DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF AEROSPACE ENGINEERING

Course	:	Stochastic Aerospace Systems (ae4-304)
Date	:	April 7, 2011 from 9:00 until 12:00 hr
Remarks	:	Write your name, initials and student number on your work Answer all questions in English or Dutch and mark all pages with your name.

The exam consists of 9 questions, a correctly answered question is rewarded with 10 points. The final mark is then:

$$1 \leq 1 + 9 * (0 \ldots 10)/10 \leq 10$$

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1. **CORRELATION FUNCTION** (10 points)

   Consider the output $\bar{y}(t)$ of a first order system to a zero-mean, Gaussian white noise signal $\bar{w}(t)$. The first order system dynamics are given by the following transfer function:

   $$H_w^y(s) = \frac{1}{1 + \gamma s}$$

   with time constant γ.

   The auto-correlation of the output signal $\bar{y}(t)$, $K_{\bar{y}\bar{y}}(\tau)$ is illustrated in Figure 1, for three values of the time constant γ (γ_1 (a), γ_2 (b), and γ_3 (c)).

   Which of the following statements is true? Explain your answer.

   [a] $\gamma_1 > \gamma_2 > \gamma_3$

   [b] $\gamma_1 < \gamma_2 < \gamma_3$

   [c] None of the auto-correlation functions represents the response of a first order system to a white noise input signal.

2. **FOURIER TRANSFORM** (10 points)

   Prove Parceval's theorem:

   $$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

3. **PROBABILITY DENSITY FUNCTION** (10 points)

   The random variable $\bar{x}$ has a probability density function $f_{\bar{x}}(x)$ as depicted in Figure 2.

   What is the probability $\Pr\{\bar{x} \geq -1\}$?

   [a] 0.125

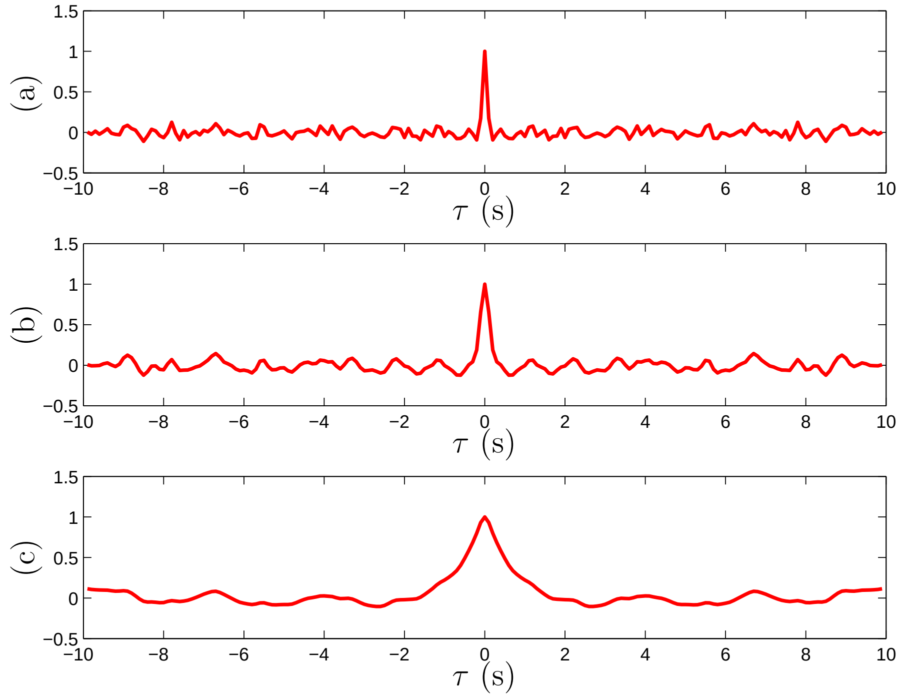

Figure 1: The auto-correlation $K_{\bar{y}\bar{y}}(\tau)$ of the output signal $\bar{y}(t)$ of a first order system driven by white noise.

