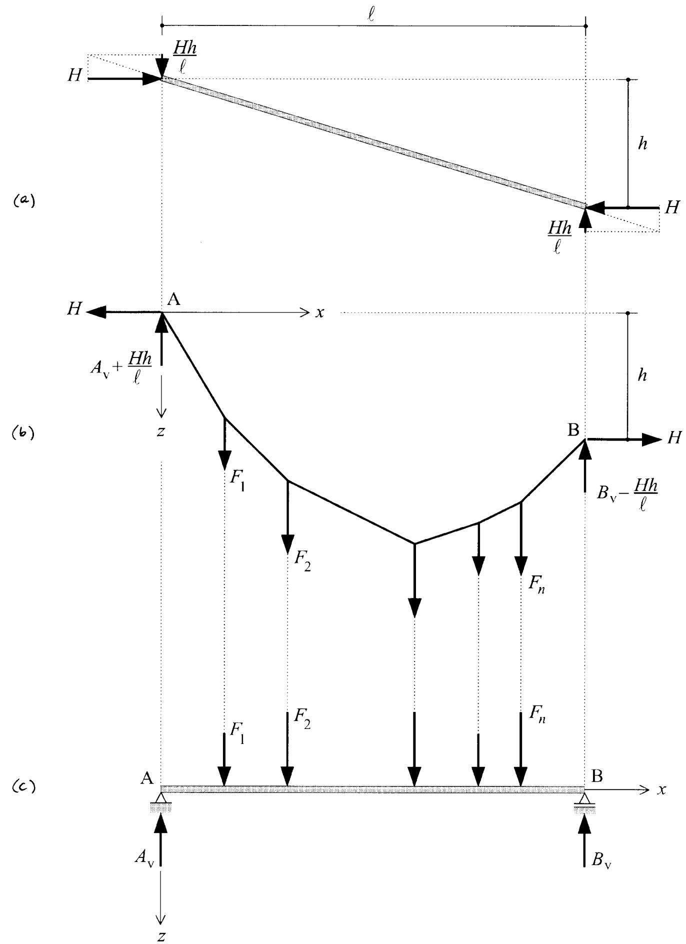


Figure 14.8 (a) The compression bar isolated from the cable. (b) The support reactions of the cable without compression bar. (c) The support reactions of a simply supported beam with the same span and load.

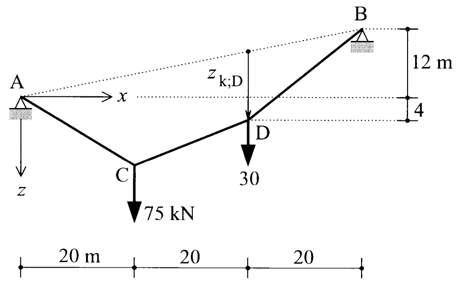


Figure 14.9 A cable loaded by two vertical forces.

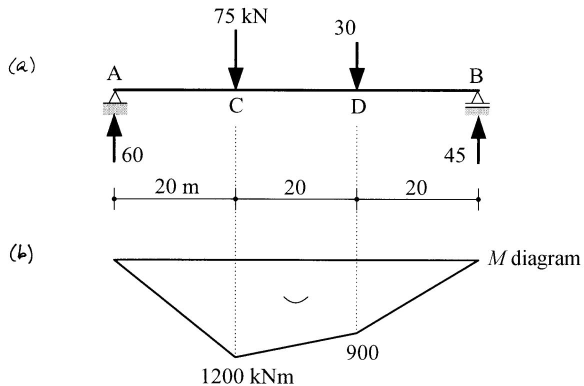


Figure 14.10 (a) A simply supported beam with the same span and load as the cable in Figure 14.9 with (b) the associated bending moment diagram.

The forces at A and B on the isolated cable in Figure 14.8b can be seen as the support reactions of a cable without compression member, supported at two fixed points.

The vertical support reactions A_v and B_v are equal to those of a beam with the same span and load only if the supports are at equal elevations.

With a difference h between both support elevations, the two horizontal support reactions H form a couple that leads to a change in the vertical support reactions of Hh/ℓ.

This is applied in the following two examples.

Example 1

The cable in Figure 14.9, supported at the fixed points A and B, is loaded in C and D by vertical forces of 75 and 30 kN respectively. Only the location of point D is given for the cable shape: it is 4 metres lower than support A.

Questions:

a. Determine the cable shape.
b. Determine the horizontal support reactions at A and B.
c. Determine the vertical support reactions at A and B.
d. Determine the maximum and minimum cable forces.

Solution:

a. Figure 14.10a shows a simply supported beam AB with the same (horizontal) span as the cable and the same vertical load. The support reactions are also shown. Figure 14.10b shows the associated M diagram. Under chord AB, the cable has the same shape as the M diagram; according to (11):

$$Hz_k = M.$$

The scale factor H is the horizontal component of the tensile force in the cable. At D, the distance from the chord AB to the cable is (see Figure 14.9)

$$z_{k;D} = \tfrac{2}{3}(12 \text{ m}) + (4 \text{ m}) = 12 \text{ m}.$$

This gives

$$H = \frac{M_D}{z_{k;D}} = \frac{900 \text{ kNm}}{12 \text{ m}} = 75 \text{ kN}.$$

At C, the distance between the chord and the cable is

$$z_{k;C} = \frac{M_C}{H} = \frac{1200 \text{ kNm}}{75 \text{ kN}} = 16 \text{ m}.$$

The cable shape is now determined (see Figure 14.11).

b. The horizontal support reactions at A and B are equal to the horizontal component H of the cable force determined above:

$$A_h = B_h = H = 75 \text{ kN}.$$

The horizontal support reactions are shown in Figure 14.11.

c. The vertical support reactions A_v and B_v in Figure 14.11 are equal to the vertical support reactions of the beam in Figure 14.10a, but since the horizontal support reactions act at different levels these have to be corrected by a force

$$\frac{Hh}{\ell} = \frac{(75 \text{ kN})(12 \text{ m})}{60 \text{ m}} = 15 \text{ kN}$$

so that

$$A_v = (60 \text{ kN}) - (15 \text{ kN}) = 45 \text{ kN},$$

$$B_v = (45 \text{ kN}) + (15 \text{ kN}) = 60 \text{ kN}.$$

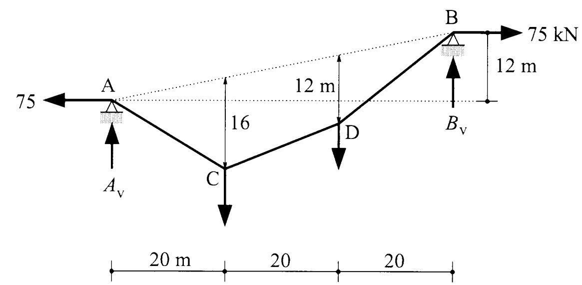


Figure 14.11 Under the chord, the cable has exactly the same shape as the bending moment diagram in Figure 14.10b. The scale factor is $H = 75$ kN.

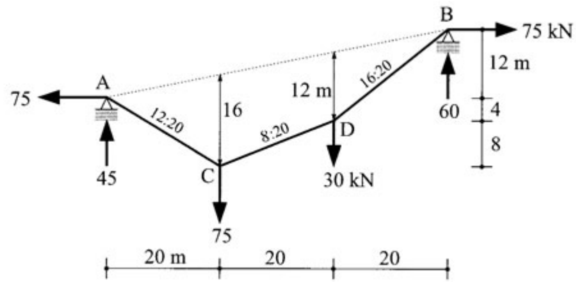


Figure 14.12 Support reactions and cable shape.

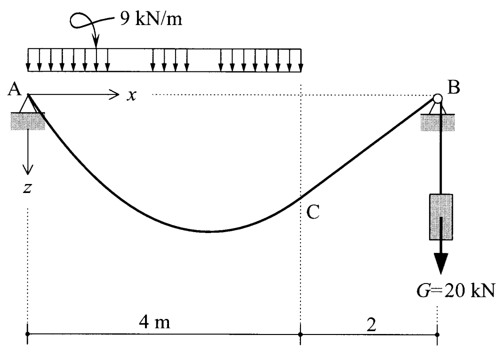


Figure 14.13 At B, cable AB passes over a pulley, and is kept under tension by a weight of 20 kN.

d. Figure 14.12 shows all the support reactions. This figure also includes the slopes of the straight cable parts. With (5)

$$N = H\sqrt{1 + (\tan\alpha)^2}$$

we find the cable force N in each of the straight cable parts:

$$N^{\mathrm{AC}} = (75\ \mathrm{kN})\sqrt{1 + (12/20)^2} = 87.5\ \mathrm{kN},$$

$$N^{\mathrm{CD}} = (75\ \mathrm{kN})\sqrt{1 + (8/20)^2} = 80.8\ \mathrm{kN},$$

$$N^{\mathrm{DB}} = (75\ \mathrm{kN})\sqrt{1 + (16/20)^2} = 96.0\ \mathrm{kN}.$$

The maximum cable force occurs in the steepest part DB:

$$N_{\mathrm{max}} = N^{\mathrm{DB}} = 96.0\ \mathrm{kN}.$$

The minimum cable force occurs in the shallowest part CD:

$$N_{\mathrm{min}} = N^{\mathrm{CD}} = 76.5\ \mathrm{kN}.$$

Check: N^{AC} and N^{DB} can also be found from the support reactions at A and B respectively:

$$N^{\mathrm{AC}} = \sqrt{A_{\mathrm{h}}^2 + A_{\mathrm{v}}^2} = \sqrt{(75\ \mathrm{kN})^2 + (45\ \mathrm{kN})^2} = 87.5\ \mathrm{kN},$$

$$N^{\mathrm{DB}} = \sqrt{B_{\mathrm{h}}^2 + B_{\mathrm{v}}^2} = \sqrt{(75\ \mathrm{kN})^2 + (60\ \mathrm{kN})^2} = 96.0\ \mathrm{kN},$$

Example 2

A uniformly distributed load of 9 kN/m acts over AC on the cable in Figure 14.13, of which the supports A and B are at equal elevations. At B, the cable runs over a frictionless pulley with negligible dimensions. A block of weight $G = 20$ kN is suspended at B.

Questions:

a. Determine the vertical component of the cable force in CB.
b. Determine the horizontal component of the cable force.
c. Determine the shape of the cable.
d. Determine the maximum and minimum forces in the cable and the places where these occur.
e. Determine the support reactions at A and B.

Solution:

a. Figure 14.14a shows a beam AB with the same span and load as the cable. Figures 14.14b and 14.14c also show the associated bending moment diagram and shear force diagram, with various details. The calculation is left to the reader.

$$V^{\text{CB}} = 12 \text{ kN}.$$

b. Since the pulley at B is frictionless, the following applies:

$$N^{\text{CB}} = \sqrt{H^2 + (V^{\text{CB}})^2} = G.$$

From this, we can find the horizontal component H of the cable force:

$$H = \sqrt{G^2 - (V^{\text{CB}})^2} = \sqrt{(20 \text{ kN})^2 - (12 \text{ kN})^2} = 16 \text{ kN}.$$

c. The cable has the same shape as the bending moment diagram in Figure 14.14b. According to (11):

$$H z_{\text{k}} = M$$

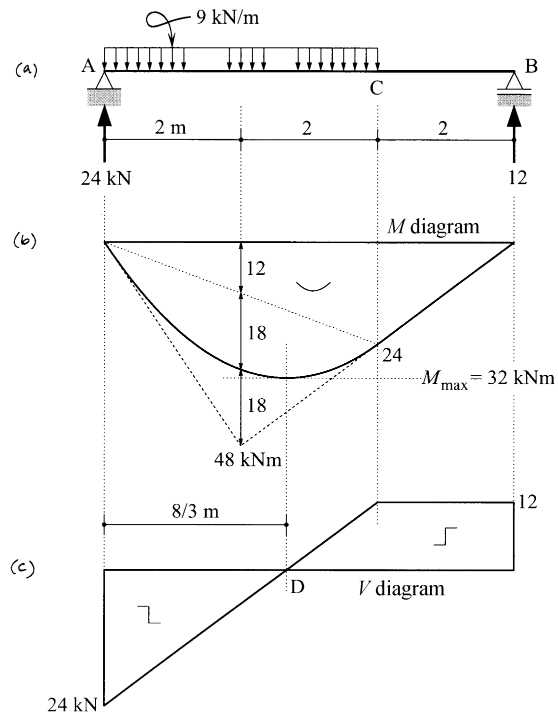


Figure 14.14 (a) A simply supported beam with the same span and load as the cable in Figure 14.13, with the associated (b) bending moment diagram and (c) shear force diagram.

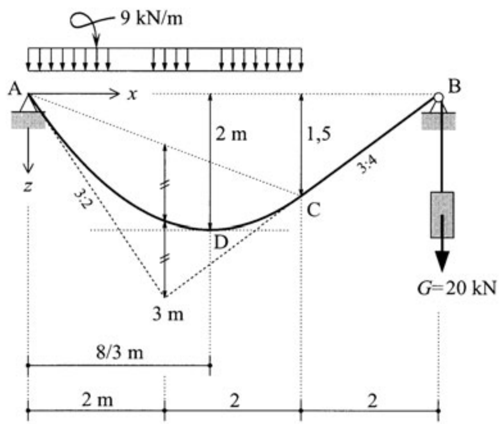


Figure 14.15 The cable is parabolic over AC. To sketch the shape of the cable, we can use the same auxiliary lines when drawing a parabolic bending moment diagram.

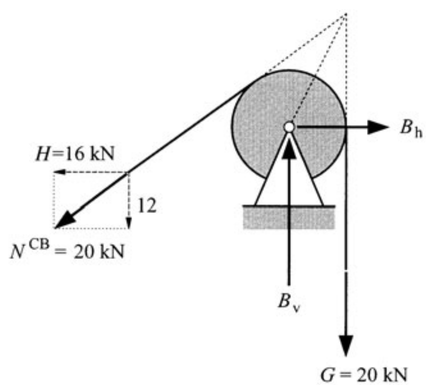


Figure 14.16 The forces acting in B on the trolley.

we find

$$z_{k;C} = z_C = \frac{M_C}{H} = \frac{24 \text{ kNm}}{16 \text{ kN}} = 1.5 \text{ m}.$$

In D, the bending moment is a maximum, and the cable sags most:

$$z_{k;D} = z_D = \frac{M_D}{H} = \frac{32 \text{ kNm}}{16 \text{ kN}} = 2 \text{ m}.$$

The cable shape over AC is parabolic. The auxiliary lines for drawing a parabolic M diagram (see Section 12.1.6) can also be used to draw the cable shape (see Figure 14.15).

d. The cable force is a maximum where the slope of the cable is a maximum. This is at A, as shown in Figure 14.15. With (5)

$$N = H\sqrt{1 + (\tan\alpha)^2}$$

we find

$$N_{max} = N_A = (16 \text{ kN})\sqrt{1 + (3/2)^2} = 28.84 \text{ kN}.$$

The cable force is a minimum at D, where the cable is horizontal and $V = 0$:

$$N_{min} = H = 16 \text{ kN}$$

e. The horizontal support reaction at A is equal to H:

$$A_h = H = 16 \text{ kN} (\leftarrow).$$

Since the cable supports are at equal elevations, the vertical support reaction at A is equal to the support reaction of the beam in Figure 14.14a:

$$A_\mathrm{v} = 24 \ \text{kN} \ (\uparrow).$$

Check: Figure 14.16 shows the forces acting on the pulley at B. The forces N^CB and G, both 20 kN, are known. The force equilibrium can be used to find the support reactions at B:

$$B_\mathrm{h} = H = 16 \ \text{kN} \ (\rightarrow),$$

$$B_\mathrm{v} = (12 \ \text{kN}) + (20 \ \text{kN}) = 32 \ \text{kN}) \ (\uparrow).$$

Check: The horizontal support reaction at B is equal to H. The vertical support reaction is equal to the support reaction at B of the beam in Figure 14.14a, increased with the vertical cable force G.

The support reactions at A and B are shown in Figure 14.17.

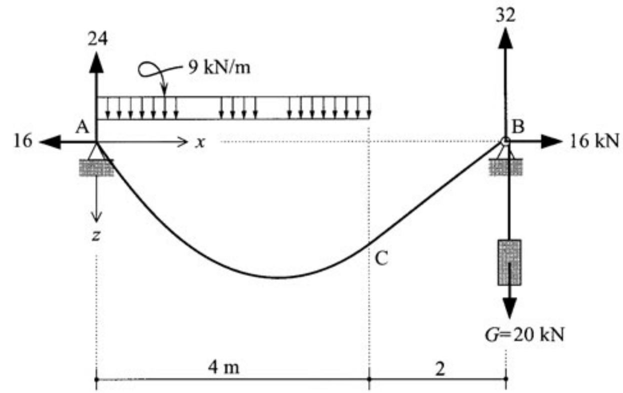


Figure 14.17 The support reactions in A and B.

14.1.3 Cable equation

Figure 14.18 shows a cable subject to a distributed load $q_z = q_z(x)$. The cable shape is $z = z(x)$.

In Figure 14.19, a small cable element of length Δx has been isolated from the deformed cable and blown up. As $\Delta x \to 0$ the distributed load q_z on the cable element can be considered to be uniformly distributed.

Assume the cable force at the left-hand section is a tensile force N, with a horizontal component H and a vertical component V.[1] The cable force at the right-hand section could have changed with respect to magnitude and

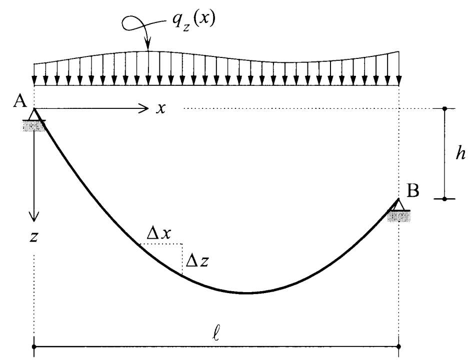


Figure 14.18 Cable with distributed load (force per horizontally measured length).

[1] V is not the transverse force in the cable, but the vertical component of the tensile force N. Instead of H and V we could also formally have written N_h and N_v.

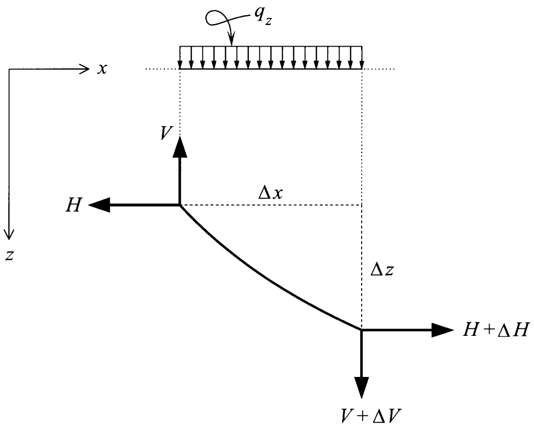


Figure 14.19 The enlarged element isolated from the cable.

direction. Assume for the right-hand part that the tensile force in the cable has the components $H + \Delta H$ and $V + \Delta V$.

There are three equilibrium equations for the cable element:

$$\sum F_x = -H + (H + \Delta H) = 0,$$

$$\sum F_z = -V + (V + \Delta V) + q_z \Delta x = 0,$$

$$\sum T_y|\text{right-hand section} = +H\Delta z - V\Delta x + q_z \Delta x \left(\tfrac{1}{2}\Delta x\right) = 0.$$

In the last equilibrium equation, the quadratic term in Δx is a degree smaller than the linear term in Δx and can be ignored as $\Delta x \to 0$. This leads to

$$\Delta H = 0,$$

$$\Delta V + q_z \Delta x = 0,$$

$$H\Delta z - V\Delta x = 0.$$

Divide each of these equations by Δx and proceed to the limit $\Delta x \to 0$; we generate three differential equations:

$$\frac{\mathrm{d}H}{\mathrm{d}x} = 0, \tag{12}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q_z, \tag{13}$$

$$H\frac{\mathrm{d}z}{\mathrm{d}x} = V. \tag{14}$$

It follows from equation (12) that

$$H = \text{constant}.$$

The horizontal component H of cable force N is constant, or in other words, independent of x. This is in line with what we derived earlier in Section 14.1.1 for a cable subject to a system of vertical forces.

From equation (14) we find that the cable force N is directed along the tangent to the cable, as in Figure 14.20:

$$\tan\alpha = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{V}{H}.$$

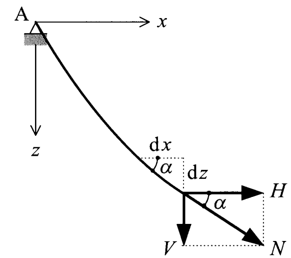


Figure 14.20 The cable force N is directed along the tangent to the cable: $\tan\alpha = \mathrm{d}z/\mathrm{d}x = V/H$.

The tensile force N in the cable is therefore

$$N = \sqrt{H^2 + V^2} = H\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2} = H\sqrt{1 + (\tan\alpha)^2}. \tag{15}$$

Tensile force N is largest where the slope $\mathrm{d}z/\mathrm{d}x$ of the cable is largest.

If we differentiate (14), in which H is constant, we find

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{\mathrm{d}V}{\mathrm{d}x}.$$

By substituting (13) in the equation above, we arrive at the so-called *cable equation*:

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -q_z. \tag{16}$$

This differential equation, derived from the equilibrium of a cable element, forms a relationship between the horizontal component H of the cable force, the cable shape $z = z(x)$, and the distributed load $q_z = q_z(x)$.

The cable shape for a certain load $q_z = q_z(x)$ is the function $z = z(x)$ that satisfies the cable equation and the boundary conditions at the ends where the cable is suspended. In order to solve the cable equation, we have to know H. Sometimes H is not given, while the length of the cable is known. Finding the solution is far more complicated in that case.

Hereafter, we assume that the horizontal component H is known.

In Section 14.1.4, using the cable equation as basis, we determine the cable shape under a uniformly distributed load (*force per horizontally measured length*). The associated cable shape is a *parabola*.

In Section 14.1.5, we calculate the cable shape due to its dead weight (*force per length measured along the cable*). The shape of the cable under its dead weight is a *catenary*.

14.1.4 Cable with uniformly distributed load; parabola

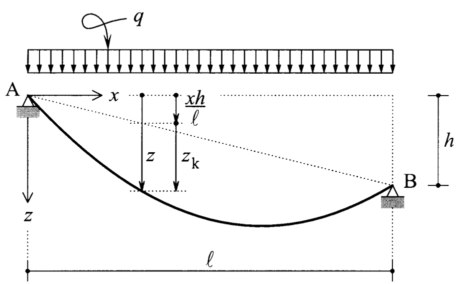


Figure 14.21 Cable with a uniformly distributed load q (force per horizontally measured length).

In Figure 14.21, cable AB, with span ℓ, carries a uniformly distributed load $q_z = q$. The difference in elevation of the supports A and B is h. From the cable equation we find

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -q.$$

After integrating once, we find

$$H\frac{\mathrm{d}z}{\mathrm{d}x} = -qx + C_1,$$

while after integrating once more we find

$$Hz = -\tfrac{1}{2}qx^2 + C_1 x + C_2.$$

The integration constants C_1 and C_2 follow from the boundary conditions at supports A and B:

$$x = 0;\ z = 0,$$
$$x = \ell;\ z = h.$$

Working out the boundary conditions gives

$$C_1 = H\frac{h}{\ell} + \tfrac{1}{2}q\ell,$$
$$C_2 = 0.$$

The cable shape is a parabola:

$$z = \frac{\tfrac{1}{2}qx(\ell - x)}{H} + \frac{h}{\ell}x.$$

This can be denoted as

$$z = z_\mathrm{k} + \frac{h}{\ell}x$$

in which z_k is the distance from the chord to the cable (see Figure 14.21):

$$z_\mathrm{k} = \frac{\tfrac{1}{2}qx(\ell - x)}{H} = \frac{M}{H}.$$

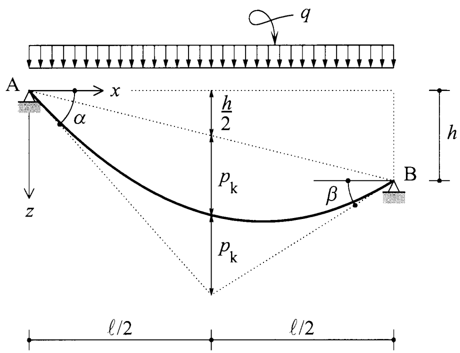


Figure 14.22 With uniformly distributed loads, the cable assumes the shape of a parabola. At A and B the tangents to the parabola are shown.

$M = \frac{1}{2}qx(\ell - x)$ is the bending moment in a simply supported beam with a uniformly distributed load (see Section 10.2.2, Example 1). The cable has the same parabolic shape under the chord as the M diagram; the *scale factor* is H.

Assume p_k is the sag of the parabola under the chord, that is the distance between the parabola and the chord at the middle of the span ℓ (see Figure 14.22):

$$p_\mathrm{k} = z_\mathrm{k}\left(x = \tfrac{1}{2}\ell\right) = \frac{\frac{1}{8}q\ell^2}{H}. \tag{17a}$$

If the sag p_k of the parabola under the chord is given, the horizontal component of the cable force follows from

$$H = \frac{\frac{1}{8}q\ell^2}{p_\mathrm{k}}. \tag{17b}$$

The slope of the cable is then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\frac{1}{2}q\ell}{H} - \frac{qx}{H} + \frac{h}{\ell}. \tag{18}$$

At the supports A ($x = 0$) and B ($x = \ell$) the slope is

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_\mathrm{A} = \frac{h}{\ell} + \frac{\frac{1}{2}q\ell}{H},$$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_\mathrm{B} = \frac{h}{\ell} - \frac{\frac{1}{2}q\ell}{H}.$$

Check: These expressions can also be determined directly from Figure 14.22, where the tangents to the parabola at A and B are shown. Using (17a) we find

$$\tan\alpha = +\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{\mathrm{A}} = \frac{2p_{\mathrm{k}} + \frac{1}{2}h}{\frac{1}{2}\ell} = \frac{\frac{1}{2}q\ell}{H} + \frac{h}{\ell}, \tag{19a}$$

$$\tan\beta = -\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{\mathrm{B}} = \frac{2p_{\mathrm{k}} - \frac{1}{2}h}{\frac{1}{2}\ell} = \frac{\frac{1}{2}q\ell}{H} - \frac{h}{\ell}, \tag{19b}$$

The maxixum sag in the cable appears where $\mathrm{d}z/\mathrm{d}x = 0$. Here the parabola has its vertex. Equation (18) gives

$$x_{\text{vertex}} = \tfrac{1}{2}\ell + \frac{Hh}{q\ell} \tag{20a}$$

or, using (17b)

$$x_{\text{vertex}} = \tfrac{1}{2}\ell + \frac{\frac{1}{8}hl}{p_{\mathrm{k}}}. \tag{20b}$$

If we select the coordinate system at the vertex of the parabola, as in Figure 14.23, the formulas are far easier. With the boundary conditions ($x = 0$; $z = 0$) and ($x = 0$; $\mathrm{d}z/\mathrm{d}x = 0$) the cable shape is

$$z = -\frac{\frac{1}{2}qx^2}{H}. \tag{21}$$

The slope of the cable is then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{qx}{H}. \tag{22}$$

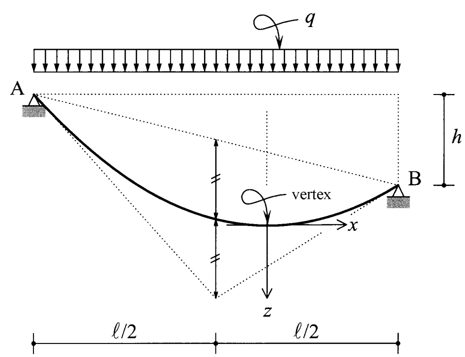


Figure 14.23 The origin of the xz coordinate system chosen at the vertex of the parabola.

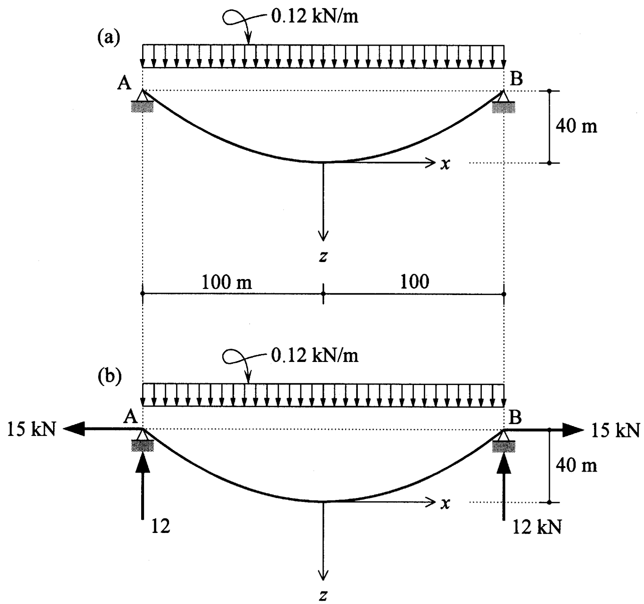


Figure 14.24 (a) Cable with the end supports at equal elevations, subject to a uniformly distributed load (force per horizontally measured length). (b) Support reactions.

The vertical component of the cable force is

$$V = H\frac{\mathrm{d}z}{\mathrm{d}x} = -qx. \tag{23}$$

For the tensile force in the cable we find

$$N = \sqrt{H^2 + V^2} = \sqrt{H^2 + (qx)^2}. \tag{24}$$

The use of these formulas is illustrated using an example.

Example

The cable in Figure 14.24a, with the end supports A and B at equal elevations, has a span of 200 metres and a sag of 40 metres at midspan. The cable carries a uniformly distributed load of 0.12 kN/m.

Questions:

a. Determine the horizontal component H of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

Solution:

a. The cable shape is a parabola of which the maximum sag, on the basis of symmetry, is at midspan. If we set the origin of the coordinate system here, then in accordance with (21)

$$H = -\frac{\frac{1}{2}qx^2}{z}.$$

Using the known coordinates of B ($x = 100$ m; $z = -40$ m) we find

$$H = -\frac{\frac{1}{2}(0.12\ \mathrm{kN/m})(100\ \mathrm{m})^2}{(-40\ \mathrm{m})} = 15\ \mathrm{kN}.$$

Of course we could also use the coordinates of A.

b. The horizontal support reactions at A and B are equal to H (see Figure 14.24b). The vertical support reactions A_v and B_v follow from the equilibrium of the cable as a whole. On the basis of symmetry, each support carries half of the total load:

$$A_v = B_v = \tfrac{1}{2}(200 \text{ m})(0.12 \text{ kN/m}) = 12 \text{ kN } (\uparrow).$$

c. According to (24), the tensile force N in the cable is

$$N = \sqrt{H^2 + (qx)^2}.$$

The tensile force is a maximum at the supports A and B, with $x = \pm 100$ m

$$N_{\max} = \sqrt{(15 \text{ kN})^2 + \{(0.12 \text{ kN/m})(\pm 100 \text{ m})\}^2} = 19.2 \text{ kN}.$$

Check:

$$N_A = N_{\max}\sqrt{A_h^2 + A_v^2} = \sqrt{(15 \text{ kN})^2 + (12 \text{ kN})^2} = 19.2 \text{ kN}.$$

Of course the same applies at B.

14.1.5 Cable subject to its dead weight; catenary

Figure 14.25 shows a cable under its uniformly distributed dead weight g. In the cable equation

$$H\frac{d^2z}{dx^2} = -q.$$

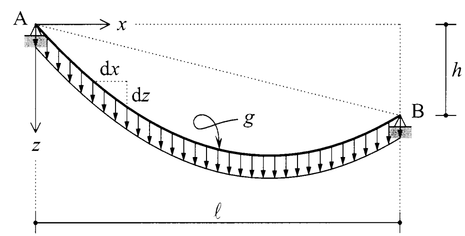


Figure 14.25 Cable loaded by its dead weight g (force per length measured along the cable).

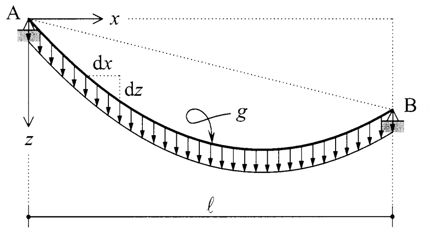


Figure 14.25 Cable loaded by its dead weight g (force per length measured along the cable).

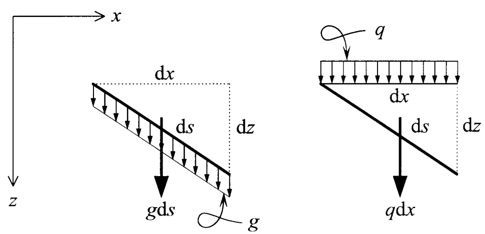


Figure 14.26 Replacing g (force per length measured along the cable) by q (force per horizontally measured length): $q = g\,\mathrm{d}s/\mathrm{d}x$.

q is a *vertical force per horizontally measured length*. The dead weight g of the cable is a *vertical force per length measured along the cable*.[1] In order to replace the dead weight g by the load q, Figure 14.26 shows an infinitesimally small cable element with length $\mathrm{d}s$. From the figure we find

$$g\,\mathrm{d}s = q\,\mathrm{d}x$$

and so

$$q = g\frac{\mathrm{d}s}{\mathrm{d}x} = g\frac{\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}z)^2}}{\mathrm{d}x} = g\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}.$$

The cable equation is now:

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -g\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}. \tag{25}$$

To solve this second degree differential equation we assume:[2]

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \sinh u \tag{26}$$

---

1 The symbol g is used for the dead weight of the cable, instead of the formal q_{dw}. By doing so, we avoid the recurring index “dw” and maintain the distinction with q (force per horizontally measured length). There should not be any confusion with the gravitational field strength g in this section.

2 The hyperbolic functions $\sinh(u)$ and $\cosh(u)$ are defined as follows:

$$\sinh(u) = \tfrac{1}{2}(\mathrm{e}^{+u} - \mathrm{e}^{-u}),$$

$$\cosh(u) = \tfrac{1}{2}(\mathrm{e}^{+u} + \mathrm{e}^{-u}).$$

in which u is a new variable. Substitute (26) in (25):

$$H\frac{\mathrm{d}}{\mathrm{d}x}(\sinh u) = -g\sqrt{1+\sinh^2 u}.$$

Calculating the terms on both sides of the equals sign gives

$$H\cosh u \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = -g\cosh u$$

so that

$$H\frac{\mathrm{d}u}{\mathrm{d}x} = -g.$$

By integrating this first degree equation in u we find

$$u = -\frac{gx}{H} + C_1.$$

Substitution in (26) gives

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \sinh u = \sinh\left(-\frac{gx}{H} + C_1\right). \tag{27}$$

Integrating with respect to x gives

$$z = -\frac{H}{g}\cosh\left(-\frac{gx}{H} + C_1\right) + C_2. \tag{28}$$

The integration constants C_1 and C_2 follow from the boundary conditions.

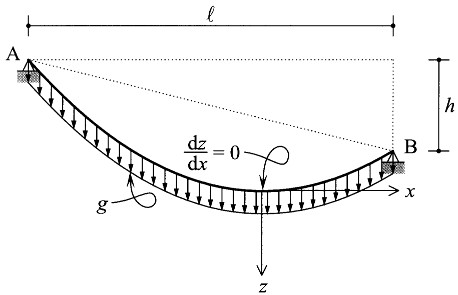


Figure 14.27 The origin of the xz coordinate system chosen at the point where $dz/dx = 0$.

If we choose the origin of the coordinate system at the point in the cable where $dz/dx = 0$, the boundary conditions are (see Figure 14.27)

$$x = 0; \quad z = 0,$$

$$x = 0; \quad \frac{dz}{dx} = 0.$$

With these boundary conditions, (27) and (28) give $C_1 = 0$ and $C_2 = H/g$ and the cable shape is[1]

$$z = -\frac{H}{g}\left(\cosh\frac{gx}{H} - 1\right). \tag{29}$$

The slope of the cable is

$$\frac{dz}{dx} = -\sinh\frac{gx}{H}. \tag{30}$$

The vertical component of the cable force is

$$V = H\frac{dz}{dx} = -H\sinh\frac{gx}{H}. \tag{31}$$

The tensile force in the cable is

$$N = \sqrt{H^2 + V^2} = H\sqrt{1 + \sinh^2\frac{gx}{H}} = H\cosh\frac{gx}{H}. \tag{32a}$$

---

1 Hereafter, we use the properties $\cosh(-u) = \cosh(+u)$ and $\sinh(-u) = -\sinh(+u)$.

According to (29)

$$H \cosh \frac{gx}{H} = H - gz$$

so that the tensile force can also be written as

$$N = H - gz. \tag{32b}$$

The use of the derived formulas is illustrated by an example.

Example
The cable in Figure 14.28a, of which the supports A and B are at equal elevations, has a span of 200 metres and a sag of 40 metres at the middle of the span. The cable is carrying only its dead weight of 0.12 kN/m.

Questions:
a. Determine the horizontal component H of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.

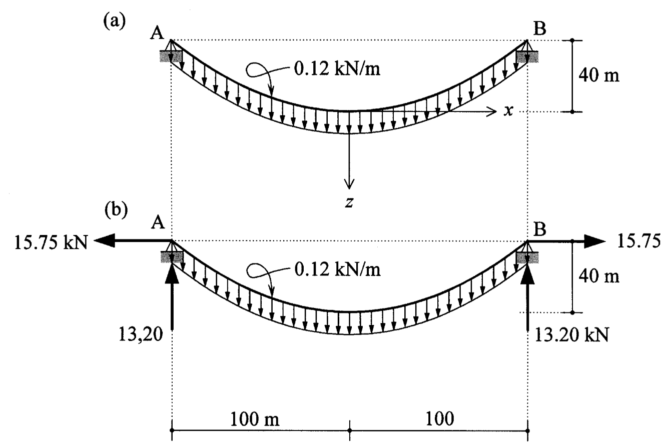


Figure 14.28 (a) Cable with the end supports at equal elevations, loaded by its dead weight (force per length measured along the cable). (b) Support reactions.

Solution:
a. On the basis of symmetry, the cable is horizontal at midspan. If we assume here the origin of the coordinate system, the cable shape according to (29) would be

$$z = -\frac{H}{g}\left(\cosh \frac{gx}{H} - 1\right).$$

By substituting the known coordinates of A or B, we obtain an equation that allows us to calculate H. With the coordinates of B ($x = 100$ m; $z = -40$ m), for example, we find

$$(-40 \text{ m}) = -\frac{H}{(0.12 \text{ kN/m})}\left(\cosh \frac{(0.12 \text{ kN/m})(100 \text{ m})}{H} - 1\right).$$

Table 14.1

H (kN)	f_1	f_2
15.00	1.320	1.337
15.25	1.315	1.326
15.50	1.310	1.315
15.75	1.305	1.305
16.00	1.300	1.295

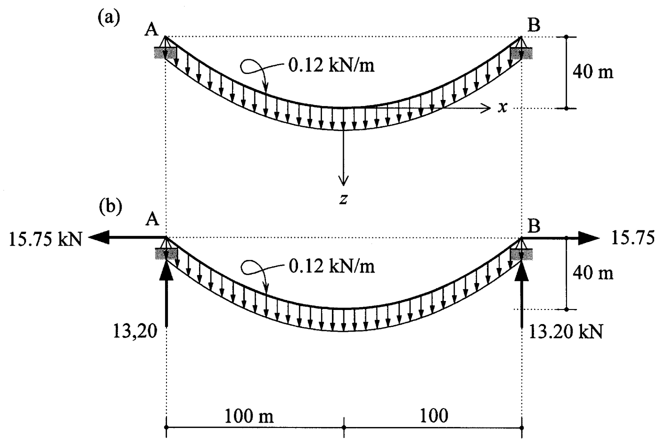


Figure 14.28 (a) Cable with the end supports at equal elevations, loaded by its dead weight (force per length measured along the cable). (b) Support reactions.

