DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF AEROSPACE ENGINEERING

Course	:	Stochastic Aerospace Systems (ae4304)
Date	:	January 22, 2016 from 9:00 until 12:00 hr
Remarks	:	Write your name, initials and student number on your work. Answer all questions in English or Dutch and mark all pages with your name.

The exam consists of 9 questions, a correctly answered question is rewarded with the number of points as indicated. To compute your grade, we add up all the points that you gain, add 10, and divide by 10.

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1. ERGODICITY (10 points)

Given a stationary ergodic stochastic process $\bar{x}$, each of whose sample functions is a sine wave of amplitude, period and mean value of 1, or, a realization $\bar{x}(t)$ is given by:

$$\bar{x}(t) = 1 + \sin(2\pi t)$$

The auto-product function $R_{\bar{x}\bar{x}}(\tau)$ of $\bar{x}$ is equal to:

$$R_{\bar{x}\bar{x}}(\tau) = 1 + \tfrac{1}{2}\cos(2\pi\tau) \qquad (*)$$

[a] Compute the auto-covariance function $C_{\bar{x}\bar{x}}(\tau)$

[b] Compute the auto-correlation function $K_{\bar{x}\bar{x}}(\tau)$

[c] Prove relationship (*) above. In your proof, you can use the fact that since $\bar{x}$ is ergodic, $R_{\bar{x}\bar{x}}(\tau)$ can be calculated by computing the time average of a realization $\bar{x}(t)$, rather than taking the ensemble average of $\bar{x}$. That is, use the relationship:

$$R_{\bar{x}\bar{x}}(\tau) = \mathrm{E}\{\bar{x}(t)\bar{x}(t+\tau)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \bar{x}(t)\bar{x}(t+\tau)\mathrm{d}t$$

2. FOURIER SERIES (10 points)

Consider the periodical signal $x(t)$, defined by:

$$x(t) = 1 - 2\sin(4\pi t) + 6\cos(9\pi t + \pi/3 + c),$$

with c a real-valued constant number.

[a] What is the average of this signal?

[b] What is the period T of this signal, in seconds? Explain your answer.

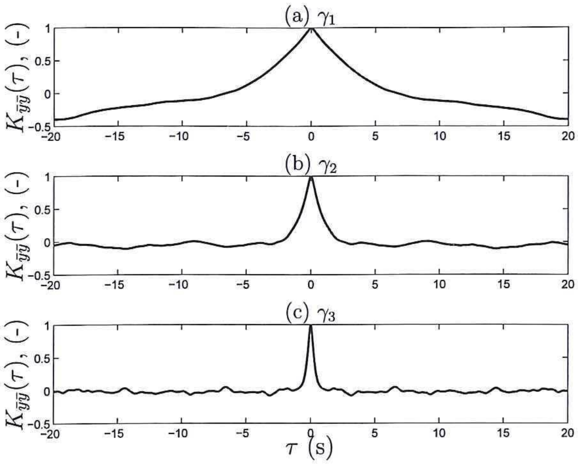

Figure 1: The auto-correlation $K_{\bar{y}\bar{y}}(\tau)$ of the output signal $\bar{y}(t)$ of a second order system driven by white noise.

[c] Assume $c = 0$. Obtain the trigonometric Fourier series coefficients of this waveform.

[d] For which value(s) of c would this Fourier series only consist of sine components? How many seconds of time-shift correspond to that(those) value(s)?

3. FOURIER TRANSFORM (5 points)

Prove the 'scaling' theorem of the Fourier Transform.

That is, when: $y(t) = x(at)$,

with: $X(\omega) = \mathcal{F}\{x(t)\}$ and $Y(\omega) = \mathcal{F}\{y(t)\}$,

then: $Y(\omega) = \frac{1}{|a|}X(\frac{\omega}{a})$,

for an arbitrary real-valued non-zero constant a.

4. CORRELATION FUNCTION (5 points)

Consider the output $\bar{y}(t)$ of a second order system to a zero-mean, Gaussian white noise signal $\bar{w}(t)$. The second order system dynamics are given by the following transfer function:

$$H_w^y(s) = \frac{1}{(1+3s)(1+\gamma s)}$$

with γ the time constant of the second lag component.

The auto-correlation of the output signal $\bar{y}(t)$, $K_{\bar{y}\bar{y}}(\tau)$, is illustrated in Figure 1, for three values of γ: γ_1 (a), γ_2 (b) and γ_3 (c).

Which of the following statements is true? Explain your answer.

[a] None of the auto-correlation functions represents the response of a second order system to a white noise input signal.

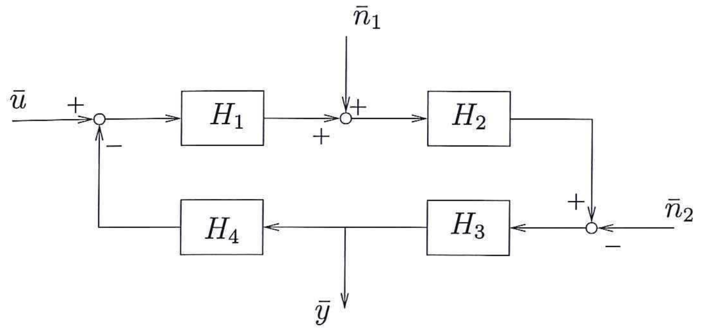

Figure 2: Closed loop system configuration.