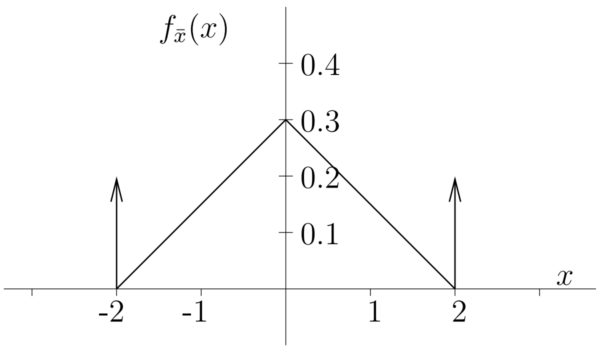

Figure 2: The probability density function $f_{\bar{x}}(x)$ of stochastic variable $\bar{x}$.

[b] 0.275

[c] 0.725

[d] 0.750

[e] 0.875

[f] Not enough data available

4. FOURIER TRANSFORM (10 points)

Given a signal $x(t)$. This signal is Fourier transformed, resulting in $X(\omega)$. The real part of $X(\omega)$ is zero, the imaginary part of $X(\omega)$ is non-zero.

Which of the following statements is true? Explain your answer.

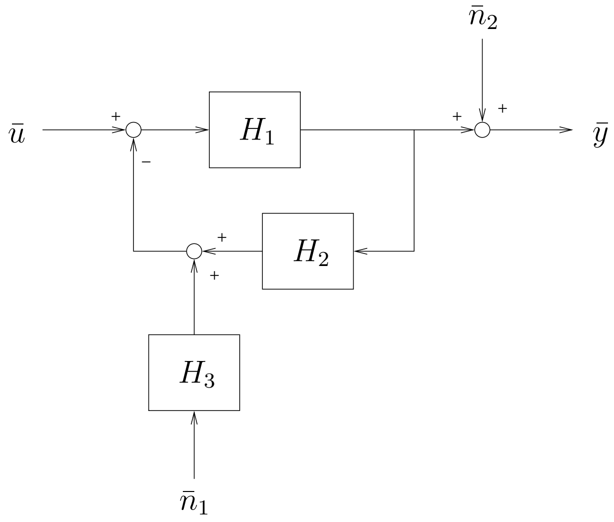

Figure 3: Closed loop system configuration.

[a] $x(t)$ is odd.

[b] $x(t)$ is even.

[c] $x(t)$ is not odd, not even.

[d] Without more information, one cannot determine whether any of the former answers are correct.

5. SYSTEM (10 points)

Given an closed loop system configuration shown in Figure 3. In this figure, H_1, H_2 and H_3 are Linear Time-Invariant systems.

Signals $\bar{n}_1$ and $\bar{n}_2$ are zero-mean, uncorrelated white noise with intensities N_1 and N_2, respectively. Signal $\bar{u}$ is an arbitrary zero-mean input signal that is uncorrelated with $\bar{n}_1$ and $\bar{n}_2$.

Derive an expression that relates the Power Spectral Density of the output signal $\bar{y}$ to the PSDs of all input signals ($\bar{u}$, $\bar{n}_1$, $\bar{n}_2$) in the closed loop.

6. DISCRETE FOURIER TRANSFORM (10 points)

Consider two time-domain signals, sampled using the discretization time $\triangle t$, $x[n] = x[n \triangle t]$ and $y[n] = y[n \triangle t]$. The number of samples in both arrays is N.

[a] Given the (auto) circular covariance function,

$$C_{xx}[r] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n+r]$$

with r the separation sample, then prove that the (auto) Periodogram $I_{xx}[k]$ equals

$$I_{xx}[k] = \frac{1}{N} X[-k]X[k] \tag{1}$$

by Discrete Fourier Transforming the (auto) circular covariance function $C_{xx}[r]$ according to

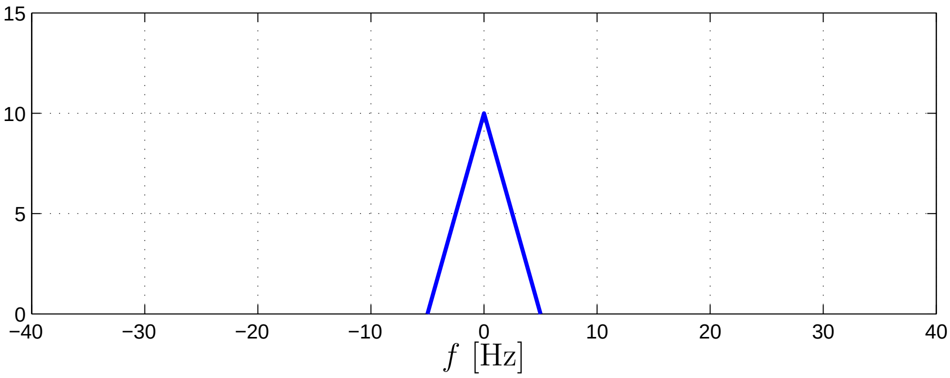

Figure 4: CT Fourier transform of CT signal $y(t)$.