This can be converted into

$$1 + \frac{4.8 \text{ kN}}{H} = \cosh \frac{12 \text{ kN}}{H}.$$

To solve H we assume

$$f_1 = 1 + \frac{4.8 \text{ kN}}{H}$$

and

$$f_2 = \cosh \frac{12 \text{ kN}}{H}.$$

For various values of H we now calculate the function values f_1 and f_2. The calculation is performed in Table 14.1.

We are looking for the value of H for which $f_1 = f_2$. The table gives

$$H = 15.75 \text{ kN}.$$

b. According to (32b), the tensile force in the cable is

$$N = H - gz.$$

The tensile force is a maximum at A and B, where $z = -40$ m:

$$N_{\max} = (15.75 \text{ kN}) - (0.12 \text{ kN/m})(-40 \text{ m}) = 20.55 \text{ kN}.$$

c. The horizontal support reactions are equal to the horizontal component H of the tensile force in the cable (see Figure 14.28b):

$$A_h = B_h = H = 15.75 \text{ kN}.$$

The vertical support reactions at A and B are equal to the vertical component V of the tensile force in the cable. According to (31)

$$V = -H \sinh \frac{gx}{H}.$$

At the supports, with $x = \pm 100$ m, we find (see Figure 14.28b)

$$A_\mathrm{v} = B_\mathrm{v} = |V_{x=\pm 100\ \mathrm{m}}|$$
$$= \left| -(15.75\ \mathrm{kN}) \sinh \frac{(0.12\ \mathrm{kN/m})(\pm 100\ \mathrm{m})}{15.75\ \mathrm{kN}} \right| = 13.20\ \mathrm{kN}\ (\uparrow).$$

Note: If we replace the uniformly distributed *load along the cable* by an equal uniformly distributed *load along the chord*, we obtain the situation of the example in Figure 14.24 (see Section 14.1.4). In that case the cable shape is a parabola. In Table 14.2, the results are compared for a parabola and a catenary, both with $\ell = 200$ m and $p_\mathrm{k} = 40$ m.

The differences are relatively minor. In the example, the ratio between the sag and span is

$$\frac{p_\mathrm{k}}{\ell} = \frac{40\ \mathrm{m}}{200\ \mathrm{m}} = 0.2.$$

The differences decrease sharply as the ratio p_k/ℓ decreases.

For *taut cables* ($p_\mathrm{k}/\ell \ll 1$), the catenary can be approximated by a parabola, for which the distributed load *along the cable* is replaced by an equal distributed load *along the chord* (see Figure 14.29).

Table 14.2

forces in kN	parabola	catenary
H	15	15.75
$V_{\max}$	12	13.20
$N_{\max}$	19.2	20.55

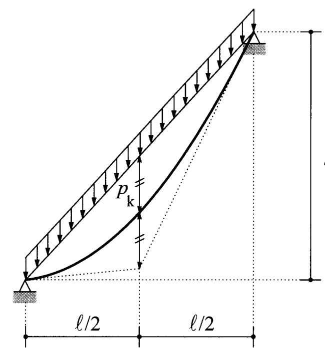


Figure 14.29 For taut cables ($p_\mathrm{k}/\ell \ll 1$) the distributed load along the cable can be approximated by an equal distributed load along the chord.

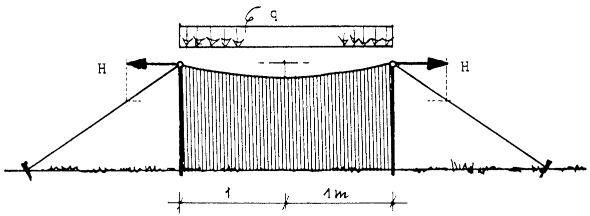


Figure 14.30 The load on the ridge of a shelter.

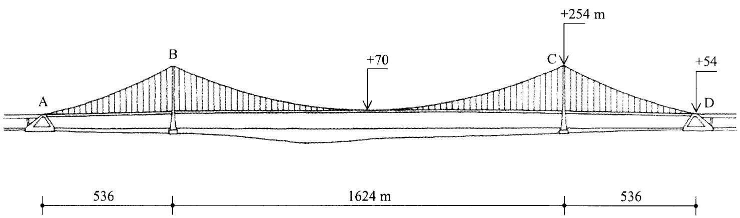


Figure 14.31 Suspension bridge over the Storebaelt (Large Belt) in Denmark.

14.1.6 Examples

Example 1

Figure 14.30 represents the longitudinal section of a shelter. The vertical load on the ridge is $q = 125$ N/m. The horizontal component of the force that the guys exert on the ridge is $H = 500$ N.

Question:
Determine the sag of the ridge in the middle of the shelter.

Solution:
The following applies for the sag:

$$p_k = \frac{\frac{1}{8}q\ell^2}{H} = \frac{\frac{1}{8}(125\text{ N/m})(2\text{ m})^2}{500\text{ N}} = 125\text{ mm}.$$

Example 2

The dimensions for the suspension bridge in Figure 14.31 are derived from the bridge over the Storebaelt (Large Belt) in Denmark. The load on the cable, consisting of the dead weight of the cable, bridge deck and traffic load is set at 250 kN/m. The structure is designed in such a way that there is no bending in the towers.

Questions:

a. Determine the horizontal component H of the cable force in middle span BC.
b. Determine the maximum cable force in the middle span.
c. Determine the forces that cables BC and CD at C exert on the tower.
d. Determine the forces that cable CD at D exerts on the foundation block.
e. Determine the maximum cable force in end span CD.
f. Determine the ratio p_k/ℓ for the middle span and the end spans.

Solution:
Since the load is a force per horizontally measured length, the cable shapes in the middle span and the end spans are parabolas.

a. For middle span BC

$$p_k^{BC} = (254\text{ m}) - (70\text{ m}) = 184\text{ m}.$$

This gives

$$H^{BC} = \frac{\frac{1}{8}q(\ell^{BC})^2}{p_k^{BC}} = \frac{\frac{1}{8}(250\text{ kN/m})(1624\text{ m})^2}{184\text{ m}} = 448\text{ MN}.$$

b. The vertical component of the tensile force in cable BC is a maximum at B and C, as the cable is steepest here:

$$V_{max}^{BC} = \tfrac{1}{2}q\ell^{BC} = \tfrac{1}{2}(250\text{ kN/m})(1624\text{ m}) = 203\text{ MN}.$$

The cable force is also a maximum here:

$$N_{max}^{BC} = \sqrt{(H^{BC})^2 + (V_{max}^{BC})^2}$$
$$= \sqrt{(448\text{ MN})^2 + (203\text{ MN})^2} = 492\text{ MN}.$$

c. In Figure 14.32, cables BC and CD have been isolated at C from the tower. If there is no bending in the tower, the resulting horizontal force on the tower must be zero. This means that the horizontal component of the cable force in an end span is equal to that in the middle span:

$$H^{CD} = H^{BC} = 448\text{ MN}.$$

In Figure 14.33, cable CD has been isolated. The resultant R of the uniformly distributed load is

$$R = (250\text{ kN/m})(536\text{ m}) = 134\text{ MN}.$$

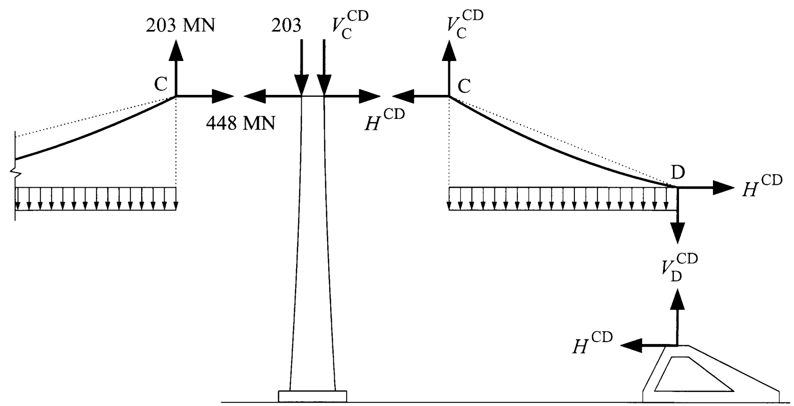


Figure 14.32 Cables BC and CD isolated from tower C and foundation block D.

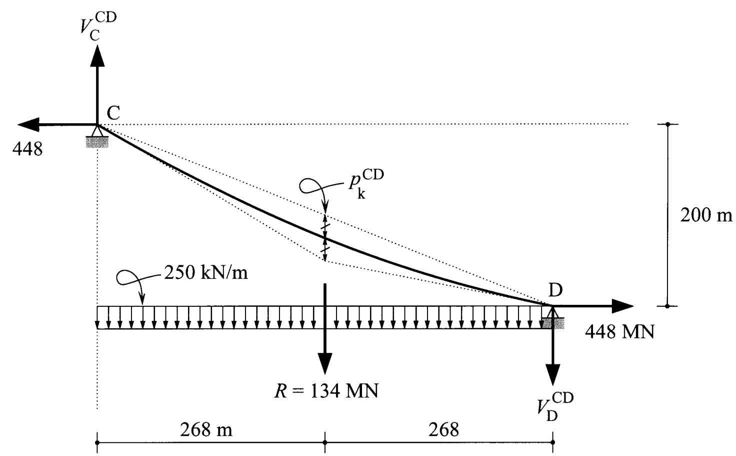


Figure 14.33 The isolated cable CD.

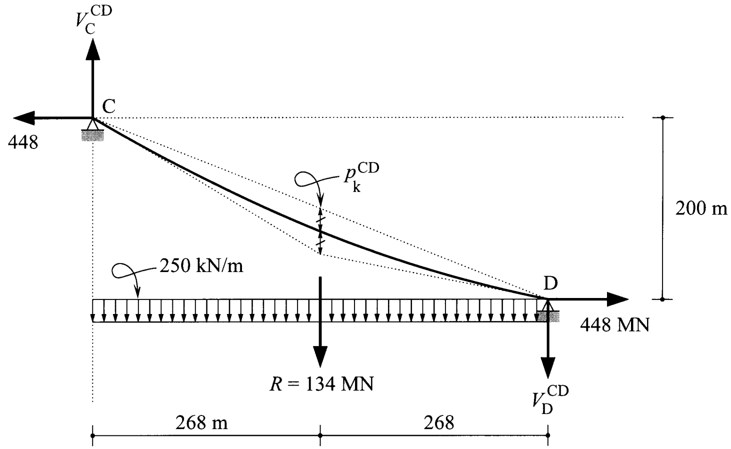


Figure 14.33 The isolated cable CD.

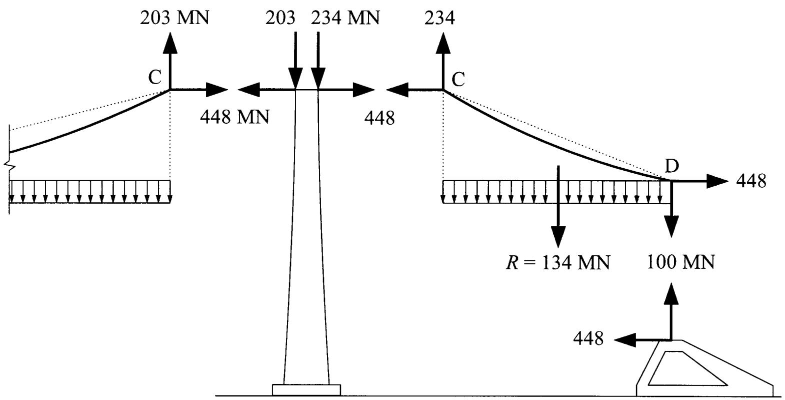


Figure 14.34 The forces on tower C and foundation block D.

The moment equilibrium of cable CD gives

$$\sum T|\mathrm{D} \circlearrowleft = (448\text{ MN})(200\text{ m}) - V_C^{CD}(536\text{ m}) + (134\text{ MN})(268\text{ m}) = 0$$

so that

$$V_C^{CD} = \frac{(448\text{ MN})(200\text{ m}) + (134\text{ MN})(268\text{ m})}{536\text{ m}} = 234\text{ MN}.$$

Next we find from the vertical force equilibrium

$$V_C^{CD} = V_C^{CD} - R = (234\text{ MN}) - (134\text{ MN}) = 100\text{ MN}.$$

In Figure 14.34, cables BC and CD have been isolated from the tower at C and the foundation block at D. The tower is loaded at C by a vertical compressive force:

$$V_C^{BC} + V_C^{CD} = (203\text{ MN}) + (234\text{ MN}) = 437\text{ MN}.$$

d. The foundation block in D is subject to a horizontal force of 448 MN and an upward vertical force of 100 MN (see Figure 14.34).

e. The maximum force in cable CD occurs at C, where the cable is steepest:

$$N_{max}^{CD} = \sqrt{(H^{CD})^2 + (V_{max}^{CD})^2} = \sqrt{(448\text{ MN})^2 + (234\text{ MN})^2} = 505\text{ MN}.$$

f. For middle span BC, it applies that

$$\frac{p_{\rm k}^{\rm BC}}{\ell^{\rm BC}} = \frac{184\ \text{m}}{1624\ \text{m}} = 0.113.$$

The maximum (vertically measured) distance from cable CD to the chord is

$$p_{\rm k}^{\rm CD} = \frac{\frac{1}{8}q(\ell^{\rm CD})^2}{H^{\rm CD}} = \frac{\frac{1}{8}(250\ \text{kN/m})(536\ \text{m})^2}{448 \times 10^3\ \text{kN}} = 20\ \text{m}$$

so that for the end spans the following applies:

$$\frac{p_{\rm k}^{\rm CD}}{\ell^{\rm CD}} = \frac{20\ \text{m}}{536\ \text{m}} = 0.037.$$

Example 3

Cable AB in Figure 14.35 has a span of 60 m and is carrying a number of pipelines with a total weight of 20 kN/m. The difference in elevation of the end supports at A and B is 12 m. C is the lowest point of the cable and is 4 m below B.

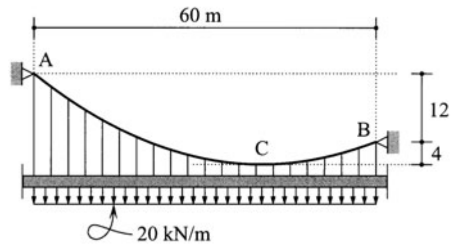


Figure 14.35 Cable AB is carrying a number of pipelines with a weight of 20 kN/m.

Questions:

a. Determine the horizontal component of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

Solution:

a. We can assume a coordinate system at A or C, and use the formulas derived above. Here we will use a different approach. In Figure 14.36, cable parts AC and BC have been isolated and all acting forces are shown. At C, the cable is horizontal and only force H acts. The lengths $\ell_{\rm A}$ and $\ell_{\rm B}$ are still unknown. The moment equilibrium of AC about A gives

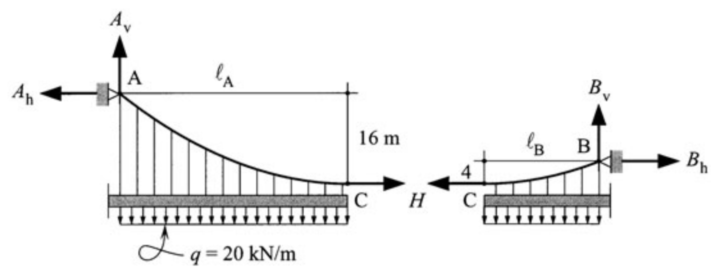


Figure 14.36 Cable parts AC and BC isolated at the lowest point C.

$$H \times (16\ \text{m}) = \tfrac{1}{2}q\ell_{\rm A}^2. \tag{a}$$

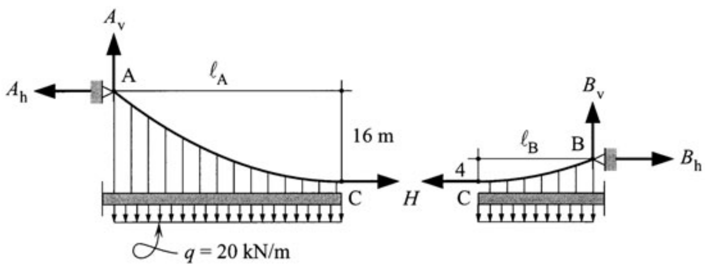


Figure 14.36 Cable parts AC and BC isolated at the lowest point C.

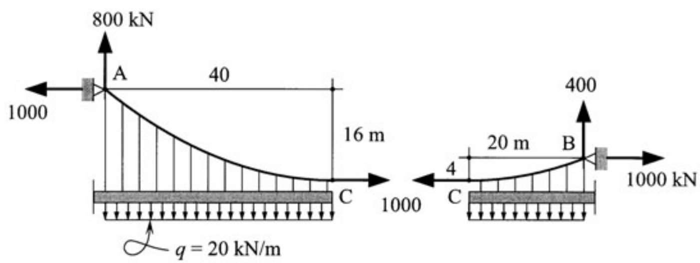


Figure 14.37 The isolated cable sections AC and BC with their dimensions and support reactions.

The moment equilibrium of BC about B gives

$$H \times (4 \text{ m}) = \tfrac{1}{2} q \ell_B^2. \qquad \text{(b)}$$

From (a) and (b) we can derive

$$\frac{\ell_A^2}{\ell_B^2} = 4 \Rightarrow \ell_A = 2\ell_B.$$

With $\ell_A + \ell_B = 60$ m we find

$$\ell_A = 40 \text{ m},$$
$$\ell_B = 20 \text{ m}.$$

Substituting $\ell_A = 40$ m in (a) gives, with $q = 20$ kN/m,

$$H = \frac{\tfrac{1}{2} q \ell_A^2}{16 \text{ m}} = \frac{\tfrac{1}{2}(20 \text{ kN/m})(40 \text{ m})^2}{16 \text{ m}} = 1000 \text{ kN}.$$

b. The horizontal support reactions follow from the horizontal force equilibrium of AC and CB (see Figure 14.37):

$$A_h = B_h = H = 1000 \text{ kN}.$$

The vertical support reactions follow from the vertical force equilibrium of AC and CB (see Figure 14.37):

$$A_v = q\ell_A = (20 \text{ kN/m})(40 \text{ m}) = 800 \text{ kN},$$
$$B_v = q\ell_B = (20 \text{ kN/m})(20 \text{ m}) = 400 \text{ kN}.$$

c. The maximum cable force occurs at A, where the slope of the cable is steepest:

$$N_{\max} = N_A = \sqrt{A_h^2 + A_v^2} = \sqrt{(1000\text{ kN})^2 + (800\text{ kN})^2} = 1281\text{ kN}.$$

Example 4

A boat has cast its anchor in 30 m deep water (see Figure 14.38). The horizontal force that the boat exerts on the anchor chain is 3.5 kN. The anchor chain has a dead weight of 24 N/m. The upward water pressure on the chain is 3 N/m.

Question:

Determine the (horizontally measured) length ℓ for which the chain is free from the bottom, and the maximum force in the chain.

Solution:

The (uniformly) distributed load on the chain is equal to the dead weight minus the upward water pressure:

$$q = (24\text{ N/m}) - (3\text{ N/m}) = 21\text{ N/m}.$$

This load is a force per length measured along the chain. The anchor chain will therefore assume the shape of a catenary.[1] In Figure 14.39, the anchor chain has been isolated. At A, the cable is tangent to the bottom, and only a horizontal force H acts. The horizontal force equilibrium gives

$$H = 3.5\text{ kN}.$$

For the further calculation, we use the formulas derived in Section 14.1.5. For the catenary in the coordinate system given in Figure 14.39, it applies

[1] We ignore the fact that the upward water pressure is missing over the small part that the chain is above the water.

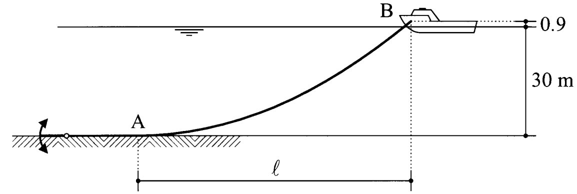


Figure 14.38 A boat has cast its anchor in 30 m deep water.

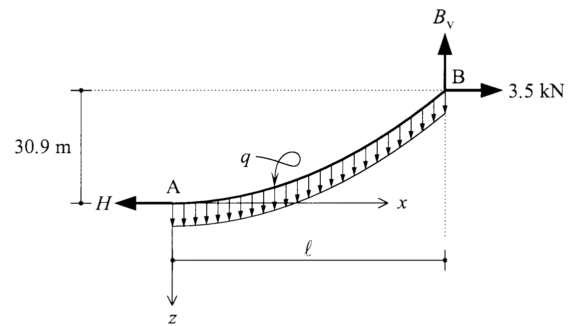


Figure 14.39 The isolated anchor chain with a uniformly distributed load along the chain. The anchor chain has the shape of a catenary.

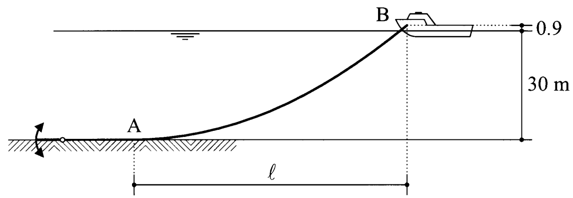


Figure 14.38 A boat has cast its anchor in 30 m deep water.

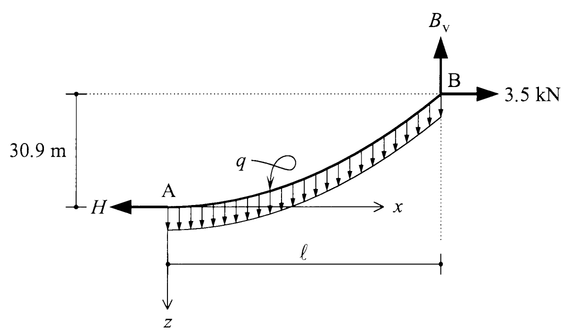


Figure 14.39 The isolated anchor chain with a uniformly distributed load along the chain. The anchor chain has the shape of a catenary.

that

$$z = -\frac{H}{q}\left(\cosh\frac{qx}{H} - 1\right)$$

or

$$\cosh\frac{qx}{H} = 1 - \frac{qz}{H}$$

so that

$$x = \frac{H}{q}\cosh^{-1}\left(1 - \frac{qz}{H}\right).$$

At B, $x = \ell$; $z = -30.9$ m applies, which gives the following (be aware of the units!):

$$\ell = \frac{3500\ \text{N}}{21\ \text{N/m}}\cosh^{-1}\left(1 - \frac{(21\ \text{N/m})(-30.9\ \text{m})}{3500\ \text{N}}\right) = 100\ \text{m}$$

For the vertical force at B

$$B_\text{v} = -V_{x=\ell} = H\sinh\frac{q\ell}{H}$$
$$= (3500\ \text{N})\sinh\left(\frac{21\ \text{N/m})(100\ \text{m})}{3500\ \text{N}}\right) = 2228\ \text{N} \approx 2.23\ \text{kN}.$$

The maximum force in the anchor chain is

$$N_\text{max} = N_\text{B} = \sqrt{H^2 + B_\text{v}^2} = \sqrt{(3.5\ \text{kN})^2 + (2.23\ \text{kN})^2} = 4.15\ \text{kN}.$$

Alternative solution:
The load q is a *vertical force measured per length along the cable*. If the an-

chor chain is taut, this load can be approximated by an equal *vertical force per length measured along the chord* (see Figure 14.40). The associated shape of the anchor chain is then a parabola.

The resultant of the distributed load is

$$R = q\sqrt{\ell^2 + h^2}.$$

The moment equilibrium of the isolated chain gives

$$\sum T|\mathrm{B} = Hh - R \cdot \tfrac{1}{2}\ell = Hh - \tfrac{1}{2}q\ell\sqrt{\ell^2 + h^2} = 0$$

so that

$$\frac{2Hh}{q} = \ell\sqrt{\ell^2 + h^2}.$$

After squaring

$$\left(\frac{2Hh}{q}\right)^2 = \ell^2(\ell^2 + h^2)$$

we find

$$\ell^4 + h^2\ell^2 - \left(\frac{2Hh}{q}\right)^2 = 0.$$

With $h = 30.9$ m, $H = 3500$ N and $q = 21$ N/m this leads to

$$\ell^4 + (30.9\text{ m})^2\ell^2 - \left(\frac{2 \times (3500\text{ N})(30.9\text{ m})}{21\text{ N/m}}\right)^2 = 0$$

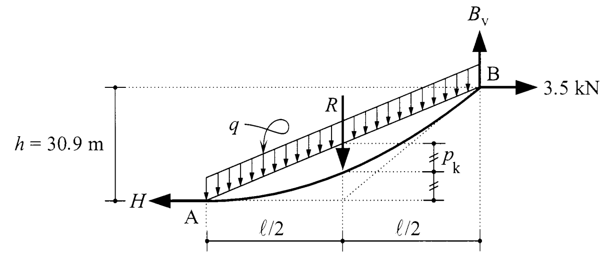


Figure 14.40 If the anchor chain is taut ($p_k/\ell \ll 1$), the distributed load along the chain can be replaced by an equal distributed load along the chord. The anchor chain now has the shape of a parabola.

so that

$$\ell^4 + (9.54.81\ \mathrm{m}^2)\ell^2 - (106.09 \times 10^6\ \mathrm{m}^4) = 0.$$

The solution is

$$\ell^2 = 9.834 \times 10^3\ \mathrm{m}^2,$$
$$\ell = 99.17\ \mathrm{m}.$$

We find for the vertical force at B:

$$B_\mathrm{v} = R = q\sqrt{\ell^2 + h^2}$$
$$= (21\ \mathrm{N/m})\sqrt{(99.17\ \mathrm{m})^2 + (30.9\ \mathrm{m})^2} = 2181\ \mathrm{N} \approx 2.18\ \mathrm{kN}.$$

This gives the maximum force in the chain:

$$N_\mathrm{max} = N_\mathrm{B} = \sqrt{H^2 + B_\mathrm{v}^2} = \sqrt{(3.5\ \mathrm{kN})^2 + (2.18\ \mathrm{kN})^2} = 4.13\ \mathrm{kN}.$$

The values found for ℓ and N_max deviate some 0.5% from those found using the exact calculation. The load *along the chord* (and a parabolic shape of the anchor chain) is therefore a good substitute for the load *along the anchor chain* (and a catenary). The ratio p_k/ℓ is

$$\frac{p_\mathrm{k}}{\ell} = \frac{(30.9\ \mathrm{m})/4}{99.17\ \mathrm{m}} = 0.078 \ll 1.$$

Example 5
A simply supported concrete beam with length $\ell = 12$ m and a rectangular cross-section $A = bh = 300 \times 800\ \mathrm{mm}^2$ carries a variable load $q_\mathrm{q} = 16$ kN/m (see Figure 14.41). The dead weight of concrete is 25 kN/m^3.

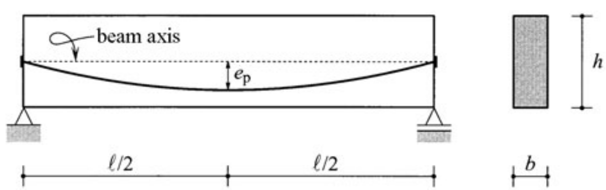


Figure 14.41 A concrete beam with a parabolic tendon.

Parabolic *post-tensioned cables* have been applied to the beam. After the concrete has been poured and hardened, the cables are tensioned by means of screw jacks, and anchored at the beam ends. The anchors are located at the beam axis. At midspan, the cables have an eccentricity of $e_p = 240$ mm with respect to the beam axis.

The (total) prestressing force in the cables is $F_p = 1050$ kN.[1]

Question:
Determine and draw the support reactions and the M diagram and V diagram the *prestressed beam* as a result of the following:

a. the dead weight only;
b. the dead weight and variable load.

Solution:
Before being tensioned, the cables are located in cylindrical canals. When the cables are tensioned, they will be pressed against the upper side of the canals (see Figure 14.42). The cables are in equilibrium because the beam exerts on the cables a uniformly distributed load q_p. In Figure 14.43 the two post-tensioned cables are replaced by a single tendon and all the forces acting on it are shown.

From the parabolic shape of the tendon we find

$$\tan\alpha = \frac{2e_p}{\frac{1}{2}\ell} = \frac{4e_p}{\ell} = \frac{4 \times (0.24\text{ m})}{12\text{ m}} = 0.08.$$

The prestressing force F_p has components:

$$F_{p;h} = F_p \cos\alpha = 0.0968 \times F_p = -0.9968 \times (1050\text{ kN}) = 1046.6\text{ kN},$$

$$F_{p;v} = F_p \sin\alpha = 0.0797 \times F_p = -0.0797 \times (1050\text{ kN}) = 83.7\text{ kN}.$$

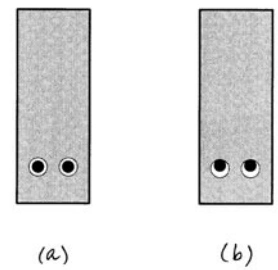


Figure 14.42 The location of the prestressing cables in their canals: (a) before and (b) after tensioning.

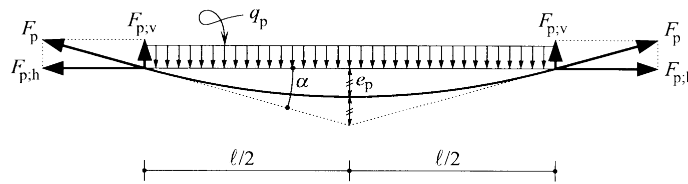


Figure 14.43 The isolated parabolic tendon with all the forces acting on it. The tangents to the tendon are shown at the ends.

[1] The index p refers to prestressing.

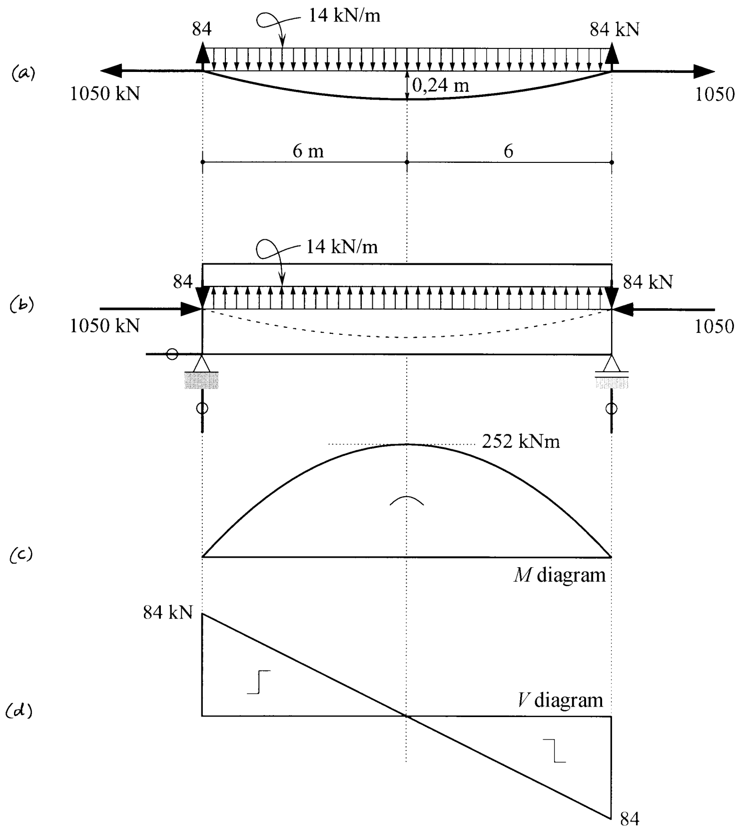


Figure 14.44 (a) The forces that the beam exerts on the tendon. (b) The forces that the tendon exerts on the beam. (c) The bending moment diagram and (d) shear force diagram of the beam due to the prestressing.

Since for prestressing cables in general $e_p/\ell \ll 1$, α is very small, so that we can make the following approximations (see Figure 14.43):

$$\cos\alpha \approx 1,$$

$$\sin\alpha \approx \tan\alpha = \frac{4e_p}{\ell} = 0.08.$$

In that case

$$F_{p;h} = F_p = 1050 \text{ kN},$$

$$F_{p;v} = \frac{4e_p}{\ell} F_p = 0.08 \times (1050 \text{ kN}) = 84 \text{ kN}.$$

We will use these values in further calculations.

In the tendon

$$F_{p;h}e_p = \tfrac{1}{8} q_p \ell^2$$

so that

$$q_p = \frac{8F_{p;h}e_p}{\ell^2} = \frac{8 \times (1050 \text{ kN})(0.24 \text{ m})}{12 \text{ m})^2} = 14 \text{ kN/m}.$$

In Figure 14.44a all the forces acting on the isolated tendon are shown again, this time with their values. All these forces are exerted by the concrete beam on the tendon: the concentrated forces via the anchors, the distributed forces directly via the beam. On the basis of the principle of action and reaction, the concrete beam is subject to equal and opposite forces (see Figure 14.44b). In Figures 14.44c and 14.44d, the associated M and V diagrams of the beam are shown. Since the forces form an equilibrium system (the vertical anchor forces are in equilibrium with the vertical distributed forces), the support reactions at A and B are zero, and not equal to the shear force here.

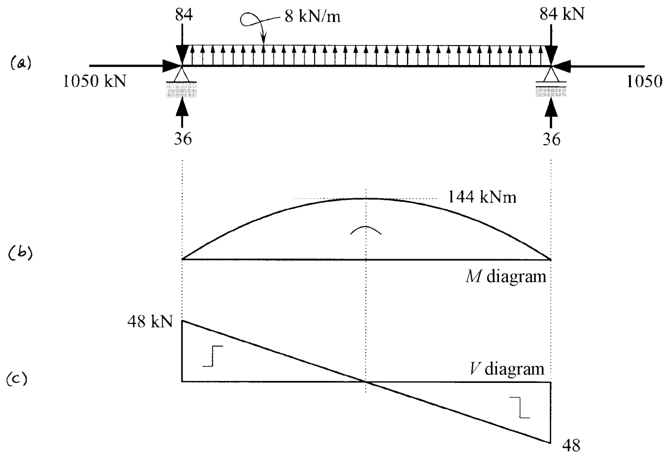


Figure 14.45 (a) All the forces acting on the beam due to the prestressing and dead weight. (b) The associated bending moment diagram and (c) the shear force diagram.

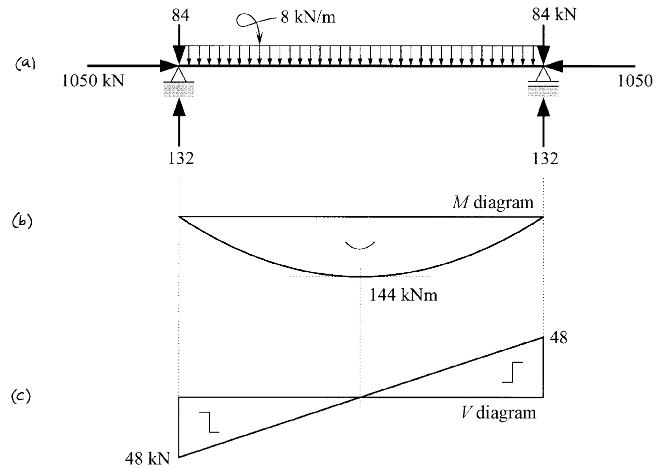


Figure 14.46 (a) All the forces acting on the beam due to the prestressing, dead weight and variable load. (b) The associated bending moment diagram and (c) the shear force diagram.

a. Figure 14.45a shows the forces due to the prestressing and dead weight for the beam modelled as a line element. For the dead weight, it applies that

$$q_{\mathrm{dw}} = (0.3\ \mathrm{m})(0.8\ \mathrm{m})(25\ \mathrm{kN/m^3}) = 6\ \mathrm{kN/m}.$$

The resulting distributed load acts upwards:

$$q_{\mathrm{p}}\ (\uparrow) + q_{\mathrm{dw}}\ (\downarrow) = (14 - 6)(\mathrm{kN/m})\ (\uparrow) = 8\ \mathrm{kN/m}\ (\uparrow).$$

Figures 14.45b and 14.45c show the associated M and V diagrams.

b. Figure 14.46a shows the forces on the beam modelled as a line element due to the prestressing, dead weight and variable load. The resulting

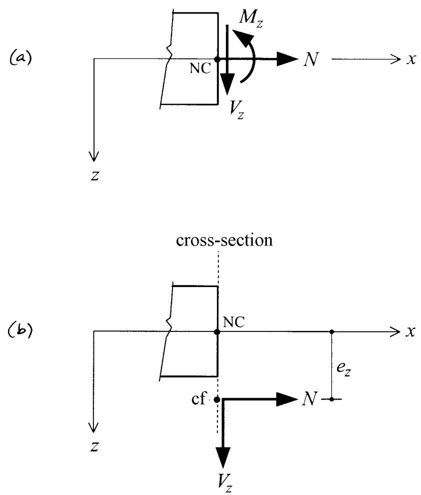


Figure 14.47 (a) The bending moment M_z and normal force N at the normal force centre NC are statically equivalent to (b) a single force N at the centre of force cf, with an eccentricity $e_z = M_z/N$ with respect to the member axis.

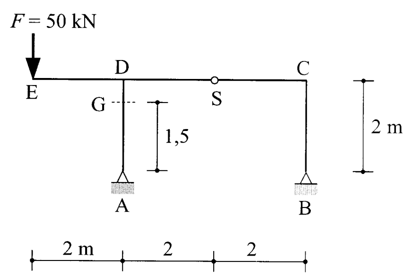


Figure 14.48 Three-hinged frame loaded by a vertical force at E.

distributed load is now acting downwards:

$$(q_{\rm dw} + q_{\rm q})\ (\downarrow) + q_{\rm p}\ (\uparrow) = (6 + 16 - 14)(\text{kN/m})\ (\downarrow) = 8\ \text{kN/m}\ (\downarrow).$$

Figures 14.46b and 14.46c show the associated M and V lines.

Note that in both Figure 14.45 and 14.46 the shear forces directly adjacent to the supports are not equal to the support reactions.

14.2 Centre of force and line of force

In a cross-section (cs) acts a bending moment M_z, normal force N and shear force V_z (see Figure 14.47a). The shear force is the resultant of all shear stresses in the cross-section; the bending moment and the normal forces are the resultants of the normal stresses (see also Section 10.1).

Assume that the resultant of all the normal stresses is a force N with eccentricity e_z with respect to the member axis (see Figure 14.47b). By shifting the normal force N normal to its line of action to the normal centre NC on the member axis, we create the bending moment M_z in Figure 14.47a:

$$M_z = Ne_z.$$

The point in the cross-section where the resultant of all the normal stresses is transferred is known as the *centre of force* (cf). The centre of force can also be described as the intersection of the cross-section with the line of action of the resultant of all the forces that the cross-section has to transfer (see Section 10.1.1). This will be clarified in the two examples at the end of the section.