[b] $\gamma_1 < \gamma_2 < \gamma_3$

[c] $\gamma_1 > \gamma_2 > \gamma_3$

5. SPECTRUM (5 points)

Given an LTI system, the transfer function of which equals:

$$H(s) = \frac{1}{1 + (2\zeta/\omega_0)s + (1/\omega_o^2)s^2},$$

with $\omega_0 = 5$ rad/s and ζ=0.7. This system is driven by a white noise input signal $\bar{w}(t)$ that has intensity $W = 1$, resulting in an output signal $\bar{y}(t)$.

[a] Make a clear sketch of what the auto-power spectral density of $\bar{y}(t)$, $S_{\bar{y}\bar{y}}(\omega)$, looks like; Use logarithmic horizontal and vertical axes.

[b] In this sketch, indicate with a dash-dotted line what $S_{\bar{y}\bar{y}}(\omega)$ looks like when the noise intensity W increases to 10.

6. SYSTEM (15 points)

Given a closed loop system configuration shown in Figure 2. In this figure, H_1, H_2, H_3 and H_4 are Linear Time-Invariant systems.

Signals $\bar{n}_1$ and $\bar{n}_2$ are zero-mean, uncorrelated white noise with intensities N_1 and N_2, respectively. Signal $\bar{u}$ is an arbitrary zero-mean input signal that is uncorrelated with $\bar{n}_1$ but that *is* correlated with $\bar{n}_2$.

Derive an expression that relates the Power Spectral Density of the output signal $\bar{y}$ to the auto/cross PSDs of all input signals to the closed loop.

7. SAMPLING (15 points)

Given a signal $x(t) = 2\cos(\omega_1 t) + 4\cos(\omega_2 t)$, with $\omega_1 = 3\pi$ rad/s and $\omega_2 = 12\pi$ rad/s.

[a] Compute $X(\omega)$, the Fourier transform of $x(t)$.

[b] What is the minimum sampling frequency in rad/s? Explain your answer.

We sample this signal $x(t)$ using an impulse-train sampling procedure, with sampling frequency 30π rad/s, resulting in the discrete time series $x[n]$.

[c] Make a clear sketch of the continuous-time Fourier transform (CTFT) of $x(n \triangle t)$ (i.e., our 'model' of the samples) from -60π rad/s to 60π rad/s.

Suppose we would conduct a discrete Fourier transform (DFT) on N samples of $x[n]$, the discrete time series that we just obtained.

[d] What would be the frequency resolution of this DFT?

[e] What is the minimum number of samples that we need to prevent DFT leakage? Clearly explain your answer.

[f] In your sketch of subquestion (c) above, clearly indicate at what range of frequencies the DFT provides us a 'view' on the CTFT of $x(n \triangle t)$.

8. DISCRETE FOURIER TRANSFORM (10 points)

[a] Prove that the Discrete Fourier Transform of a constant b equals

$$DFT\{b\} = \sum_{n=0}^{N-1} be^{-j\frac{2\pi kn}{N}} = Nb\delta[k],$$

with $\delta[k]$ the Kronecker delta function ($\delta[k]$ equals 1 for $k = 0$ and it equals 0 for all other values of k).

[b] Prove that the periodogram $I_{yy}[k]$ of the signal $y[n] = ax[n] + b$ equals:

$$I_{yy}[k] = a^2 I_{xx}[k] + (2a\text{Re}\{X[k]\} + Nb)b\delta[k],$$

with:

$$I_{xx}[k] = \frac{1}{N}X[-k]X[k],$$

and Re$\{X[k]\}$ the real part of the Fourier transform of $x[n]$.

9. LYAPUNOV (15 points)

Given a Linear Time-Invariant system with transfer function $H(s)$:

$$H(s) = \frac{(1 + s\tau_1)}{(1 + s\tau_2)(1 + s\tau_3)}$$

This system is driven by a signal that is zero for $t < 0$ and that equals zero-mean white noise $\bar{w}$ for $t > 0$ (the intensity W equals 2). The system is at rest at $t = 0$.

Prove that the "steady state" variance of the output signal $\bar{y}$, $\sigma_{\bar{y}}^2$ equals:

$$\frac{1}{(\tau_2 + \tau_3)}\left(1 + \frac{\tau_1^2}{\tau_2\tau_3}\right).$$

Tip: create the system state-space description (A, B, C, D), then solve the Lyapunov equation $AC_{\bar{x}\bar{x},ss} + C_{\bar{x}\bar{x},ss}A^T + BWB^T = 0$, with $C_{\bar{x}\bar{x},ss}$ the steady-state covariance of the system state vector, and then compute $C_{\bar{y}\bar{y},ss}$.