DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF AEROSPACE ENGINEERING

Course	:	Stochastic Aerospace Systems (ae4-304)
Date	:	April 10, 2012 from 14:00 until 17:00 hr
Remarks	:	Write your name, initials and student number on your work Answer all questions in English or Dutch and mark all pages with your name.

The exam consists of 9 questions, a correctly answered question is rewarded with 10 points. The final mark is then:

$$1 \leq 1 + 9 * (0 \ldots 10)/10 \leq 10$$

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1. PRODUCT, COVARIANCE, CORRELATION (10 points)

For each of the following statements, check whether it is true. Explain your answer.

[a] $R_{\bar{x}\bar{y}}(\tau) = R_{\bar{y}\bar{x}}(-\tau)$

[b] $C_{\bar{x}\bar{y}}(\tau) = C_{\bar{y}\bar{x}}(-\tau)$

[c] $K_{\bar{x}\bar{x}}(\tau) = K_{\bar{x}\bar{x}}(-\tau)$

[d] $K_{\bar{x}\bar{x}}(0) = 1$

2. FOURIER SERIES (10 points)

Prove the Fourier Series (FS) expansion.

That is, prove mathematically that a periodic signal $\bar{x}$(t) can be approximated by the Fourier series expansion $\tilde{\bar{x}}(t)$ (as in the formula page) with coefficients a_k and b_k (as in the formula page). In your proof, use the fact that all basic cosine and sine functions (with frequencies an integer number times of the fundamental frequency ω_0) are orthogonal:

$$\int_{t_0}^{t_0+T} \sin k\omega_0 t \cos \ell\omega_0 t \, dt = 0$$

$$\int_{t_0}^{t_0+T} \sin k\omega_0 t \sin \ell\omega_0 t \, dt = \begin{cases} 0 \text{ if } k \neq \ell \\ \frac{T}{2} \text{ if } k = \ell \end{cases}$$

$$\int_{t_0}^{t_0+T} \cos k\omega_0 t \cos \ell\omega_0 t \, dt = \begin{cases} 0 \text{ if } k \neq \ell \\ \frac{T}{2} \text{ if } k = \ell \end{cases}$$

3. FOURIER TRANSFORM (10 points)

Prove that a convolution in the time domain becomes a multiplication in the frequency domain.

I.e. when: $z(t) = x(t) \star y(t)$

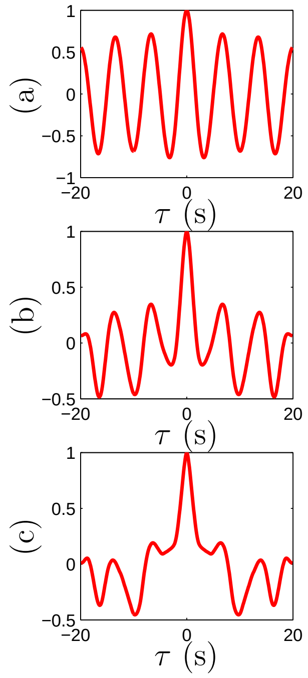

Figure 1: The auto-correlation $K_{\bar{y}\bar{y}}(\tau)$ of the output signal $\bar{y}(t)$ of a second order system driven by white noise.

then: $$Z(\omega) = X(\omega)Y(\omega),$$

with $\star$ the convolution operator.

4. CORRELATION FUNCTION (5 points)

Consider the output $\bar{y}(t)$ of a second order system to a zero-mean, Gaussian white noise signal $\bar{w}(t)$. The second order system dynamics are given by the following transfer function:

$$H_w^y(s) = \frac{1}{1 + 2\frac{\zeta}{\omega_n}s + \frac{1}{\omega_n^2}s^2}$$

with ζ the damping and ω_n the natural frequency of the second order system, respectively.

The auto-correlation of the output signal $\bar{y}(t)$, $K_{\bar{y}\bar{y}}(\tau)$ is illustrated in Figure 1, for three values of the damping ζ (ζ_1 (a), ζ_2 (b), and ζ_3 (c)) and a natural frequency ω_n of 1.0 rad/s.

Which of the following statements is true? Explain your answer.

[a] $\zeta_1 > \zeta_2 > \zeta_3$

[b] $\zeta_1 < \zeta_2 < \zeta_3$

[c] None of the auto-correlation functions represents the response of a second order system to a white noise input signal.