For the z coordinate of the centre of force, indicated by means of e_z, it holds that

$$e_z = \frac{M_z}{N}.$$

The centres of force at all consecutive cross-sections together form a line known as the *line of force*.

If the normal force is a tensile force ($N > 0$), we also refer to *centres of tension* and *lines of tension* instead of centres of force and lines of force. For a compressive force ($N < 0$), we refer to *centres of pressure* and *lines of pressure*.

The following can be said about lines of force:

- There is no line of force in areas where the normal force is zero. For $N \to 0$ it always holds that $e_z \to \infty$, and the centre of force is at infinity.
- If the bending moment is zero, $e_z = 0$ applies, and the centre of force is on the member axis. This means that in structures with hinges, the line of force always passes through the hinges, as the bending moment is zero there.
- Where the line of force intersects the member axis (or coincides with it), $e_z = 0$ and the bending moment is zero.

Example 1

Figure 14.48 shows a three-hinged portal frame, loaded by a vertical force $F = 50$ kN at E. Figure 14.49a shows the support reactions and Figures 14.49b and 14.49c shows the M and N diagrams. The calculation is left to the reader.

In Figure 14.49a, the lines of action of force F and the support reactions at A and B are shown. These lines of action intersect in one point. This offers a graphical check of the moment equilibrium of the three-hinged frame (see Section 5.3, Example 1).

Questions:

a. Determine the centre of force in cross-section G of post AD, 1.50 m above support A.
b. Determine the line of force for the parts AD, THE, DSC and BC.

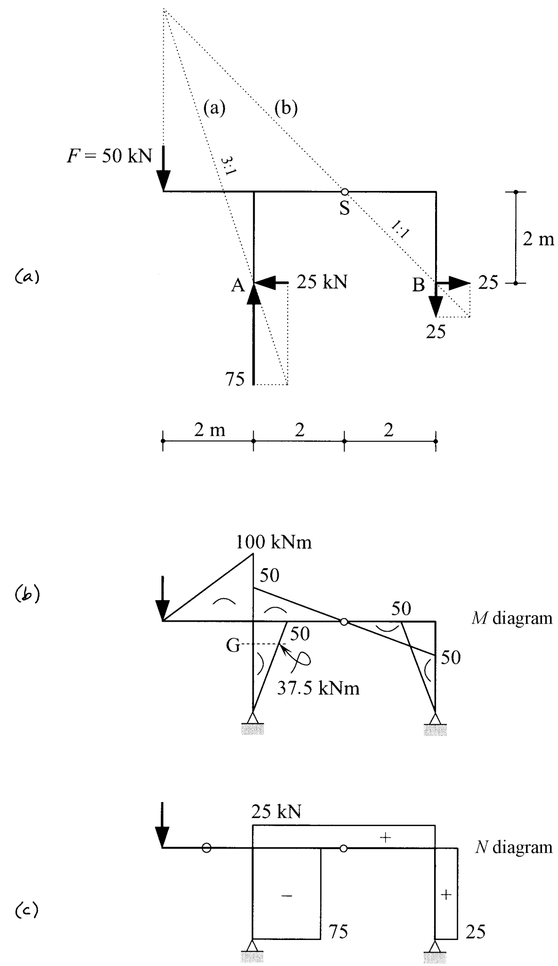


Figure 14.49 (a) Support reactions and line of action figure. (b) Bending moment diagram. (c) Normal force diagram.

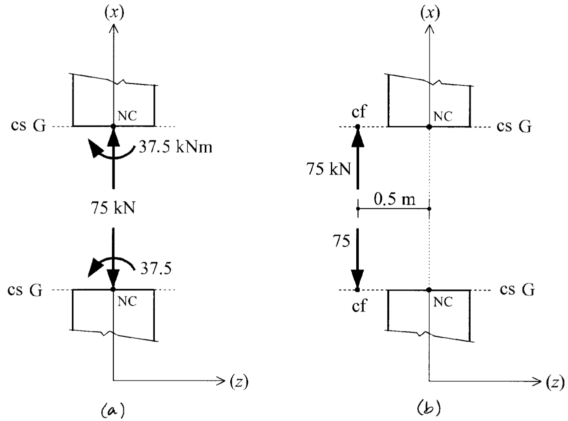


Figure 14.50 (a) Bending moment and normal force at cross-section G. They are statically equivalent to (b) the eccentric compressive forces to the left of the member axis.

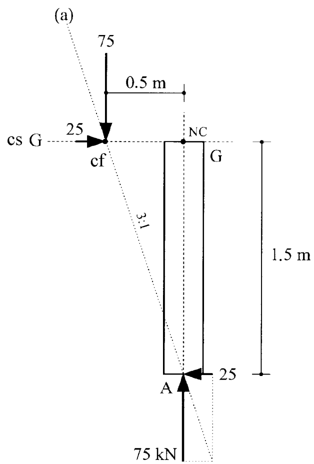


Figure 14.51 Isolated part AG. The centre of force cf in cross-section G is located on the line of action of the support reaction at A; that is, the force cross-section G has to transfer.

Solution:

a. Figure 14.50a shows the bending moment and the normal force at cross-section (cs) G. From the M diagram for post AD we can derive that the bending moment at the cross-section is $(1.5/2.0) \times (50\text{ kNm}) = 37.5\text{ kNm}$, with tension at the "inside" of the frame. From the N diagram it follows that there is a compressive force of 75 kN at the cross-section.

The section forces in Figure 14.50a are statically equivalent to the eccentric compressive forces in Figure 14.50b. The centre of force (cf) will be to the left of the member axis since a compressive force to the left of the member axis causes (with respect to the normal force centre NC) a moment that has the same direction as the bending moment in Figure 14.50a. The magnitude of the eccentricity e is:

$$e = \frac{|M|}{|N|} = \frac{37.5\text{ kNm}}{75\text{ kN}} = 0.5\text{ m}.$$

The location of the centre of force can also be calculated formally in a local coordinate system. In order to find the correct sign for e_z we do have to use the correct signs for N and M_z. For the xz coordinate system in Figure 14.50

$$M_z = +37.5\text{ kNm} \quad \text{and} \quad N = -75\text{ kN}$$

so that

$$e_z = \frac{M_z}{N} = \frac{+37.5\text{ kNm}}{-75\text{ kN}} = -0.5\text{ m}.$$

The centre of force is indeed to the left of the member axis.

b. In Figure 14.51, the part AG has been isolated. The support reactions act at A. If there is a equilibrium, a force has to act at cross-section G that has the same magnitude as the resulting force at A, and has the same line of action, but has to act in the opposite direction. In other words, the centre of force at cross-section G is located on the line of action (a) of the resulting force at A (see also Figure 14.49a). This leads to the following statement: the centre of force is the intersection of the cross-section with the line of action of the resultant of all forces that the cross-section has to transfer.

Since all the cross-sections in AD have to transfer the same resulting force at A, the centres of force in those cross-sections are on the same line of action (a). The line of force for AD therefore coincides with line of action (a) (see Figure 14.52a). Since the normal force in AC is a compressive force, the line of force is a line of pressure.

In the same way, the line of force of BC coincides with line of action (b) of the support reaction at B (see Figure 14.52a). Since the normal force is a tensile force, the line of force is here a line of tension.

If we look at a section in girder DSC, this, seen from the left, has to transfer the resultant of force F and the support reactions at A, and, seen from the right, the support reaction at B. Both have the same line of action (b), as shown by the line of action figure (see Figure 14.52b). The line of force for DSC coincides with line of action (b) and is a line of tension.

Since the normal force is zero, there exists no line of force for DE. All cross-sections between D and E have to transfer the same vertical force F. The line of action of F is parallel to the cross-sections, so that there are no intersections and therefore no centres of force.

Note: If the normal force in a beam is constant, the figure that is enclosed between the line of force and the member axis has the same shape as the M diagram. This is not surprising, for[1]

$$M = Ne.$$

[1] Without taking into account the coordinate system and signs.

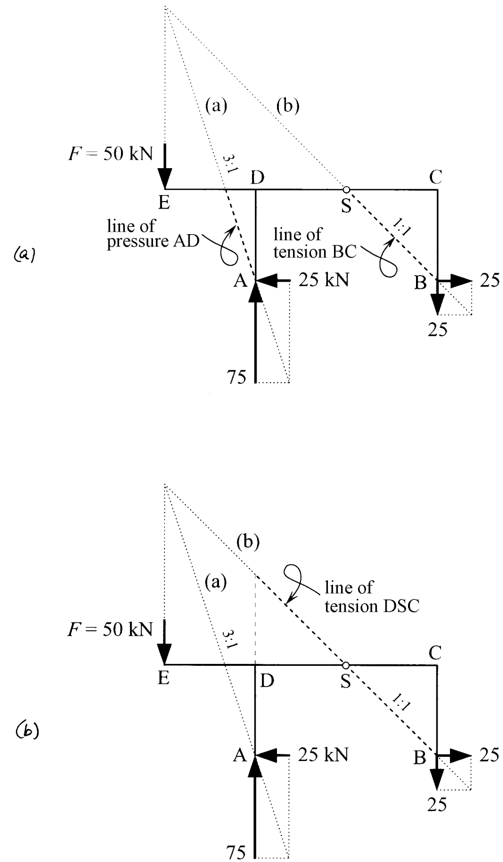


Figure 14.52 (a) The lines of force for AD and BC coincide with the lines of action of the support reactions at A and B respectively. (b) The line of force for DSC coincides with the line of action of the support reaction at B.

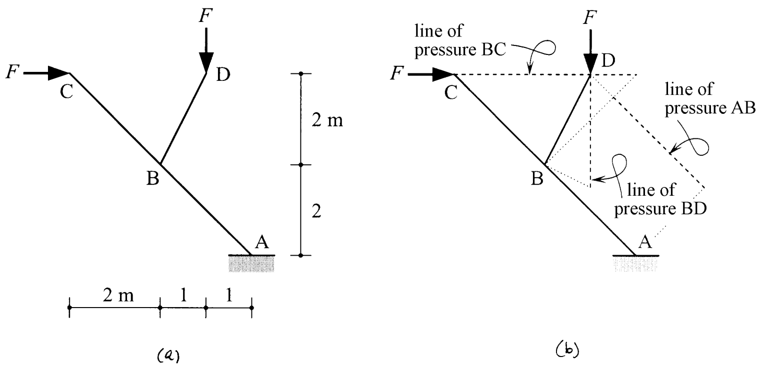


Figure 14.53 (a) Fixed bar type structure, loaded by two forces, with (b) the lines of force.

The scale factor is N. If N is a tensile force, the line of force is on the same side of the member axis as the M diagram. If N is a compressive force, the line of force and the M diagram are not on the same side. It is up to the reader to verify this, using the bending moment diagram in Figure 14.49b and the lines of force in Figure 14.52.

Example 2

The structure ABCD in Figure 14.53a is fixed at A, and loaded by a horizontal force F at C and a vertical force F at D.

Question:
Determine the lines of force for AB, BC and BD.

Solution:
The lines of force are shown in Figure 14.53b.

All cross-sections between C and B have to transfer the horizontal force F. The line of force for BC therefore coincides with the line of action of this horizontal force.

All cross-sections between D and B have to transfer the vertical force F. The line of force for BD coincides with the line of action of this vertical force.

All cross-sections between B and A have to transfer the resultant of the forces F at C and D. The line of force for AB coincides with the line of action of this resultant. The line of action figure shows that this line of action passes through D and is parallel to ABC.

Since the normal force is a compressive force everywhere, all the lines of force are lines of pressure.

14.3 Relationship between cable, line of force and structural shape

The bending moment and the normal force are the resultants of the normal stresses in a cross-section. Figure 14.54 shows the stress distribution due to a bending moment M and a normal force N.[1]

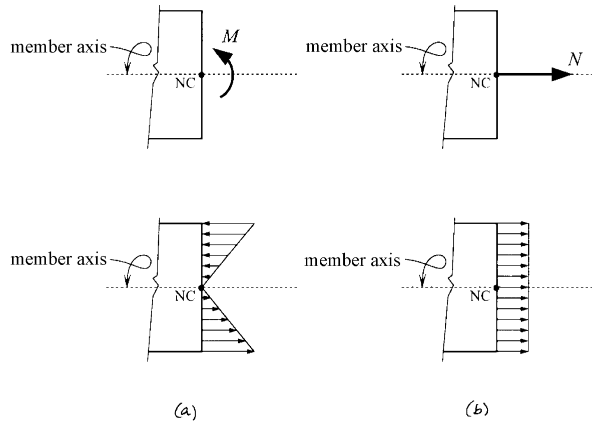


Figure 14.54 The distribution of normal stresses in a cross-section due to (a) a bending moment M and (b) a normal force N.

A characteristic of stress distribution in bending is that the outermost fibres of the cross-section are most heavily loaded, while the fibres in the environment of the member axis are virtually unloaded. In contrast, the stresses due to a normal force are constant over the cross-section. In extension, all fibres are therefore loaded equally.

If we compare the stress distributions due to bending and extension, the material in the cross-section is used far more efficiently in extension than in bending. With bending, the strength capacity of the fibres around the member axis is not used, and the small stresses only marginally contribute to the bending moment. For beams loaded by bending, one often sees an adaptation of the cross-section by omitting the less active material in the cross-section. In this way, a rectangular cross-section may become a *tubular section* or an I-section (see Figure 14.55).

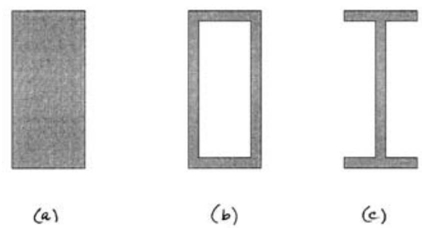


Figure 14.55 If a beam with (a) a rectangular cross-section is loaded by bending, the fibres around the member axis remain virtually unloaded. With (b) a tubular section or (c) an I-section the material is more effectively distributed across the cross-section.

In addition, when designing structures, designers look for shapes in which the bending moments remain as small as possible, and in which the force flow preferably occurs by extension. This is achieved by ensuring the member axis and line of force coincide as much as possible.

Since a cable cannot transfer bending moments, it assumes a shape in which the line of force coincides with its axis everywhere. Taking the cable shape and line of force as basis, in the following four examples, we look for

[1] In Volume 2, *Stresses, Deformations, Displacements*, we take a closer look at the exact development of the normal stresses in a cross-section and at the conditions under which the stress distribution in Figure 14.54 applies.

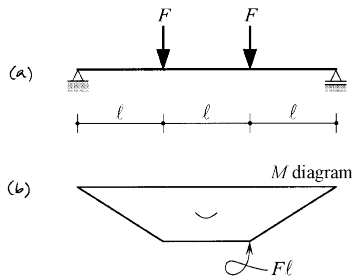


Figure 14.56 (a) A beam subject to bending by two forces with (b) the associated bending moment diagram.

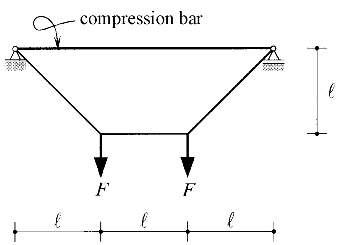


Figure 14.57 Cable with compression bar loaded by two forces. All the parts are subject to extension (tension and compression).

structural shapes in which the bending moments are as small as possible.

Example 1

The beam in Figure 14.56a is subject to bending by the two forces F. The M diagram is shown in Figure 14.56b.

In Figure 14.57, the same load is carried by a cable with compression bar. The cable and compression bar transfer normal forces only. The cable has the same shape as the M diagram in Figure 14.56b. With a cable sag the scale factor is

$$H = \frac{F\ell}{\ell} = F.$$

H is the (compressive) force in the bar that is equal to the horizontal component of the (tensile) force in the cable.

In Figure 14.58a the straight cable parts have been replaced by bars. Plus and minus signs indicate whether the bar forces are tensile or compressive. The structure can be considered a kind of *arch under tension* that is kept together by a *compression bar*. If the bar structure in Figure 14.58a is "turned over" with equal loads as shown in Figure 14.58b, all the signs in the bars change. The structure has now changed into an *arch under compression* with tension bar (*tie rod*).

In the position shown in Figure 14.58b, the bar structure is in equilibrium. However, the equilibrium is unstable (unreliable): a small change in position will cause the equilibrium to fail and the bar structure will collapse.[1] The bar structure is kinematically indeterminate. This collapse can be pre-

[1] To prove this we have to investigate the equilibrium of the structure in its deformed state. However, this topic is beyond the scope of this book. Here we assume that the reader is acquainted with this phenomenon of instbility on the basis of some practical experience.

vented by making the structure kinematically determinate, for example by introducing bracing members, and changing the bar structure into a truss (see Figures 14.59 and 14.60b). If we calculate the member forces for the given load, we find that all the interior members are zero members.

The cable in Figure 14.57 and the bar structure in Figure 14.58a are also kinematically indeterminate. The equilibrium is stable (reliable) in this case as the load makes the structure go back into the original equilibrium position after a disruption.

In Figure 14.60a, the bar structure in Figure 14.58a has been changed into a kinematically determinate truss. All the interior members are zero. As in contrast to the cable, the truss has the benefit that the shape does not change when the load changes.

In Figure 14.61, the forces on the truss are shifted to the horizontal plane through the supports. The verticals are no longer zero members. These

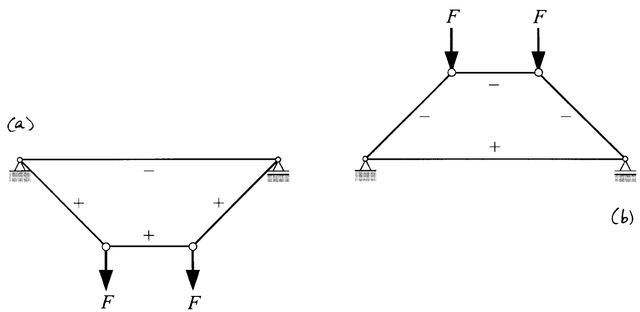


Figure 14.58 (a) The cable replaced by a bar structure. The bar structure is kinematically indeterminate, but the equilibrium is stable (reliable). (b) If the bar structure is folded over, the signs change in all the bars. The equilibrium is now unreliable (unstable).

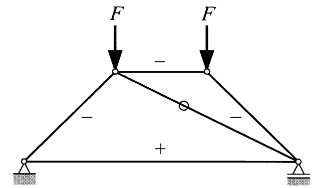


Figure 14.59 By applying an additional bar, the kinematically indeterminate bar structure changes into a kinematically and statically determinate truss.

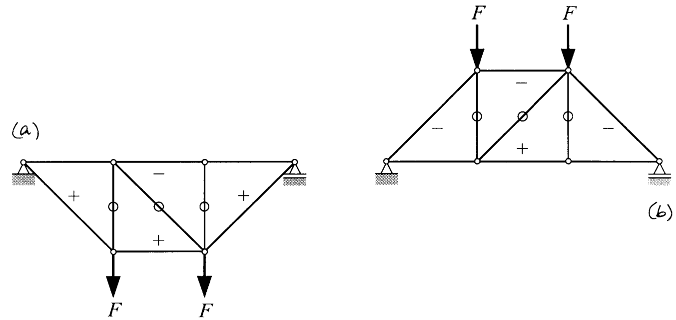


Figure 14.60 The bar structures in Figure 14.58 changed into trusses.

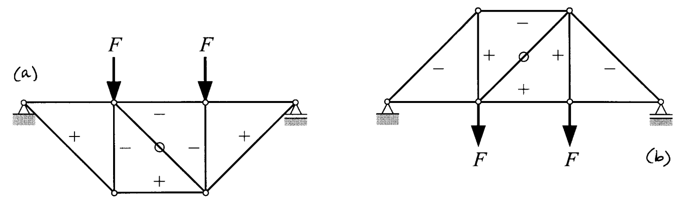


Figure 14.61 These trusses are an alternative for the beam subject to bending in Figure 14.56.

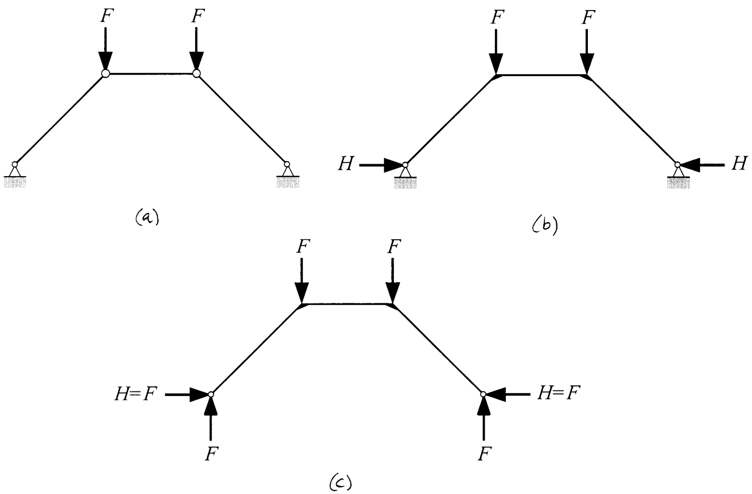


Figure 14.62 (a) Kinematically indeterminate bar structure, is (b) changed into a two-hinged frame. (c) The support reactions of the statically indeterminate two-hinged frame if axial deformation is ignored; the line of force coincides with the bent member axis: there is no bending.

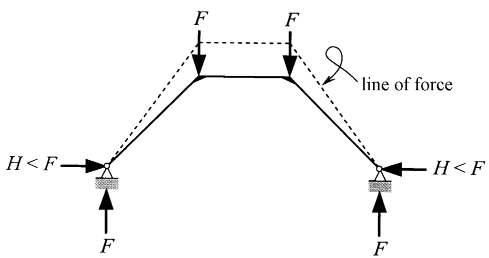


Figure 14.63 If we take into account the axial deformation, the horizontal support reactions are smaller and the line of force no longer coincides with the (bent) member axis: bending is generated.

trusses, in which all the members are subject to extension (or are zero members), can be an alternative for the beam subject to bending in Figure 14.56.

Figure 14.62a shows the bar structure from Figure 14.58b, but now without tension member. This kinematically indeterminate structure can be made kinematically determinate not only by changing it into a truss, but also by replacing the hinged joints between the bars by rigid joints. The structure then becomes a bent member, recognisable in Figure 14.62b as a two-hinged frame.

One problem is that the frame is statically indeterminate to the first degree. As such, it is not possible to determine the horizontal support reaction H directly from the equilibrium. The deformation of the frame also has to be taken into account. If the deformation by normal forces is ignored (as it was in the cable), it is possible to show that no bending occurs under the given load, and that the normal forces in the frame are equal to the forces in the two-force members in Figure 14.62a. In Figure 14.62c, the frame has been isolated and all the forces acting on it are shown. The line of force coincides everywhere with the bent member axis and there is no bending anywhere.

In reality, there is always some axial deformation due to normal forces. As such, the horizontal support reactions are somewhat smaller and, because the vertical support reactions remain equal, the line of force no longer coincides with the bent member axis (see Figure 14.63). Axial deformation therefore induces bending in the two-hinged frame.

Since statically indeterminate structures are more sensitive to settling and temperature, statically determinate structures are generally preferable, because the force distribution is more manageable. In Figure 14.64, the statically indeterminate two-hinged frame has been changed into a statically determined three-hinged frame. With the given load, the line of force coincides everywhere with the bent member axis and there is no bending anywhere.

Example 2
On girder CSD, the three-hinged portal frame in Figure 14.65a is carrying a uniformly distributed load. In Figures 14.65b and 14.65c, the support reactions and the bending moment diagram are shown. The calculation is left to the reader.

Question:
How can one reduce the bending moment in the frame, without changing the given load?

Solution:
Figure 14.65d shows the line of force for girder CSD. Cross-section C has to transfer the support reaction at A; the centre of force for cross-section C is therefore at A. In the same way, the centre of force for cross-section D is at B. The line of force passes through hinge S. The line of force for CSD has the same shape as a cable under a uniformly distributed load, which is a parabola. Since the normal force in girder CSD is a compressive force, “the parabolic cable is upside down” and the line of force is a line of pressure.

Check: Regard the line of force as an “upside-down cable” (see Figure 14.65d):

$$H = \frac{\frac{1}{8}q\ell^2}{p} = \frac{\frac{1}{8} \times (4\ \text{kN/m})(12\ \text{m})^2}{4\ \text{m}} = 18\ \text{kN}.$$

This force is indeed equal to the horizontal support reactions at A and B (see Figure 14.65b).

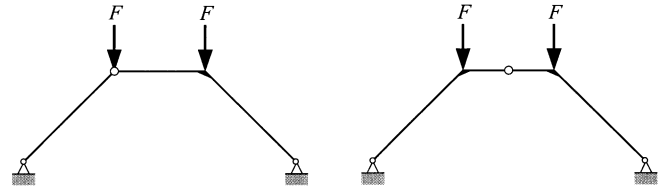


Figure 14.64 Three-hinged frames are statically determinate and therefore the force flow is less sensitive to axial deformations, settling and the influence of temperature.

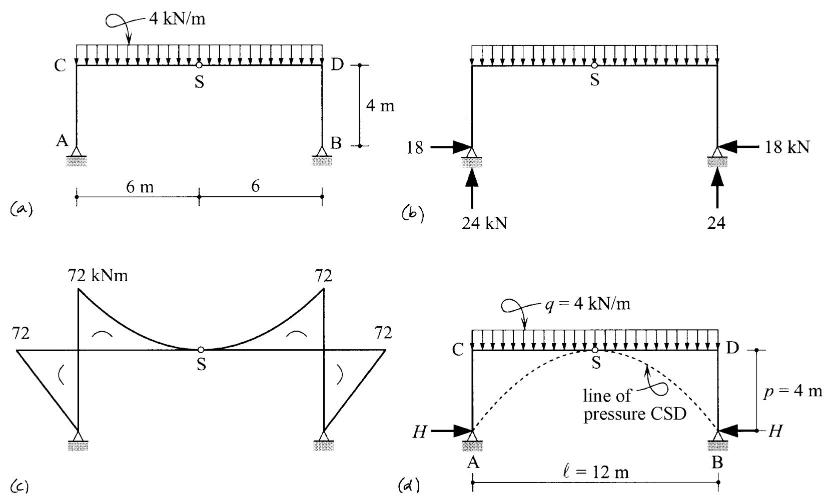


Figure 14.65 (a) Three-hinged frame with uniformly distributed load on girder CSD. (b) Support reactions. (c) Bending moment diagram. (d) The line of pressure for girder CSD is a parabola through A, S and B.

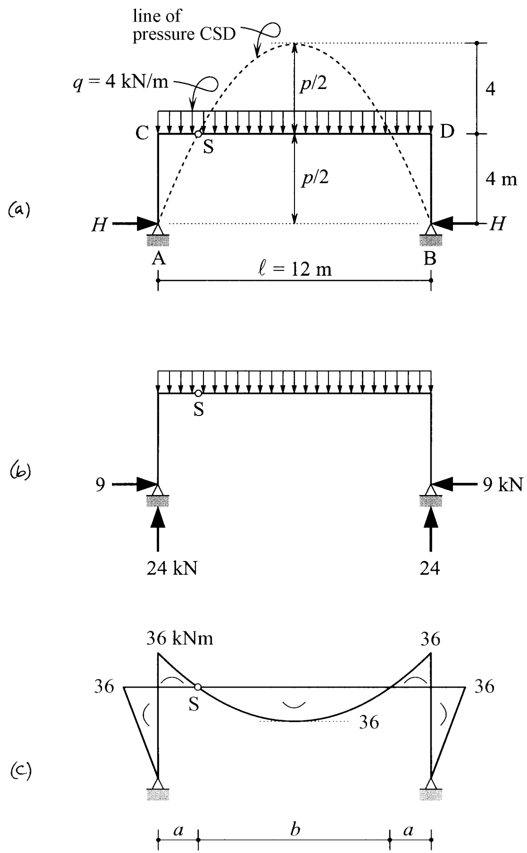


Figure 14.66 (a) The line of pressure can be changed by changing the location of hinge S. (b) The support reactions when the line of pressure is equally above and below the girder. (c) Bending moment diagram.

The bending moment in the girder can be influenced by the location of hinge S (see Figure 14.66a). The line of force maintains the shape of an "upside-down parabolic cable" through A, B and S.

Distance e from the line of pressure to the girder is a measure for the magnitude of the bending moment. It always holds that

$$|M| = |Ne| = |He|.$$

The moment distribution is most favourable when, in contrast to Figure 14.65d, the line of pressure is as much above the girder as below it. This situation is shown in Figure 14.66a. Here it holds that

$$H = \frac{\frac{1}{8}q\ell^2}{p} = \frac{\frac{1}{8} \times (4\ \text{kN/m})(12\ \text{m})^2}{8\ \text{m}} = 9\ \text{kN}.$$

Figures 14.66b and 14.66c show the support reactions and the bending moment diagram respectively. The distance b between the moment zeros follows from

$$\tfrac{1}{8}qb^2 = 36\ \text{kNm}$$

in which $q = 4$ kN/m. We find

$$b = \sqrt{\frac{8 \times (36\ \text{kNm})}{4\ \text{kN/m}}} = \sqrt{72\ \text{m}^2} \approx 8.5\ \text{m}.$$

The location of hinge S is then

$$a = \frac{\ell - b}{2} \approx 1.75\ \text{m}.$$

By moving hinge S we adapt the line of pressure to the shape of the frame. Alternatively, the shape of the frame can also be adapted to the line of pressure. This is shown in Figure 14.67a.[1]

For the horizontal support reaction H it holds that

$$H = \frac{\frac{1}{8}q\ell^2}{p} = \frac{\frac{1}{8} \times (4\ \text{kN/m})(12\ \text{m})^2}{4\ \text{m}} = 18\ \text{kN}.$$

In Figure 14.67b the support reactions are shown. They are of equal magnitude to the support reactions of the three-hinged frame in Figure 14.65. Figure 14.67c shows the bending moment diagram.

The moment distribution is determined below for girder SD. For this reason, in Figure 14.68 a part directly adjacent to hinge S has been isolated. Only the horizontal compressive force H acts at hinge S; there is no vertical force. In the other section, there is a bending moment M, shear force V and normal force N. From the moment equilibrium about this section we find

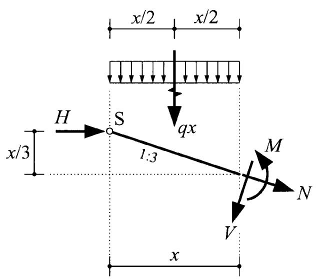


Figure 14.68 The forces on the isolated part of the frame to the right of S.

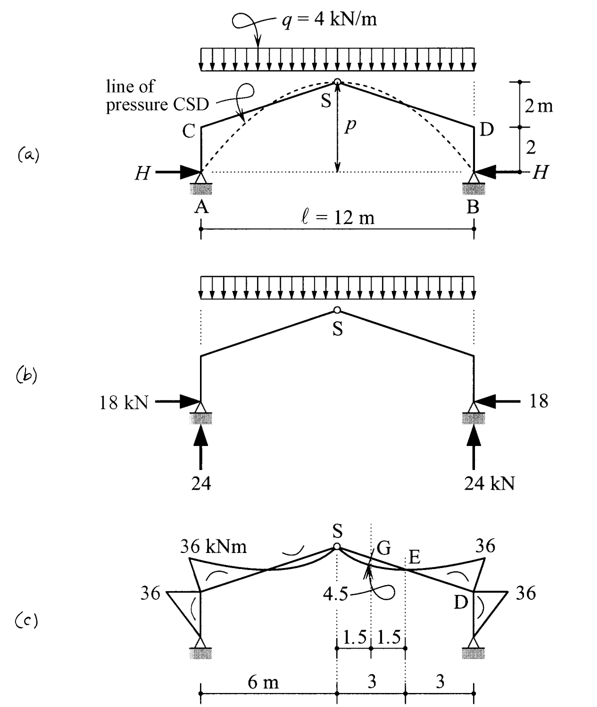


Figure 14.67 (a) A three-hinged frame whose shape has been somewhat adapted to the shape of the line of pressure. (b) Support reactions. (c) Bending moment diagram.

1 Here we assume that the distributed load is still a force per horizontally measured length.

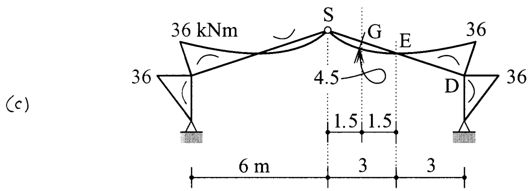


Figure 14.67 (c) Bending moment diagram.

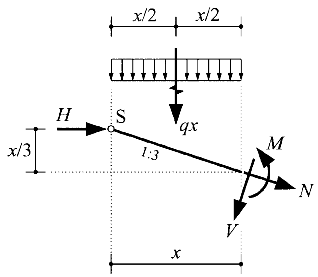


Figure 14.68 The forces on the isolated part of the frame to the right of S.

$$M = H \cdot \tfrac{1}{3}x - qx \cdot \tfrac{1}{2}x = \tfrac{1}{3}Hx - \tfrac{1}{2}qx^2 \tag{a}$$

in which $H = 18$ kN and $q = 4$ kN/m.

Checking expression (a) for the bending moment at D, with $x = 6$ m

$$M = M_{\mathrm{D}} = \tfrac{1}{3} \times (18\text{ kN})(6\text{ m}) - \tfrac{1}{2} \times (4\text{ kN/m})(4\text{ m})^2 = -36\text{ kNm}.$$

This minus sign indicates that the bending moment at D acts opposite to the direction shown in Figure 14.68. This is in agreement with the M diagram in Figure 14.67c.

The bending moment is zero at E (see Figure 14.67c). With $M = M_{\mathrm{E}} = 0$, it follows from (a) that:

$$x = x_{\mathrm{E}} = \frac{2H}{3q} = \frac{2 \times (18\text{ kN})}{3 \times (4\text{ kN/m})} = 3\text{ m}.$$

The field moment in SD is a maximum at G. Here $\mathrm{d}M/\mathrm{d}x = 0$. Differentiating expression (a) gives

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \tfrac{1}{3}H - qx = 0$$

so that

$$x = x_{\mathrm{G}} = \frac{\tfrac{1}{3}H}{q} = \frac{\tfrac{1}{3} \times (18\text{ kN})}{4\text{ kN/m}} = 1.5\text{ m}.$$

From here, (a) gives

$$M = M_{\mathrm{G}} = \tfrac{1}{3} \times (18\text{ kN})(1.5\text{ m}) - \tfrac{1}{2} \times (4\text{ kN/m})(1.5\text{ m})^2 = 4.5\text{ kNm}.$$

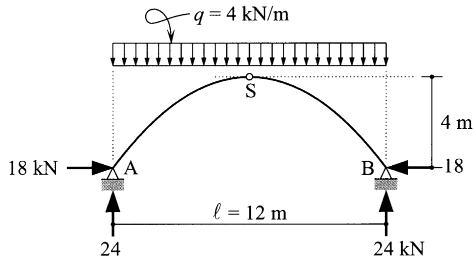


Figure 14.69 The adaptation to the line of force is optimal if the frame is in the shape of a parabola.

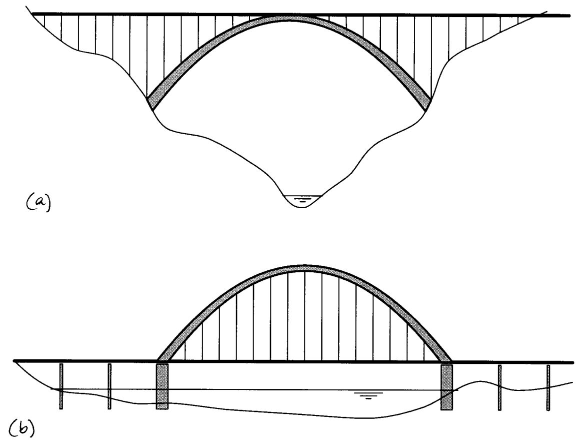


Figure 14.70 (a) Arch bridge with upper deck. The horizontal forces in the arch are directly transferred to the foundation. (b) Arch bridge with lower deck. The horizontal forces from the arch are transferred via the structure of the bridge deck, which acts as a tie rod.

The adaptation to the line of force is optimal if we give the frame the shape of a parabolic arch (see Figure 14.69). The line of force now coincides everywhere with the axis of the arch, and there is no bending anywhere. The support reactions are equal to those of the frames in Figures 14.65 and 14.67.

In Figure 14.70a, the arch has been used in a *bridge with upper deck*. This type of bridge is generally found in mountainous regions. The good foundation ground, generally rock, is capable of transferring the horizontal support reactions. In Figure 14.70b, the arch is used in a *bridge with lower deck*. By using the structure of the bridge deck as a *tie rod*, the piers are not subject to the horizontal forces from the arch.

Example 3

The third example demonstrating the concept of line of force concerns structures with *brickwork*. Brickwork effectively resists pressure, but is very poor at transferring tensile stresses. Since there is little tensile strength, this must not be relied on; the tensile strength has to be neglected in the calculation. Structures made of brickwork must therefore be designed so that no tension occurs.

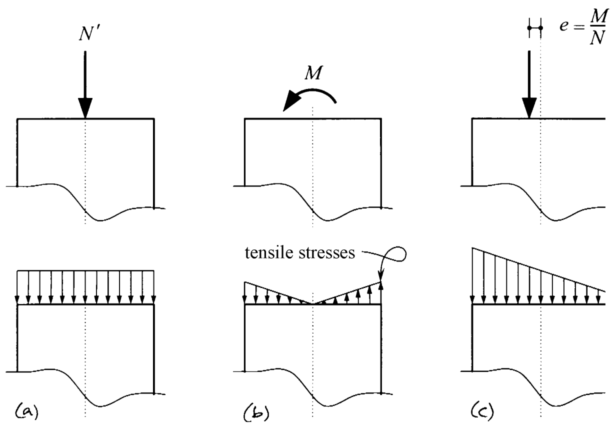


Figure 14.71 The distribution of the normal stresses in a rectangular cross-section due to (a) a centric compressive force, (b) a bending moment and (c) an eccentric compressive force.

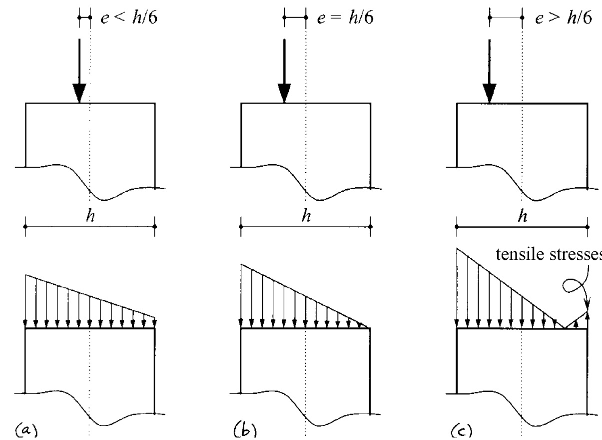


Figure 14.72 The normal stress distribution in a rectangular cross-section due to an eccentric compressive force. (a) With minor eccentricity, the entire cross-section is subject to compression. (b) If $e = h/6$, the normal stress is zero at the least compressed edge. (c) With major eccentricity, tensile stresses occur in the cross-section.