$$I_{xx}[k] = \sum_{r=0}^{N-1} C_{xx}[r]e^{-j\frac{2\pi kr}{N}} \tag{2}$$

with in Eq. (1) $X[k]$ the DFT of $x[n]$, and in Eqs. (1) and (2) k the frequency counter.

[b] Prove/explain that the auto Periodogram $I_{xx}[k]$ is real-valued for all k.

[c] Now consider the cross Periodogram $I_{xy}[k]$ with

$$I_{xy}[k] = \frac{1}{N}X[-k]Y[k] \tag{3}$$

Prove that the cross Periodogram $I_{xy}[k]$ is complex-valued for all k.

[d] Briefly explain how the cross Periodogram $I_{xy}[k]$ may be used in practical applications such as system-identification in the frequency-domain (that is, the estimation of the frequency response function $H[k] \approx H(\omega)$ with ω the circular frequency (in rad/s)).

[e] What is the definition of the discrete frequency array ω_k? Use the frequency counter k, the number of samples N and the discretization time $\triangle t$ for the definition.

7. SAMPLING (10 points)

A continuous-time (CT) signal $y(t)$ is sampled, with a sampling frequency of 20 Hz, resulting in a discrete-time (DT) signal $y[k]$.

The original CT Fourier transform of the signal, $Y(\omega)$, is illustrated in Figure 4.

Which of the following statements is true? Please explain your answer.

[a] The CT Fourier transform of the discretized signal $y[k]$ is shown in Figure 5(a).

[b] The CT Fourier transform of the discretized signal $y[k]$ is shown in Figure 5(b).

[c] The CT Fourier transform of the discretized signal $y[k]$ is shown in Figure 5(c).

[d] None of the above is correct.

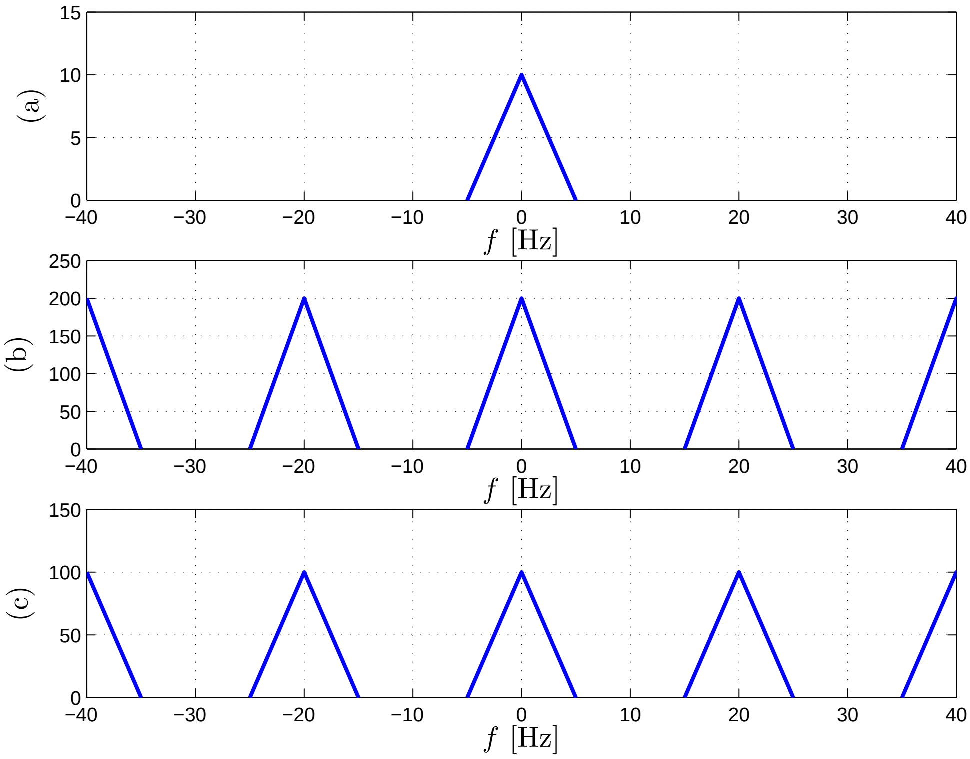

Figure 5: Three possible CT Fourier transforms of DT signal $y[k]$.