5. DISCRETE FOURIER TRANSFORM (15 points)

Consider two time-domain signals, sampled using the discretization time $\triangle t$, $x[n] = x(n \triangle t)$ and $y[n] = y(n \triangle t)$. The number of samples in both arrays is N.

[a] Given the (auto) circular covariance function,

$$C_{xx}[r] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n+r],$$

with r the separation sample, then prove that the (auto) Periodogram $I_{xx}[k]$ equals:

$$I_{xx}[k] = \frac{1}{N} X[-k]X[k], \tag{1}$$

by Discrete Fourier Transforming the (auto) circular covariance function $C_{xx}[r]$ according to:

$$I_{xx}[k] = \sum_{r=0}^{N-1} C_{xx}[r] e^{-j\frac{2\pi k r}{N}}, \tag{2}$$

with in Eq. (1) $X[k]$ the DFT of $x[n]$, and in Eqs. (1) and (2) k the frequency counter.

[b] Prove/explain that the auto Periodogram $I_{xx}[k]$ is real-valued for all k.

[c] Now consider the cross Periodogram $I_{xy}[k]$ with:

$$I_{xy}[k] = \frac{1}{N} X[-k]Y[k] \tag{3}$$

Prove/explain that the cross Periodogram $I_{xy}[k]$ is complex-valued for all k.

[d] Briefly explain how the cross Periodogram $I_{xy}[k]$ may be used in practical applications such as system-identification in the frequency-domain (that is, the estimation of the frequency response function $H[k] \approx H(\omega)$ with ω the radial frequency (in rad/s)).

[e] What is the definition of the discrete frequency array ω_k? Use the frequency counter k, the number of samples N and the discretization time $\triangle t$ for the definition.

6. SAMPLING (10 points)

A continuous-time (CT) signal $y(t)$ is sampled, with a sampling frequency of 5 Hz, resulting in a discrete-time (DT) signal $y[k]$.

The original CT Fourier transform of the signal, $Y(\omega)$, is illustrated in Figure 2.

In a clear drawing, sketch the CT Fourier transform of the discretized signal $y[k]$.

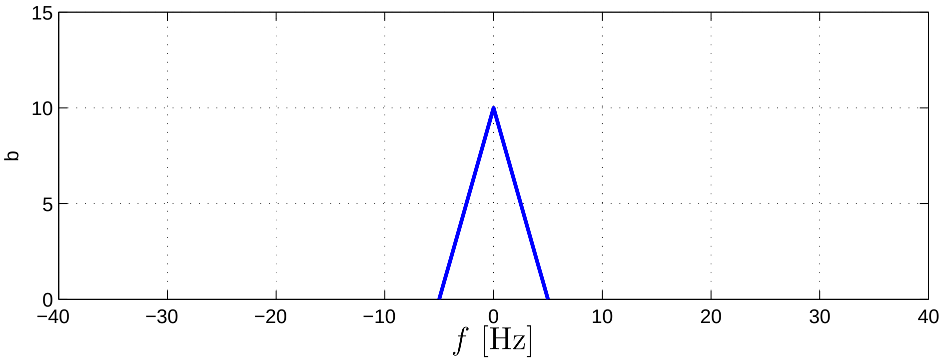

Figure 2: CT Fourier transform of CT signal $y(t)$.

7. SPECTRUM (10 points)

Given an LTI system, the transfer function of which equals:

$H(s) = \frac{K}{1 + \tau s}$,

with $K = 1$ and τ=2.0 seconds.

This system is driven by a white noise input signal $\bar{w}$ that has intensity $W = 2$, resulting in an output signal $\bar{y}$.

Figure 3 shows four possible Power Spectral Densities $S_{\bar{y}\bar{y}}(\omega)$.

Which of the following statements is true? Explain your answer!

[**a**] PSD (1) is the correct spectrum.

[**b**] PSD (2) is the correct spectrum.

[**c**] PSD (3) is the correct spectrum.

[**d**] PSD (4) is the correct spectrum.

[**e**] None of the spectra in Figure 3 is correct.

8. SYSTEM (10 points)

Given an closed loop system configuration shown in Figure 4. In this figure, H_1, H_2 and H_3 are Linear Time-Invariant systems.

Signal $\bar{n}$ is zero-mean noise signal with intensity N. Signal $\bar{u}$ is an arbitrary input signal (zero mean) that is correlated with $\bar{n}$.