In Figures 14.71a and 14.71b the distribution of the normal stresses is shown for a rectangular cross-section due to a centric compressive force N' and a bending moment M. The centric compressive force and the bending moment are together statically equivalent to an eccentric compressive force N' (see Figure 14.71c). Due to the eccentricity of the compressive force, the compressive stresses increase on one side of the cross-section and decrease on the other.

When the eccentricity e of the compressive force is equal to one sixth of the depth h of the cross-section ($e = h/6$), the stress diagram is triangular (see Figure 14.72b). At one of the sides, the normal stress is zero. If the eccentricity is larger ($e > h/6$), then tensile stresses occur at that side (see

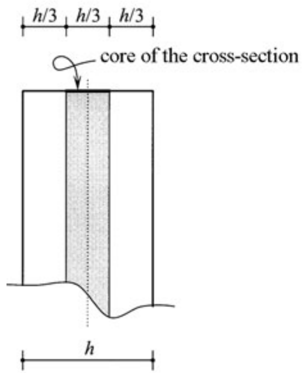


Figure 14.73 There are no tensile stresses if the centre of pressure is inside the core of the cross-section. With rectangular cross-sections this is the middle third.

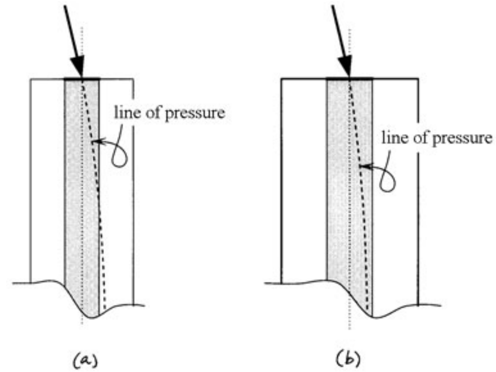


Figure 14.74 Due to the dead weight of a column, the line of pressure tends increasingly towards a line parallel to the member axis. (a) Tensile stresses occur where the line of pressure reaches outside the core (b) Increasing the column cross-section has two positive effects: the core increases in size while, thanks to the larger weight of the column, the eccentricity of the line of force decreases.

Figure 14.72c). In brickwork the joints cannot transfer tensile stresses and cracks will form. The cohesion of the cross-section is lost so that there is a danger of collapse.

To prevent tensile stresses in the cross-section, the centre of pressure (the point of application of the compressive force in the cross-section) for a rectangular cross-section must lie within the middle third of the cross-section. This area is known as the *core of the cross-section*[1] (see Figure 14.73).

Figure 14.74a shows a brickwork column, loaded at the top by an oblique force. At first, the line of force has the direction of the oblique force. Since the normal force increases downwards due to the weight of the column, the line of force is curved and increasingly tends towards a line parallel to the

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1 In Volume 2, *Stresses, Deformations, Displacements*, we shall address this issue in further detail.

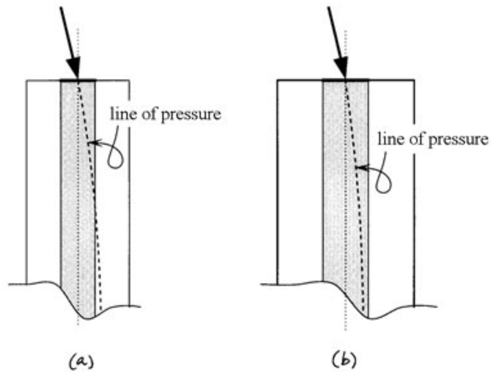


Figure 14.74 Due to the dead weight of a column, the line of pressure tends increasingly towards a line parallel to the member axis. (a) Tensile stresses occur where the line of pressure reaches outside the core (b) Increasing the column cross-section has two positive effects: the core increases in size while, thanks to the larger weight of the column, the eccentricity of the line of force decreases.

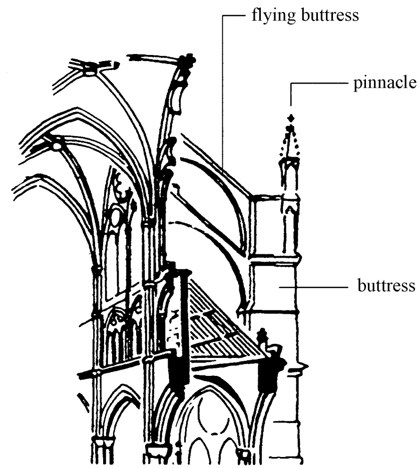


Figure 14.75 In gothic cathedrals, we can see how the tensile stresses in the buttresses are suppressed by using the weight of pinnacles.

column axis.

When there is a danger that the line of force will extend beyond the core of the cross-section, the column cross-section can be enlarged. First, due to the dead weight, the normal force will increase and the eccentricity of the centre of force will decrease: the line of pressure thereby moves towards the column axis. In addition, the core of the cross-section increases, and therefore the area in which the centre of pressure can lie also increases (see Figure 14.74b).

Increasing the compressive force N' in the cross-section, thereby reducing the eccentricity e of the line of force ($e = M/N'$), is also possible by increasing the height of the column to a greater height than is strictly necessary, or by introducing additional weight on the column by means of heavy statues or pinnacles. The latter is often used in gothic cathedrals (see Figure 14.75). Due to the introduction of aisles, flying buttresses were needed to support the main structure obliquely. The forces are transferred to buttresses, weighed down by pinnacles.

We have only looked at brickwork. There are many other materials that are unable to resist tension, such as *concrete*. Here, the issue of tensile stresses is approached differently.

In *reinforced concrete*, steel bars are placed in the area of tension. These reinforcement bars transfer the tensile stresses in the cross-section (see Figure 14.76).

In *prestressed concrete*, the tensile stresses are "suppressed" by introducing a prestress (see Figure 14.77). The prestress achieves the same effect as the weight of the pinnacles on the buttresses in Figure 14.75 (see also Section 13.1.5 and Section 14.1.6, Example 5).

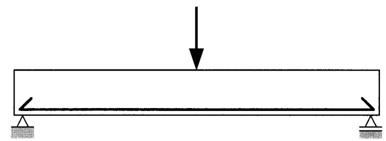


Figure 14.76 In reinforced concrete, the tensile stresses in the cross-section are transferred by reinforcement bars.

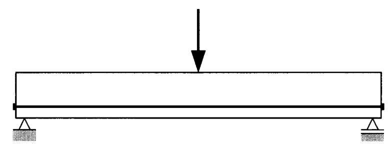


Figure 14.77 In prestressed concrete, the tensile stresses in the cross-section are "suppressed" by applying tendons

(a)

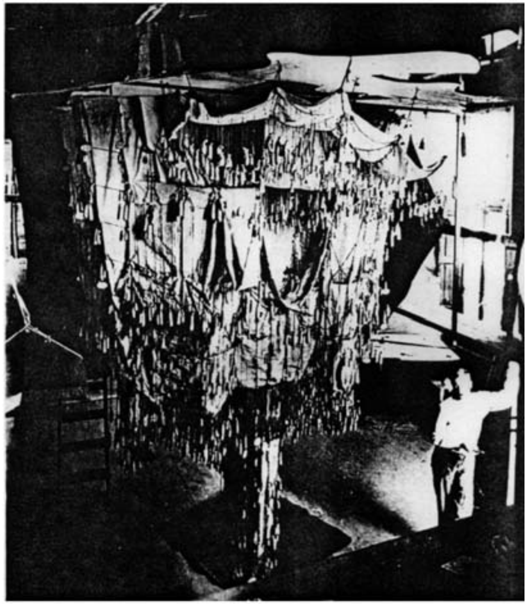
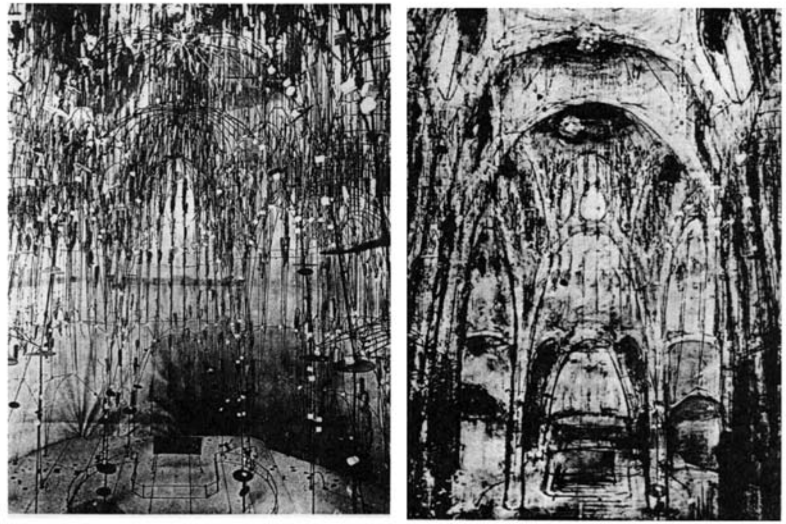


Figure 14.78 (a) Original photograph of Gaudi's suspension model for the Colonia Güell.

(b)



(c)

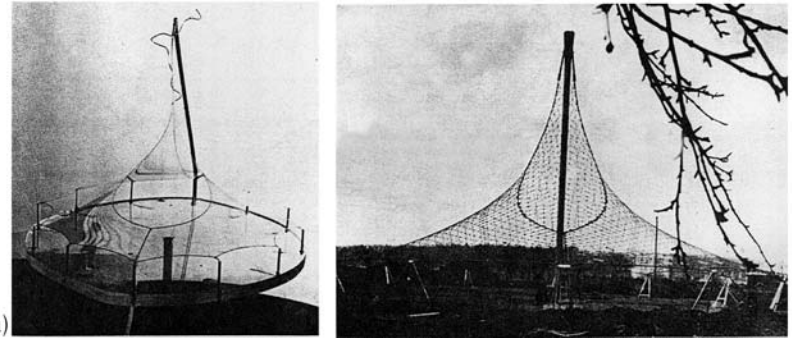
Figure 14.78 (b) Photograph of the inside of the model, rotated through 180°. The upside-down cables change into arches. (c) Interior sketch by Gaudi on the basis of the model.

Example 4

For the last example on lines of force, we refer back to the start: the cable shape.

The Spanish architect Gaudi[1] is known for his whimsical vaults in the Colonia Güell and the Sagrada Familia, two churches in Barcelona. He determined the shape of the vaults using a "suspension model". Figure 14.78a shows an original photo of the model for the Colonia Güell, which was built between 1898 and 1908. Using bags of lead suspended from ropes, representing the dead weight, Gaudi determined the preferred shape of the arches. Figure 14.78b is a photo of the inside of the model, rotated through 180°. The upside-down cables are transformed into arches. Using this model, Gaudi drew the interior sketch in Figure 14.78c. Due to a lack of funds, the construction of the Colonia Güell had to be abandoned in 1914. Only the crypt was completed.

When designing cable structures subject to tension (cable networks) Frei Otto and his staff used the same experiment over 50 years later. In their method, soap membranes were the most important tool. To translate a *soap membrane model* in Figure 14.79a into an actual structure, it was meticulously photographed and measured. Figure 14.79b shows the actual structure being built. One of Frei Otto's most famous buildings is the roof of the Olympic Stadium in Munich (1972) as a tent with saddle roof shapes (see Figure 14.80).



(a) (b)

Figure 14.79 (a) Frei Otto used the soap membrane model as a tool for designing cable networks. (b) The soap membrane model changed into an actual structure.

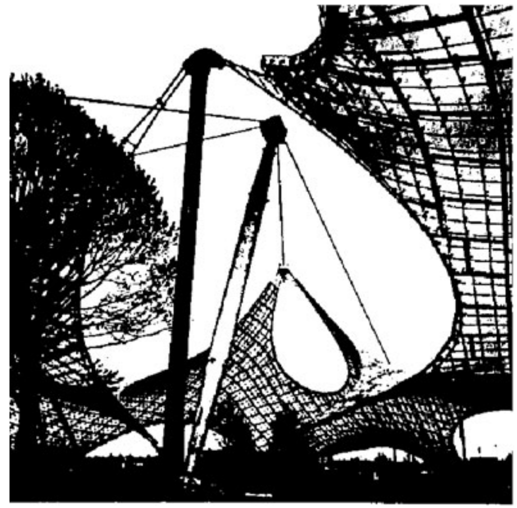


Figure 14.80 One of Frei Otto's most famous buildings is the roof of the Olympic stadium in Munich (1972) in the shape of a tent with saddle roof shapes.

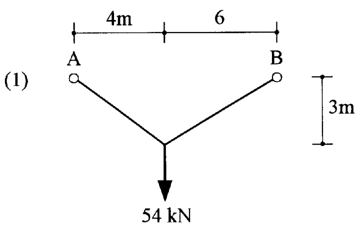
1 Antoni Gaudi i Cornet (1852–1926), Spanish architect, studied and worked in Barcelona. He derived the shapes for his buildings from nature; this became known as the organic style. His most famous creation is the Sagrada Familia in Barcelona (construction started in 1883, not completed).

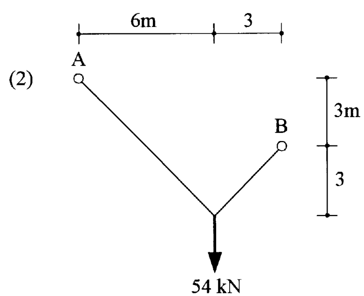
14.4 Problems

Unless indicated otherwise, the dead weight of the cable is ignored in the problems.

Cables, mixed problems (Section 14.1)

14.1: 1–2 Given two different cables, hung from fixed points A and B, are loaded by a single force.

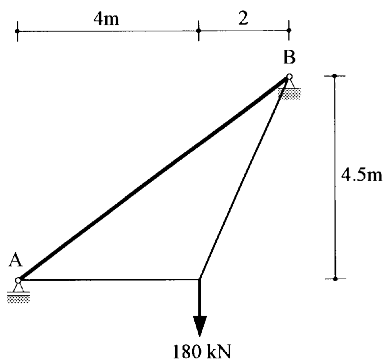




Questions:

a. Determine the vertical support reaction at A.
b. Determine the horizontal support reaction at B.
c. Determine the maximum cable force.

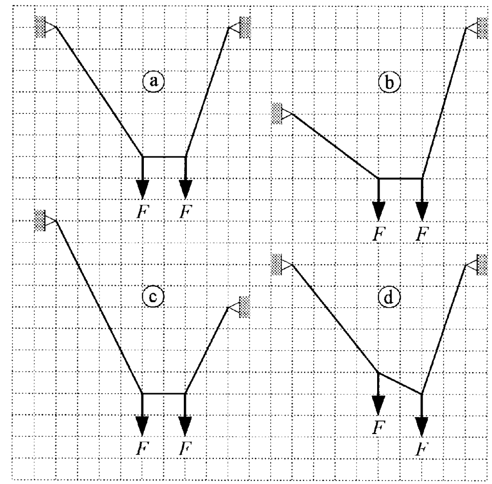
14.2 Given a cable with compression bar, loaded by a force of 180 kN.



Questions:

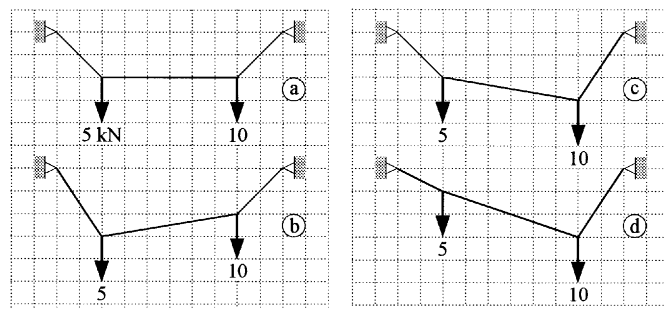
a. Determine the vertical component of the force in the compression bar.
b. Determine the horizontal component of the force in the compression bar.

14.3 A cable is loaded by two equal forces F.



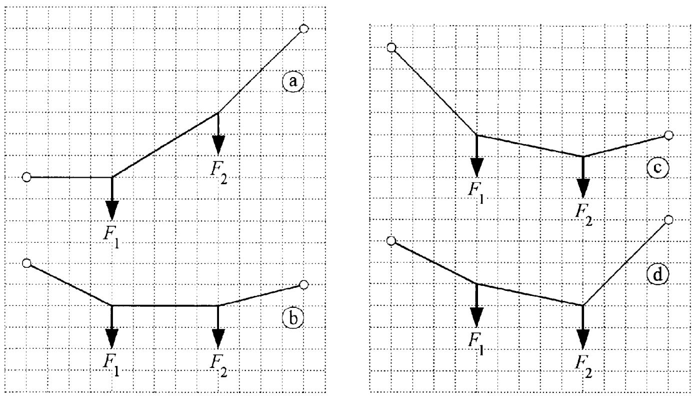
Question:
Which cable shape fits the load?

14.4 Two weights of respectively 5 and 10 kN are suspended from a cable.



Question:
Which cable shape fits the load?

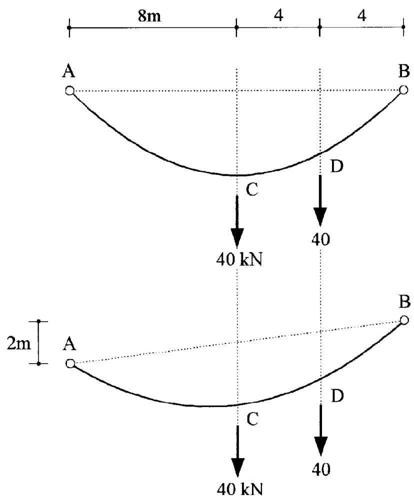
14.5 A cable, suspended between two fixed points, is subject to the forces F_1 and F_2, with $F_1 > F_2$.



Question:
Which cable shape does not fit this load?

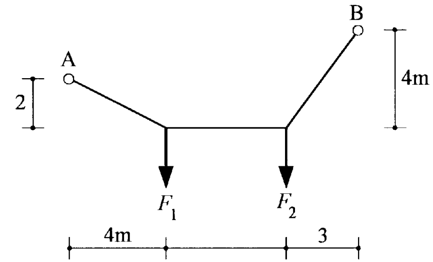
14.6: 1–2 A cable suspended between fixed points A and B is loaded at C and D by two forces of 40 kN. The horizontal measured lengths are those of the final position.

Question:
Find the ratio between the vertical distances from chord AB to the points C and D on the cable.



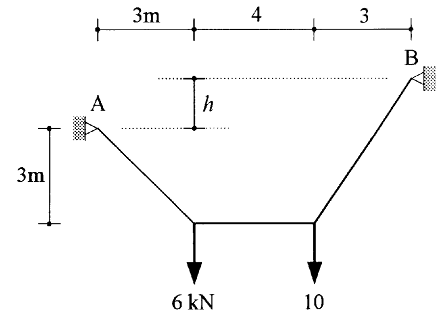
14.7 The cable shown is loaded by forces F_1 and F_2 in such a way that the middle part is horizontal.

Question:
Find the ratio F_1/F_2.



14.8 A cable is loaded by two forces of respectively 6 and 10 kN. The cable is horizontal between these forces.

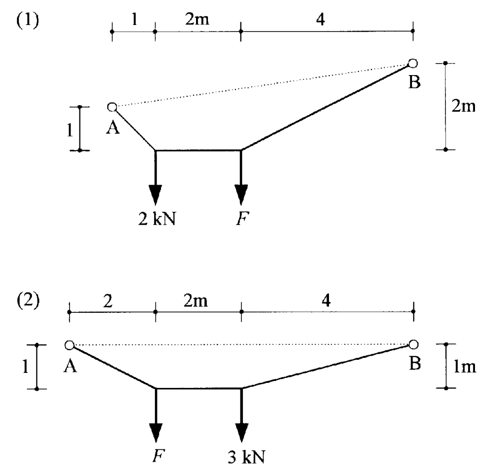
Question:
Determine the difference in height h between the suspension points A and B.



14.9: 1–2 Two cables, suspended at the fixed points A and B, are loaded by two forces. The magnitude of force F is unknown.

Questions:

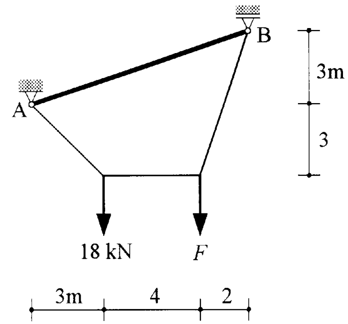
a. Determine the vertical support reaction at A.
b. Determine the horizontal component of the cable force.
c. Determine the magnitude of F.
d. Determine the maximum cable force.



14.10 Given a cable with compression bar. The cable is loaded by a force of 18 kN and an unknown force F, and assumes the shape shown.

Questions:

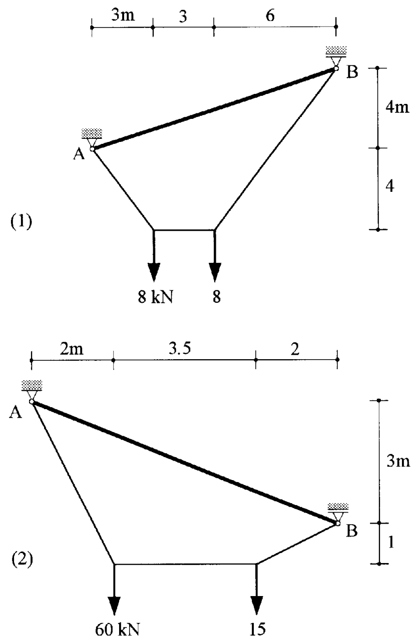
a. Determine the horizontal component of the cable force.
b. Determine the normal force in compression bar AB.
c. Determine the magnitude of F.



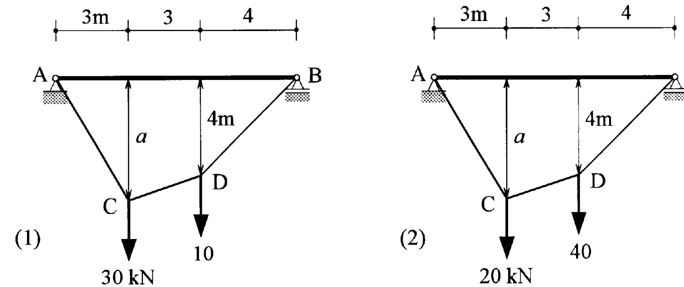
14.11: 1–2 Two cables with compression member are loaded in such a way that the middle part of the cable is horizontal.

Questions:

a. Determine the support reactions at A and B.
b. Determine the vertical component of the force in member AB.
c. Determine the maximum cable force.



14.12: 1–2 Two different cables with compression bar AB are loaded at C and D by forces.



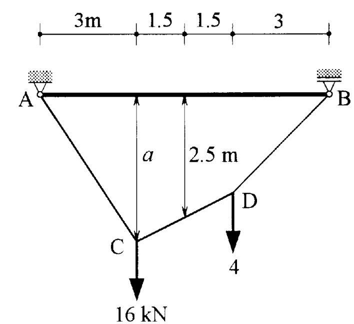
Questions:

a. Determine the force in compression bar AB.
b. Determine the distance a from point of application C to bar AB.
c. Determine the maximum cable force.

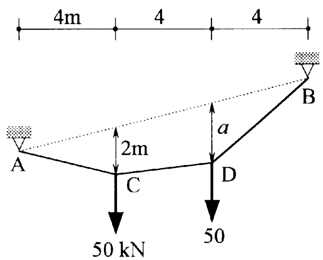
14.13 A cable with compression bar is loaded at C and D by two forces of 16 and 4 kN respectively. At midspan the distance between the cable and compression bar AB is 2.5 metres.

Questions:

a. Determine the force in the compression member.
b. Determine the distance a from point of application C to member AB
c. Determine the maximum cable force.



14.14 A cable suspended between A and B is loaded at C and D by two equal forces of 50 kN. The distance from point of application C to chord AB is 2 metres.



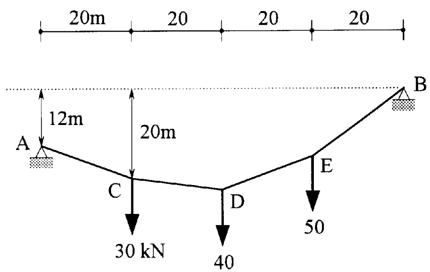
Questions:

a. Determine the horizontal component of the cable force.
b. Determine the distance a from point of application D to chord AB.

14.15 The cable shown is loaded at C, D and E by forces of 30, 40 and 50 kN respectively. The difference in elevation of the end supports at A and B is 12 metres. The distance from point of application C to the horizontal through B is 20 metres.

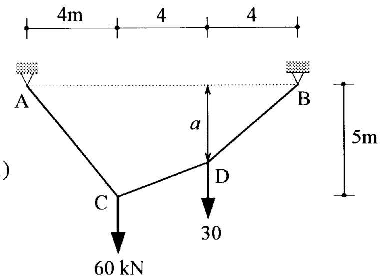
Questions:

a. Determine the distance from point of application D to the horizontal through B.
b. Determine the distance from point of application E to the horizontal through B.
c. Determine the support reactions at A.
d. Determine the support reactions at B.

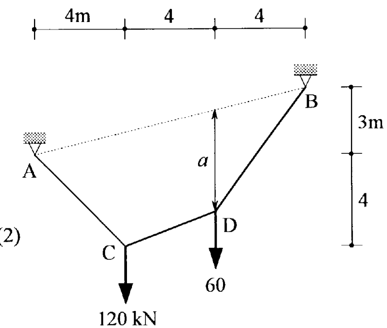


14.16: 1–2 Two cables are suspended between points A and B, and are loaded in C and D by forces. The horizontal measured lengths are equal in both cases.

(1)



(2)



Questions:

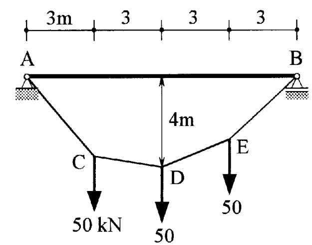
a. Determine the horizontal component of the cable force.
b. Determine the distance a of point of application D to chord AB.
c. Determine the maximum cable force.

14.17: 1–2 Given two cables with compression bar, loaded by three forces of 50 kN.

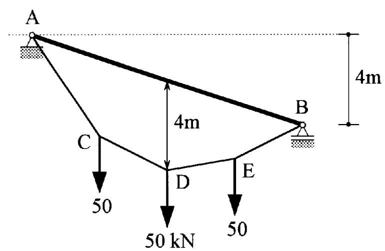
Questions:

a. Determine the force in the compression bar.
b. Draw the cable shape to scale.
c. Determine the minimum cable force.
d. Determine the maximum cable force.

(1)



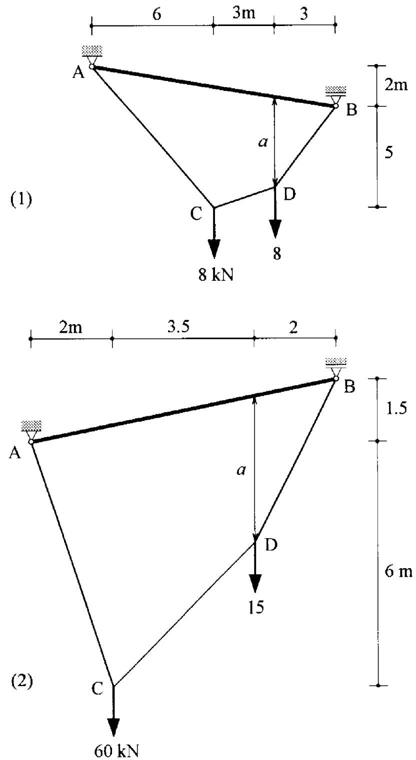
(2)



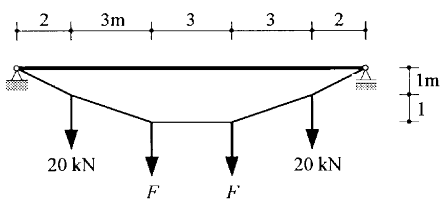
14.18: 1–2 Two cables with compression bar AB are loaded at C and D by forces.

Questions:

a. Determine the horizontal component of the cable force.
b. Determine the normal force in member AB.
c. Determine the distance a from point of application D to bar AB.
d. Determine the maximum cable force.



14.19 A cable with compression bar, loaded by two forces of 20 kN and two unknown forces F.

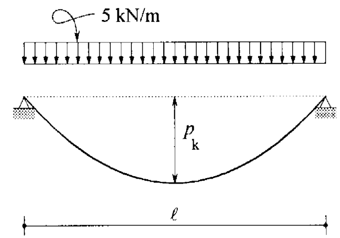


Questions:

a. Determine the force in the compression bar.
b. Determine the magnitude of forces F.

14.20: 1–3 A cable is carrying a uniformly distributed load of 5 kN/m and thereby assumes the shape shown.

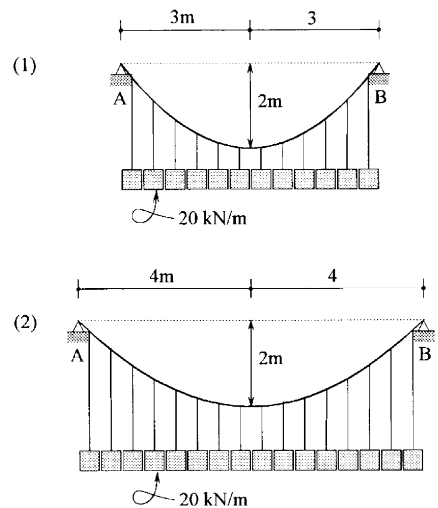
Question:
Determine the maximum cable force when:
(1) $\ell = 16$ m and $p_k = 3.20$ m.
(2) $\ell = 18$ m and $p_k = 4.50$ m
(3) $\ell = 20$ m and $p_k = 4$ m.



14.21: 1–2 Two cables with a uniformly distributed load of 20 kN/m.

Questions:

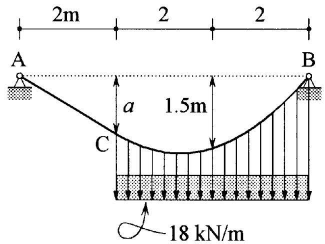
a. Determine the horizontal component of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.



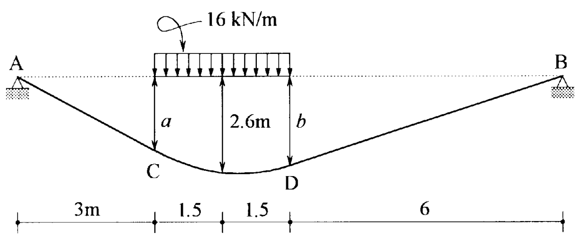
14.22 Due to the uniformly distributed load of 18 kN/m on CB cable AB assumes the shape shown.

Questions:

a. Determine the horizontal component of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.
d. Determine the sag a at C.
e. Determine the maximum sag of the cable.



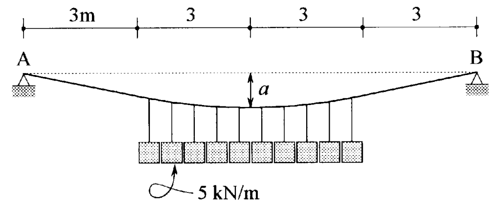
14.23 A uniformly distributed load of 16 kN/m is acting on cable AB between C and D. The cable sags 2.6 metres at the middle of the distributed load.



Questions:

a. Determine the horizontal component of the cable force.
b. Determine the sag a at C.
c. Determine the sag b at D.
d. Determine the maximum sag of the cable.

14.24 In the middle of cable AB a uniformly distributed load $q = 5$ kN/m is acting over a length of 6 metres. The horizontal component of the cable force is 75 kN.



Questions:

a. Determine the support reactions at A and B.
b. Determine the maximum cable force.
c. Determine the sag a in the middle of the cable.

14.25 See the figure for problem 14.24. In the middle part of cable AB a uniformly distributed load $q = 5$ kN/m acts over a length of 6 metres. The cable assumes the shape shown with $a = 1.25$ m.

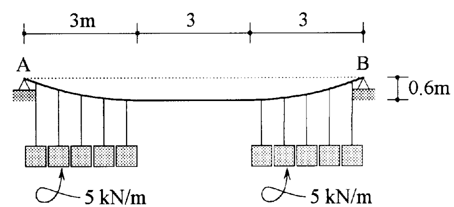
Questions:

a. Determine the horizontal component of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

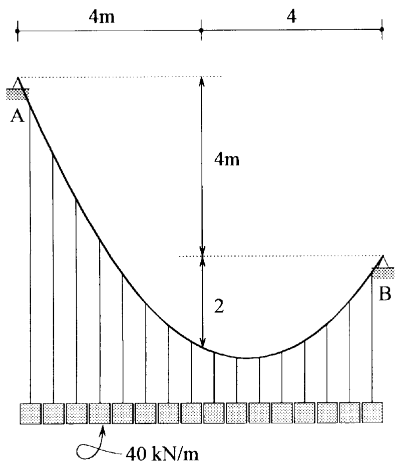
14.26 A uniformly distributed load $q = 5$ kN/m acts on cable AB at both ends over a length of 3 metres. The middle part is unloaded. The cable assumes the shape shown.

Questions:

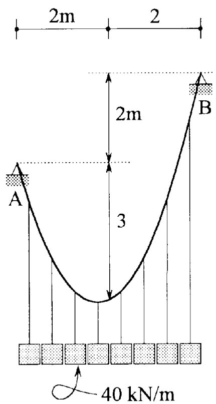
a. Determine the horizontal component of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.



14.27: 1–2 Given two cables with a uniformly distributed load of 40 kN/m.



(1)

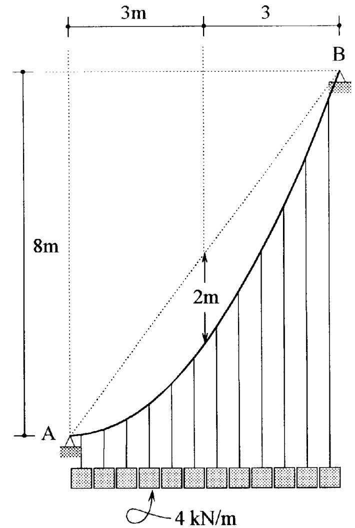


(2)

Questions:

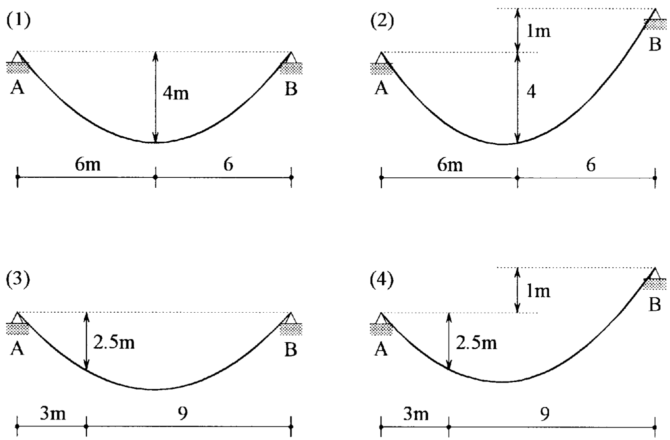
a. Determine the horizontal component of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.
d. Where is the lowest point of the cable?
e. Determine the difference in height between support A and the lowest point of the cable.

14.28 Given a cable with a uniformly distributed load of 4 kN/m.



Question:
Determine the support reactions at A and B.

14.29: 1–4 Given four cables that under the influence of a uniformly distributed load $q = 36$ kN/m assume the shape shown.



Questions:

a. Determine the horizontal component of the force in the cable.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

14.30: 1–4 As problem 14.29, but now with a uniformly distributed load q of 7.2 kN/m.

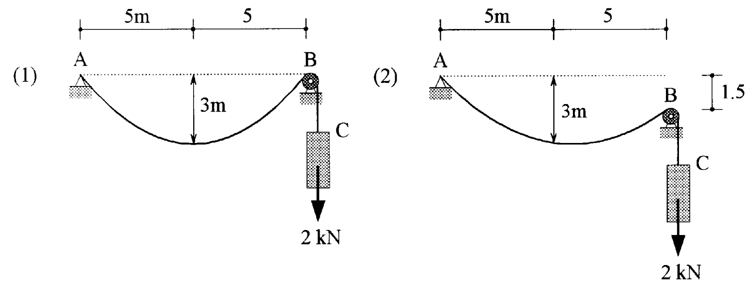
14.31: 1–4 Given four cables that under the influence of a uniformly distributed load q assume the shape in problem 14.29. The horizontal component of the cable force is 54 kN.

Questions:

a. Determine the magnitude of the distributed load q.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

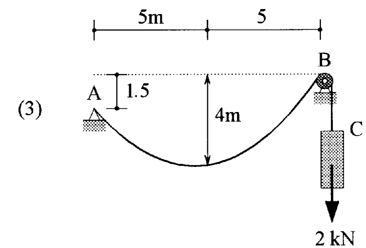
14.32: 1–4 As problem 14.31, but now the horizontal component of the cable force is 135 kN.

14.33: 1–3 At B, cable ABC passes over a pulley and is kept in equilibrium by a counterweight of 2 kN at C. The pulley is frictionless and has negligibly small dimensions. Assume that the cable has a parabolic shape due to of its dead weight. Ignore the dead weight of cable part BC.



Questions:

a. Determine the dead weight of the cable AB.
b. Determine the support reactions at A.
c. Determine the support reactions at B.



14.34 Given a cable.

Question:
Which load has to act on the cable to give it the following shapes:

a. Parabola.
b. Catenary.
c. Circle.

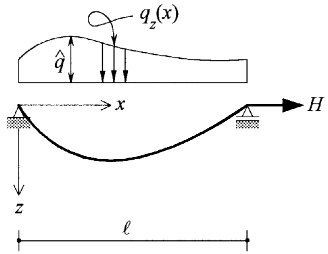
14.35: 1–4 A cable with span ℓ is loaded by four different distributed loads $q_z(x)$. Here $\hat{q}$ is the top value of the distributed load. The horizontal component of the cable force is H. In your calculation use $\ell = 15$ m, $\hat{q} = 8$ kN/m and $H = 75$ kN.

(1) $q_z = \hat{q}\dfrac{x}{\ell}$

(2) $q_z = 4\hat{q}\dfrac{x}{\ell}\left(1 - \dfrac{x}{\ell}\right)$

(3) $q_z = \hat{q}\sin\dfrac{\pi x}{\ell}$

(4) $q_z = \hat{q}\left(1 - \sin\dfrac{x}{\ell}\right)$



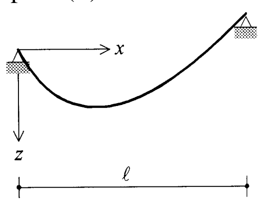
Questions:
a. Draw the load diagram.
b. Determine the cable shape as a function of x.
c. Determine the maximum sag in the cable.
d. Determine the maximum cable force.

14.36: 1–4 As problem 14.35, but now with values $\ell = 20$ m, $\hat{q} = 6$ kN/m and $H = 120$ kN.