8. LYAPUNOV (10 points)

Given a Linear Time-Invariant system with transfer function $H(s)$:

$$H(s) = \frac{1}{(1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2)}.$$

This system is driven by a signal that is zero for $t < 0$ and equals zero-mean white noise $\bar{w}$ for $t > 0$ (the white noise intensity W equals 1). The system is at rest at $t = 0$.

Prove that the "steady state" variance of the output signal $\bar{y}$, $\sigma_{\bar{y}}^2$ equals $\frac{\omega_0}{4\zeta}$.

> Tip: create the system state-space description (A, B, C, D), then solve the Lyapunov equation $AC_{xx,ss} + C_{xx,ss}A^T + BWB^T = 0$, with $C_{xx,ss}$ the steady-state covariance of the state equation, and then compute $C_{yy,ss}$.

9. SPECTRUM (10 points)

Given an LTI system, the transfer function of which equals:

$$H(s) = \frac{1}{1 + \frac{2\zeta}{\omega_0}s + \frac{1}{\omega_o^2}s^2},$$

with $\omega_0 = 1$ rad/s and ζ=0.7.

This system is driven by a white noise input signal $\bar{w}$ that has intensity $W = 2$,

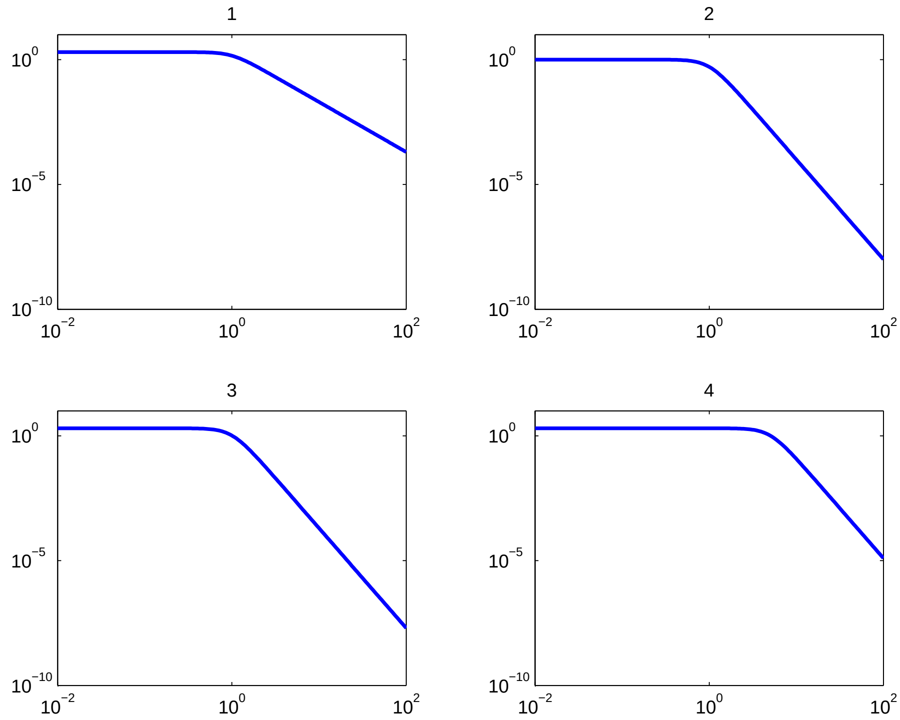

Figure 6: Power spectral densities $S_{\bar{y}\bar{y}}(\omega)$.

resulting in an output signal $\bar{y}$.

Figure 6 shows four possible Power Spectral Densities $S_{\bar{y}\bar{y}}(\omega)$.

Which of the following statements is true? Explain your answer!

[**a**] PSD (1) is the correct spectrum.

[**b**] PSD (2) is the correct spectrum.

[**c**] PSD (3) is the correct spectrum.

[**d**] PSD (4) is the correct spectrum.

[**e**] None of the spectra in Figure 6 is correct.