Derive an expression that relates the Power Spectral Density of the output signal $\bar{y}$ to the PSDs of all other signals ($\bar{u}$, $\bar{n}$) in the closed loop.

9. LYAPUNOV (10 points)

Given a Linear Time-Invariant system with transfer function $H(s)$:

$$H(s) = \frac{1}{(1 + s\tau_1)(1 + s\tau_2)}.$$

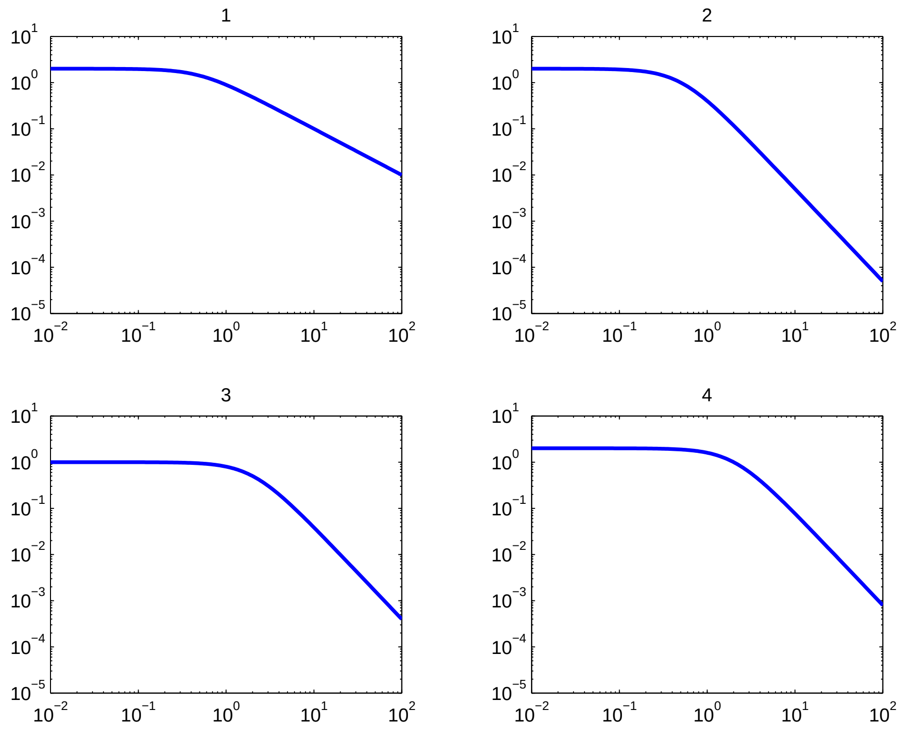

Figure 3: Power spectral densities $S_{\bar{y}\bar{y}}(\omega)$.

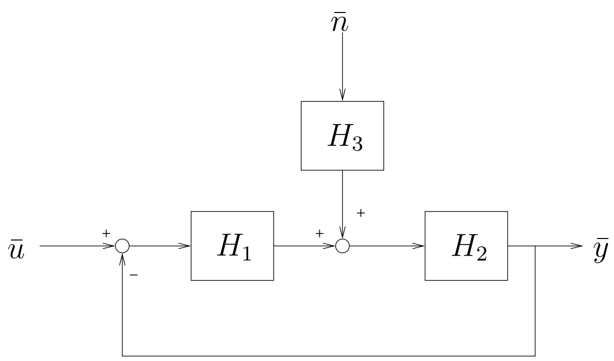

Figure 4: Closed loop system configuration.

This system is driven by a signal that is zero for $t < 0$ and that equals zero-mean white noise $\bar{w}$ for $t > 0$ (the white noise intensity W equals 1). The system is at rest at $t = 0$.

Prove that the "steady state" variance of the output signal $\bar{y}$, $\sigma_{\bar{y}}^2$ equals $\frac{1}{2(\tau_1 + \tau_2)}$.

> Tip: create the system state-space description (A, B, C, D), then solve the Lyapunov equation $AC_{\bar{x}\bar{x},ss} + C_{\bar{x}\bar{x},ss}A^T + BWB^T = 0$, with $C_{\bar{x}\bar{x},ss}$ the steady-state covariance of the state equation, and then compute $C_{\bar{y}\bar{y},ss}$.