14.37: 1–2 Under the influence of a distributed load $q_z(x)$ a cable with span ℓ assumes one of the following cable shapes $z(x)$:

(1) $z(x) = -(7.5 \text{ m})\left(\dfrac{x}{\ell}\right)^2 + (5 \text{ m})\left(\dfrac{x}{\ell}\right)$

(2) $z(x) = -(8 \text{ m})\left(\dfrac{x}{\ell}\right)^3 + (6 \text{ m})\left(\dfrac{x}{\ell}\right)$

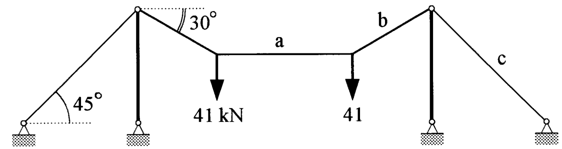


The horizontal component of the cable force is H. In your calculation use $\ell = 12$ m and $H = 60$ kN.

Questions:
a. Draw the cable shape.
b. Determine the maximum sag of the cable.
c. Determine the distributed load as a function of x.
d. Determine the maximum cable force.

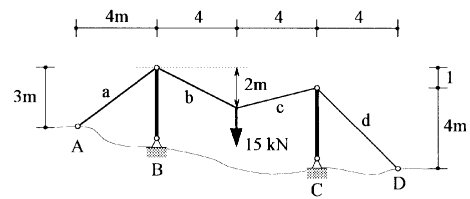
14.38: 1–2 As problem 14.37, but now with $\ell = 15$ m and $H = 90$ kN.

14.39 A symmetrical cable structure consists of two bar supports and a number of cables. The structure is loaded as shown by two forces of 41 kN.



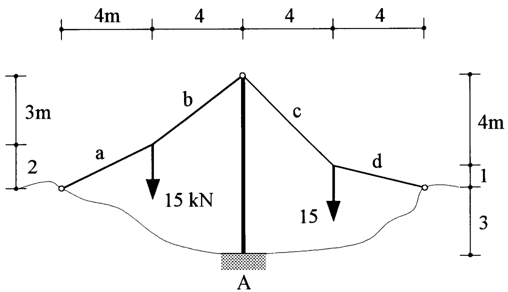
Questions:
a. Which of the cables a to c is most heavily loaded?
b. Determine the force in the most heavily loaded cable.
c. Determine the normal force in a bar support.

14.40 The cable structure shown consists of two bar supports and the four cables a, b, c and d. A force of 15 kN acts at midspan.



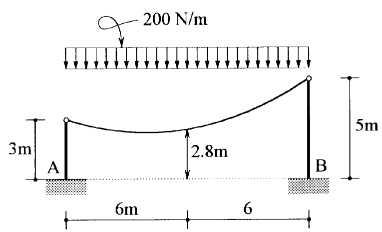
Questions:
a. Which cable is most heavily loaded?
b. Determine the force in the most heavily loaded cable.
c. Determine the support reactions at A to D and check the force and moment equilibrium of the structure as a whole.

14.41 The cable structure shown consists of a column fixed at A and the four cables a, b, c and d. The structure is loaded as shown by two forces of 15 kN.



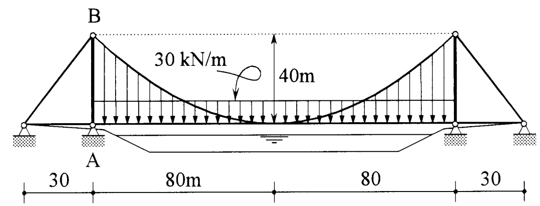
Questions:
a. Which cable is most heavily loaded?
b. Determine the support reactions at A, and draw them as they are acting in reality.

14.42 Party decorations made of coloured lamps and flags are suspended on a cable between two fixed columns of varying height. With a uniformly distributed load of 200 N/m the cable assumes the shape as shown.



Questions:
a. Determine the largest cable force.
b. Determine the support reactions at A. Draw them as they are acting in reality.
c. Determine the support reactions at B. Draw them as they are acting in reality.
d. Determine the free height under the lowest point of the cable.

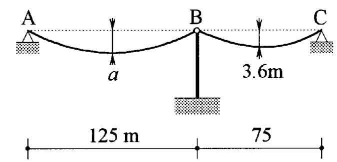
14.43 Using the cable structure shown, a number of pipelines are led across a river. The load on the cable is uniformly distributed and is 30 kN/m.



Question:
Determine the normal force in tower AB.

14.44 Two electricity cables, each with the same dead weight, (force per length), are attached to a mast at B. The maximum sag in field BC is 3.60 metres. We can assume that the cables have a parabolic shape due to their dead weight.

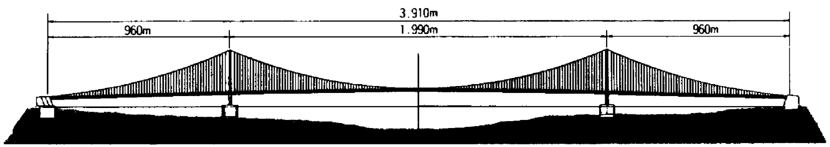
Questions:
a. Determine the maximum sag a in field AB so that the total horizontal force that the cables at B exert on the mast is zero.
b. Determine the support reactions at A and C in case the dead weight of the cables is 12 N/m.



14.45 The Akashi Kaikyo Bridge in Japan was the largest suspension bridge in the world when it was opened in 1998/1999. The main span is 1990 m, and the side spans are 960 m. The following values apply for the position of the cable with respect to the average seawater level in Tokyo Bay:

Location	Level in metres
End anchors	+53
Towers	+297
Centre main span	+96

The load on the cable, consisting of the dead weight (of the cable, hangers and deck structure) and traffic loading is 450 kN/m.



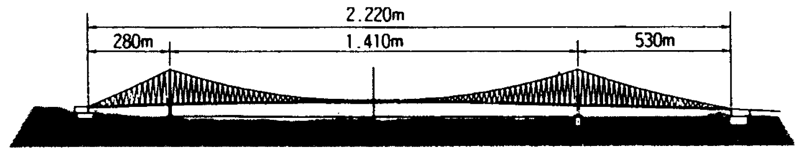
Questions:

a. Determine the forces that the cable in the middle span exerts on the towers.
b. Determine the forces that the cables in the end spans exert on the towers.
c. Determine the total load on a tower.
d. Determine the forces that the cables in the end spans exert on the anchor blocks.
e. Determine the maximum cable force in the middle span.
f. Determine the maximum cable force in the end spans.

14.46 On 17 July 1981, the suspension bridge over the Humber in Hull, England was opened, the longest suspension bridge in the world at the time. The bridge has a main span of 1410 m and end spans of respectively 290 and 530 m. A remarkable feature of this bridge is the major difference in length between both end spans. The following (estimated) values apply for the location of the cable:

Location	Level in metres
End anchors	+32.5
Towers	+162.5
Centre main span	+60.5

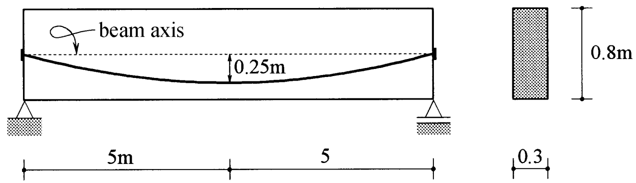
Assume that the load on the cable, consisting of the dead weight (of cable, hangers and deck structure) and the traffic loading, is 200 kN/m.



Questions:

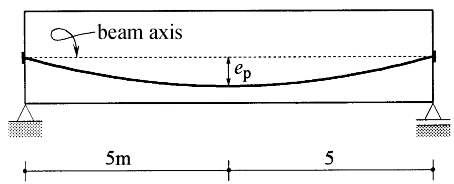
a. Determine the forces that the cable in middle field exerts on the towers.
b. Determine the forces that the cable in the left-hand end field exerts on the tower.
c. Determine the total load on the left-hand tower.
d. Determine the forces that the cable in the left-hand end field exerts on the anchor block.
e. Determine the forces that the cable in the right-hand end field exerts on the tower.
f. Determine the total load on the right-hand tower.
g. Determine the forces that the cable in the right-hand end field exerts on the anchor block.
h. Determine the maximum cable force in the middle field.
i. Determine the maximum cable force in the left-hand end field.
j. Determine the maximum cable force in the right-hand end field.
k. The cable in the short end field was designed stronger than the cable in the other fields. Is this in line with your calculation?

14.47 A simply supported prestressed beam has a length of 10 metres and a rectangular cross-section of 0.3×0.8 m^2. The prestressing cable is parabolic with a maximum eccentricity of 0.25 m. The prestressing force is 1000 kN. The dead weight of the beam is 2500 kN/m^3. The variable load is 30 kN/m.



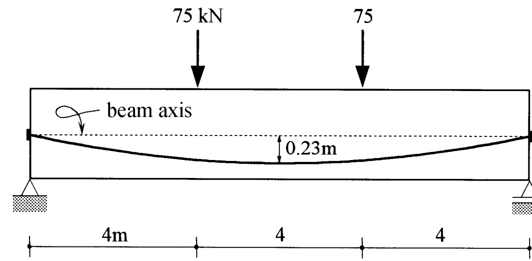
Questions:
a. Determine the forces that the prestressing cable exerts on the beam.
b. Determine the M and V diagrams for the beam resulting from only the prestressing.
c. Determine the M and V diagrams due to the prestressing and dead weight.
d. Determine the M and V diagrams due to the prestressing, dead weight and variable load.

14.48 A simply supported prestressed beam with a length of 10 metres has a dead weight of 6 kN/m. The variable load is 24 kN/m. The parabolic tendon has a maximum eccentricity e_p. The prestressing force is F_p.



Questions:
a. How large must the product $F_p e_p$ be so that the maximum bending moment due to the prestressing and dead weight (in an absolute sense) is equal to the maximum bending moment due to the prestressing, dead weight and variable load?
b. How large must the product $F_p e_p$ be so that the maximum bending moment due to the prestressing and dead weight (in an absolute sense) is 5/7 the magnitude of the maximum bending moment due to the prestressing, dead weight and variable load?

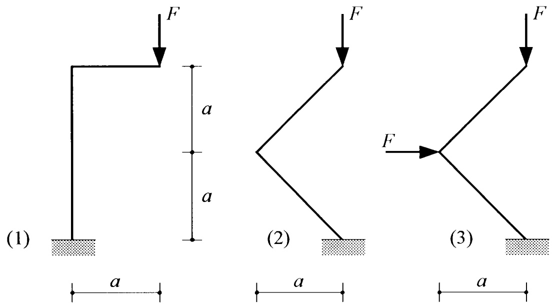
14.49 A simply supported prestressed beam with a length of 12 metres has a dead weight of 7 kN/m. The variable load consists of two forces of 75 kN. The parabolic prestressing cable has an eccentricity of 0.23 m. The prestressing force is 1200 kN.



Questions:
a. Determine the forces that the prestressing cable exerts on the beam.
b. Determine the M and V diagrams for the beam due to only the prestressing.
c. Determine the M and V diagrams due to the prestressing and dead weight.
d. Determine the M and V diagrams due to the prestressing, dead weight and variable load.

Centre of point and line of force (Section 14.2)

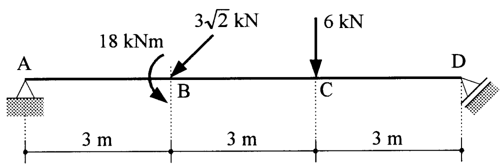
14.50: 1–3 Three fixed and bent beams are loaded by forces F.



Questions:

a. For each part of the structure draw the line of force and indicate whether it is a line of tension or a line of pressure.
b. From the line of force derive the variation of the normal force, shear force and bending moment. Draw the N, V and M diagrams for the entire structure.

14.51 Beam ABCD is supported at A on a hinge and at D on a roller; the roller track is inclined at an angle of 45°. A couple acts at B, as well as a 45° force. A vertical force acts at C. Dimensions and load can be found in the figure.



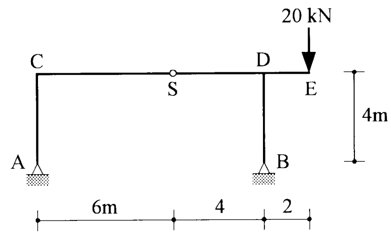
Questions:

a. Calculate and draw the support reactions at A and D.
b. Calculate and draw the M diagram, V diagram and N diagram, with the deformation symbols. Write down the relevant values.
c. For ABCD, draw the line(s) of force, and indicate whether they refer to tension or compression.
d. Where in cross-section C is the centre of force?

14.52 A three-hinged frame with overhang, is loaded by a force of 20 kN on the overhang.

Questions:

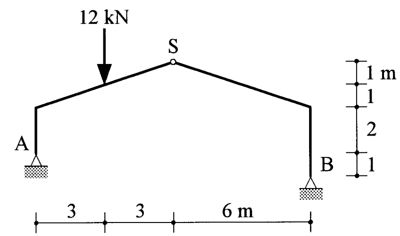
a. Determine the line of force for all parts of the frame and indicate whether they refer to tension or compression.
b. Determine the centre of force at cross-section C on column AC.
c. Determine the centre of force at cross-section C of girder CS.



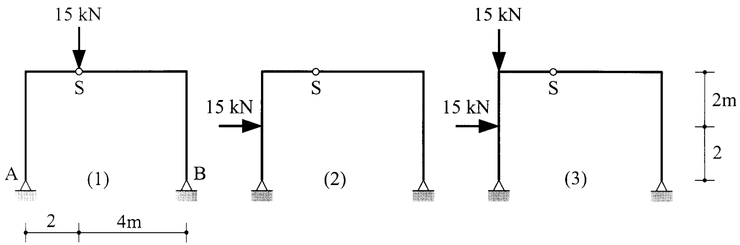
14.53 Pitched roof portal frame ASB is loaded by a vertical force of 12 kN.

Questions:

a. Determine the support reactions at A and B. Draw them as they act in reality and include relevant values.
b. Draw the M diagram, with the deformation symbols. Include relevant values.
c. Draw the lines of force for all parts of the frame. Indicate clearly whether they refer to tension or compression.



14.54: 1–3 The same three-hinged frame ASB is loaded in various ways by forces of 15 kN.



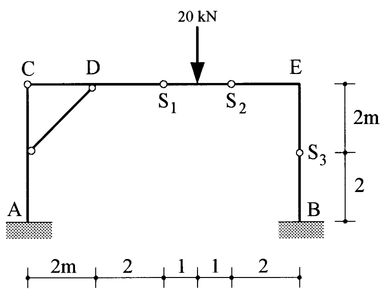
Questions:

a. Draw the lines of force for all parts of the frame. Indicate clearly whether they refer to tension or compression.
b. Draw the M and N diagrams for all parts of the frame.

14.55 Given a compound frame.

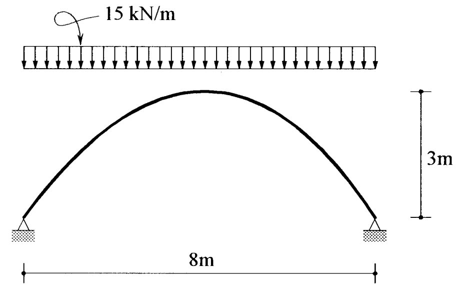
Questions:

a. Draw the M diagram for CDE.
b. Draw the line of force for DE.
c. How large is the normal force in DE?
d. Draw the line of force for CD.



Relationship between cable, line of force and structural shape (Section 14.3)

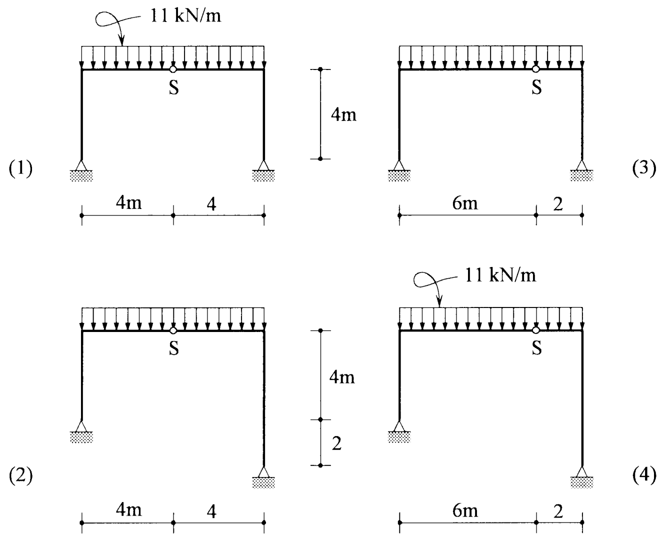
14.56 A parabolic arch has a uniformly distributed load. There is no axial deformation.



Questions:

a. Draw the line of force for the arch. Does it refer to tension or compression?
b. Determine and draw the horizontal support reactions.
c. Determine and draw the vertical support reactions.
d. Determine the maximum normal force in the arch.

14.57: 1–4 Four three-hinged frames have the same uniformly distributed load of 11 kN/m.



Questions:

a. Sketch the line of force for the girder.
b. Determine the normal force in the girder from this line of force.
c. Determine and draw the support reactions.
d. Draw the M, V and N diagrams for the entire structure.

15 Virtual Work

In this chapter we deal with the virtual work equation: an often used alternative for the equilibrium equations.

In Section 15.1, we first introduce the concepts *work* and *strain energy*. Performing work can be seen as a mechanical process of a body exchanging energy with its environment. To illustrate the concept, we have included a separate section on strain energy. The strain energy concept plays an important role in calculations that are based on energy considerations, but is not covered further in this chapter.

The concept of work returns in the *virtual work equation*. In Sections 15.2 to 15.4 we show for a particle, a rigid body and a mechanism respectively that the virtual work equation is equivalent to the equilibrium equations. Support reactions and section forces can be derived directly from the equilibrium equations, but also with the *principle of virtual work*. We provide a number of examples in Section 15.5.

The virtual work equation is especially useful for determining the influence lines for support reactions and section forces in statically determinate bar type structures; this is covered in Chapter 16.

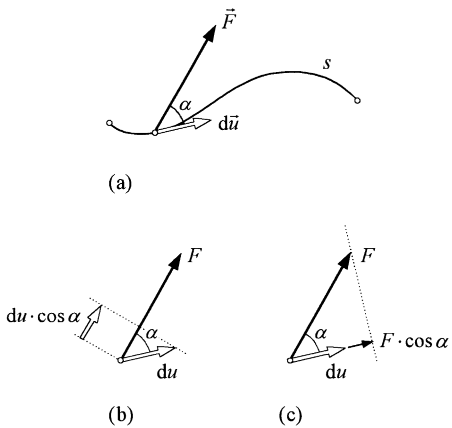


Figure 15.1 Work is defined as the inner product of force $\vec{F}$ and displacement $\mathrm{d}\vec{u}$:
$\mathrm{d}A = \vec{F} \cdot \vec{u} = |\vec{F}| \cdot |\mathrm{d}\vec{u}| \cdot \cos\alpha = F \cdot \mathrm{d}u \cos\alpha = F \cos\alpha \cdot \mathrm{d}u$.

15.1 Work and strain energy

In this section we look at the concepts work and strain energy.

15.1.1 Work

If the point of application of force $\vec{F}$ in Figure 15.1a undergoes an infinitesimal displacement $\mathrm{d}\vec{u}$ along path s, this is referred to as the force performing an (infinitesimal) amount of *work* $\mathrm{d}A$, defined as the *inner product* of the vectors $\vec{F}$ and $\mathrm{d}\vec{u}$:

$$\begin{aligned} \mathrm{d}A &= \vec{F} \cdot \mathrm{d}\vec{u} \\ &= F_x\,\mathrm{d}u_x + F_y\,\mathrm{d}u_y + F_z\,\mathrm{d}u_z. \end{aligned}$$

Work is a *scalar quantity*.

The inner product of two vectors can also be calculated as the product of the magnitude (modulus) of both vectors and the cosine of the enclosed angle:

$$\mathrm{d}A = |\vec{F}| \cdot |\mathrm{d}\vec{u}| \cdot \cos\alpha = F \cdot \mathrm{d}u \cdot \cos\alpha.$$

This can be seen as the product of the force and the component of the displacement in the direction of the force, F and $\mathrm{d}u \cos\alpha$ respectively (see Figure 15.1b). It can also be seen as the product of the displacement and the component of the force in the direction of the displacement, $\mathrm{d}u$ and $F\cos\alpha$ respectively (see Figure 15.1c).

Note that the force F does not perform any work if it is normal to the displacement $\mathrm{d}u$ (in that case $\alpha = \pm\pi/2$ and $\cos\alpha = 0$).

If the point of application of the force moves a finite distance along path s, the total amount of work is equal to the sum of the contributions of all the infinitesimal displacements. Mathematically this corresponds to integrating over the path length s:

$$A = \int_s \vec{F} \cdot \mathrm{d}\vec{u}.$$

The magnitude and direction of the force F may depend on the location on the route. If $\vec{F}$ is constant (for a vector that means constant in magnitude and direction), then $\vec{F}$ can be excluded from the integration symbol. The total amount of work performed is then (see Figure 15.2):

$$A = \int_s \vec{F} \cdot \mathrm{d}\vec{u} = \vec{F} \int_s \mathrm{d}\vec{u} = \vec{F} \cdot \vec{u}.$$

In this case, the total amount of work A depends only on the position of the starting and end points and not on the shape of the route followed.

In full

$$A = F_x u_x + F_y u_y + F_z y_z = F u \cos\alpha.$$

Note that no work is performed if $\vec{F}$ and $\vec{u}$ are normal to one another (in that case $\alpha = \pm\pi/2$ and $\cos\alpha = 0$).

Forces that are constant in magnitude and direction include gravitational forces.

The dimension of work is force multiplied by distance. The applicable SI unit is the *joule*, denoted as J:

$$\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2.$$

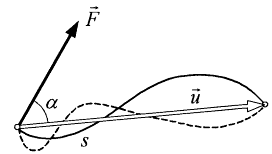


Figure 15.2 If $\vec{F}$ is constant, the total amount of work to be performed depends only on the location of the starting point and end point, and not on the route followed.

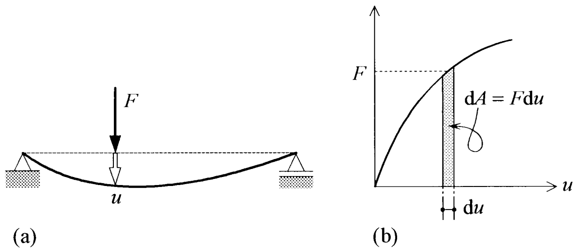


Figure 15.3 (a) A beam subject to bending with (b) the associated load-displacement diagram.

15.1.2 Strain energy

Consider the simply supported beam in Figure 15.3a, with a point load F. Due to a load, the beam will bend. The sag at the concentrated load is u. The relationship between the load F and displacement u can be shown in a *load-displacement diagram* (see Figure 15.3b). The shape of the diagram depends on the properties of the material. The shape is not important at this stage.

If with an increasing load the displacement u increases by an amount $\mathrm{d}u$, the force F performs work

$$\mathrm{d}A = F\,\mathrm{d}u.$$

When the load and displacement have reached their final value, the total amount of work performed is:

$$A = \int_0^u F\,\mathrm{d}u.$$

The total amount of work performed is equal to the area under the load-displacement diagram.

Performing work can be seen as a *mechanical process of energy exchange* between a body and its environment. If the load performs positive work, energy is extracted from the environment and transferred to the structure. If there is no exchange of heat from the structure to its environment, and the structure is and remains at rest (the energy added is not converted into kinetic energy), then the energy transferred is absorbed as *strain energy*. Strain energy is the energy required to deform the structure.

The work performed A is equal to the increase in strain energy E_v:

$$A = \Delta E_\mathrm{v}.$$

In general it is assumed that the strain energy is zero in the undeformed state. In that case, it holds for the deformed state that

$$A = E_\mathrm{V}.$$

The strain energy stored in the beam is equal to the area under the load displacement diagram (see Figure 15.4).

The SI unit for strain energy is *joule*.

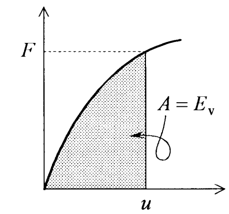


Figure 15.4 The strain energy E_V stored in the beam is equal to the total amount of work performed A, and is equal to the area under the load-displacement diagram.

15.2 Virtual work equation for a particle

In this section, we show that the virtual work equation for a particle is just another form of the equilibrium equations.

When particles are compelled to follow a particular path, the *virtual displacements* that are in line with the (limited) degree of freedom of the particle are known as *kinematically admissible virtual displacements*. These displacements are subject to special demands: they must be geometrically linear. Physically this can be translated into the demand that the virtual displacements must be very small.

15.2.1 Equilibrium

Assume a particle subject to forces (see Figure 15.5). The particle is in equilibrium if the equations for the force equilibrium in the x, y and z directions respectively are satisfied:

$$\sum F_x = 0,$$

$$\sum F_y = 0,$$

$$\sum F_z = 0.$$

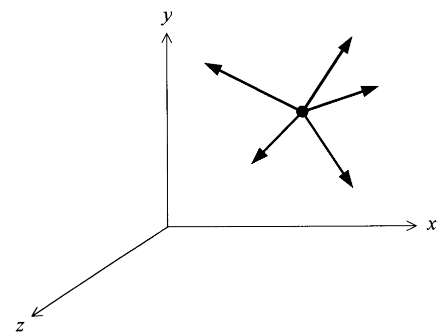


Figure 15.5 A particle subject to forces.

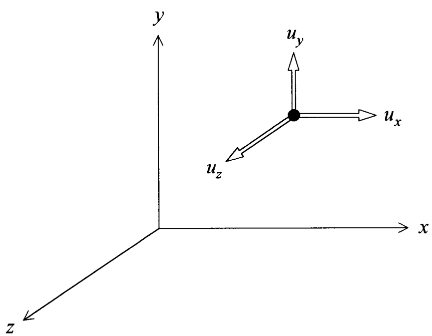


Figure 15.6 The displacement components of a particle.

These three equilibrium equations can also be formulated otherwise.

Assume G is a new quantity, defined as follows:

$$G = \lambda_1 \sum F_x + \lambda_2 \sum F_y + \lambda_3 \sum F_z.$$

In this equation, λ_1; λ_2; λ_3 are *arbitrary quantities* that *cannot all equal zero concurrently*.

The demand that $G = 0$ for each arbitrary combination of λ_1; λ_2; λ_3 (not all equal to zero concurrently) is equivalent to the three equations for the force equilibrium. For example, the combination could be

$$\lambda_1 \neq 0; \;\; \lambda_2 = 0; \;\; \lambda_3 = 0$$

in which case

$$G = \lambda_1 \sum F_x + 0 \times \sum F_y + 0 \times \sum F_z = \lambda_1 \sum F_x$$

and G can be equal to zero only if

$$\sum F_x = 0.$$

Likewise, the combination $\lambda_1 = 0$; $\lambda_2 \neq 0$; $\lambda_3 = 0$ leads to $\sum F_y = 0$. The combination $\lambda_1 = 0$; $\lambda_2 = 0$; $\lambda_3 \neq 0$ leads to $\sum F_z = 0$.

15.2.2 Virtual work equation

The quantities that are to be chosen arbitrarily λ_1; λ_2; λ_3 can also be considered to be arbitrary (imagined) displacements u_x; u_y; u_z (see Figure 15.6), so that

$$G = u_x \sum F_x + u_y \sum F_y + u_z \sum F_z.$$

In this case, G can be interpreted as the (imagined) work A done by the forces acting on the particle.

Since we are not talking about actual but rather *imagined displacements*, of *arbitrary magnitude*, they are referred to as *virtual displacements* and they are denoted by δu_x; δu_y; δu_z (see Figure 15.7). In mathematics, δ is known as the *variation symbol*. A virtual displacement therefore stands for a *variation* of the displacement.

The work due to the virtual displacements is known as *virtual work*. This is denoted by δA:

$$\delta A = \delta u_x \sum F_x + \delta u_y \sum F_y + \delta u_z \sum F_z.$$

If the displacement of a particle in equilibrium is varied, the virtual work done by the forces acting on it is zero. The converse is also true: if one varies the displacement of a particle, and the virtual work is zero, then the particle is in equilibrium.

Conclusion: *A particles is in equilibrium only if the virtual work performed by the forces acting on it is zero for any virtual displacement*:

$$\delta A = \delta u_x \sum F_x + \delta u_y \sum F_y + \delta u_z \sum F_z = 0.$$

This is known as the *principle of virtual work*.

The principle of virtual work combines the three independent equilibrium equations into one virtual work equation. The virtual work equation is just another form of the equilibrium equations.

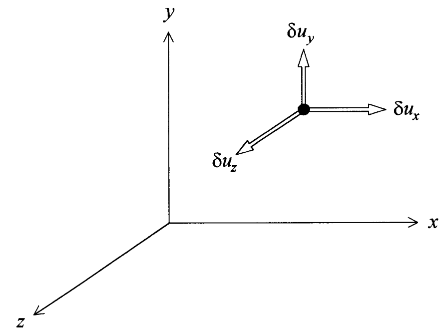


Figure 15.7 The virtual displacements of a particle.

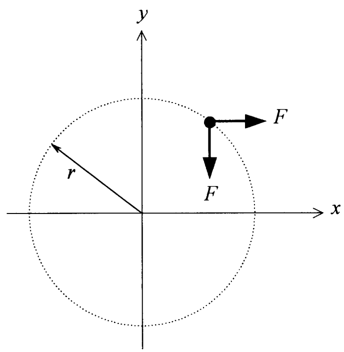


Figure 15.8 A particle that is compelled to follow a circular path with radius r is loaded by two forces F.

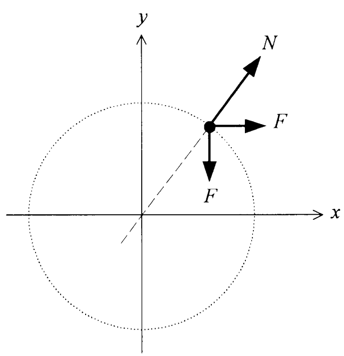


Figure 15.9 The isolated particle with all the forces acting on it.

15.2.3 Kinematically admissible virtual displacements

A particle is compelled to follow a circular path with diameter r in the xy plane. The path can be defined as

$$f(x, y) = x^2 + y^2 - r^2 = 0.$$

There is no friction. The particle is loaded by a horizontal and vertical force F, as shown in Figure 15.8.

Using the principle of virtual work, we now look for the positions on the circle at which the particle is in equilibrium.

In Figure 15.9 the particle has been isolated. Since there is no friction, the interaction force N is normal to the circular path.

If we are not interested in the interaction force N between the particle and its path, we can choose a virtual displacement along the prescribed path. Since N is normal to the path, it does not appear in the virtual work equation.

A virtual displacement that is conform with the (limited) freedom of movement of the particle is referred to as a *kinematically admissible virtual displacement*.

The virtual work δA due to a kinematically admissible virtual displacement is

$$\delta A = +F\delta u_x - F\delta u_y. \tag{a}$$

δu_x and δu_y are the virtual displacements in the x and y directions respectively.

For the path it holds that

$$f(x, y) = x^2 + y^2 - r^2 = 0. \tag{b}$$

After the virtual displacement, it applies that

$$f(x + \delta u_x, y + \delta u_y) = (x + \delta u_x)^2 + (y + \delta u_y)^2 - r^2 = 0. \tag{c}$$

If we combine equations (c) and (b) we find the following relationship between δu_x and δu_y:

$$2x\delta u_x + (\delta u_x)^2 + 2y\delta u_y + (\delta u_y)^2 = 0. \tag{d}$$

Since this equation is determined by the geometry of the prescribed path, it is known as a *geometric equation*.

During the variation of the displacement, the forces do not change direction. The same is true for N. Since N does not perform work, the virtual displacement has to occur along the tangent of the prescribed path, as shown in Figure 15.10. This means that the geometric relationship between δu_x and δu_y has to be linear. Ignoring the quadratic (higher order) terms in the geometric equation (d) can be physically interpreted as a demand that the virtual displacements have to be small.

If we remove the quadratic terms in (d)[1] we find

$$2x\delta u_x + 2y\delta u_y = 0$$

or

$$\delta u_y = -\frac{x}{y}\delta u_x. \tag{e}$$

When the particle is in equilibrium, $\delta A = 0$. With (a) and (e) the virtual work equation becomes

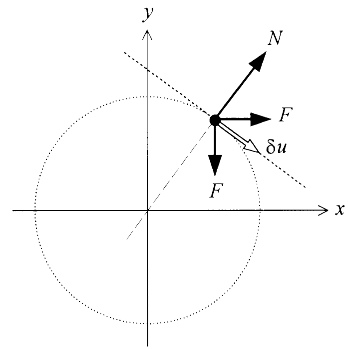


Figure 15.10 During the change in displacement the forces do not change direction. Since N performs no work the virtual displacement δu has to take place along the tangent to the prescribed path.

[1] This is referred to as the linearisation of the geometric equation (d).

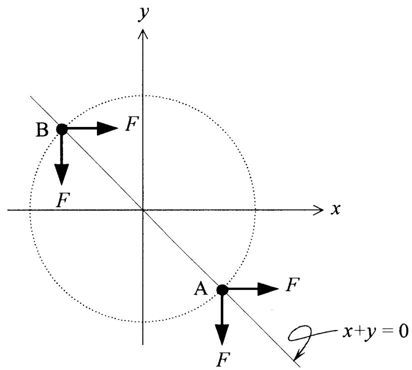


Figure 15.11 There are two locations on the circular path where the particle is in equilibrium under the influence of forces F: A and B.

$$\delta A = +F\delta u_x - F\delta u_y = F\delta u_x + F\frac{x}{y}\delta u_x = \left(1+\frac{x}{y}\right)F\delta u_x = 0.$$

Since $F \neq 0$ and $\delta u_x \neq 0$ the solution is

$$1+\frac{x}{y} = 0 \quad \text{or} \quad x+y=0.$$

The equilibrium positions are therefore on the line $x+y=0$. This leads to two solutions: the particle under the influence of forces F is in equilibrium at either A or B (see Figure 15.11).

Comment: The principle of virtual work says nothing about the *state of the equilibrium*. It cannot be used to discover that the equilibrium at A is reliable (stable) and that the equilibrium at B is unreliable (unstable).[1]

15.2.4 Virtual displacements in a mathematical sense

The fact that the geometric equations between the varied displacements have to be linear means, from a mathematical perspective, that we have to consider the so-called *first-order variation*. The first-order variation of a function $f(x, y)$ is defined as

$$\delta f(x,y) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

---

[1] If the particle at B loses its equilibrium as a result of a small disruption it will become increasingly far removed from the original equilibrium position due to forces F: the equilibrium at B is unreliable (unstable equilibrium). At A the forces F compel the particle to return to its original equilibrium position after disruption: the equilibrium at A is reliable (stable equilibrium). Examining the reliability of equilibrium (*stability investigation*) is beyond the scope of this book.

or, with $\delta x = \delta u_x$ and $\delta y = \delta u_y$,

$$\delta f(x, y) = \frac{\partial f}{\partial x}\delta u_x + \frac{\partial f}{\partial y}\delta u_y.$$

For the function

$$f(x, y) = x^2 + y^2 - r^2 = 0$$

that describes the circular path of the particle we find

$$\begin{aligned}\delta f(x, y) &= \frac{\partial}{\partial x}(x^2 + y^2 - r^2)\delta u_x + \frac{\partial}{\partial y}(x^2 + y^2 - r^2)\delta u_y \\ &= 2x\delta u_x + 2y\delta u_y = 0.\end{aligned}$$

This leads directly to the expression (e) we are looking for:

$$\delta u_y = -\frac{x}{y}\delta u_x.$$

The method used here is far simpler than the approach in Section 15.2.3, where we first derived the geometric equation (d) and then *linearised* it (by removing all the non-linear terms).

15.3 Virtual work equation for a rigid body

For rigid bodies, the complete equilibrium equations can also be replaced by a single virtual work equation. Deriving the virtual work equation for a rigid body shows again that the geometrical relationship between the virtual displacements has to be linear.

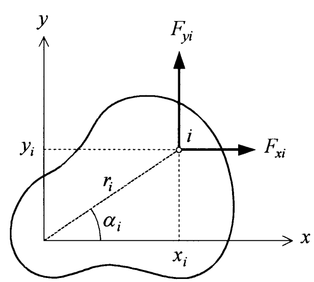


Figure 15.12 The components of force F at point i.

15.3.1 Equilibrium

In a plane[1] there are three equilibrium equations, two for the force equilibrium:

$$\sum F_x = 0,$$
$$\sum F_y = 0$$

and one for the moment equilibrium:

$$\sum T_z = 0.$$

There is equilibrium when all three conditions are satisfied.

The requirement

$$G = \lambda_1 \sum F_x + \lambda_2 \sum F_y + \lambda_3 \sum T_z = 0$$

for all arbitrary combinations of $\lambda_1; \lambda_2; \lambda_3$ (not all concurrently zero) is an alternative for the three equilibrium equations:

- The combination $\lambda_1 \neq 0; \lambda_2 = 0; \lambda_3 = 0$ leads to $\sum F_x = 0$.
- The combination $\lambda_1 = 0; \lambda_2 \neq 0; \lambda_3 = 0$ leads to $\sum F_y = 0$.
- The combination $\lambda_1 = 0; \lambda_2 = 0; \lambda_3 \neq 0$ leads to $\sum T_z = 0$.

Assume a number of forces are acting on the body (see Figure 15.12):

$F_{xi}; F_{yi}$ are the components of F_i at point i with coordinates $x_i; y_i$;
$F_{xj}; F_{yj}$ are the components of F_j at point j with coordinates $x_j; y_j$; etc.

[1] For simplicity we will look only at rigid bodies in the xy plane.

To keep the picture simple, Figure 15.12 includes only the components of the force at point i.

The body is in equilibrium when the following condition is satisfied:

$$G = \lambda_1 \sum_i F_{xi} + \lambda_2 \sum_i F_{yi} + \lambda_3 \sum_i (x_i F_{yi} - y_i F_{xi}) = 0$$

for each arbitrary choice of λ_1; λ_2; λ_3 (not all concurrently zero).

15.3.2 Displacement of a point on a rigid body

The displacement of a rigid body in a plane is defined by the displacement of, for example, point O to O′ and a rotation about O. Assume that the components of the *translation* (displacement) are u_{x0}; u_{y0} and the *rotation* is φ_{z0}.

Instead of using its coordinates x_i; y_i, we can define the location of an arbitrary point i also by its angle α_i and the radius r_i. The displacement of point i is (see Figure 15.13)

$$u_{xi} = u_{x0} - a,$$

$$u_{yi} = u_{y0} + b.$$

a and b are the result of the rotation φ_{z0}. Due to the rotation, point i moves through an arc length $\varphi_{z0} r_i$ along the circle with radius r_i and centre O′. For a and b it applies that

$$a = r_i \left\{ \cos\alpha_i - \cos(\alpha_i + \varphi_{z0}) \right\},$$

$$b = r_i \left\{ \sin(\alpha_i + \varphi_{z0}) - \sin\alpha_i \right\}.$$

The expressions are much simplified when the rotation is small. If $\varphi_{z0} \ll 1$

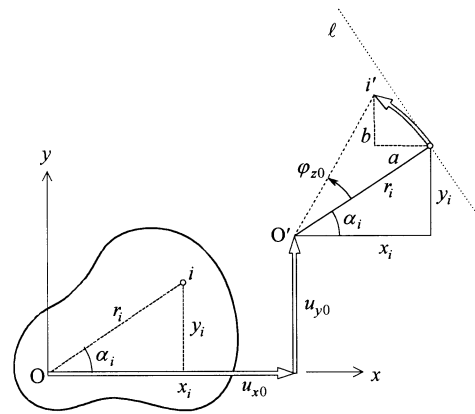


Figure 15.13 The displacement of point i due to a translation u_{x0}; u_{y0} and a *large* rotation φ_{z0}.

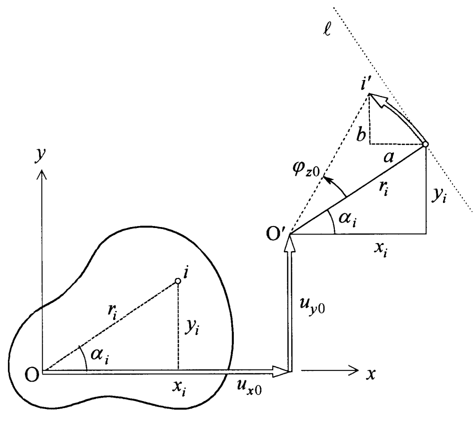


Figure 15.13 The displacement of point i due to a translation u_{x0}; u_{y0} and a *large* rotation φ_{z0}.

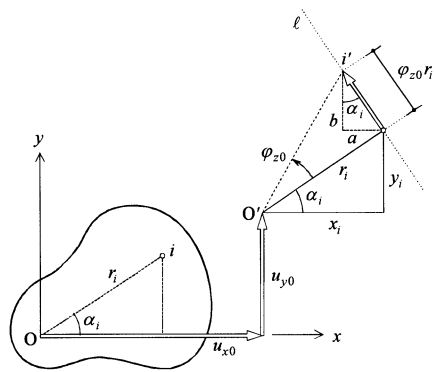


Figure 15.14 The displacement of point i due to a translation u_{x0}; u_{y0} and a *small* rotation φ_{z0}.

the circle can be replaced by its tangent ℓ (see Figure 15.14). The displacement of point i due to the rotation is then $\varphi_{z0} r_i$ with the following components:

$$a = \varphi_{z0} r_i \sin\alpha_i = \varphi_{z0} y_i,$$

$$b = \varphi_{z0} r_i \cos\alpha_i = \varphi_{z0} x_i.$$

For small rotations the following applies (ignoring the signs):

- *the horizontal displacement is equal to "rotation × vertical distance to the centre of rotation"*;
- *the vertical displacement is equal to "rotation × horizontal distance to the centre of rotation"*.

Conclusion: *Due to a translation u_{x0}; u_{y0} and a small rotation φ_{z0} of the body the displacement of point i is*

$$u_{xi} = u_{x0} - a = u_{x0} - \varphi_{z0} y_i,$$

$$u_{yi} = u_{x0} + b = u_{y0} + \varphi_{z0} x_i.$$

Note that the geometric relationships between the various displacement quantities are linear for small rotations.

15.3.3 Virtual work equation

When a body is given a virtual displacement, the virtual work performed by all forces acting on it is

$$\delta A = \sum_i F_{xi} \delta u_{xi} + \sum_i F_{yi} \delta u_{yi}.$$

The virtual displacements δu_{xi}; δu_{yi} of point i can be expressed by three independent virtual displacements δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$ of the body. The equation for δA assumes the form of expression G (see Section 15.3.1) only if the relationships between δu_{xi}; δu_{yi} and δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$ are linear.

The geometric relationships appeared to be linear only for bodies subject to minor rotations. We can also say that the virtual displacements have to be very small.

In that case, the virtual work performed by the force at point i is

$$F_{xi}\delta u_{xi} + F_{yi}\delta u_{yi} = F_{xi}(\delta u_{x0} - y_i\delta\varphi_{z0}) + F_{yi}(\delta u_{y0} + x_i\delta\varphi_{z0}).$$

The total work performed by all the forces at the points $i, j, \ldots$ is

$$\delta A = \sum_i F_{xi}(\delta u_{x0} - y_i\delta\varphi_{z0}) + \sum_i F_{yi}(\delta u_{y0} + x_i\delta\varphi_{z0}).$$

This gives

$$\delta A = \delta u_{x0}\sum_i F_{xi} + \delta u_{y0}\sum_i F_{yi} + \delta\varphi_{z0}\sum_i (x_i F_{yi} - y_i F_{xi}).$$

The expression for virtual work δA is now in the same form as the expression for G; the quantities λ_1; λ_2; λ_3 have been replaced by the virtual displacements δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$.

The demand

$$\delta A = \delta u_{x0}\sum_i F_{xi} + \delta u_{y0}\sum_i F_{yi} + \delta\varphi_{z0}\sum_i (x_i F_{yi} - y_i F_{xi}) = 0$$

for all arbitrary combinations of δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$ (not all equal to zero) is a necessary and sufficient condition for equilibrium and is known as the *principle of virtual work.*

For simplicity, concentrated couples were not addressed. A concentrated couple can be replaced by a statically equivalent pair of forces. Assuming

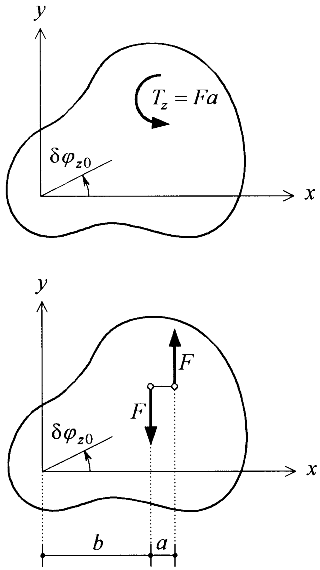


Figure 15.15 A concentrated couple can be replaced by a statically equivalent pair of forces. The virtual work performed by the couple is equal to the product of couple and rotation.

the pair of forces in Figure 15.15 we find

$$\delta A = -F \cdot b\delta\varphi_{z0} + F \cdot (a+b)\delta\varphi_{z0} = Fa \cdot \delta\varphi_{z0} = T_z \cdot \delta\varphi_{z0}.$$

The work performed by a couple is equal to the product of couple and rotation. The work is positive when the couple and rotation are in the same direction.

15.3.4 Virtual rotations in a mathematical sense

In deriving the virtual work equation, we found that the geometric relationships had to be linear in the virtual displacements. Mathematically, this means that the *first-order variation* has to be assumed for these virtual displacements.

We previously deduced (see Figure 15.13) that the following applies for the displacement of an arbitrary point i, due to a translation u_{x0}; u_{y0} and a rotation φ_{z0}:

$$\begin{aligned} u_{xi} &= u_{x0} - r_i\{\cos\alpha_i - \cos(\alpha_i + \varphi_{z0})\}, \\ u_{yi} &= u_{y0} + r_i\{\sin(\alpha_i + \varphi_{z0}) - \sin\alpha_i\}. \end{aligned}$$

The first-order variation of u_{xi} is defined as

$$\delta u_{xi} = \frac{\partial u_{xi}}{\partial u_{x0}}\delta u_{x0} + \frac{\partial u_{xi}}{\partial \varphi_{z0}}\delta\varphi_{z0}.$$

In the same way, the first-order variation of u_{yi} is defined as

$$\delta u_{yi} = \frac{\partial u_{yi}}{\partial u_{y0}}\delta u_{y0} + \frac{\partial u_{yi}}{\partial \varphi_{z0}}\delta\varphi_{z0}.$$

Elaborating these expressions (for $\varphi_{z0} = 0$) indeed leads to

$$\delta u_{xi} = \delta u_{x0} - r_i \sin\alpha_i \delta\varphi_{z0} = \delta u_{x0} - y_i \delta\varphi_{z0},$$

$$\delta u_{yi} = \delta u_{y0} + r_i \cos\alpha_i \delta\varphi_{z0} = \delta u_{y0} + x_i \delta\varphi_{z0}.$$

15.4 Virtual work equation for mechanisms

For mechanisms, the complete equilibrium equations can also be replaced by a virtual work equation. When the virtual displacement is chosen conform the freedom of movement at the joints (a *kinematically admissible virtual displacement*), the work performed by the interaction forces in the joints is zero and the virtual work equation includes only the work performed by the external forces.

When drawing the virtual displacements, one has to imagine that in the drawing the dimensions of the structure are considerably reduced and that the (very small) virtual displacements are considerably blown up. The magnitude of the angles (of rotation) is indicated by the so-called *orthogonal value*.

15.4.1 Virtual work equation

Mechanisms are systems of interconnected rigid bodies in which the joints are such that the bodies still have a certain degree of freedom with respect to one another.

Consider a system of two mutually hinged rigid bodies (1) and (2), loaded by a number of forces (see Figure 15.16a). There are acting interaction forces at the joint (joining forces). These occur always in pairs. In this case,

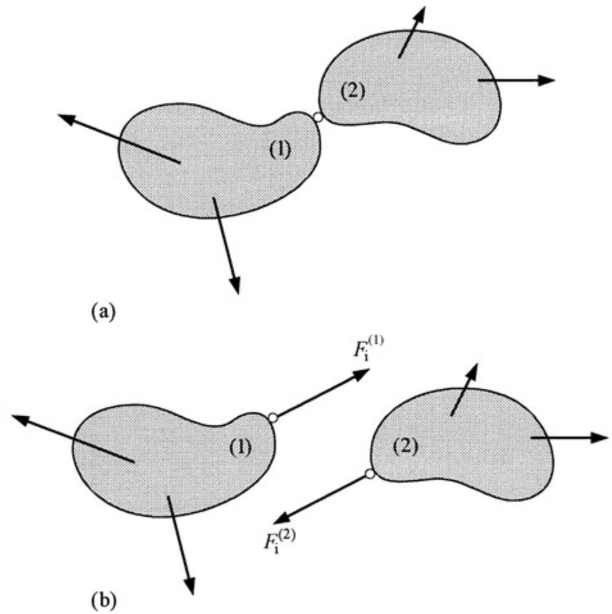


Figure 15.16 (a) Two bodies connected in a hinge. (b) The interaction forces at the hinged joint.

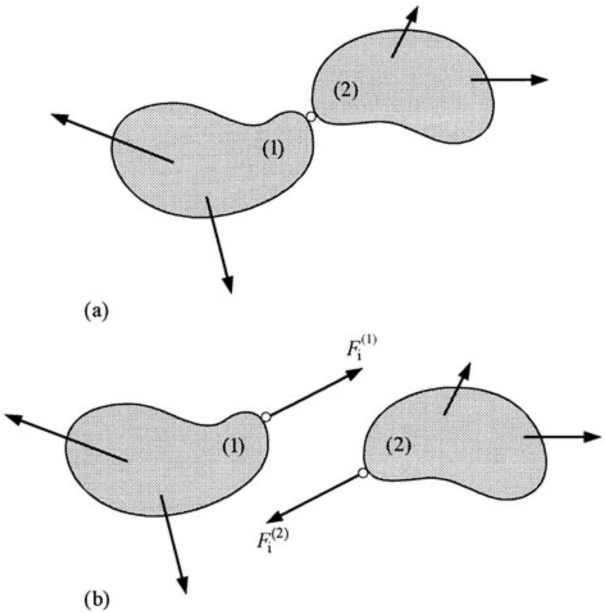


Figure 15.16 (a) Two bodies connected in a hinge. (b) The interaction forces at the hinged joint.

these forces are the two equal and opposite forces $F_{\mathrm{i}}^{(1)}$ and $F_{\mathrm{i}}^{(2)}$ (see Figure 15.16b).[1]

To investigate the equilibrium, the two bodies can be isolated from one another (see Figure 15.16b). The principle of virtual work can then be applied on each body. The virtual work $\delta A^{(1)}$ for a virtual displacement of body (1) is split into a part $\delta A_{\mathrm{e}}^{(1)}$ due to the external forces[2] on body (1) and a part $\delta A_{\mathrm{i}}^{(1)}$ due to the interaction force on body (1):

$$\delta A^{(1)} = \delta A_{\mathrm{e}}^{(1)} + \delta A_{\mathrm{i}}^{(1)} = 0.$$

The same applies for body (2):

$$\delta A^{(2)} = \delta A_{\mathrm{e}}^{(2)} + \delta A_{\mathrm{i}}^{(2)} = 0.$$

We can choose a virtual displacement for the connected bodies that is consistent with the freedom of movement in the joint. Such a displacement is referred to as a *kinematically admissible virtual displacement*. The virtual work equation is then

$$\delta A = \delta A^{(1)} + \delta A^{(2)} = \delta A_{\mathrm{e}}^{(1)} + \delta A_{\mathrm{e}}^{(2)} + \delta A_{\mathrm{i}}^{(1)} + \delta A_{\mathrm{i}}^{(2)} = 0.$$

The benefit of a kinematically admissible virtual displacement is that the interaction forces $F_{\mathrm{i}}^{(1)}$ and $F_{\mathrm{i}}^{(2)}$ together perform zero work. The points of application of these two equal and opposite forces always undergo the same

1 The upper index refers to the body, the lower index i refers to the interaction.

2 External in this sense does not refer to "the outside", but rather to the cause of the force "from the outside". The so-called external forces (also called loads), are independent forces in contrast to the interaction forces or joining forces (also referred to as internal forces), that are dependent forces.

displacement, so that

$$\delta A_{\mathrm{i}}^{(1)} + \delta A_{\mathrm{i}}^{(2)} = 0.$$

The virtual work equation now includes only the work performed by the external forces:

$$\delta A = \delta A_{\mathrm{e}}^{(1)} + \delta A_{\mathrm{e}}^{(2)} = 0.$$

Conclusion: *Due to a kinematically admissible virtual displacement of a mechanism in equilibrium, the virtual work performed by the (external) load equals zero. The fact that the work performed by the load is zero is a necessary and sufficient condition for the equilibrium of a mechanism.*

The approach can easily be expanded to include mechanisms of more than two bodies, as well as other than hinged joints.

15.4.2 The magnitude of the virtual displacements

Figure 15.17 shows a slide or a so-called *shear force hinge* with the interaction forces M; N, and a kinematically admissible virtual displacement δu; $\delta\varphi$.

We can see immediately that the pair of bending moments M does not perform any work. How different is it for the normal force pair N, that as a couple $N\delta u$ undergoes a (small) rotation $\delta\varphi$ and therefore, taking into account the directions shown in the figure, performs the work $-N\delta u\delta\varphi$. The geometric significance of $\delta u\delta\varphi$ can be seen on the figure.

When deriving the virtual work equation for a rigid body, the demand arose that the geometric relationships have to be linear in the virtual displacements. To achieve that, the virtual displacements have to be very small. Quadratic terms in the virtual displacements are therefore a degree smaller in the virtual work equation and can be discarded. Mathematically, this

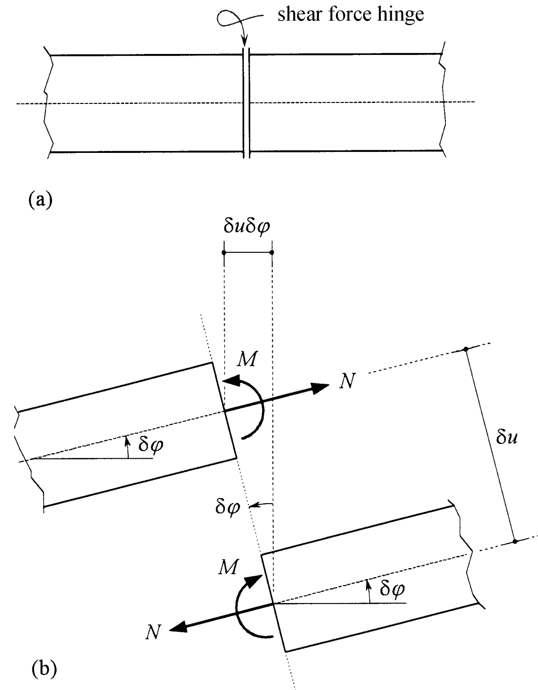


Figure 15.17 (a) A slide or a so-called shear force hinge with (b) the interaction forces after a kinematically admissible virtual displacement.

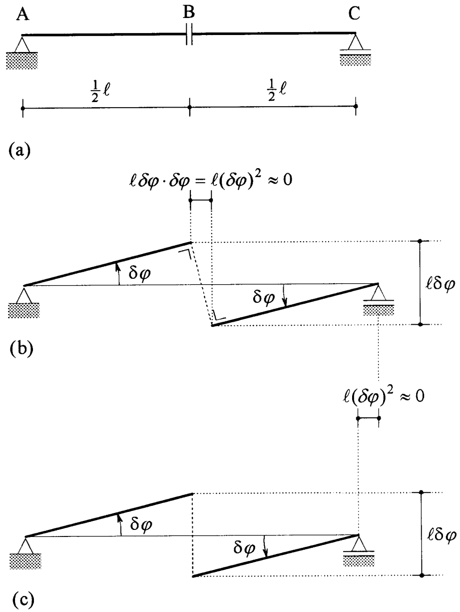


Figure 15.18 (a) The simply supported beam AC with a slide or shear force hinge at midspan B. (b) A kinematically admissible virtual displacement: the segments to the right and left of the slide remain parallel to one another. (c) The displacements in case the rotation $\delta\varphi$ is small and the quadratic terms in $\delta\varphi$ can be neglected.

demand is formulated by looking only at the first-order variation of the displacements in the virtual work equation (in other words, only the linear terms in the virtual displacements). This limitation with respect to the virtual displacements means that the $\delta u \delta\varphi$ is neglected, and the normal force pair N performs no virtual work.

In diagrams, the dimensions of the structure are greatly reduced and the displacements, even though they are infinitesimally small, are greatly enlarged. This can give rise to problems at first sight.

As an example, consider the simply supported beam ABC modelled as a line element in Figure 15.18a, with a *slide* or *shear force hinge* at midspan B. The mechanism has one degree of freedom. For a kinematically admissible virtual displacement, the displacement must be consistent with the freedom of movement at the supports and the slide. The latter means that beam segments AB and BC must remain parallel to one another on both sides of the shear force hinge. Figures 15.18b and 15.18c show the virtual displacement for the mechanism in two different ways. Figure 15.18b would seem to be the correct one, but this is not so.

One has to imagine that the virtual rotation $\delta\varphi$ is very small. In that case the horizontal displacement of the beam ends with respect to one another at B, $\ell(\delta\varphi)^2$, is a degree smaller than the vertical displacement $\ell\delta\varphi$. So the displaced beam ends at the shear force hinge B can be drawn directly above one another. Figure 15.18c gives therefore the correct representation of the virtual displacements in the mechanism.

Another example is beam ABC in Figure 15.19a, that has been changed into a mechanism by the introduction of the hinge at S. Figure 15.19b shows a virtual displacement: a bend $\delta\theta$ occurs at hinge S. The small angle (rotation) $\delta\theta$, that is drawn to a large scale, can be defined by the ratio $\delta a/\ell$. This value is known as the *orthogonal value*. The orthogonal value is not equal to the sine or tangent of the angle, nor to the value in radians.

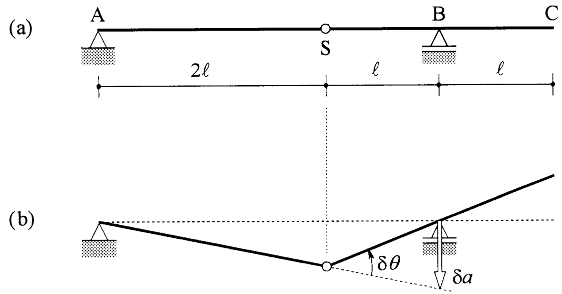


Figure 15.19 (a) Mechanism with (b) a kinematically admissible virtual displacement.

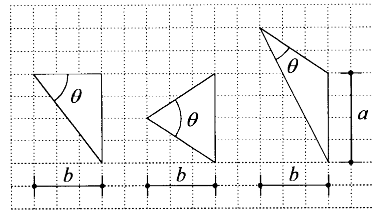


Figure 15.20 Three angles θ with the same orthogonal value a/b.

Figure 15.20 shows three angles θ that differ in magnitude from a trigonometric perspective, but for which the orthogonal value a/b is the same. Referring to Section 15.3.2 for the displacement due to a small rotation, it holds for all three cases that the vertical displacement a is equal to "*angle (of rotation)* $\theta \times$ *horizontal distance b to the centre of rotation*".

15.5 Calculating forces using virtual work

With statically determinate bar type structures, the support reactions and section forces follow directly from the *equilibrium*. When determining these forces, we can also use the *principle of virtual work* instead of the equilibrium equations. To do so, we have to change the structure into a mechanism in such a way that the force to be determined can perform work if the mechanism is subject to a virtual displacement. This is explained below by means of a number of examples.

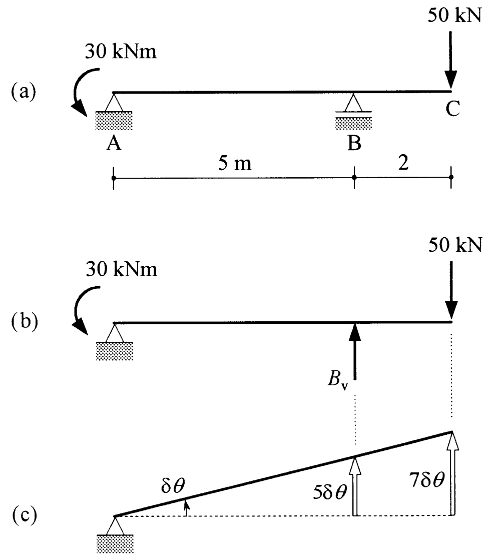


Figure 15.21 (a) Beam with (b) a mechanism for determining the vertical support reaction at B and (c) a kinematically admissible virtual displacement of the mechanism (displacements in m).

15.5.1 Support reactions

Example 1 – support reaction

We will now determine the vertical support reaction at B for the beam in Figure 15.21a using the principle of virtual work.

Solution:

Assume B_v, the vertical support reaction at B, has the direction as shown in Figure 15.21b. By removing the support at B we form a mechanism in which B_v can perform work. The mechanism has one degree of freedom.[1] Apply a virtual rotation $\delta\theta$ at A. Figure 15.21c shows the virtual displacement of the mechanism. The virtual displacements at B and C are equal to "*rotation* × *horizontal distance to centre of rotation A*" (see Section 15.3.2); they are respectively $5\delta\theta$ and $7\delta\theta$ m.

Note that the deformation of the beam is not taken into account.

Due to the virtual displacement, the couple and the forces perform virtual work. The total amount of work performed is

$$\delta A = +(30 \text{ kNm}) \times \delta\theta + B_v \times (5\delta\theta \text{ m}) - (50 \text{ kN}) \times (7\delta\theta \text{ m}).$$

The couple performs positive work, as does the support reaction B_v; the force at C in contrast performs negative work. For equilibrium the following applies

$$\delta A = 0$$

so that

[1] The position of the mechanism is fully defined by a single parameter, such as the rotation at A or the vertical displacement at B.

$$B_\mathrm{v} = \frac{-(30\text{ kNm}) \times \delta\theta + (50\text{ kN}) \times (7\delta\theta\text{ m})}{(5\delta\theta\text{ m})} = +64\text{ kN}.$$

As expected, the support reaction B_v turns out to be independent of the magnitude and direction of the virtual rotation $\delta\theta$.

Example 2 – Fixed-end moment

The hinged beam in Figure 15.22a carries a uniformly distributed load over its entire length. Below we will determine the fixed-end moment using the principle of virtual work.

Solution:

Assume the fixed-end moment A_m acts in the direction shown in Figure 15.22b. If we replace the fixed-end support by a hinged support, we create a mechanism in which the, as yet unknown, fixed-end moment A_m can perform work.

We select the vertical displacement of the hinge at S as a degree of freedom. Assume that the hinge is subject to a virtual displacement δu. Figure 15.22c shows the virtual displacement of the mechanism. Using "*vertical displacement* = *rotation* × *horizontal distance*" we can express the virtual rotation $\delta\theta$ at A in the virtual displacement δu:

$$\delta\theta = \frac{\delta u}{a}.$$

In equilibrium equations, a load on a rigid body can be replaced by its resultant. This also applies to the formulation of the virtual work equation. The distributed loads on AS and SB are replaced by their resultants:

$$R^\mathrm{AS} = qa \;\text{ and }\; R^\mathrm{SB} = 2qa.$$

The virtual displacements at the point of these resultants are easy to determine, and are both $\frac{1}{2}\delta u$.

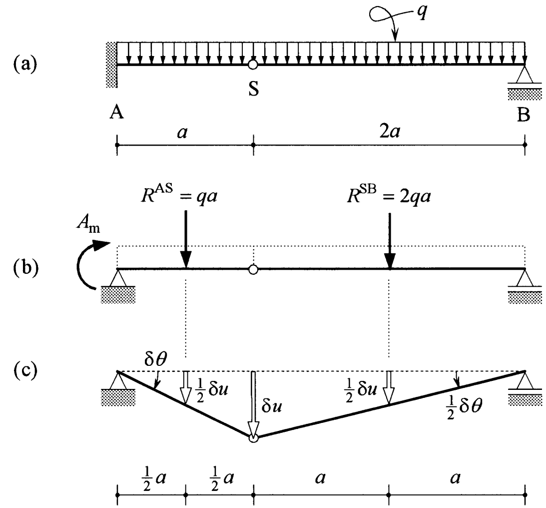


Figure 15.22 (a) Hinged beam with (b) a mechanism for determining the fixed-end moment at A and (c) a kinematically admissible virtual displacement of the mechanism.

For equilibrium the virtual work is zero:

$$\delta A = +A_{\mathrm{m}} \cdot \delta\theta + R^{\mathrm{AS}} \cdot \tfrac{1}{2}\delta u + R^{\mathrm{SB}} \cdot \tfrac{1}{2}\delta u$$
$$= +A_{\mathrm{m}} \cdot \frac{\delta u}{a} + qa \cdot \tfrac{1}{2}\delta u + 2qa \cdot \tfrac{1}{2}\delta u = 0$$

so that

$$A_{\mathrm{m}} = -\tfrac{3}{2}qa^2.$$

In reality, A_{m} therefore acts in the direction opposite to the one assumed in Figure 15.22b.

Note: It is incorrect to use the resultant of the total distributed load over ASB. This gives a different (and incorrect) result. Check it!

15.5.2 Section forces

Example 1 – Bending moment
The simply supported beam in Figure 15.23a carries a uniformly distributed load over its entire length. Here we will calculate the bending moment M_{C} at cross-section C, at a third of the span.

Solution:
Change the structure into a mechanism (Figure 15.23b) by applying a hinge at C. Since a hinge cannot transfer bending moments, the bending moment M_{C} at C is applied to the mechanism as an external load. We have to take into account that the bending moment is an interaction force and therefore occurs as a pair of moments: one moment M_{C} acts on the left-hand part and another moment M_{C}, equal and opposite, acts on the right-hand part. The direction of M_{C} in Figure 15.23b is an assumption.

Assume C is subjected to a vertical virtual displacement δu. Figure 15.23c shows the virtual displacement of the mechanism. AC undergoes a rotation

$\delta\theta^{AC}$, and CB undergoes a rotation $\delta\theta^{CB}$. Both rotations can be expressed in terms of δu:

$$\delta\theta^{AC} = \frac{\delta u}{a} \text{ and } \delta\theta^{CB} = \frac{\delta u}{2a}.$$

Having replaced the distributed loads on AC and CB by their resultants, we find the virtual work equation:

$$\delta A = -M_C \cdot \delta\theta^{AC} - M_C \cdot \delta\theta^{CB} + qa \cdot \tfrac{1}{2}\delta u + 2qa \cdot \tfrac{1}{2}\delta u$$
$$= -M_C \cdot \frac{\delta u}{a} - M_C \cdot \frac{\delta u}{2a} + qa \cdot \tfrac{1}{2}\delta u + 2qa \cdot \tfrac{1}{2}\delta u = 0$$

so that

$$M_C = +qa^2.$$

The plus sign indicates that the bending moment acts in the direction assumed in Figure 15.23b.

Of course, the result is the same if we select a virtual displacement δu at C upwards instead of downwards. We have to realise that the virtual displacement of the mechanism has nothing to do with the actual deformation of the beam. In the virtual work equation, the actual deformation of the beam is neglected and all beam segments are considered entirely rigid.

In the deformed mechanism, beam segments AC and CB bend with respect to one another at the joint by $\delta\theta_C$. This is also referred to as a "*gap*".

From the geometry of the deformed mechanism in Figure 15.23c we find

$$\delta\theta_C = \delta\theta^{AC} + \delta\theta^{CB}.$$

Looking back to the virtual work equation, we see that the contribution by

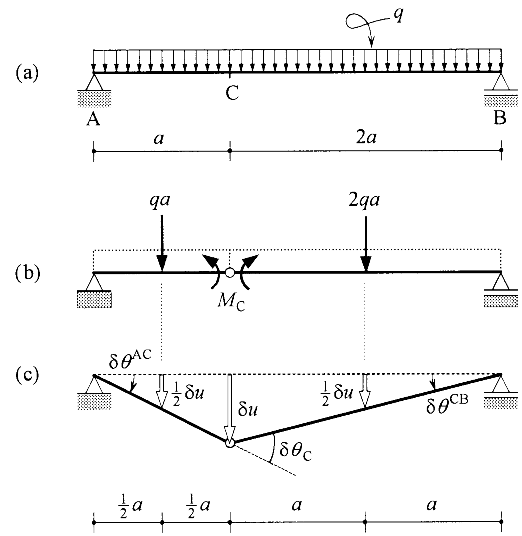


Figure 15.23 (a) Simply supported beam with (b) a mechanism for determining the bending moment at C, and (c) a kinematically admissible virtual displacement of the mechanism.

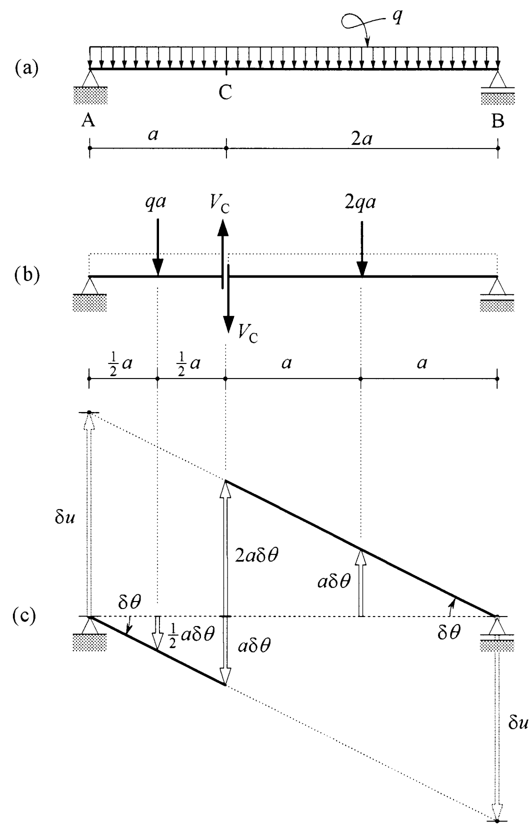


Figure 15.24 (a) Simply supported beam with (b) a mechanism for determining the shear force at C and (c) a kinematically admissible virtual displacement of the mechanism.

the pair of moments M_C, regardless of the sign, is equal to

$$\delta A(\text{due to } M_C) = \text{“bending moment} \times \text{gap”}$$
$$= M_C \cdot \delta\theta_C.$$

The sign is determined by the directions in which we choose the bending moment and the virtual displacement.

Example 2 – Shear force
We will now derive the shear force V_C at C for the same beam as in Example 1 (see Figure 15.24a).

Solution:
Change the structure into a mechanism (Figure 15.24b) by creating a *slide* or *shear force hinge* at C that cannot transfer shear forces.

The shear force is applied to the mechanism at C as an (external) load. Since the shear force is an interaction force it acts as a *pair of forces*: one shear force V_C acts on the left-hand segment and another equal and opposite shear force V_C acts on the right-hand segment. The direction of V_C in Figure 15.24b is an assumption.

Let the mechanism undergo a virtual displacement by displacing beam segments AC and CB at the shear force hinge over a distance δu with respect to one another. Both beam segments remain parallel to one another and are subject to the same rotation $\delta\theta$. We find the relationship between δu and $\delta\theta$ from the geometry of the deformed mechanism in Figure 15.24c:

$$\delta u = 3a\delta\theta.$$

After replacing the distributed loads on AC and CB by their resultants, we find the following for the virtual work equation:

$$\delta A = -V_C \cdot a\delta\theta - V_C \cdot 2a\delta\theta + qa \cdot \tfrac{1}{2}a\delta\theta - 2qa \cdot a\delta\theta = 0$$

so that

$$V_C = -\tfrac{1}{2}qa.$$

The minus sign indicates that the direction of V_C is opposite to the direction assumed in Figure 15.24b.

In the virtual work equation, the contribution of the *shear force* V_C, regardless of the sign, is equal to

$$\delta A(\text{due to } V_C) = \text{“shear force} \times \text{displacement in the shear force hinge”}$$
$$= V_C \cdot \delta u.$$

The sign is determined by the directions in which we assume the shear force and the virtual displacement.

Example 3 – Forces in a hinged beam

For the hinged beam in Figure 15.25a we will look for the mechanisms to determine the support moment at B, the shear force directly to the left of B and the shear force directly to the right of B.

Solution:

In order to find the support at B we convert the beam into a mechanism by introducing a hinge at B (Figure 15.25b). The bending moment is allowed to act on the mechanism as a load (as a *pair of moments*). The direction of M_B and the direction of the virtual displacement $\delta\theta$ can be chosen arbitrarily.

Figure 15.25c shows the mechanism for the shear force directly to the left of B. At this point, a slide or shear force hinge has been fitted into the beam. The shear force V_B has been applied on the mechanism as a load (a pair of

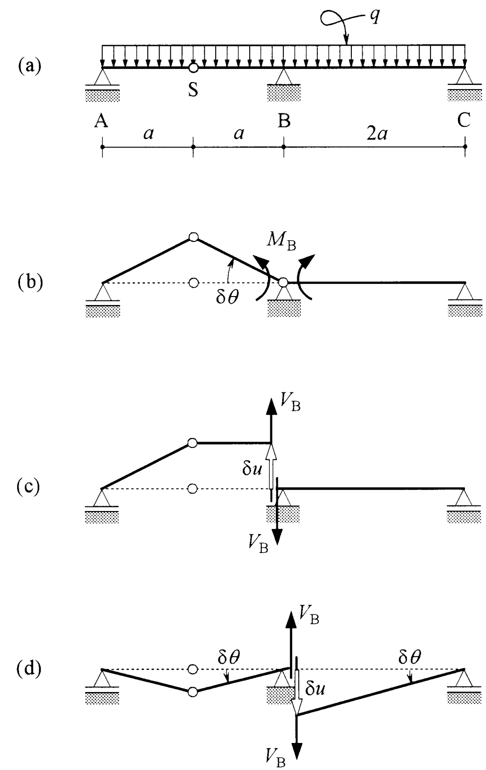


Figure 15.25 (a) Hinged beam with the mechanisms for determining (b) the support moment at B, (c) the shear force directly to the left of B and (d) the shear force directly to the right of B.

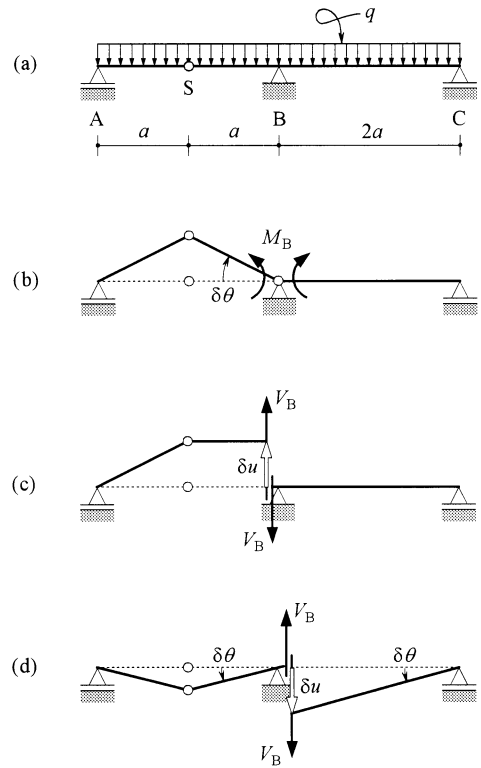


Figure 15.25 (a) Hinged beam with the mechanisms for determining (b) the support moment at B, (c) the shear force directly to the left of B and (d) the shear force directly to the right of B.

forces). At the shear force hinge, segments SB and BC can displace only with respect to one another, and cannot turn with respect to one another. Segments SB and BC therefore remain parallel. BC is fixed in a horizontal position due to the supports at B and C. With the vertical displacement δu, SB therefore remains horizontal.

Figure 15.25d gives the mechanism for the shear force directly to the right of B. Here too, SB and BC remain parallel. Due to the displacement δu at the shear force hinge, BC undergoes a rotation $\delta\theta = \delta u/2a$. SB undergoes the same rotation.

If the beam carries a uniformly distributed load q over its entire length (Figure 15.25a) the support moment at B is

$$M_B = -qa^2.$$

The shear force directly to the left of B is

$$V_B = +\tfrac{3}{2}qa.$$

The shear force directly to the right of B is

$$V_B = -\tfrac{3}{2}qa.$$

Check the answers using these mechanisms.

Example 4 – Normal force

Here we will derive the normal force in the truss in Figure 15.26a for the member DE using the principle of virtual work.

Solution:

Convert the truss into a mechanism by introducing a connection in member DE that cannot transfer normal forces. Such a telescopic connection is also referred to as a *normal force hinge*. At the normal force hinge, the normal force N is applied to the mechanism as a load (a pair of forces). In

Figure 15.26b it has been assumed that the normal force is a tensile force.

Figure 15.26c shows the virtual displacement for the mechanism. The mechanism consists of two self-contained bodies ACD and BCE that can respectively rotate about A and B and are hinged at C.

With the rules

"*horizontal displacement* = *rotation* × *vertical distance*", and

"*vertical displacement* = *rotation* × *horizontal distance*",

we can determine the rotation of the parts ACD and BCE and the displacements of the joints. Figure 15.26c all the relevant quantities are expressed in terms of the vertical displacement δw of joint C.

The member ends in the normal force hinge move with respect to one another over a distance δu that is equal to the distance that the joints D and E move towards one another:

$$\delta u = \tfrac{1}{4}\delta w + \tfrac{1}{2}\delta w = \tfrac{3}{4}\delta w.$$

We write down the virtual work equation:

$$F \cdot \tfrac{1}{2}\delta w + N \cdot \delta u = F \cdot \tfrac{1}{2}\delta w + N \cdot \tfrac{3}{4}\delta w = 0$$

so that

$$N = -\tfrac{2}{3}F.$$

The normal force in member DE is therefore a tensile force.

In the virtual work equation, the contribution by the *normal force* N, regardless of the sign, is equal to

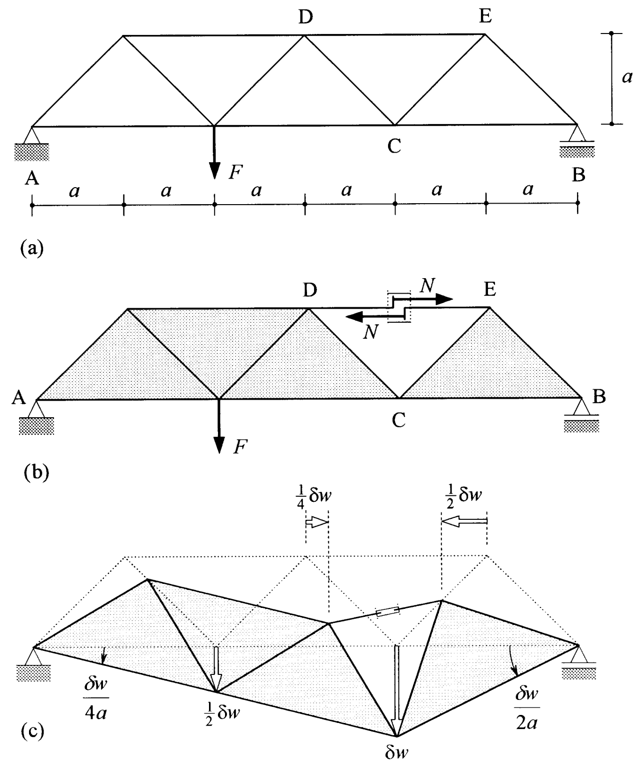


Figure 15.26 (a) Truss with (b) a mechanism for determining the normal force in DE and (c) a kinematically admissible virtual displacement of the mechanism.

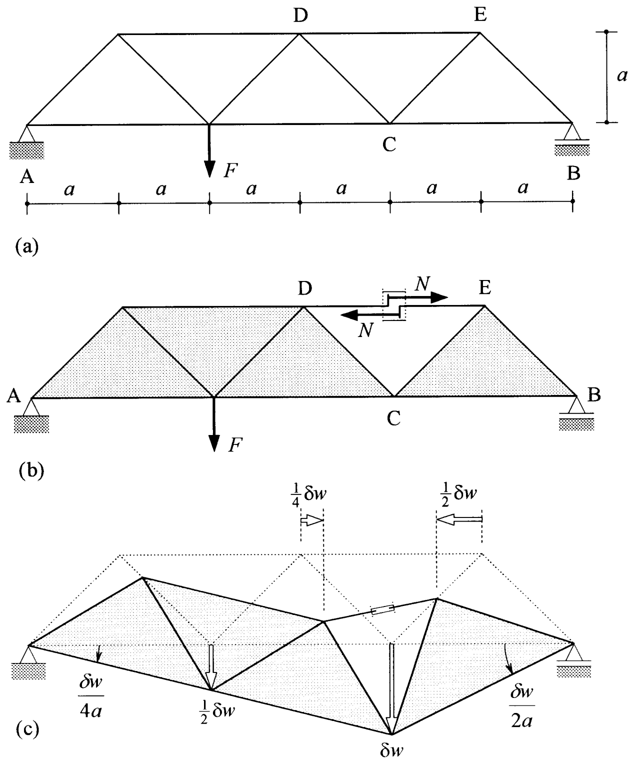


Figure 15.26 (a) Truss with (b) a mechanism for determining the normal force in DE and (c) a kinematically admissible virtual displacement of the mechanism.

$$\delta A(\text{due to } N) = \text{“normal force} \times \text{displacement in the normal force hinge”}$$

$$= N \cdot \delta u.$$

The sign is determined by the directions of the normal force and virtual displacement.

This example shows that the initial simplicity of the virtual work equation to replace the equilibrium equations is somewhat overshadowed by the more complicated geometry of the deformed mechanism.

15.6 Problems

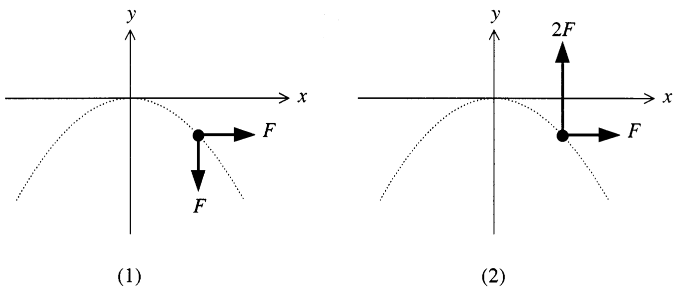
General comment: All calculations must be performed using virtual work.

Virtual work – mixed problems

15.1: 1–2 A particle is compelled to follow the following parabolic path in the xy plane:

$$y = -\frac{x^2}{2a}.$$

The particle is loaded by a horizontal and vertical force, as shown in the figure. There is no friction.



Questions:

a. At which point(s) is the particle in equilibrium? Give the coordinates for this/these point(s).
b. Can you (intuitively) say anything about the reliability (stability) of the equilibrium at this/these point(s)?

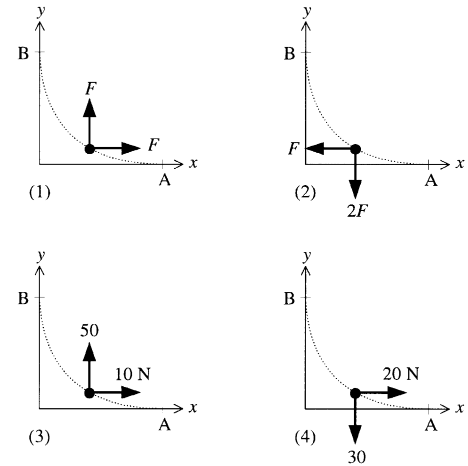
15.2: 1–4 A particle is compelled to follow a frictionless path in the xy plane between A and B with the following definition:

$$\sqrt{x} + \sqrt{y} = \sqrt{a}.$$

Here $a = 9$ m. The particle is loaded by the forces shown in the figure.

Questions:

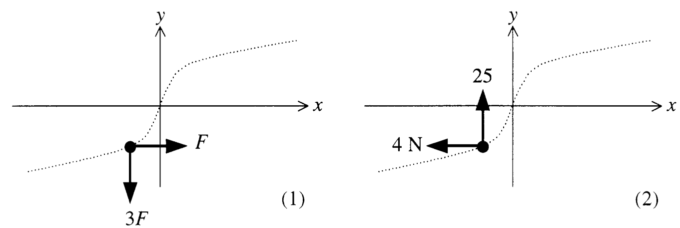
a. At which position(s) is the particle in equilibrium between A and B? Give the coordinates for this/these point(s).
b. Can you (intuitively) say anything about the reliability (stability) of the equilibrium at this/these position(s)?



15.3: 1–2 A particle is compelled to follow the frictionless path of a cubic parabola:

$$y = \sqrt[3]{x}.$$

The particle is loaded by the forces shown in the figure.



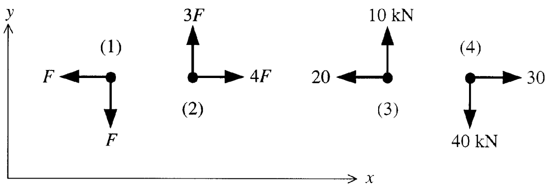
Questions:

a. At which position(s) is the particle in equilibrium? Give the coordinates for this/these point(s).
b. Can you (intuitively) say anything about the reliability (stability) of the equilibrium at this/these position(s)?

15.4: 1–4 A particle is compelled to follow an elliptical path:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

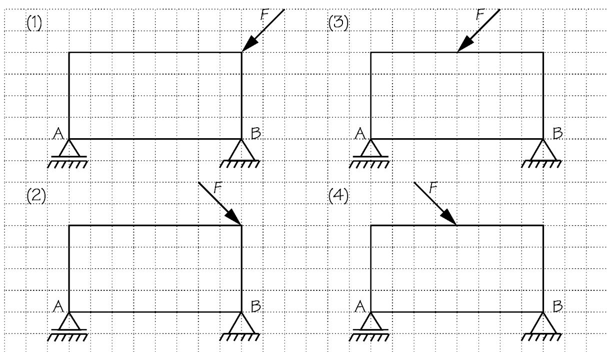
with $a = 2$ m and $b = 4$ m. The path is frictionless. The particle is loaded by the forces shown in the figure.



Questions:

a. Draw the path of the particle.
b. At which position(s) is the particle in equilibrium? Give the coordinates for this/these point(s).
c. Can you (intuitively) say anything about the reliability (stability) of the equilibrium in this/these position(s)?

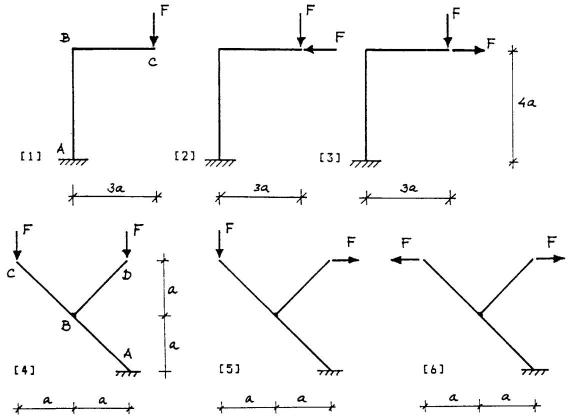
15.5: 1–4 A block is supported on a roller at A and a hinge at B, and is loaded by a force $F = 20\sqrt{2}$ kN in four different ways.



Questions:

a. Determine the support reaction at A.
b. Determine the vertical component of the support reaction at B.
c. Determine the horizontal component of the support reaction at B.

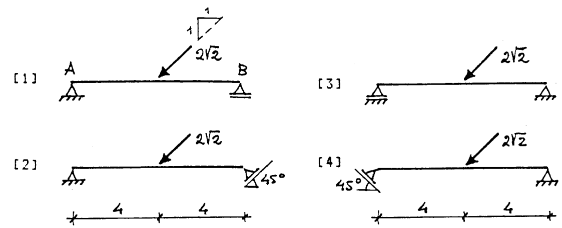
15.6: 1–6 You are given a number of structures fixed at A.



Questions:

a. Determine the horizontal support reaction at A.
b. Determine the vertical support reaction at A.
c. Determine the fixed-end moment at A.

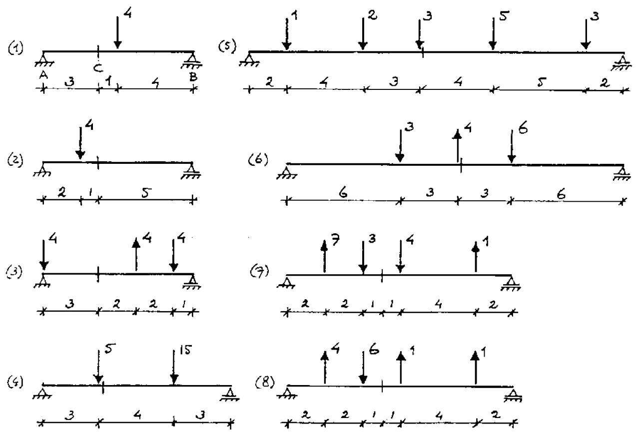
15.7: 1–4 A number of beams are supported on a hinge and a roller. The dimensions are in m, the forces are in kN.



Questions:

a. Determine (the components of) the support reaction at A.
b. Determine (the components of) the support reaction at B.

15.8: 1–8 A number of beams, simply supported at A and B, are composed of segments AC and BC that are rigidly joined at C. The location of joint C is shown in the figure by means of a vertical dash. The forces are given in kN, the lengths are in m.



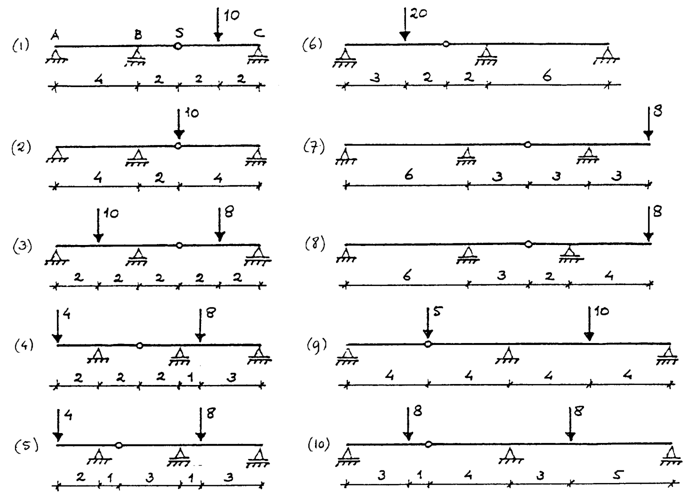
Questions:

a. Determine the support reaction at A.
b. Determine the support reaction at B
c. Determine the shear force at C.
d. Determine the bending moment at C.

15.9: 1–10 For hinged beam ABC you are given the lengths in metres and forces in kN.

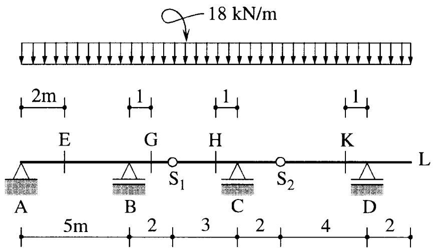
Questions:

a. Determine the support reaction at A.
b. Determine the support reaction at B.
c. Determine the support reaction at C.
d. Determine the support moment at B.
e. Determine the shear force directly to the left of B.
f. Determine the shear force directly to the right of B.



15.10: 1–10 As problem 15.9, but replace the concentrated loads by a uniformly distributed load of 10 kN/m over the entire length of the beam.

15.11 The hinged beam shown is subject to a uniformly distributed load of 18 kN/m.



Questions:

a. Determine the shear force at E.
b. Determine the bending moment at E.
c. Determine the shear force at G.
d. Determine the bending moment at G.
e. Determine the shear force at H.
f. Determine the bending moment at H.
g. Determine the shear force at K.
h. Determine the bending moment at K.

16 Influence Lines

In many structures, the support reactions and section forces depend not only on the magnitudes of the loads, but also on their placement. This is particularly true for bridges, where an important part of the load consists of moving vehicles. Other examples are assembly halls with crane runways or warehouses in which the placement of loads can change.

If we want to choose the dimensions of a structural element to check it for strength and rigidity, it is important to know the location at which the load or set of loads generate the most severe effects. Important tools for finding the most unfavourable placement of loads are the so-called *influence lines*. Influence lines are graphic representations of the magnitude of a support reaction or section force at a fixed location due to a single point load with variable position.[1]

In this chapter we look at how to determine influence lines for forces in statically determinate structures, and how to use them.

There are two methods for determining influence lines: directly from the equilibrium equations (Section 16.1), or by means of virtual work (Section 16.2). We will demonstrate both methods by means of examples.

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[1] There are also influence lines for displacements and rotations. We do not cover those here.

Finally, we will show how to use influence lines to determine the placement of loads to have the maximum effect (Section 16.3).

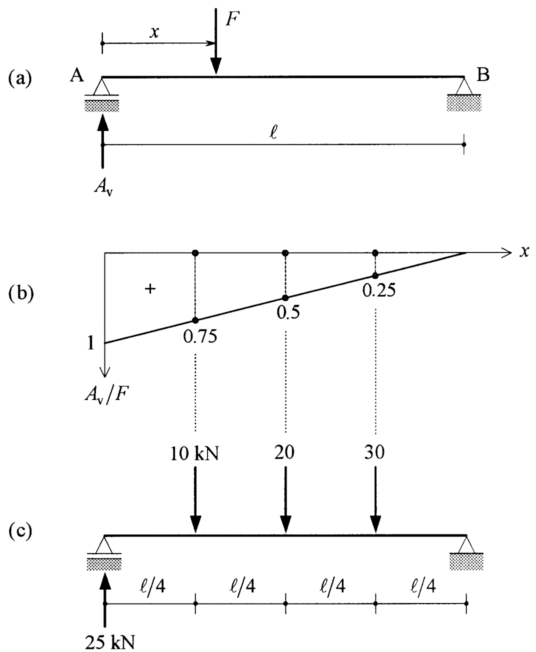


Figure 16.1 (a) Simply supported beam with (b) the influence line for the vertical support reaction at A and (c) the support reaction A_v due to a set of loads.

16.1 Influence lines using equilibrium equations

Here, for a simply supported beam, we will derive the influence lines for a support reaction, bending moment and shear force directly from the equilibrium equations. As you will notice, this method will become already laborious for a hinged beam.

16.1.1 Simply supported beam

Example 1 – Influence line for a support reaction

The principle of influence lines is discussed on the basis of the simply supported beam AB in Figure 16.1a. The beam is loaded by a moving point load F.

Question:
How does the vertical support reaction A_v at A change as the point load moves from A to B?

Solution:
Assume A_v acts in the direction indicated in Figure 16.1a. If the point load is placed at a distance x from A, it follows from the moment equilibrium about B that

$$A_v = F\frac{\ell - x}{\ell}.$$

If the position x of the point load is assumed to be variable, and we draw A_v/F as a function of x (see Figure 16.1b), we obtain the *influence line for the support reaction* A_v. It is the convention to plot the positive values of A_v/F in the (positive) direction of F.

For influence lines, one does not plot A_v, but rather A_v/F as a function of the position of the point load. We also can interpret the influence line as the variation of the support reaction due to a moving unit load (e.g. $F = 1$ kN).

The magnitude of the support reaction, or rather the *influence* A_v/F, can be read from the influence line at the position of the load. In this way, using the influence line, the support reaction can be quickly derived if the beam is subject to a set of loads. For example, for the case in Figure 16.1c:

$$A_v = +0.75 \times (10\text{ kN}) + 0.5 \times (20\text{ kN}) + 0.25 \times (30\text{ kN}) = +25\text{ kN}.$$

From the influence line we can see that at the position of the force of 10 kN the influence is 0.75, or in other words: the contribution of this force to A_v is:

$$+0.75 \times (10\text{ kN}) = +7.5\text{ kN}.$$

The total support reaction is found by superposing all the individual contributions.

Example 2 – Influence line for a bending moment
We will now determine the bending moment M_C at midspan C for the beam in Example 1 (see Figure 16.2a).

Solution:
Assume the bending moment M_C is positive if it causes tension at C at the underside of the beam. The magnitude of M_C follows from the moment equilibrium of AC (or CB) about C. It should be remembered that it makes a difference in the equilibrium equations whether the load is to the left or right of the cross-section under consideration.

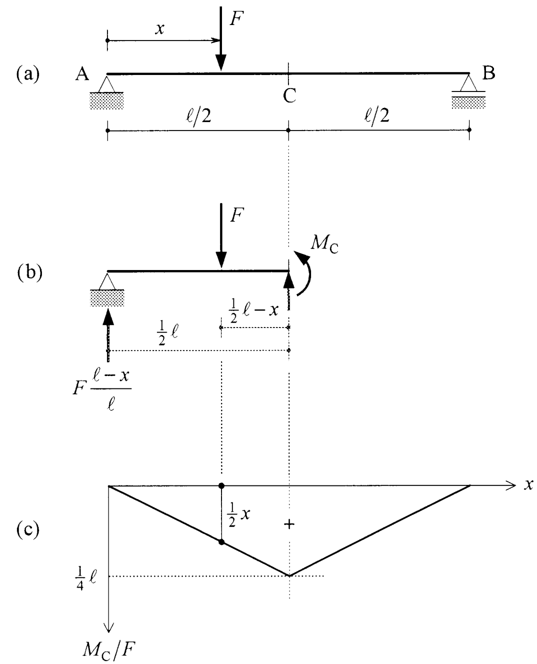


Figure 16.2 (a) Simply supported beam with (b) the isolated left-hand segment and (c) the influence line for the bending moment at C.

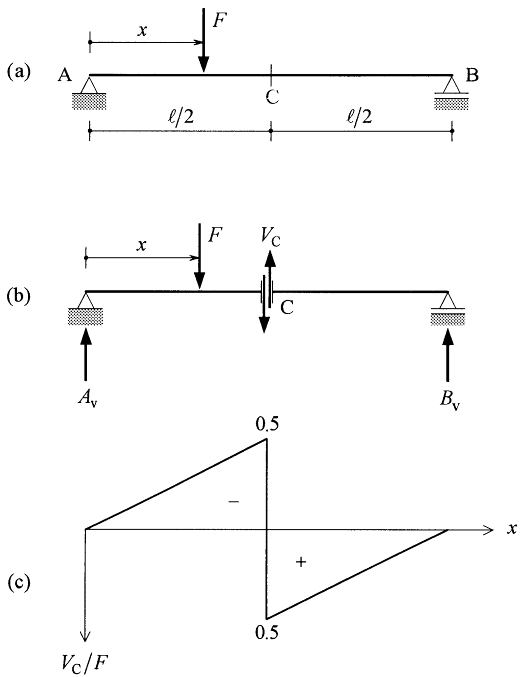


Figure 16.3 (a) Simply supported beam with (b) the positive direction assumed for shear force V_C and (c) the influence line for the shear force at C.

If the load is to the left of C ($0 \leq x < \frac{1}{2}\ell$) then the bending moment at C equals (see Figure 16.2b)

$$M_C = +F\frac{\ell - x}{\ell} \cdot \tfrac{1}{2}\ell - F \cdot \left(\tfrac{1}{2}\ell - x\right) = +\tfrac{1}{2}Fx.$$

If the load is to the right of C ($\frac{1}{2}\ell < x \leq \ell$) then the bending moment at C equals

$$M_C = +F\frac{\ell - x}{\ell} \cdot \tfrac{1}{2}\ell = +\tfrac{1}{2}F(\ell - x).$$

The variation of M_C/F as a function of x, as shown graphically in Figure 16.2c, is referred to as the *influence line for the bending moment at C*.

The influence line has its maximum in the middle. This means that the bending moment at C is a maximum when the point load F is at midspan:

$$M_C = \left(+\tfrac{1}{4}\ell\right) \times F = +\tfrac{1}{4}F\ell.$$

Example 3 – Influence line for a shear force
For the same beam we will now determine the influence line for the shear force V_C at C (see Figure 16.3a).

Solution:
The direction assumed for V_C is shown in Figure 16.3b. The shear force V_C can be found from the vertical force equilibrium of AC (or CB).

If the load is to the left of C ($\frac{1}{2}\ell < x \leq \ell$) the equilibrium of AC gives

$$V_C = A_v - F = F\frac{\ell - x}{\ell} - F = -F\frac{x}{\ell}.$$

If the load is to the right of C ($\frac{1}{2}\ell < x \leq \ell$) then the shear force at C is equal to the support reaction A_v at A:

$$V_C = A_v = F\frac{\ell - x}{\ell}.$$

This determines the *influence line for the shear force at C*. The influence line is shown in Figure 16.3c.

If the point load is to the left of C the influence line is negative. The shear force acts in the direction opposite to the one assumed in Figure 16.3b.

The shear force at C is a maximum if the point load F is directly to the left or right of C:

$$V_C = (\pm 0.5) \times F = \pm\tfrac{1}{2}F.$$

A change of sign occurs at C.

16.1.2 Hinged beam

Figure 16.4 shows the influence lines for a hinged beam. The positive direction of the support reaction B_v at B is shown in Figure 16.4a. For the bending moment and the shear force, the positive directions according to the xz coordinate system are given in Figure 16.4f.

We do not address the calculation of each of these influence lines in detail here. They can be found by, for various positions of the point load, using the equilibrium equations to calculate the magnitude of the various quantities. In this case, that leads to a fair amount of work as there are so many positions of the load to be considered.

It is preferable to investigate a number of characteristic positions of the point load (e.g. just above the support loads, or at the hinges) and to remember that the influence line between certain points has to be linear.

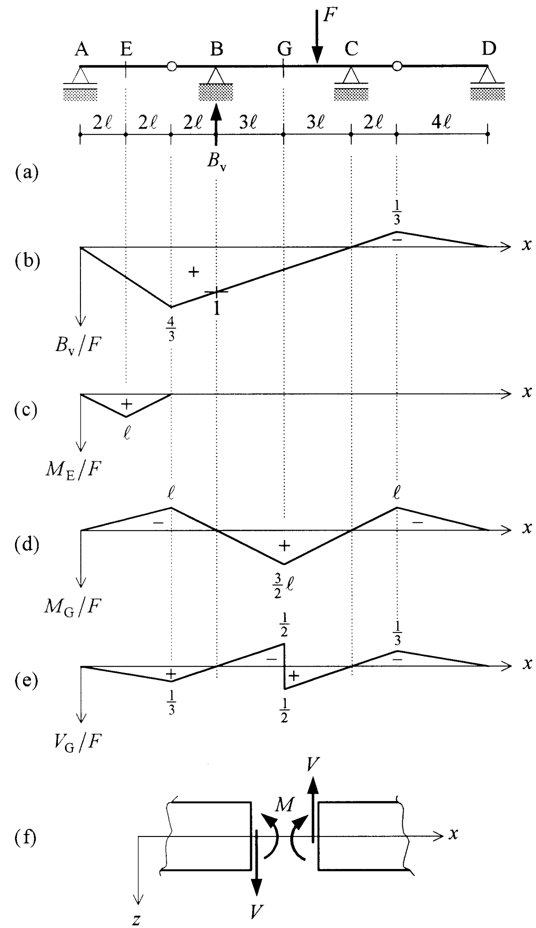


Figure 16.4 (a) Hinged beam with influence lines for (b) the vertical support reaction at B, (c) the bending moment at E, (d) the bending moment at G and (e) the shear force at G; (f) positive directions for bending moment and shear force.

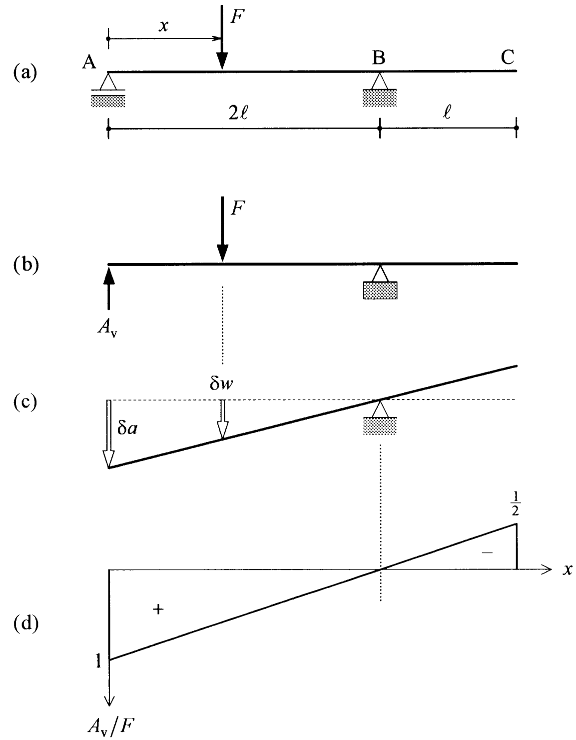


Figure 16.5 (a) Beam with (b) mechanism for determining the vertical support reaction at A, (c) virtual displacement for which A_v performs negative work and (d) the influence line for A_v.

In the next section we introduce an alternative method, based on virtual work.

16.2 Influence lines using virtual work

In Chapter 15 we showed that the virtual work equation offers an alternative formulation of equilibrium equations. When determining influence lines, we can replace the equilibrium equations by a single virtual work equation.

The alternative method, based on *virtual work*, provides the shape of the influence line more quickly with less calculation. It is necessary, however, to take three rules into account. We describe the method, and the rules, below.

16.2.1 Simply supported beam

Example 1 – Influence line for a support reaction
To determine the influence line for the vertical support reaction A_v at A using virtual work, we convert the beam in Figure 16.5a into a mechanism by removing the roller support at A. The unknown support reaction A_v is applied to the mechanism as a load (see Figure 16.5b).

We now apply a virtual displacement to the mechanism by displacing A over a distance δa so that A_v performs *negative work* (first rule) for the directions assumed in Figure 16.5b (see Figure 16.5c). Assume that the point of application of F undergoes a virtual displacement δw and that δw *is positive in the direction of* F (second rule).

For equilibrium, the virtual work is zero:

$$\delta A = -A_v \delta a + F \delta w = 0$$

so that

$$A_{\text{v}} = F\frac{\delta w}{\delta a} \quad \text{or} \quad \frac{A_{\text{v}}}{F} = \frac{\delta w}{\delta a}.$$

The virtual displacement δw (positive in the direction of F) is dependent on the position of the point load. It turns out that A_{v}/F, the ordinate of the influence line, is proportional to the virtual displacement δw. This means that on a certain scale (δa) the deflection line of the mechanism due to virtual displacement is identical with the influence line we are looking for. The influence line for A_{v} is shown in Figure 16.5d. The value of the ordinate at A is equal to 1 as there $\delta w = \delta a$.

The fact that the influence line and the deflection line of the mechanism are the same shape means that the signs of both quantities δw and δa are the same. This is a consequence of the fact that the virtual displacement has been chosen in such a way that the quantity we are looking for (A_{v}) performs negative work (rule 1).

If, for the scale factor δa, we chose *a displacement that is equal to the unit of length* – this is known as a *unit displacement* – (third rule), then the influence line is identical with the deflection line of the mechanism.

Conclusion: *If the force sought performs* negative work *(rule 1) over a* unit displacement *(rule 3), then the influence line is equal to the deflection line of the mechanism. The influence line is positive where the displacement is in the direction of F (segment AB) and negative where the displacement is in the opposite direction of F (segment BC) (rule 2).*

Example 2 – Influence line for a bending moment
The second example relates to the influence line for the bending moment at cross-section D of the beam in Figure 16.6a.

Again we convert the beam into a *mechanism* by introducing a hinge at D. The action of the bending moment at D is replaced by the pair of moments M_{D} that are applied to the mechanism at either side of the hinge (see Figure 16.6b). The direction of M_{D} can be chosen arbitrarily.

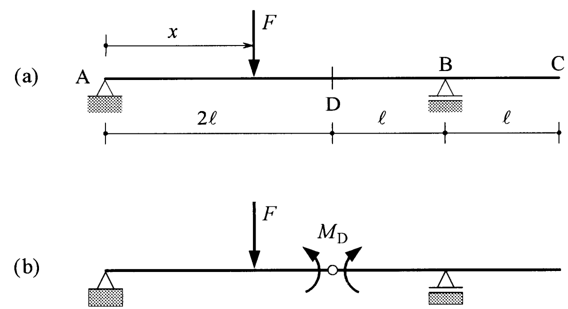


Figure 16.6 (a) Beam with (b) mechanism for determining the bending moment at D.

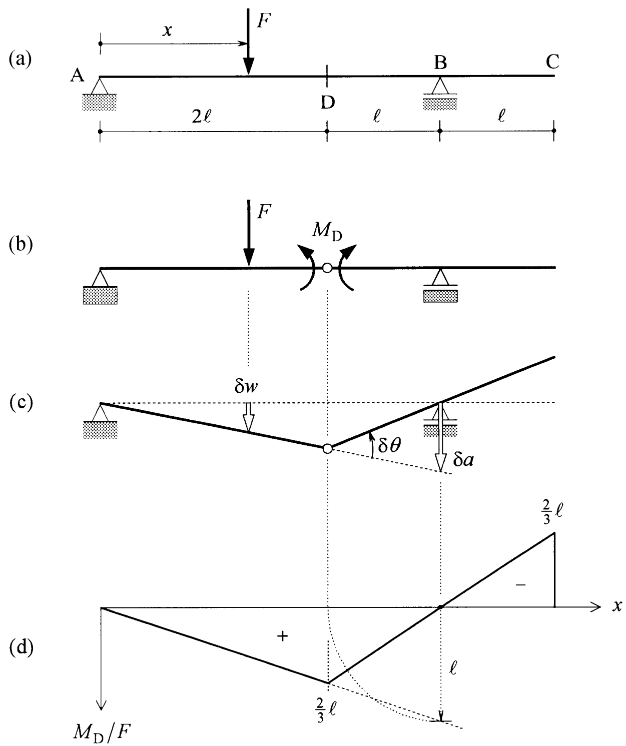


Figure 16.6 (a) Beam with (b) mechanism for determining the bending moment at D. (c) Virtual displacement for which M_D performs negative work and (d) the influence line for M_D.

We apply a virtual displacement to the mechanism by rotating beam segments AD and DBC at D through an angle $\delta\theta$ with respect to one another, but in such a way that M_D *performs negative work* (rule 1) (see Figure 16.6c). Assume that load F is displaced by a distance δw, and that δw *is positive in the direction of* F (rule 2).

For equilibrium, the virtual work is zero:

$$-M_D\delta\theta + F\delta w = 0$$

so that

$$\frac{M_D}{F} = \frac{\delta w}{\delta\theta}.$$

The bending moment M_D is proportional to the displacement δw. The influence line for M_D therefore has the same shape as the deflection line of the mechanism. The scale factor is the angle $\delta\theta$.

For a virtual displacement, δw and $\delta\theta$ are infinitesimally small, but their ratio is finite. The influence line can therefore be drawn on an enlarged scale. $\delta\theta$ can be defined as (see also Section 15.4.2):

$$\delta\theta = \frac{\delta a}{\ell}.$$

If we select δa equal to ℓ, it is said that the angle $\delta\theta$ has the *orthogonal unit value* – this is also referred to as a *unit rotation* (rule 3). In that case the influence line is exactly the same as the deflection line of the mechanism (see Figure 16.6d).

Conclusion: *If we apply a* unit rotation *to the hinge, so that the bending moment performs* negative work, *then the influence line for that bending moment is the same as the deflection line of the mechanism.*

Arcs can be used to construct an orthogonal unit angle as shown in Figure 16.7. In that figure, two arcs are drawn, but one arc is actually sufficient, as appears from the plot of the influence line in Figure 16.6d.

Example 3 – Influence line for a shear force
The third example relates to the influence line for the shear force in cross-section D of the beam in Figure 16.8a.

The procedure is identical to that in the previous examples. The beam is transformed into a mechanism by introducing a *slide joint* or *shear force hinge* at D and replacing the action of the shear force at D by the pair of forces V_D that are applied on the mechanism at either side of the slide joint (see Figure 16.8b). The direction of V_D can be chosen arbitrarily.

Subsequently, the mechanism is subjected to a virtual displacement by displacing segments AD and DBC at D over a distance δu with respect to one another, but in such a way that V_D *performs negative work* (rule 1) (see Figure 16.8c). In the deformed mechanism, beam segments AD and DBC remain parallel to one another.

Assume load F moves over a distance δw, and δw *is positive in the direction of* F (rule 2).

For equilibrium, the virtual work is zero:

$$-V_D \delta u + \delta w = 0$$

so that

$$\frac{V_D}{F} = \frac{\delta w}{\delta u}.$$

The shear force V_D is proportional to the deflection δw. The influence line for the shear force at D is the same shape as the deflection line of the deformed mechanism. The scale factor is the displacement δu.

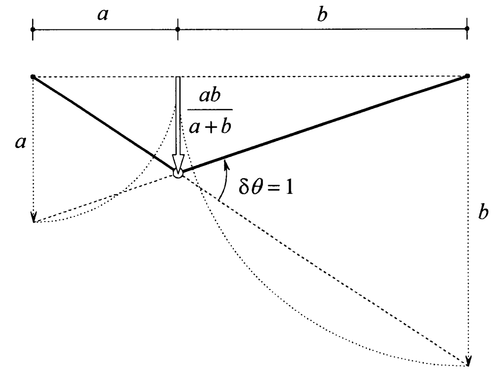


Figure 16.7 The construction of an angle $\delta\theta$ with orthogonal unit value.

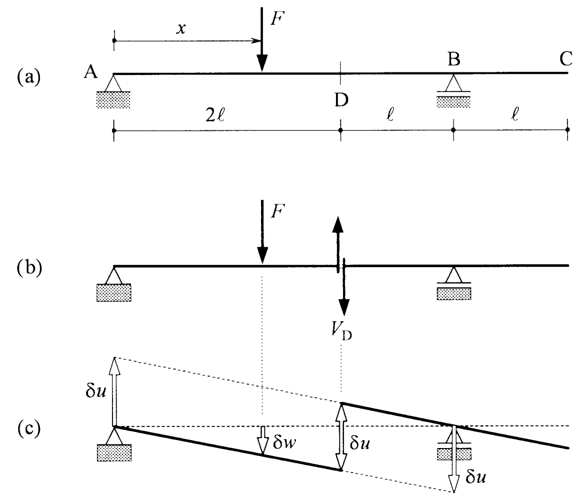


Figure 16.8 (a) Beam with (b) mechanism for determining the shear force at D, (c) virtual displacement for which V_D performs negative work.

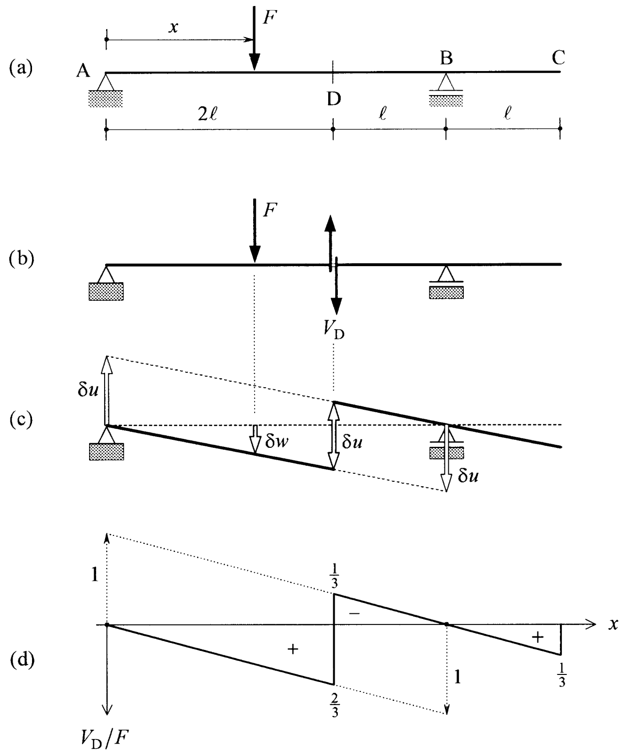


Figure 16.8 (a) Beam with (b) mechanism for determining the shear force at D, (c) virtual displacement for which V_D performs negative work. (d) The influence line for V_D.

If we choose δu as a *unit length displacement* (rule 3), then the influence line is identical to the deflection line of the mechanism (see Figure 16.8d).

Conclusion: *If the shear force performs* negative work *over a* unit displacement, *then the influence line is the same as the deflection line of the mechanism. Where the displacement is in the direction of F (segments AD and BC), the influence line is positive and the shear force acts in the assumed direction. The influence line is negative where the displacement is in the direction opposite to that of F (segment DB); here the shear force acts opposite to the direction assumed.*

16.2.2 General procedure for the method of virtual work

Determining the influence line for a force quantity[1] using the method of virtual work requires the same procedure each time:

- Convert the structure into a mechanism by creating a joint (hinge) that cannot transfer the force quantity in question.
- Allow the force quantity to act on the mechanism as a load.
- Apply a virtual displacement to the mechanism such that the force quantity performs negative work.
- Select a unit displacement or unit rotation for the displacement or rotation respectively over which the force performs work. The deflection line of the deformed mechanism is the influence line.
- The force quantity in question is positive when the displacement is in the direction of the force quantity and negative when it is opposite to the direction of the force quantity.

---

[1] Generalisation for support reactions and section forces.

16.2.3 Hinged beam

On the basis of a few examples relating to hinged beams, we demonstrate that the method of virtual work provides a quick and easy way of plotting influence lines.

Example 1 – Influence line for a fixed-end moment
The hinged beam in Figure 16.9a has hinges at S_1 and S_2, and is fixed at D. We will determine the influence line for the fixed-end moment M_D at D.

The influence line is found by changing the fixed-end support at D into a hinged support, and there applying a unit rotation such that M_D performs negative work.[1] The deformed mechanism in Figure 16.9b is then the influence line in Figure 16.9c.

The influence line shows that the fixed-end moment for the direction assumed for M_D in Figure 16.9b has a maximum positive value when load F is at S_1:

$$M_D = +2Fa.$$

The most negative fixed-end moment occurs when the load is at A, the end of the overhang:

$$M_D = -4Fa.$$

The zeros in the influence lines allow us to check: the fixed-end moment is always zero when the load is placed at one of the supports B, C or D.

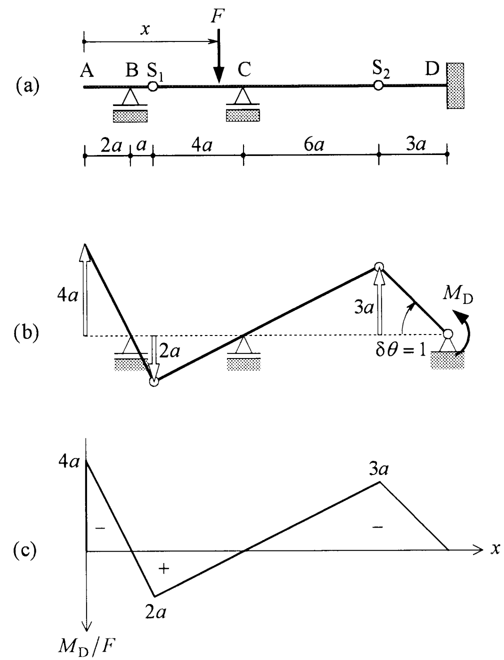


Figure 16.9 (a) Hinged beam with (b) mechanism for determining fixed-end moment M_D at D and (c) the influence line for M_D.

[1] Note: the unit rotation is applied at D and not at S_2!

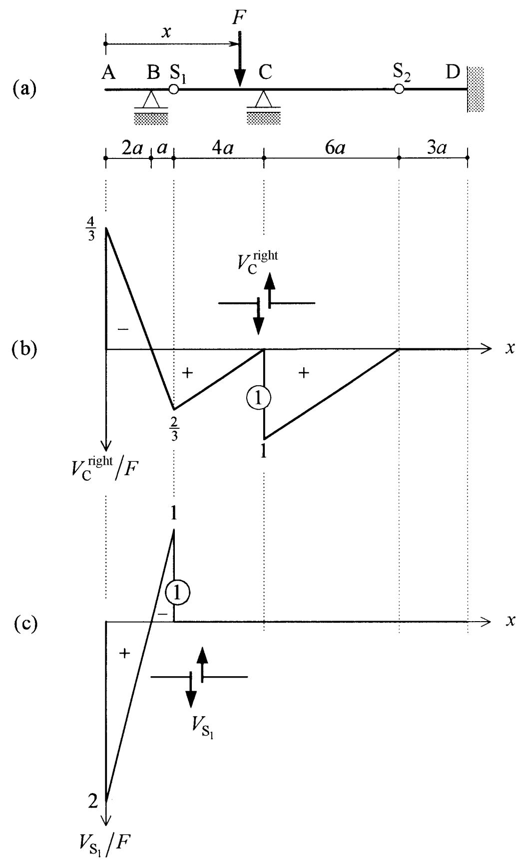


Figure 16.10 (a) Hinged beam with (b) the influence line for the shear force directly to the right of C and (c) the influence line for the shear force at S_1.

Example 2 – Influence line for shear forces
The beam from Example 1 is again shown in Figure 16.10a.

Figure 16.10b shows the influence line for the shear force V_C^{right} directly to the right of C. Figure 16.10c shows the influence line for the shear force V_{S_1} at hinge S_1.

The influence lines are found by introducing a slide joint directly to the right of C, and at S_1, respectively, and there applying a unit displacement such that the shear force performs negative work. The deformed mechanism is then the influence line we are looking for.

The (assumed) positive direction of the shear force is shown separately in the figures.

The mechanisms are not shown separately. However, a ① shows where in the mechanism a unit displacement was applied.

In the influence line for V_C^{right} (Figure 16.10b) the paths through S_1C and CS_2 are parallel. After all, in the mechanism, segments S_1C and CS_2 can displace only with respect to one another, and cannot rotate with respect to one another.

In the influence line for V_{S_1} (Figure 16.10c) the paths through ABS_1 and S_1C are not parallel, as in accordance with the mechanism the segments ABS_1 and S_1C can rotate with respect to one another due to the hinge at S_1.

Example 3 – Various influence lines
Figures 16.11b to 16.11e show various influence lines for the hinged beam in Figure 16.11a. The positive direction of the support reaction B_v at B is shown in Figure 16.11a. For the bending moment and the shear force, the positive directions are related to the xz coordinate system in Figure 16.11f.

These influence lines are also shown in Section 16.1.2. There, we did not address the amount of arithmetic needed for a direct calculation from the equilibrium equation.

The method of virtual work gives the same result with far less effort. If the correct mechanism is selected, and the virtual displacement is applied in such a way that the required quantity performs negative work over a unit displacement or unit rotation, the deformed mechanism is the influence line we are looking for.

The mechanisms are not shown separately. However, a ① shows where in the mechanism a unit displacement or unit rotation was applied.

It is recommended to check the influence line by calculating the value and the sign at a few relevant points from the equilibrium equations.

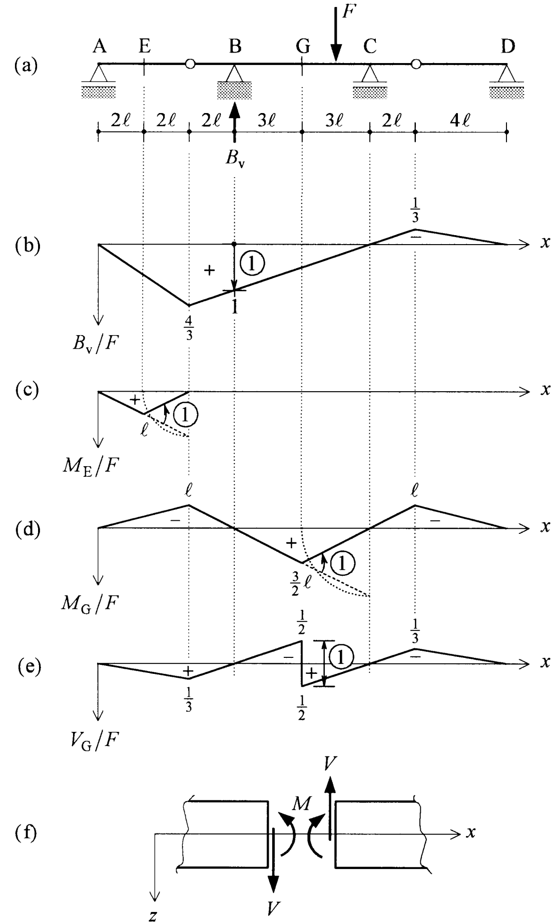


Figure 16.11 (a) Hinged beam with influence lines for (b) the vertical support reaction at B, (c) the bending moment at E, (d) the bending moment at G and (e) the shear force at G; (f) positive directions for bending moment and shear force.

16.3 Working with influence lines

We have found that the method of virtual work provides the easiest way to find influence lines.

In this section we address working with influence lines, and we do not discuss how they are found. Using a number of examples, we show how to determine the force quantity in question (support reaction, section force) using an influence line for a *set of loads* and a *uniformly distributed load*.

Influence lines are often used for determining the *most unfavourable placement of the load*, the placement where the load has the most severe effect on the quantity in question. We also provide a number of examples of this.

16.3.1 Calculating forces using a given influence line

Example 1 – Set of loads

From an influence line, we can read off the influence of a point load with a variable position on a certain quantity (support reaction, section force) at a

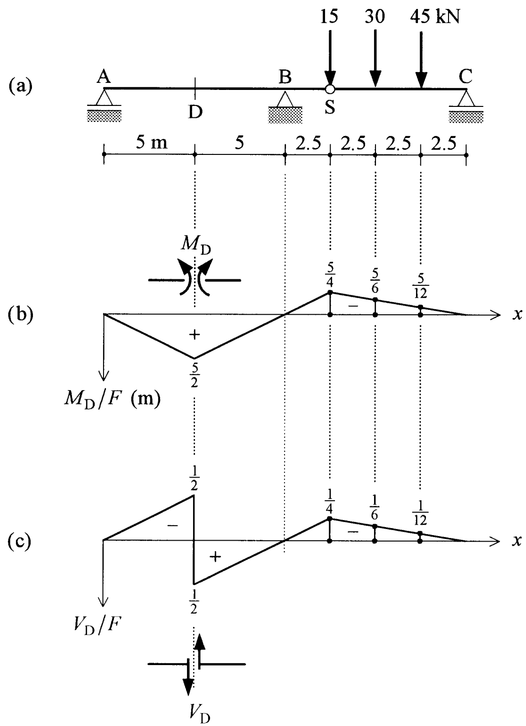


Figure 16.12 (a) Hinged beam with set of loads and the influence lines for (b) the bending moment at D and (c) the shear force at D.

fixed location. We can also say that the influence line gives the variation of a certain quantity due to a movable unit load. The value of the quantity is found at the position of the point load.

Figure 16.12 shows a hinged beam with the influence lines for the bending moment and the shear force at D. The positive directions of M_D and V_D are shown with the influence lines.

The hinged beam is loaded in field BC by a set of forces. The length of the beam, and the positions and magnitudes of the forces are shown in the figure. For each of the point loads, the influence line shows the associated influence quantity at the position of the load. The influence quantity for the bending moment (M_D/F) has the dimension of length. Figure 16.12b therefore includes the values in metres. The influence quantity for the shear force (V_D/F) is dimensionless.

The influence lines give the following for the bending moment:

$$M_D = -\left(\tfrac{5}{4}\text{ m}\right) \times (15\text{ kN}) - \left(\tfrac{5}{6}\text{ m}\right) \times (30\text{ kN}) - \left(\tfrac{5}{12}\text{ m}\right) \times (45\text{ kN})$$
$$= -62.5\text{ kNm}$$

and for the shear force

$$V_D = -\tfrac{1}{4} \times (15\text{ kN}) - \tfrac{1}{6} \times (30\text{ kN}) - \tfrac{1}{12} \times (45\text{ kN}) = -12.5\text{ kN}.$$

The correctness of these values can be checked by considering the equilibrium equations.

Example 2 – Uniformly distributed load

In Figure 16.13a, the beam in the previous example is loaded along its entire length by a uniformly distributed load $q = 80$ kN/m. Here too we will determine the bending moment and the shear force at D.

First we calculate the bending moment. The influence quantity M_D/F, which is a function of x, is hereafter for simplicity denoted by $f(x)$. The contribution dM_D to the bending moment M_D of the distributed load q over a small length dx is found by multiplying the small resulting force $q\,dx$ by the associated value of $f(x)$ of the influence line:

$$dM_D = f(x) \cdot q\,dx.$$

The bending moment at D due to the distributed load between $x = x_1$ and $x = x_2$ is found by summing up all the contributions, or in other words, by integrating:

$$M_D = \int_{x_1}^{x_2} f(x) q\,dx.$$

Since the distributed load is constant here, q can be taken outside the integration symbol:

$$M_D = q \int_{x_1}^{x_2} f(x)\,dx.$$

The integral represents the area of the influence line between x_1 and x_2 (see Figure 16.13b).

The bending moment due to a uniformly distributed load q is therefore equal to the load q, multiplied by the area of the influence line for the part where the load is acting. The signs have to be taken into account when determining the magnitude of the areas.

The bending moment at D due to the uniformly distributed full load is found from the influence line in Figure 16.13b:

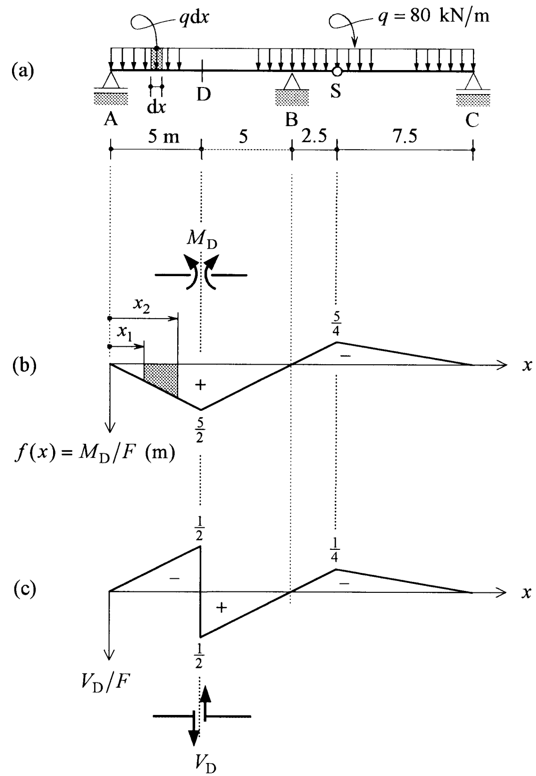


Figure 16.13 (a) Hinged beam with uniformly distributed full load and the influence lines for (b) the bending moment at D and (c) the shear force at D.

$$M_D = \left\{+\tfrac{1}{2} \times (10\text{ m}) \times \left(\tfrac{5}{2}\text{ m}\right) - \tfrac{1}{2} \times (10\text{ m}) \times \left(\tfrac{5}{4}\text{ m}\right)\right\} \times (80\text{ kN/m})$$
$$= 500\text{ kNm}.$$

In the same way, the shear force at D is found from the area of the influence line in Figure 16.13c. Since it is immediately clear that the total area of the influence line over field AB is zero, we have only to determine the area over field BC:

$$V_D = -\tfrac{1}{2} \times (10\text{ m}) \times \tfrac{1}{4} \times (80\text{ kN/m}) = -100\text{ kN}.$$

16.3.2 Most unfavourable placements of loads

If the load consists of a single point load, the influence line shows directly where the force has the maximum effect. Also for a uniformly distributed load the most unfavourable placement is rather easy to find. With a set of loads, however, this is no longer the case, and several positions will have to be investigated.

Here we look at a case in which we can rather easily calculate the most unfavourable position for a set of loads. The second example relates to a uniformly distributed load.

Figure 16.14 shows the influence line for the bending moment at C for beam AB. The maximum value is m. The beam is subject to a set of loads. Part of the loads is in field AC; another part is in field CB. If the set of loads moves over a distance Δx, the ordinate of the influence line at the position of the loads in field AC increases by

$$\Delta x \tan\alpha = \Delta x \frac{m}{a}.$$

Therefore, the bending moment at C increases by

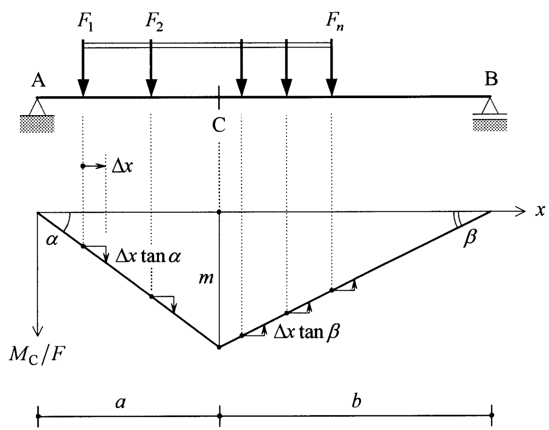


Figure 16.14 Beam with the influence line for the bending moment at C, with a set of loads that moves over a distance Δx.

$$\Delta M_{\mathrm{C}} = \sum_{\mathrm{AC}} F_i \cdot \Delta x \frac{m}{a} = R^{(\mathrm{AC})} \frac{m}{a} \Delta x.$$

$R^{(\mathrm{AC})}$ is the resultant of all the loads in field AC.

At the same time, the ordinate of the influence line at the position of each of the loads in field CB decreases by

$$\Delta x \tan \beta = \Delta x \frac{m}{b}.$$

This changes the bending moment at C by

$$\Delta M_{\mathrm{C}} = -\sum_{\mathrm{CB}} F_i \cdot \Delta x \frac{m}{b} = -R^{(\mathrm{CB})} \frac{m}{b} \Delta x.$$

$R^{(\mathrm{CB})}$ is the resultant of all the loads in field CB.

Due to the displacement of the set of loads by a distance Δx the total increase of the bending moment at C is

$$\Delta M_{\mathrm{C}} = \left(\frac{R^{(\mathrm{AC})}}{a} - \frac{R^{(\mathrm{CB})}}{b} \right) \cdot m \Delta x = (q^{(\mathrm{AC})} - q^{(\mathrm{CB})}) \cdot m \Delta x.$$

Here, $q^{(\mathrm{AC})} = R^{(\mathrm{AC})}/a$ and $q^{(\mathrm{CB})} = R^{(\mathrm{CB})}/b$ can be seen as the average loads in fields AC and CB respectively. As long as $q^{(\mathrm{AC})}$ is larger than $q^{(\mathrm{CB})}$ the bending moment increases if the set of loads moves in the positive x direction.

If one of the loads passes the location C, the average field loads change. The bending moment M_{C} is a maximum for that load at C for which $(q^{(\mathrm{AC})} - q^{(\mathrm{CB})})$ is zero or changes sign.

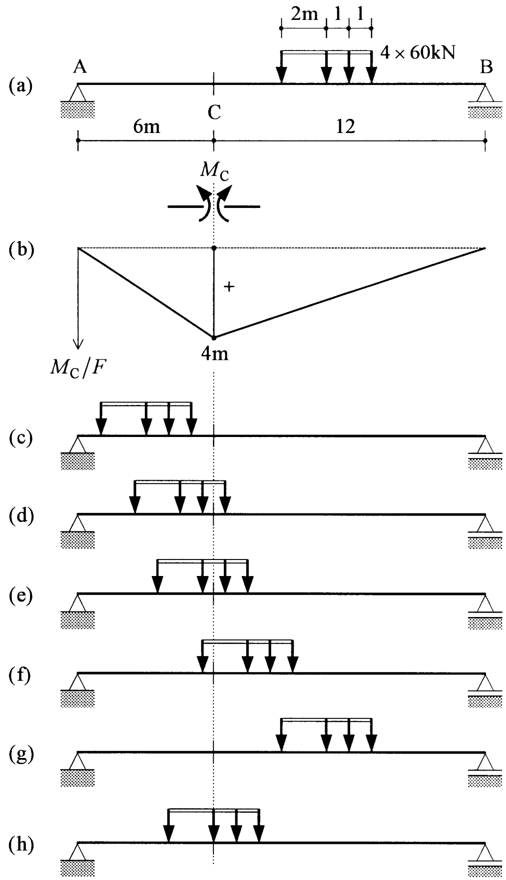


Figure 16.15 (a) Beam with a set of loads; (b) influence line for the bending moment at C; (c) to (g) positions of the set of loads to be investigated; (h) most unfavourable position of the set of loads.

Example 1 – Most unfavourable placement of a set of loads
The beam in Figure 16.15a carries a moving set of loads consisting of four forces of 60 kN for which the mutual distances are shown in the figure. Figure 16.15b shows the influence line for the bending moment at C. We will calculate the maximum bending moment at C due to the set of loads.

Solution:
The maximum bending moment at C occurs when the placement of the set of loads is such that the average loads of fields AC and CB are zero or change sign. Figures 16.15c to 16.15g show five consecutive positions by moving a load from field AC to BC. The average field loads are shown in Table 16.1 for each of the positions.

For the change from position (e) to position (f) in Figure 16.15, a change in sign occurs in $(q^{(AC)} - q^{(CB)})$. Figure 16.15h therefore gives the most unfavourable position of the set of loads in relation to the bending moment at C.

Table 16.1

Position load	$q^{(AC)}$ (kN/m)	$q^{(CB)}$ kN/m)	$q^{(AC)} - q^{(CB)}$
Figure 16.15c	$\frac{4\times60}{6} = 40$	0	> 0
Figure 16.15d	$\frac{3\times60}{6} = 30$	$\frac{1\times60}{12} = 5$	> 0
Figure 16.15e	$\frac{2\times60}{6} = 20$	$\frac{2\times60}{12} = 10$	> 0
Figure 16.15f	$\frac{1\times60}{6} = 10$	$\frac{3\times60}{12} = 15$	< 0
Figure 16.15g	0	$\frac{4\times60}{12} = 20$	< 0

The maximum bending moment is

$$M_{\mathrm{C}} = \left(\tfrac{8}{3} + 4\tfrac{44}{12} + \tfrac{40}{12}\right)\,(\mathrm{m}) \times (60\ \mathrm{kN}) = 820\ \mathrm{kNm}.$$

The first term between brackets includes the influence values to be found from the influence line at the position of each of the point loads.

Example 2 – Most unfavourable placement of a uniformly distributed load

The hinged beam in Figure 16.16a is a model of a bridge. The traffic load on the bridge, consisting of a large number of vehicles in a line, is modelled as a uniformly distributed moving load of 90 kN/m. Figure 16.16b shows the influence line for the shear force at E. We will determine the maximum shear force at E in an absolute sense.

Solution:

Since gaps may appear in traffic jams, the uniformly distributed load is sometimes interrupted. The maximum positive shear force occurs when the uniformly distributed load is present in all fields where the influence line is positive as indicated in Figure 16.16c. This gives

$$V_{\mathrm{E}} = \left\{\tfrac{1}{2} \times (30\ \mathrm{m}) \times \left(+\tfrac{1}{3}\right) + \tfrac{1}{2} \times (20\ \mathrm{m}) \times \left(+\tfrac{2}{3}\right)\right\} \times (90\ \mathrm{kN/m})$$
$$= 1050\ \mathrm{kN}.$$

The maximum negative shear force is found for the load in Figure 16.16d:

$$V_{\mathrm{E}} = \left\{\tfrac{1}{2} \times (10\ \mathrm{m}) \times \left(-\tfrac{1}{3}\right) + \tfrac{1}{2} \times (30\ \mathrm{m}) \times \left(-\tfrac{1}{3}\right)\right\} \times (90\ \mathrm{kN/m})$$
$$= -600\ \mathrm{kN}.$$

The maximum shear force in an absolute sense is therefore 1050 kN and occurs with the load shown in Figure 16.16c.

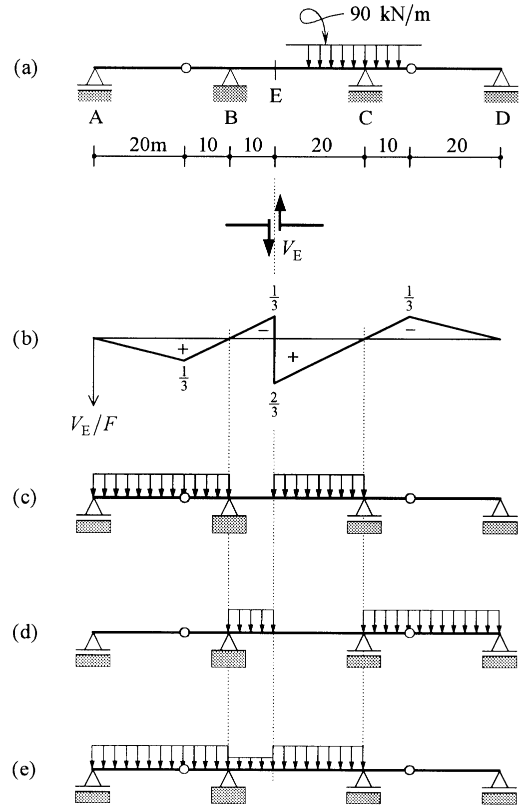


Figure 16.16 (a) Bridge modelled as a hinged beam with uniformly distributed movable load; (b) influence line for the shear force at E; (c) the load that causes the maximum positive shear force; (d) the load that causes the maximum negative shear force; (e) for railway bridges, trains are an uninterrupted load.

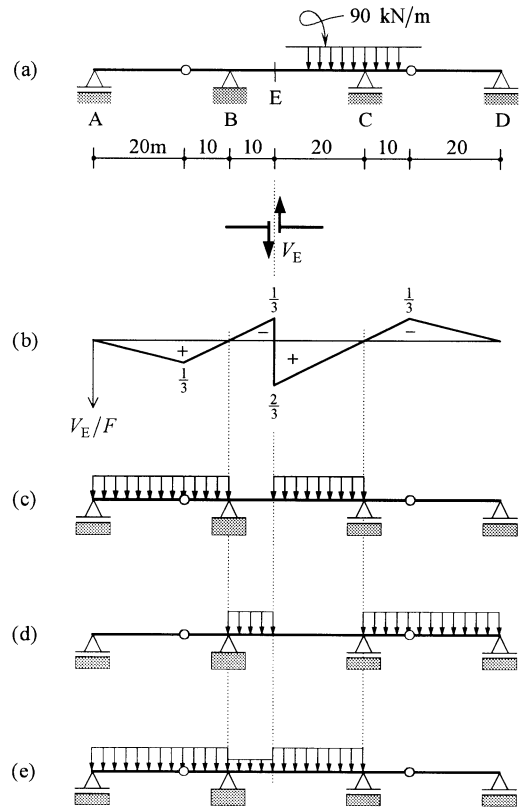


Figure 16.16 (a) Bridge modelled as a hinged beam with uniformly distributed movable load; (b) influence line for the shear force at E; (c) the load that causes the maximum positive shear force; (d) the load that causes the maximum negative shear force; (e) for railway bridges, trains are an uninterrupted load.

The fact that the positive shear force is predominant is clear from the influence line: the positive area under the influence line is larger than the negative area.

In contrast to bridges for standard traffic, loads for trains on railway bridges are uninterrupted loads, that may consist partly of empty carriages, for which one then assumes a lesser load. For a railway bridge, the maximum shear force at E in an absolute sense occurs for the load given in Figure 16.16e. Assume the uniformly distributed load is again 90 kN/m, but now with a minimum of 15 kN/m for the empty carriages. The maximum shear force is then

$$\begin{aligned} V_E &= \left\{\tfrac{1}{2} \times (30\text{ m}) \times \left(+\tfrac{1}{3}\right) + \tfrac{1}{2} \times (20\text{ m}) \times \left(+\tfrac{2}{3}\right)\right\} \times (90\text{ kN/m}) + \\ &\quad + \tfrac{1}{2} \times (10\text{ m}) \times \left(-\tfrac{1}{3}\right) \times (15\text{ kN/m}) \\ &= 1025\text{ kN}. \end{aligned}$$

It is up to the reader to check that neither a distributed load over the entire length AD nor a distributed load over BD are predominant.

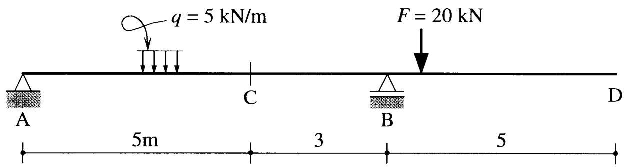
16.4 Problems

General comment: If the influence line of a quantity X is requested, the positive direction of this quantity must be stated beforehand. You must then indicate the corresponding direction of X on the influence line by means of plus and minus signs (or deformation symbols).

Influence lines using virtual work (Sections 16.2 and 16.3)

16.1: 1–5 A point load $F = 20$ kN and a uniformly distributed load $q = 5$ kN/m can move across a beam. The same questions are asked for the following five quantities X:

1. $X =$ bending moment at B.
2. $X =$ bending moment at C.
3. $X =$ shear force directly to the left of B.
4. $X =$ shear force directly to the right of B.
5. $X =$ shear force at C.

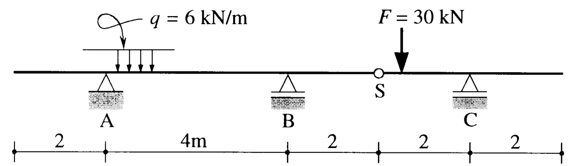


Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X when the uniformly distributed load q acts only on CD.
e. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
f. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.2: 1–6 A point load $F = 30$ kN and a uniformly distributed load $q = 6$ kN/m can move over the hinged cantilever beam. The same questions are asked for six different quantities X:

1. $X =$ vertical support reaction at A.
2. $X =$ vertical support reaction at B.
3. $X =$ shear force directly to the left of B.
4. $X =$ shear force directly to the right of B.
5. $X =$ bending moment in the middle of AB.
6. $X =$ bending moment at B.

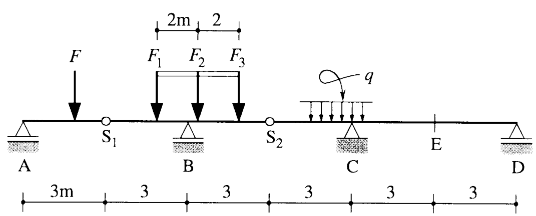


Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
e. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
f. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.3: 1–9 A point load F, a set of loads F_1; F_2; F_3 and a uniformly distributed load q can move across the hinged beam in Figure 16.3. Use $F = 40$ kN, $F_1 = 20$ kN, $F_2 = 50$ kN, $F_3 = 30$ kN and $q = 8$ kN/m. The same questions are asked for nine different quantities X:

1. $X =$ vertical support reaction at B.
2. $X =$ vertical support reaction at C.
3. $X =$ vertical support reaction at D.
4. $X =$ bending moment at B.
5. $X =$ bending moment at C.
6. $X =$ bending moment at E.
7. $X =$ shear force directly to the left of C.
8. $X =$ shear force directly to the right of C.
9. $X =$ shear force at S_2.

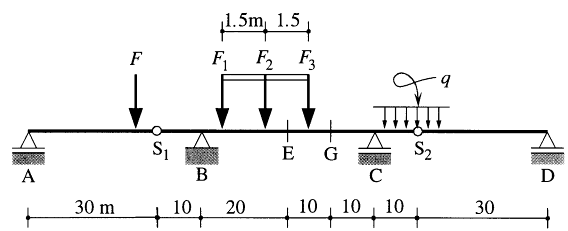


Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X due to the set of loads when F_1 is at S_1.
e. Using the influence line, determine the value of X due to the set of loads when F_2 is at S_2.
f. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
g. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
h. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.4: 1–10 A point load F, a set of loads F_1; F_2; F_3 and a uniformly distributed load q can move across the hinged beam in Figure 16.4. Use $F = F_1 = F_2 = F_3 = 200$ kN and $q = 24$ kN/m. The same questions are asked for 10 different quantities X:

1. $X =$ vertical support reaction at A.
2. $X =$ vertical support reaction at B.
3. $X =$ bending moment at B.
4. $X =$ bending moment at E.
5. $X =$ bending moment at G.
6. $X =$ shear force at S_1.
7. $X =$ shear force at E.
8. $X =$ shear force at G.
9. $X =$ shear force directly to the left of C.
10. $X =$ shear force directly to the right of C.



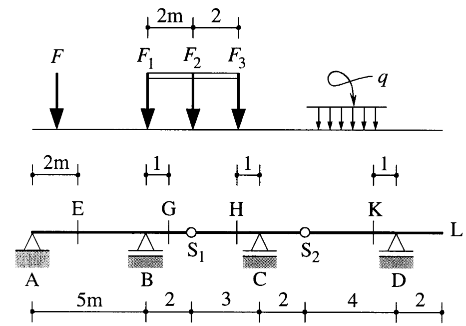
Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.

d. Using the influence line, determine the value of X due to the set of loads when F_1 is at S_1.
e. Using the influence line, determine the value of X due to the set of loads when F_2 is at E.
f. Using the influence line, determine the value of X when the uniformly distributed load q acts only on BD.
g. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
h. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
i. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.5: 1–18 A point load F, a set of loads F_1; F_2; F_3 and a uniformly distributed load q can move across the hinged beam in Figure 16.5. Use $F = 80$ kN, $F_1 = 30$ kN, $F_2 = 50$ kN, $F_3 = 20$ kN and $q = 18$ kN/m. The same questions are asked for 18 different quantities X:

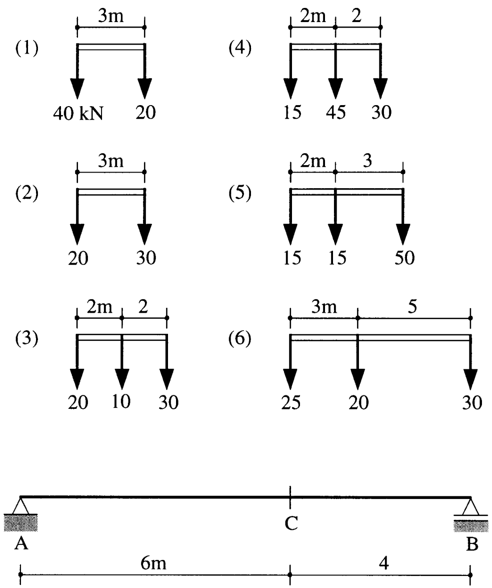
1. X = vertical support reaction at A.
2. X = vertical support reaction at B.
3. X = vertical support reaction at C.
4. X = vertical support reaction at D.
5. X = shear force at E.
6. X = shear force at G.
7. X = shear force at S_1.
8. X = shear force at H.
9. X = shear force directly to the left of C.
10. X = shear force directly to the right of C.
11. X = shear force at S_2.
12. X = shear force at K.
13. X = bending moment at B.
14. X = bending moment at C.
15. X = bending moment at E.
16. X = bending moment at G.
17. X = bending moment at H.
18. X = bending moment at K.



Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X due to the set of loads when F_1 is at E.
e. Using the influence line, determine the value of X due to the set of loads when F_2 is at G.
f. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
g. Using the influence line, determine the value of due to a uniformly distributed load q between S_1 and S_2 and between K and L.
h. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
i. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

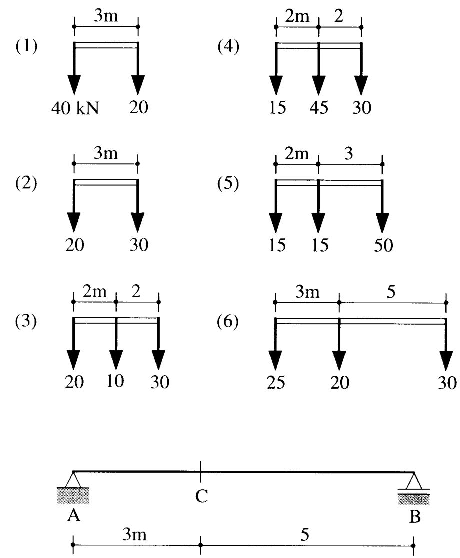
16.6: 1–6 Given the same simply supported beam and six different sets of loads.



Questions:

a. Determine the influence line for the support reaction at A.
b. Determine the maximum value of the support reaction at A due to the set of loads.
c. Determine the influence line for bending moment at C.
d. Determine the maximum value of the bending moment at C due to the set of loads.
e. Determine the influence line for the shear force at C.
f. Determine the maximum value for the shear force at C due to the set of loads.

16.7: 1–6 Given the same simply supported beam and six different sets of loads. Answer the same questions as in problem 16.6.



Index

C

G

H

M

N

O

P

T

U

Latin capitals

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
A	Work	Nm (J)
A	Area	m^2
A_h	Horizontal component of the support reaction at A[2]	N
A_m	Fixed-end moment at A[2]	Nm
A_v	Vertical component of the support reaction at A[2]	N
C_1; C_2	Integration constants	–
E_v	Strain energy	Nm (J)
F	Concentrated force, point load	N
$\vec{F}$	Force vector	N
F_a	Force along line of action a[3]	N
F_h	Horizontal component of the force F	N
F_v	Vertical component of the force F	N
F_x; F_y; F_z	Components of the force F	N
F_p	Prestressing force	N
G	Gravitational load	N
H	Horizontal component of the cable force	N

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
M	Bending moment	Nm
M_t	Torsional moment	Nm
M_y	Bending moment in the xy plane	Nm
M_z	Bending moment in the xz plane	Nm
N	Normal force	N
R	Resulting force, resultant	N
T	Concentrated couple (external moment)	Nm
$\vec{T}$	Moment vector of a couple	Nm
T_x	Moment about the x axis	Nm
T_y	Moment about the y axis	Nm
T_z	Moment about the z axis	Nm
V	Shear force	N
V	Vertical component of the cable force	N
V_y	Shear force in the xy plane	N
V_z	Shear force in the xz plane	N

[1] Expressed in the basic units.
[2] The kernel A refers to the location and could therefore also be B, C, etc.
[3] The index a refers to the line of action and could therefore also be b, c, etc.

Latin lower case letters

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
a	Distance, length	m
a	Acceleration	m/s^2
b	Distance, width	m
d	Diameter	m
d	Depth	m
$\vec{d}$	(non-standardised) direction vector	–
d_x; d_y; d_z	Components of the direction vector	–
e	Eccentricity	m
e_p	Eccentricity of the prestressing force	m
$\vec{e}$	Unit vector	–
e_x; e_y; e_z	Components of the unit vector	–
e_z	z coordinate of the centre of force in the cross-section	m
g	Distributed permanent load	N/m
g	Gravitational acceleration	m/s^2
g	Gravitational field strength	N/kg
h	Height	m
ℓ	Span, length	m
m	Mass	kg
$\vec{n}$	Unit normal vector	–

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
$n_x; n_y; n_z$	Components of the unit normal vector	–
p	Surface load (pressure, stress)	$\mathrm{N/m^2}$ (Pa)
p	Rise or sag of the parabola at midspan	m
p_k	Distance between parabola and chord at midspan	m
$\vec{p}$	Stress vector	$\mathrm{N/m^2}$ (Pa)
$p_x; p_y; p_z$	Components of the stress vector	$\mathrm{N/m^2}$ (Pa)
q	Line load	N/m
q	Distributed variable load	N/m
$\hat{q}$	Top value of the line load	N/m
q_x	Distributed load in the x direction (the direction of the member axis)	N/m
q_z	Distributed load in the z direction (a direction normal to the member axis)	N/m
r	Radius	m
$\vec{r}$	Location vector	m
$r_x; r_y; r_z$	Components of the location vector	m
s	Path length	m

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
t	Time	s
u	Displacement in the x direction	m
$\vec{u}$	Displacement vector	m
$u_x; u_y; u_z$	Components of the displacement vector	m
v	Velocity	m/s
v	Displacement in the y direction	m
w	Displacement in the z direction	m
x	Rectangular coordinate	m
y	Rectangular coordinate	m
z	Rectangular coordinate	m

[1]Expressed in the basic units.

Greek letters

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
α	Angle	rad
β	Angle	rad
γ	Angle	rad
γ	Specific weight	$\mathrm{N/m^3}$
ϑ	Angle, change in angle due to rotation	rad
ρ	(mass) density, specific mass	$\mathrm{kg/m^3}$

Quantity		SI unit
Symbol	**Name**	**Symbol**[1]
σ	Stress, normal stress	$\mathrm{N/m^2}$ (Pa)
σ_{ij}	Stress on a plane with the normal in the i direction ($i = x, y, z$), and acting in the j direction ($j = x, y, z$); normal stress when $i = j$ and shear stress when $i \neq j$	$\mathrm{N/m^2}$ (Pa)
τ	Shear stress	$\mathrm{N/m^2}$ (Pa)
φ	Angle, change in angle due to rotation	rad
φ_x	Rotation about the x axis	rad
φ_y	Rotation about the y axis	rad
φ_z	Rotation about the z axis	rad

[1]Expressed in basic units.

A number of other signs and sign combinations

Δ	Change, increase
ΔM	Increase in M
$\sum$	Summation symbol
$\sum T\|\mathrm{B}$	Moment sum with respect to point B
δ	Variation symbol
δA	Virtual work
δu	Virtual displacement
$\delta\varphi$	Virtual rotation
F_A^BC	Force F at A (sub-index) on body BC (upper index)